# Numerical integration of turbulence models

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# K-Epsilon turbulence model

## Differential equations of the k-epsilon model

For the purpose of this paper, we will concentrate on the k-epsilon-turbulence formulation. The model equations are



Here, means the turbulent production rate, and it is determined by



The term  is the norm of the matrix of the velocity gradient.

A similar expression, , is dedicated to turbulent buoyancy effects.

The turbulent viscosity is a function of the turbulent quantities k and epsilon, its quantification is



The given constants are .

## Numerical evolution scheme and time integration of the k-epsilon model

The numerical evolution scheme is



which just arises by replacing the spatial derivatives by its FPM-MLS operators.

For better numerical analysis, we can rewrite this scheme by replacing  by its formal expression together with and, for simplicity, omitting the term 



From system , we derive a singularity formulation, which is either



or



If not both values  and  are zero, we can provide numerical mean values (ref. section 1.3 Analytical evaluation of the mean values of the singular terms)



and



It remains to provide a possibly precise numerical time integration of the scheme where we avoid singularities by using the mean values and . Thus, the numerical evolution scheme is



For the scheme , we can now apply two schemes that guaranty the positivity of the terms  and .

The first scheme is based on semi-implicit first order time stepping, which is



Which finally leads to



and more easily to



where



The second idea is to analytically integrate the system of equations , which can be rewritten in a simpler way as



where



The analytical solution of is given by



This integration scheme is not yet implemented, however the scheme is implemented in FLIQUID\_KEPSILON\_explicit.F

## Analytical evaluation of the mean values of the singular terms

The singular terms  and  need to be evaluated analytically if we assume negligibility of the diffusion term  in the equation . This equation then reduces to



and a bit more simple



where  ,  , and 

The analytical solution to the differential equation is



In the same fashion,



is also a solution to the same differential equation. Thus the time evolution of x is given by solving and for the variable x, i.e.



as well as



The arctanh is defined between -1 and 1, the arccoth is defined from 1 to infinity and -1 to -infinity. Hence, if



then equation has to be used, if



then equation has to be used, and finally if



then the limit of the differential equation is reached and hence



## Boundary conditions for solid walls

The solid walls particles in FPM can be treated like interior particles, with one exception: the wall particles are assumed to be shifted to the interior of the flow domain by a small value . The value of  is called the wall layer thickness (stored in ind\_WallLayer or in the common-variable WallLayer). Thus, in the model , the term  needs to have an additional contribution, namely the contribution that comes from the fact that the velocity drops to zero exactly at the wall. So, we have the enhanced model



In reference to the FPM-code, we call



Moreover, also the production rate  has to be extended by a term, which is in the order of magnitude of



where  = velocity, =velocity of the wall movement, = boundary normal

## Incorporation of turbulent wall tension into the velocity boundary conditions

The turbulent wall stress is computed by the following set of equations:



where the definitions are given by





 are the measured values of the tangential velocity and the turbulent kinetic energy at some location with the distance to the wall. We assume this value to be governed by the non-dimensionalized parameter  (see equations ff.). In fact we define



with  being the smoothing length.  is also referred to as "wall layer".

With the known values of  (Viscosity, density) we are able to resolve for the turbulent boundary tension . The boundary tension finally will be part of the momentum and energy balance of the sliding wall particles. That is done in the following sense: If  is the direction of the velocity along the wall, then we can write down the momentum balance in this direction by



where the viscous term  is comprised of the physical and turbulent viscosity. The discretization of equation , based on a control element of the thickness of the wall layer , is straight forward in the sense:



or equivalently



which is, in other words, a momentum balance on a control element of the thickness  in the direction of . The term  are the viscous stresses on the free-flow-side and  are the viscous stresses at the wall side of the layer-control element. In order to provide a numerically stable formulation, we employ a trick such that  acts as a viscous force that counteracts the velocity, that is



The representative velocity  would most preferably have to be , however then the equation becomes nonlinear and difficult to solve numerically. Actually, we choose . Finally, the whole solution of the boundary velocity can be rewritten as



# Further analytical considerations of the k-epsilon model

The following small analysis of the k epsilon model is based on simply the production and dissipation rates of the model, however we neglect the diffusion terms. This is a simplification that signifies a homogeneous turbulence distribution, for instance in a mixing unit without big gradients in the turbulent quantities.

## Reduced model without diffusion



Here, . The consideration of the k-epsilon system without viscous terms seems to be useful as we can always imagine cases where k and epsilon are evenly distributed, for instance a liquid in a turbulent mixing unit.

## Limit of the relation k and epsilon

The time derivative of the term  is given by the scheme above





From equation we learn that the limit of the value  as time goes to infinity is



## Relation of time change of diffusion

The turbulent diffusion is given by



The time change rate is



And finally



So, of course we can also determine the change rate of the turbulent viscosity at infinite time:



We can now also ask for a dedicated change of the turbulent viscosity during a given time step, i.e.



For given k, we can ask what should be epsilon, ergo



## Paradoxon of the characteristic turbulent length scale

We remember, from literature, that the turbulent length scale is defined as



The time change rate is now easily derived by



Replacing the time derivatives of k and epsilon yield



Division by the length scale itself gives



Therefore, for the nondimensionalized change rate of the turbulent length scale, we also find a limit as time goes to infinity:



Equation reveals a paradoxon as the term  is strictly positive for all (). This means that the turbulent length scale rises even under high shear deformation of the fluid, which is not consistent in itself. We would expect a shrinkage of this quantity under high shear rates.

## Strategies of initializing k and epsilon at high shear rates

Due to the paradoxon described in the section above, and also due to the fact that small starting values of k and epsilon might lead to extremely long times until reaching significant k and epsilon, we might want to initialize the values of k and epsilon at high shear rates.

### Strategy 1:

Assume the infinity ratio of k and epsilon and assume an initial turbulent viscosity equal to physical viscosity. I.e.



And



With the so computed initial value of k and equation we are able to compute the initial value of epsilon.

### Strategy 2:

We can also assume a certain initial value for k by



With values for , then we can find the initial value for epsilon by either the well known relation or by forcing the initial turbulent viscosity again to physical viscosity, i.e.



If we force equation , then we can compute the initialized viscosity by



Plugging in the values of the constants we have



If we choose  the we find



which is exactly the turbulent viscosity proposed be the Smagorinsky-Turbulence model. In this way, for turbulent production, Smagorinsky and k-epsilon are similar. The are different in the decay of the turbulences.

The turbulent length scale produced by this is



That means by choosing alpha we have the freedom to either influence the turbulent viscosity or the turbulent length scale to a desired value.

Strategy 2 is, bw the way, the one implemented right now in FPM.