Simple Harmonic Motion

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Contents

1 Oscillation and Vibration

1.1 Oscillation

- Oscillation is the repetitive variation, typically in time, of some measure about a central value (often a point of equilibrium) or between two or more different states.
- The term vibration is precisely used to describe mechanical oscillation.
- Familiar examples of oscillation include a swinging pendulum and alternating current.

1.2 Vibration

- Vibration is a mechanical phenomenon whereby oscillations occur about an equilibrium point.
- The word comes from Latin vibrationem ("shaking, brandishing").
- The oscillations may be periodic, such as the motion of a pendulum—or random, such as the movement of a tire on a gravel road.

1.3 Differences between Oscillation and Vibration

- Oscillation is the definite displacement of a body in terms of distance or time, whereas vibration is the movement brought about in a body due to oscillation.
- Oscillation takes place in physical, biological systems, and often in our society, but vibrations is associated with mechanical systems only.
- Oscillation is about a single body, whereas vibration is the result of collective oscillation of atoms in the body.
- All vibrations are oscillations, but not all oscillations are vibrations.

2 Simple Harmonic Motion

2.1 Definition

Definition 2.1.1: Simple Harmonic Motion

Simple harmonic motion is a type of periodic motion where the restoring force is directly proportional to the displacement and acts in the direction opposite to that of displacement.

A particle is said to execute SHM when it will

(a) Trace and retrace the same path over and over again.

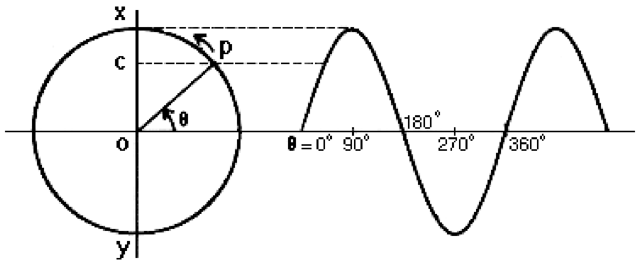
(b) Change direction at a regular interval of time.

(c) Move along a straight line.

(d) Have acceleration proportional to its displacement from the mean position.

A particle which satisfies the condition (a) only is said to execute **periodic motion**. A particle which satisfies condition (a) and (b) is said to execute **vibratory motion**.

Let P be a particle moving on the circumference of a circle of radius r with a uniform velocity v. Let angular velocity be $\omega = v/r$.



Displacement of the particle from the mean position is given by $y = r \sin \omega t$
So, velocity of the particle is given by $v = \dfrac{dy}{dt} = \omega r \cos \omega t$
And acceleration of the particle is given by $a = \dfrac{dv}{dt} = -\omega^2 r \sin \omega t = -\omega^2 y$

Angle	Position of vibrating particle	Displacement $y = r \sin \omega t$	Velocity $\frac{\mathrm{d}y}{\mathrm{d}t} = \omega r \cos \omega t$	Acceleration $-\omega^2 r \sin \omega t = -\omega^2 y$
0	O	0	ωr	0
$\pi/2$	X	r	0	$-\omega^2 a$
π	O	0	$-\omega r$	0
$3\pi/2$	Y	$-r$	0	$\omega^2 r$
2π	O	0	ωr	0

2.2 Differential Equation of SHM

Let y be the displacement of the particle from the mean position at time t, r be the amplitude, and α be the epoch of the vibrating particle

$$y = r \sin(\omega t + \alpha) \tag{1}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = r\omega \cos(\omega t + \alpha) \tag{2}$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = -r\omega^2 \sin(\omega t + \alpha) \tag{3}$$

Hence the differential equation of SHM is

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \omega^2 y = 0 \tag{4}$$

2.3 Solution of the Differential Equation of SHM

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \omega^2 y = 0 \tag{5}$$

Here,

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}y}$$

$$v\,dv + \omega^2 y\,dy = 0$$
$$\int v\,dv + \omega^2 \int y\,dy = 0$$
$$\frac{v^2}{2} + \frac{\omega^2 y^2}{2} = C'$$
$$\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \omega^2 y^2 = C^2$$

At maximum displacement, $y = r$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = 0$
So, $C^2 = \omega^2 r^2$

$$\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = \omega^2 (r^2 - y^2)$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = \omega\sqrt{r^2 - y^2}$$
$$\int \frac{1}{\sqrt{r^2 - y^2}}\,dy = \int \omega\,dt$$
$$\sin^{-1}\frac{y}{r} = \omega t + \alpha$$
$$\boxed{y = r\sin(\omega t + \alpha)} \tag{6}$$

By expanding equation (6), we get

$$y = r\sin\omega t\cos\alpha + r\cos\omega t\sin\alpha \tag{7}$$

If $y = 0$ at $t = 0$, then $\alpha = 0$

$$y = r\sin\omega t \tag{8}$$

If $y = r$ at $t = 0$, then $\alpha = \pi/2$

$$y = r\cos\omega t \tag{9}$$

Hence, the general solution of the differential equation of SHM is

$$\boxed{y = A\sin\omega t + B\cos\omega t} \tag{10}$$

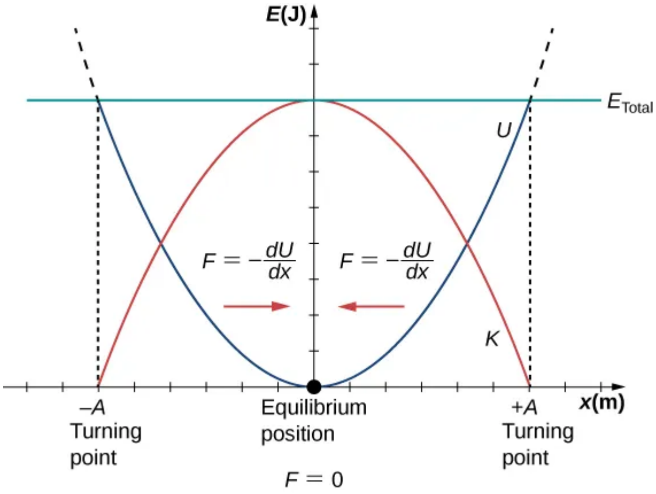
3 Energy in SHM

3.1 Total Energy of a Vibrating Particle

$$\text{Kinetic Energy} = \frac{1}{2}m\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2$$
$$= \frac{1}{2}m\omega^2 r^2\cos^2(\omega t + \alpha)$$
$$\text{Potential Energy} = \frac{1}{2}ky^2$$
$$= \frac{1}{2}m\omega^2 r^2$$
$$= \frac{1}{2}m\omega^2 r^2\sin^2(\omega t + \alpha)$$

Thus, the total energy of the vibrating particle is

$$\boxed{E = \frac{1}{2}kr^2 = \frac{1}{2}m\omega^2 r^2} \tag{11}$$



3.2 Average Kinetic Energy

Kinetic energy of the particle is given by

$$K = \frac{1}{2}m\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = \frac{1}{2}m\omega^2 r^2 \cos^2(\omega t + \alpha) \tag{12}$$

Hence, average kinetic energy is

$$\begin{aligned}
\overline{K} &= \frac{1}{T}\int_0^T K\,dt \\
&= \frac{1}{T}\int_0^T \frac{1}{2}m\omega^2 r^2 \cos^2(\omega t + \alpha)\,dt \\
&= \frac{1}{2}m\omega^2 r^2 \frac{1}{T}\int_0^T \cos^2(\omega t + \alpha)\,dt \\
&= \frac{1}{2}m\omega^2 r^2 \frac{1}{T}\int_0^T \frac{1 + \cos 2(\omega t + \alpha)}{2}\,dt \\
&= \frac{1}{2}m\omega^2 r^2 \frac{1}{T}\left[\frac{t}{2} + \frac{\sin 2(\omega t + \alpha)}{4\omega}\right]_0^T \\
&= \frac{1}{2}m\omega^2 r^2 \frac{1}{T}\left[\frac{T}{2} + \frac{\sin 2(\omega T + \alpha)}{4\omega} - \frac{\sin 2\alpha}{4\omega}\right] \\
&= \frac{1}{2}m\omega^2 r^2 \frac{1}{T}\left[\frac{T}{2} + \frac{\sin 2\alpha}{4\omega} - \frac{\sin 2\alpha}{4\omega}\right] \\
&= \frac{1}{4}m\omega^2 r^2
\end{aligned}$$

$$\boxed{\overline{K} = \frac{1}{4}m\omega^2 r^2 = \frac{1}{4}kr^2 = \frac{1}{2}E} \tag{13}$$

3.3 Average Potential Energy

Potential energy of the particle is given by

$$U = \frac{1}{2}ky^2 = \frac{1}{2}m\omega^2 r^2 \sin^2(\omega t + \alpha) \tag{14}$$

Hence, average potential energy is

$$\overline{U} = \frac{1}{T}\int_0^T U\,dt$$

$$
\begin{aligned}
&= \frac{1}{T}\int_0^T \frac{1}{2}m\omega^2 r^2 \sin^2(\omega t+\alpha)\,dt \\
&= \frac{1}{2}m\omega^2 r^2 \frac{1}{T}\int_0^T \sin^2(\omega t+\alpha)\,dt \\
&= \frac{1}{2}m\omega^2 r^2 \frac{1}{T}\int_0^T \frac{1-\cos 2(\omega t+\alpha)}{2}\,dt \\
&= \frac{1}{2}m\omega^2 r^2 \frac{1}{T}\left[\frac{t}{2}-\frac{\sin 2(\omega t+\alpha)}{4\omega}\right]_0^T \\
&= \frac{1}{2}m\omega^2 r^2 \frac{1}{T}\left[\frac{T}{2}-\frac{\sin 2(\omega T+\alpha)}{4\omega}+\frac{\sin 2\alpha}{4\omega}\right] \\
&= \frac{1}{2}m\omega^2 r^2 \frac{1}{T}\left[\frac{T}{2}+\frac{\sin 2\alpha}{4\omega}-\frac{\sin 2\alpha}{4\omega}\right] \\
&= \frac{1}{4}m\omega^2 r^2
\end{aligned}
$$

$$\boxed{\overline{U} = \frac{1}{4}m\omega^2 r^2 = \frac{1}{4}kr^2 = \frac{1}{2}E} \tag{15}$$

4 Composition of Two SHMs of Same Frequency

4.1 Same Direction

Let y_1 and y_2 be the displacements of two SHM of same frequency ω, amplitude r_1 and r_2, and phases α_1 and α_2 respectively.

$$y_1 = r_1 \sin(\omega t+\alpha_1) \tag{16}$$
$$y_2 = r_2 \sin(\omega t+\alpha_2) \tag{17}$$

If the two SHMs are in the same direction, then the resultant displacement is

$$
\begin{aligned}
y &= y_1 + y_2 \\
&= r_1 \sin(\omega t+\alpha_1) + r_2 \sin(\omega t+\alpha_2) \\
&= r_1 \sin\omega t\cos\alpha_1 + r_1\cos\omega t\sin\alpha_1 + r_2\sin\omega t\cos\alpha_2 + r_2\cos\omega t\sin\alpha_2
\end{aligned}
$$

$$\therefore y = (r_1\cos\alpha_1 + r_2\cos\alpha_2)\sin\omega t + (r_1\sin\alpha_1 + r_2\sin\alpha_2)\cos\omega t \tag{18}$$

In equation (18), let

$$r_1\cos\alpha_1 + r_2\cos\alpha_2 = A\cos\varphi \tag{19}$$
$$r_1\sin\alpha_1 + r_2\sin\alpha_2 = A\sin\varphi \tag{20}$$

Then, equation (18) becomes

$$y = A\cos\varphi\sin\omega t + A\sin\varphi\cos\omega t \tag{21}$$
$$y = A\sin(\omega t+\varphi) \tag{22}$$

Here, A is the amplitude of the resultant SHM and φ is the phase of the resultant SHM.
From equations (19) and (20), we get

$$A^2 = A^2\sin^2\varphi + A^2\cos^2\varphi$$

$$
\begin{aligned}
A^2 &= r_1^2 \sin^2 \alpha_1 + r_2^2 \sin^2 \alpha_2 + 2r_1 r_2 \sin \alpha_1 \sin \alpha_2 \\
&\quad + r_1^2 \cos^2 \alpha_1 + r_2^2 \cos^2 \alpha_2 + 2r_1 r_2 \cos \alpha_1 \cos \alpha_2 \\
A^2 &= r_1^2 + r_2^2 + 2r_1 r_2 (\sin \alpha_1 \sin \alpha_2 + \cos \alpha_1 \cos \alpha_2) \\
A^2 &= r_1^2 + r_2^2 + 2r_1 r_2 \cos (\alpha_1 - \alpha_2)
\end{aligned}
$$

$$
\boxed{A = \sqrt{r_1^2 + r_2^2 + 2r_1 r_2 \cos (\alpha_1 - \alpha_2)}} \tag{23}
$$

And,

$$
\boxed{\varphi = \tan^{-1} \frac{r_1 \sin \alpha_1 + r_2 \sin \alpha_2}{r_1 \cos \alpha_1 + r_2 \cos \alpha_2}} \tag{24}
$$

4.1.1 Special Cases

(I) Same phase : $\alpha_1 = \alpha_2$

In this case, equation (21) becomes

$$
A = \sqrt{r_1^2 + r_2^2 + 2r_1 r_2 \cos 0} = \sqrt{(r_1 + r_2)^2} = r_1 + r_2
$$

$$
\boxed{A = r_1 + r_2} \tag{25}
$$

And,

$$
\varphi = \tan^{-1} \frac{r_1 \sin \alpha + r_2 \sin \alpha}{r_1 \cos \alpha + r_2 \cos \alpha} = \tan^{-1} \left(\frac{r_1 + r_2}{r_1 + r_2} \tan \alpha \right) = \tan^{-1} (\tan \alpha)
$$

$$
\boxed{\varphi = \alpha} \tag{26}
$$

(II) Opposite phase : $\alpha_1 - \alpha_2 = (2n+1)\pi$**, where** $n = 0, 1, 2, \cdots$
In this case, equation (21) becomes

$$
A = \sqrt{r_1^2 + r_2^2 + 2r_1 r_2 \cos (\alpha_2 + \pi - \alpha_2)} = \sqrt{(r_1 - r_2)^2} = r_1 - r_2
$$

$$
\boxed{A = r_1 - r_2} \tag{27}
$$

And,

$$
\varphi = \tan^{-1} \frac{r_1 \sin (\alpha + \pi) + r_2 \sin \alpha}{r_1 \cos (\alpha + \pi) + r_2 \cos \alpha} = \tan^{-1} \left(\frac{r_1 \sin \alpha - r_2 \sin \alpha}{r_1 \cos \alpha - r_2 \cos \alpha} \right) = \tan^{-1} (-\tan \alpha)
$$

$$
\boxed{\varphi = \alpha + \pi} \tag{28}
$$

4.2 Right Angle

Let x and y be the displacements of two SHM of same frequency ω, amplitude a and b respectively, and phase difference α, acting at right angle to each other.

$$
x = a \sin (\omega t + \alpha) \tag{29}
$$

$$
y = b \sin (\omega t) \tag{30}
$$

Or,

$$
\frac{x}{a} = \sin (\omega t + \alpha) \tag{31}
$$

$$\frac{y}{b} = \sin(\omega t) \tag{32}$$

Thus, we get

$$\begin{aligned}
\frac{x}{a} &= \sin\omega t\cos\alpha + \cos\omega t\sin\alpha \\
\frac{x}{a} &= \frac{y}{b}\sqrt{1-\sin^2\alpha} + \sqrt{1-\frac{y^2}{b^2}}\sin\alpha \\
\frac{x}{a} - \frac{y}{b}\sqrt{1-\sin^2\alpha} &= \sqrt{1-\frac{y^2}{b^2}}\sin\alpha \\
\frac{x^2}{a^2} + \frac{y^2}{b^2}(1-\sin^2\alpha) - \frac{2xy}{ab}\sqrt{1-\sin^2\alpha} &= \left(1-\frac{y^2}{b^2}\right)\sin^2\alpha
\end{aligned}$$

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\sqrt{1-\sin^2\alpha} = \sin^2\alpha} \tag{33}$$

Equation (33) represents the general equation of the resultant SHM of the two perpendicular SHMs. The resulting curves are also known as Lissajous firgures.

4.2.1 Special Cases

(I) If $\alpha = 0$ or 2π

$$\cos\alpha = 1, \qquad \sin\alpha = 0$$

Then equation (33) becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab} = 0$$

Or,

$$\frac{x}{a} = \frac{y}{b}$$

This represents a straight line passing through the origin.

(II) If $\alpha = \pi$

$$\cos\alpha = -1, \qquad \sin\alpha = 0$$

Then equation (33) becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{2xy}{ab} = 0$$

Or,

$$\frac{x}{a} = -\frac{y}{b}$$

This represents a straight line with negative slope passing through the origin.

(III) If $\alpha = \pi/2$ or, $3\pi/2$

$$\cos\alpha = 0, \qquad \sin\alpha = 1$$

Then equation (33) becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

This represents an ellipse.

(IV) If $\alpha = \pi/2$ or, $3\pi/2$, and $a = b$

$$\cos\alpha = 0, \qquad \sin\alpha = 1$$

Then equation (33) becomes

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$$

Or,

$$x^2 + y^2 = a^2$$

This represents a circle of radius a.

(V) If $\alpha = \pi/4$ or, $7\pi/4$

$$\cos\alpha = \frac{1}{\sqrt{2}}, \qquad \sin\alpha = \frac{1}{\sqrt{2}}$$

Then equation (33) becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\frac{1}{\sqrt{2}} = \frac{1}{2}$$

This represents an oblique ellipse.

(VI) If $\alpha = 3\pi/4$ or, $5\pi/4$

$$\cos\alpha = -\frac{1}{\sqrt{2}}, \qquad \sin\alpha = \frac{1}{\sqrt{2}}$$

Then equation (33) becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{2xy}{ab}\frac{1}{\sqrt{2}} = \frac{1}{2}$$

This represents an oblique ellipse (negative slope).

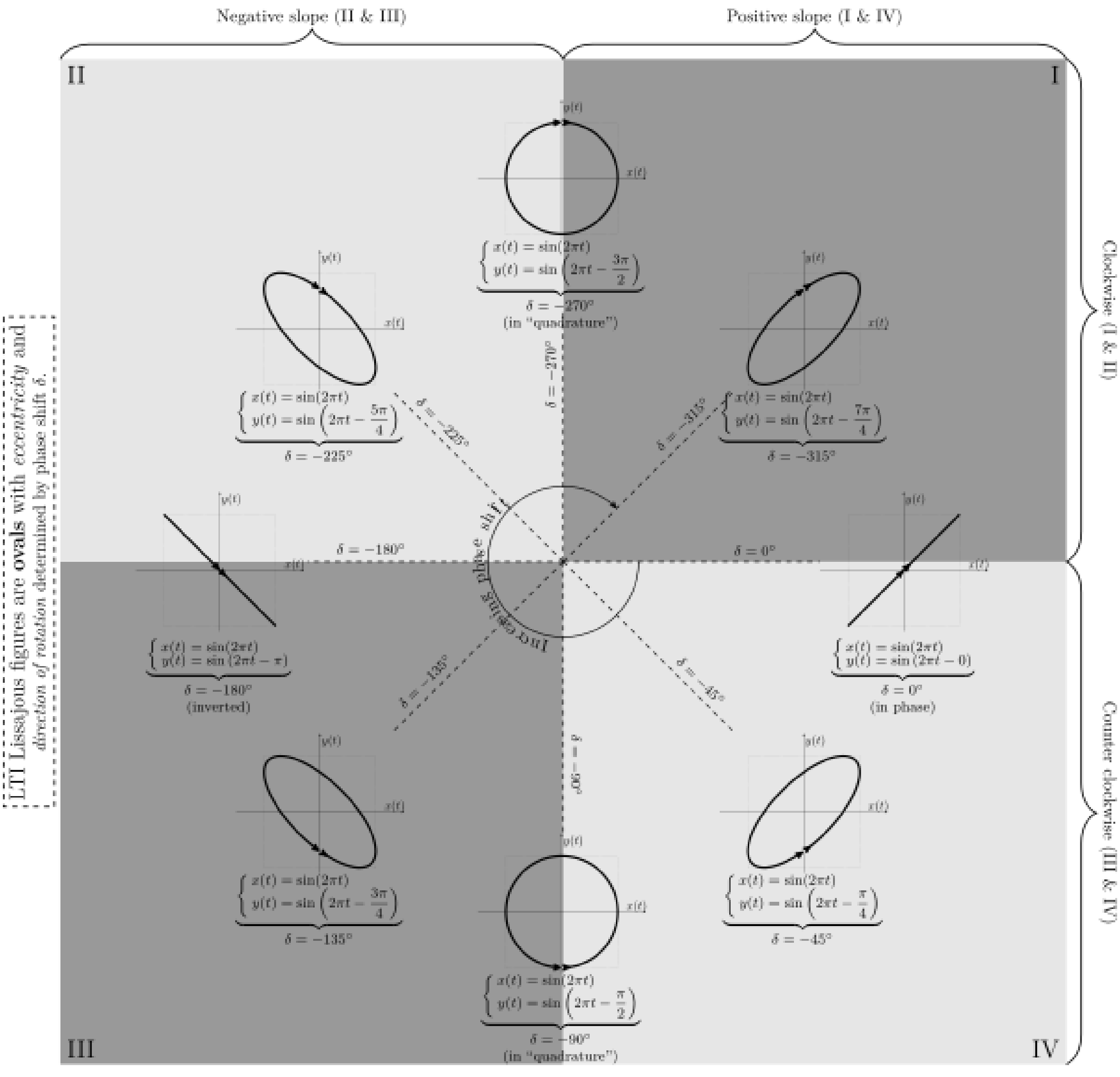


Figure 1: Lissajous figures