Nonlocality of Interaction of Scales in the Dynamics of 2D Incompressible Fluids

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(Received 27 April 1999)

We use numerical simulations to prove that the dynamics of the small-scale vorticity in two-dimensional incompressible fluids is nearly passive and that the feedback of the small scales onto the large-scale dynamics is important only for the part of the large-scale range which is adjacent or close to the small-scale range.

PACS numbers: 47.27.Eq, 47.11.+j, 47.27.Gs, 47.32.Cc

The dynamics of high Reynolds number turbulent flows usually couples a broad range of scales, from the energy injection scale to the dissipative scale. This range is too wide to be fully resolved by direct numerical simulations of the Navier-Stokes equations. This motivated a number of attempts to describe separately the behavior of the "energy containing" scales (corresponding to either the large scales or the scales of the mean motions) via modeling of the action of the "dissipative" small or random scales of motions. In most cases, the modeling is empirical and based on hand waving or dimensional arguments about the small-scale dynamics. There are a few attempts to derive a model from asymptotic expansions based on the Navier-Stokes equations. Traditionally, these models are based on two different expansion parameters, based on the ratio of the typical length ℓ and time τ of the small scale motions, and the typical length L and time scale T of the energy containing large-scale motions. This results in two parameters: $\epsilon = \ell/L$ and $\eta = \tau/T$. For example, the eddy-viscosity description of Dubrulle and Frisch [1] requires $\epsilon \ll 1$, $\eta \ll 1$ but $\epsilon/\eta \gg 1$. The classical rapid distortion theory (RDT) of Batchelor and Proudman [2] describes the dynamic of the small scales via linearized equations of motions and requires $\epsilon = 0$ and $\eta \gg 1$. More recently, Dubrulle and Nazarenko [3] introduced a dynamical subgrid model for two-dimensional turbulence based on RDT for weakly inhomogeneous mean flows by assuming $\epsilon \ll 1$ and $\eta \gg 1$. These assumptions correspond to a developed stage of the 2D turbulence in which an energy condensation into large eddies is observed, so that evolution at the small scales is dominated by the nonlocal interaction with these large eddies rather than by the local interaction of the small eddies among themselves. This nonlocality property has been reported in a number of papers on the numerical simulation of 2D turbulence [4]. In this Letter, we specifically focus on studying the mechanisms of the interaction between large and small scales of the system and, particularly, on studying the nonlocality property. Consider the two-dimensional Euler equation in the vorticity form:

$$\begin{aligned} \mathrm{div}\mathbf{u} &= 0\,, \\ \partial_t \omega + \mathrm{div}(\mathbf{u}\omega) &= 0\,, \end{aligned} \tag{1}$$

where $\omega = \mathrm{curl}\mathbf{u}$. We now define the large- and small-scale part of the vorticity and of the velocity, via a given filter G as follows:

$$\mathbf{U}(\mathbf{x}) = \langle \mathbf{u} \rangle = \int G(\mathbf{x}-\mathbf{x}')\mathbf{u}(\mathbf{x}')\,d\mathbf{x}', \qquad \mathbf{u}'(\mathbf{x}) = \mathbf{u} - \mathbf{U}\,,$$

$$\begin{aligned} &\Omega(\mathbf{x}) = \langle \omega \rangle = \int G(\mathbf{x}-\mathbf{x}')\omega(\mathbf{x}')d\mathbf{x}', \\ &\omega'(\mathbf{x}) = \omega - \Omega\,. \end{aligned} \tag{2}$$

The filter G can be of a cutoff type or a smoother one, but such that the resulting large-scale and small-scale fields would contain mostly low wave number and high wave number Fourier components, correspondingly. Inserting this decomposition into the Euler Eq. (1) and taking its large- and small-scale component, we get a set of two coupled equations:

$$\partial_t \Omega + \mathrm{div}\langle \mathbf{U}\Omega \rangle + \mathrm{div}\langle \mathbf{u}'\Omega \rangle + \mathrm{div}\langle \mathbf{U}\omega' \rangle + \mathrm{div}\langle \mathbf{u}'\omega' \rangle = 0, \tag{3}$$

and

$$\begin{aligned} \partial_t \omega' + \mathrm{div}\mathbf{U}\Omega + \mathrm{div}\mathbf{u}'\Omega + \mathrm{div}\mathbf{U}\omega' + \\ \mathrm{div}\mathbf{u}'\omega' - \mathrm{div}\langle \mathbf{U}\Omega \rangle - \mathrm{div}\langle \mathbf{u}'\Omega \rangle - \\ \mathrm{div}\langle \mathbf{U}\omega' \rangle - \mathrm{div}\langle \mathbf{u}'\omega' \rangle = 0\,. \end{aligned} \tag{4}$$

Not all of the terms in the above system of coupled equations for the small and large scales are equally important dynamically, and our first aim is to study their relative importance using an *a priori* test based on a direct numerical simulation (DNS). The data used for the test are from a 1024^2 DNS of decaying turbulence using real (i.e., not hyper-) viscosity after 36 turnover times with $R_\lambda = 105$. Here, R_λ is the Reynolds number based on the Taylor scales. The initial condition is a vorticity field with most of the energy concentrated at the wave number $k = 40$. After 36 turnover times, the spectrum develops an inertial range between $k = 8$ and $k = 40$ with a slope close to k^{-3} (see inset of Fig. 1). A cutoff filter at $k_c = 32$ was applied to the DNS data to separate the flow into large- and small-scale components. This was used to compute the four different nonlinear terms appearing in the equation [namely, $\mathrm{div}(U\Omega)$, $\mathrm{div}(U\omega')$, $\mathrm{div}(u'\Omega)$, and $\mathrm{div}(u'\omega')$]. The cutoff scale was chosen so that it lies in the inertial range, and it allows for a reasonable amount of large-scale energy, so that our hypothesis of nonlocality is valid. These are the only restrictions pertaining to the choice of the cutoff scale. In Fig. 1, we plotted the square modulus of the Fourier component of the four nonlinear terms as a function of the wave number. One sees that for a wave number less than the cutoff wave number (corresponding to the large-scale equation), the large-scale–large-scale nonlinear interaction $\mathrm{div}(U\Omega)$ provides the dominant term, followed by the large-scale–small-scale interaction term $\mathrm{div}(U\omega')$ which becomes comparable to $\mathrm{div}(U\Omega)$ close to the cutoff wave number. The other two terms, $\mathrm{div}(u'\Omega)$ and $\mathrm{div}(u'\omega')$, are both negligible throughout the large-scale range of wave numbers. Note that both linear terms $\mathrm{div}(U\omega')$ and $\mathrm{div}(u'\Omega)$ asymptote to the same value at the low wave number end of the large-scale range [although both of them are much less than $\mathrm{div}(U\Omega)$]. The nonlinear small-scale–small-scale interaction term is much less than all the other terms. For wave numbers greater than the cutoff (the small-scale equation), the ordering of the terms changes and the dominant one becomes the contribution from $U\omega'$, followed by that of $U\Omega$ up to approximately $2k_c$, where by construction $U\Omega$ vanishes. For larger wave numbers, $\mathrm{div}(u'\omega')$ becomes the second dominant term but it remains 1 or 2 orders of magnitude smaller than the leading order term $\mathrm{div}(U\omega')$ which is

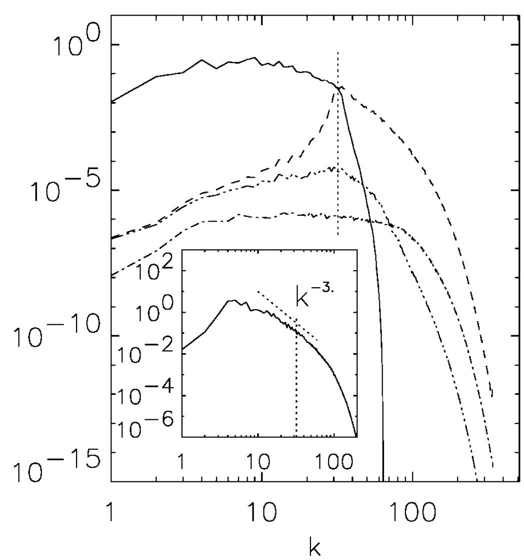

FIG. 1. Square modulus of the four Reynolds tensor components for a decaying turbulence: continuous line, $\mathrm{div}(\mathbf{U}\Omega)$; $--$, $\mathrm{div}(\mathbf{U}\omega')$; $-\cdot-$, $\mathrm{div}(\mathbf{u}'\omega')$; and $-\cdot\cdot\cdot-$, $\mathrm{div}(\mathbf{u}'\Omega)$. The initial field is the result of a DNS 1024^2 for decaying turbulence. The energy spectrum and the separation scale are shown in the inset.

kept in our model. We have also performed the same test using different initial fields and different filters and always found the same qualitative behavior, as long as the cutoff scales allow for sufficient large-scale energy.

Based on the above results, we can write a reduced system of equations for the large and small scales by retaining only the dominant terms in the original system (3) and (4):

$$\partial_t \Omega + \mathrm{div}\langle \mathbf{U}\Omega\rangle + \mathrm{div}\langle \mathbf{U}\omega'\rangle = 0\,, \tag{5a}$$

$$\partial_t \omega' + \mathrm{div}\mathbf{U}\omega' = \mathrm{div}\langle \mathbf{U}\Omega\rangle - \mathrm{div}\mathbf{U}\Omega + \mathrm{div}\langle \mathbf{U}\omega'\rangle\,. \tag{5b}$$

Note that, in the case of turbulence with a spectral gap, the mixed average $\langle U\omega'\rangle$ and $\langle u'\Omega\rangle$ vanish and the only remaining contribution from the small scales is $\mathrm{div}\langle u'\omega'\rangle$. This was the case considered in [3]. There are some observations indicating the existence of a spectral gap in the atmosphere [5]. However, this spectral gap is not obtained in general 2D numerical simulations of decaying or forced turbulence, so we shall not consider this possibility in the present paper. Had we assumed that our average is performed in a statistical sense (e.g., with respect to the realization), we would also have obtained a mean flow equation forced only by the small-scale Reynolds stress term, as in [3]. The fluctuating ("small-scale") equation would appear even more simplified, since the only relevant terms up to order epsilon will be $\partial_t\omega' + \mathrm{div}(\mathbf{U}\omega') = 0$. This situation corresponds to the classical RDT, where vorticity (in 2D) is considered to be passively advected by the mean flow.

The set of Eqs. (5) describes an interesting coupled system between large and small scales. By virtue of (5), the dynamics of the large scales is determined first by their mutual (nonlinear) interaction, but also by the "mixed" interaction between them and the small scales. Note that this is markedly different from standard Reynolds stress phenomenology, where it is usually assumed that the main interaction comes from the small-scale–small-scale term. In that sense, our coupled model is not based on an eddy-viscosity assumption, and captures behavior which could not be parametrized via, for example, a Smagorinsky description.

According to (5b), the small scales evolve under the conjugate action of two processes, the first being their advection by the large-scale velocity field and the second their growth by the continuous building of small scales via nonlinear interaction between large and small scales (related to the enstrophy cascade). In that respect, the small scales behave like a forced passive scalar, where the forcing depends on the large-scale strain. However, the small scales are not completely passive since they are allowed to react back over the large scale via the nonlinear coupling between the large and small scales.

This feedback of small scales on the large scales, as well as the forcing of the small scales via the nonlinear term, makes an essential difference between the present model and RDT, in which the small scales are only passively advected by the main stream. In such a situation, the small-scale behavior is directly related to the initial and boundary conditions, an *a priori* undesirable feature to describe the fully developed stage of a turbulence, which should be independent of initial and boundary conditions [6]. By allowing continuous injection of vorticity (forcing) at small scale, we do, in contrast, allow a loss of memory from initial conditions, since after long enough time the "transient linear solution" (solution of the homogeneous small-scale equation, i.e., the classical RDT equation) will eventually decay and the long term behavior of the solution will be governed by the forcing, i.e., the nonlinear coupling between large and small scales. Note, however, that we do conserve an essential desirable feature of the RDT description, which is the linearity of the small-scale equation. This allows explicit analytical solution of the small-scale equations in terms of the large-scale flow, in certain simple cases such as when the large scales are made of a dipole [3], a pure strain [7], or a shear flow [8].

Our choice of the reduced model was motivated by the *a priori* DNS test measuring relative importance of terms in the original system of equations at a given instant of time. However, averaged characteristics may differ from their instantaneous values. Indeed, collective phase effects (decorrelation) among terms could lead to a relative depletion or enhancement of the effective contribution of the various terms over a long period of time. We therefore conducted a second series of (dynamical) tests by a direct numerical integration of the reduced model (5), in several typical 2D situations: decaying turbulence, forced turbulence, and vortex merging. To get an inertial range as large as possible, we performed all the tests by replacing the dissipation term by a hyperdissipation $\nu \Delta^p \mathbf{u}$ with $p = 8$. In preliminary tests, we have checked that the hyperviscous numerical simulation (HNS) is a very good approximation at large scales of a DNS at a much larger resolution. Since we are mostly interested in the large-scale behavior of our model, we felt that it was safe to use the HNS to perform the tests, allowing a substantial economy of computational resources. The filter used here is a cutoff in the Fourier space, at $k = 21$ for the three cases tested with this dynamical test. It is small enough to allow enough energy at large scale for the nonlocality assumption to be valid. The comparison was made for the same initial conditions and resolutions (512^2). Both our model and the HNS were performed using a pseudospectral code over a $2\pi \times 2\pi$ periodic domain.

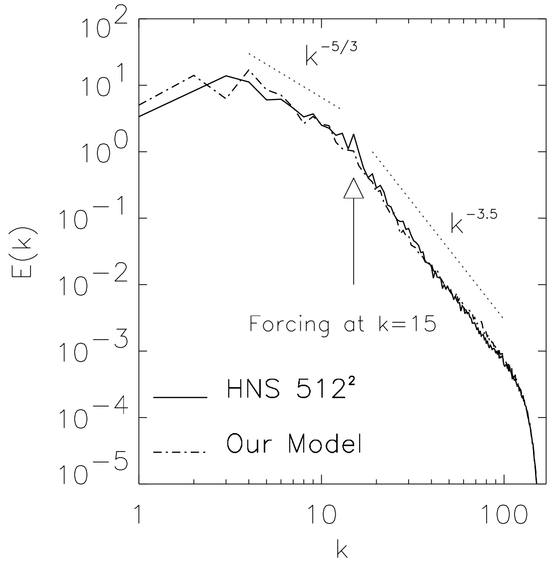

FIG. 2. Energy spectra of turbulence forced at $k = 15$. The HNS (512^2) is compared with our model with the same resolution and the same hyperviscous dissipation. The separation scale is at $k = 21$.

For forced turbulence, the results are presented in Fig. 2. The initial condition for the two simulations is a vorticity field with a small total energy compared to the final state. The forcing was produced by keeping constant in time the energy at a given wave number ($k_x = 15, k_y = 0$). One can see that the energy spectra

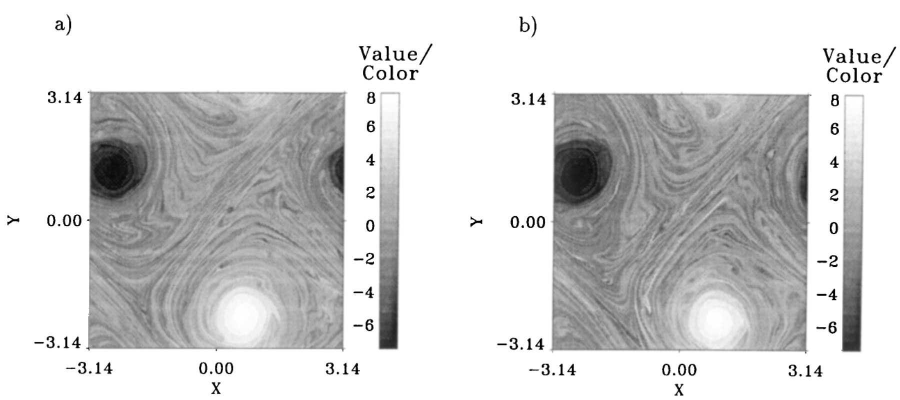

FIG. 3. Vorticity field of decaying turbulence obtained by (a) HNS 512^2, and (b) our model 512^2 with a separation scale at $k = 21$.

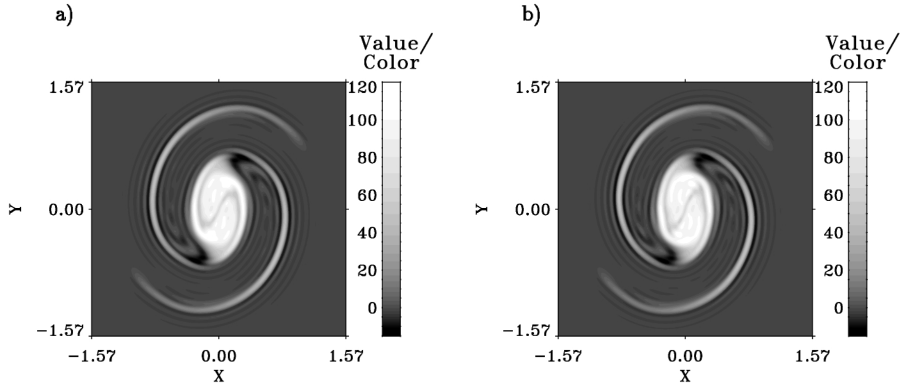

FIG. 4. Vorticity field of a vortex merging process obtained by (a) HNS 512^2, and (b) our model 512^2 with a separation scale at $k = 21$ (only $1/4$ of the total domain is represented).

are in very good agreement with the hyperviscous numerical simulation data for all scales. For decaying turbulence, the vorticity fields obtained after seven turnover times are shown in Fig. 3. This time corresponds to approximately $160T_w$, where T_w is the time of the maximum enstrophy. The agreement between the final state of our model and the final state of the hyperviscous simulation is excellent, even after several turnover times. A small difference between the dipole location is due to the intrinsic chaoticity of Navier-Stokes equations. However, both fields present the same physical characteristics at large and small scales. Note that in our model we can consider scale as small as the smallest scale in the HNS because we do actually compute the small scales up to the hyperviscous cutoff. We also checked the very good agreement of the energy spectra at all scales. To check that the good agreement was indeed due to our good description of the small scales, we ran, as an additional test, a 32^2 HNS (our small-scale models were thus replaced by hyperviscosity). The vorticity fields and the energy spectra at large scales were quite different from the fully resolved HNS, showing the importance of the small-scale parametrization. The third test was performed on a vortex merging. This process is very sensitive to the small-scale parametrization, and, therefore, provides a rather stringent test of our model. The initial vorticity field is made of two vortices separated by a distance δ. The corresponding vorticity distribution is given, in polar coordinates, by

$$\omega(r,\theta) = 50\{1 - \tanh[(r - R_0)/\Delta]\}, \qquad (6)$$

with $R_0 = 0.05 \times 2\pi$, $\Delta = 0.01 \times 2\pi$, and $\delta = 0.15 \times 2\pi$. The total vorticity during the merging is shown in Fig. 4 at approximately $12T_w$. Note the very good agreement of the merged vortex orientation, and of the filamentary structure. This indicates that our model reproduces very well the dynamics of the full system.

The present validation of our model is interesting for several reasons. First, it justifies *a posteriori* any analytical development based on the two simplified coupled Eqs. (5). This is important because the linearity of the small-scale equations allows, as in classical rapid distortion theory, exact analytical or semianalytical results. Since we have shown that this linear description is a very good approximation of the real dynamics, we may thereby obtain approximate nonperturbative solutions of our approximate system which are a more accurate solution of the total dynamics than those obtained by standard perturbative methods. Also, our result is of interest for large eddy simulations of 2D turbulence. Indeed, our set of coupled equations provides a basis for a dynamical subgrid scale model, where the small-scale contribution is computed from a dynamical equation rather than postulated [9]. In contrast with traditional subgrid scale models (such as Smagorinsky [10]), such dynamical subgrid scale models take into account such an important physical property of the 2D small-scale dynamics as the nonlocality of interaction (linearity of the small-scale equation). This improvement comes at a slightly larger computational cost with respect to traditional subgrid scale models but it gives more accurate results obtained without any adjustable parameters. These results will be reported in a separate paper [9].

B. D. was supported financially by NATO.

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