האוניברסיטה הפתוחה
The Open University of Israel
الجامعة المفتوحة

Welcome to 20944

Algorithmic Robotics

Version 0001

Guy Tordjman

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Algorithmic Robotics

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Course Introduction

Chapter 2- Configuration Space

Chapter 3- Rigid-Body Motions

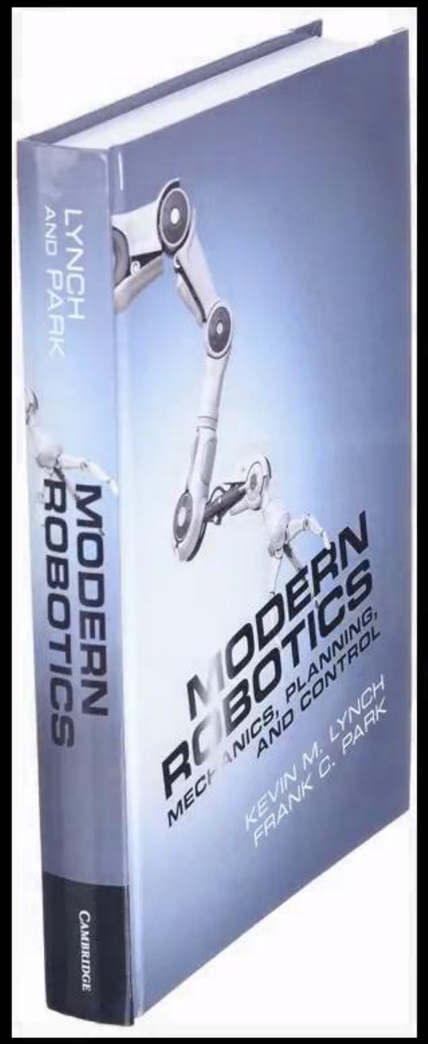
→ Chapter 4- Forward Kintematics (Lecture 4)

Administrative

- Dr. Elishai Ezra Tsur - Course manager. email : elishai@nbel-lab.com
- Guy Tordjman - Course Host. email : turgibot@gmail.com. cell : 053-7203788
- 4 assignments - must submit at least 3 @ 5 points each.
- Assignments contribute max 15 or 20 pts. exam contributes 85 or 80 pts.

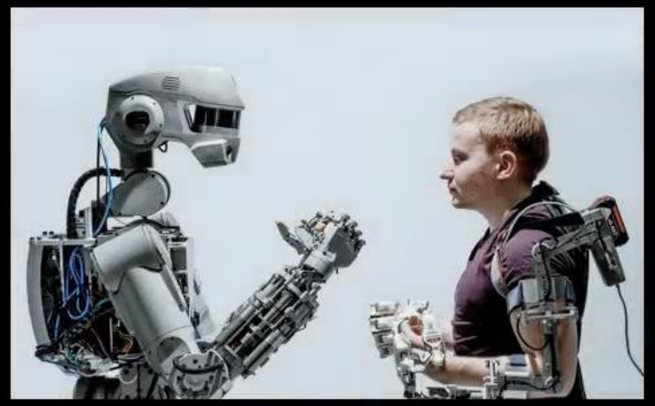
Course Book

- Modern Robotics - by Kevin M. Lynch and Frnack C. Park
- Our course deals with 6 chapters
- Alongsided by short video lessons from Dr. Elishai
- Excerises mostly from the book
- Plenty of online resources including wiki, Coursera and GitHub



How to study and succeed in the course

- Read the relevant material in the book and course material.
- Watch Elishai's videos.
- Practise a few questions from the book before class.
- Zoom and participate in lessons.(sessions are recorded and uploaded)
- Complete all the assignments
- Ask anything during lecture. email anytime or call me on Tuesday 13-14 at 053-7203788



Course Overview

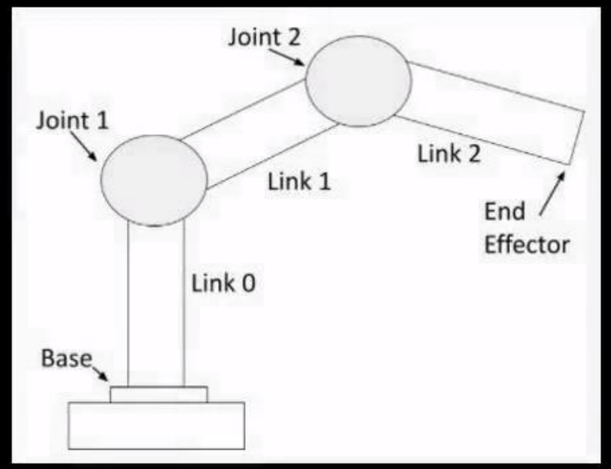
The course subjects 6 chapters in the book:

1. Chapter 2: Configuration Space.
2. Chapter 3: Rigid-Body Motions.
3. Chapter 4: Forward Kinematics.
4. Chapter 5: Velocity Kinematics and Statics.
5. Chapter 6: Inverse Kinematics.
6. Chapter 8: Dynamics of Open Chains.

Chapter 2: Configuration Space

Links, Joints, End-Effector

1. A robot is mechanically constructed by connecting a set of bodies, called links, to each other using various types of joints.
2. Actuators, such as electric motors, deliver forces or torques that cause the robot's links to move.
3. Usually an end-effector, such as a gripper or hand for grasping and manipulating objects, is attached to a specific link.
4. All the robots considered in this course have links that can be modeled as rigid bodies



Chapter 2: Configuration Space

- The configuration of a robot is a complete specification of the position of every point of the robot.
- The number of Degrees of freedom (DOF) of a robot is the minimum number n of real-valued coordinates needed to represent the configuration.
- The Configuration space or C-space, is an n-dimensional space containing all possible configurations of the robot.
- The configuration of a robot is represented by a point in its C-space.

Chapter 2: Configuration Space

2.1 Degrees of Freedom of a Rigid Body

DOF : The minimum number n of real-valued coordinates needed to represent a configuration.

Chapter 2: Configuration Space

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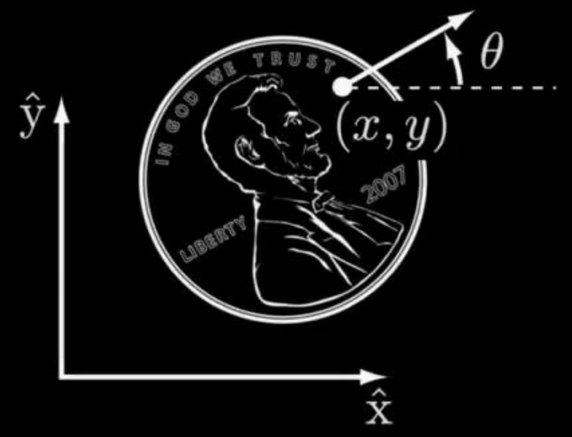


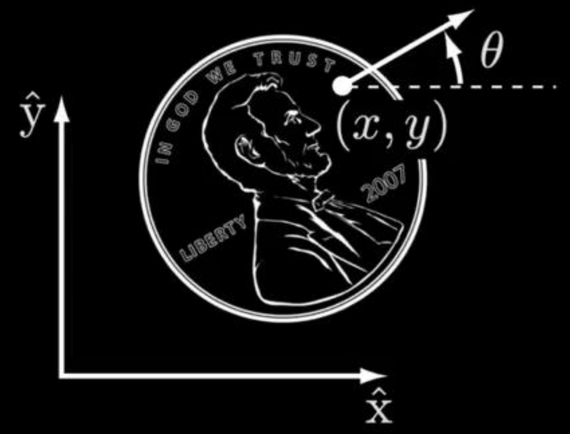
coin on a plane

#DOF = m =

Chapter 2: Configuration Space

2.1 Degrees of Freedom of a Rigid Body

DOF : The minimum number n of real-valued coordinates needed to represent a configuration.



coin on a plane

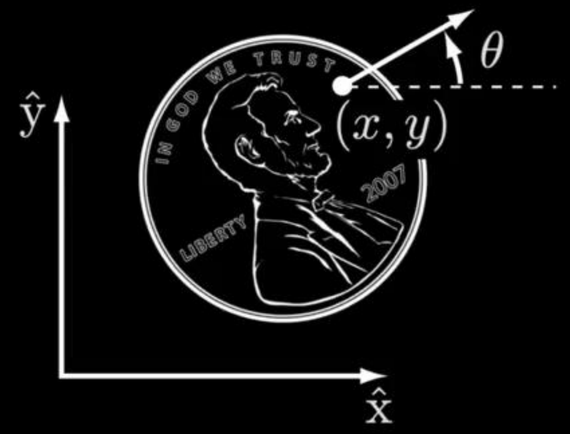
$(x, y, \theta) + heads/tails$

#DOF = m =

Chapter 2: Configuration Space

2.1 Degrees of Freedom of a Rigid Body

DOF : The minimum number n of real-valued coordinates needed to represent a configuration.



coin on a plane

$(x, y, \theta) + heads/tails$

NOT A REAL NUMBER

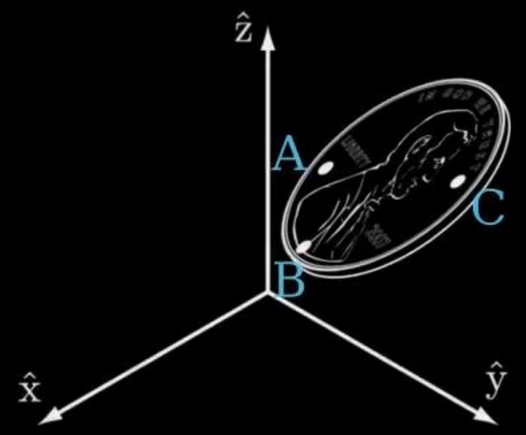
#DOF = m = 3

Chapter 2: Configuration Space

2.1 Degrees of Freedom of a Rigid Body

DOF : The minimum number n of real-valued coordinates needed to represent a configuration.

DOF = (sum of freedoms of the bodies) - (number of independent constraints)



coin in space

#DOF = m =

Chapter 2: Configuration Space

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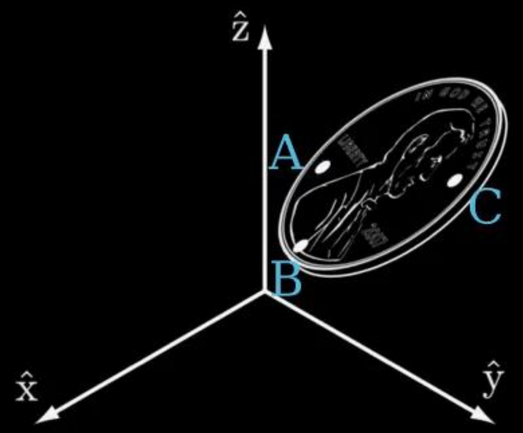

coin in space

Point	#DOF	#Constraints	m
A			
B			
C			

#DOF = m =

Chapter 2: Configuration Space

2.1 Degrees of Freedom of a Rigid Body

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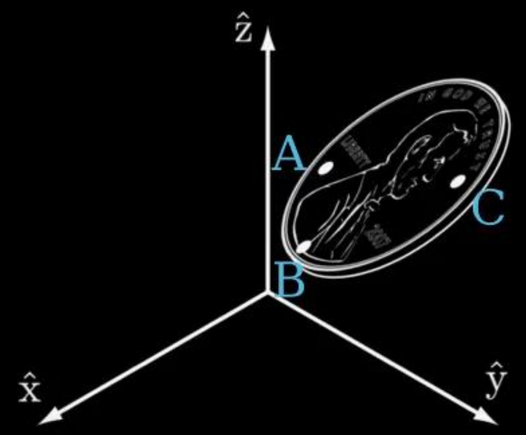


coin in space

Point	#DOF		#Constraints		m
A	3	-	0	=	3
B	3	-	1	=	2
C	3	-	2	=	1

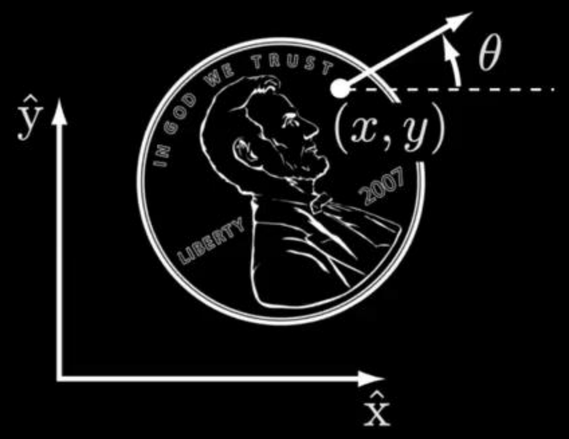
#DOF = m = 3+2+1 = 6

Chapter 2: Configuration Space

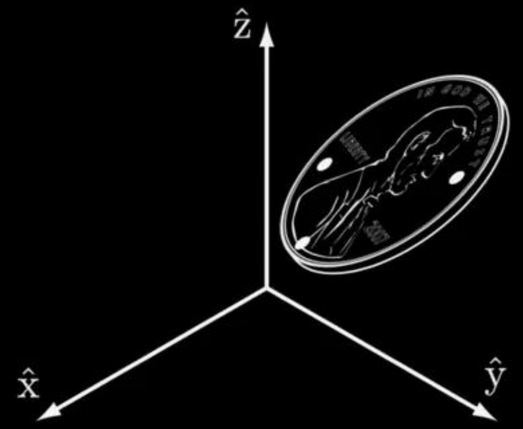
2.1 Degrees of Freedom of a Rigid Body

A rigid body moving in a 3D space, (spatial rigid body), has 6 DOF.

A rigid body moving in a 2D plane (planar rigid body), has 3 DOF.



coin on a plane

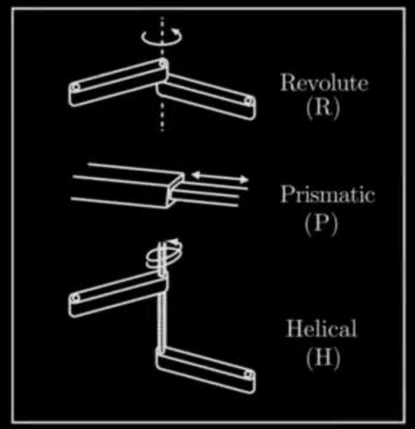


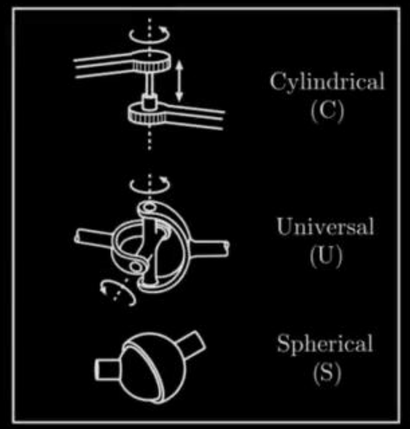
coin in space

Chapter 2: Configuration Space

2.2 Degrees of Freedom of a Robot

Joint types and their DOF





Joint type	dof f	Constraints c between two planar rigid bodies	Constraints c between two spatial rigid bodies
Revolute (R)	1	2	5
Prismatic (P)	1	2	5
Helical (H)	1	N/A	5
Cylindrical (C)	2	N/A	4
Universal (U)	2	N/A	4
Spherical (S)	3	N/A	3

Chapter 2: Configuration Space

2.2 Degrees of Freedom of a Robot

Revolute (R) joint

constraints:	5
freedoms:	1

Chapter 2: Configuration Space

2.2 Degrees of Freedom of a Robot

Revolute (R) joint

constraints:	5
freedoms:	1

Chapter 2: Configuration Space

2.2 Degrees of Freedom of a Robot

Prismatic (P) joint

constraints:	5
freedoms:	1

Chapter 2: Configuration Space

2.2 Degrees of Freedom of a Robot

Prismatic (P) joint

constraints:	5
freedoms:	1

Chapter 2: Configuration Space

2.2 Degrees of Freedom of a Robot

Spherical (S) joint

constraints:	3
freedoms:	3

Chapter 2: Configuration Space

2.2 Degrees of Freedom of a Robot

Spherical (S) joint

constraints:	3
freedoms:	3

Chapter 2: Configuration Space

2.2 Degrees of Freedom of a Robot

Universal (U) joint

constraints:	4
freedoms:	2

Chapter 2: Configuration Space

2.2 Degrees of Freedom of a Robot

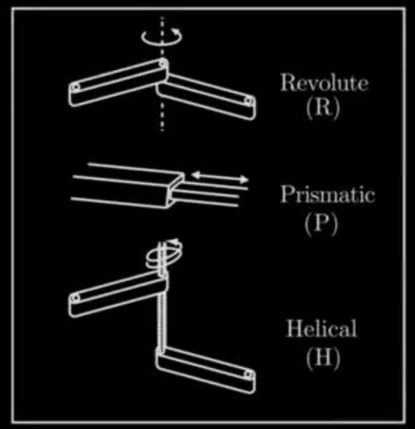
Universal (U) joint

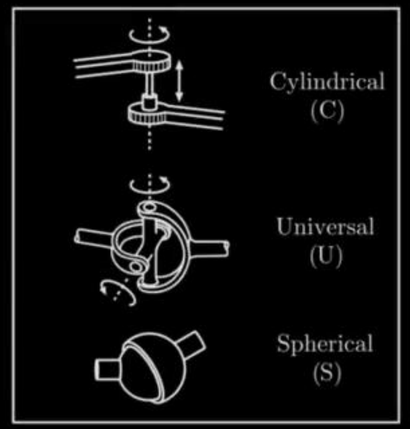
constraints:	4
freedoms:	2

Chapter 2: Configuration Space

2.2 Degrees of Freedom of a Robot

Joint types and their DOF





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Chapter 2: Configuration Space

2.2.2 Grübler's Formula

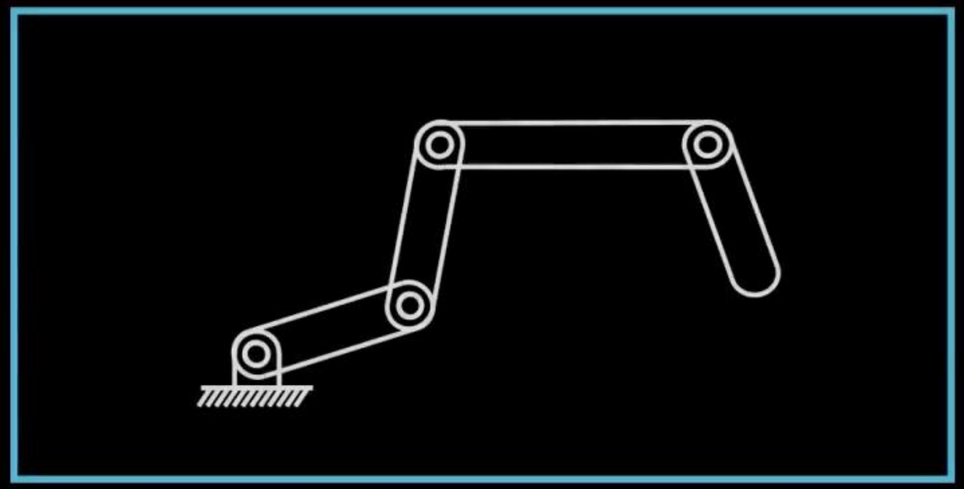
- Let N be the number of links (Ground is also regarded as a link).
- Let J be the number of joints.
- Let m be the number of DOF of a rigid body. (3 planar 6 spatial mechanism)
- Let f_i be the number of freedoms provided by joint i.
- Let c_i be the number of constraints provided by joint i.

$$\begin{aligned}\text{dof} &= \underbrace{m(N-1)}_{\text{rigid body freedoms}} - \underbrace{\sum_{i=1}^{J} c_i}_{\text{joint constraints}} \\ &= m(N-1) - \sum_{i=1}^{J}(m - f_i) \\ &= m(N-1-J) + \sum_{i=1}^{J} f_i.\end{aligned}$$

Chapter 2: Configuration Space

2.2.2 Grübler's Formula - Example 1

$$dof = m(N-1-J) + \sum_{i=1}^{J} f_i$$



k-link planar serial chain

Chapter 2: Configuration Space

2.2.2 Grübler's Formula - Example 1

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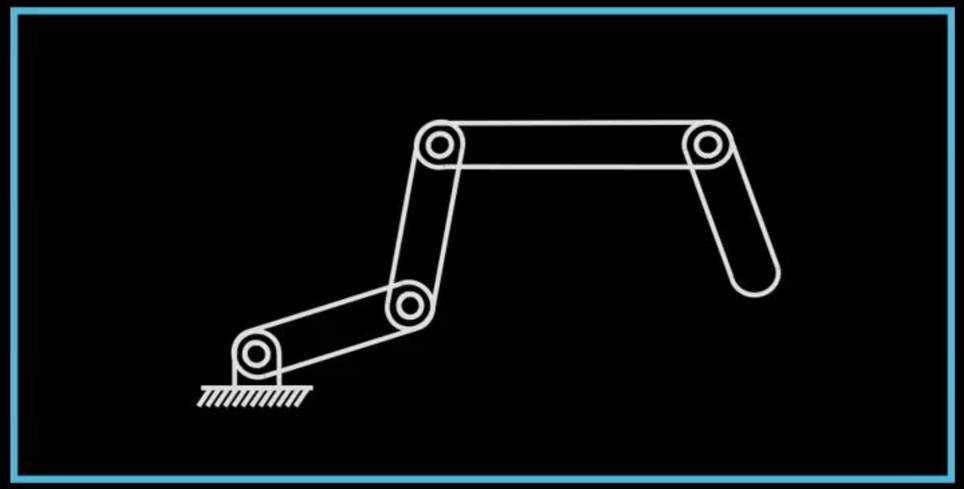


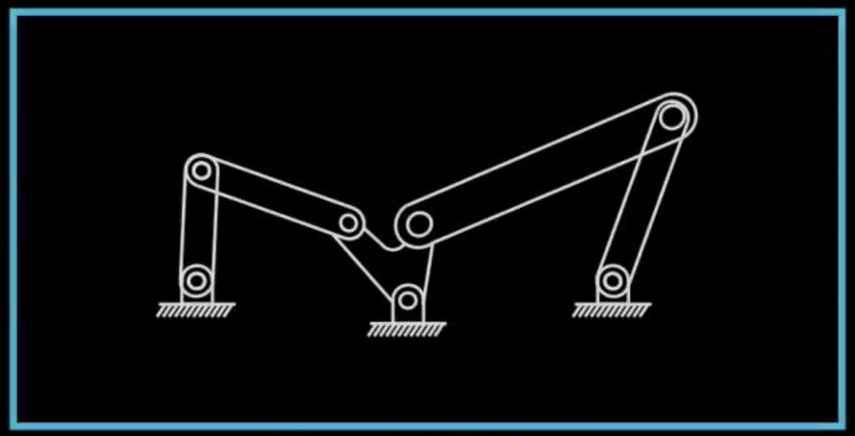
k-link planar serial chain

$$dof = 3((k+1)-1-k)+k = k$$

Chapter 2: Configuration Space

2.2.2 Grübler's Formula - Example 2

$$dof = m(N - 1 - J) + \sum_{i=1}^{J} f_i$$



Watt six-bar linkage

Chapter 2: Configuration Space

2.2.2 Grübler's Formula - Example 2

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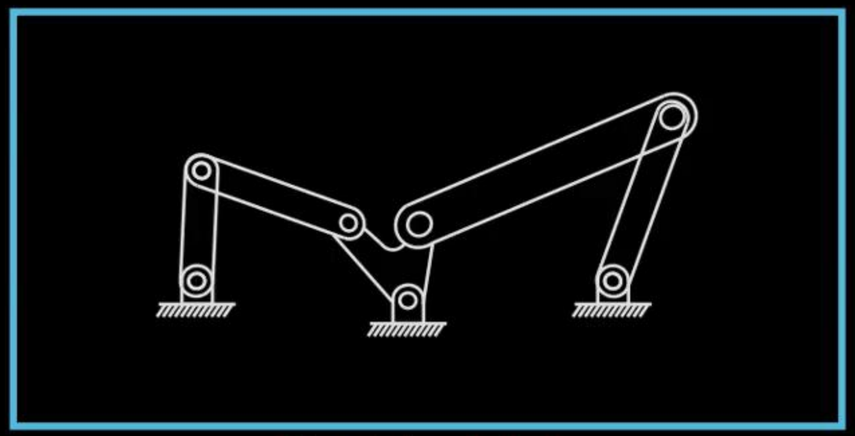


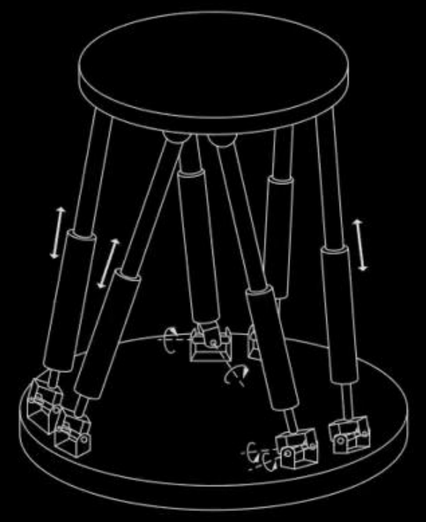
Watt six-bar linkage

$$m = 3, N = 6, J = 7, f_i = 1$$

$$dof = 3((6) - 1 - 7) + 7 = 1$$

2.2.2 Grübler's Formula - Example 3

$$dof = m(N - 1 - J) + \sum_{i=1}^{J} f_i$$

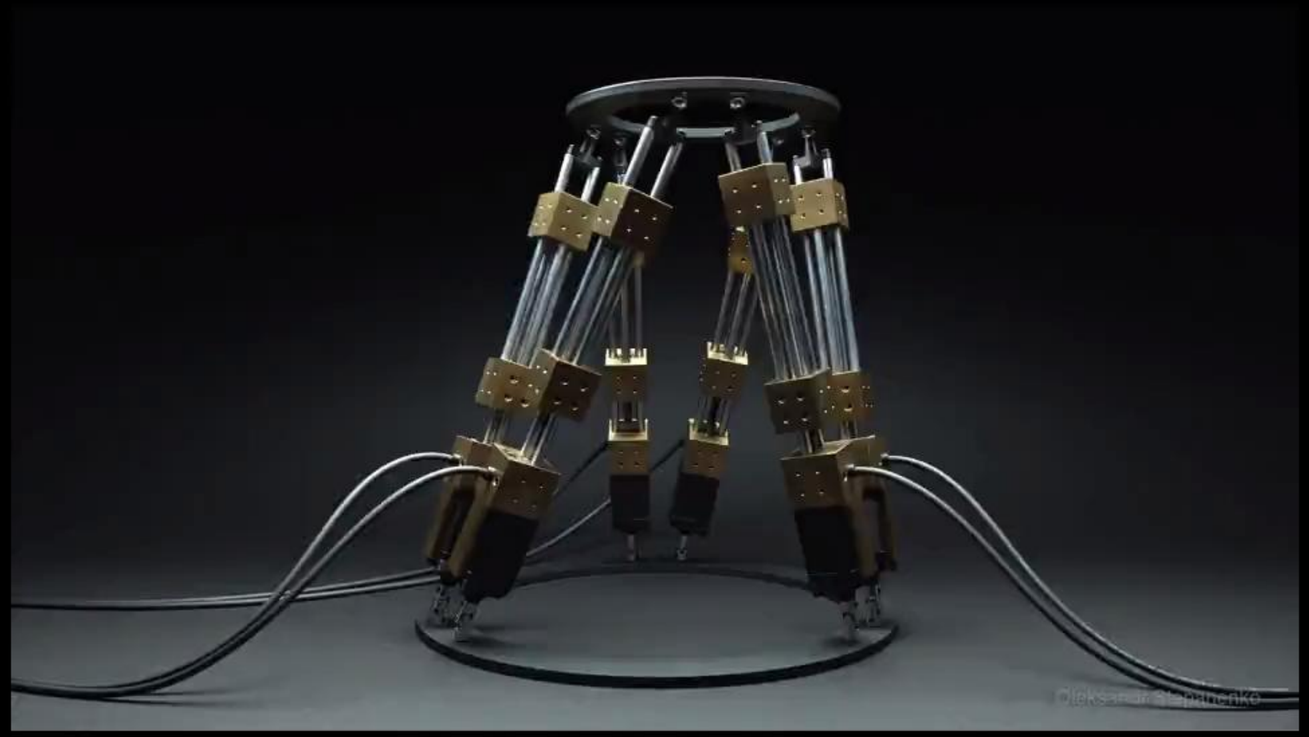


Stewart–Gough platform

Chapter 2: Configuration Space

2.2.2 Grübler's Formula - Example 3

$$dof = m(N-1-J) + \sum_{i=1}^{J} f_i$$



Chapter 2: Configuration Space

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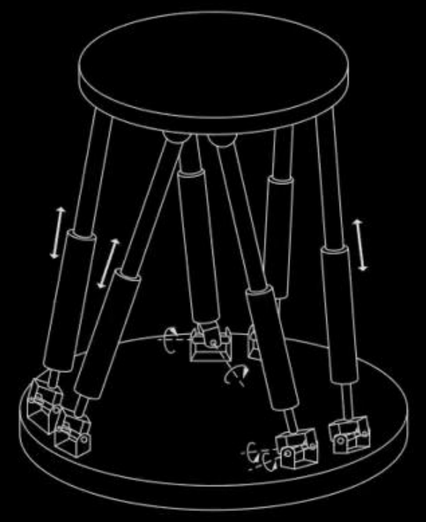

Stewart–Gough platform

Chapter 2: Configuration Space

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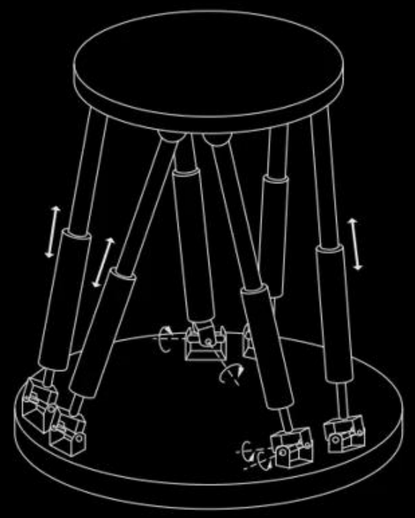


Stewart–Gough platform

$$m = 6, N = 14, J = 18$$

$$6 \cdot (univeral) : f_i = 2$$

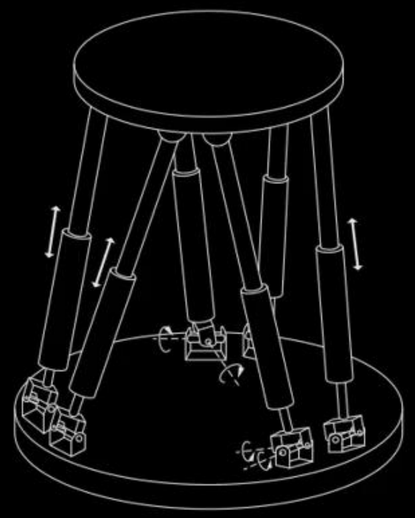
$$6 \cdot (spherical) : f_i = 3$$

$$6 \cdot (prismatic) : f_i = 1$$

Chapter 2: Configuration Space

2.2.2 Grübler's Formula - Example 3

$$dof = m(N-1-J) + \sum_{i=1}^{J} f_i$$



Stewart–Gough platform

$$m = 6, N = 14, J = 18$$

$$6 \cdot (univeral) : f_i = 2$$

$$6 \cdot (spherical) : f_i = 3$$

$$6 \cdot (prismatic) : f_i = 1$$

$$dof = 6(14-1-18) + 6(1) + 6(2) + 6(3) = 6$$

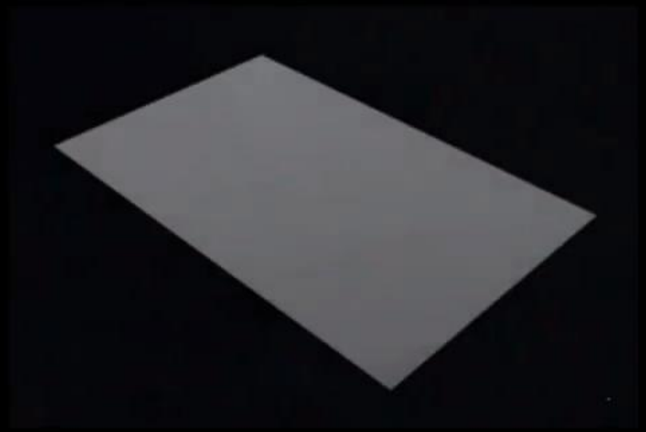
Chapter 2: Configuration Space

2.3: Topology and Representation

dof is the dimension of the C-space, toplogy is its shape

Two C-spaces may have the same dof but differ in other ways.

The 2D surface of a sphere



The 2D surface of a plane

Chapter 2: Configuration Space

2.3: Topology and Representation

The topology of a space is independent of how we represent it.

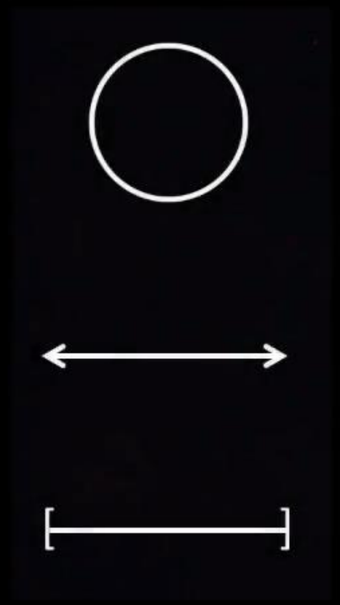
Two spaces are topologically equivalent if one can be smoothly deformed to the other without cutting or pasting.

Chapter 2: Configuration Space

2.3: Topology and Representation

1D topologically distinct shapes:.

$\mathbb{S}^1$



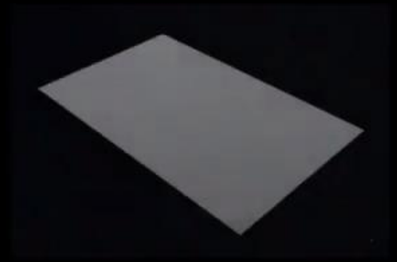
$\mathbb{E}^1$

2D topologically distinct shapes:

$\mathbb{S}^2$

$\mathbb{T}^2$

$\mathbb{E}^2$



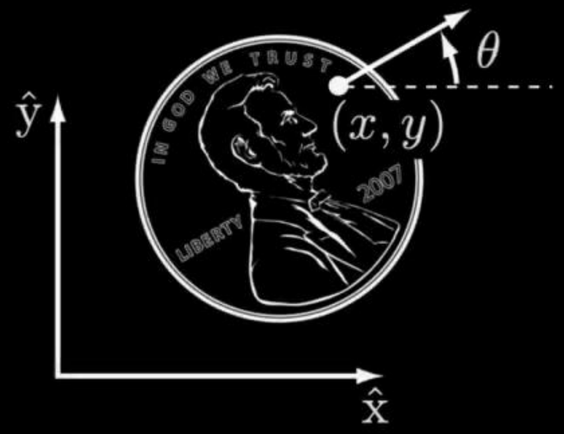
$\mathbb{E}^1 \times \mathbb{S}^1$

Chapter 2: Configuration Space

2.3: Topology and Representation

Some spaces are Cartesian products of spaces of lower dimension, e.g.,

$\mathbb{E}^1 \times \mathbb{S}^1$

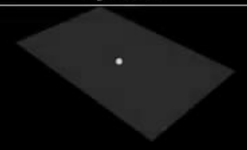
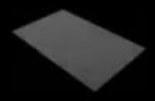
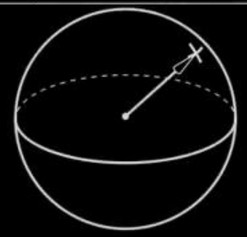
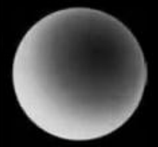
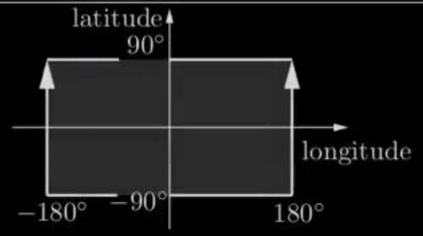
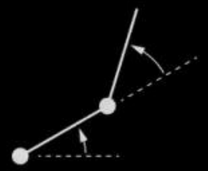
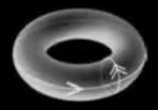
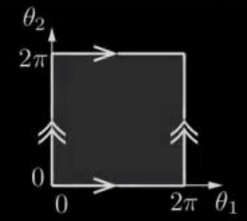
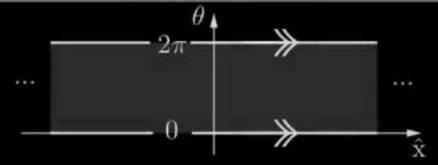


coin on a plane

$\mathbb{E}^2 \times \mathbb{S}^1$

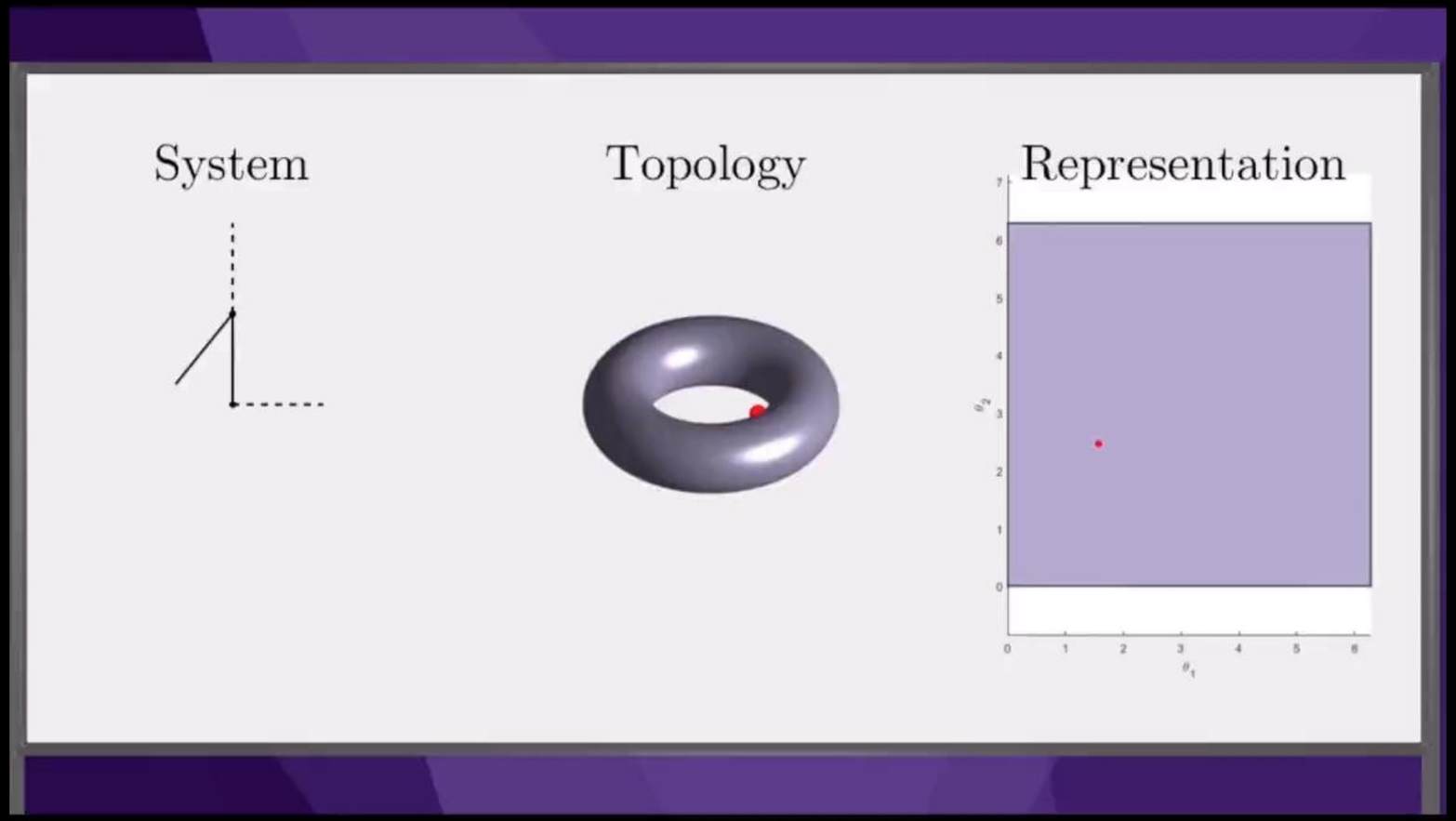
Chapter 2: Configuration Space

2.3: Topology and Representation

system	topology	sample representation
 point on a plane	 $\mathbb{E}^2$	$\mathbb{R}^2$
 spherical pendulum	 S^2	 $[-180°, 180°) \times [-90°, 90°]$
 2R robot arm	 $T^2 = S^1 \times S^1$	 $[0, 2\pi) \times [0, 2\pi)$
rotating sliding knob	$\mathbb{E}^1 \times S^1$	 $\mathbb{R}^1 \times [0, 2\pi)$

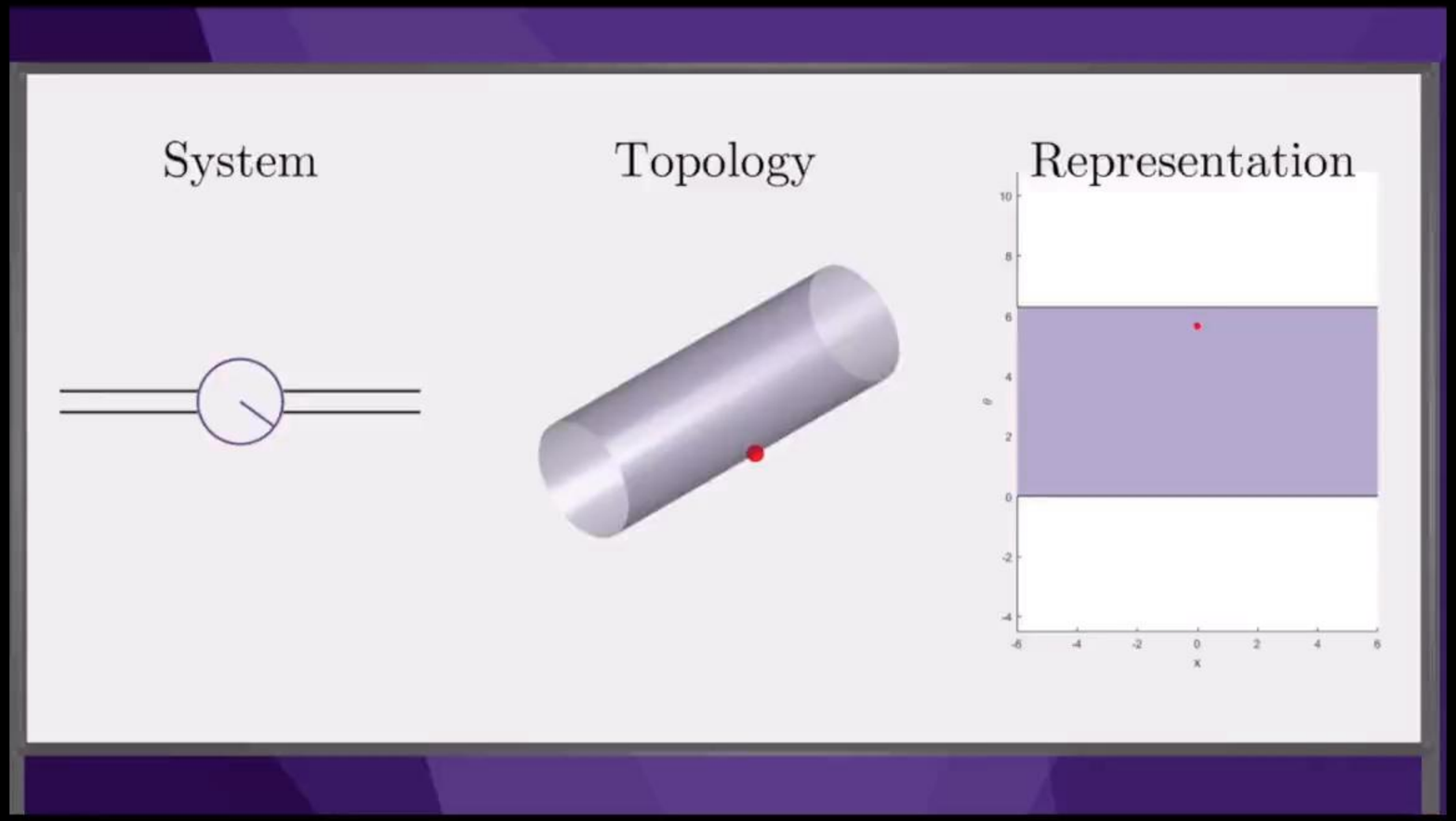
Chapter 2: Configuration Space

2.3: Topology and Representation



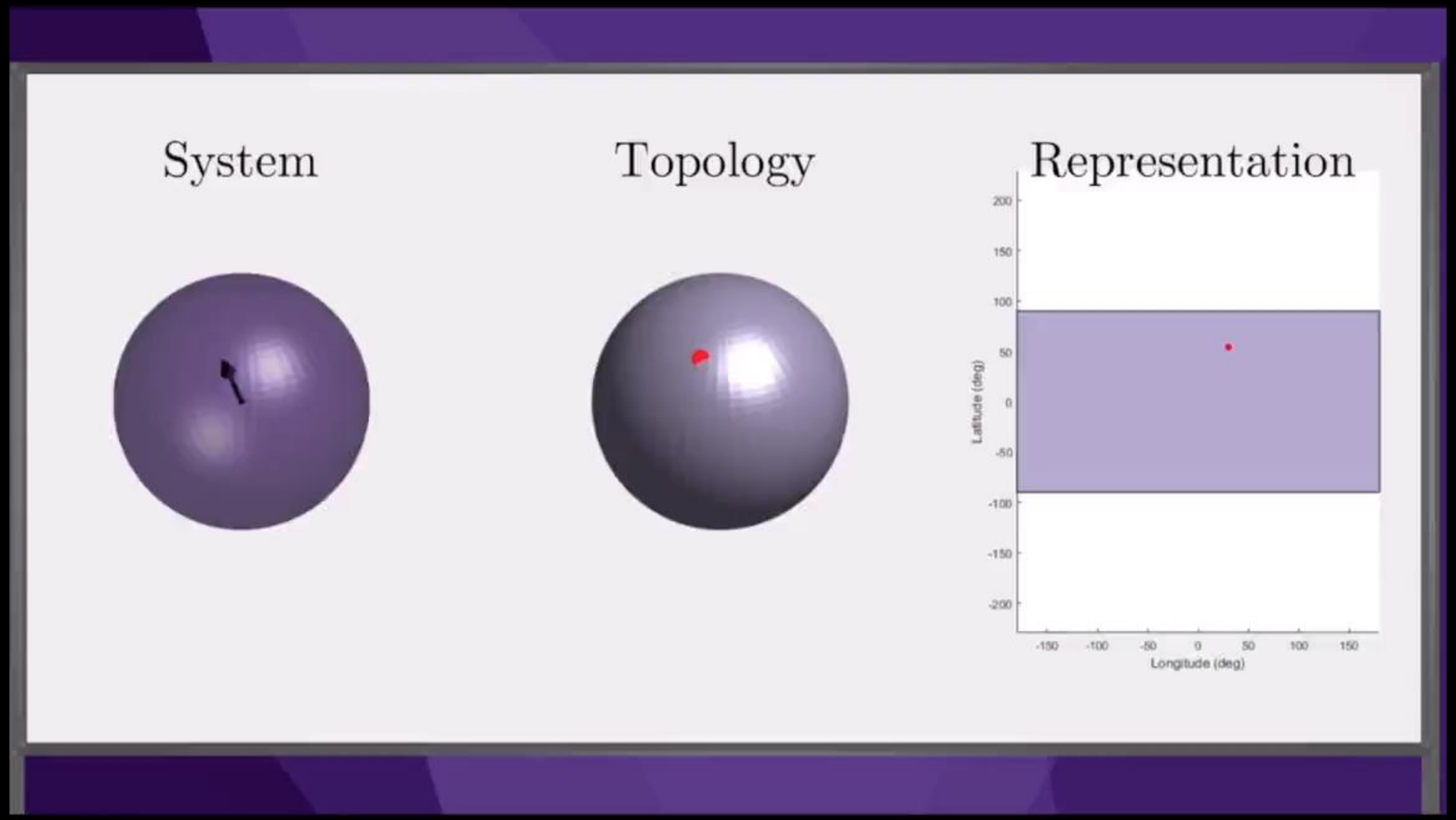
Chapter 2: Configuration Space

2.3: Topology and Representation



Chapter 2: Configuration Space

2.3: Topology and Representation



Chapter 2: Configuration Space

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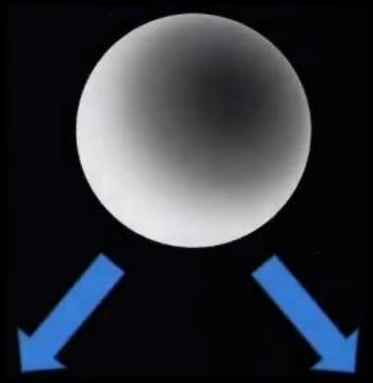
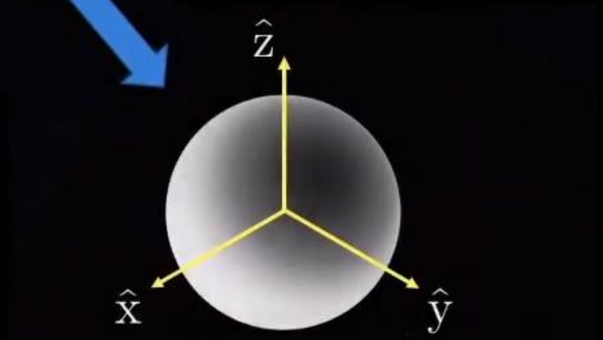


Chapter 2: Configuration Space

2.3: Topology and Representation

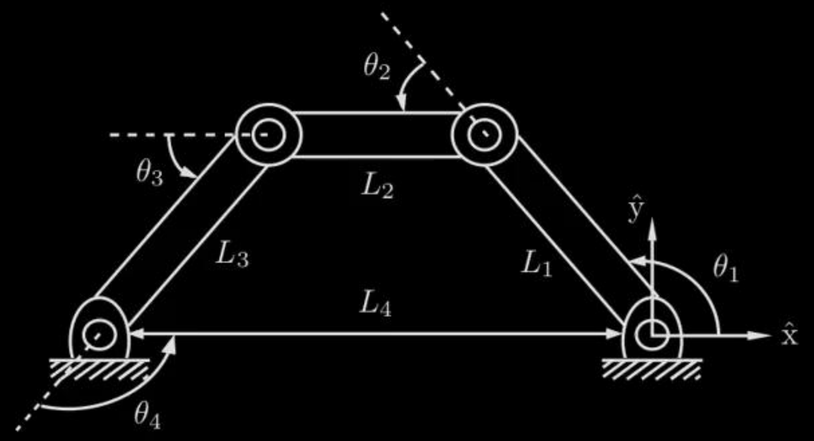


explicit parametrization: minimum number of coordinates, e.g., latitude, longitude

implicit representation: a surface embedded in a higher-dimensional space, e.g., (x, y, z) such that $x^2 + y^2 + z^2 = 1$

Chapter 2: Configuration Space

2.4: Configuration and Velocity Constraints



The four bar linkage.

dof =

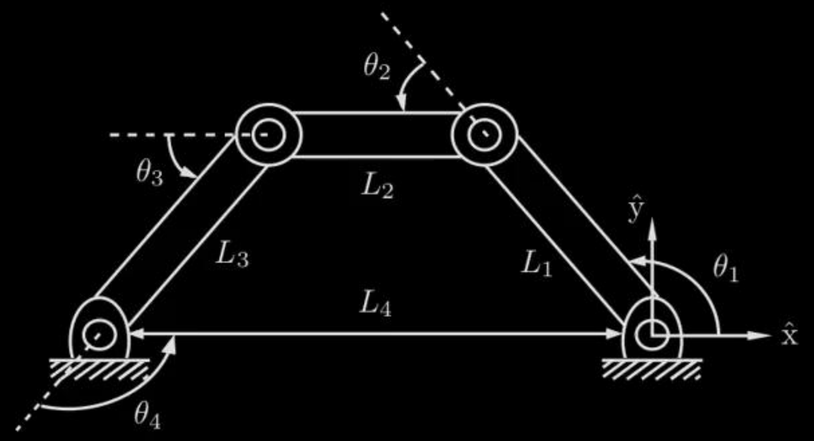
Topology :

Explicit representaion :

Implicit representaion :

Chapter 2: Configuration Space

2.4: Configuration and Velocity Constraints



The four bar linkage.

dof = 3(4-1-4)+4 = 1

Topology : $\mathbb{E}^1$

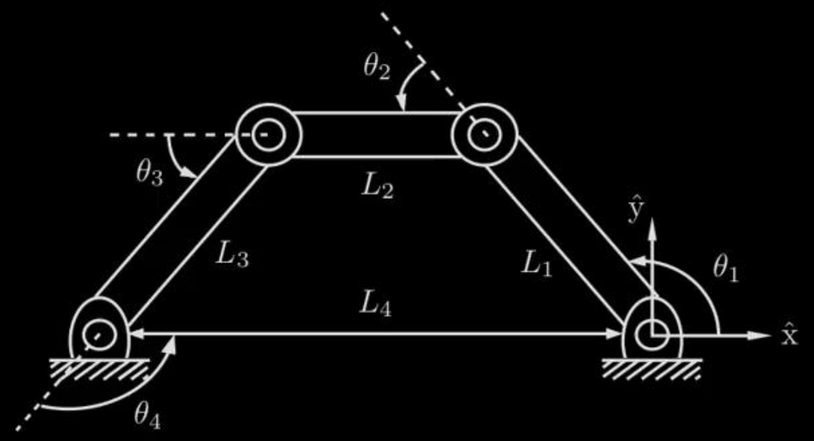
Explicit representaion : $\mathbb{R}^1$

Implicit representaion : $\mathbb{R}^4$

Chapter 2: Configuration Space

2.4: Configuration and Velocity Constraints

k independent **holonomic constraints** on n unknown variables reduce an n-dim C-space to n-k dof



The four bar linkage.

$$
\begin{aligned}
L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) + \cdots + L_4 \cos(\theta_1 + \cdots + \theta_4) &= 0, \\
L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2) + \cdots + L_4 \sin(\theta_1 + \cdots + \theta_4) &= 0, \\
\theta_1 + \theta_2 + \theta_3 + \theta_4 - 2\pi &= 0.
\end{aligned}
$$

loop-closure equations

Chapter 2: Configuration Space

2.4: Configuration and Velocity Constraints

k independent **holonomic constraints** on n unknown variables reduce an n-dim C-space to n-k dof

A general form, for $\theta = [\theta_1, ..., \theta_n]^T$

$$g(\theta) = \begin{bmatrix} g_1(\theta_1, \ldots, \theta_n) \\ \vdots \\ g_k(\theta_1, \ldots, \theta_n) \end{bmatrix} = 0,$$

Chapter 2: Configuration Space

2.4: Configuration and Velocity Constraints

Suppose that a closed-chain robot with such loop-closure equations is in motion.

Pfaffian constraints: are constraints on velocity:

$$\frac{d}{dt}g(\theta(t)) = 0; \qquad \begin{bmatrix} \frac{\partial g_1}{\partial \theta_1}(\theta) & \cdots & \frac{\partial g_1}{\partial \theta_n}(\theta) \\ \vdots & \ddots & \vdots \\ \frac{\partial g_k}{\partial \theta_1}(\theta) & \cdots & \frac{\partial g_k}{\partial \theta_n}(\theta) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \vdots \\ \dot{\theta}_n \end{bmatrix} = 0, \qquad \frac{\partial g}{\partial \theta}(\theta)\dot{\theta} = 0.$$

The pfaffian constraints can be written as $A(\theta)\dot{\theta} = 0,$

If pfaffian constraints can be integrated to equivalent configuration constraints, they are holonomic.

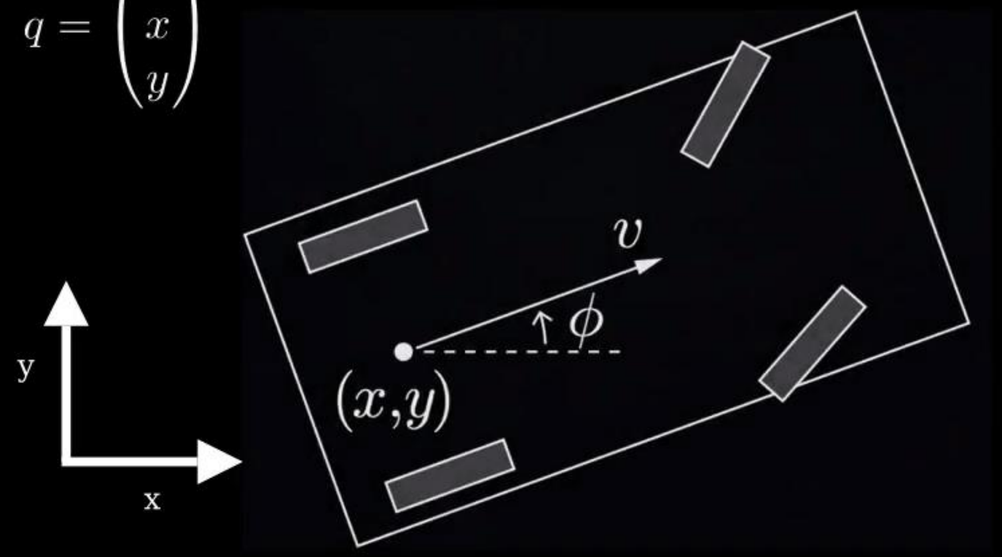
Chapter 2: Configuration Space

2.4: Configuration and Velocity Constraints

If pfaffian constraints can NOT be integrated to equivalent configuration constraints, they are non-holonomic.

They reduce the dimension of the feasible velocities, but not the dimension of the C-space.

$$q = \begin{pmatrix} \phi \\ x \\ y \end{pmatrix}$$



A car on a plane.

Chapter 2: Configuration Space

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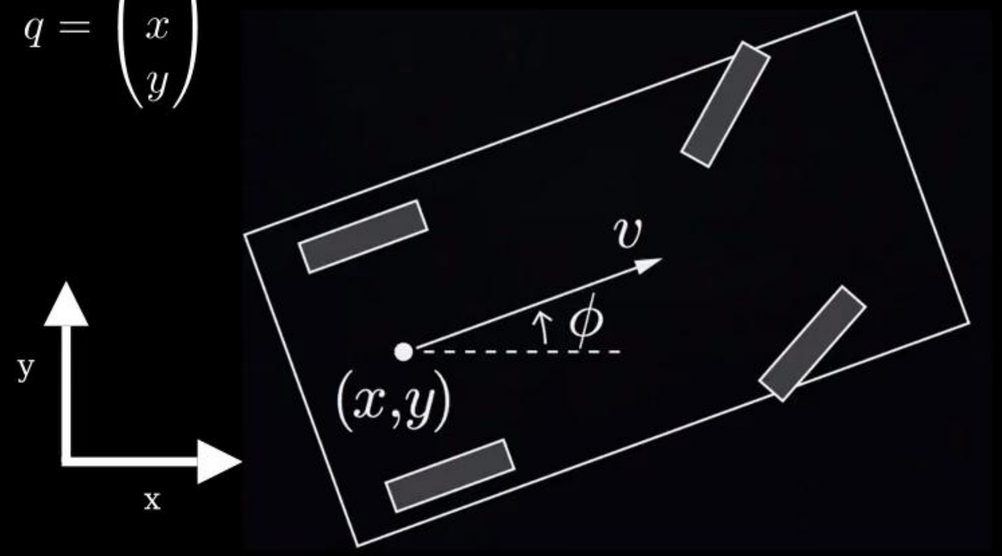

$$\dot{y} = v sin(\phi)$$

$$\dot{x} = v cos(\phi)$$

A car on a plane.

Chapter 2: Configuration Space

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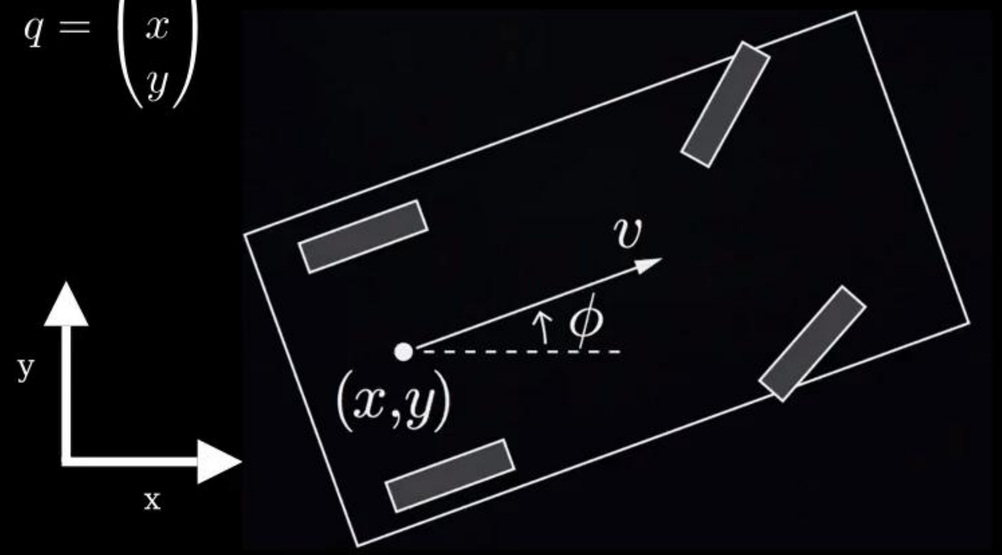

A car on a plane.

$$\dot{y} = vsin(\phi) \quad \rightarrow v = \frac{\dot{y}}{sin(\phi)}$$

$$\dot{x} = vcos(\phi) \quad = \frac{\dot{y}}{sin(\phi)} cos(\phi)$$

Chapter 2: Configuration Space

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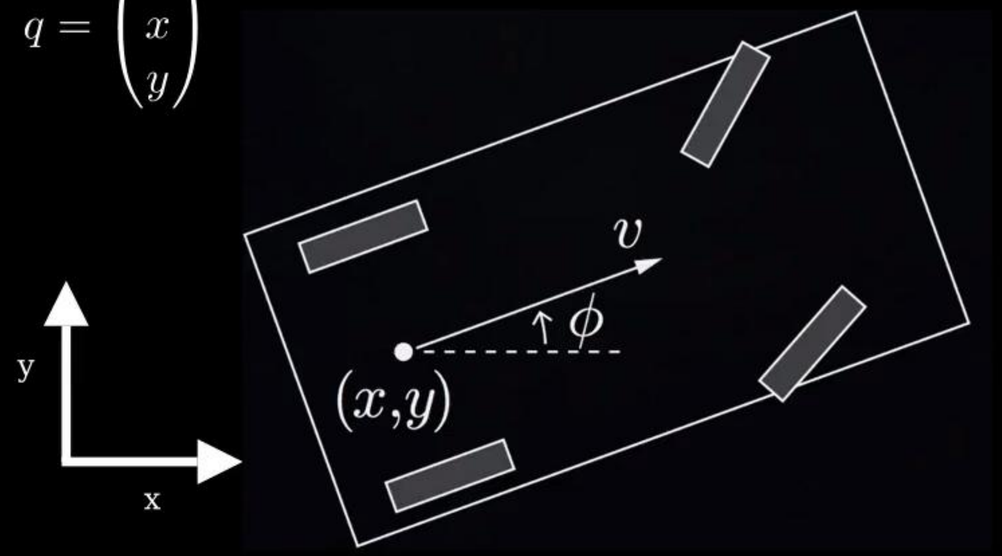

A car on a plane.

$$\dot{y} = v sin(\phi) \rightarrow v = \frac{\dot{y}}{sin(\phi)}$$

$$\dot{x} = v cos(\phi) = \frac{\dot{y}}{sin(\phi)} cos(\phi) \quad \rightarrow \dot{x} sin(\phi) - \dot{y} cos(\phi) = 0$$

Chapter 2: Configuration Space

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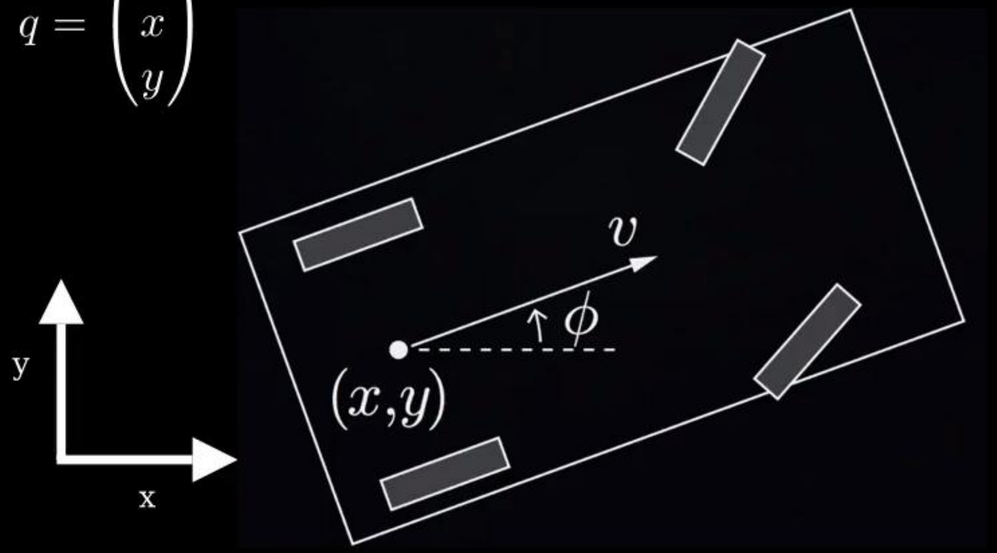
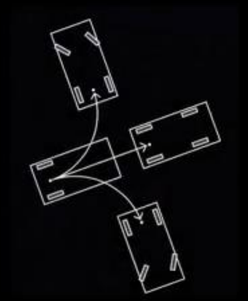


A car on a plane.

$$\dot{y} = v sin(\phi) \rightarrow v = \frac{\dot{y}}{sin(\phi)}$$

$$\dot{x} = v cos(\phi) = \frac{\dot{y}}{sin(\phi)} cos(\phi) \rightarrow \dot{x} sin(\phi) - \dot{y} cos(\phi) = 0$$



$$A(q)\dot{q} = \begin{pmatrix} 0 & sin(\phi) & cos(\phi) \end{pmatrix} \cdot \begin{pmatrix} \dot{\phi} \\ \dot{x} \\ \dot{y} \end{pmatrix} = 0$$

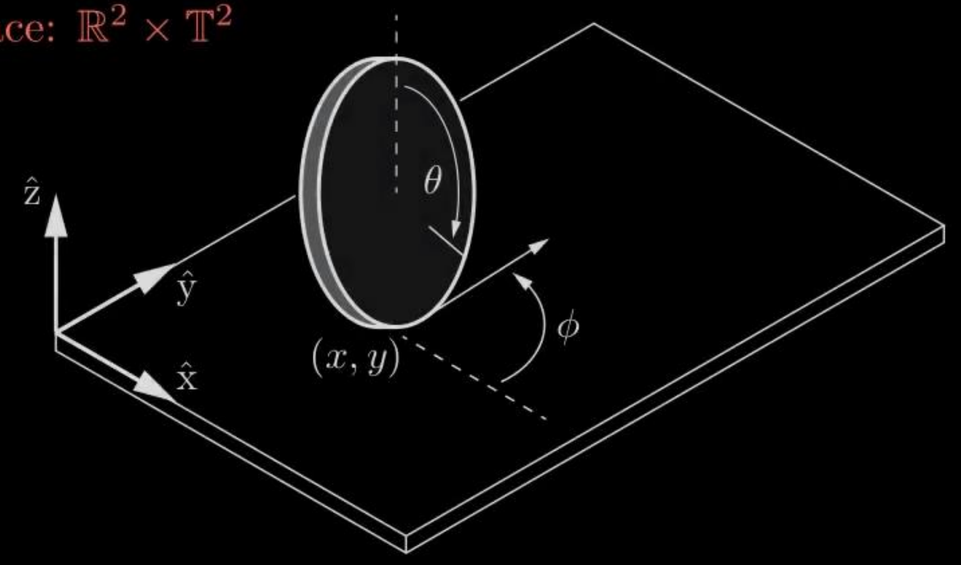
Chapter 2: Configuration Space

2.4: Configuration and Velocity Constraints

If pfaffian constraints can NOT be integrated to equivalent configuration constraints, they are non-holonomic.

They reduce the dimension of the feasible velocities, but not the dimension of the C-space.

C-space: $\mathbb{R}^2 \times \mathbb{T}^2$



A coin rolling on a plane without slipping.

Chapter 2: Configuration Space

2.4: Configuration and Velocity Constraints

If pfaffian constraints can NOT be integrated to equivalent configuration constraints, they are non-holonomic.

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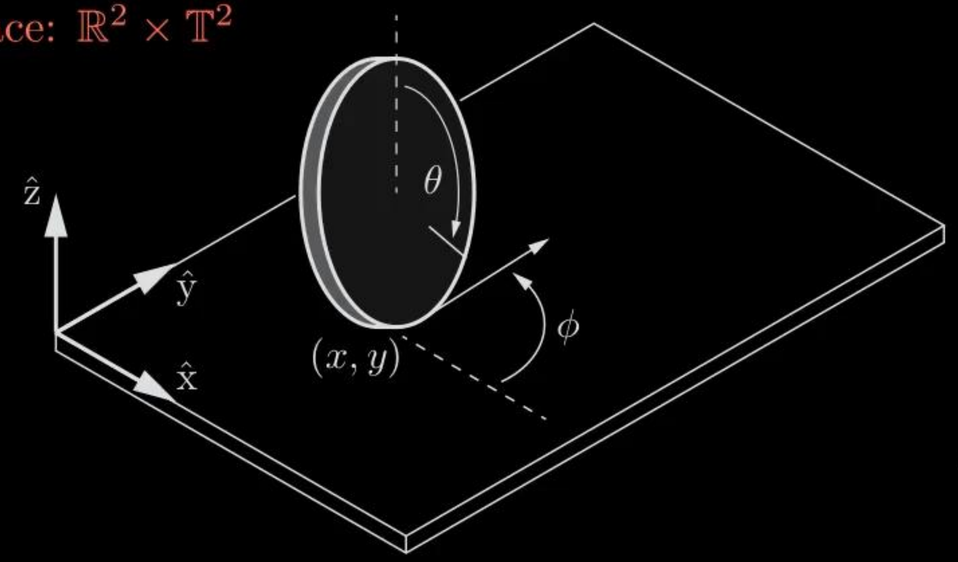
C-space: $\mathbb{R}^2 \times \mathbb{T}^2$



A coin rolling on a plane without slipping.

The coin must always roll in the direction indicated by $(\cos\phi, \sin\phi)$, with forward speed $r\dot{\theta}$:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = r\dot{\theta} \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix}$$

Collecting the four C-space coordinates into a single vector $q = [q_1\ q_2\ q_3\ q_4]^{\mathrm{T}} = [x\ y\ \phi\ \theta]^{\mathrm{T}} \in \mathbb{R}^2 \times T^2$, the above no-slip rolling constraint can then be expressed in the form

$$\begin{bmatrix} 1 & 0 & 0 & -r\cos q_3 \\ 0 & 1 & 0 & -r\sin q_3 \end{bmatrix} \dot{q} = 0. \tag{2.10}$$

These constraints are not integrable;

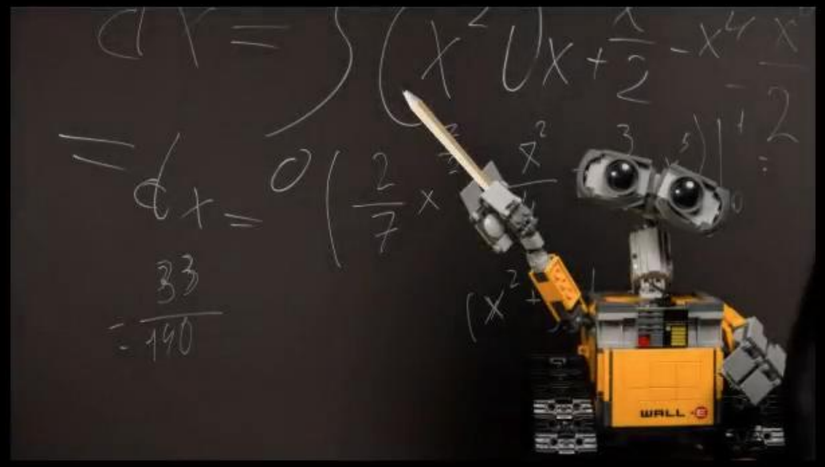
Chapter 2: Configuration Space

2.5: Task Space and Workspace

The task space is a space in which the robot's task can be naturally expressed.

For example, if the task is to control the position of the tip of a marker on a board, then task space is the Euclidean plane.

You only have to know about the task, not the robot, to define the task space.

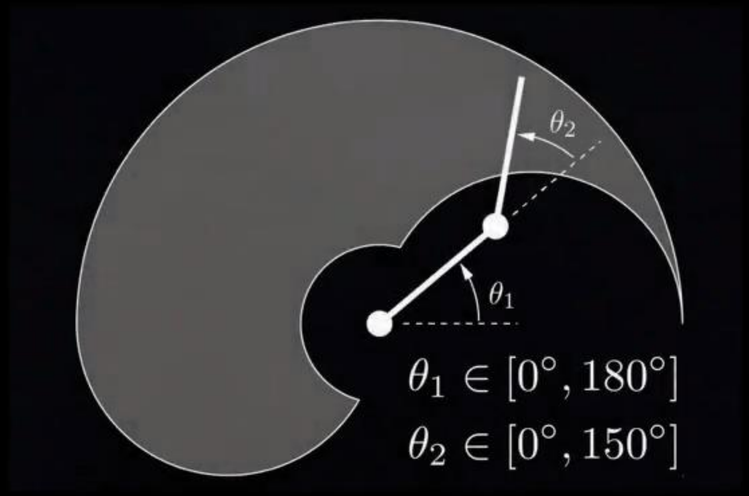


2.5: Task Space and Workspace

The workspace is a specification of the configurations that the end-effector of the robot can reach.

It has nothing to do with a particular task.

For example, a planar robot with 2 revolute joints, limited to ranges of motion of 180 and 150 degrees, has the workspace:



Chapter 2: Configuration Space

2.5: Task Space and Workspace

The SCARA robot

Chapter 2: Configuration Space

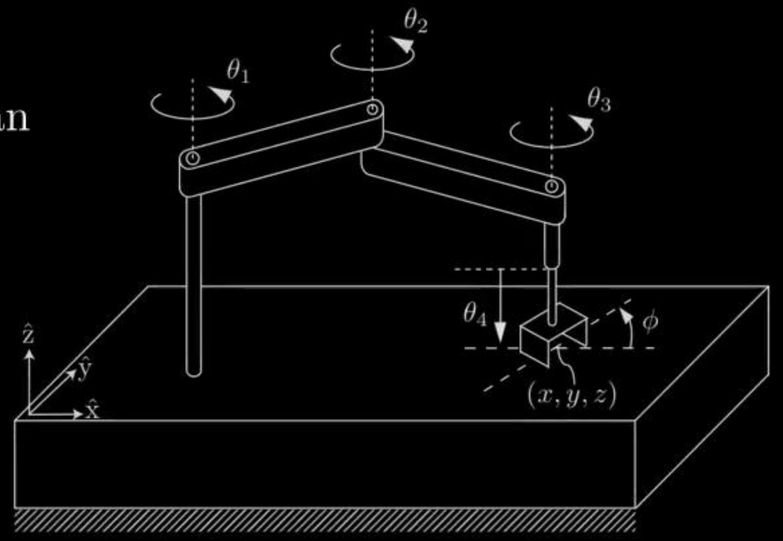
2.5: Task Space and Workspace

The SCARA robot is an RRRP open chain that is widely used for tabletop pick-and-place tasks.

The end-effector configuration is described by the four parameters (x, y, z, ϕ).

Task space: $\mathbb{R}^3 \times \mathbb{S}^1$

The workspace is defined as the EE reachable points in (x, y, z) Cartesian space, since all orientations can be achieved



Chapter 2: Configuration Space

2.5: Task Space and Workspace

The task space and workspace relate to configuration of the EE and are distinct from the robot's C-space.

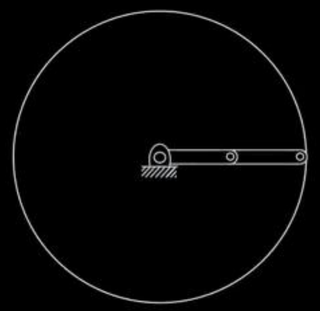
Some points in the task space may not be reachable at all by the robot

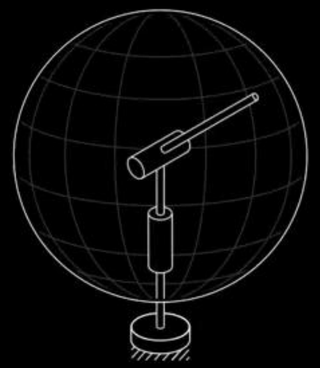
All points in the workspace are reachable by at least one configuration of the robot.

A point in the task space or the workspace may be achievable by more than one robot configuration.

Some freedoms of the EE (for ex. orientation), do not have to be represented for the task and workspace.

Two mechanisms with the same C-space may also have diferent workspaces.





Course Introduction

Chapter 2- Configuration Space

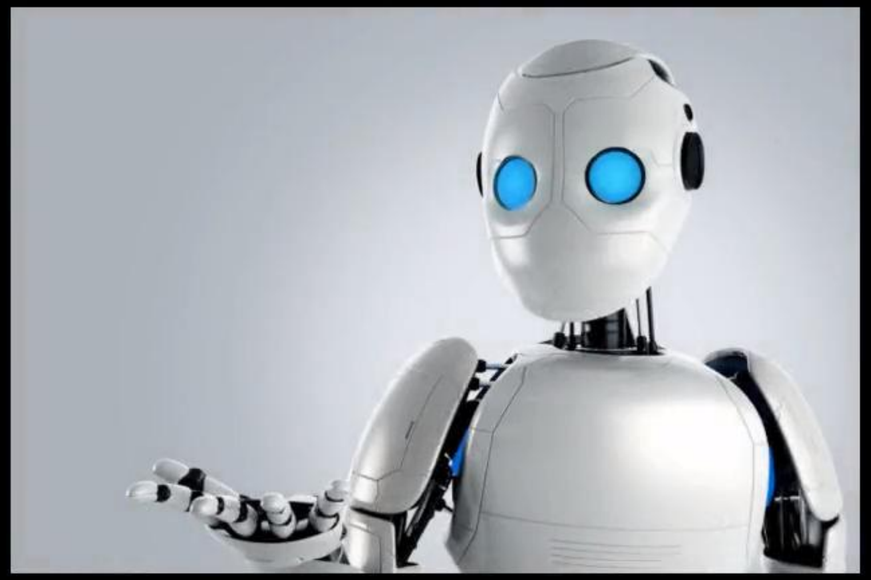
→ Chapter 3- Rigid-Body Motions

Chapter 4- Forward Kintematics (Lecture 4)

Chapter 3: Rigid-Body Motions

In this chapter we will ...

- Learn the mathematical description of rigidbody configuration, rotation, velocity and forces.
- Adopt the usage of an implicit representation.
- Attach a reference frame to a rigid body as the origin of its 3 axis space (or 2 in 2d).
- Use the right hand rule.



Chapter 3: Rigid-Body Motions

The right hand rule

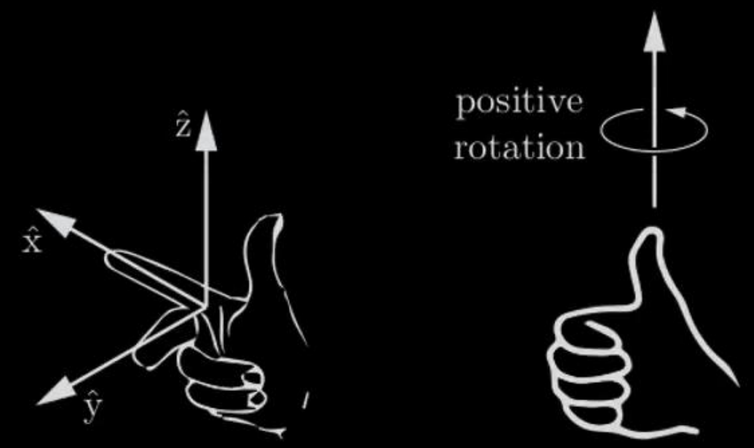
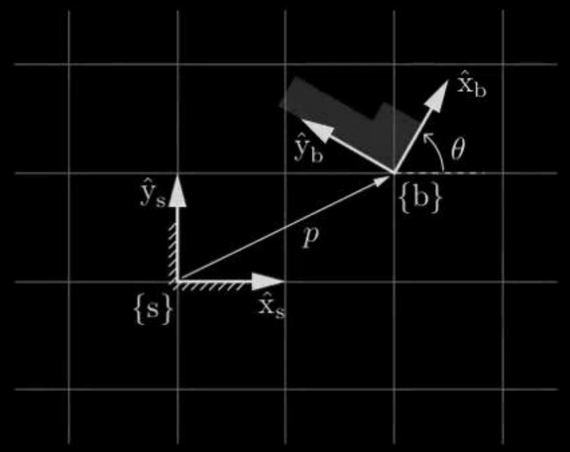


Figure 3.2: (Left) The $\hat{x}$, $\hat{y}$, and $\hat{z}$ axes of a right-handed reference frame are aligned with the index finger, middle finger, and thumb of the right hand, respectively. (Right) A positive rotation about an axis is in the direction in which the fingers of the right hand curl when the thumb is pointed along the axis.

Chapter 3: Rigid-Body Motions

3.1: Rigid-Body Motions in the Plane

- Given a fixed frame {s} and a body frame {b} as shown in the figure.
- How would you describe the configuration of planar body in respect to {s} ?



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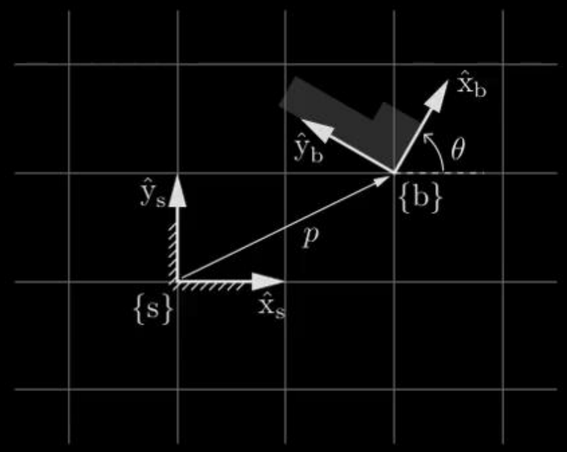
Translation

Orientation



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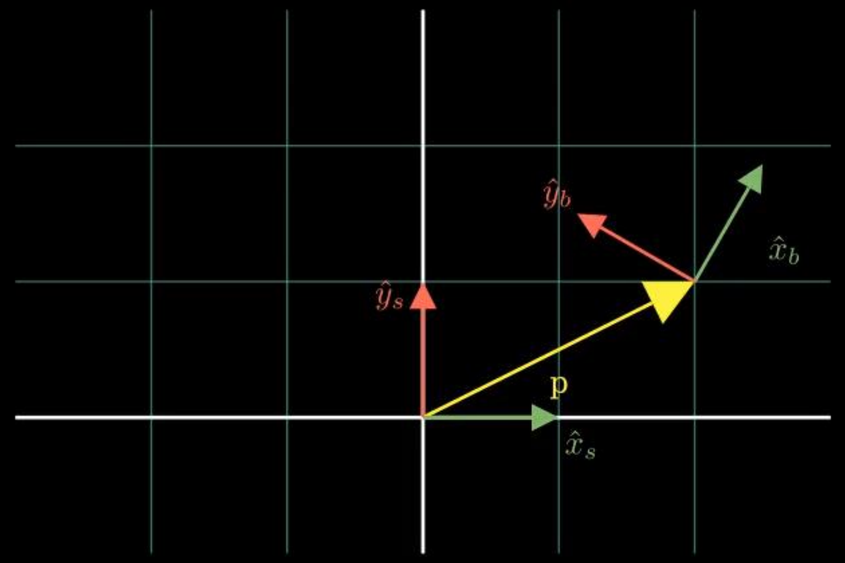

Orientation

Chapter 3: Rigid-Body Motions

3.1: Rigid-Body Motions in the Plane

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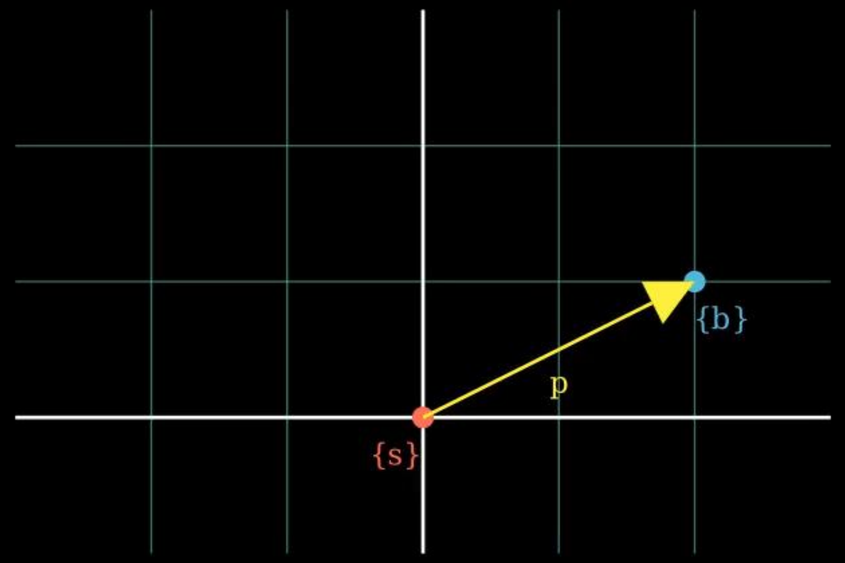
Translation

$$p = p_x \hat{x}_s + p_y \hat{y}_s$$

$$p = \begin{bmatrix} p_x \\ p_y \end{bmatrix}$$



Orientation

Chapter 3: Rigid-Body Motions

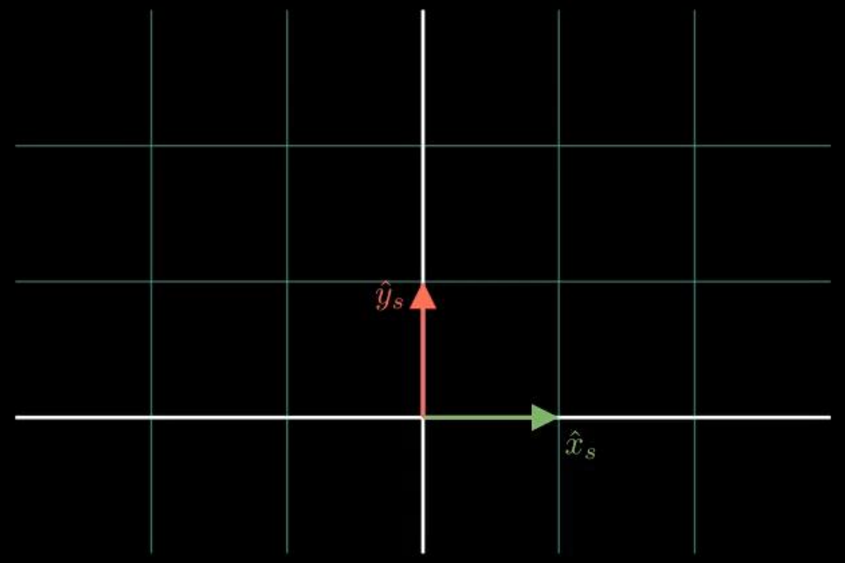
3.1: Rigid-Body Motions in the Plane

- Given a fixed frame {s} and a body frame {b} as shown in the figure.
- How would you describe the configuration of planar body in respect to {s} ?

Translation

$$p = p_x \hat{x}_s + p_y \hat{y}_s$$

$$p = \begin{bmatrix} p_x \\ p_y \end{bmatrix}$$



Orientation

Chapter 3: Rigid-Body Motions

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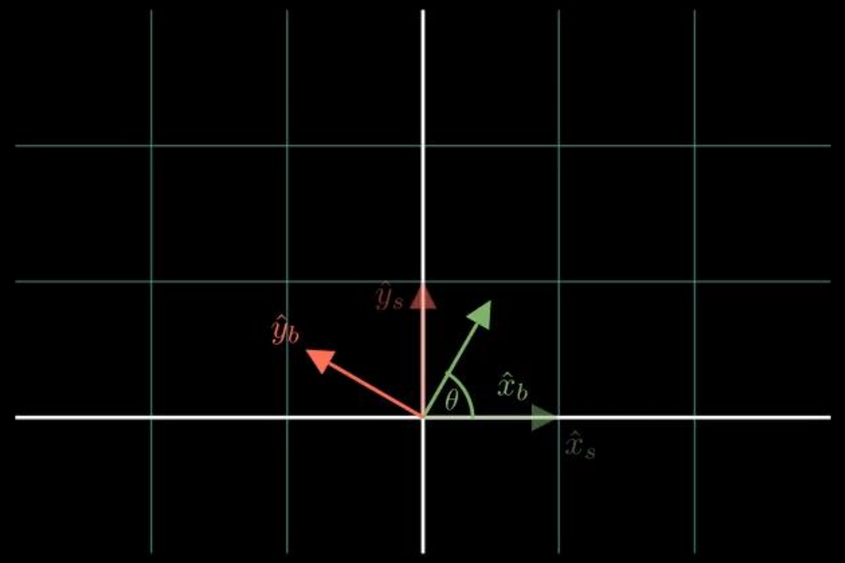

Orientation

Chapter 3: Rigid-Body Motions

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- Given a fixed frame {s} and a body frame {b} as shown in the figure.
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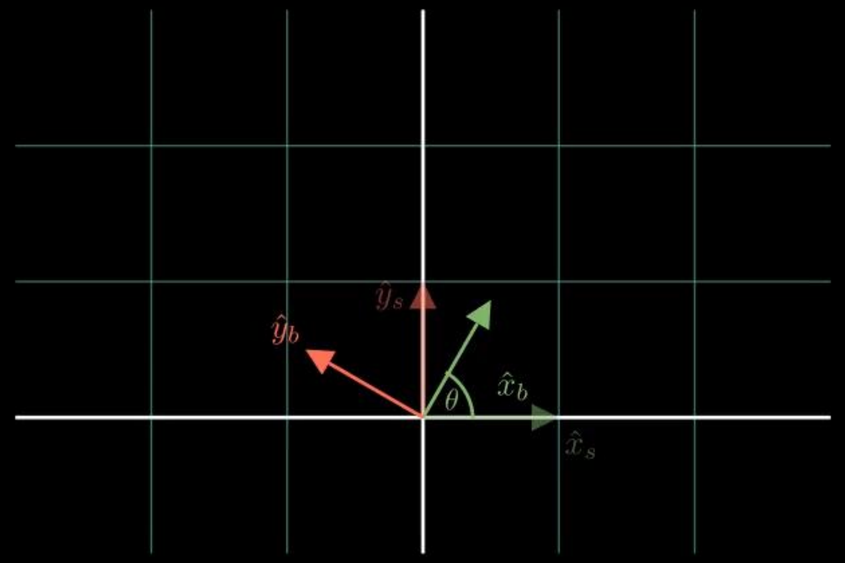


Orientation

$$P = [\hat{\mathrm{x}}_\mathrm{b} \;\; \hat{\mathrm{y}}_\mathrm{b}] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Chapter 3: Rigid-Body Motions

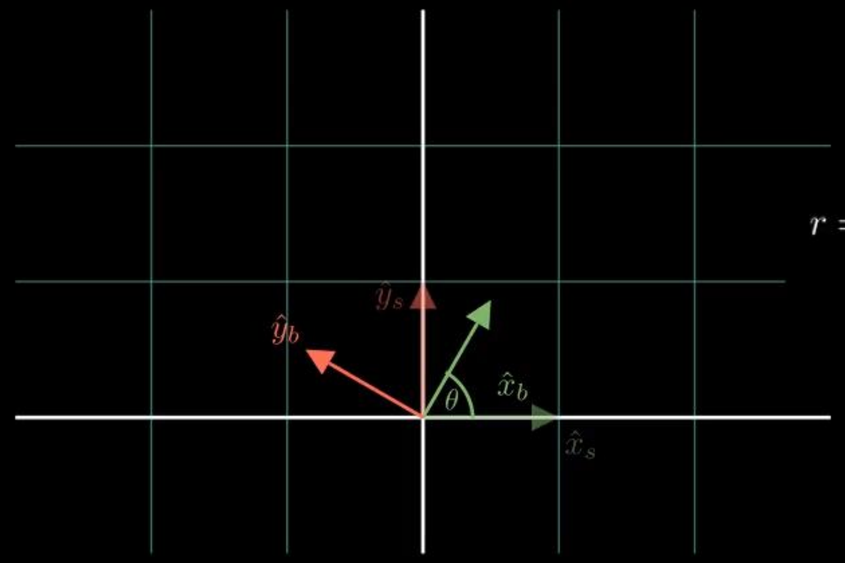
3.1: Rigid-Body Motions in the Plane

- Given a fixed frame {s} and a body frame {b} as shown in the figure.
- How would you describe the configuration of planar body in respect to {s} ?

Translation

$$p = p_x \hat{x}_s + p_y \hat{y}_s$$

$$p = \begin{bmatrix} p_x \\ p_y \end{bmatrix}$$



Translation & Rotation

$$r = \begin{bmatrix} r_x \\ r_y \end{bmatrix}, \qquad R = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}.$$

Chapter 3: Rigid-Body Motions

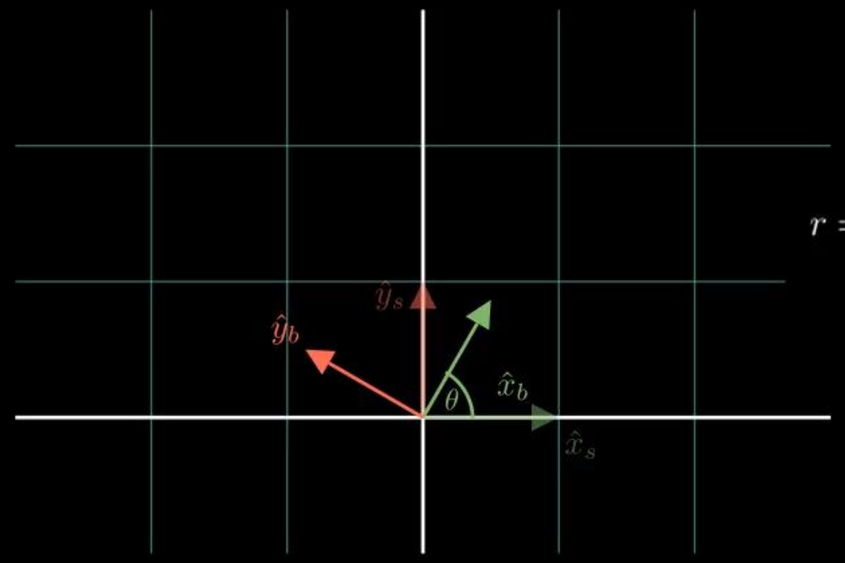
3.1: Rigid-Body Motions in the Plane

- We Used 4 parameters to describe the rotation.
- What's the actual DOF of the rotation?

Rotation Constraints

$$||\hat{x}|| = 1$$
$$||\hat{y}|| = 1$$

$$\hat{x} \cdot \hat{y} = 0$$



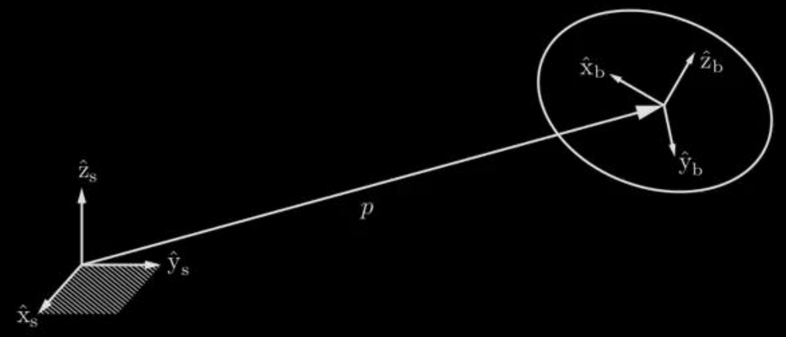
Translation & Rotation

$$r = \begin{bmatrix} r_x \\ r_y \end{bmatrix}, \qquad R = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}.$$

Chapter 3: Rigid-Body Motions

3.1: Rigid-Body Motions in the Plane

- Section 3.1 shows ...
- how to represent a 2D configuration of a rigid body in {s}
- how to change the reference frame in which a 2D vector or frame is represented
- how to displace a 2D vector or a frame
- an example of 2D screw motion and twist
- We will cover it in 3D in section 3.2



Chapter 3: Rigid-Body Motions

3.2.1: Rotation Matrices

- There are three major uses for a 3x3 rotation matrix R:
- represent an orientation
- to change the reference frame in which a vector or a frame is represented
- to rotate a vector or a frame

$$R = \begin{bmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\gamma \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta\cos\gamma - \cos\alpha\sin\gamma \\ -\sin\beta & \cos\beta\sin\gamma & \cos\beta\cos\gamma \end{bmatrix}$$

Chapter 3: Rigid-Body Motions

3.2.1: Rotation Matrices

- We Used 9 parameters to describe the rotation.
- What's the actual DOF of the rotation?

$$R = \begin{bmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\gamma \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta\cos\gamma - \cos\alpha\sin\gamma \\ -\sin\beta & \cos\beta\sin\gamma & \cos\beta\cos\gamma \end{bmatrix}$$

Chapter 3: Rigid-Body Motions

3.2.1: Rotation Matrices

$$\mathrm{R} = \begin{bmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\gamma \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta\cos\gamma - \cos\alpha\sin\gamma \\ -\sin\beta & \cos\beta\sin\gamma & \cos\beta\cos\gamma \end{bmatrix}$$

(a) The unit norm condition: $\hat{\mathrm{x}}_\mathrm{b}$, $\hat{\mathrm{y}}_\mathrm{b}$, and $\hat{\mathrm{z}}_\mathrm{b}$ are all unit vectors, i.e.,

$$\begin{aligned} r_{11}^2 + r_{21}^2 + r_{31}^2 &= 1, \\ r_{12}^2 + r_{22}^2 + r_{32}^2 &= 1, \\ r_{13}^2 + r_{23}^2 + r_{33}^2 &= 1. \end{aligned} \tag{3.17}$$

(b) The orthogonality condition: $\hat{\mathrm{x}}_\mathrm{b} \cdot \hat{\mathrm{y}}_\mathrm{b} = \hat{\mathrm{x}}_\mathrm{b} \cdot \hat{\mathrm{z}}_\mathrm{b} = \hat{\mathrm{y}}_\mathrm{b} \cdot \hat{\mathrm{z}}_\mathrm{b} = 0$ (here $\cdot$ denotes the inner product), or

$$\begin{aligned} r_{11}r_{12} + r_{21}r_{22} + r_{31}r_{32} &= 0, \\ r_{12}r_{13} + r_{22}r_{23} + r_{32}r_{33} &= 0, \\ r_{11}r_{13} + r_{21}r_{23} + r_{31}r_{33} &= 0. \end{aligned} \tag{3.18}$$

These six constraints can be expressed more compactly as a single set of constraints on the matrix R,

$$R^\mathrm{T} R = I, \tag{3.19}$$

Chapter 3: Rigid-Body Motions

3.2.1: Rotation Matrices

Definition 3.1. The **special orthogonal group** $SO(3)$, also known as the group of rotation matrices, is the set of all 3×3 real matrices R that satisfy (i) $R^{\mathrm{T}}R = I$ and (ii) $\det R = 1$.

The set of 2×2 rotation matrices is a subgroup of $SO(3)$ and is denoted $SO(2)$.

Definition 3.2. The **special orthogonal group** $SO(2)$ is the set of all 2×2 real matrices R that satisfy (i) $R^{\mathrm{T}}R = I$ and (ii) $\det R = 1$.

From the definition it follows that every $R \in SO(2)$ can be written

$$R = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

where $\theta \in [0, 2\pi)$. The elements of $SO(2)$ represent planar orientations and the elements of $SO(3)$ represent spatial orientations.

Chapter 3: Rigid-Body Motions

3.2.1.1: Properties of Rotation Matrices

- **closure:** AB is also in the group.
- **associativity:** $(AB)C = A(BC)$.
- **identity element existence:** There exists an element I in the group (the identity matrix for $SO(n)$) such that $AI = IA = A$.
- **inverse element existence:** There exists an element A^{-1} in the group such that $AA^{-1} = A^{-1}A = I$.

Proposition 3.3. *The inverse of a rotation matrix $R \in SO(3)$ is also a rotation matrix, and it is equal to the transpose of R, i.e., $R^{-1} = R^{\mathrm{T}}$.*

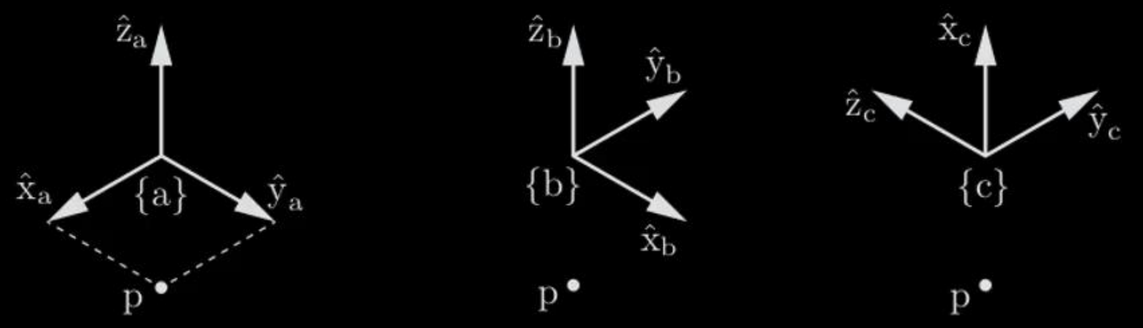
Proposition 3.4. *The product of two rotation matrices is a rotation matrix.*

Proposition 3.5. *Multiplication of rotation matrices is associative, $(R_1R_2)R_3 = R_1(R_2R_3)$, but generally not commutative, $R_1R_2 \neq R_2R_1$. For the special case of rotation matrices in $SO(2)$, rotations commute.*

Proposition 3.6. *For any vector $x \in \mathbb{R}^3$ and $R \in SO(3)$, the vector $y = Rx$ has the same length as x.*

Chapter 3: Rigid-Body Motions

3.2.1.2: Uses of Rotation Matrices



Translation

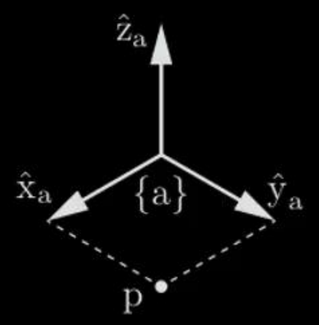
$$p_a = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad p_b = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad p_c = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$$

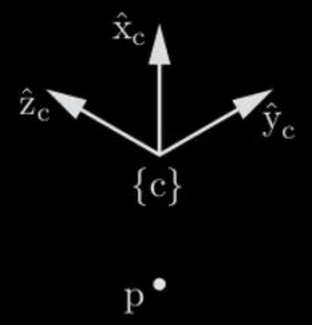
Orientation

$$R_a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R_b = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R_c = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

Chapter 3: Rigid-Body Motions

3.2.1.2: Uses of Rotation Matrices



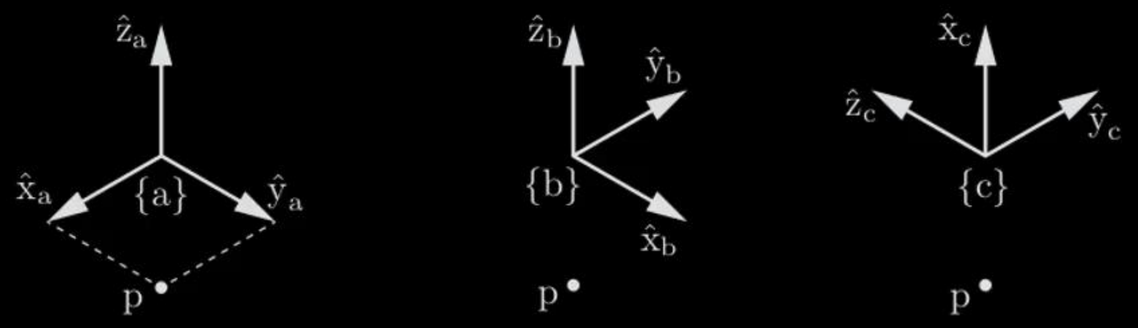


Representing an orientation

$$R_{ac} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}, \qquad R_{ca} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}.$$

Chapter 3: Rigid-Body Motions

3.2.1.2: Uses of Rotation Matrices



Changing the reference frame

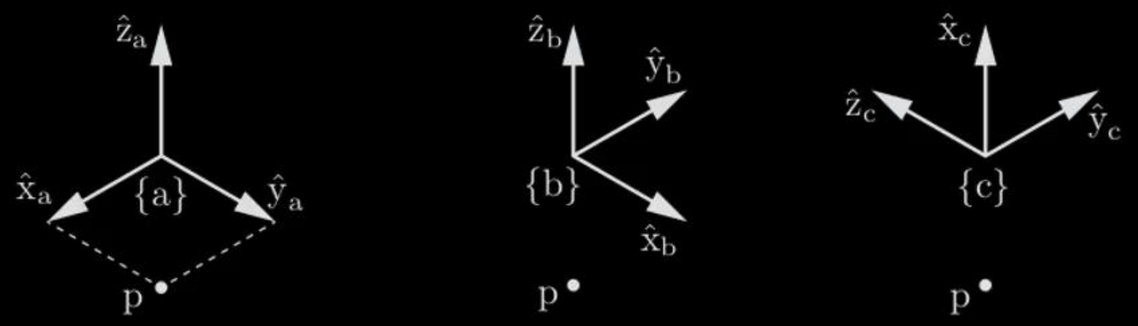
let R_{ab} represent the orientation of b in a

let R_{bc} represent the orientation of c in b

How would you extract the orientation of c in a?

Chapter 3: Rigid-Body Motions

3.2.1.2: Uses of Rotation Matrices



Changing the reference frame

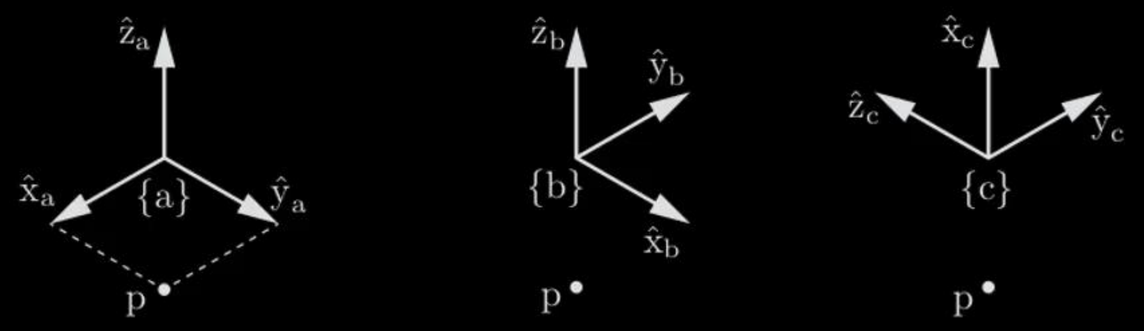
let R_{ab} represent the orientation of b in a

let R_{bc} represent the orientation of c in b

How would you extract the orientation of c in a? $R_{ac} = R_{ab}R_{bc}$

Chapter 3: Rigid-Body Motions

3.2.1.2: Uses of Rotation Matrices



Changing the reference frame

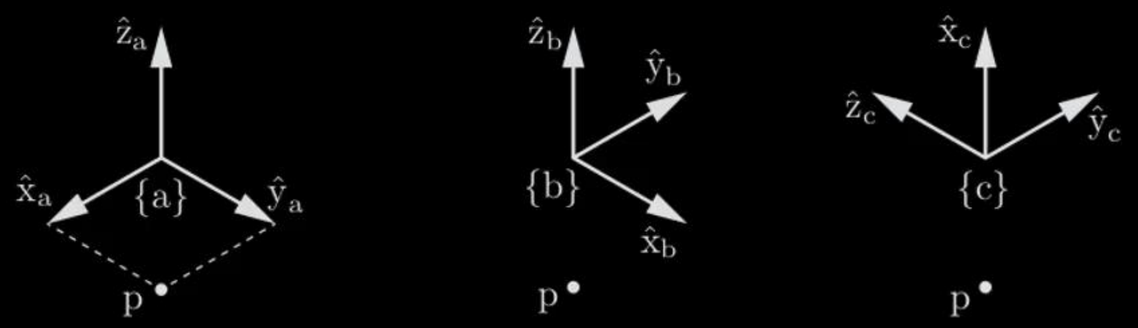
let R_{ab} represent the orientation of b in a

let p_b represent the location of p in b

How would you extract the location of p in a?

Chapter 3: Rigid-Body Motions

3.2.1.2: Uses of Rotation Matrices



Changing the reference frame

let R_{ab} represent the orientation of b in a

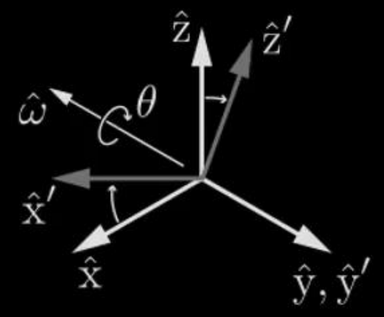
let p_b represent the location of p in b

How would you extract the location of p in a? $p_a = R_{ab} p_b$

Chapter 3: Rigid-Body Motions

3.2.1.2: Uses of Rotation Matrices

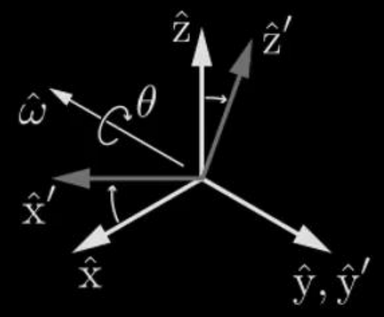
Rotating a vector or a frame



Chapter 3: Rigid-Body Motions

3.2.1.2: Uses of Rotation Matrices

Rotating a vector or a frame



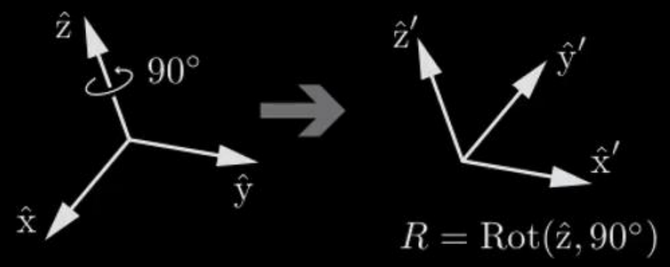
for $\hat{\omega} = (\hat{\omega}_1, \hat{\omega}_2, \hat{\omega}_3)$,

$$\mathrm{Rot}(\hat{\omega}, \theta) = \begin{bmatrix} c_\theta + \hat{\omega}_1^2(1 - c_\theta) & \hat{\omega}_1\hat{\omega}_2(1 - c_\theta) - \hat{\omega}_3 s_\theta & \hat{\omega}_1\hat{\omega}_3(1 - c_\theta) + \hat{\omega}_2 s_\theta \\ \hat{\omega}_1\hat{\omega}_2(1 - c_\theta) + \hat{\omega}_3 s_\theta & c_\theta + \hat{\omega}_2^2(1 - c_\theta) & \hat{\omega}_2\hat{\omega}_3(1 - c_\theta) - \hat{\omega}_1 s_\theta \\ \hat{\omega}_1\hat{\omega}_3(1 - c_\theta) - \hat{\omega}_2 s_\theta & \hat{\omega}_2\hat{\omega}_3(1 - c_\theta) + \hat{\omega}_1 s_\theta & c_\theta + \hat{\omega}_3^2(1 - c_\theta) \end{bmatrix},$$

Chapter 3: Rigid-Body Motions

3.2.1.2: Uses of Rotation Matrices

Rotating a vector or a frame



$$\mathrm{Rot}(\hat{x},\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \quad \mathrm{Rot}(\hat{y},\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix},$$

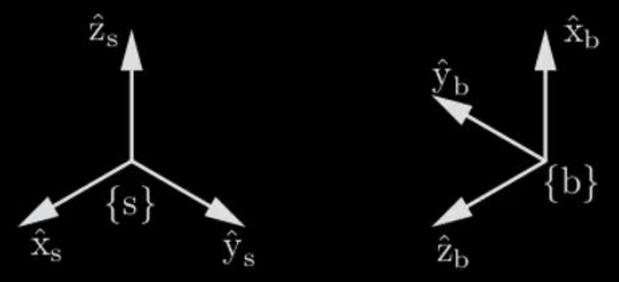
$$\mathrm{Rot}(\hat{z},\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Chapter 3: Rigid-Body Motions

3.2.1.2: Uses of Rotation Matrices

R_{sb} represents frame b relative to s.
Rotate b by θ about a unit axis $\hat{w} = \hat{z}$

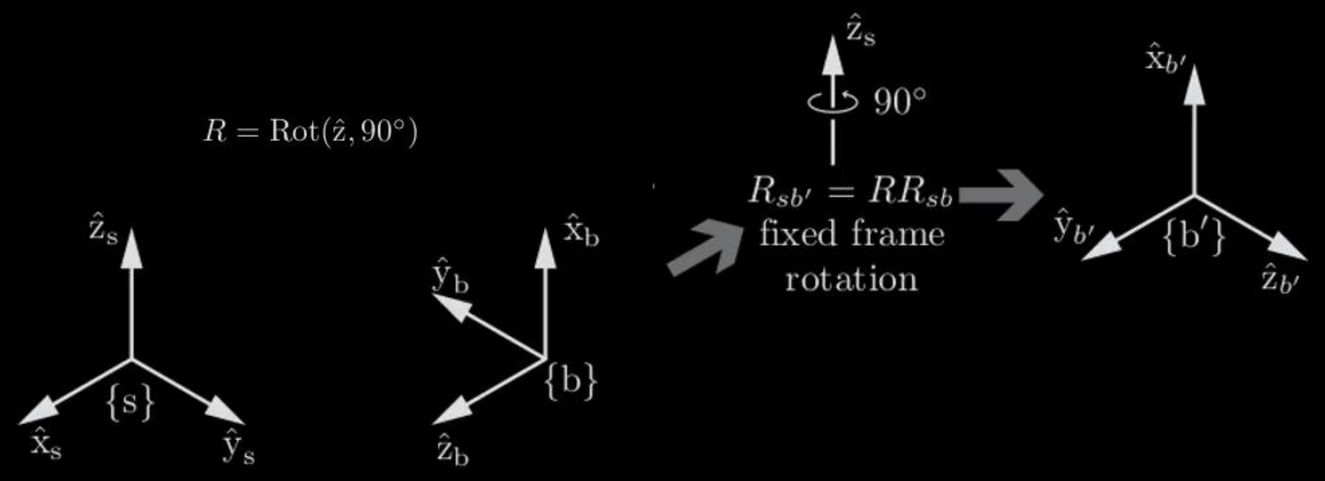
$R = \mathrm{Rot}(\hat{z}, 90°)$



Chapter 3: Rigid-Body Motions

3.2.1.2: Uses of Rotation Matrices

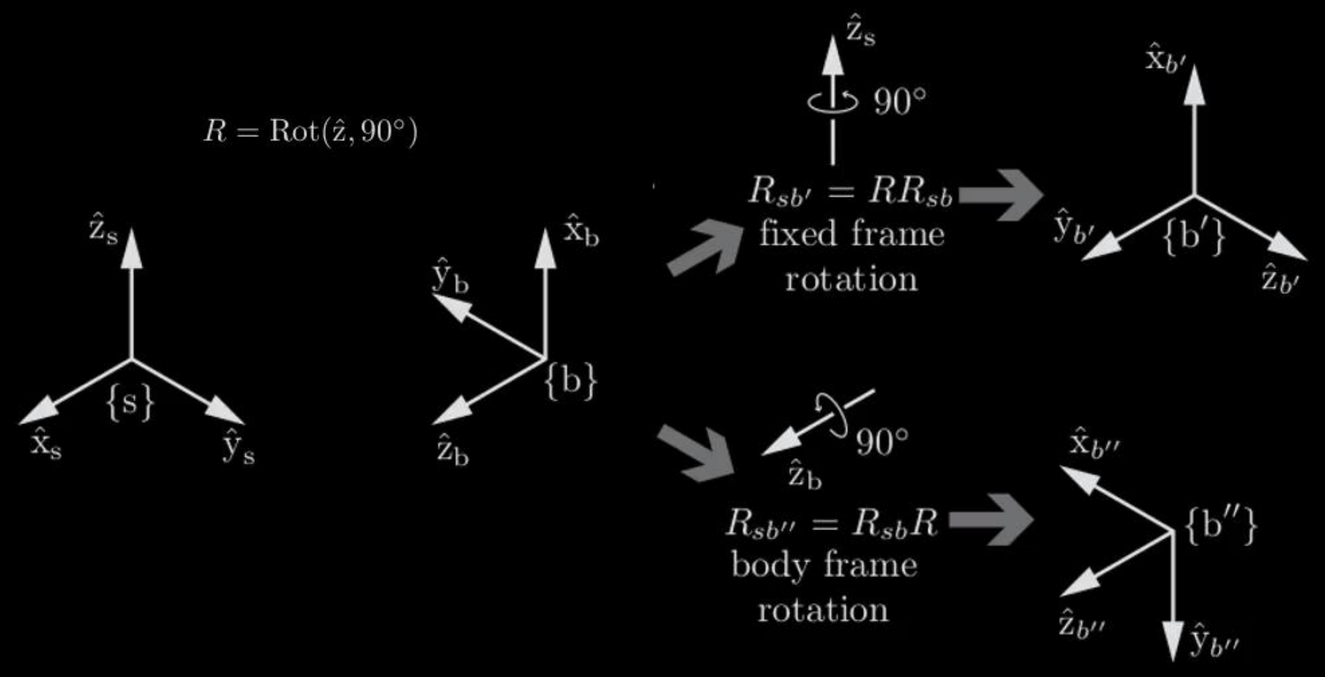
R_{sb} represents frame b relative to s.
Rotate b by θ about a unit axis $\hat{w} = \hat{z}$



Chapter 3: Rigid-Body Motions

3.2.1.2: Uses of Rotation Matrices

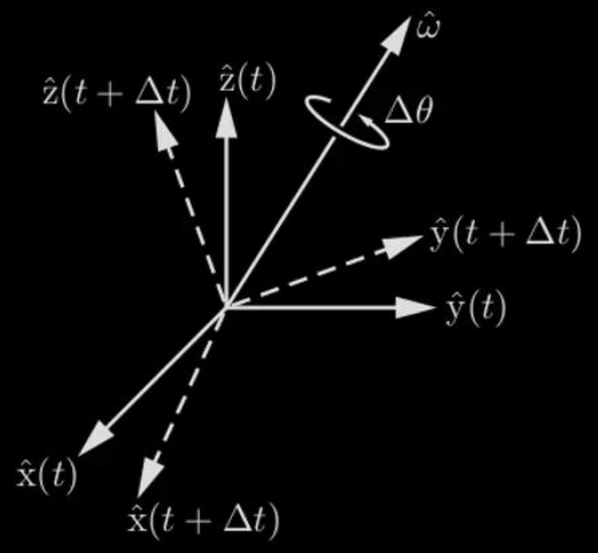
R_{sb} represents frame b relative to s.
Rotate b by θ about a unit axis $\hat{w} = \hat{z}$



Chapter 3: Rigid-Body Motions

3.2.2: Angular Velocities

In this section we'll see how to respresent rotation using angular velocity



Chapter 3: Rigid-Body Motions

3.2.2: Angular Velocities

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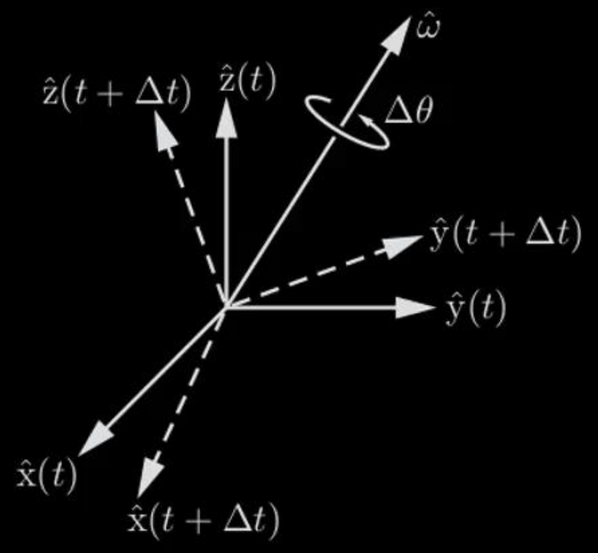
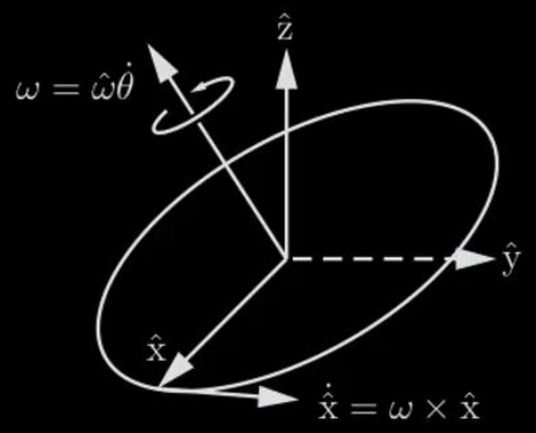




$$\begin{aligned} \dot{\hat{x}} &= w \times \hat{x}, \\ \dot{\hat{y}} &= w \times \hat{y}, \\ \dot{\hat{z}} &= w \times \hat{z}. \end{aligned}$$

Chapter 3: Rigid-Body Motions

3.2.2: Angular Velocities

Let $R(t)$ be the rotation matrix describing the orientation of the body frame with respect to the fixed frame at time t.

let $w_s \in \mathbb{R}^3$ be the angular velocity w expressed in fixed-frame coordinates

$$\dot{r}_i = \omega_s \times r_i, \qquad i = 1, 2, 3.$$

$$\dot{R} = [\omega_s \times r_1 \;\; \omega_s \times r_2 \;\; \omega_s \times r_3] = \omega_s \times R.$$

Chapter 3: Rigid-Body Motions

3.2.2: Angular Velocities

Let $R(t)$ be the rotation matrix describing the orientation of the body frame with respect to the fixed frame at time t.

let $w_s \in \mathbb{R}^3$ be the angular velocity w expressed in fixed-frame coordinates

$$\dot{r}_i = \omega_s \times r_i, \qquad i = 1, 2, 3.$$

$$\dot{R} = [\omega_s \times r_1 \;\; \omega_s \times r_2 \;\; \omega_s \times r_3] = \omega_s \times R.$$

The cross product is eliminated by using skew symetric matrix multiplication.

$$\dot{R} = w_s \times R = [w_s]R$$

$$[x] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

Chapter 3: Rigid-Body Motions

3.2.2: Angular Velocities

Some usuful properties:(see book for proff):

$$R[w]R^T = [Rw]$$

$$\dot{R} = [w_s]R$$

$$\dot{R}R^{-1} = [w_s]$$

$$w_s = R_{sb}w_b \rightarrow w_b = R_{sb}^{-1}w_s = R^{-1}w_s = R^T w_s$$

Proposition 3.9. *Let $R(t)$ denote the orientation of the rotating frame as seen from the fixed frame. Denote by* w *the angular velocity of the rotating frame. Then*

$$\dot{R}R^{-1} = [\omega_s], \quad (3.37)$$

$$R^{-1}\dot{R} = [\omega_b], \quad (3.38)$$

Chapter 3: Rigid-Body Motions

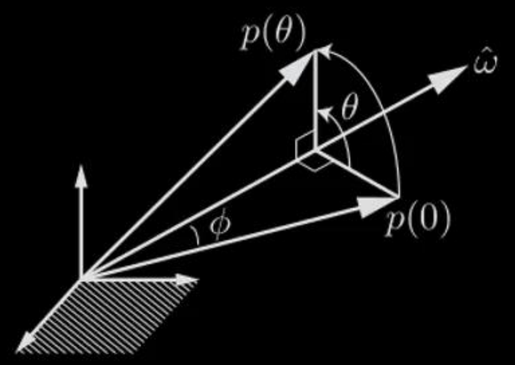
3.2.3.2: Exponential Coordinates of Rotations

$p(0)$ is rotated by θ about $\hat{w}$ to $p(\theta)$:

Assume that: $\dot{\theta} = 1rad/sec$, then $w = \hat{w}\dot{\theta} = \hat{w}$

$$\dot{p} = \hat{w} \times p$$

$$\dot{p} = [\hat{w}]p$$



Chapter 3: Rigid-Body Motions

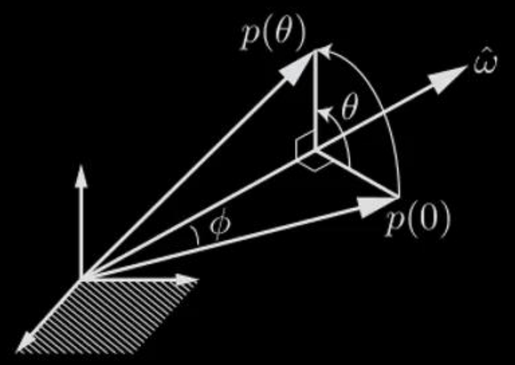
3.2.3.2: Exponential Coordinates of Rotations

$p(0)$ is rotated by θ about $\hat{w}$ to $p(\theta)$:

Assume that: $\dot{\theta} = 1rad/sec$, then $w = \hat{w}\dot{\theta} = \hat{w}$

$$\dot{p} = \hat{w} \times p$$

$$\dot{p} = [\hat{w}]p$$



This is a linear differential equation of the form $\dot{x} = Ax$

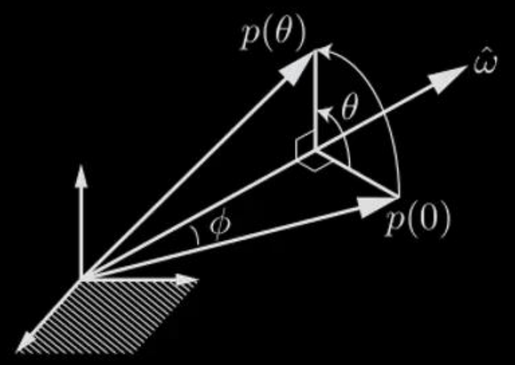
Its solution id given by: $p(t) = e^{[\hat{w}]t}p(0)$

Change parameter and get: $p(\theta) = e^{[\hat{w}]\theta}p(0) = Rot(\hat{w}, \theta)p(0)$

Chapter 3: Rigid-Body Motions

3.2.3.2: Exponential Coordinates of Rotations

The vector $\hat{w}\theta \in \mathbb{R}^3$ is the three-parameter exponential coordinate representation of the rotation



The Rodrigues' formula for rotations:

$$Rot(\hat{w}, \theta) = e^{[\hat{w}]\theta} = I + sin\theta[\hat{w}] + (1 - cos\theta)[\hat{w}]^2$$

$R' = Rot(\hat{w}, \theta)R$ is the orientation achieved by the rotation R by θ about the axis $\hat{w}$ in the fixed frame

$R'' = RRot(\hat{w}, \theta)$ is the orientation achieved by the rotation R by θ about the axis $\hat{w}$ in the body frame

Chapter 3: Rigid-Body Motions

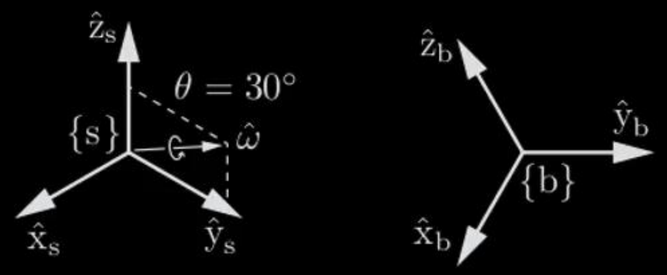
3.2.3.2: Exponential Coordinates of Rotations

Example 3.2:

The frame b is obtained by rotation from an initial orientation aligned with the fixed frame s about a unit axis $\hat{w}_1 = (0, 0.866, 0.5)$ by an angle of

$$\theta_1 = 30 = 0.524 rad$$

a) Calculate the rotation matrix representation of b



Chapter 3: Rigid-Body Motions

3.2.3.2: Exponential Coordinates of Rotations

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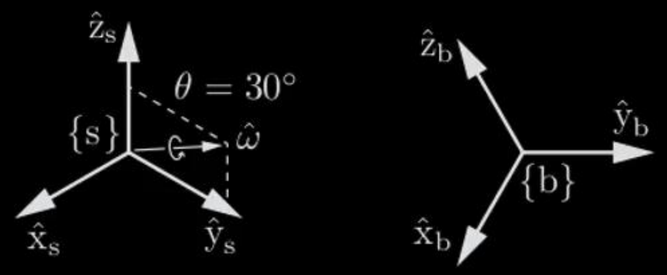

a) Calculate the rotation matrix representation of b

$$\begin{aligned}
R &= e^{[\hat{\omega}_1]\theta_1} \\
&= I + \sin\theta_1[\hat{\omega}_1] + (1-\cos\theta_1)[\hat{\omega}_1]^2 \\
&= I + 0.5\begin{bmatrix} 0 & -0.5 & 0.866 \\ 0.5 & 0 & 0 \\ -0.866 & 0 & 0 \end{bmatrix} + 0.134\begin{bmatrix} 0 & -0.5 & 0.866 \\ 0.5 & 0 & 0 \\ -0.866 & 0 & 0 \end{bmatrix}^2 \\
&= \begin{bmatrix} 0.866 & -0.250 & 0.433 \\ 0.250 & 0.967 & 0.058 \\ -0.433 & 0.058 & 0.899 \end{bmatrix}.
\end{aligned}$$

Chapter 3: Rigid-Body Motions

3.2.3.2: Exponential Coordinates of Rotations

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The frame b is obtained by rotation from an initial orientation aligned with the fixed frame s about a unit axis $\hat{w}_1 = (0, 0.866, 0.5)$ by an angle of

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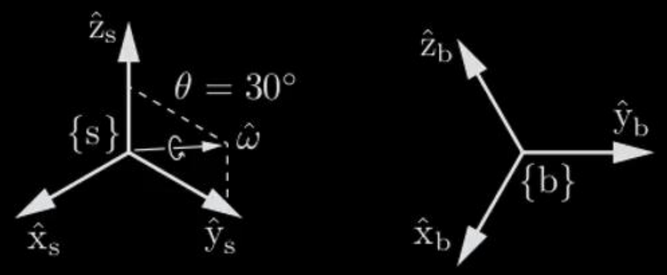
a) Calculate the rotation matrix representation of b

b) b got rotated again by θ_2 about a fixed frame axis $\hat{w}_2 \neq \hat{w}_1$



Chapter 3: Rigid-Body Motions

3.2.3.2: Exponential Coordinates of Rotations

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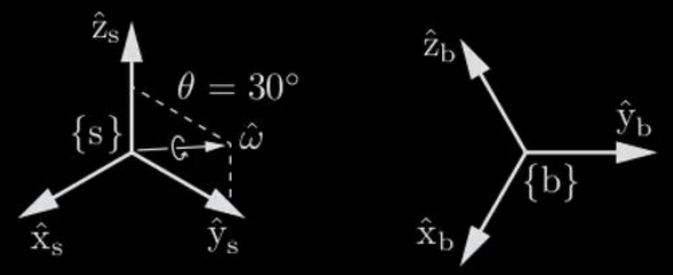
$$\theta_1 = 30 = 0.524 rad$$

a) Calculate the rotation matrix representation of b

b) b got rotated again by θ_2 about a fixed frame axis $\hat{w}_1 \neq \hat{w}_1$

$$R' = Rot(\hat{w}_2, \theta_2)R$$



Chapter 3: Rigid-Body Motions

3.2.3.2: Exponential Coordinates of Rotations

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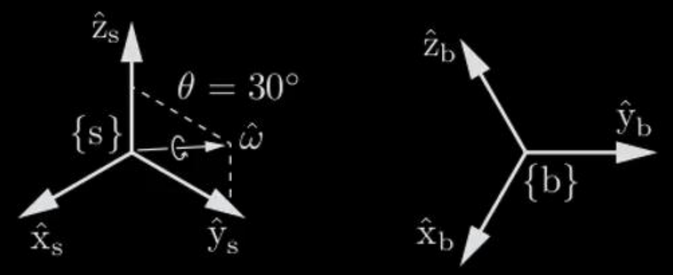
$$\theta_1 = 30 = 0.524 rad$$

a) Calculate the rotation matrix representation of b

b) b got rotated again by θ_2 about a fixed frame axis $\hat{w}_1 \neq \hat{w}_1$

c) same as b) but the rotation is about the body frame



Chapter 3: Rigid-Body Motions

3.2.3.2: Exponential Coordinates of Rotations

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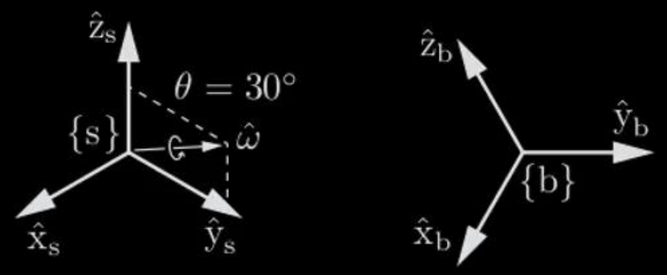


a) Calculate the rotation matrix representation of b

b) b got rotated again by θ_2 about a fixed frame axis $\hat{w}_1 \neq \hat{w}_1$

c) same as b) but the rotation is about the body frame

$$R' = RRot(\hat{w}_2, \theta_2)$$

3.2.3.3: Matrix Logarithm of Rotations

$$\begin{aligned} \exp: \quad & [\hat{\omega}]\theta \in so(3) \quad \rightarrow \quad R \in SO(3), \\ \log: \quad & R \in SO(3) \quad \rightarrow \quad [\hat{\omega}]\theta \in so(3). \end{aligned}$$

Chapter 3: Rigid-Body Motions

3.2.3.3: Matrix Logarithm of Rotations

Algorithm: Given $R \in SO(3)$, find a $\theta \in [0, \pi]$ and a unit rotation axis $\hat{\omega} \in \mathbb{R}^3, \|\hat{\omega}\| = 1$, such that $e^{[\hat{\omega}]\theta} = R$. The vector $\hat{\omega}\theta \in \mathbb{R}^3$ comprises the exponential coordinates for R and the skew-symmetric matrix $[\hat{\omega}]\theta \in so(3)$ is the matrix logarithm of R.

(a) If $R = I$ then $\theta = 0$ and $\hat{\omega}$ is undefined.

(b) If $\operatorname{tr} R = -1$ then $\theta = \pi$. Set $\hat{\omega}$ equal to any of the following three vectors that is a feasible solution:

$$\hat{\omega} = \frac{1}{\sqrt{2(1 + r_{33})}} \begin{bmatrix} r_{13} \\ r_{23} \\ 1 + r_{33} \end{bmatrix} \tag{3.58}$$

or

$$\hat{\omega} = \frac{1}{\sqrt{2(1 + r_{22})}} \begin{bmatrix} r_{12} \\ 1 + r_{22} \\ r_{32} \end{bmatrix} \tag{3.59}$$

or

$$\hat{\omega} = \frac{1}{\sqrt{2(1 + r_{11})}} \begin{bmatrix} 1 + r_{11} \\ r_{21} \\ r_{31} \end{bmatrix}. \tag{3.60}$$

(Note that if $\hat{\omega}$ is a solution, then so is $-\hat{\omega}$.)

(c) Otherwise $\theta = \cos^{-1}\left(\frac{1}{2}(\operatorname{tr} R - 1)\right) \in [0, \pi)$ and

$$[\hat{\omega}] = \frac{1}{2\sin\theta}(R - R^{\mathrm{T}}). \tag{3.61}$$

Chapter 3: Rigid-Body Motions

3.3.1: Homogeneous Transformation Matrices

Definition 3.13. The **special Euclidean group** $SE(3)$, also known as the group of **rigid-body motions** or **homogeneous transformation matrices** in $\mathbb{R}^3$, is the set of all 4×4 real matrices T of the form

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_1 \\ r_{21} & r_{22} & r_{23} & p_2 \\ r_{31} & r_{32} & r_{33} & p_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{3.62}$$

where $R \in SO(3)$ and $p \in \mathbb{R}^3$ is a column vector.

Definition 3.14. The special Euclidean group $SE(2)$ is the set of all 3×3 real matrices T of the form

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}, \tag{3.63}$$

where $R \in SO(2)$, $p \in \mathbb{R}^2$, and 0 denotes a row vector of two zeros.

A matrix $T \in SE(2)$ is always of the form

$$T = \begin{bmatrix} r_{11} & r_{12} & p_1 \\ r_{21} & r_{22} & p_2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & p_1 \\ \sin\theta & \cos\theta & p_2 \\ 0 & 0 & 1 \end{bmatrix},$$

where $\theta \in [0, 2\pi)$.

Chapter 3: Rigid-Body Motions

3.3.1.1: Properties of Transformation Matrices

Proposition 3.15. *The inverse of a transformation matrix $T \in SE(3)$ is also a transformation matrix, and it has the following form:*

$$T^{-1} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} R^{\mathrm{T}} & -R^{\mathrm{T}}p \\ 0 & 1 \end{bmatrix}. \tag{3.64}$$

$$T \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} Rx + p \\ 1 \end{bmatrix}$$

Chapter 3: Rigid-Body Motions

3.3.1.2: Uses of Transformation Matrices

As was the case for rotation matrices, there are three major uses for a transformation matrix T:

(a) to represent the configuration (position and orientation) of a rigid body;

(b) to change the reference frame in which a vector or frame is represented;

(c) to displace a vector or frame.

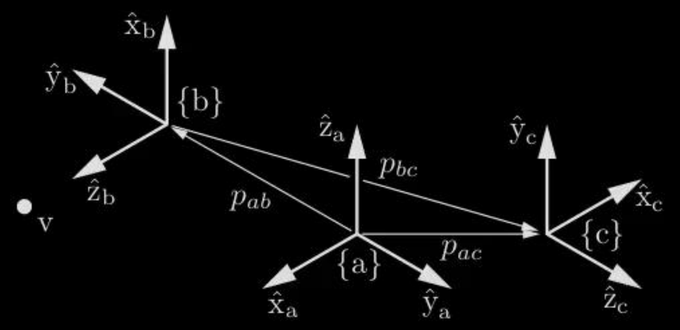
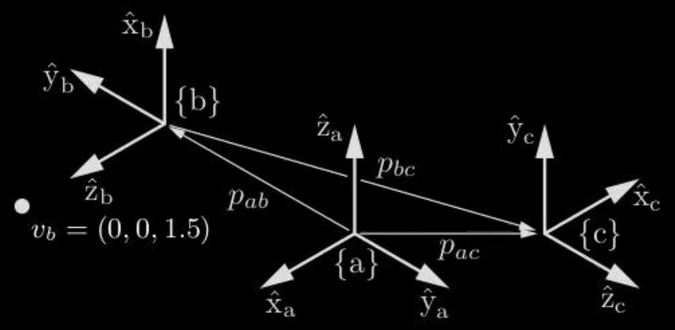


Figure 3.14: Three reference frames in space, and a point v that can be represented in {b} as $v_b = (0, 0, 1.5)$.

Chapter 3: Rigid-Body Motions

3.3.1.2: Uses of Transformation Matrices

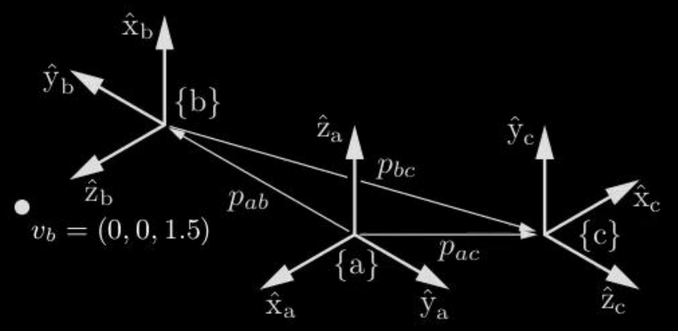
(a) Representation of configuration



Chapter 3: Rigid-Body Motions

3.3.1.2: Uses of Transformation Matrices

(a) Representation of configuration



(R_{sb}, p_{sb}), and $T_{sc} = (R_{sc}, p_{sc})$, respectively, can be expressed relative to $\{s\}$ by the rotations

$$R_{sa} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad R_{sb} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad R_{sc} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

The location of the origin of each frame relative to $\{s\}$ can be written

$$p_{sa} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad p_{sb} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}, \qquad p_{sc} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

Since $\{a\}$ is collocated with $\{s\}$, the transformation matrix T_{sa} constructed from (R_{sa}, p_{sa}) is the identity matrix.

Chapter 3: Rigid-Body Motions

3.3.1.2: Uses of Transformation Matrices

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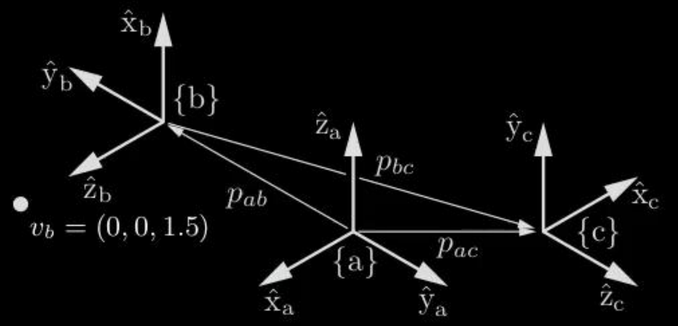

Chapter 3: Rigid-Body Motions

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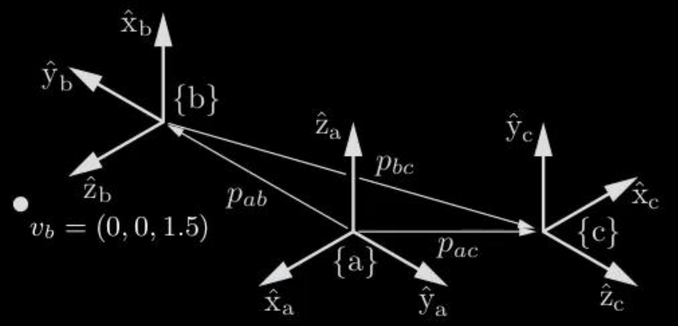

How would you represent T_{sc}?

Chapter 3: Rigid-Body Motions

3.3.1.2: Uses of Transformation Matrices

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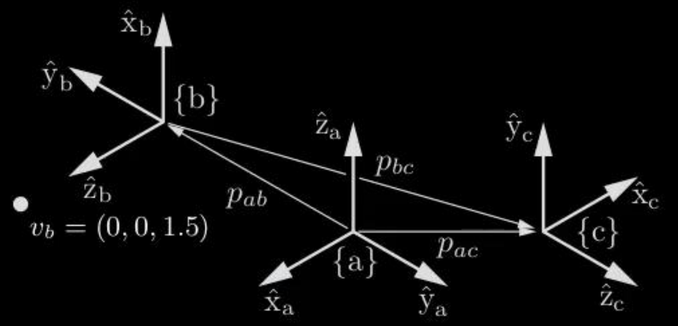

How would you represent T_{sc}? $\quad T_{sc} = \begin{bmatrix} R_{sc} & p_{sc} \\ 0 & 1 \end{bmatrix}$

Chapter 3: Rigid-Body Motions

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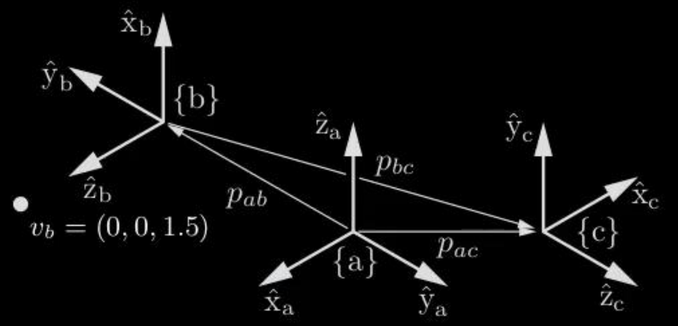

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Since {a} is collocated with {s}, the transformation matrix T_{sa} constructed from (R_{sa}, p_{sa}) is the identity matrix.

How would you represent T_{sc}? $\quad T_{sc} = \begin{bmatrix} R_{sc} & p_{sc} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Chapter 3: Rigid-Body Motions

3.3.1.2: Uses of Transformation Matrices

(b) Change of refrence frame

(R_{sb}, p_{sb}), and $T_{sc} = (R_{sc}, p_{sc})$, respectively, can be expressed relative to {s} by the rotations

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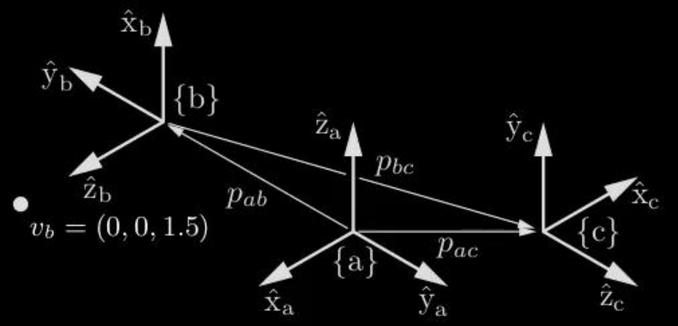

How would you represent T_{bc}?

Chapter 3: Rigid-Body Motions

3.3.1.2: Uses of Transformation Matrices

(b) Change of refrence frame

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The location of the origin of each frame relative to {s} can be written

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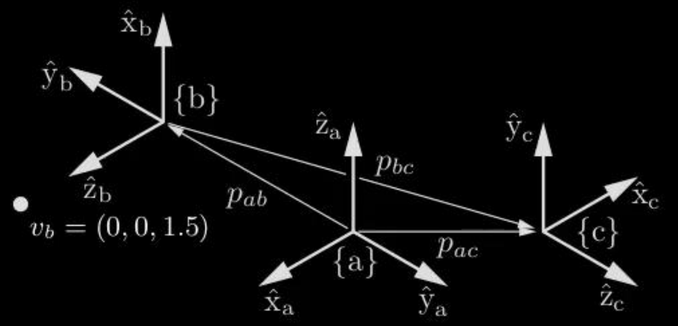

How would you represent T_{bc}? $\quad T_{bc} = T_{bs}T_{sc} = (T_{sb})^{-1}T_{sc}$

Chapter 3: Rigid-Body Motions

3.3.1.2: Uses of Transformation Matrices

(b) Change of refrence frame

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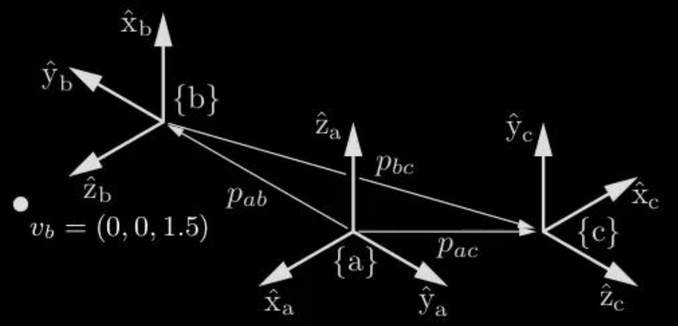


The location of the origin of each frame relative to {s} can be written

$$p_{sa} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad p_{sb} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}, \qquad p_{sc} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

Since {a} is collocated with {s}, the transformation matrix T_{sa} constructed from (R_{sa}, p_{sa}) is the identity matrix.

How would you represent T_{bc}? $\quad T_{bc} = T_{bs}T_{sc} = (T_{sb})^{-1}T_{sc} \quad = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

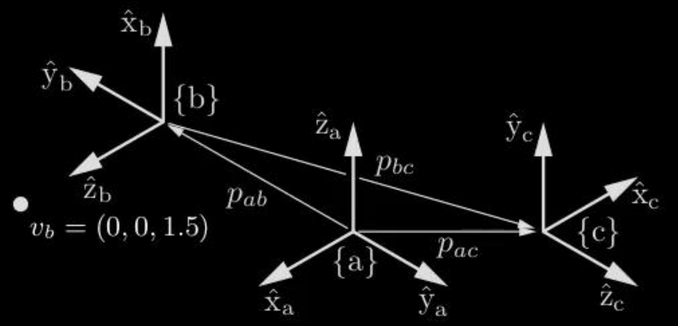
Chapter 3: Rigid-Body Motions

3.3.1.2: Uses of Transformation Matrices

(b) Change of refrence frame

(R_{sb}, p_{sb}), and $T_{sc} = (R_{sc}, p_{sc})$, respectively, can be expressed relative to {s} by the rotations

$$R_{sa} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad R_{sb} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad R_{sc} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$



The location of the origin of each frame relative to {s} can be written

$$p_{sa} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad p_{sb} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}, \qquad p_{sc} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

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Chapter 3: Rigid-Body Motions

3.3.1.2: Uses of Transformation Matrices

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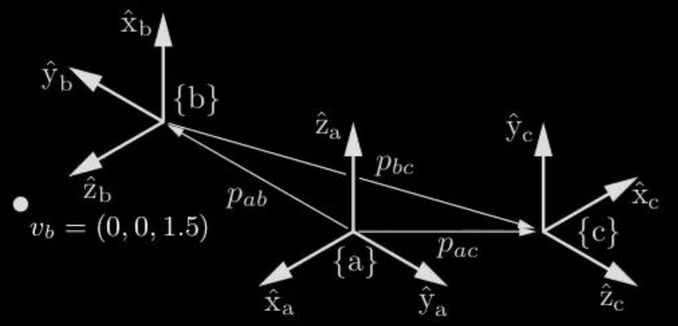

How would you represent v_a?

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3.3.1.2: Uses of Transformation Matrices

(b) Change of refrence frame

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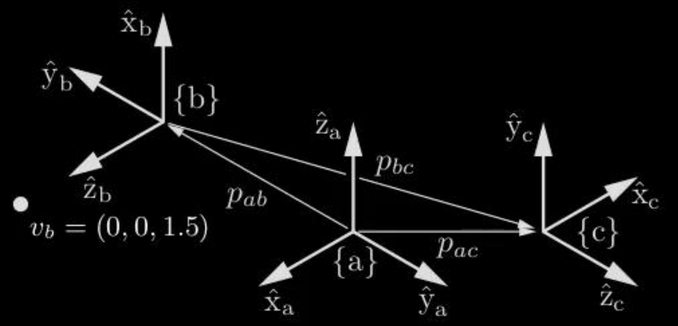

How would you represent v_a? $\quad v_a = T_{ab} v_b$

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3.3.1.2: Uses of Transformation Matrices

(b) Change of refrence frame

(R_{sb}, p_{sb}), and $T_{sc} = (R_{sc}, p_{sc})$, respectively, can be expressed relative to {s} by the rotations

$$R_{sa} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad R_{sb} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \qquad R_{sc} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

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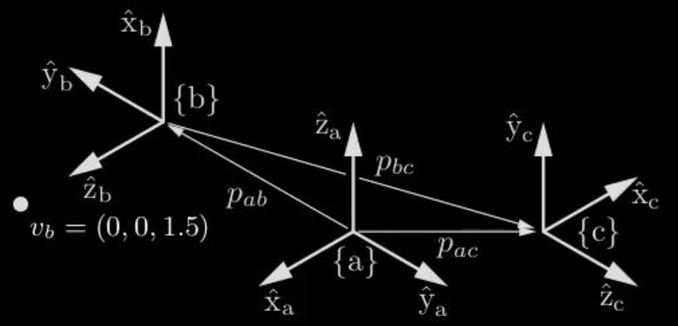

How would you represent v_a? $\quad v_a = T_{ab} v_b \quad = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1.5 \\ 1 \end{bmatrix}$

Chapter 3: Rigid-Body Motions

3.3.1.2: Uses of Transformation Matrices

(b) Change of refrence frame

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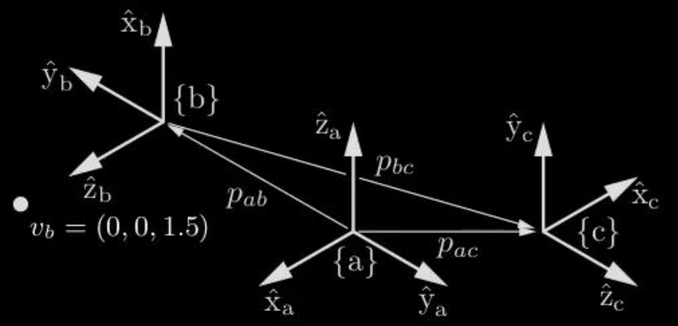

How would you represent v_a? $\quad v_a = T_{ab} v_b \quad = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.5 \\ -2 \\ 0 \\ 1 \end{bmatrix}$

Chapter 3: Rigid-Body Motions

3.3.1.2: Uses of Transformation Matrices

(b) Change of refrence frame

Changing the reference frame of a vector or a frame. By a subscript cancellation rule analogous to that for rotations, for any three reference frames {a}, {b}, and {c}, and any vector v expressed in {b} as v_b,

$$T_{ab}T_{bc} = T_{a\cancel{b}}T_{\cancel{b}c} = T_{ac}$$
$$T_{ab}v_b = T_{a\cancel{b}}v_{\cancel{b}} = v_a,$$

where v_a is the vector v expressed in {a}.

Chapter 3: Rigid-Body Motions

3.3.1.2: Uses of Transformation Matrices

(c) Displacing a vector or a frame

$$T = (R, p) = (Rot(\hat{w}, \theta), p)$$

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3.3.1.2: Uses of Transformation Matrices

(c) Displacing a vector or a frame

$$T = (R, p) = (Rot(\hat{w}, \theta), p)$$

$$\mathrm{Rot}(\hat{\omega}, \theta) = \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathrm{Trans}(p) = \begin{bmatrix} 1 & 0 & 0 & p_x \\ 0 & 1 & 0 & p_y \\ 0 & 0 & 1 & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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3.3.1.2: Uses of Transformation Matrices

(c) Displacing a vector or a frame

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Whether we pre-multiply or post-multiply T_{sb} by $T = (R, p)$ determines whether the $\hat{\omega}$-axis and p are interpreted as in the fixed frame {s} or in the body frame {b}:

$$\begin{aligned} T_{sb'} &= TT_{sb} = \mathrm{Trans}(p)\,\mathrm{Rot}(\hat{\omega}, \theta)T_{sb} \qquad \text{(fixed frame)} \\ &= \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_{sb} & p_{sb} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} RR_{sb} & Rp_{sb} + p \\ 0 & 1 \end{bmatrix} \end{aligned} \tag{3.66}$$

$$\begin{aligned} T_{sb''} &= T_{sb}T = T_{sb}\,\mathrm{Trans}(p)\,\mathrm{Rot}(\hat{\omega}, \theta) \qquad \text{(body frame)} \\ &= \begin{bmatrix} R_{sb} & p_{sb} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{sb}R & R_{sb}p + p_{sb} \\ 0 & 1 \end{bmatrix}. \end{aligned} \tag{3.67}$$

Chapter 3: Rigid-Body Motions

3.3.1.2: Uses of Transformation Matrices

(c) Displacing a vector or a frame

Transformation T with $\hat{w} = (0, 0, 1)$, $\theta = 90$ and $p = (0, 2, 0)$

$$T = (\mathrm{Rot}(\hat{\omega}, \theta), p) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Chapter 3: Rigid-Body Motions

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The frame $\{b\}$ represnted in frame $\{s\}$ by:

$$T_{sb} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Chapter 3: Rigid-Body Motions

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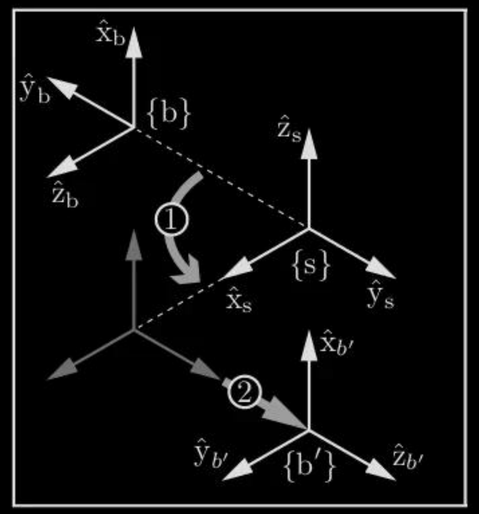
Fixed frame transformation T'_{sb}



Chapter 3: Rigid-Body Motions

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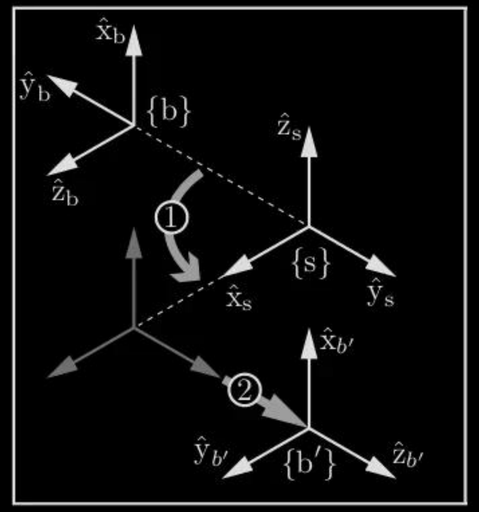

$$TT_{sb} = T_{sb'} = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Chapter 3: Rigid-Body Motions

3.3.1.2: Uses of Transformation Matrices

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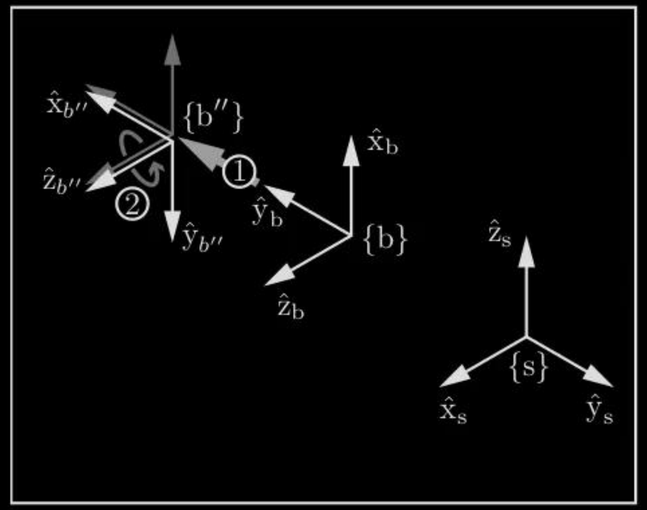
Body frame transformation T''_{sb}



Chapter 3: Rigid-Body Motions

3.3.1.2: Uses of Transformation Matrices

(c) Displacing a vector or a frame

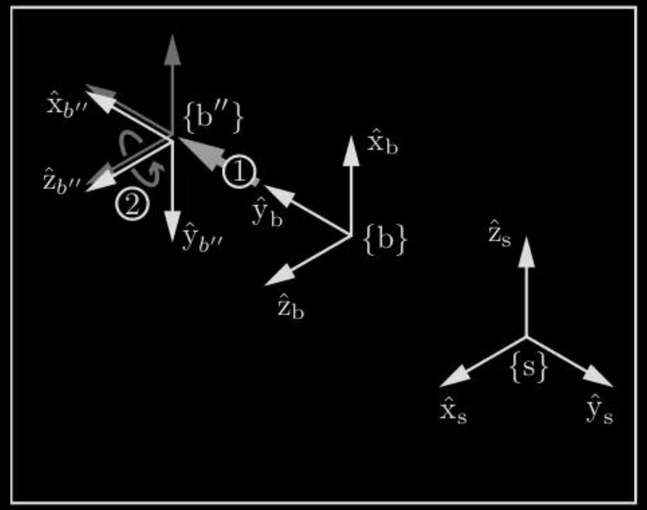
Transformation T with $\hat{w} = (0, 0, 1)$, $\theta = 90$ and $p = (0, 2, 0)$

$$T = (\mathrm{Rot}(\hat{\omega}, \theta), p) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The frame $\{b\}$ represnted in frame $\{s\}$ by:

$$T_{sb} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Body frame transformation T''_{sb}



$$T_{sb}T = T_{sb''} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & -4 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Chapter 3: Rigid-Body Motions

3.3.1.2: Uses of Transformation Matrices

Example 3.19

Given these Transformation matrices:

Whats the configuration of object {e} in respect to the robot EE ?

$$T_{db} = \begin{bmatrix} 0 & 0 & -1 & 250 \\ 0 & -1 & 0 & -150 \\ -1 & 0 & 0 & 200 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$T_{de} = \begin{bmatrix} 0 & 0 & -1 & 300 \\ 0 & -1 & 0 & 100 \\ -1 & 0 & 0 & 120 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$T_{ad} = \begin{bmatrix} 0 & 0 & -1 & 400 \\ 0 & -1 & 0 & 50 \\ -1 & 0 & 0 & 300 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$T_{bc} = \begin{bmatrix} 0 & -1/\sqrt{2} & -1/\sqrt{2} & 30 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} & -40 \\ 1 & 0 & 0 & 25 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

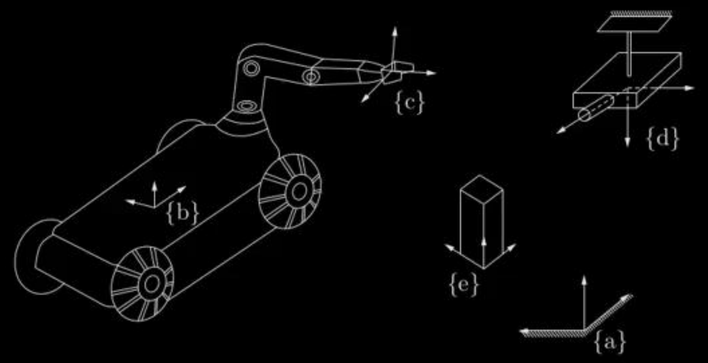


Figure 3.16: Assignment of reference frames.

Chapter 3: Rigid-Body Motions

3.3.1.2: Uses of Transformation Matrices

Example 3.19

Given these Transformation matrices:

$$T_{db} = \begin{bmatrix} 0 & 0 & -1 & 250 \\ 0 & -1 & 0 & -150 \\ -1 & 0 & 0 & 200 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

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Whats the configuration of object {e} in respect to the robot EE ?

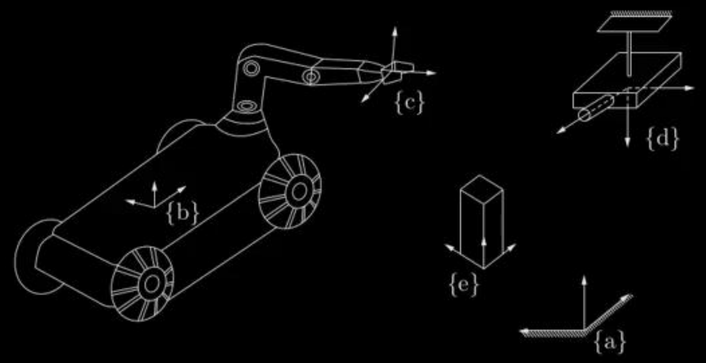


Figure 3.16: Assignment of reference frames.

In order to calculate how to move the robot arm so as to pick up the object, the configuration of the object relative to the robot hand, T_{ce}, must be determined. We know that

$$T_{ab}T_{bc}T_{ce} = T_{ad}T_{de},$$

where the only quantity besides T_{ce} not given to us directly is T_{ab}. However, since $T_{ab} = T_{ad}T_{db}$, we can determine T_{ce} as follows:

$$T_{ce} = \left(T_{ad}T_{db}T_{bc}\right)^{-1} T_{ad}T_{de}.$$

$$T_{ce} = \begin{bmatrix} 0 & 0 & 1 & -75 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & -260/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} & 0 & 130/\sqrt{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Chapter 3: Rigid-Body Motions

3.3.2: Twists

Twist is an $\mathbb{R}^6$ vector that describes the velocity of a rigid body

$$V = \begin{bmatrix} w \\ v \end{bmatrix} \quad \begin{matrix} \text{Angular velocity} \\ \text{Linear velocity} \end{matrix}$$

Chapter 3: Rigid-Body Motions

3.3.2: Twists

Twist is an $\mathbb{R}^6$ vector that describes the velocity of a rigid body

$$V = \begin{bmatrix} w \\ v \end{bmatrix} \quad \begin{matrix} \text{Angular velocity} \\ \text{Linear velocity} \end{matrix}$$

Let

$$T_{sb}(t) = T(t) = \begin{bmatrix} R(t) & p(t) \\ 0 & 1 \end{bmatrix}$$

Let us first see what happens when we pre-multiply $\dot{T}$ by T^{-1}:

$$\begin{aligned} T^{-1}\dot{T} &= \begin{bmatrix} R^{\mathrm{T}} & -R^{\mathrm{T}}p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} R^{\mathrm{T}}\dot{R} & R^{\mathrm{T}}\dot{p} \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} [\omega_b] & v_b \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

This is the matrix representation of a BODY twist

Chapter 3: Rigid-Body Motions

3.3.2: Twists

Proposition 3.22. *Given a fixed (space) frame* {s}*, a body frame* {b}*, and a differentiable* $T_{sb}(t) \in SE(3)$*, where*

$$T_{sb}(t) = \begin{bmatrix} R(t) & p(t) \\ 0 & 1 \end{bmatrix}, \tag{3.80}$$

then

$$T_{sb}^{-1}\dot{T}_{sb} = [\mathcal{V}_b] = \begin{bmatrix} [\omega_b] & v_b \\ 0 & 0 \end{bmatrix} \in se(3) \tag{3.81}$$

is the matrix representation of the **body twist**, *and*

$$\dot{T}_{sb}T_{sb}^{-1} = [\mathcal{V}_s] = \begin{bmatrix} [\omega_s] & v_s \\ 0 & 0 \end{bmatrix} \in se(3) \tag{3.82}$$

is the matrix representation of the **spatial twist**. *The twists* $\mathcal{V}_s$ *and* $\mathcal{V}_b$ *are related by*

$$\mathcal{V}_s = \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} = [\mathrm{Ad}_{T_{sb}}]\mathcal{V}_b, \tag{3.83}$$

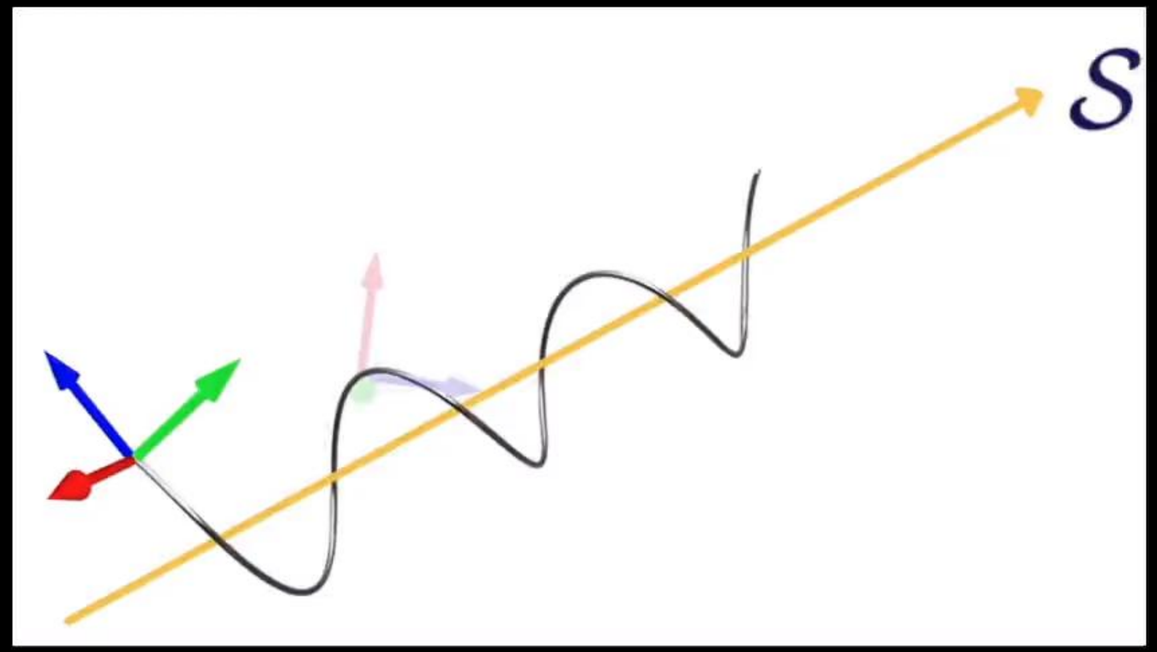
$$\mathcal{V}_b = \begin{bmatrix} \omega_b \\ v_b \end{bmatrix} = \begin{bmatrix} R^{\mathrm{T}} & 0 \\ -R^{\mathrm{T}}[p] & R^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \omega_s \\ v_s \end{bmatrix} = [\mathrm{Ad}_{T_{bs}}]\mathcal{V}_s. \tag{3.84}$$

More generally, for any two frames {c} *and* {d}*, a twist represented as* $\mathcal{V}_c$ *in* {c} *is related to its representation* $\mathcal{V}_d$ *in* {d} *by*

$$\mathcal{V}_c = [\mathrm{Ad}_{T_{cd}}]\mathcal{V}_d, \qquad \mathcal{V}_d = [\mathrm{Ad}_{T_{dc}}]\mathcal{V}_c.$$

Chapter 3: Rigid-Body Motions

3.3.2.2: The Screw Interpretation of a Twist



Chapter 3: Rigid-Body Motions

3.3.2.2: The Screw Interpretation of a Twist

S is the collection $\{q, \hat{s}, h\}$

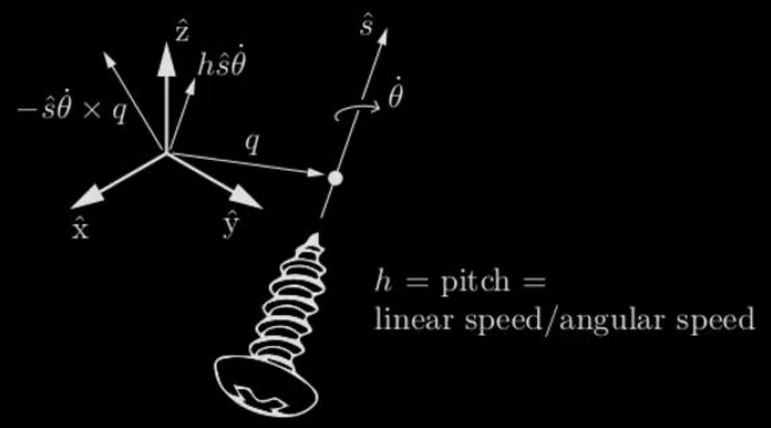


Figure 3.19: A screw axis $\mathcal{S}$ represented by a point q, a unit direction $\hat{s}$, and a pitch h.

Chapter 3: Rigid-Body Motions

3.3.2.2: The Screw Interpretation of a Twist

S is the collection $\{q, \hat{s}, h\}$

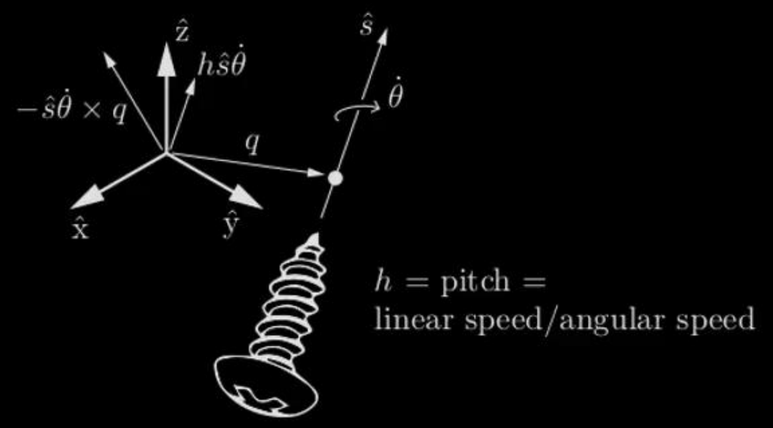


Figure 3.19: A screw axis $\mathcal{S}$ represented by a point q, a unit direction $\hat{s}$, and a pitch h.

so a twist V corresponding to an angular velocity $\dot{\theta}$ about a screw S is: $\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} \hat{s}\dot{\theta} \\ -\hat{s}\dot{\theta} \times q + h\hat{s}\dot{\theta} \end{bmatrix}$

Chapter 3: Rigid-Body Motions

3.3.2.2: The Screw Interpretation of a Twist

Instead of using $\{q, \hat{s}, h\}$, the screw axis S is described using a normalized version of a twist:

(a) If $\omega \neq 0$ then $\mathcal{S} = \mathcal{V}/\|\omega\| = (\omega/\|\omega\|, v/\|\omega\|)$. The screw axis $\mathcal{S}$ is simply $\mathcal{V}$ normalized by the length of the angular velocity vector. The angular velocity about the screw axis is $\dot{\theta} = \|\omega\|$, such that $\mathcal{S}\dot{\theta} = \mathcal{V}$.

(b) If $\omega = 0$ then $\mathcal{S} = \mathcal{V}/\|v\| = (0, v/\|v\|)$. The screw axis $\mathcal{S}$ is simply $\mathcal{V}$ normalized by the length of the linear velocity vector. The linear velocity along the screw axis is $\dot{\theta} = \|v\|$, such that $\mathcal{S}\dot{\theta} = \mathcal{V}$.

This leads to the following definition of a "unit" (normalized) screw axis:

Chapter 3: Rigid-Body Motions

3.3.2.2: The Screw Interpretation of a Twist

This leads to the following definition of a "unit" (normalized) screw axis:

Definition 3.24. For a given reference frame, a **screw axis** $\mathcal{S}$ is written as

$$\mathcal{S} = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6,$$

where either (i) $\|\omega\| = 1$ or (ii) $\omega = 0$ and $\|v\| = 1$. If (i) holds then $v = -\omega \times q + h\omega$, where q is a point on the axis of the screw and h is the pitch of the screw ($h = 0$ for a pure rotation about the screw axis). If (ii) holds then the pitch of the screw is infinite and the twist is a translation along the axis defined by v.

Since a screw axis $\mathcal{S}$ is just a normalized twist, the 4×4 matrix representation $[\mathcal{S}]$ of $\mathcal{S} = (\omega, v)$ is

$$[\mathcal{S}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \in se(3), \qquad [\omega] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \in so(3), \qquad (3.85)$$

where the bottom row of $[\mathcal{S}]$ consists of all zeros. Also, a screw axis represented as $\mathcal{S}_a$ in a frame {a} is related to the representation $\mathcal{S}_b$ in a frame {b} by

$$\mathcal{S}_a = [\mathrm{Ad}_{T_{ab}}]\mathcal{S}_b, \qquad \mathcal{S}_b = [\mathrm{Ad}_{T_{ba}}]\mathcal{S}_a.$$

Chapter 3: Rigid-Body Motions

3.3.3.1: Exponential Coordinates of Rigid-Body Motions

By analogy to the exponential coordinates $\hat{\omega}\theta$ for rotations, we define the six-dimensional **exponential coordinates of a homogeneous transformation** T as $\mathcal{S}\theta \in \mathbb{R}^6$, where $\mathcal{S}$ is the screw axis and θ is the distance that must be traveled along the screw axis to take a frame from the origin I to T. If the pitch of the screw axis $\mathcal{S} = (\omega, v)$ is finite then $\|\omega\| = 1$ and θ corresponds to the angle of rotation about the screw axis. If the pitch of the screw is infinite then $\omega = 0$ and $\|v\| = 1$ and θ corresponds to the linear distance traveled along the screw axis.

Proposition 3.25. *Let $\mathcal{S} = (\omega, v)$ be a screw axis. If $\|\omega\| = 1$ then, for any distance $\theta \in \mathbb{R}$ traveled along the axis,*

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} e^{[\omega]\theta} & \left(I\theta + (1-\cos\theta)[\omega] + (\theta - \sin\theta)[\omega]^2\right) v \\ 0 & 1 \end{bmatrix}. \tag{3.88}$$

If $\omega = 0$ and $\|v\| = 1$, then

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix}.$$

Chapter 3: Rigid-Body Motions

3.3.3.2: Matrix Logarithm of Rigid-Body Motions

$$\begin{array}{rlcl} \exp: & [\mathcal{S}]\theta \in se(3) & \rightarrow & T \in SE(3), \\ \log: & T \in SE(3) & \rightarrow & [\mathcal{S}]\theta \in se(3). \end{array}$$

The Problem: Given a transformation matrix (R, p) $\in SE(3)$, one can find a screw axis $S = (w, v)$ and a scalar θ such that:

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix},$$

i.e., the matrix

$$[\mathcal{S}]\theta = \begin{bmatrix} [\omega]\theta & v\theta \\ 0 & 0 \end{bmatrix} \in se(3)$$

is the matrix logarithm of $T = (R, p)$.

Chapter 3: Rigid-Body Motions

3.3.3.2: Matrix Logarithm of Rigid-Body Motions

Algorithm: Given (R, p) written as $T \in SE(3)$, find a $\theta \in [0, \pi]$ and a screw axis $\mathcal{S} = (\omega, v) \in \mathbb{R}^6$ (where at least one of $\|\omega\|$ and $\|v\|$ is unity) such that $e^{[\mathcal{S}]\theta} = T$. The vector $\mathcal{S}\theta \in \mathbb{R}^6$ comprises the exponential coordinates for T and the matrix $[\mathcal{S}]\theta \in se(3)$ is the matrix logarithm of T.

(a) If $R = I$ then set $\omega = 0$, $v = p/\|p\|$, and $\theta = \|p\|$.

(b) Otherwise, use the matrix logarithm on $SO(3)$ to determine ω (written as $\hat{\omega}$ in the $SO(3)$ algorithm) and θ for R. Then v is calculated as

$$v = G^{-1}(\theta)p \tag{3.91}$$

where

$$G^{-1}(\theta) = \frac{1}{\theta}I - \frac{1}{2}[\omega] + \left(\frac{1}{\theta} - \frac{1}{2}\cot\frac{\theta}{2}\right)[\omega]^2. \tag{3.92}$$

Course Introduction

Chapter 2- Configuration Space

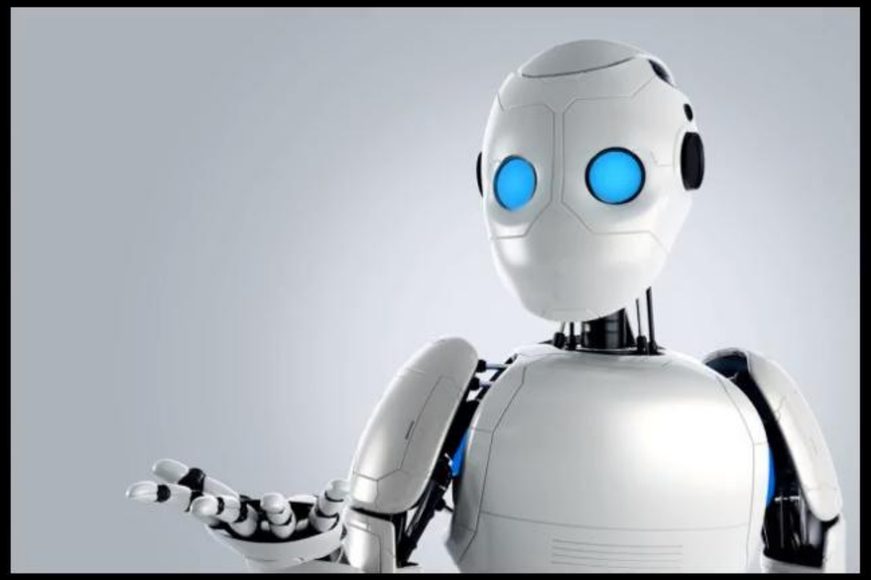
Chapter 3- Rigid-Body Motions

→ Chapter 4- Forward Kintematics (Lecture 4)

Chapter 4: Forward Kintematics

Roadmap :

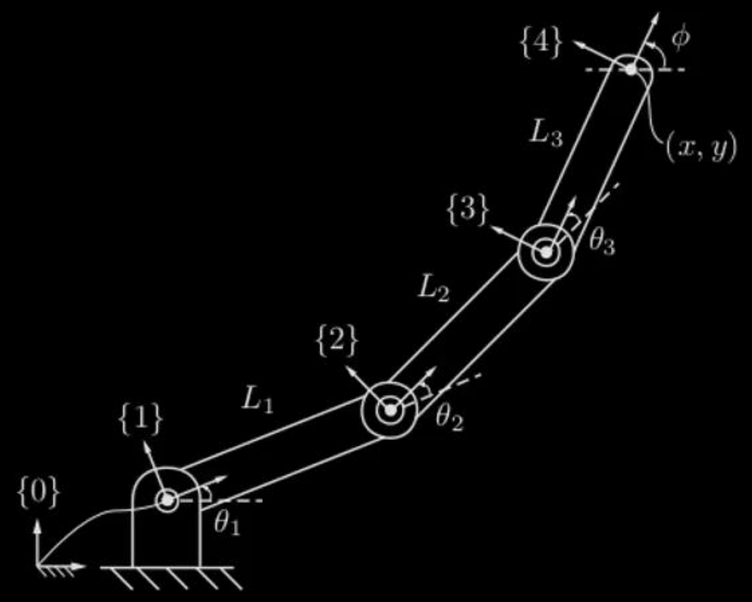
1. Configuration space, joint, links, DOF, grublers formula, task and workspaces ...
2. Rigid body motion, rotation matrices, twist, transformation matrices, screw motion, exponential coordinates
3. Today: Forward kinematics - the calculation of the position and orientation of the EE from its joint coordinates



Chapter 4: Forward Kintematics

The Problem :

Given a set of joint variables θ, what is the configuration of the EE?



Given an open chain with a fixed reference frame {s} and a reference frame {b} attached to some point on its last link – this frame is denoted the end-effector frame – the forward kinematics is the mapping $T(\theta)$ from the joint values θ to the position and orientation of {b} in {s}.

Chapter 4: Forward Kintematics

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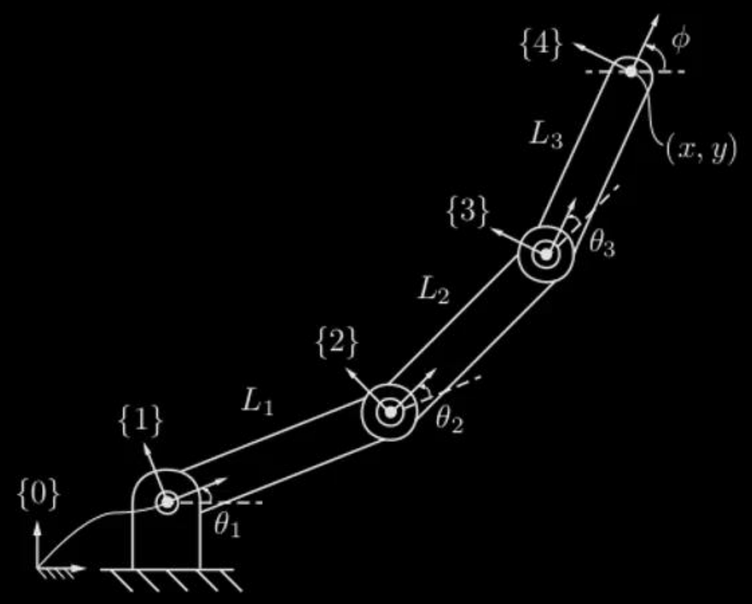

Solution 1: Trigonometrically

$$
\begin{aligned}
x &= L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) + L_3 \cos(\theta_1 + \theta_2 + \theta_3), \\
y &= L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2) + L_3 \sin(\theta_1 + \theta_2 + \theta_3), \\
\phi &= \theta_1 + \theta_2 + \theta_3.
\end{aligned}
$$

Chapter 4: Forward Kintematics

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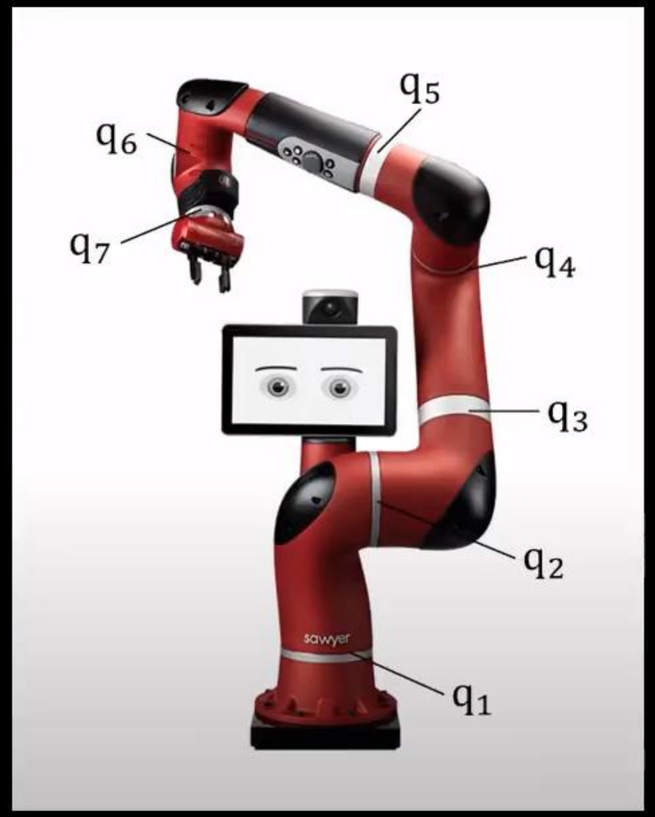

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Chapter 4: Forward Kintematics

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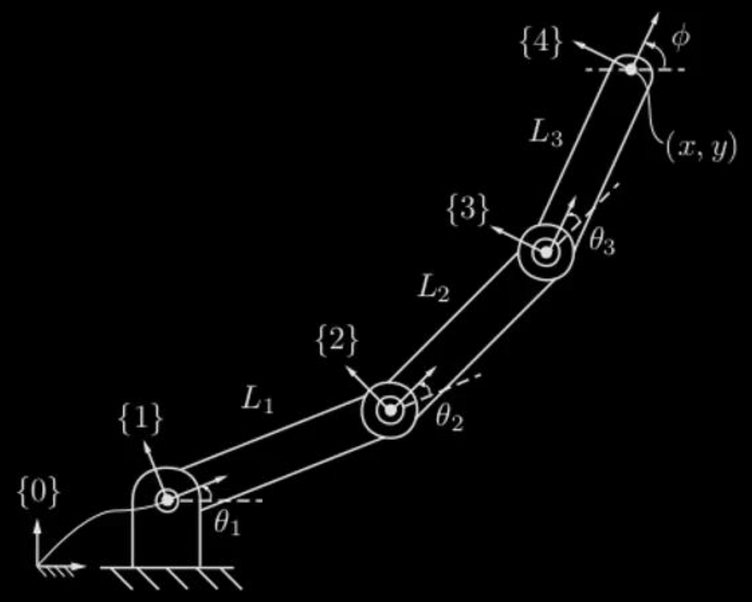

Chapter 4: Forward Kintematics

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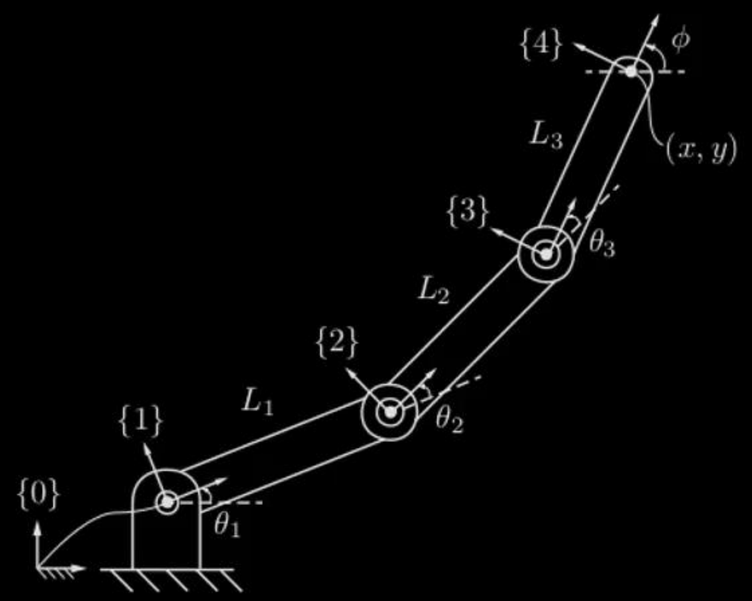


Solution 2: Transformation matrix multiplication

$$T_{04} = T_{01}T_{12}T_{23}T_{34},$$

where

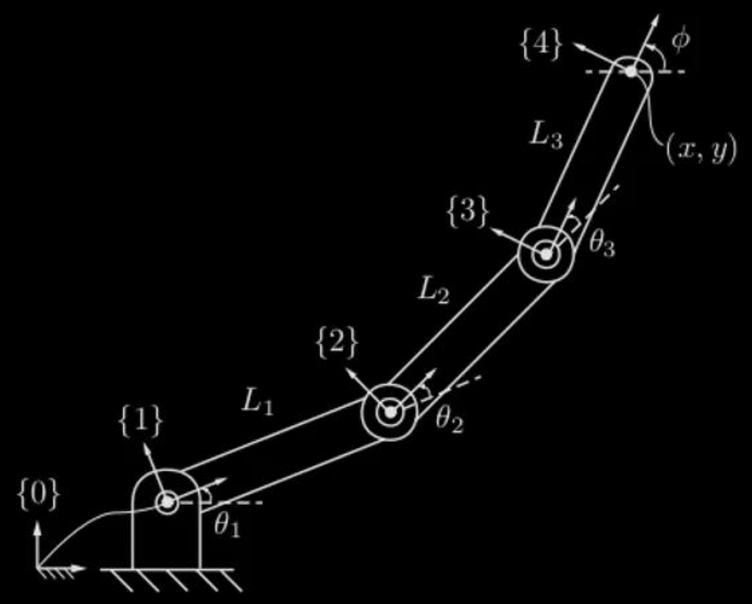
$$T_{01} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad T_{12} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & L_1 \\ \sin\theta_2 & \cos\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$T_{23} = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & L_2 \\ \sin\theta_3 & \cos\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad T_{34} = \begin{bmatrix} 1 & 0 & 0 & L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (4.5)$$

Chapter 4: Forward Kintematics

The Problem :

Given a set of joint variables θ, what is the configuration of the EE?



Solution 3: Product of Exponentials

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_n]\theta_n} M,$$

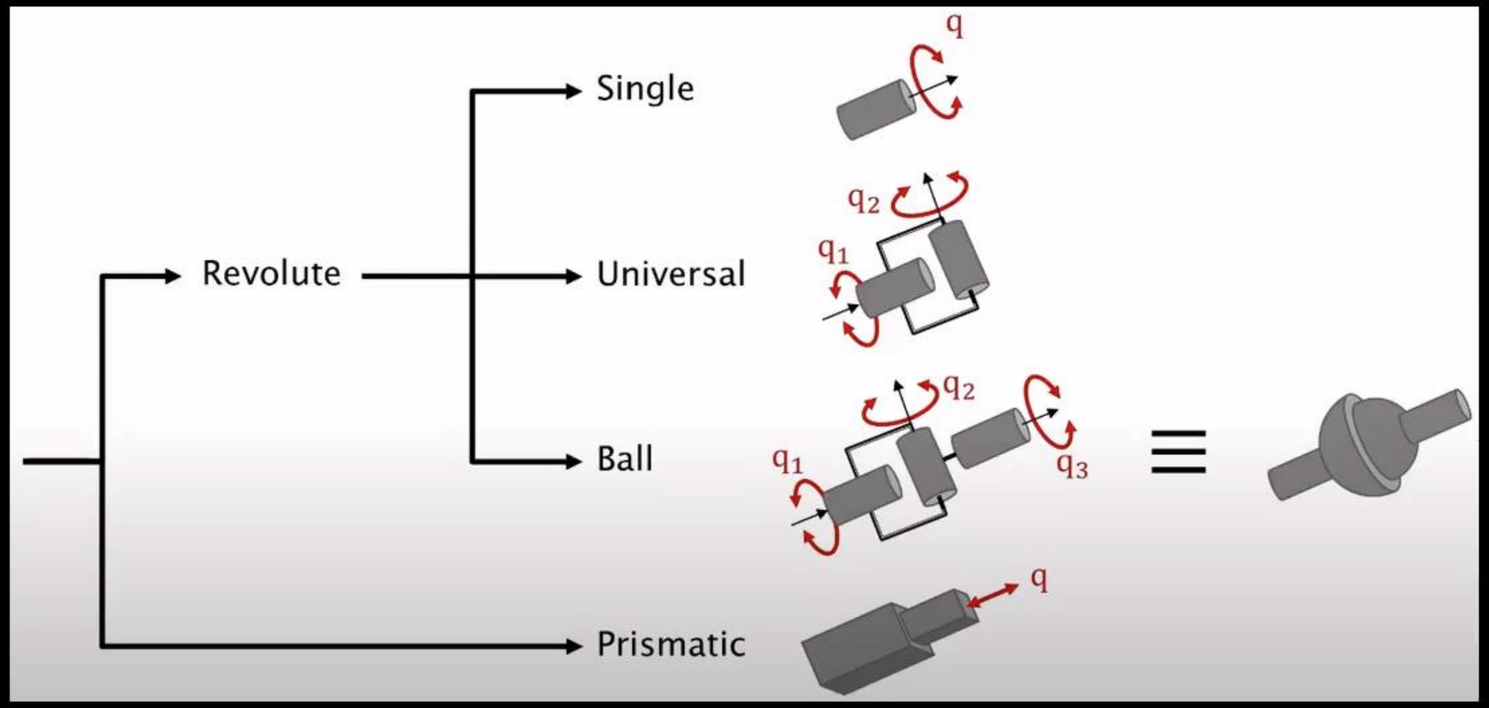
$$T(\theta) = M e^{[\mathcal{B}_1]\theta_1} \cdots e^{[\mathcal{B}_n]\theta_n}.$$

Chapter 4: Forward Kintematics

In this chapter

We only deal with open chain robots where the joints to consider are:

Joint Types



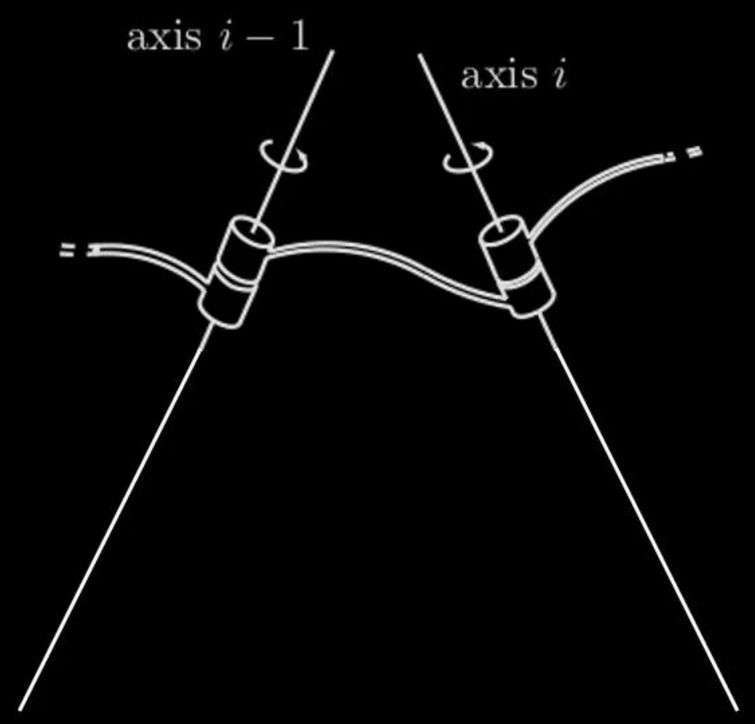
Appendix C: Denavit–Hartenberg Parameters

F.K. using D.H. method:

1. Assign frames according to the D.H. convention rules
2. Create the D.H. parameters table.
3. Plug the table values into the corresponding H.T.M.
4. Multiply the H.T.M together in order

Appendix C: Denavit–Hartenberg Parameters

Frame assignment:



Appendix C: Denavit–Hartenberg Parameters

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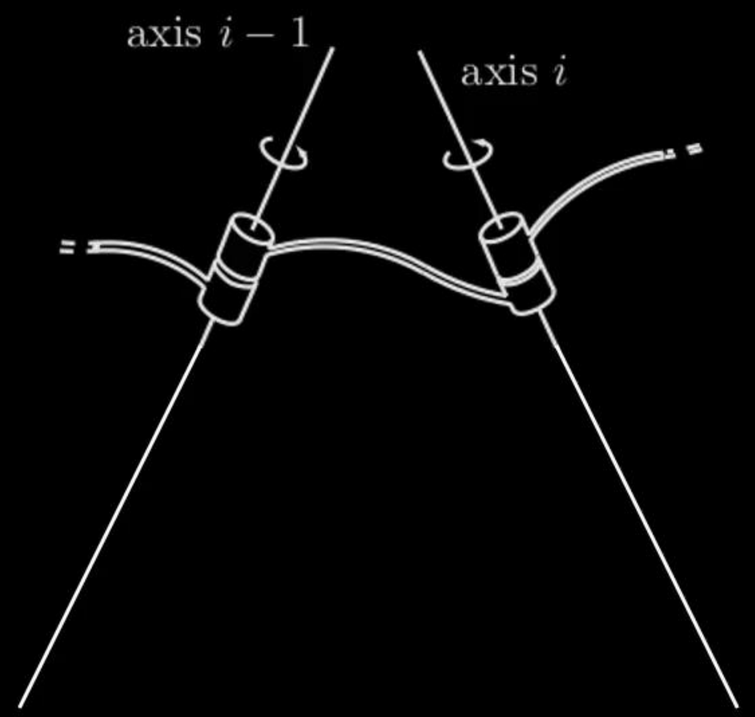

Rule 1: The $\hat{z}_i$ axis coincides with joint axis i and the $\hat{z}_{i-1}$ axis coincides with joint axis $i-1$.

Appendix C: Denavit–Hartenberg Parameters

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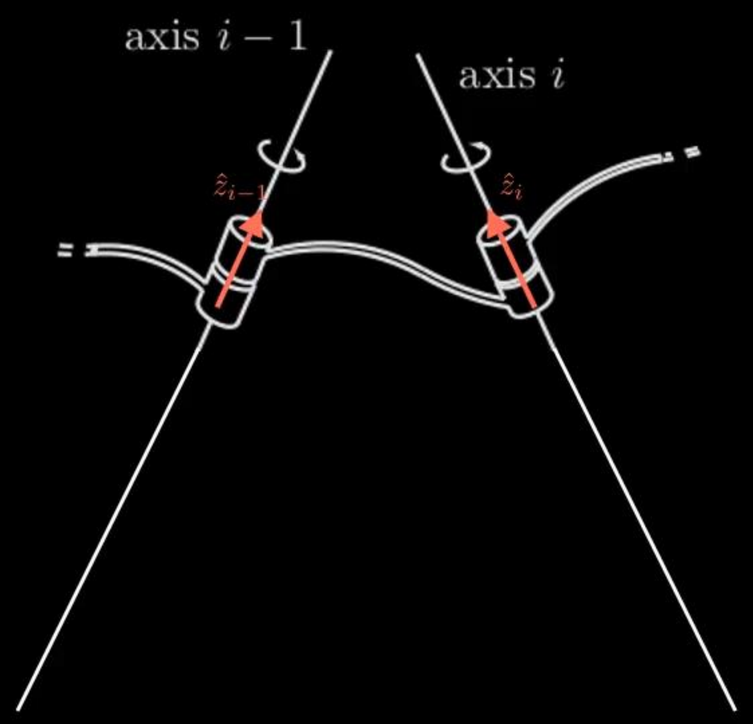

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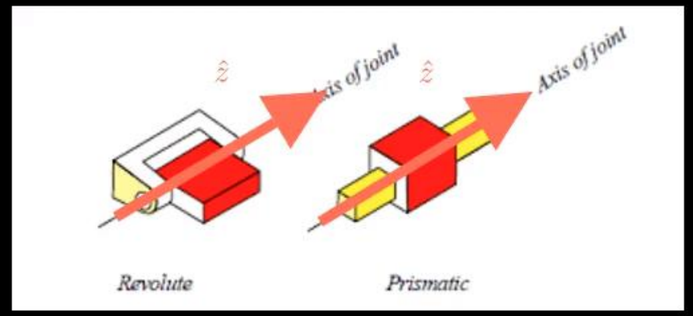
For prismatic joints, the $\hat{z}$-direction of the link reference frame is along the positive direction of translation.



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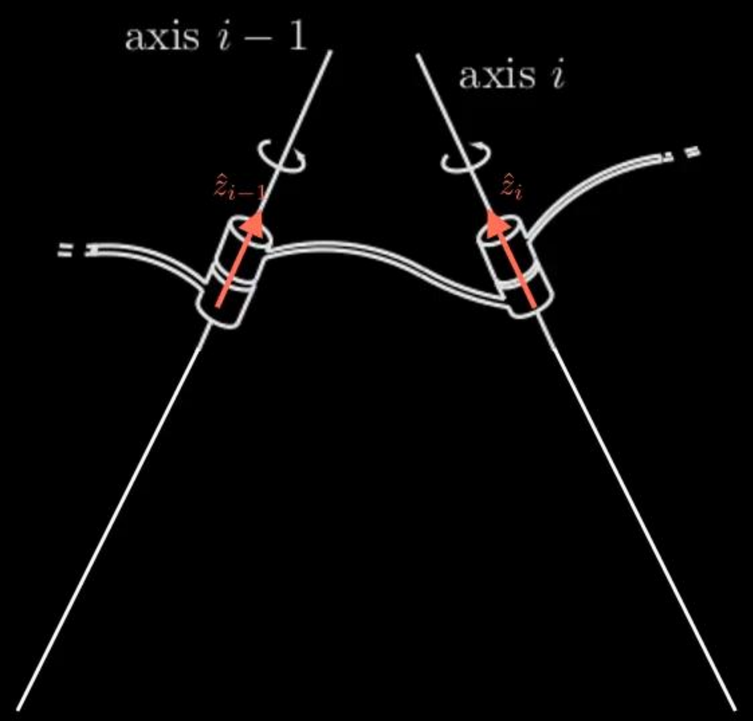

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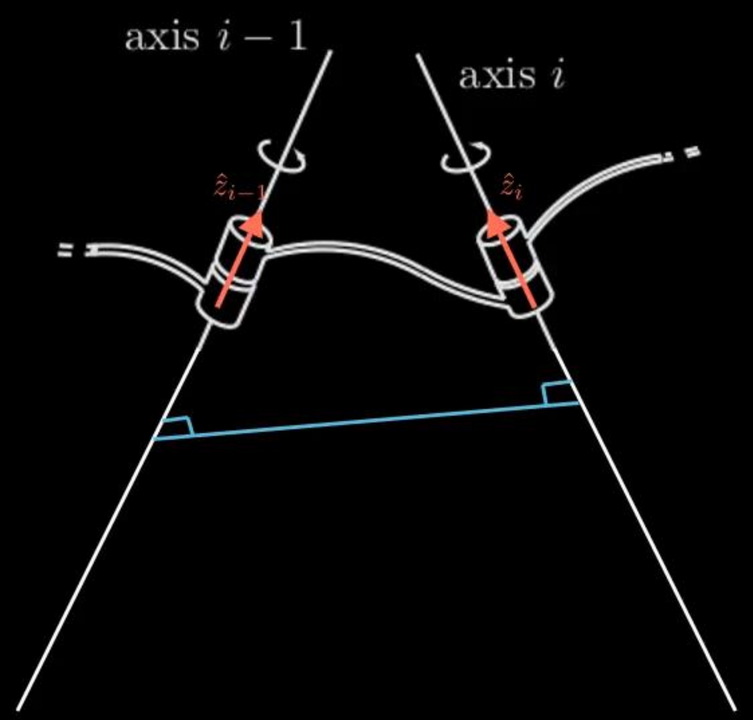

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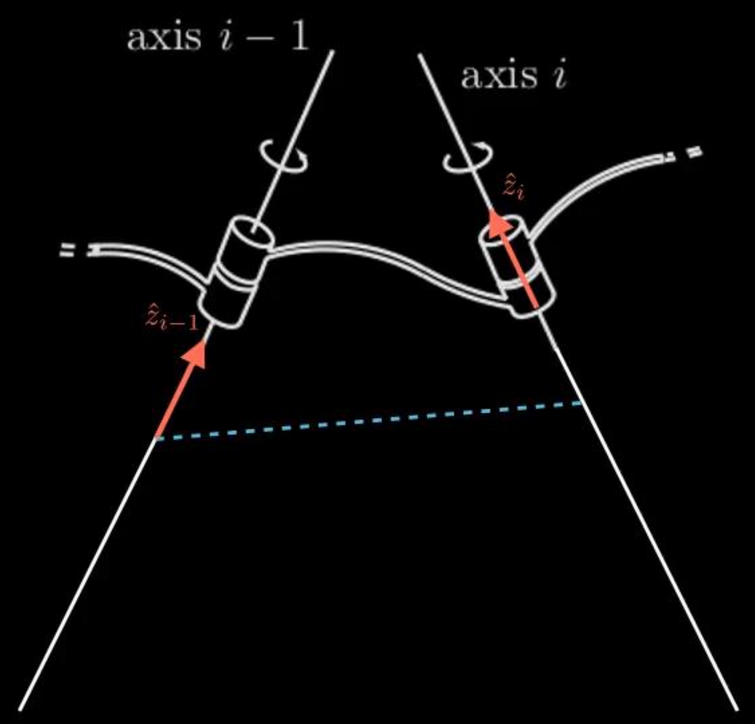

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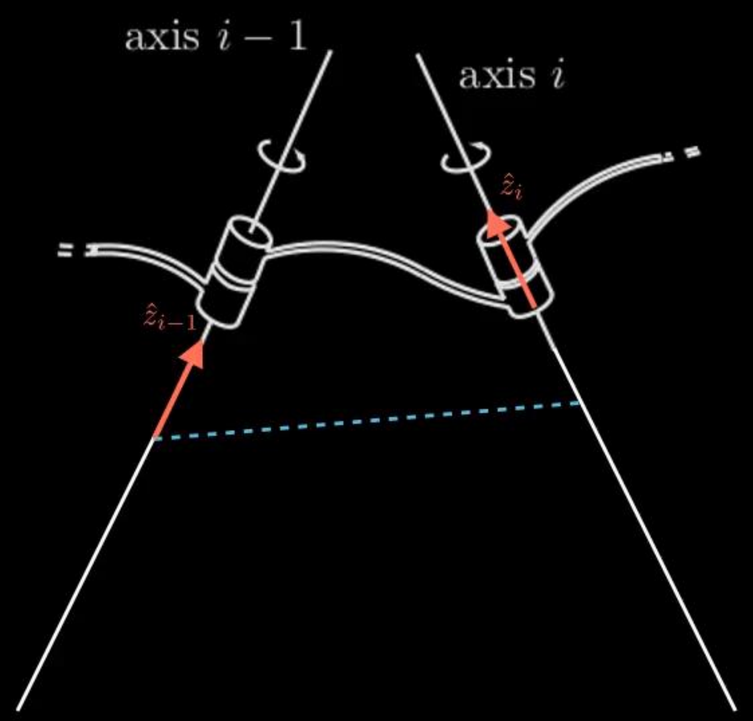

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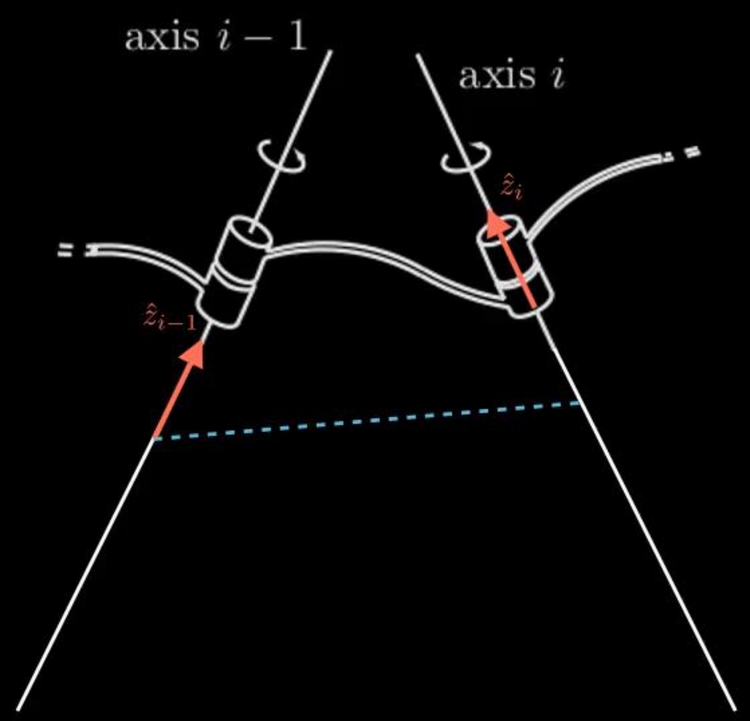

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Appendix C: Denavit–Hartenberg Parameters

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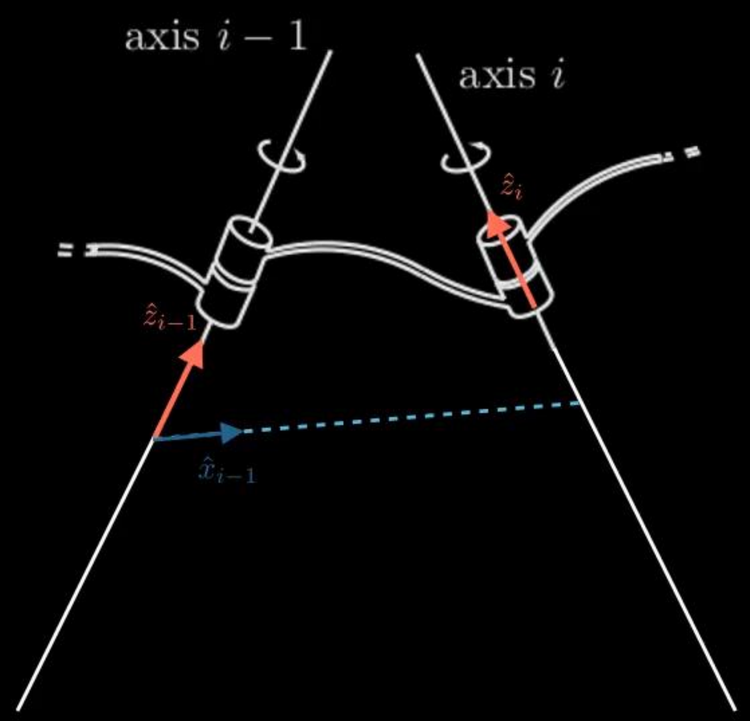

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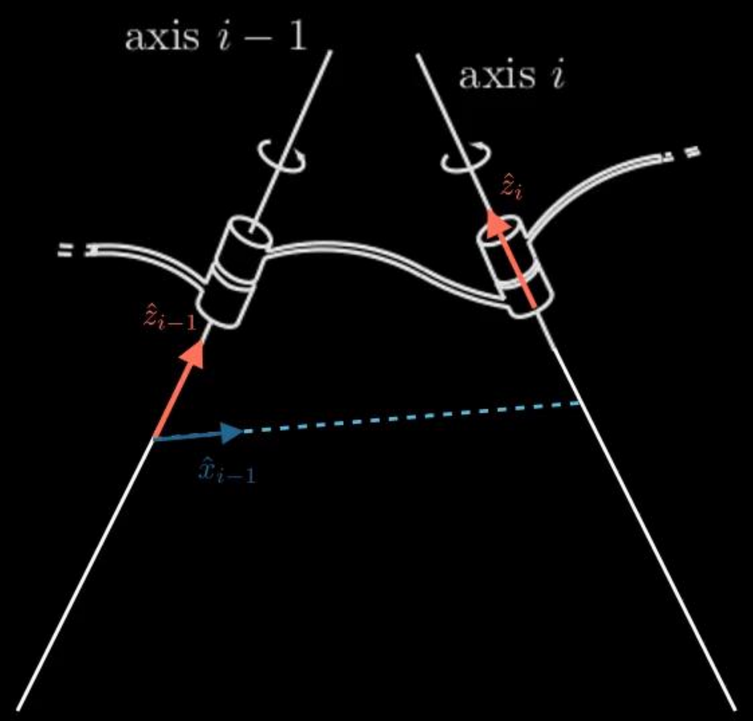

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Appendix C: Denavit–Hartenberg Parameters

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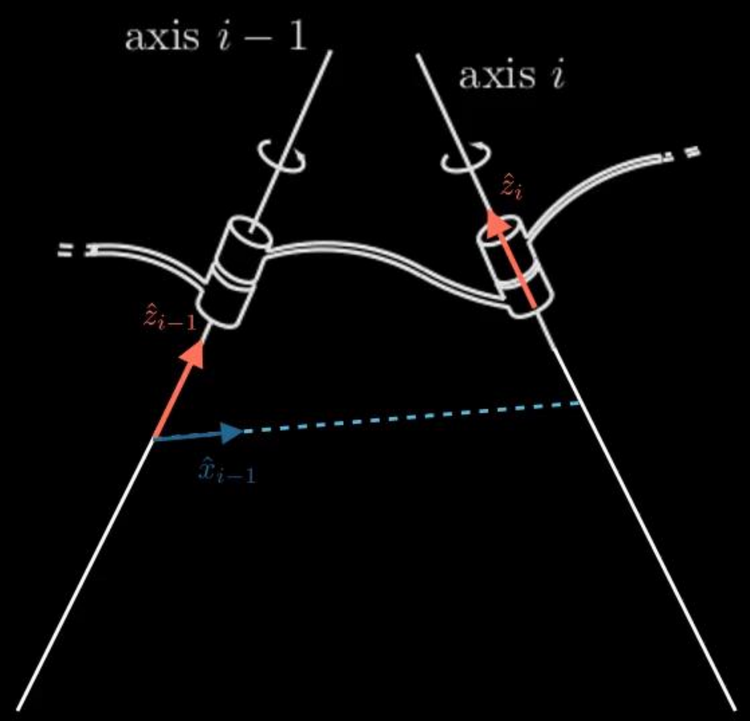

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Appendix C: Denavit–Hartenberg Parameters

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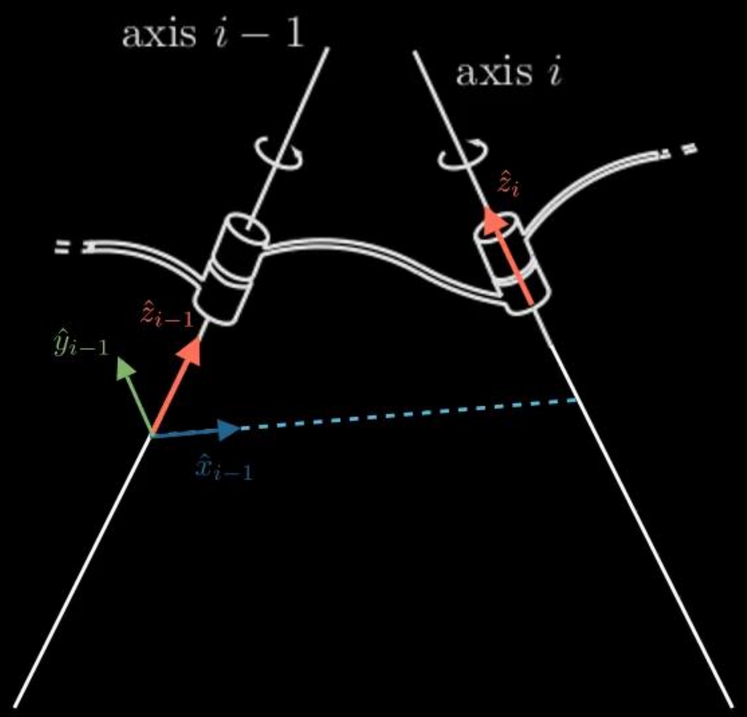

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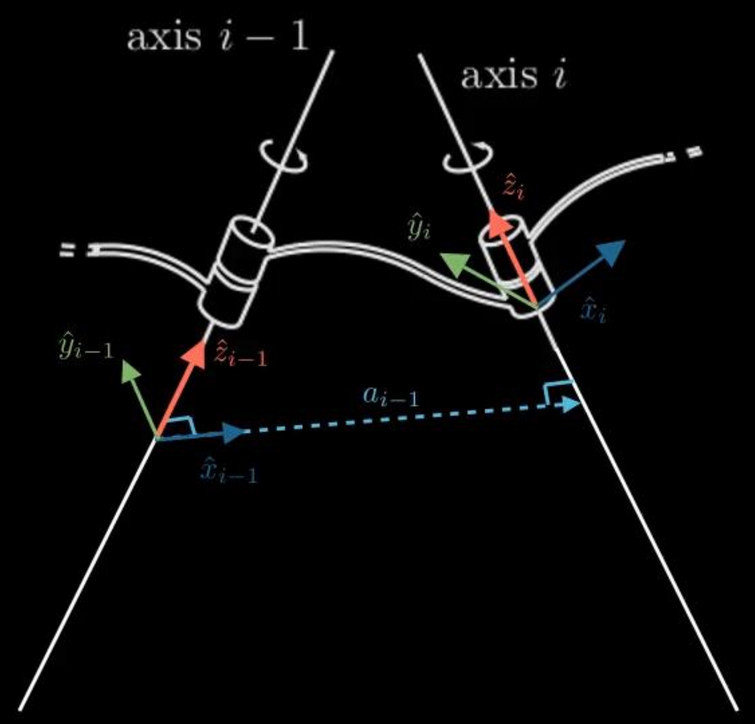
Appendix C: Denavit–Hartenberg Parameters

The 4 parameters:

a_{i-1} is the link length, which is the length of the mutually perpendicular line $\hat{a}_{i-1}$.

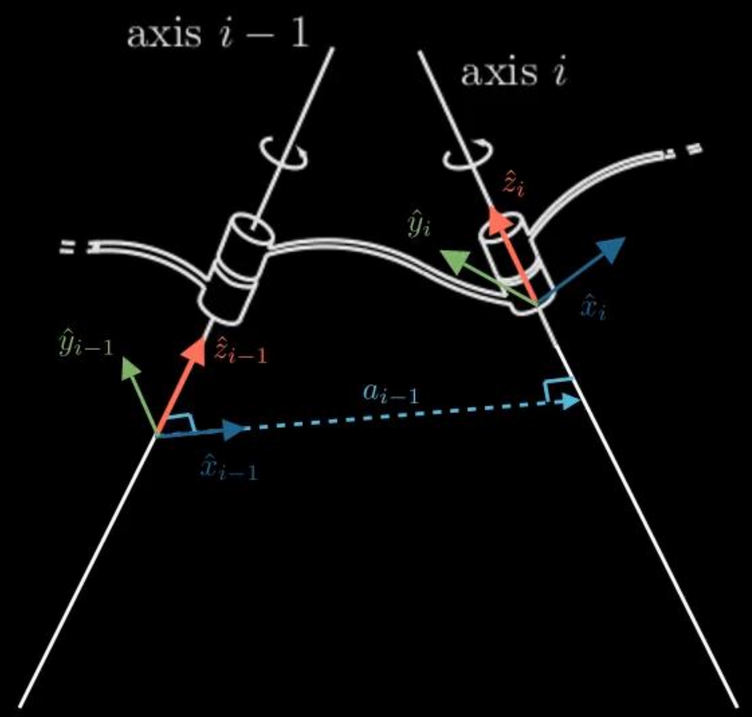


Appendix C: Denavit–Hartenberg Parameters

The 4 parameters:

a_{i-1} is the link length, which is the length of the mutually perpendicular line $\hat{a}_{i-1}$.

α_{i-1} is called the link twist, which is the angle from $\hat{z}_{i-1}$ to $\hat{z}_i$ measured about $\hat{x}_{i-1}$.



Appendix C: Denavit–Hartenberg Parameters

The 4 parameters:

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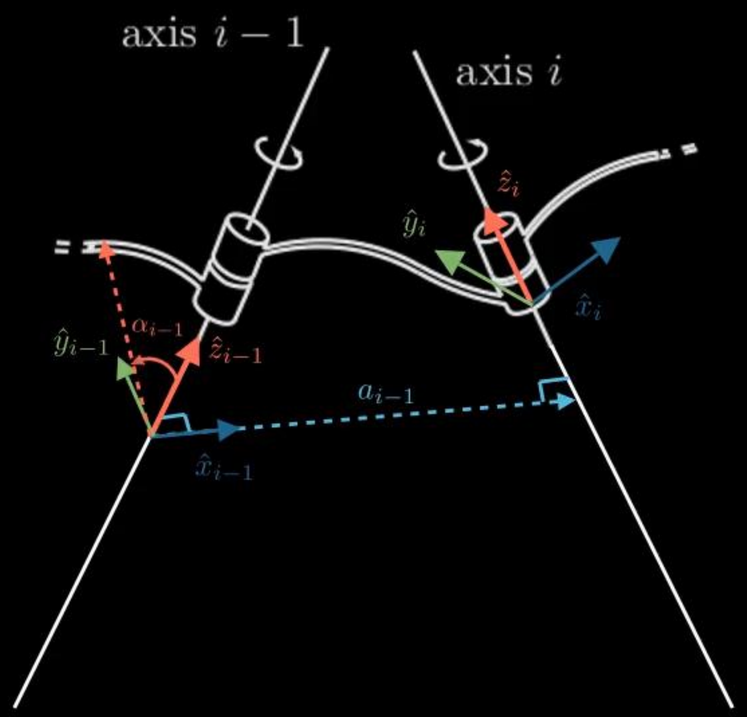

Appendix C: Denavit–Hartenberg Parameters

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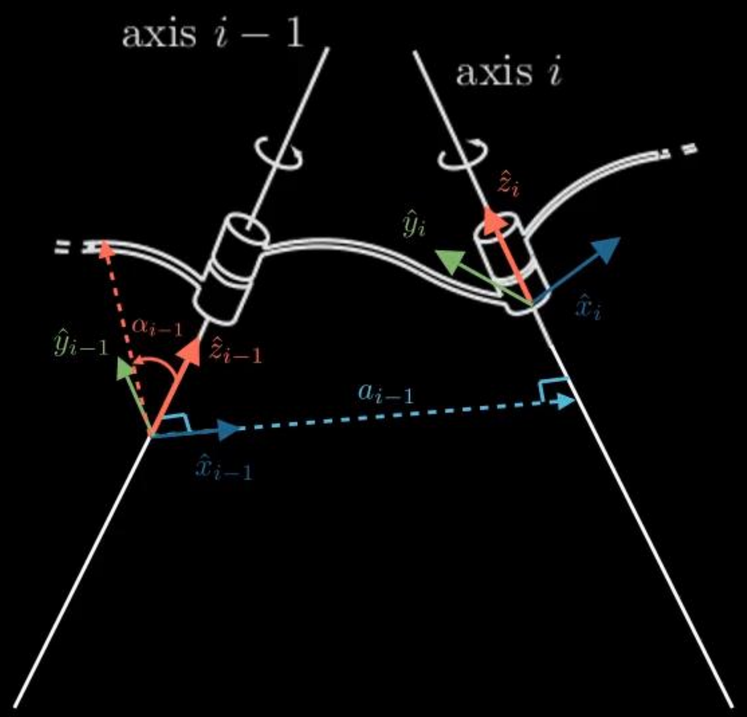

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Appendix C: Denavit–Hartenberg Parameters

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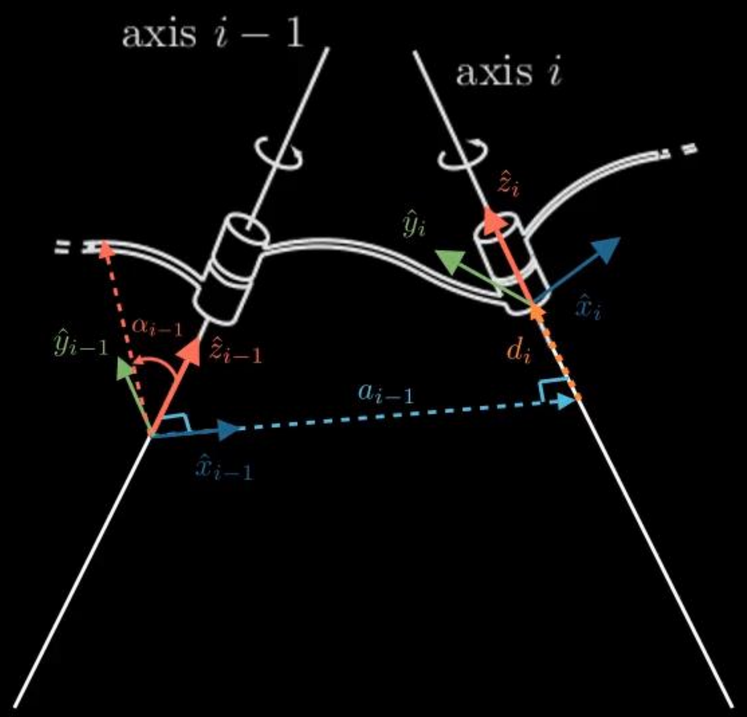

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Appendix C: Denavit–Hartenberg Parameters

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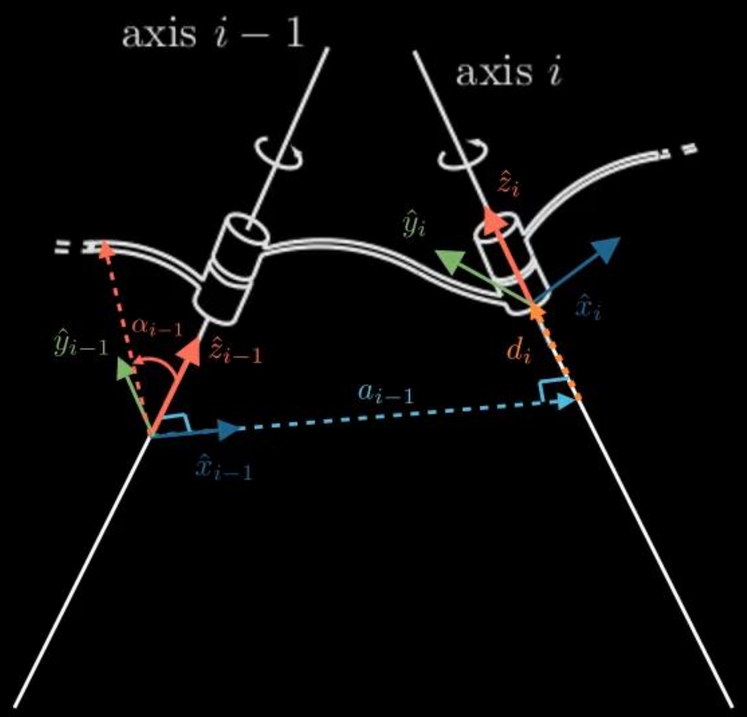

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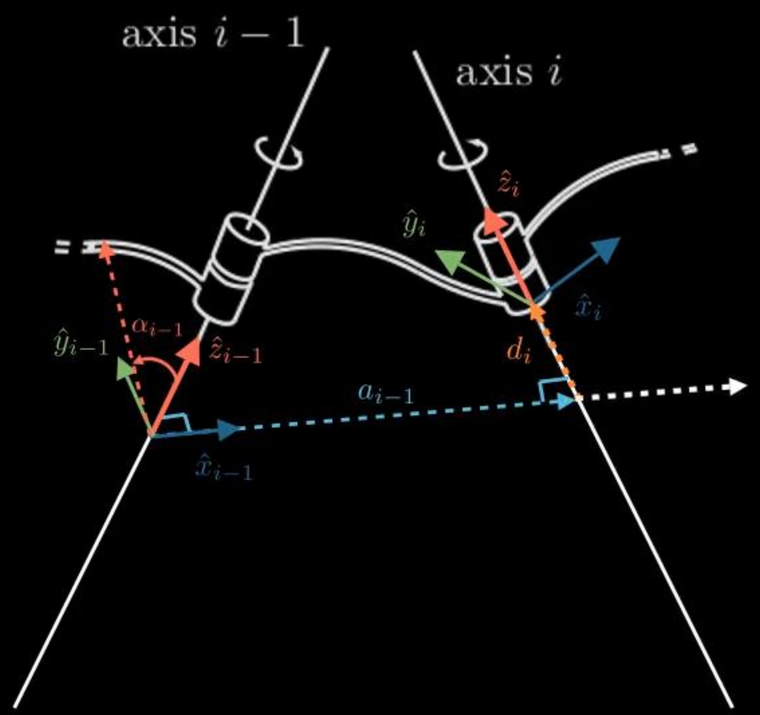
ϕ_i: is called the joint angle, which is the angle from $\hat{x}_{i-1}$ to $\hat{x}_i$ measured about the $\hat{z}_i$-axis.

Appendix C: Denavit–Hartenberg Parameters

The 4 parameters:



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Appendix C: Denavit–Hartenberg Parameters

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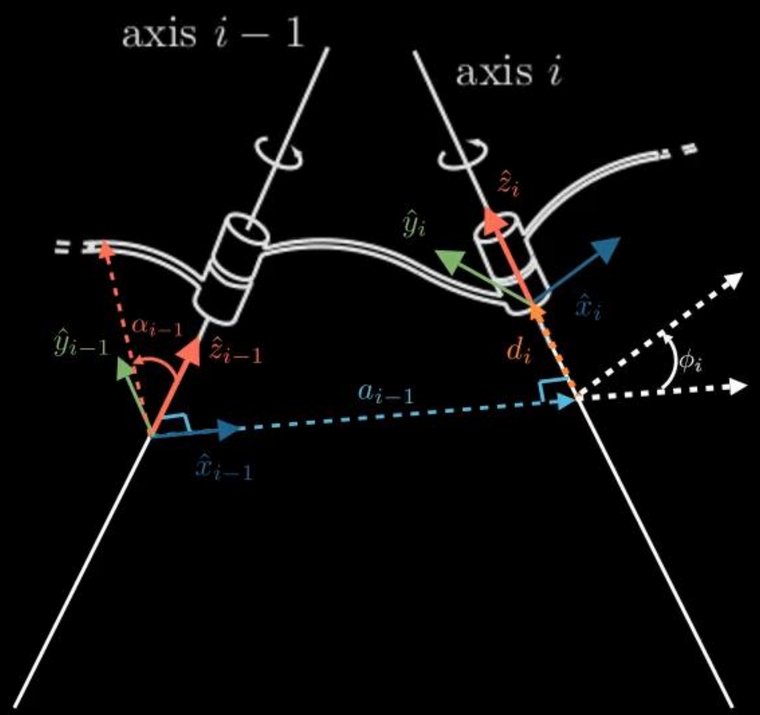

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Appendix C: Denavit–Hartenberg Parameters

Assigning the Ground and End-Effector Frames:

We would like to simplify as many of the D.H. parametes as possible.

The ground frame is usually chosen to coincide with the link-1 frame in its zero (rest) position.

If the joint is revolute this choice forces $a_0 = \alpha_0 = d_1 = 0$.

If the joint is prismatic joint we have $a_0 = \alpha_0 = \phi_1 = 0$.

The end-effector frame is attached to some reference point on the end-effector, usually at a location that makes the description of the task intuitive and natural and also simplifies as many of the D–H parameters as possible

Appendix C: Denavit–Hartenberg Parameters

Manipulator Forward Kinematics:

$$T_{i-1,i} = \mathrm{Rot}(\hat{\mathrm{x}}, \alpha_{i-1})\mathrm{Trans}(\hat{\mathrm{x}}, a_{i-1})\mathrm{Trans}(\hat{\mathrm{z}}, d_i)\mathrm{Rot}(\hat{\mathrm{z}}, \phi_i) = \begin{bmatrix} \cos\phi_i & -\sin\phi_i & 0 & a_{i-1} \\ \sin\phi_i\cos\alpha_{i-1} & \cos\phi_i\cos\alpha_{i-1} & -\sin\alpha_{i-1} & -d_i\sin\alpha_{i-1} \\ \sin\phi_i\sin\alpha_{i-1} & \cos\phi_i\sin\alpha_{i-1} & \cos\alpha_{i-1} & d_i\cos\alpha_{i-1} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathrm{Rot}(\hat{\mathrm{x}}, \alpha_{i-1}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha_{i-1} & -\sin\alpha_{i-1} & 0 \\ 0 & \sin\alpha_{i-1} & \cos\alpha_{i-1} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathrm{Trans}(\hat{\mathrm{x}}, a_{i-1}) = \begin{bmatrix} 1 & 0 & 0 & a_{i-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

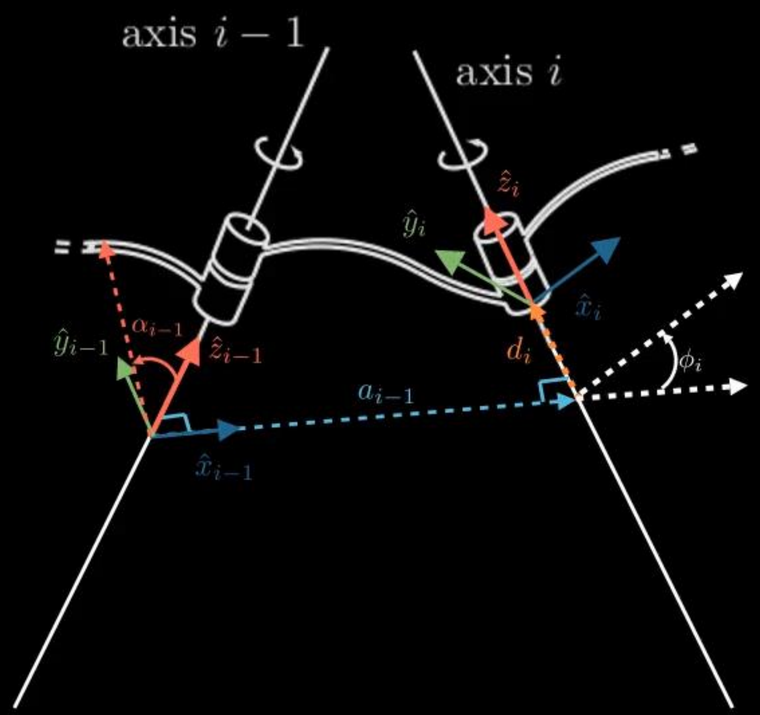
$$\mathrm{Trans}(\hat{\mathrm{z}}, d_i) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathrm{Rot}(\hat{\mathrm{z}}, \phi_i) = \begin{bmatrix} \cos\phi_{i-1} & -\sin\phi_{i-1} & 0 & 0 \\ \sin\phi_{i-1} & \cos\phi_{i-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Appendix C: Denavit–Hartenberg Parameters

Manipulator Forward Kinematics:

$$T_{i-1,i} = \mathrm{Rot}(\hat{\mathrm{x}}, \alpha_{i-1})\mathrm{Trans}(\hat{\mathrm{x}}, a_{i-1})\mathrm{Trans}(\hat{\mathrm{z}}, d_i)\mathrm{Rot}(\hat{\mathrm{z}}, \phi_i)$$



A useful way to visualize $T_{i,i-1}$ is that it transports frame $\{i-1\}$ to frame $\{i\}$ via the following sequence of four transformations:

(a) A rotation of frame $\{i-1\}$ about its $\hat{\mathrm{x}}$-axis by an angle α_{i-1}.

(b) A translation of this new frame along its $\hat{\mathrm{x}}$-axis by a distance a_{i-1}.

(c) A translation of the new frame formed by (b) along its $\hat{\mathrm{z}}$-axis by a distance d_i.

(d) A rotation of the new frame formed by (c) about its $\hat{\mathrm{z}}$-axis by an angle ϕ_i.

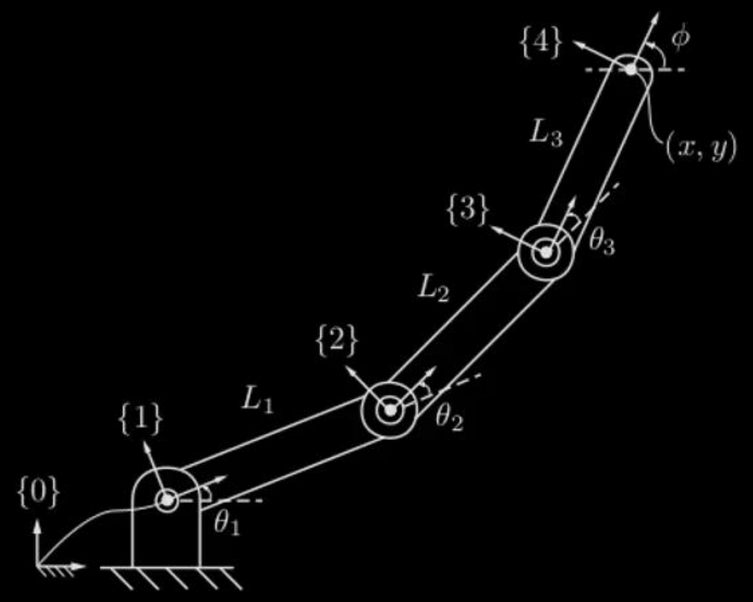
Note that switching the order of the first and second steps will not change the final form of $T_{i-1,i}$. Similarly, the order of the third and fourth steps can also be switched without affecting $T_{i-1,i}$.

Exercises …

Chapter 4: Forward Kintematics

4.1 Product of Exponentials:

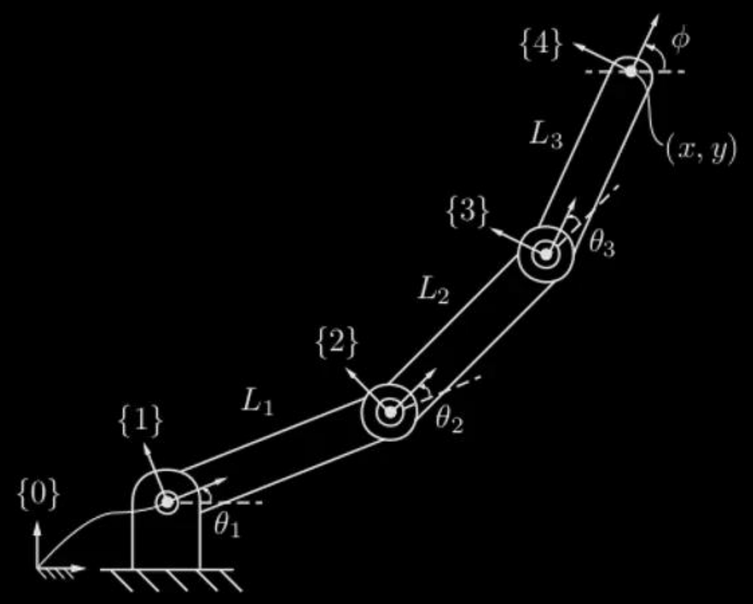
Home or Zero position of a robot is when all the joints are set to zero



Chapter 4: Forward Kintematics

4.1 Product of Exponentials:

Define M to be the position and orientation of frame {4} when the robot is at its home position.

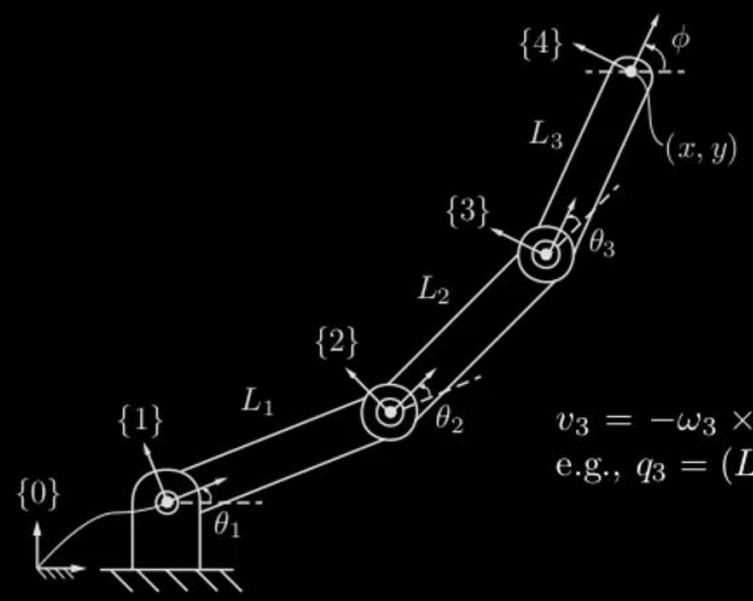


$$M = \begin{bmatrix} 1 & 0 & 0 & L_1 + L_2 + L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Chapter 4: Forward Kintematics

4.1 Product of Exponentials:

If θ_1 and θ_2 are held at their zero position then the screw axis corresponding to rotating about joint 3 can be expressed in the $\{0\}$ frame as:

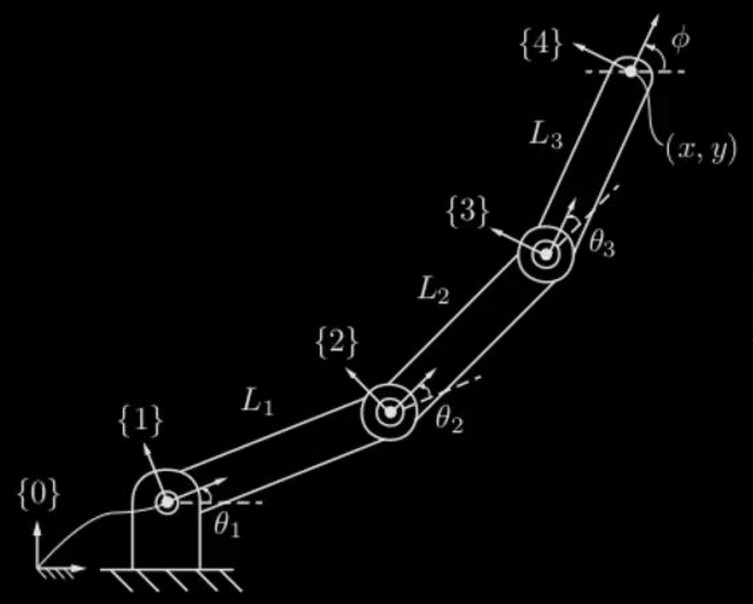


$$\mathcal{S}_3 = \begin{bmatrix} \omega_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -(L_1 + L_2) \\ 0 \end{bmatrix}$$

$v_3 = -\omega_3 \times q_3$, where q_3 is any point on the axis of joint 3 expressed in $\{0\}$, e.g., $q_3 = (L_1 + L_2, 0, 0)$.

Chapter 4: Forward Kintematics

4.1 Product of Exponentials:



The screw axis $\mathcal{S}_3$ can be expressed in $se(3)$ matrix form as

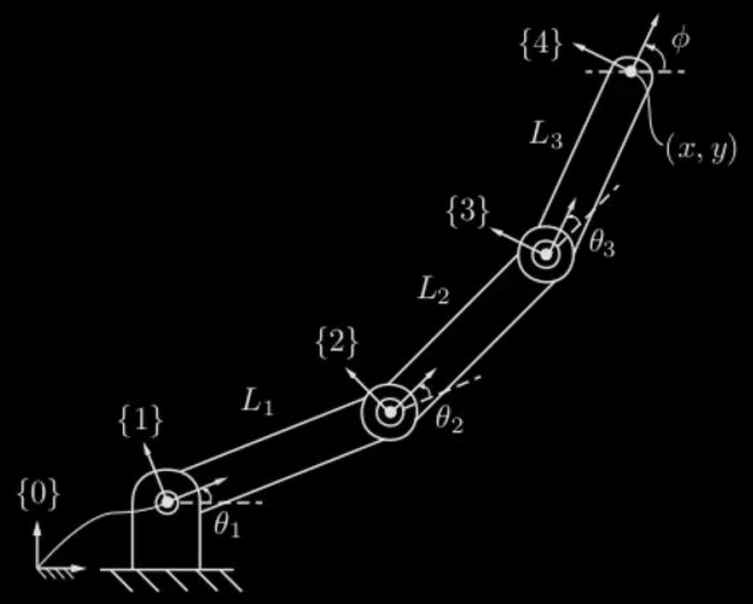
$$[\mathcal{S}_3] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -(L_1 + L_2) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, for any θ_3, the matrix exponential representation for screw motions from the previous chapter allows us to write

$$T_{04} = e^{[\mathcal{S}_3]\theta_3} M \qquad (\text{for } \theta_1 = \theta_2 = 0). \tag{4.7}$$

Chapter 4: Forward Kintematics

4.1 Product of Exponentials:



Now, for $\theta_1 = 0$ and any fixed (but arbitrary) θ_3, rotation about joint 2 can be viewed as applying a screw motion to the rigid (link 2)/(link 3) pair, i.e.,

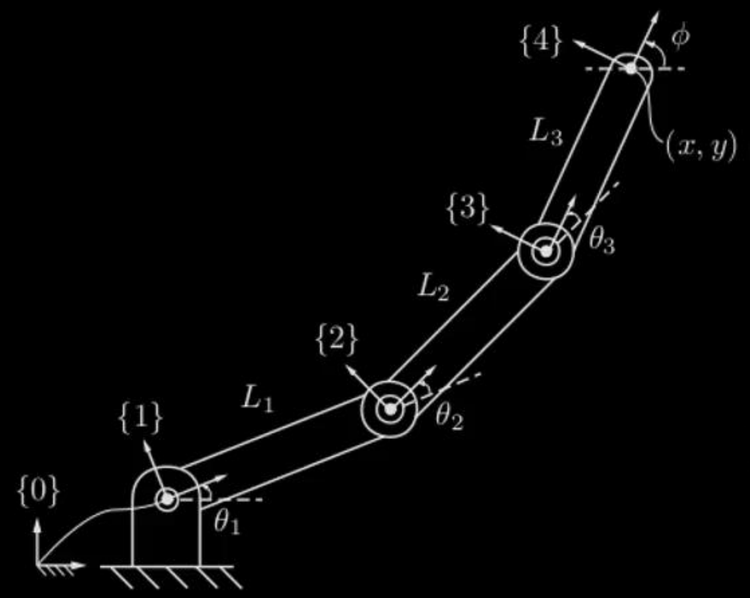
$$T_{04} = e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M \qquad (\text{for } \theta_1 = 0), \tag{4.8}$$

where $[\mathcal{S}_3]$ and M are as defined previously, and

$$[\mathcal{S}_2] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -L_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{4.9}$$

Chapter 4: Forward Kintematics

4.1 Product of Exponentials:



Finally, keeping θ_2 and θ_3 fixed, rotation about joint 1 can be viewed as applying a screw motion to the entire rigid three-link assembly. We can therefore write, for arbitrary values of $(\theta_1, \theta_2, \theta_3)$,

$$T_{04} = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M, \tag{4.10}$$

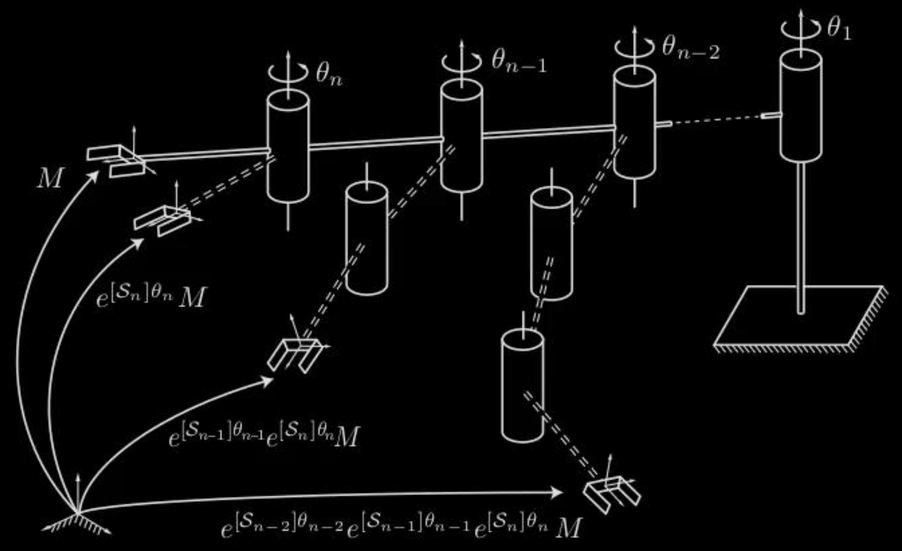
where

$$[\mathcal{S}_1] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{4.11}$$

Thus the forward kinematics can be expressed as a product of matrix exponentials, each corresponding to a screw motion. Note that this latter derivation of the forward kinematics does not use any link reference frames; only $\{0\}$ and M must be defined.

Chapter 4: Forward Kintematics

4.1 Product of Exponentials:



$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} \cdots e^{[\mathcal{S}_{n-1}]\theta_{n-1}} e^{[\mathcal{S}_n]\theta_n} M. \tag{4.14}$$

This is the product of exponentials formula describing the forward kinematics of an n-dof open chain. Specifically, we call Equation (4.14) the **space form** of the product of exponentials formula, referring to the fact that the screw axes are expressed in the fixed space frame.

To summarize, to calculate the forward kinematics of an open chain using the space form of the PoE formula (4.14), we need the following elements:

(a) the end-effector configuration $M \in SE(3)$ when the robot is at its home position;

(b) the screw axes $\mathcal{S}_1, \ldots, \mathcal{S}_n$ expressed in the fixed base frame, corresponding to the joint motions when the robot is at its home position;

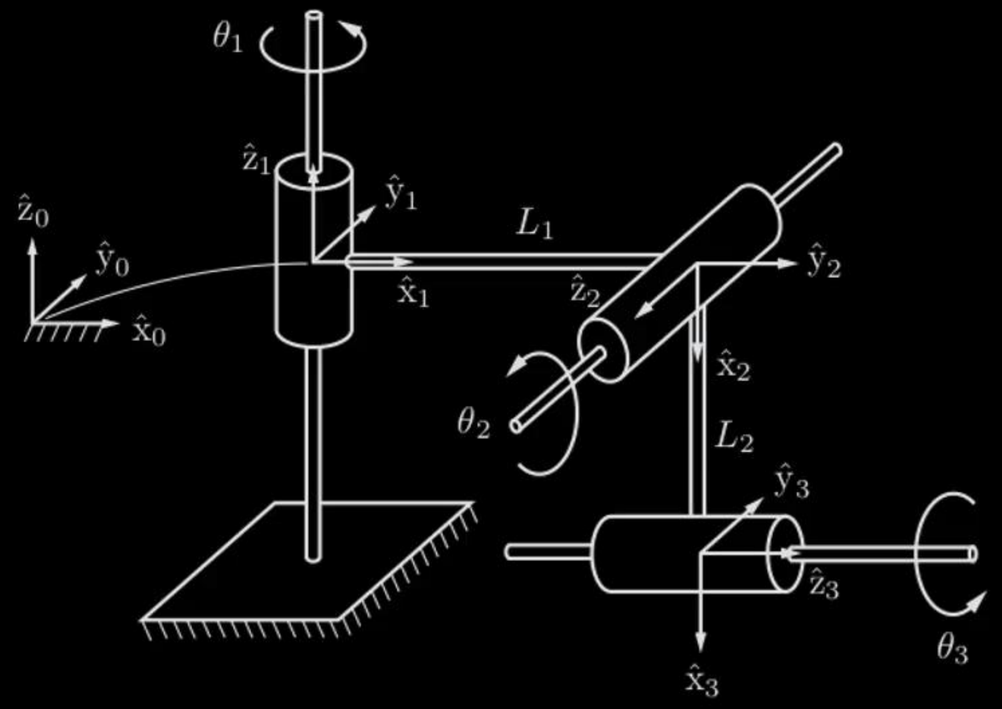
(c) the joint variables $\theta_1, \ldots, \theta_n$.

Unlike the D–H representation, no link reference frames need to be defined.

Chapter 4: Forward Kintematics

Example 4.1:

Derive the forward kinematics for the spatial open chain robot using the PoE formula



Chapter 4: Forward Kintematics

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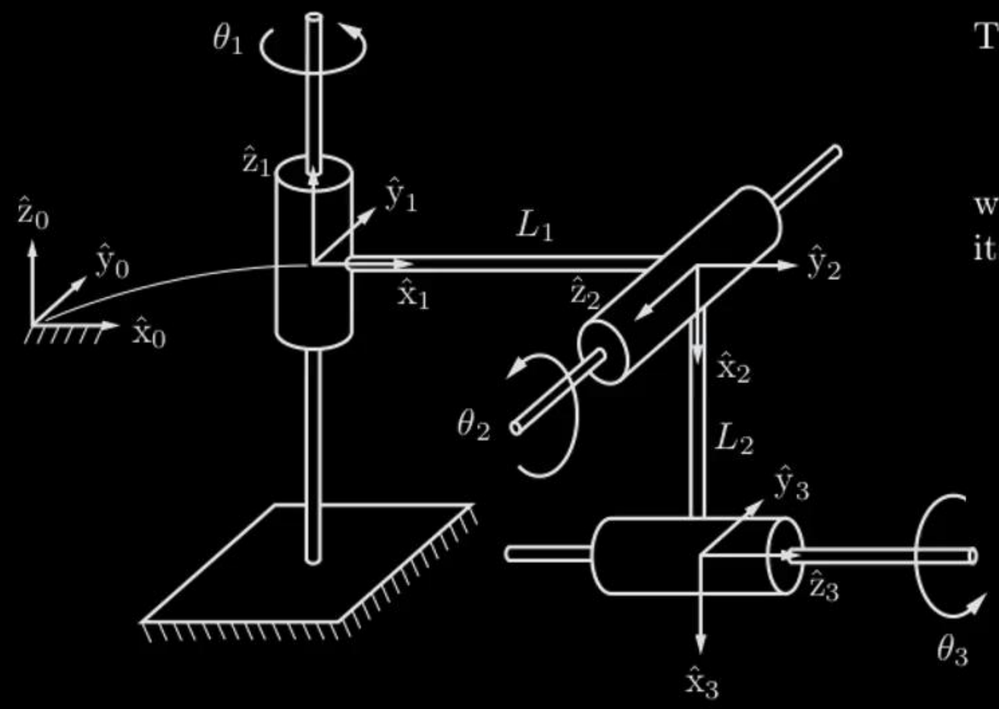

The forward kinematics has the form

$$T(\theta) =$$

where $M \in SE(3)$ is the end-effector frame configuration when the robot is in its zero position. By inspection M can be obtained as

$$M = \begin{bmatrix} \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \end{bmatrix}.$$

Chapter 4: Forward Kintematics

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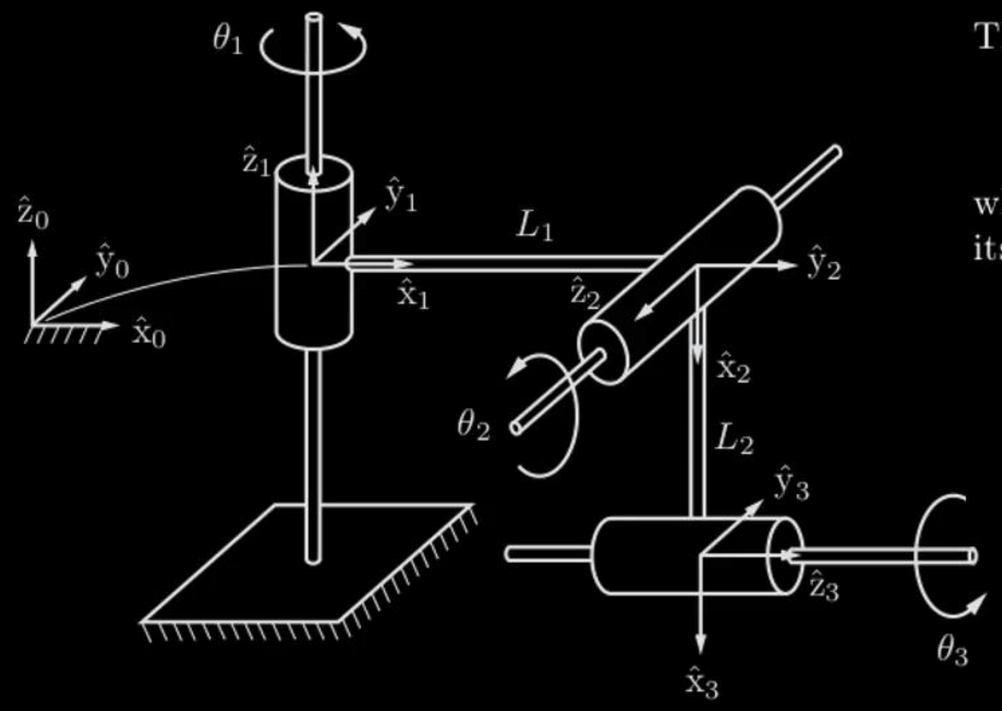
Derive the forward kinematics for the spatial open chain robot using the PoE formula



The forward kinematics has the form

$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} M,$$

where $M \in SE(3)$ is the end-effector frame configuration when the robot is in its zero position. By inspection M can be obtained as

$$M = \begin{bmatrix} 0 & 0 & 1 & L_1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Chapter 4: Forward Kintematics

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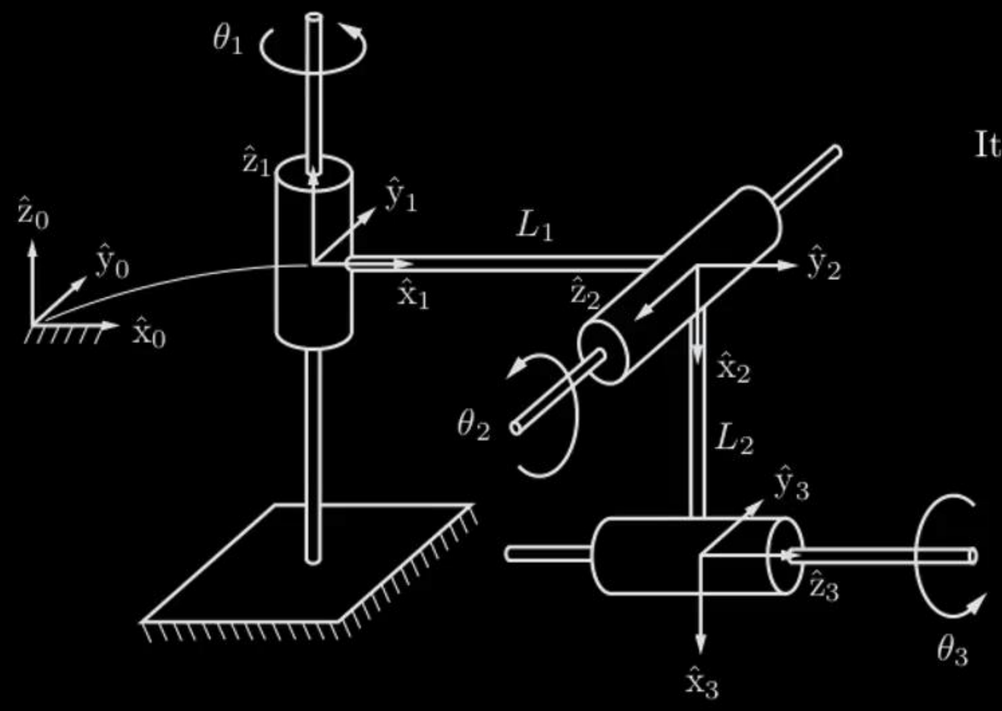
Derive the forward kinematics for the spatial open chain robot using the PoE formula



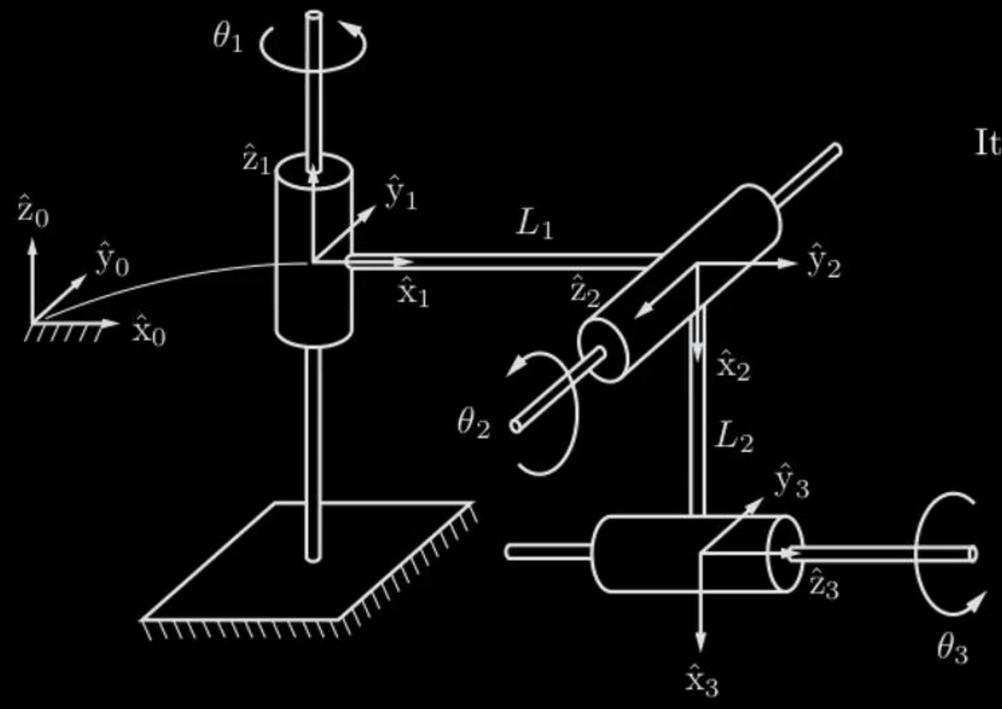
It will be more convenient to list the screw axes in the following tabular form:

i	ω_i	v_i
1		
2		
3		

Chapter 4: Forward Kintematics

Example 4.1:

Derive the forward kinematics for the spatial open chain robot using the PoE formula



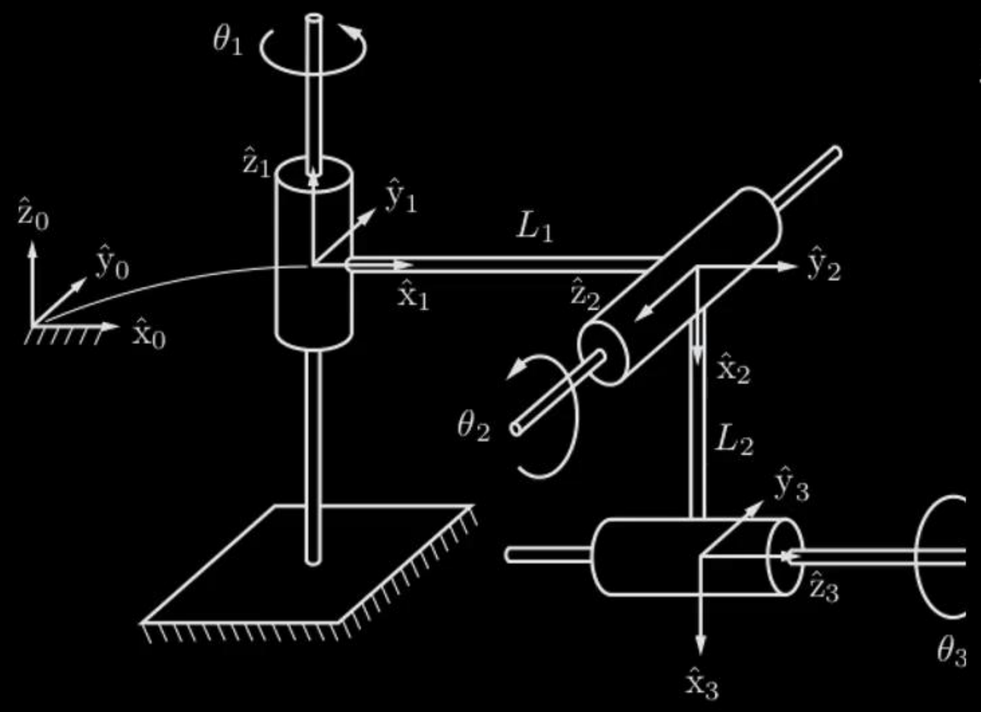
It will be more convenient to list the screw axes in the following tabular form:

i	ω_i	v_i
1	$(0,0,1)$	$(0,0,0)$
2	$(0,-1,0)$	$(0,0,-L_1)$
3	$(1,0,0)$	$(0,L_2,0)$

Chapter 4: Forward Kintematics

Example 4.1:

Derive the forward kinematics for the spatial open chain robot using the PoE formula



In summary, we have the following 4×4 matrix representations for the three joint screw axes $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$:

$$[\mathcal{S}_1] = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

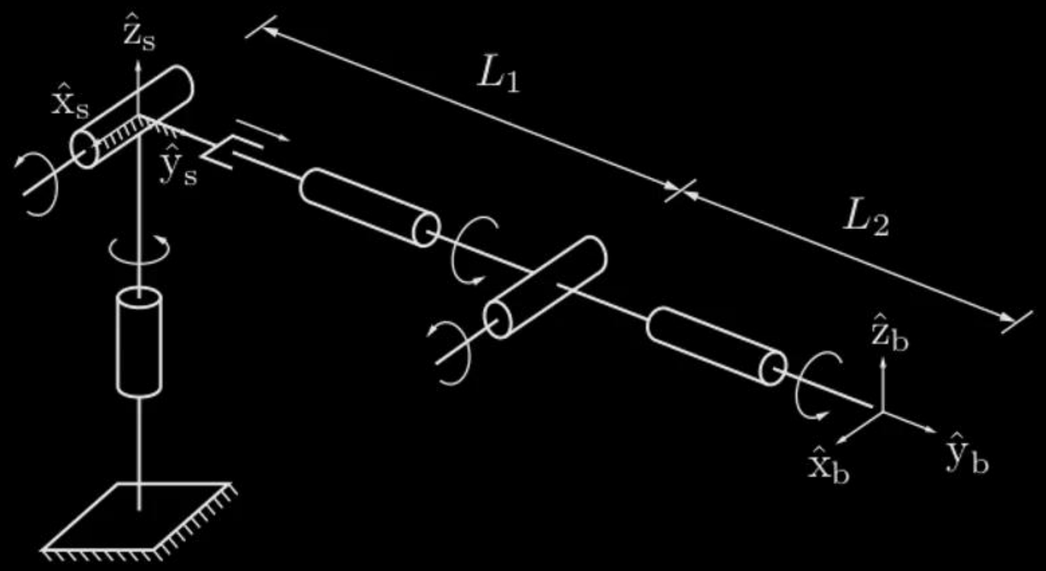
$$[\mathcal{S}_2] = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -L_1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$[\mathcal{S}_3] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -L_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Chapter 4: Forward Kintematics

Example 4.4:

Derive the forward kinematics for the spatial open chain robot using the PoE formula



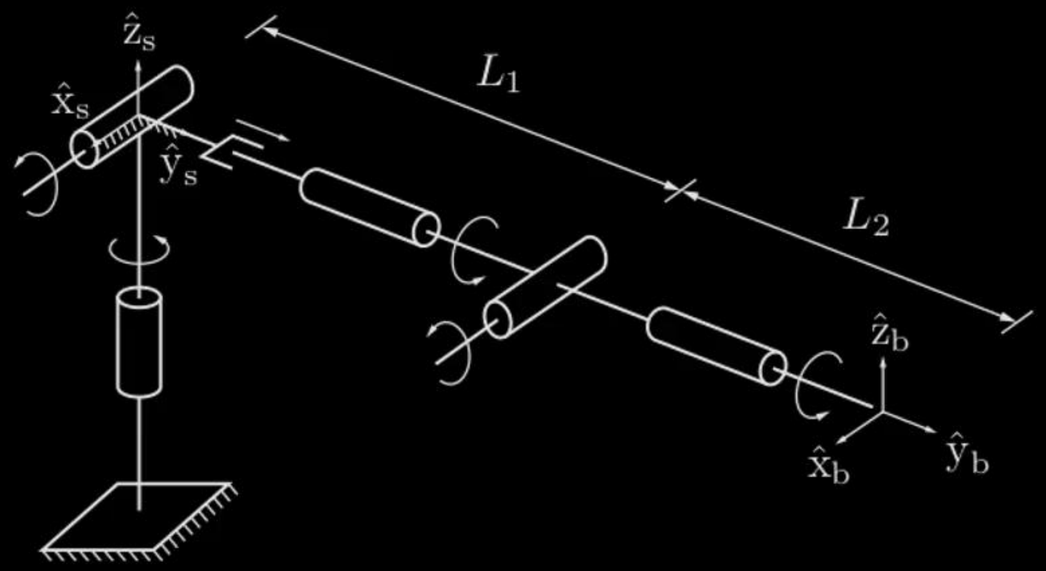
$$T(\theta) =$$

$$M = \begin{bmatrix} \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \\ \quad & \quad & \quad & \quad \end{bmatrix}.$$

Chapter 4: Forward Kintematics

Example 4.4:

Derive the forward kinematics for the spatial open chain robot using the PoE formula

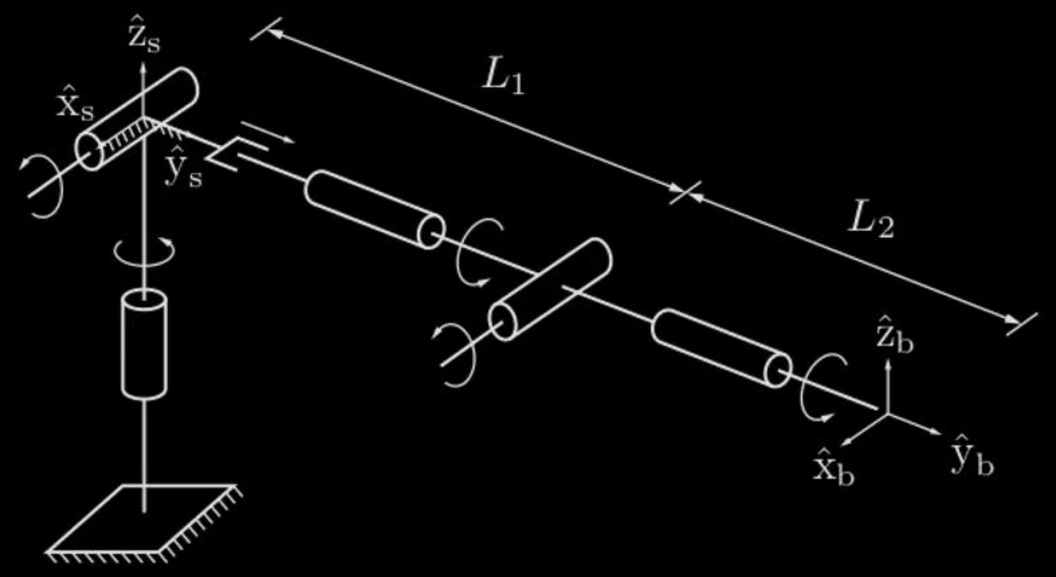


$$T(\theta) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} e^{[\mathcal{S}_3]\theta_3} e^{[\mathcal{S}_4]\theta_4} e^{[\mathcal{S}_5]\theta_5} e^{[\mathcal{S}_6]\theta_6} M$$

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & L_1 + L_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Example 4.4:

Derive the forward kinematics for the spatial open chain robot using the PoE formula



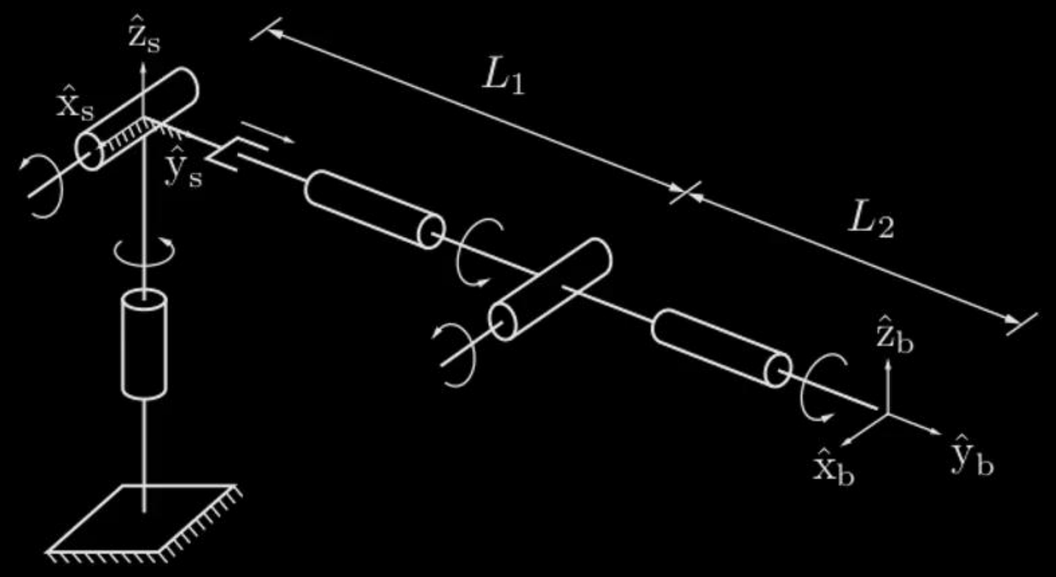
The screw axes $\mathcal{S}_i = (\omega_i, v_i)$ are listed in the following table:

i	ω_i	v_i
1		
2		
3		
4		
5		
6		

Chapter 4: Forward Kintematics

Example 4.4:

Derive the forward kinematics for the spatial open chain robot using the PoE formula



The screw axes $\mathcal{S}_i = (\omega_i, v_i)$ are listed in the following table:

i	ω_i	v_i
1	$(0,0,1)$	$(0,0,0)$
2	$(1,0,0)$	$(0,0,0)$
3	$(0,0,0)$	$(0,1,0)$
4	$(0,1,0)$	$(0,0,0)$
5	$(1,0,0)$	$(0,0,-L_1)$
6	$(0,1,0)$	$(0,0,0)$

4.1.3 Second Formulation: Screw Axes in the End-Effector Frame

The PoE formula can also be represented in the body form:

$$
\begin{aligned}
T(\theta) &= e^{[\mathcal{S}_1]\theta_1} \dots e^{[\mathcal{S}_n]\theta_n} M \\
&= e^{[\mathcal{S}_1]\theta_1} \dots M e^{M^{-1}[\mathcal{S}_n]M\theta_n} \\
&= e^{[\mathcal{S}_1]\theta_1} \dots M e^{M^{-1}[\mathcal{S}_{n-1}]M\theta_{n-1}} e^{M^{-1}[\mathcal{S}_n]M\theta_n} \\
&= M e^{M^{-1}[\mathcal{S}_1]M\theta_1} \dots e^{M^{-1}[\mathcal{S}_{n-1}]M\theta_{n-1}} e^{M^{-1}[\mathcal{S}_n]M\theta_n} \\
&= M e^{[\mathcal{B}_1]\theta_1} \dots e^{[\mathcal{B}_{n-1}]\theta_{n-1}} e^{[\mathcal{B}_n]\theta_n},
\end{aligned}
$$