Fourier Analysis

Turja Roy

ID: 2108052

Contents

1 Introduction

1.1 Periodic Functions

Definition 1.1.1: Periodic Functions

A function $f(x)$ is said to be have a *period* P or to be *periodic* with period P if for all x, $f(x+P) = f(x)$ where P is a positive constant. The least value of $P > 0$ is called the *least period* or simply the *period* of $f(x)$.

Example 1.1: Some examples of periodic functions

1. $\sin x$ has periods $2\pi, 4\pi, 6\pi, \cdots$ and $-\pi, -3\pi, -5\pi, \cdots$ and hence the least period is 2π.
2. $\cos x$ has the least period 2π.
3. $\tan x$ has the least period π.

Some other examples:

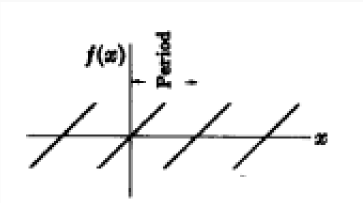

Figure 1.1.1

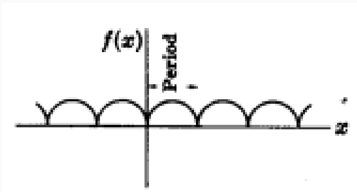

Figure 1.1.2

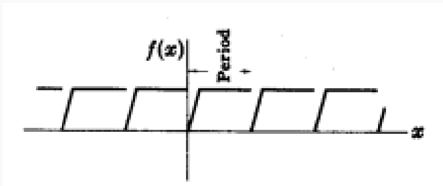

Figure 1.1.3

1.2 Piecewise Continuous Functions

Definition 1.2.1: Piecewise Continuous Functions

A function $f(x)$ is said to be *piecewise continuous* in the interval $[a, b]$ if $f(x)$ is continuous in the interval (a, b) and has a finite number of finite discontinuities in the interval $[a, b]$.

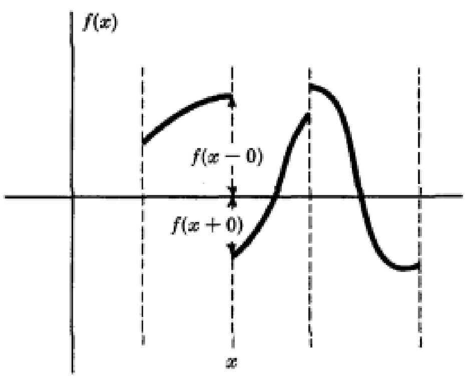

Figure 1.2.1

The right-hand limit of $f(x)$ is often denoted by $\lim_{\epsilon\to 0} f(x+\epsilon) = f(x+0)$, where $\epsilon > 0$.

Similarly, the left-hand limit of $f(x)$ is denoted by $\lim_{\epsilon\to 0} f(x-\epsilon) = f(x-0)$, where $\epsilon > 0$. The values of $f(x+0)$ and $f(x-0)$ at the point x in (1.2.1) are as indicated.

2 Fourier Expansion

2.1 Definition

Definition 2.1.1: Fourier Expansion

Let $f(x)$ be defined in the interval $(-L, L)$ and determined outside of this interval by $f(x+2L) = f(x)$, i.e. assume that $f(x)$ has the period $2L$. The *Fourier series* or *Fourier expansion* corresponding to $f(x)$ is defined to be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right] \tag{2.1.1}$$

where the *Fourier coefficients* a_n and b_n are given by

$$\begin{cases} a_0 = \dfrac{1}{L} \displaystyle\int_{-L}^{L} f(x) dx \\ a_n = \dfrac{1}{L} \displaystyle\int_{-L}^{L} f(x) \cos\left(\dfrac{n\pi x}{L}\right) dx \qquad n = 0, 1, 2, \cdots \\ b_n = \dfrac{1}{L} \displaystyle\int_{-L}^{L} f(x) \sin\left(\dfrac{n\pi x}{L}\right) dx \end{cases} \tag{2.1.2}$$

2.2 Some pre-derivations

$$\begin{aligned} I &= \int_{-L}^{L} \sin^2 \frac{n\pi x}{L}\, dx \\ &= \sin\frac{n\pi x}{L} \cdot \frac{L}{n\pi} (\cos n\pi - \cos n\pi) + \frac{n\pi}{L} \cdot \frac{L}{n\pi} \int_{-L}^{L} \cos\frac{n\pi x}{L} \cos\frac{n\pi x}{L}\, dx \\ &= \int_{-L}^{L} \cos^2 \frac{n\pi x}{L}\, dx \\ &= \frac{1}{2} \int_{-L}^{L} \left(\cos\frac{2n\pi x}{L} + 1 \right) dx \\ &= \frac{1}{2} \int_{-L}^{L} \cos\frac{2n\pi x}{L}\, dx + \frac{1}{2} \int_{-L}^{L} dx \\ &= 0 + \frac{1}{2} \cdot 2L \\ &= L \end{aligned}$$

$$
\begin{aligned}
I_1 &= \int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx \qquad [m \neq 0] \\
&= \cos\frac{m\pi x}{L}\int_{-L}^{L}\cos\frac{n\pi x}{L}\,dx + \frac{m}{n}\int_{-L}^{L}\sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx \\
&= \frac{L}{n\pi}\cos\frac{m\pi x}{L}(\sin n\pi + \sin n\pi) + \frac{m}{n}I_2 \\
&= 0 + \frac{m}{n}\left[\int_{-L}^{L}\sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx\right] \\
&= \frac{m}{n}\left[\sin\frac{m\pi x}{L}\int_{-L}^{L}\sin\frac{n\pi x}{L}\,dx + \frac{m}{n}\int_{-L}^{L}\cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx\right] \\
&= \frac{m}{n}\left[\frac{L}{n\pi}\sin\frac{m\pi x}{L}(-\cos n\pi + \cos n\pi) + \frac{m}{n}\right] \\
&= 0 + \frac{m^2}{n^2}I_1 \\
I_1 &= 0 = I_2
\end{aligned}
$$

To summarize, we have

$$\int_{-L}^{L}\sin^2\frac{n\pi x}{L}\,dx = \int_{-L}^{L}\cos^2\frac{n\pi x}{L}\,dx = L \tag{2.2.1}$$

$$\int_{-L}^{L}\cos mx\,dx = \int_{-L}^{L}\sin mx\,dx = 0 \tag{2.2.2}$$

$$\int_{-L}^{L}\sin\frac{n\pi x}{L}\cos\frac{n\pi x}{L}\,dx = 0 \tag{2.2.3}$$

$$\int_{-L}^{L}\sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \int_{-L}^{L}\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = 0 \qquad [m \neq n] \tag{2.2.4}$$

2.3 Derivation of a_0

Taking integral on both sides of (2.1.1) from $-L$ to L, we get

$$
\begin{aligned}
\int_{-L}^{L} f(x)\,dx &= \frac{a_0}{2}\int_{-L}^{L}dx + \sum_{n=1}^{\infty}\int_{-L}^{L}\left[a_n\cos\frac{m\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right]\,dx \\
&= \frac{a_0}{2}\cdot 2L \qquad \text{[All the other terms are 0 according to equation (2.2.2)]}
\end{aligned}
$$

$$\boxed{a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx}$$

2.4 Derivation of a_n

Multiplying both sides of (2.1.1) by $\cos\dfrac{m\pi x}{L}$ and integrating from $-L$ to L, we get

$$
\begin{aligned}
\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx &= \frac{a_0}{2}\int_{-L}^{L}\cos\frac{m\pi x}{L}\,dx \\
&\quad + \sum_{n=1}^{\infty}\int_{-L}^{L}\left[a_n\cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\right]dx \\
&= a_n\int_{-L}^{L}\cos^2\frac{m\pi x}{L}\,dx \\
&= a_n\cdot L \qquad \text{[All the other terms are 0 according to equation (2.1.2)]}
\end{aligned}
$$

$$
\boxed{a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx}
$$

2.5 Derivation of b_n

Multiplying both sides of (1) by $\sin\dfrac{m\pi x}{L}$ and integrating from $-L$ to L, we get

$$
\begin{aligned}
\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\,dx &= \frac{a_0}{2}\int_{-L}^{L}\sin\frac{m\pi x}{L}\,dx \\
&\quad + \sum_{n=1}^{\infty}\int_{-L}^{L}\left[a_n\sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\right]dx \\
&= a_n\int_{-L}^{L}\sin^2\frac{m\pi x}{L}\,dx \\
&= a_n\cdot L \qquad \text{[All the other terms are 0 according to equation (2.1.2)]}
\end{aligned}
$$

$$
\boxed{b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\,dx}
$$

2.6 Examples

Example 2.1: Obtain the fourier series for $f(x) = x - x^2$ in the interval $(-\pi, \pi)$ and hence evaluate

$$
\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots
$$

$$
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)
$$

where

$$
a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)dx = \frac{1}{\pi}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-\pi}^{\pi} = -\frac{2\pi^2}{3}
$$

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\cos nx dx \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\cos nx\, dx - \int_{-\pi}^{\pi} x^2\cos nx\, dx\right] \\
&= -\frac{2}{\pi}\int_{0}^{\pi} x^2\cos nx\, dx \\
&= -\frac{2}{\pi}\left[\frac{x^2}{n}\sin nx - \frac{2}{n}\int x\sin nx\, dx\right]_0^{\pi} \\
&= \frac{4}{n\pi}\left[-\frac{x}{n}\cos nx + \frac{1}{n^2}\sin nx\right]_0^{\pi} \\
&= -\frac{4}{n^2}(-1)^n
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\cos nx\, dx \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\sin nx\, dx - \int_{-\pi}^{\pi} x^2\sin nx\, dx\right] \\
&= \frac{2}{\pi}\int_{0}^{\pi} x\sin nx\, dx \\
&= \frac{2}{\pi}\left[-\frac{x}{n}\cos nx + \frac{1}{n^2}\sin nx\right]_0^{\pi} \\
&= -\frac{2}{n}(-1)^n
\end{aligned}
$$

$$
\therefore f(x) = -\frac{\pi^2}{3} - \sum_{n=1}^{\infty}\left(\frac{4}{n^2}(-1)^n\cos nx + \frac{2}{n}(-1)^n\sin nx\right)
$$

For $x = 0$, we get

$$
\begin{aligned}
0 &= -\frac{\pi^2}{3} - \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n \\
\frac{\pi^2}{12} &= -\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^2}
\end{aligned}
$$

$$
\boxed{\therefore 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}}
$$

Example 2.2: Find a fourier series to represent the function $f(x) = e^x$ for $-\pi < x < \pi$ and hence derive a series for $\dfrac{\pi}{\sinh\pi}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x dx = \frac{1}{\pi} e^x \Big|_{-\pi}^{\pi} = \frac{e^{\pi} - e^{-\pi}}{\pi} = \frac{2\sinh x}{\pi}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \cos nx \, dx \\
&= \frac{1}{\pi} \left[e^x \frac{\sin nx}{n} - \frac{1}{n} \int e^x \sin nx \, dx \right]_{-\pi}^{\pi} \\
&= -\frac{1}{n\pi} \left[-e^x \frac{\cos nx}{n} + \frac{1}{n} \int e^x \cos nx \, dx \right]_{-\pi}^{\pi} \\
a_n \left(1 + \frac{1}{n^2} \right) &= \frac{(-1)^n}{n^2 \pi} \left(e^{\pi} - e^{-\pi} \right) \\
a_n &= \frac{(-1)^n}{n^2 \pi} \left(e^{\pi} - e^{-\pi} \right) \left(1 + \frac{1}{n^2} \right)^{-1} \\
a_n &= 2 \frac{(-1)^n}{n^2 \pi} \sinh x \left(1 + \frac{1}{n^2} \right)^{-1}
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \sin nx \, dx \\
&= \frac{1}{\pi} \left[-e^x \frac{\cos nx}{n} + \frac{1}{n} \int e^x \cos nx \, dx \right]_{-\pi}^{\pi} \\
&= \frac{1}{\pi} \left[-e^x \frac{\cos nx}{n} + \frac{1}{n} \left\{ e^x \frac{\sin nx}{n} - \frac{1}{n} \int e^x \sin nx \, dx \right\} \right]_{-\pi}^{\pi} \\
&= \frac{1}{\pi} \left[-e^x \frac{\cos nx}{n} - \frac{1}{n^2} \int e^x \sin nx \, dx \right] \\
b_n &= -\frac{(-1)^n}{n\pi} \left(e^{\pi} - e^{-\pi} \right) \left(1 + \frac{1}{n^2} \right)^{-1} \\
b_n &= -2 \frac{(-1)^n}{n\pi} \sinh x \left(1 + \frac{1}{n^2} \right)^{-1}
\end{aligned}$$

$$\boxed{f(x) = e^x = 2 \frac{\sinh x}{\pi} \left[\frac{1}{2} + \sum_{n=1}^{\infty} \left(1 + \frac{1}{n^2} \right)^{-1} \frac{(-1)^n}{n} \left(\frac{1}{n} \cos nx - \sin nx \right) \right]}$$

For $x = 0$, we get

$$\begin{aligned}
\frac{\pi}{\sinh x} &= 1 + 2 \sum_{n=1}^{\infty} \left(1 + \frac{1}{n^2} \right)^{-1} \frac{(-1)^n}{n^2} \\
&= 1 + 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + 1}
\end{aligned}$$

$$\boxed{\frac{\pi}{\sinh x} = 1 + 2 \left(-\frac{1}{2} + \frac{1}{5} - \frac{1}{10} + \cdots \right)}$$

3 Fourier Integral

3.1 Definition

Definition 3.1.1: Fourier Integral

The Fourier integral of a function f defined on the interval $(-\infty, \infty)$ is given by

$$f(x) = \frac{1}{\pi} \int_0^\infty \left[A(\alpha)\cos\alpha x + B(\alpha)\sin\alpha x\right] d\alpha \tag{3.1.1}$$

where the coefficients $A(\alpha)$ and $B(\alpha)$ are given by

$$\begin{cases} A(\alpha) = \displaystyle\int_{-\infty}^{\infty} f(x)\cos\alpha x\, dx \\ \\ B(\alpha) = \displaystyle\int_{-\infty}^{\infty} f(x)\sin\alpha x\, dx \end{cases} \tag{3.1.2}$$

The Fourier integral can also be written in the form

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)\, dt\, d\lambda \tag{3.1.3}$$

where $\lambda = \dfrac{n\pi}{L}$

Fourier series were used to represent a function f defined on the finite interval $(-L, L)$ or $(0, L)$. It converged to f and to its periodic extension. In this sense, Fourier series is assosiated with periodic functions.

Fourier integral represents a certain type of non-periodic functions that are defined on $(-\infty, \infty)$ or $(0, \infty)$.

3.2 Derivation

Let a function f be defined on $(-L, L)$. The fourier series of the function is then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \tag{3.2.1}$$

where the coefficients are given by

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} f(t)\, dt \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}\, dt \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}\, dt \end{aligned}$$

Now, let $a_n = \dfrac{n\pi}{L}$,
then $\Delta\alpha = \alpha_{n+1} - \alpha_n = \dfrac{\pi}{L}$

So, we get

$$\Delta f(x) = \frac{1}{2\pi}\left(\int_{-L}^{L} f(t)\,dt\right)\Delta\alpha + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\left(\int_{-L}^{L} f(t)\cos\alpha_n t\,dt\right)\cos\alpha_n x + \left(\int_{-L}^{L} f(t)\sin\alpha_n t\,dt\right)\sin\alpha_n x\right]\Delta\alpha \tag{3.2.2}$$

We now expand the interval $(-L, L)$ by taking $L \to \infty$, which implies that $\Delta\alpha \to 0$. Consequently, we get

$$\lim_{\Delta\alpha\to 0}\sum_{n=1}^{\infty} F(\alpha_n)\Delta\alpha \to \int_0^{\infty} F(\alpha)\,d\alpha \tag{3.2.3}$$

Thus, the limit of the first term in the Fourier series $\int_{-L}^{L} f(t)\,dt$ vanishes, and the limit of the sum becomes

$$\boxed{f(x) = \frac{1}{\pi}\int_0^{\infty}\left[\left(\int_{-\infty}^{\infty} f(t)\cos\alpha t\,dt\right)\cos\alpha x + \left(\int_{-\infty}^{\infty} f(t)\sin\alpha t\,dt\right)\sin\alpha x\right]d\alpha} \tag{3.2.4}$$

This is the Fourier integral of f on the interval $(-\infty, \infty)$.

3.3 Alternative Derivation

Substituting the values of a_0, a_n and b_n in (3.1.1), we get

$$\begin{aligned} f(x) &= \frac{1}{2L}\int_{-L}^{L} f(t)\,dt + \frac{1}{L}\sum_{n=1}^{\infty}\left[\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}\cos\frac{n\pi x}{L}\,dt + \int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}\sin\frac{n\pi x}{L}\,dt\right] \\ &= \frac{1}{2L}\int_{-L}^{L} f(t)\,dt + \frac{1}{L}\sum_{n=1}^{\infty}\left[\int_{-L}^{L} f(t)\left\{\cos\frac{n\pi t}{L}\cos\frac{n\pi x}{L} + \sin\frac{n\pi t}{L}\sin\frac{n\pi x}{L}\right\}dt\right] \end{aligned}$$

$$f(x) = \frac{1}{2L}\int_{-L}^{L} f(t)\,dt + \frac{1}{L}\sum_{n=1}^{\infty}\int_{-L}^{L} f(t)\cos\frac{n\pi}{L}(t-x)\,dt \tag{3.3.1}$$

Now, if we assume that $\int_{-\infty}^{\infty} |f(x)|\,dx$ converges, the first term on the right side of (3.2.2) approaches 0 as $L \to \infty$, since

$$\left|\frac{1}{2L}\int_{-L}^{L} f(t)\,dt\right| \le \frac{1}{2L}\int_{-\infty}^{\infty} |f(t)|\,dt$$

The second term on the right side of (3.2.2) approaches

$$\begin{aligned} &\lim_{L\to\infty}\frac{1}{L}\sum_{n=1}^{\infty}\int_{-\infty}^{\infty} f(t)\cos\frac{n\pi}{L}(t-x)\,dt \\ &= \lim_{\delta\lambda\to 0}\frac{1}{\pi}\sum_{n=1}^{\infty}\delta\lambda\int_{-\infty}^{\infty} f(t)\cos n\delta\lambda(t-x)\,dt \end{aligned}$$

where $\lambda = \dfrac{n\pi}{L}$ which implies that $\delta\lambda = \dfrac{\pi}{L}$.

We know,

$$\lim_{\delta\lambda \to 0} \sum_{n=1}^{\infty} \delta\lambda F(\lambda_n) = \int_0^{\infty} F(\lambda)\, d\lambda$$

Thus we get

$$\boxed{f(x) = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(t) \cos \lambda(t - x)\, dt\, d\lambda} \tag{3.3.2}$$

This is another form of the Fourier integral.

3.4 Fourier Sine and Cosine Integrals

We can rewrite (3.1.1) as

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \cos \lambda x \int_{-\infty}^{\infty} f(t) \cos \lambda t\, dt\, d\lambda + \frac{1}{\pi} \int_0^{\infty} \sin \lambda x \int_{-\infty}^{\infty} f(t) \sin \lambda t\, dt\, d\lambda \tag{3.4.1}$$

If $f(x)$ is an odd function, $f(t)\cos\lambda t$ is also an odd function while $f(t)\sin\lambda t$ is even. Then the first term on the right side of (3.4.1) vanishes and we get

$$\boxed{f(x) = \frac{2}{\pi} \int_0^{\infty} \sin \lambda x \int_0^{\infty} f(t) \sin \lambda t\, dt\, d\lambda} \tag{3.4.2}$$

This is known as the Fourier sine integral.

Similarly, if $f(x)$ is an even function, (3.4.1) becomes

$$\boxed{f(x) = \frac{2}{\pi} \int_0^{\infty} \cos \lambda x \int_0^{\infty} f(t) \cos \lambda t\, dt\, d\lambda} \tag{3.4.3}$$

This is known as the Fourier cosine integral.

3.5 Complex Form of Fourier Integral

Equation (3.1.3) can be written as

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \cos \lambda(t - x)\, dt\, d\lambda \tag{3.5.1}$$

because $\cos\lambda(t-x)$ is an even function f λ. Also, since $\sin\lambda(t-x)$ is an odd function of λ, we have

$$0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \sin \lambda(t - x)\, dt\, d\lambda \tag{3.5.2}$$

Now, multiplying (3.5.2) by i and adding it to (3.5.1), we get

$$\boxed{f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) e^{i\lambda(t-x)}\, dt\, d\lambda} \tag{3.5.3}$$

This is the complex form of the Fourier integral.

4 Fourier Transforms

4.1 Definition

Definition 4.1.1: Fourier Transform

The Fourier transform of a function f defined on the interval $(-\infty, \infty)$ is given by

$$F(\lambda) = \int_{-\infty}^{\infty} f(x) e^{-i\lambda x}\, dx \tag{4.1.1}$$

The inverse Fourier transform is given by

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\lambda) e^{-i\lambda x}\, d\lambda \tag{4.1.2}$$

4.2 Derivation

We know that

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) e^{i\lambda(t-x)}\, dt\, d\lambda \tag{4.2.1}$$

We can rewrite (4.2.1) as follows

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda x}\, d\lambda \int_{-\infty}^{\infty} f(t) e^{i\lambda t}\, dt$$

It follows that if

$$\boxed{F(\lambda) = \int_{-\infty}^{\infty} f(t) e^{i\lambda t}\, dt} \tag{4.2.2}$$

then

$$\boxed{f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\lambda) e^{-i\lambda x}\, d\lambda} \tag{4.2.3}$$

Here, $F(\lambda)$ is called the **Fourier transform** of $f(x)$ and $f(x)$ is called the **inverse Fourier transform** of $F(\lambda)$.

4.3 Fourier Sine and Cosine Transforms

We know

$$f(x) = \frac{2}{\pi} \int_{0}^{\infty} \sin \lambda x \int_{0}^{\infty} f(t) \sin \lambda t\, dt\, d\lambda \tag{4.3.1}$$

It follows that if

$$F_s(\lambda) = \int_{0}^{\infty} f(x) \sin \lambda x\, dx \tag{4.3.2}$$

then

$$f(x) = \frac{2}{\pi} \int_{0}^{\infty} F_s(\lambda) \sin \lambda x\, d\lambda \tag{4.3.3}$$

Here, $F_s(\lambda)$ is called the **Fourier sine transform** of $f(x)$ in $0 < x < \infty$. Also the function $f(x)$ is known as the **inverse Fourier sine transform** of $F_s(\lambda)$.

Again, we know

$$f(x) = \frac{2}{\pi} \int_{0}^{\infty} \cos \lambda x \int_{0}^{\infty} f(t) \cos \lambda t\, dt\, d\lambda \tag{4.3.4}$$

Similarly, if

$$F_c(\lambda) = \int_0^\infty f(x) \cos \lambda x \, dx \tag{4.3.5}$$

then

$$f(x) = \frac{2}{\pi} \int_0^\infty F_c(\lambda) \cos \lambda x \, d\lambda \tag{4.3.6}$$

Here, $F_c(\lambda)$ is called the **Fourier cosine transform** of $f(x)$ in $0 < x < \infty$. Also the function $f(x)$ is known as the **inverse Fourier cosine transform** of $F_c(\lambda)$.

4.4 Finite Fourier Sine and Cosine Transforms

These transforms are usefor for BVPs where at least two of the boundaries are parallel and separated by a finite distance.
The **finite Fourier sine transform** of $f(x)$ in $0 < x < L$ is given by

$$\boxed{F_s(\lambda) = \int_0^L f(x) \sin \lambda x \, dx} \tag{4.4.1}$$

and the **inverse finite Fourier sine transform** of $F_s(\lambda)$ is given by

$$\boxed{f(x) = \frac{2}{L} \sum_{n=1}^{\infty} F_s(\lambda_n) \sin \lambda_n x} \tag{4.4.2}$$

The **finite Fourier cosine transform** of $f(x)$ in $0 < x < L$ is given by

$$\boxed{F_c(\lambda) = \int_0^L f(x) \cos \lambda x \, dx} \tag{4.4.3}$$

and the **inverse finite Fourier cosine transform** of $F_c(\lambda)$ is given by

$$\boxed{f(x) = \frac{2}{L} \sum_{n=1}^{\infty} F_c(\lambda_n) \cos \lambda_n x} \tag{4.4.4}$$