Homework: Particle Physics

Yingsheng Huang

November 2, 2016

1. The force range and characteristic interaction time for all three types of interaction but gravity are

	Force Range	Characteristic interaction time
Strong interaction	1 fm	1×10^{-23} s
Electromagnetic interaction	∞	1×10^{-16} s
Weak interaction	1/400 fm	1×10^{-10} s

2. For electron in an EM field, Dirac equation can be written as [1]

$$[i\frac{\partial}{\partial t} + e\phi - \boldsymbol{\alpha} \cdot (\mathbf{P} + e\mathbf{A}) - m\beta]\psi = 0 \tag{1}$$

where $\mathbf{P} = -i\nabla$. Set

$$\psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix} e^{-imt}$$

so that the certain part of electron rest mass can be removed. Then we have

$$\begin{aligned} i\frac{\partial}{\partial t}\varphi &= \boldsymbol{\sigma} \cdot (\mathbf{P} + e\mathbf{A})\chi - e\phi\varphi \\ i\frac{\partial}{\partial t}\chi &= \boldsymbol{\sigma} \cdot (\mathbf{P} + e\mathbf{A})\varphi - e\phi\chi - 2m\chi \end{aligned}$$

At nonrelativistic limit, we have

$$\chi \approx \frac{1}{2m}\boldsymbol{\sigma} \cdot (\mathbf{P} + e\mathbf{A})\varphi \tag{2}$$

where $\chi/\varphi \ll 1$. Then

$$i\frac{\partial}{\partial t}\varphi = \frac{1}{2m}[\boldsymbol{\sigma} \cdot (\mathbf{P} + e\mathbf{A})]^2\varphi - e\phi\varphi$$

Using

$$\begin{aligned} [\boldsymbol{\sigma} \cdot (\mathbf{P} + e\mathbf{A})]^2 &= (\mathbf{P} + e\mathbf{A})^2 + i\boldsymbol{\sigma} \cdot [(\mathbf{P} + e\mathbf{A}) \times (\mathbf{P} + e\mathbf{A})] \\ &= (\mathbf{P} + e\mathbf{A})^2 + ie\boldsymbol{\sigma} \cdot [\mathbf{P} \times \mathbf{A} + \mathbf{A} \times \mathbf{P}] \\ &= (\mathbf{P} + e\mathbf{A})^2 + e\boldsymbol{\sigma} \cdot (\nabla \times \mathbf{A}) \\ &= (\mathbf{P} + e\mathbf{A})^2 + e\boldsymbol{\sigma} \cdot \end{aligned}$$

Then (2) becomes

$$
\begin{aligned}
i\frac{\partial}{\partial t}\varphi &= \left[\frac{1}{2m}(\mathbf{P}+e\mathbf{A})^2 + \frac{e}{2m}\boldsymbol{\sigma}\cdot\ - e\phi\right]\varphi \\
&= \left[\frac{1}{2m}(\mathbf{P}+e\mathbf{A})^2 - \boldsymbol{\mu}\cdot\ - e\phi\right]\varphi
\end{aligned}
$$

where $\boldsymbol{\mu} = -\frac{e\hbar}{2mc}\boldsymbol{\sigma}$ (in our previous calculation $\hbar = c = 1$) is the intrinsic magnetic moment of electron.

3. From the definition of Breit-Wigner formula, we have the distribution function for decay to a specific quantum state

$$
P(E) = \frac{1}{2\pi}\frac{\Gamma}{(E-M)^2 + \Gamma^2/4} \tag{3}
$$

and also

$$
\Gamma = \frac{1}{\tau}
$$

(3) can be proved by (using some inturtive fourier transformations)

$$
\psi(M) = \int_{-\infty}^{\infty}\frac{\mathrm{d}t}{\sqrt{2\pi}}e^{iMt}\left|t\right\rangle = \int_{-\infty}^{\infty}\frac{\mathrm{d}t}{\sqrt{2\pi}}e^{iMt}e^{-i(m-i\frac{\Gamma}{2})t}\left|0\right\rangle = \frac{1}{\sqrt{2\pi}}\frac{1}{(M-m)+i\frac{\Gamma}{2}}
$$

and the mass distribution function would be

$$
\rho(M) \equiv \Gamma\psi(M)\psi^*(M) = \frac{1}{2\pi}\frac{\Gamma}{(E-M)^2+\Gamma^2/4}
$$

More strict prove can be found in B.R.Martin's *Particle Physics* which I briefly listed in the end of this problem.

With the rest mass M and the lifetime τ of the particle, the distribution function would be

$$
P(E) = \frac{1}{2\pi}\frac{\Gamma}{(E-M)^2 + 1/(4\tau^2)} \tag{4}
$$

Prove [2] of (3): We choose the wavefunctions of possible final states to be orthonormal, the wavefunction describes state of any timestamp can be expanded by

$$
\boldsymbol{\Psi}(\mathbf{r},t) = \sum_{n=0}^{\infty} a_n(t)e^{-iE_nt}\psi_n(\mathbf{r})
$$

where

$$
a_0(0) = 1, \qquad a_n(0) = 0 (n \geq 1)
$$

and

$$
E_n = H_{nn} = \int \psi_n^* H \psi_n \mathrm{d}x
$$

To determine the time dependency of $a_n(t)$, we use the Schrödinger equation

$$
i\frac{\partial\boldsymbol{\Psi}}{\partial t} = H\boldsymbol{\Psi}
$$

and

$$\sum_m \left\{ i\frac{\mathrm{d}a_m}{\mathrm{d}t} e^{-iE_m t}\psi_m + E_m a_m e^{-iE_m t}\psi_m \right\} = \sum_m a_m e^{-iE_m t} H\psi_m$$

which gives (assuming terms when $n \neq m$ are small except for the first)

$$i\frac{\mathrm{d}a_n}{\mathrm{d}t} = H_{n0} e^{-i(E_0 - E_n)t} a_0$$

and also we evaluate this problem in the rest frame for the decaying particle, and asuume $a_0(t) = e^{-\Gamma t/2}$ which consist with the exponential decay law. Now we have

$$ia_n = -iH_{n0} \left\{ \frac{e^{-i[(M-E_n)-i\Gamma/2]t} - 1}{(E_n - M) + i\Gamma/2} \right\} \xlongequal{t \gg 1/\Gamma} \frac{iH_{n0}}{(E_n - M) + i\Gamma/2}$$

Now the probability would be

$$P_n = a_n^2 = \frac{|H_{n0}|^2}{(E_n - M)^2 + \Gamma^2/4} = \frac{2\pi}{\Gamma} |H_{n0}|^2 P(E_n - M)$$

where

$$P(E_n - M) = \frac{\Gamma/2\pi}{(E - M)^2 + \Gamma^2/4}$$

We can verify it's normalized.

4. For unstable particles with long enough lifetime to be tracked, their lifetime can be calculated using simple Lorentz invariance property

$$m\tau = Et - pL = E\frac{L}{v} - pL = E^2\frac{L}{p} - pL = \frac{m^2 L}{p}$$

which means we can estimate the particle lifetime by $\tau = \frac{mL}{p}$.

5. The mass of J/ψ particle is 3.1GeV, and which of electron is 0.5MeV. To produce J/ψ particle in e^+e collision, we have

$$M = 3.1\text{GeV}, \quad m = 0.5\text{MeV}$$

and in center-of-mass frame

$$4E^2 \geq M^2$$

where E is the energy of one electron/positron. Note that

$$E = \gamma m = \frac{1}{\sqrt{1-\beta^2}} m$$

So we have

$$\frac{1}{\sqrt{1-\beta^2}} m \geq 2M$$

which gives

$$\beta \geq \sqrt{1 - \frac{m^2}{4M^2}} \approx 0.99999999024453693539961460987023$$

And that's the minimum velocity electron/positron must go.

6. The Mandelstam variables can be expressed by

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2 \tag{5}$$

$$t = (p_1 - p_3)^2 = (p_2 - p_4)^2 \tag{6}$$

$$u = (p_1 - p_4)^2 = (p_2 - p_3)^2 \tag{7}$$

From which we have (in CM frame)

$$\begin{aligned} s + t + u &= (p_1 + p_2)^2 + (p_1 - p_3)^2 + (p_1 - p_4)^2 \\ &= 3p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2p_1p_2 - 2p_1p_3 - 2p_1p_4 \end{aligned}$$

Assuming

$$p_1 = (E_1, \mathbf{p_1}), \;\; p_2 = (E_2, \mathbf{p_2}), \;\; p_3 = (E_3, \mathbf{p_3}), \;\; p_4 = (E_4, \mathbf{p_4})$$
$$E_1 + E_2 = E_3 + E_4, \;\; \mathbf{p_1} + \mathbf{p_2} = \mathbf{p_3} + \mathbf{p_4}$$

and

$$\begin{aligned} s + t + u &= 3p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2p_1p_2 - 2p_1p_3 - 2p_1p_4 \\ &= 3p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2E_1(E_2 - E_3 - E_4) - 2\mathbf{p_1}(\mathbf{p_2} - \mathbf{p_3} - \mathbf{p_4}) \\ &= 3p_1^2 + p_2^2 + p_3^2 + p_4^2 - 2E_1^2 + 2|\mathbf{p_1}|^2 \\ &= p_1^2 + p_2^2 + p_3^2 + p_4^2 \\ &= m_1^2 + m_2^2 + m_3^2 + m_4^2 \end{aligned}$$

7. For branch $A \to BC$, the partial width is [2]

$$\Gamma(A \to BC) = \frac{g_{ABC}^2}{8\pi} \frac{|p_B|}{M_A^2}$$

If A is an unstable particle with width Γ, from Breit-Wigner formula, we have

$$P_f(E) = \frac{1}{2\pi} \frac{\Gamma_f}{(E - M)^2 + \Gamma^2/4} \tag{8}$$

which satisfies normalization $\int_{-\infty}^{\infty} \mathrm{d}E P_f(E) = 1$. Considering the momentum of particle B is (using the fact that B and C have the same mass, which means they have opposite momentums, and we choose the centre-of-mass frame of A)

$$p_B = \sqrt{\left(\frac{E}{2}\right)^2 - M_B^2}$$

Then the decay width becomes

$$\begin{aligned} \Gamma &= \frac{g_{ABC}^2}{8\pi} \int_{-\infty}^{\infty} \mathrm{d}E P_f(E) \frac{p_B}{E^2} \\ &= \frac{g_{ABC}^2}{8\pi} \int_{-\infty}^{\infty} \mathrm{d}E \frac{1}{2\pi} \frac{\Gamma_f}{(E - M_A)^2 + \Gamma^2/4} \frac{p_B}{E^2} \\ &= \frac{g_{ABC}^2}{16\pi^2} \int_{-\infty}^{\infty} \mathrm{d}E \frac{\Gamma_f}{(E - M_A)^2 + \Gamma^2/4} \frac{\sqrt{\left(\frac{E}{2}\right)^2 - M_B^2}}{E^2} \end{aligned}$$

8. 3-particle decay phase space integrals can be derived as follows [3]:

First we know that the gengeral non-relativistic expression for N-body phase space is

$$\mathrm{d}n = (2\pi)^3 \prod_{i=1}^{N} \frac{\mathrm{d}^3\mathbf{p}_i}{(2\pi)^3} \delta^3\left(\mathbf{p}_a - \sum_{i=1}^{N} \mathbf{p}_i\right) \tag{9}$$

where $\mathbf{p}_a$ is the momentum of the decaying particle. (This expression can be derived easily from the phase space volume of each particle.) According to Fermi's golden rule (and notice the $(2E_i)^{1/2}$ ratio difference between $\mathcal{M}_{fi}$ and T_{fi}), the decay rate can be written as

$$\Gamma_{fi} = \frac{(2\pi)^4}{2E_a} \int |\mathcal{M}_{fi}|^2 \delta(E_a - \sum_{i=1}^{N} E_i) \delta^3(\mathbf{p}_a - \sum_{i=1}^{N} \mathbf{p}_i) \prod_{i=1}^{N} \frac{\mathrm{d}^3\mathbf{p}_i}{(2\pi)^3 2E_i} \tag{10}$$

So for 3-particle decay the decay rate can be written as

$$\Gamma_{fi} = \frac{(2\pi)^4}{2E_a} \int |\mathcal{M}_{fi}|^2 \delta(E_a - E_1 - E_2 - E_3) \delta^3(\mathbf{p}_a - \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) \frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3 2E_1} \frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3 2E_2} \frac{\mathrm{d}^3\mathbf{p}_3}{(2\pi)^3 2E_3}$$

Now we consider it in the centre-of-mass frame of the decaying particle A, which means $E_a = m_a$ and $\mathbf{p}_a = 0$. Through the integration of delta function and $\mathrm{d}^3\mathbf{p}_3$, we have

$$\begin{aligned}
\Gamma_{fi} &= \frac{(2\pi)^4}{2m_a} \int |\mathcal{M}_{fi}|^2 \delta(m_a - E_1 - E_2 - E_3) \delta^3(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3) \frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3 2E_1} \frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3 2E_2} \frac{\mathrm{d}^3\mathbf{p}_3}{(2\pi)^3 2E_3} \\
&= \frac{(2\pi)^4}{2m_a} \int |\mathcal{M}_{fi}|^2 \delta(m_a - E_1 - E_2 - E_3) \frac{\mathrm{d}^3\mathbf{p}_1}{(2\pi)^3 2E_1} \frac{\mathrm{d}^3\mathbf{p}_2}{(2\pi)^3 2E_2} \frac{1}{(2\pi)^3 2E_3} \\
&= \frac{1}{2^4 m_a (2\pi)^5} \int \frac{|\mathcal{M}_{fi}|^2}{E_1 E_2 E_3} \delta(m_a - E_1 - E_2 - E_3) \mathrm{d}^3\mathbf{p}_1 \mathrm{d}^3\mathbf{p}_2 \\
&= \frac{1}{2^4 m_a (2\pi)^5} \int \frac{|\mathcal{M}_{fi}|^2}{E_1 E_2 E_3} \delta(m_a - E_1 - E_2 - E_3) \mathrm{d}^3\mathbf{p}_1 \mathrm{d}^3\mathbf{p}_2 \\
&= \frac{1}{2^4 m_a (2\pi)^5} \int \frac{|\mathcal{M}_{fi}|^2 |\mathbf{p}_1|^2 |\mathbf{p}_2|^2 \mathrm{d}p_1 \, \mathrm{d}(\cos\theta_1) \, \mathrm{d}\phi_1 \mathrm{d}p_2 \, \mathrm{d}(\cos\theta_2) \, \mathrm{d}\phi_2}{E_1 E_2 \sqrt{|\mathbf{p}_1|^2 + |\mathbf{p}_2|^2 + 2|\mathbf{p}_1||\mathbf{p}_2| \cos\theta_2 + m_3^2}} \\
&\qquad\qquad \delta(m_a - E_1 - E_2 - \sqrt{|\mathbf{p}_1|^2 + |\mathbf{p}_2|^2 + 2|\mathbf{p}_1||\mathbf{p}_2| \cos\theta_2 + m_3^2})
\end{aligned}$$

where θ_1, ϕ_1 and ϕ_2 are independent of the integral and therefore can be integrated first. Note that

$$\frac{\mathrm{d}|\mathbf{p}_i|}{\mathrm{d}E_i} = \frac{E_i}{|\mathbf{p}_i|}$$

which means

$$\mathrm{d}|\mathbf{p}_i| = \frac{E_i}{|\mathbf{p}_i|} \mathrm{d}E_i$$

and mark the kernel of δ function as

$$f(\cos\theta_2) \equiv m_a - E_1 - E_2 - \sqrt{|\mathbf{p}_1|^2 + |\mathbf{p}_2|^2 + 2|\mathbf{p}_1||\mathbf{p}_2| \cos\theta_2 + m_3^2}$$

we have

$$f'(\cos\theta_2) = -\frac{2|\mathbf{p}_1||\mathbf{p}_2|}{2\sqrt{|\mathbf{p}_1|^2 + |\mathbf{p}_2|^2 + 2|\mathbf{p}_1||\mathbf{p}_2| \cos\theta_2 + m_3^2}}$$

and the real root of $f(\cos\theta_2)=0$ is

$$\cos\theta_2' = \frac{(m_a - E_1 - E_2)^2 - m_3^2 - |\mathbf{p}_1|^2 - |\mathbf{p}_2|^2}{2|\mathbf{p}_1||\mathbf{p}_2|}$$

So we have

$$\begin{aligned}
&\delta(m_a - E_1 - E_2 - \sqrt{|\mathbf{p}_1|^2 + |\mathbf{p}_2|^2 + 2|\mathbf{p}_1||\mathbf{p}_2|\cos\theta_2 + m_3^2}) \\
=&\frac{\delta(\cos\theta_2 - \cos\theta_2')}{\frac{2|\mathbf{p}_1||\mathbf{p}_2|}{2\sqrt{|\mathbf{p}_1|^2 + |\mathbf{p}_2|^2 + 2|\mathbf{p}_1||\mathbf{p}_2|\cos\theta_2' + m_3^2}}}
\end{aligned}$$

And the original formula becomes

$$\begin{aligned}
\Gamma_{fi} &= \frac{8\pi^2}{2^4 m_a (2\pi)^5}\int \frac{|\mathcal{M}_{fi}|^2|\mathbf{p}_1|^2|\mathbf{p}_2|^2 \mathrm{d}p_1\mathrm{d}p_2\,\mathrm{d}(\cos\theta_2)}{E_1E_2\sqrt{|\mathbf{p}_1|^2 + |\mathbf{p}_2|^2 + 2|\mathbf{p}_1||\mathbf{p}_2|\cos\theta_2 + m_3^2}}\frac{\delta(\cos\theta_2 - \cos\theta_2')}{\frac{2|\mathbf{p}_1||\mathbf{p}_2|}{2\sqrt{|\mathbf{p}_1|^2 + |\mathbf{p}_2|^2 + 2|\mathbf{p}_1||\mathbf{p}_2|\cos\theta_2' + m_3^2}}} \\
&= \frac{1}{2^3 m_a (2\pi)^3}\int \frac{|\mathcal{M}_{fi}|^2|\mathbf{p}_1|^2|\mathbf{p}_2|^2 \mathrm{d}p_1\mathrm{d}p_2}{E_1E_2\sqrt{|\mathbf{p}_1|^2 + |\mathbf{p}_2|^2 + 2|\mathbf{p}_1||\mathbf{p}_2|\cos\theta_2' + m_3^2}}\frac{2\sqrt{|\mathbf{p}_1|^2 + |\mathbf{p}_2|^2 + 2|\mathbf{p}_1||\mathbf{p}_2|\cos\theta_2' + m_3^2}}{2|\mathbf{p}_1||\mathbf{p}_2|} \\
&= \frac{1}{2^3 m_a (2\pi)^3}\int \frac{|\mathcal{M}_{fi}|^2|\mathbf{p}_1||\mathbf{p}_2|\mathrm{d}p_1\mathrm{d}p_2}{E_1E_2} \\
&= \frac{1}{2^3 m_a (2\pi)^3}\int \frac{|\mathcal{M}_{fi}|^2|\mathbf{p}_1||\mathbf{p}_2|\frac{E_1}{|\mathbf{p}_1|}\mathrm{d}E_1\frac{E_2}{|\mathbf{p}_2|}\mathrm{d}E_2}{E_1E_2} \\
&= \frac{1}{8 m_a (2\pi)^3}\int |\mathcal{M}_{fi}|^2 \mathrm{d}E_1\mathrm{d}E_2
\end{aligned}$$

which is exactly the form of square Dalitz plot. Transform it a little bit and we have the standard form of Dalitz plot (note that $s_2 = (p_2+p_3)^2 = (p_a - p_1)^2 \to \mathrm{d}s_2 = -2m_a\mathrm{d}E_1$ and similar for s_3)

$$\Gamma_{fi} = \frac{1}{32 m_a (2\pi)^3}\int |\mathcal{M}_{fi}|^2 \mathrm{d}s_2\mathrm{d}s_3$$

Now let's review another form of the standard Dalitz form

$$\frac{\mathrm{d}\Gamma_{fi}}{\mathrm{d}s_2\mathrm{d}s_3} = \frac{1}{32 m_a (2\pi)^3}|\mathcal{M}_{fi}|^2$$

and its physical meaning is obvious: the density of data points on a Dalitz plot is proportional to the decay matrix element.

References

[1] 曾谨言. 量子力学 (卷 II), 第 5 版. 北京：科学出版社, 2013.

[2] Martin, Brian R., and Graham Shaw. Particle physics. John Wiley & Sons, 2013.

[3] Thomson, Mark. Modern particle physics. Cambridge University Press, 2013.