

Fourier Analysis

Turja Roy

ID: 2108052

Contents

1	Introduction	2
1.1	Periodic Functions	2
1.2	Piecewise Continuous Functions	2
2	Fourier Expansion	3
2.1	Definition and Derivation	3
2.1.1	Definition	3
2.1.2	Some pre-derivations	3
2.1.3	Derivation of a_0	4
2.1.4	Derivation of a_n	4
2.1.5	Derivation of b_n	5
2.2	Examples	5
3	Fourier Transforms	8
3.1	Fourier Integral	8
3.1.1	From Fourier Series to Fourier Integral	8
3.1.2	Derivation	8
3.1.3	Alternative Derivation	9
3.2	Definition	10

1 Introduction

1.1 Periodic Functions

Definition 1.1.1: Periodic Functions

A function $f(x)$ is said to be have a *period* P or to be *periodic* with period P if for all x , $f(x + P) = f(x)$ where P is a positive constant. The least value of $P > 0$ is called the *least period* or simply the *period* of $f(x)$.

Example 1.1: Some examples of periodic functions

1. $\sin x$ has periods $2\pi, 4\pi, 6\pi, \dots$ and $-\pi, -3\pi, -5\pi, \dots$ and hence the least period is 2π .
2. $\cos x$ has the least period 2π .
3. $\tan x$ has the least period π .

Some other examples:

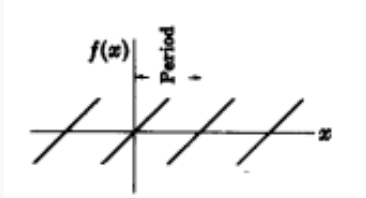


Figure 1.1.1

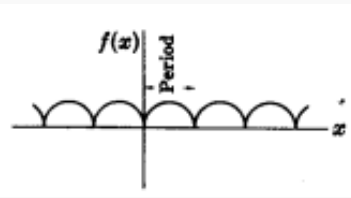


Figure 1.1.2

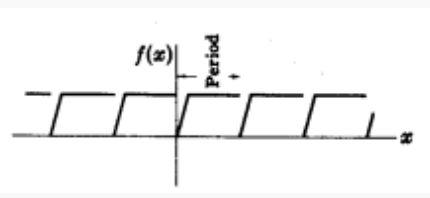


Figure 1.1.3

1.2 Piecewise Continuous Functions

Definition 1.2.1: Piecewise Continuous Functions

A function $f(x)$ is said to be *piecewise continuous* in the interval $[a, b]$ if $f(x)$ is continuous in the interval (a, b) and has a finite number of finite discontinuities in the interval $[a, b]$.

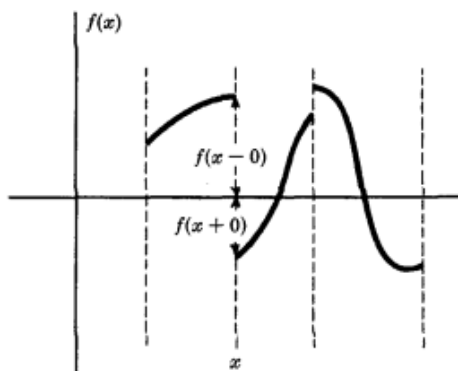


Figure 1.2.1

The right-hand limit of $f(x)$ is often denoted by $\lim_{\epsilon \rightarrow 0} f(x + \epsilon) = f(x + 0)$, where $\epsilon > 0$.

Similarly, the left-hand limit of $f(x)$ is denoted by $\lim_{\epsilon \rightarrow 0} f(x - \epsilon) = f(x - 0)$, where $\epsilon > 0$. The values of $f(x + 0)$ and $f(x - 0)$ at the point x in (1.2.1) are as indicated.

2 Fourier Expansion

2.1 Definition and Derivation

2.1.1 Definition

Definition 2.1.1: Fourier Expansion

Let $f(x)$ be defined in the interval $(-L, L)$ and determined outside of this interval by $f(x+2L) = f(x)$, i.e. assume that $f(x)$ has the period $2L$. The *Fourier series* or *Fourier expansion* corresponding to $f(x)$ is defined to be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \left(\frac{n\pi x}{L} \right) + b_n \sin \left(\frac{n\pi x}{L} \right) \right] \quad (1)$$

where the *Fourier coefficients* a_n and b_n are given by

$$\begin{cases} a_0 = \frac{1}{L} \int_{-L}^L f(x) dx \\ a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \left(\frac{n\pi x}{L} \right) dx \\ b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \left(\frac{n\pi x}{L} \right) dx \end{cases} \quad n = 0, 1, 2, \dots \quad (2)$$

2.1.2 Some pre-derivations

$$\begin{aligned} I &= \int_{-L}^L \sin^2 \frac{n\pi x}{L} dx \\ &= \sin \frac{n\pi x}{L} \cdot \frac{L}{n\pi} (\cos n\pi - \cos n\pi) + \frac{n\pi}{L} \cdot \frac{L}{n\pi} \int_{-L}^L \cos \frac{n\pi x}{L} \cos \frac{n\pi x}{L} dx \\ &= \int_{-L}^L \cos^2 \frac{n\pi x}{L} dx \\ &= \frac{1}{2} \int_{-L}^L \left(\cos \frac{2n\pi x}{L} + 1 \right) dx \\ &= \frac{1}{2} \int_{-L}^L \cos \frac{2n\pi x}{L} dx + \frac{1}{2} \int_{-L}^L dx \\ &= 0 + \frac{1}{2} \cdot 2L \\ &= L \end{aligned}$$

$$\begin{aligned}
I_1 &= \int_{-L}^L \cos \frac{m\pi x}{L} \cos \frac{n\pi x}{L} dx \quad [m \neq 0] \\
&= \cos \frac{m\pi x}{L} \int_{-L}^L \cos \frac{n\pi x}{L} dx + \frac{m}{n} \int_{-L}^L \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L} dx \\
&= \frac{L}{n\pi} \cos \frac{m\pi x}{L} (\sin n\pi + \sin n\pi) + \frac{m}{n} I_2 \\
&= 0 + \frac{m}{n} \left[\int_{-L}^L \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L} dx \right] \\
&= \frac{m}{n} \left[\sin \frac{m\pi x}{L} \int_{-L}^L \sin \frac{n\pi x}{L} dx + \frac{m}{n} \int_{-L}^L \cos \frac{m\pi x}{L} \cos \frac{n\pi x}{L} dx \right] \\
&= \frac{m}{n} \left[\frac{L}{n\pi} \sin \frac{m\pi x}{L} (-\cos n\pi + \cos n\pi) + \frac{m}{n} \right] \\
&= 0 + \frac{m^2}{n^2} I_1 \\
I_1 &= 0 = I_2
\end{aligned}$$

To summarize, we have

$$\int_{-L}^L \sin^2 \frac{n\pi x}{L} dx = \int_{-L}^L \cos^2 \frac{n\pi x}{L} dx = L \quad (3)$$

$$\int_{-L}^L \cos mx dx = \int_{-L}^L \sin mx dx = 0 \quad (4)$$

$$\int_{-L}^L \sin \frac{n\pi x}{L} \cos \frac{n\pi x}{L} dx = 0 \quad (5)$$

$$\int_{-L}^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx = \int_{-L}^L \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} dx = 0 \quad [m \neq n] \quad (6)$$

2.1.3 Derivation of a_0

Taking integral on both sides of (1) from $-L$ to L , we get

$$\begin{aligned}
\int_{-L}^L f(x) dx &= \frac{a_0}{2} \int_{-L}^L dx + \sum_{n=1}^{\infty} \int_{-L}^L \left[a_n \cos \frac{m\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right] dx \\
&= \frac{a_0}{2} \cdot 2L \quad [\text{All the other terms are 0 according to equation (4)}]
\end{aligned}$$

$$a_0 = \frac{1}{L} \int_{-L}^L f(x) dx$$

2.1.4 Derivation of a_n

Multiplying both sides of (1) by $\cos \frac{m\pi x}{L}$ and integrating from $-L$ to L , we get

$$\begin{aligned}
\int_{-L}^L f(x) \cos \frac{m\pi x}{L} dx &= \frac{a_0}{2} \int_{-L}^L \cos \frac{m\pi x}{L} dx \\
&\quad + \sum_{n=1}^{\infty} \int_{-L}^L \left[a_n \cos \frac{m\pi x}{L} \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \cos \frac{m\pi x}{L} \right] dx \\
&= a_n \int_{-L}^L \cos^2 \frac{m\pi x}{L} dx \\
&= a_n \cdot L \quad [\text{All the other terms are 0 according to equation (2)}]
\end{aligned}$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{m\pi x}{L} dx$$

2.1.5 Derivation of b_n

Multiplying both sides of (1) by $\sin \frac{m\pi x}{L}$ and integrating from $-L$ to L , we get

$$\begin{aligned}
\int_{-L}^L f(x) \sin \frac{m\pi x}{L} dx &= \frac{a_0}{2} \int_{-L}^L \sin \frac{m\pi x}{L} dx \\
&\quad + \sum_{n=1}^{\infty} \int_{-L}^L \left[a_n \sin \frac{m\pi x}{L} \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \right] dx \\
&= a_n \int_{-L}^L \sin^2 \frac{m\pi x}{L} dx \\
&= a_n \cdot L \quad [\text{All the other terms are 0 according to equation (2)}]
\end{aligned}$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{m\pi x}{L} dx$$

2.2 Examples

Example 2.1: Obtain the fourier series for $f(x) = x - x^2$ in the interval $(-\pi, \pi)$ and hence evaluate

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (x - x^2) dx = \frac{1}{\pi} \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_{-\pi}^{\pi} = -\frac{2\pi^2}{3}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (x - x^2) \cos nx dx \\
&= \frac{1}{\pi} \left[\int_{-\pi}^{\pi} x \cos nx dx - \int_{-\pi}^{\pi} x^2 \cos nx dx \right] \\
&= -\frac{2}{\pi} \int_0^{\pi} x^2 \cos nx dx \\
&= -\frac{2}{\pi} \left[\frac{x^2}{n} \sin nx - \frac{2}{n} \int x \sin nx dx \right]_0^{\pi} \\
&= \frac{4}{n\pi} \left[-\frac{x}{n} \cos nx + \frac{1}{n^2} \sin nx \right]_0^{\pi} \\
&= -\frac{4}{n^2} (-1)^n
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} (x - x^2) \sin nx dx \\
&= \frac{1}{\pi} \left[\int_{-\pi}^{\pi} x \sin nx dx - \int_{-\pi}^{\pi} x^2 \sin nx dx \right] \\
&= \frac{2}{\pi} \int_0^{\pi} x \sin nx dx \\
&= \frac{2}{\pi} \left[-\frac{x}{n} \cos nx + \frac{1}{n^2} \sin nx \right]_0^{\pi} \\
&= -\frac{2}{n} (-1)^n
\end{aligned}$$

$$\therefore f(x) = -\frac{\pi^2}{3} - \sum_{n=1}^{\infty} \left(\frac{4}{n^2} (-1)^n \cos nx + \frac{2}{n} (-1)^n \sin nx \right)$$

For $x = 0$, we get

$$0 = -\frac{\pi^2}{3} - \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n$$

$$\frac{\pi^2}{12} = -\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$$

$$\boxed{\therefore 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}}$$

Example 2.2: Find a fourier series to represent the function $f(x) = e^x$ for $-\pi < x < \pi$ and hence derive a series for $\frac{\pi}{\sinh \pi}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x dx = \frac{1}{\pi} e^x \Big|_{-\pi}^{\pi} = \frac{e^{\pi} - e^{-\pi}}{\pi} = \frac{2 \sinh x}{\pi}$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \cos nx \, dx \\ &= \frac{1}{\pi} \left[e^x \frac{\sin nx}{n} - \frac{1}{n} \int e^x \sin nx \, dx \right]_{-\pi}^{\pi} \\ &= -\frac{1}{n\pi} \left[-e^x \frac{\cos nx}{n} + \frac{1}{n} \int e^x \cos nx \, dx \right]_{-\pi}^{\pi} \end{aligned}$$

$$\begin{aligned} a_n \left(1 + \frac{1}{n^2} \right) &= \frac{(-1)^n}{n^2 \pi} (e^{\pi} - e^{-\pi}) \\ a_n &= \frac{(-1)^n}{n^2 \pi} (e^{\pi} - e^{-\pi}) \left(1 + \frac{1}{n^2} \right)^{-1} \\ a_n &= 2 \frac{(-1)^n}{n^2 \pi} \sinh x \left(1 + \frac{1}{n^2} \right)^{-1} \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \sin nx \, dx \\ &= \frac{1}{\pi} \left[-e^x \frac{\cos nx}{n} + \frac{1}{n} \int e^x \cos nx \, dx \right]_{-\pi}^{\pi} \\ &= \frac{1}{\pi} \left[-e^x \frac{\cos nx}{n} + \frac{1}{n} \left\{ e^x \frac{\sin nx}{n} - \frac{1}{n} \int e^x \sin nx \, dx \right\} \right]_{-\pi}^{\pi} \\ &= \frac{1}{\pi} \left[-e^x \frac{\cos nx}{n} - \frac{1}{n^2} \int e^x \sin nx \, dx \right] \\ b_n &= -\frac{(-1)^n}{n\pi} (e^{\pi} - e^{-\pi}) \left(1 + \frac{1}{n^2} \right)^{-1} \\ b_n &= -2 \frac{(-1)^n}{n\pi} \sinh x \left(1 + \frac{1}{n^2} \right)^{-1} \end{aligned}$$

$$\boxed{f(x) = e^x = 2 \frac{\sinh x}{\pi} \left[\frac{1}{2} + \sum_{n=1}^{\infty} \left(1 + \frac{1}{n^2} \right)^{-1} \frac{(-1)^n}{n} \left(\frac{1}{n} \cos nx - \sin nx \right) \right]}$$

For $x = 0$, we get

$$\begin{aligned} \frac{\pi}{\sinh x} &= 1 + 2 \sum_{n=1}^{\infty} \left(1 + \frac{1}{n^2} \right)^{-1} \frac{(-1)^n}{n^2} \\ &= 1 + 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + 1} \end{aligned}$$

$$\boxed{\frac{\pi}{\sinh x} = 1 + 2 \left(-\frac{1}{2} + \frac{1}{5} - \frac{1}{10} + \cdots \right)}$$

3 Fourier Transforms

3.1 Fourier Integral

3.1.1 From Fourier Series to Fourier Integral

Fourier series were used to represent a function f defined on the finite interval $(-L, L)$ or $(0, L)$. It converged to f and to its periodic extension. In this sense, Fourier series is associated with periodic functions.

Fourier integral represents a certain type of non-periodic functions that are defined on $(-\infty, \infty)$ or $(0, \infty)$.

3.1.2 Derivation

Let a function f be defined on $(-L, L)$. The Fourier series of the function is then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \quad (7)$$

where the coefficients are given by

$$\begin{aligned} a_0 &= \frac{1}{L} \int_{-L}^L f(t) dt \\ a_n &= \frac{1}{L} \int_{-L}^L f(t) \cos \frac{n\pi t}{L} dt \\ b_n &= \frac{1}{L} \int_{-L}^L f(t) \sin \frac{n\pi t}{L} dt \end{aligned}$$

Now, let $a_n = \frac{n\pi}{L}$,

then $\Delta\alpha = \alpha_{n+1} - \alpha_n = \frac{\pi}{L}$

So, we get

$$\Delta f(x) = \frac{1}{2\pi} \left(\int_{-L}^L f(t) dt \right) \Delta\alpha + \frac{1}{\pi} \sum_{n=1}^{\infty} \left[\left(\int_{-L}^L f(t) \cos \alpha_n t dt \right) \cos \alpha_n x + \left(\int_{-L}^L f(t) \sin \alpha_n t dt \right) \sin \alpha_n x \right] \Delta\alpha \quad (8)$$

We now expand the interval $(-L, L)$ by taking $L \rightarrow \infty$, which implies that $\Delta\alpha \rightarrow 0$. Consequently, we get

$$\lim_{\Delta\alpha \rightarrow 0} \sum_{n=1}^{\infty} F(\alpha_n) \Delta\alpha \rightarrow \int_0^{\infty} F(\alpha) d\alpha \quad (9)$$

Thus, the limit of the first term in the Fourier series $\int_{-L}^L f(t) dt$ vanishes, and the limit of the sum becomes

$$\boxed{f(x) = \frac{1}{\pi} \int_0^{\infty} \left[\left(\int_{-\infty}^{\infty} f(t) \cos \alpha t dt \right) \cos \alpha x + \left(\int_{-\infty}^{\infty} f(t) \sin \alpha t dt \right) \sin \alpha x \right] d\alpha} \quad (10)$$

This is the Fourier integral of f on the interval $(-\infty, \infty)$.

3.1.3 Alternative Derivation

Substituting the values of a_0 , a_n and b_n in (7), we get

$$\begin{aligned}
 f(x) &= \frac{1}{2L} \int_{-L}^L f(t) dt + \frac{1}{L} \sum_{n=1}^{\infty} \left[\int_{-L}^L f(t) \cos \frac{n\pi t}{L} \cos \frac{n\pi x}{L} dt + \int_{-L}^L f(t) \sin \frac{n\pi t}{L} \sin \frac{n\pi x}{L} dt \right] \\
 &= \frac{1}{2L} \int_{-L}^L f(t) dt + \frac{1}{L} \sum_{n=1}^{\infty} \left[\int_{-L}^L f(t) \left\{ \cos \frac{n\pi t}{L} \cos \frac{n\pi x}{L} + \sin \frac{n\pi t}{L} \sin \frac{n\pi x}{L} \right\} dt \right] \\
 f(x) &= \frac{1}{2L} \int_{-L}^L f(t) dt + \frac{1}{L} \sum_{n=1}^{\infty} \int_{-L}^L f(t) \cos \frac{n\pi}{L}(t-x) dt
 \end{aligned} \tag{11}$$

Now, if we assume that $\int_{-\infty}^{\infty} |f(x)| dx$ converges, the first term on the right side of (11) approaches 0 as $L \rightarrow \infty$, since

$$\left| \frac{1}{2L} \int_{-L}^L f(t) dt \right| \leq \frac{1}{2L} \int_{-\infty}^{\infty} |f(t)| dt$$

The second term on the right side of (11) approaches

$$\begin{aligned}
 &\lim_{L \rightarrow \infty} \frac{1}{L} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} f(t) \cos \frac{n\pi}{L}(t-x) dt \\
 &= \lim_{\delta\lambda \rightarrow 0} \frac{1}{\pi} \sum_{n=1}^{\infty} \delta\lambda \int_{-\infty}^{\infty} f(t) \cos n\delta\lambda(t-x) dt
 \end{aligned}$$

where $\lambda = \frac{n\pi}{L}$ which implies that $\delta\lambda = \frac{\pi}{L}$.

We know,

$$\lim_{\delta\lambda \rightarrow 0} \sum_{n=1}^{\infty} \delta\lambda F(\lambda_n) = \int_0^{\infty} F(\lambda) d\lambda$$

Thus we get

$$\boxed{f(x) = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(t) \cos \lambda(t-x) dt d\lambda} \tag{12}$$

This is another form of the Fourier integral.

3.2 Definition

Definition 3.2.1: Fourier Integral

The Fourier integral of a function f defined on the interval $(-\infty, \infty)$ is given by

$$f(x) = \frac{1}{\pi} \int_0^{\infty} [A(\alpha) \cos \alpha x + B(\alpha) \sin \alpha x] d\alpha \quad (13)$$

where the coefficients $A(\alpha)$ and $B(\alpha)$ are given by

$$\begin{cases} A(\alpha) = \int_{-\infty}^{\infty} f(x) \cos \alpha x dx \\ B(\alpha) = \int_{-\infty}^{\infty} f(x) \sin \alpha x dx \end{cases} \quad (14)$$

The Fourier integral can also be written in the form

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(t) \cos \lambda(t - x) dt d\lambda \quad (15)$$

where $\lambda = \frac{n\pi}{L}$