Laplace Transform

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Contents

1 Definition, Existence, and Basic Properties of the Laplace Transform

1.1 Definition and Existence

Definition 1.1.1: Laplace Transform

Let F be a real-valued function of the real variable t, defined for $t > 0$. Let s be a variable that we shall assume to be real, and consider the function f defined by

$$f(s) = \int_0^\infty e^{-st} F(t)\, dt \tag{1}$$

for all values of s for which this integral exists. The function f defined by the integral (1) is called the Laplace Transform of the function F. We shall denote the Laplace transform of F by $\mathscr{L}\{F(t)\}$.
Thus the Laplace transform of a function f is given by

$$\mathscr{L}\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\, dt = \lim_{R\to\infty} \int_0^R e^{-st} F(t)\, dt \tag{2}$$

Some ways to write Laplace transforms:

$$\mathscr{L}F(t) = f(s) = \int_0^\infty e^{-st} F(t)\, dt$$
$$\mathscr{L}G(t) = g(s)$$
$$\mathscr{L}u(t) = \tilde{u}(s)$$

$F(t)$	$\mathscr{L}\{F(t)\} = f(s)$	$F(t)$	$\mathscr{L}\{F(t)\} = f(s)$
1	$\frac{1}{s}$	n	$\frac{n}{s}$
t	$\frac{1}{s^2}$	t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$	e^{-at}	$\frac{1}{s+a}$
$\sin at$	$\frac{a}{s^2+a^2}$	$\sinh at$	$\frac{a}{s^2-a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$	$\cosh at$	$\frac{s}{s^2-a^2}$

Table 1: Functions and their Laplace Transform

Proofs :

Let $F(t) = n$, for $t > 0$
Then

$$\begin{aligned}
\mathscr{L}\{n\} &= \int_0^\infty e^{-st} \cdot n \, dt \\
&= n \frac{-e^{st}}{s} \bigg|_0^\infty \\
&= \frac{n}{s} \quad \square
\end{aligned}$$

Let $F(t) = t$, for $t > 0$
Then

$$\begin{aligned}
\mathscr{L}\{t\} &= \int_0^\infty e^{-st} \cdot t \, dt \\
&= -t \frac{e^{-st}}{s} + \int_0^\infty \frac{e^{-st}}{s} \, dt \\
&= -e^{-st} \frac{t}{s} \bigg|_0^\infty - e^{-st} \frac{1}{s^2} \bigg|_0^\infty \\
&= \frac{1}{s^2} \quad \square
\end{aligned}$$

Let $F(t) = t^n$, for $t > 0$
Then

$$\begin{aligned}
\mathscr{L}\{t^n\} &= \int_0^\infty e^{-st} t^n \, dt \\
&= -t^n \frac{e^{st}}{s} + \int_0^\infty n t^{n-1} \frac{e^{-st}}{s} \, dt \\
&= -n t^{n-1} \frac{e^{-st}}{s^2} \bigg|_0^\infty + \int_0^\infty n(n-1) t^{n-2} \frac{e^{-st}}{s^2} \, dt \\
&= -n(n-1) t^{n-2} \left(\frac{e^{-st}}{s^3} \right) \bigg|_0^\infty + \int_0^\infty n(n-1)(n-2) t^{n-3} \frac{e^{-st}}{s^3} \, dt \\
&= \cdots \\
&= n! t^{n-n} \frac{e^{-st}}{s^{n+1}} \bigg|_0^\infty + \int_0^\infty n(n-1) \cdots (n-n) \frac{e^{-st}}{s^{n+1}} \, dt \\
&= \frac{n!}{s^{n+1}} \quad \square
\end{aligned}$$

Let $F(t) = e^{at}$, for $t > 0$
Then

$$\begin{aligned}
\mathscr{L}\{e^{at}\} &= \int_0^\infty e^{-st} e^{at} \, dt \\
&= \int_0^\infty e^{(a-s)t} \, dt \\
&= \frac{e^{(a-s)t}}{a-s} \bigg|_0^\infty \\
&= \frac{1}{s-a} \quad \square
\end{aligned}$$

Let $F(t) = e^{-at}$, for $t > 0$
Then

$$\begin{aligned}
\mathscr{L}\{e^{-at}\} &= \int_0^\infty e^{-st} e^{-at} \, dt \\
&= \int_0^\infty e^{-(a+s)t} \, dx \\
&= \frac{e^{-(a+s)t}}{s+a} \bigg|_0^\infty \\
&= \frac{1}{s+a} \quad \square
\end{aligned}$$

Let $F(t) = \sin at$, for $t > 0$
Then

$$
\begin{aligned}
\mathscr{L}\{\sin at\} &= \int_0^\infty e^{-st \sin at}\, dt \\
&= -\frac{e^{-st}}{s^2+a^2}\left(s \sin at + a \cos at\right)\Big|_0^\infty \\
&= \frac{a}{s^2+a^2} \quad \square
\end{aligned}
$$

Let $F(t) = \cos at$, for $t > 0$
Then

$$
\begin{aligned}
\mathscr{L}\{\cos at\} &= \int_0^\infty e^{-st} \cos at\, dt \\
&= \frac{e^{-st}}{s^2+a^2}\left(-s \cos at + a \sin at\right)\Big|_0^\infty \\
&= \frac{s}{s^2+a^2} \quad \square
\end{aligned}
$$

Theorem 1.1.2: Hypothesis: *Let F be a real function that has the following properties:*

1. *F is a piecewise continuous in every finite closed interval $0 \le t \le a$ $(b > 0)$.*
2. *F is of exponential order, i.e, there exists α, $M > 0$, and $t_0 > 0$ such that*

$$e^{-\alpha t}|F(t)| < M \text{ for } t > t_0$$

Conclusion: *The Laplace transform of F exists for $s > \alpha$.*

$$\mathscr{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt$$

1.2 Basic Properties of the Laplace Transform

Theorem 1.2.1* (The Linear Property)*:
Let F_1 and F_2 be functions whose Laplace transform exist, and let c_1 and c_2 be constants. Then

$$\mathscr{L}\{c_1F_1(t) + c_2F_2(t)\} = c_1\mathscr{L}\{F_1(t)\} + c_2\mathscr{L}\{F_2(t)\}$$

Example 1.1:

$$\mathscr{L}\{4t^2 - 3\cos 2t + 5e^{-t}\}$$

$$
\begin{aligned}
\mathscr{L}\{4t^2 - 3\cos 2t + 5e^{-t}\} &= 4\mathscr{L}\{t^2\} - 3\mathscr{L}\{\cos 2t\} + 5\mathscr{L}\{e^{-t}\} \\
&= 4 \cdot \frac{2!}{s^3} - 3 \cdot \frac{s}{s^2+4} + 5 \cdot \frac{1}{s+1} \\
&= \frac{8}{s^3} - \frac{3s}{s^2+4} + \frac{5}{s+1}
\end{aligned}
$$

***Theorem 1.2.2* (First Translation of Shifting Property):**
If

$$\mathscr{L}\{F(t)\} = f(s)$$

then

$$\mathscr{L}\{e^{at}F(t)\} = f(s-a)$$

Example 1.2:

$$\mathscr{L}\{e^{-t}\cos 2t\}$$

Since $\mathscr{L}\{\cos 2t\} = \dfrac{s}{s^2+4}$, we have

$$\mathscr{L}\{e^{-t}\cos 2t\} = \frac{s+1}{(s+1)^2+4} = \frac{s+1}{s^2+2s+5}\}$$

***Theorem 1.2.3* (Second Translation or Shifting Property):**
If

$$\mathscr{L}\{F(t)\} = f(s)$$

and

$$G(t) = \begin{cases} F(t-a) & \text{for } t > a \\ 0 & \text{for } t < a \end{cases}$$

then

$$\mathscr{L}\{G(t)\} = e^{-as}f(s)$$

Example 1.3: Find $\mathscr{L}\{G(t)\}$ **where** $G(t) = \begin{cases} \cos t - \frac{2\pi}{3} & \textbf{for } t > \frac{2\pi}{3} \\ 0 & \textbf{for } t < \frac{2\pi}{3} \end{cases}$

$$\begin{aligned} \mathscr{L}\{G(t)\} &= e^{-\frac{2\pi}{3}s} \cdot \mathscr{L}\{\cos t\} \\ &= e^{-\frac{2\pi}{3}s} \cdot \frac{s}{s^2+1} \\ &= \frac{se^{-\frac{2\pi}{3}s}}{s^2+1} \end{aligned}$$

***Theorem 1.2.4* (Change of Scale Property):**
If

$$\mathscr{L}\{F(t)\} = f(s)$$

then

$$\mathscr{L}\{F(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right)$$

Example 1.4: Evaluate $\mathscr{L}\{\sin 3t\}$

$$
\begin{aligned}
\mathscr{L}\{\sin 3t\} &= \frac{1}{3} \cdot \frac{1}{\left(\frac{s}{3}\right) + 1} \\
&= \frac{1}{3} \cdot \frac{3^2}{s^2 + 3^2} \\
&= \frac{3}{s^2 + 9}
\end{aligned}
$$

***Theorem 1.2.5* (Multiplication by t^n):**
If

$$\mathscr{L}\{F(t)\} = f(s)$$

then

$$\mathscr{L}\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s)$$

Example 1.5: Find $\mathscr{L}\{t^2 \cos at\}$

$$
\begin{aligned}
\mathscr{L}\{t^2 \cos at\} &= (-1)^2 \cdot \frac{d^2}{dx^2} \left(\frac{s}{s^2 + a^2} \right) \\
&= \frac{\mathrm{d}}{\mathrm{d}s} \left[\frac{s^2 + a^2 - 2s^2}{(s^2 + a^2)^2} \right] \\
&= \frac{\mathrm{d}}{\mathrm{d}s} \left[\frac{a^2 - s^2}{(s^2 + a^2)^2} \right] \\
&= \frac{(s^2 + a^2)^2(-2s) - (-s^2 + a^2)2(s^2 + a^2) \cdot 2s}{(s^2 + a^2)^4} \\
&= \frac{2s^3 - 6a^2 s}{(s^2 + a^2)^3}
\end{aligned}
$$

***Theorem 1.2.6* (Division by t):**
If

$$\mathscr{L}\{F(t)\} = f(s)$$

then

$$\mathscr{L}\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(x)\, dx$$

Example 1.6: Find $\mathscr{L}\{\frac{\sin t}{t}\}$

$$
\begin{aligned}
\mathscr{L}\left\{\frac{\sin t}{t}\right\} &= \int_s^\infty \frac{1}{s^2+1}\,du \\
&= \tan^{-1} u \Big|_s^\infty \\
&= \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s \\
&= \tan^{-1}\frac{1}{s}
\end{aligned}
$$

Theorem 1.2.7* (Laplace transform of Integral)*:
If

$$\mathscr{L}\{F(t)\} = f(s)$$

then

$$\mathscr{L}\left\{\int_0^t F(x)\,dx\right\} = \frac{1}{s}f(s)$$

Example 1.7: Evaluate $\mathscr{L}\{\int_0^t \frac{\sin u}{u}\,du\}$

$$\mathscr{L}\left\{\frac{\sin u}{u}\right\} = \tan^{-1}\frac{1}{s}$$

$$\mathscr{L}\left\{\int_0^t \frac{\sin u}{u}\,du\right\} = \frac{f(s)}{s} = \frac{1}{s}\tan^{-1}\frac{1}{s}$$

Let $F(t) = \frac{\sin t}{t}$

$$
\begin{aligned}
\mathscr{L}\left\{\frac{\sin t}{t}\right\} &= \int_s^\infty \frac{1}{u^2+1}\,du \\
&= \tan^{-1} u \Big|_s^\infty \\
&= \tan^{-1}\frac{1}{s}
\end{aligned}
$$

$$\therefore \mathscr{L}\left\{\int_0^t \frac{\sin u}{u}\,du\right\} = \frac{1}{s}\cdot\tan^{-1}\frac{1}{s}$$

Example 1.8: Evaluate $\mathscr{L}\{\int_0^t \sin 2u\,du\}$

$$\mathscr{L}\{\sin 2t\} = \frac{2}{s^2+4}$$

$$\therefore \mathscr{L}\left\{\int_0^t \sin 2u\,du\right\} = \frac{2}{s^3+4s}$$

***Theorem 1.2.8* (Periodic functions):**
If

$$\mathscr{L}\{F(t)\} = f(s)$$

then

$$\mathscr{L}\{F(t)\} = \frac{1}{1-e^{-Ts}} \int_0^T e^{-st} F(t)\, dt$$

where T is the period of $F(t)$.

Example 1.9: Find $\mathscr{L}\{F(t)\}$ for

$$F(t) = \begin{cases} \sin t & \textbf{for } 0 < t < \pi \\ 0 & \textbf{for } \pi < t < 2\pi \end{cases}$$

$$\begin{aligned} \mathscr{L}\{F(t)\} &= \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{st} \sin t\, dt \\ &= \frac{1}{1-e^{-2\pi s}} \int_0^{\pi} e^{-st} \sin t\, dt \end{aligned}$$

Example 1.10: Find $\mathscr{L}\{F(t)\}$ for

$$F(t) = \begin{cases} t & \textbf{for } 0 < t < 1 \\ 0 & \textbf{for } 1 < t < 2 \end{cases}$$

$$\begin{aligned} \mathscr{L}\{F(t)\} &= \frac{1}{1-e^{-2s}} \int_0^1 t\, dt \\ &= \frac{t^2 \Big|_0^1}{2-2e^{-2s}} \\ &= \frac{1}{2-2e^{-2s}} \end{aligned}$$

***Theorem 1.2.9* (Laplace transform of derivatives):**
If

$$\mathscr{L}\{F(t)\} = f(s)$$

then

$$\mathscr{L}\{F'(t)\} = sf(s) - F(0)$$

$$\mathscr{L}\{F''(t)\} = s^2F(s) - sF(0) - F'(0)$$

$$\mathscr{L}\{F'''(t) = s^3f(s) - s^2F(0) - sF'(0) - F''(0)\}$$

$$\boxed{\mathscr{L}\{F^{(n)}(t)\} = s^nf(s) - s^{n-1}F(0) - s^{n-2}F'(0) - \cdots - F^{(n-1)}(0)}$$

$$\boxed{\mathscr{L}\{F^{(n)}(t)\} = s^nf(s) - \sum_{i=n-1}^{0}\sum_{j=0}^{n-1} s^iF^{(j)}(0)}$$