

# MATH-181

## Calculus

Note taken by: Turja Roy  
ID: 2108052

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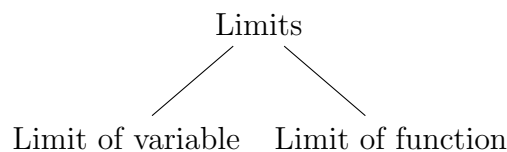
# Intro



## Definition: Function

A rule associating a unique output to a given input.

## 1 Limit



## Definition: Limit of variable

A constant  $a$  is said to be the limit of the variable  $x$ , if

$$0 < |x - a| < \delta$$

where  $\delta$  is a pre-assigned positive quantity as small as we please.

### 1.1 Limit of a Sequence

#### Definition 1.1.1: Sequence

A *sequence* is a function whose domain is  $\mathbb{N}$ .

Given a function  $f : \mathbb{N} \rightarrow \mathbb{R}$ ,  $f(n)$  is the  $n$ th term on the list. The notation for sequences reinforces this familiar Understanding.

**Example 1.1:** Each of the following are common ways to describe a sequence

- (i)  $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$
- (ii)  $(\frac{1+n}{n})_{n=1}^{\infty} = (\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \dots)$
- (iii)  $(a_n)$ , where  $a_n = 2^n$  for each  $n \in \mathbb{N}$
- (iv)  $(x_n)$ , where  $x_1 = 2$  and  $x_{n+1} = \frac{x_n+1}{2}$

### Definition 1.1.2: Convergence of a Sequence

A sequence  $(a_n)$  *converges* to a real number  $a$  if

$$\forall \epsilon, \exists N \in \mathbb{N} : n \geq N \implies |a_n - a| < \epsilon$$

To indicate that  $(a_n)$  converges to  $a$ , we write either  $\lim a_n = a$  or  $(a_n) \rightarrow a$ .

### Definition 1.1.3: $\epsilon$ -neighbourhood and $\delta$ -neighbourhood

**$\epsilon$ -neighbourhood:**

$$\forall \epsilon > 0, U_\epsilon(a) = \{x \in \mathbb{R} : |x - a| < \epsilon\} = (a - \epsilon, a + \epsilon)$$

**$\delta$ -neighbourhood:**

$$U_\delta(a) = \{x \in \mathbb{R} : |x - a| < \delta\} = (a - \delta, a + \delta)$$

### Definition 1.1.4: Convergence of a Sequence: Topological Version

A sequence  $(a_n)$  converges to  $a$  if, given any  $\epsilon$ -neighbourhood  $U_\epsilon(a)$  of  $a$ , there exists a point in the sequence after which all of the terms are in  $U_\epsilon(a)$ . In other words, every  $\epsilon$ -neighbourhood contains all but a finite number of the terms  $(a_n)$ .

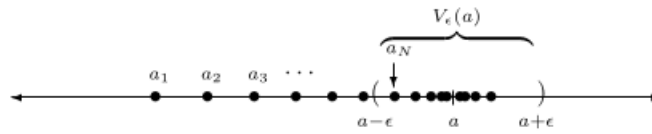


Figure 1: Limit of a sequence

The natural number  $N$  in the original version of the definition is the point where the sequence  $(a_n)$  enters  $U_\epsilon(a)$ , never to leave. It should be apparent that *the value of  $N$  depends on the choice of  $\epsilon$* . The smaller the  $\epsilon$ -neighbourhood, the larger the  $N$  may have to be.

#### Example 1.2: Prove that

$$\lim \left( \frac{n+1}{n} \right) = 1$$

Let's consider an arbitrary  $\epsilon > 0$ . Choose  $N \in \mathbb{N}$  with  $N > \frac{1}{\epsilon}$ . To verify that the choice of  $N$  is appropriate, let  $n \in \mathbb{N}$  satisfy  $n \geq N$ . Then,  $n \geq N$  implies  $n > \frac{1}{\epsilon}$ , which is the same as saying  $\frac{1}{n} < \epsilon$ . Finally this means

$$\left| \frac{n+1}{n} - 1 \right| < \epsilon \quad \text{as desired.} \quad \square$$

#### Note:-

To prove the previous example, let's work in the backward direction. To prove the conver-

gence, we have to make sure (according to the definition)  $|a_n - a| < \epsilon$ , i.e

$$\left| \frac{n+1}{n} - 1 \right| < \epsilon \text{ or, } \frac{1}{n} < \epsilon \text{ or, } n > \frac{1}{\epsilon}$$

So, we have to find out when or where  $n > \frac{1}{\epsilon}$  is true. If we choose  $N > \frac{1}{\epsilon}$ , our statement holds, and thus we prove the convergence.

**Example 1.3:** Verify, using the definition of convergence of a sequence, that the following sequences converge to the proposed limit.

(a)  $\lim_{n \rightarrow \infty} \frac{1}{6n^2+1} = 0$

(b)  $\lim_{n \rightarrow \infty} \frac{3n+1}{2n+5} = \frac{3}{2}$

(c)  $\lim_{n \rightarrow \infty} \frac{2}{\sqrt{n+3}} = 0$

(a) Let's consider  $\epsilon > 0$ . Choose  $N \in \mathbb{N}$  with  $N > \frac{1}{\sqrt{6\epsilon}}$ . For any  $n \geq N$

$$n > \frac{1}{\sqrt{6\epsilon}} \text{ or, } 6n^2 > \frac{1}{\epsilon} \text{ or, } \epsilon > \frac{1}{6n^2}$$

which finally gives

$$\left| \frac{1}{6n^2+1} - 0 \right| < \epsilon \quad \square$$

(b) For  $\epsilon > 0$ , let's choose  $N \in \mathbb{N}$  such that  $N > \frac{13}{4\epsilon}$ . Now, let there be  $n \in \mathbb{N}$  and  $n \geq N$ . Hence,

$$\begin{aligned} n &> \frac{13}{4\epsilon} \text{ or, } \epsilon > \frac{13}{4n} \\ \epsilon &> \frac{13}{4n+10} = \frac{13}{2(2n+5)} \\ \epsilon &> -\frac{13}{2(2n+5)} \\ \epsilon &> -\frac{13}{2(2n+5)} + \frac{3}{2} - \frac{3}{2} \\ \epsilon &> \frac{-13+6n+15}{2(2n+5)} - \frac{3}{2} \\ \epsilon &> \frac{3n+1}{2n+5} - \frac{3}{2} \\ \therefore \left| \frac{3n+1}{2n+5} - \frac{3}{2} \right| &< \epsilon \quad \square \end{aligned}$$

(c) For  $\epsilon > 0$ , let's choose  $N > \frac{4}{\epsilon^2}$ . Let there be  $n \geq N$ . Hence,

$$\begin{aligned} n &> \frac{4}{\epsilon^2} \\ \epsilon^2 &> \frac{4}{n} \\ \epsilon &> \frac{2}{\sqrt{n}} \\ \therefore \left| \frac{2}{\sqrt{n+3}} - 0 \right| &< \epsilon \quad \square \end{aligned}$$

**Note:-**

Scratch works for solving previous examples

(a)

$$\begin{aligned} \left| \frac{1}{6n^2 + 1} - 0 \right| &< \varepsilon \\ 6n^2 + 1 &> \frac{1}{\varepsilon} \\ n^2 &> \frac{1}{6\varepsilon} - \frac{1}{6} \\ n &> \frac{1}{\sqrt{6\varepsilon}} \end{aligned}$$

(b)

$$\begin{aligned} \left| \frac{3n+1}{2n+5} - \frac{3}{2} \right| &< \varepsilon \\ \left| -\frac{13}{4n+10} \right| &< \varepsilon \\ 4n+10 &> \frac{13}{\varepsilon} \\ n &> \frac{13}{4\varepsilon} \end{aligned}$$

(c)

$$\begin{aligned} \left| \frac{2}{\sqrt{n+3}} - 0 \right| &< \varepsilon \\ n+3 &> \frac{4}{\varepsilon^2} \\ n &> \frac{4}{\varepsilon^2} - 3 \\ n &> \frac{4}{\varepsilon^2} \end{aligned}$$

### Definition 1.1.5: Bounded Sequence

A sequence  $(x_n)$  is bounded if

$$\exists M > 0 : \forall n \in \mathbb{N}, |x_n| \leq M$$

Geometrically, this means that we can find an interval  $[-M, M]$  that contains every term in the sequence  $(x_n)$ .

**Theorem 1.1.6:** Every convergent sequence is bounded

## 1.2 Functional Limit

Considering a function  $f : A \rightarrow \mathbb{R}$ , a limit point  $c$  of  $A$  is a point with the property that every  $\epsilon$ -neighbourhood  $U_\epsilon(c)$  intersects  $A$  in some point other than  $c$ . Equivalently,  $c$  is a limit point of  $A$  if and only if  $c = \lim x_n$  for some sequence  $(x_n) \subseteq A$  with  $x_n \neq c$ .

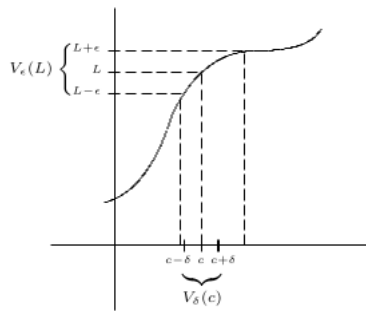


Figure 2: Definition of Functional Limit

**Definition 1.2.1:  $\epsilon - \delta$  Version**

Let  $f : A \rightarrow \mathbb{R}$  and let  $c$  be a limit point of the domain  $A$ .

$$\lim_{x \rightarrow c} f(x) = L \iff (\forall \epsilon > 0, \exists \delta > 0 : \forall x \in A, 0 < |x - c| < \delta \implies |f(x) - L| < \epsilon)$$

Here,

$$|f(x) - L| < \epsilon \text{ is equivalent to } f(x) \in U_\epsilon(L)$$

Likewise,

$$|x - c| < \delta \iff x \in U_\delta(c)$$

**Definition 1.2.2: Topological Version**

Let  $f : A \rightarrow \mathbb{R}$  and let  $c$  be a limit point of the domain of  $A$ . We can say,

$$\lim_{x \rightarrow c} f(x) = L \iff (\forall U_\epsilon(L), \exists U_\delta(c) : \forall x \in A, x \in U_\delta(c) \wedge x \neq c \implies f(x) \in U_\epsilon(L))$$

**Example 1.4: (i)** If  $f(x) = 3x + 1$ , prove that  $\lim_{x \rightarrow 2} f(x) = 7$

**(ii)** Show that  $\lim_{x \rightarrow 2} g(x) = 4$  where  $g(x) = x^2$ .

**(i)** Let  $\epsilon > 0$ . The definition requires we produce a  $\delta > 0$  so that  $0 < |x - 2| < \delta$  leads to the conclusion  $|f(x) - 7| < \epsilon$

$$|f(x) - 7| = |(3x + 1) - 7| = |3x - 6| = 3|x - 2|$$

Thus, if we choose  $\delta = \epsilon/3$ , then  $0 < |x - 2| < \delta$  implies  $|f(x) - 7| < 3(\epsilon/3) = \epsilon$

**(ii)** Given an arbitrary  $\epsilon > 0$ , our goal is to prove  $|g(x) - 4| < \epsilon$  by restricting  $|x - 2|$  to be smaller than some carefully chosen  $\delta$ .

$$|g(x) - 4| = |x^2 - 4| = |x + 2||x - 2|$$

We can make  $|x - 2|$  as small as we like, but we need an upper bound on  $|x + 2|$  in order to know how small to choose  $\delta$ . If we agree that our  $\delta$ -neighbourhood around  $c = 2$  must have radius no bigger than  $\delta = 1$ , then we get the upper bound  $|x + 2| \leq |3 + 2| = 5$  for all  $x \in U_\delta(c)$ .

Now, for  $\delta = \min(1, \epsilon/5)$ , if  $0 < |x - 2| < \delta$ , then

$$|x^2 - 4| = |x + 2||x - 2| < (5)\frac{\epsilon}{5} = \epsilon$$

and hence, the limit is proved.

**Example 1.5:** From the definition of  $(\delta, \epsilon)$  show that  $\lim_{x \rightarrow 3} (2x^3 - 3x^2 - 18x + 29) = 2$

Let for a given  $\epsilon > 0$ , however small  $0 < |x - 3| < \lambda < 1$  ———(1)

$$\begin{aligned}\therefore |2x^3 - 3x^2 - 18x + 29 - 2| &= |2x^3 - 3x^2 - 18x + 27| \\ &= |2(x - 3)^3 + 15(x - 3)^2 + 18(x - 3)| \\ &\leq 2(x - 3)^3 + 15(x - 3)^2 + 18(x - 3) \\ &\leq 2\lambda^3 + 15\lambda^2 + 18\lambda \\ &< 35\lambda \quad [\because \lambda^3 < \lambda^2 < \lambda < 1]\end{aligned}$$

Now,  $|2x^3 - 3x^2 - 18x + 29| < \epsilon$ , where  $\epsilon = 35\lambda$  ———(2)  
or,  $\lambda = \epsilon/35$ . Therefore, we can determine a small positive number  $\delta$  depending on  $\epsilon$  such that the limit is established. Here  $\delta = \epsilon/35$ . Hence it's proved that

$$\lim_{x \rightarrow 3} (2x^3 - 3x^2 - 18x + 29) = 2 \quad [\text{From (1) and (2)}]$$

### 1.2.1 Left hand limit and Right hand limit

Informally speaking, we can say  $L^-$  is the LHL of the function  $f$  at a point  $a$  if we can get  $f(x)$  as close as we want to  $L$  by taking  $a$  to the left of  $x$  and close to  $x$ , but not equal to  $a$ , and we write

$$\lim_{x \rightarrow a^-} f(x) = L^-$$

In the same way, we can say  $L^+$  is the RHL of the function if we take  $a$  to the right, and we write

$$\lim_{x \rightarrow a^+} f(x) = L^+$$

#### Definition 1.2.3: Left Hand Limit

$$L^- = \lim_{x \rightarrow a^-} f(x) \iff (\forall \epsilon > 0, \exists \delta : \forall x, 0 < (a - x) < \delta \implies |f(x) - L^-| < \epsilon)$$

#### Definition 1.2.4: Right Hand Limit

$$L^+ = \lim_{x \rightarrow a^+} f(x) \iff (\forall \epsilon > 0, \exists \delta : \forall x, 0 < (x - a) < \delta \implies |f(x) - L^+| < \epsilon)$$

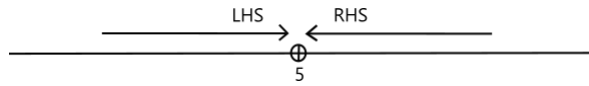


Figure 3: Left and Right Hand Limit

$$\begin{aligned}|x - 5| &< \epsilon \\ -\epsilon &< x - 5 < \epsilon \\ 5 - \epsilon &< x < 5 + \epsilon\end{aligned}$$

**Theorem 1.2.5 (Relation between one-sided and two-sided limits):** Limit of a function at point  $a$  exists if:

(i)  $\lim_{x \rightarrow a^-} f(x)$  exists

(ii)  $\lim_{x \rightarrow a^+} f(x)$  exists

(iii)  $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x)$

That is,

$$\lim_{x \rightarrow a} f(x) = L \iff \lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = L$$

### Use L'Hôpital Rule

The 7 Indeterminate forms

Put  $x=1$

$$x \rightarrow 1 \quad \frac{x^2 - 1}{x - 1} \rightarrow \frac{0}{0} \rightarrow \text{Apply L'Hôpital}$$

$$x \rightarrow 1 \quad \frac{x^2 - 1}{x + 1} \rightarrow \frac{0}{2} \rightarrow \text{Limit exists}$$

$$x \rightarrow 1 \quad \frac{x^2 + 1}{x^2 - 1} \rightarrow \frac{2}{0} \rightarrow \text{Limit doesn't exist}$$

(1)  $\frac{0}{0}$

(2)  $\frac{\infty}{\infty}$

(3)  $0 \times \infty$

(4)  $\infty - \infty$

(5)  $1^\infty$

(6)  $0^0$

(7)  $\infty^0$

**Example 1.6:** Investigate the following function at  $x = 0$  and  $x = \frac{1}{2}$

$$f(x) = \begin{cases} 1 + 2x, & \text{for } -\frac{1}{2} \leq x < 0 \\ 1 - 2x, & \text{for } 0 \leq x < \frac{1}{2} \\ -1 + 2x, & \text{for } x > \frac{1}{2} \end{cases}$$

For  $x = 0$  :

$$\text{LHL} = \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \{1 + 2 \times 0\} = 1$$

$$\text{RHL} = \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \{1 - 2 \times 0\} = 1$$

$$f(0) = 1 - 2 \times 0 = 1$$

Thus we get,  $\lim_{x \rightarrow 0} f(x) = 1 = f(0)$ . i.e The function is continuous at  $x = 0$

For  $x = \frac{1}{2}$  :

$$\text{LHL} = \lim_{x \rightarrow \frac{1}{2}^-} f(x) = \lim_{x \rightarrow \frac{1}{2}^-} \left\{ 1 - 2 \left( \frac{1}{2} \right) \right\} = 0$$

$$\text{RHL} = \lim_{x \rightarrow \frac{1}{2}^+} f(x) = \lim_{x \rightarrow \frac{1}{2}^+} \left\{ -1 + 2 \left( \frac{1}{2} \right) \right\} = 0$$



As LHL=RHL, we can say that limit exists at  $x = \frac{1}{2}$ . However, the function is discontinuous at  $x = \frac{1}{2}$  since  $f\left(\frac{1}{2}\right)$  doesn't exist.

**Theorem 1.2.6 (Fundamental Theorems on Limits):** If  $\lim_{x \rightarrow a} f(x) = l$  and  $\lim_{x \rightarrow a} g(x) = m$ , where  $l$  and  $m$  are finite quantities and  $c$  is a constant,

- (i)  $\lim_{x \rightarrow a} \{f(x) \pm g(x)\} = l \pm m$
- (ii)  $\lim_{x \rightarrow a} \{f(x)g(x)\} = lm$
- (iii)  $\lim_{x \rightarrow a} \left\{ \frac{f(x)}{g(x)} \right\} = \frac{l}{m}$ , provided  $g(x) \neq 0$
- (iv)  $\lim_{x \rightarrow a} c = c$
- (v)  $\lim_{x \rightarrow a} cf(x) = c \lim_{x \rightarrow a} f(x)$
- (vi)  $\lim_{x \rightarrow a} (f(x))^n = (\lim_{x \rightarrow a} f(x))^n$
- (vii)  $\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)}$ , provided  $\lim_{x \rightarrow a} f(x) > 0$

**Theorem 1.2.7 (Sequential Criterion for Functional Limits):** Given a function  $f : A \rightarrow \mathbb{R}$  and a limit point  $c$  of  $A$ , the following two statements are equivalent:

- (i)  $\lim_{x \rightarrow c} f(x) = L$
- (ii)  $\forall (x_n) \subseteq A : x_n \neq c, (x_n) \rightarrow c \implies f(x_n) \rightarrow L$

**Theorem 1.2.8 (The Squeeze/Sandwich Theorem):** If there is a  $c > 0$  such that

$$f(x) \leq g(x) \leq h(x)$$

for all  $x \in (a - c, a) \cup (a, a + c)$  (i.e  $x$  in a deleted neighbourhood of  $a$ ), if

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L, \text{ then } \lim_{x \rightarrow a} g(x) = L$$



Figure 4: Squeeze Theorem

## 1.2.2 Asymptotes

Greek *asymptotos* means ‘nonintersecting’. Asymptote is infinitesimally close to a curve, but never touches it.

### Definition 1.2.9: Asymptote

An asymptote of a curve  $f(x)$  is a line such that the distance between the curve and the line approaches zero as one or both of the  $x$  and  $y$  coordinates approach infinity.

There are 3 types of asymptotes:

1. Vertical asymptote
2. Horizontal asymptote
3. Oblique asymptote

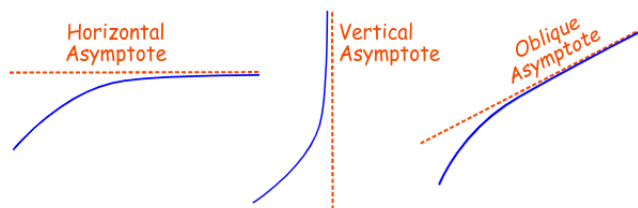


Figure 5: Types of asymptotes

### Definition 1.2.10: Vertical asymptote

The line  $x = a$  is a vertical asymptote of the function  $f(x)$  if  $f$  approaches infinity as  $x$  approaches  $a$ . That is, either

$$\lim_{x \rightarrow a^-} f(x) = \pm\infty, \text{ or } \lim_{x \rightarrow a^+} f(x) = \pm\infty, \text{ or both.}$$

### Definition 1.2.11: Horizontal asymptote

The horizontal line  $y = c$  is a horizontal asymptote of the function  $g(x)$  if

$$\lim_{x \rightarrow -\infty} g(x) = c, \text{ or } \lim_{x \rightarrow +\infty} g(x) = c$$

### Definition 1.2.12: Oblique asymptote

A function  $f(x)$  is asymptotic to the straight line  $y = mx + b$  ( $m \neq 0$ ) if

$$\lim_{x \rightarrow +\infty} [f(x) - (mx + b)] = 0, \text{ or } \lim_{x \rightarrow -\infty} [f(x) - (mx + b)] = 0$$

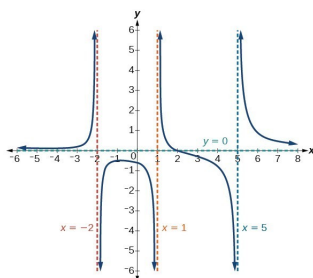


Figure 6: Vertical asymptote

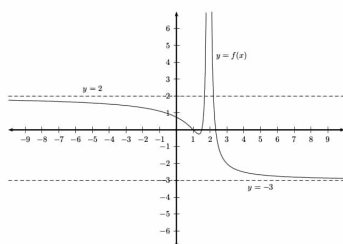


Figure 7: Horizontal asymptote

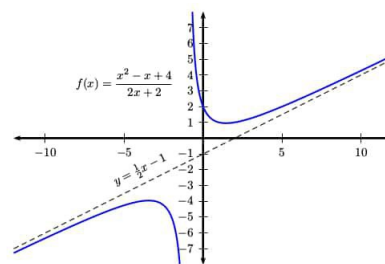


Figure 8: Oblique asymptote

### 1.2.3 End Behavior Model

For numerically large values of  $x$ , we can sometimes model the behavior of a complicated function by a simpler one that acts virtually in the same way.

**Example 1.7:** Let  $f(x) = 3x^4 - 2x^3 + 3x^2 - 5x + 6$  and  $g(x) = 3x^4$ . Show that while  $f$  and  $g$  are quite different for numerically small values of  $x$ , they are virtually identical for  $|x|$  large.

We can test the claim that  $g(x)$  models  $f(x)$  for numerically large values of  $x$  by taking the ratio of the functions as  $x \rightarrow \pm\infty$ .

$$\begin{aligned}\lim_{x \rightarrow \pm\infty} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow \pm\infty} \frac{3x^4 - 2x^3 + 3x^2 - 5x + 6}{3x^4} \\ &= \lim_{x \rightarrow \pm\infty} \left( 1 - \frac{2}{3x} + \frac{1}{x^2} - \frac{5}{3x^3} + \frac{2}{x^4} \right) \\ &= 1\end{aligned}$$

convincing evidence that  $f$  and  $g$  behave alike for  $|x|$  large.

#### Definition 1.2.13: End Behavior Model

The function  $g(x)$  is a

(a) left end behavior model for  $f(x)$  if and only if

$$\lim_{x \rightarrow -\infty} \frac{f(x)}{g(x)} = 1$$

(b) right end behavior model for  $f(x)$  if and only if

$$\lim_{x \rightarrow +\infty} \frac{f(x)}{g(x)} = 1$$

If  $g(x)$  provides both a left and a right end behavior model, it is simply called an **End Behavior Model**.

#### Example 1.8: Find an end behavior model for

(a)  $f(x) = \frac{2x^5 + x^4 - x^2 + 1}{3x^2 - 5x + 7}$

(b)  $g(x) = \frac{2x^3 - x^2 + x - 1}{5x^3 + x^2 + x - 5}$

(a)  $2x^5$  is an end behavior model for the numerator of  $f$ , and  $3x^2$  for the denominator. This makes

$$\frac{2x^5}{3x^2} = \frac{2}{3}x^3$$

an end behavior model for  $f$ .

(b)  $2x^3$  is an end behavior model for the numerator of  $g$ , and  $5x^3$  for the denominator. This makes

$$\frac{2x^3}{5x^3} = \frac{2}{5}$$

an end behavior model for  $g$ .

**Example 1.9:** Let  $f(x) = x + e^{-x}$ . Show that  $g(x) = x$  is a right end behavior model for  $f$  while  $h(x) = e^{-x}$  is a left end behavior model for  $f$ .

On the right,

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{x + e^{-x}}{x} = \lim_{x \rightarrow \infty} \left( 1 + \frac{e^{-x}}{x} \right) = 1$$

On the left,

$$\lim_{x \rightarrow -\infty} \frac{f(x)}{h(x)} = \lim_{x \rightarrow -\infty} \frac{x + e^{-x}}{e^{-x}} = \lim_{x \rightarrow -\infty} \left( \frac{x}{e^{-x}} + 1 \right) = 1$$

## 2 Continuity and Discontinuity

### Discussion (A deeper dive): Examples of Dirichlet and Thomae<sup>1</sup>

Given a function  $f$  with domain  $A \subseteq \mathbb{R}$ , we want to define continuity at a point  $c \in A$  to mean that if  $x \in A$  is chosen *near*  $c$ , then  $f(x)$  will be near  $f(c)$ .

Symbolically, we will say  $f$  is continuous at  $c$  if

$$\lim_{x \rightarrow c} f(x) = f(c)$$

However, the problem is that, at present, it's not entirely clear what is meant by  $\lim_{x \rightarrow c} f(x)$ . Let's define a function (Idea of German mathematician Peter Lejuene Dirichlet)  $g$  like this:

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

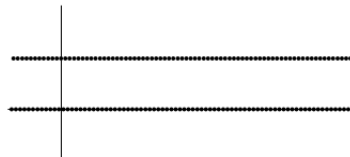


Figure 9: Dirichlet's Function,  $g(x)$

Does it make sense to attach a value to the expression  $\lim_{x \rightarrow 1/2} g(x)$ ? One idea is to consider a sequence  $(x_n) \rightarrow 1/2$ . Using our notion of limit of a sequence if we try to define  $g(x_n)$  as simply the limit of the sequence  $g(x_n)$ , the limit depends on how the sequence  $(x_n)$  is chosen. If each  $x_n$  is rational, then

$$\lim_{n \rightarrow \infty} g(x_n) = 1$$

And if  $x_n$  is irrational for each  $n$ , then

$$\lim_{n \rightarrow \infty} g(x_n) = 0$$

Generally speaking, we want the value of  $\lim_{x \rightarrow c} g(x)$  to be independent of how we approach  $c$ . In this particular case, the definition of a functional limit that we agree on should lead to the conclusion that

$$\lim_{x \rightarrow 1/2} g(x) \text{ does not exist}$$

We can also realize that Dirichlet's function is not continuous at  $c = 1/2$ . In fact, the real significance of the function is that there's nothing unique about the point  $c=1/2$ . Because both  $\mathbb{Q}$  and  $\mathbb{Q}'$  are dense in the real line, it follows that for any  $z \in \mathbb{R}$  we can find sequences  $(x_n) \subseteq \mathbb{Q}$  and  $(y_n) \subseteq \mathbb{Q}'$  such that

$$\lim x_n = \lim y_n = z$$

Because

$$\lim g(x_n) \neq \lim g(y_n)$$

the same line of reasoning reveals that  $g(x)$  is not continuous at  $z$ . In fact, Dirichlet's function is a *nowhere-continuous* on  $\mathbb{R}$ . How about if we redefine the Dirichlet's function in the following way?

$$h(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

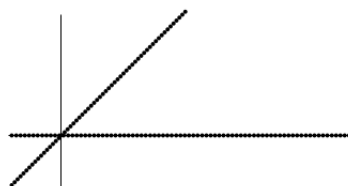


Figure 10: Modified Dirichlet's Function,  $h(x)$

<sup>1</sup>Stephen Abbott - Understanding Analysis 4.1

$h$  is not continuous at every point  $c \neq 0$  in the same way.

But, when  $c = 0$ , these two limits are both equal to  $h(0) = 0$ . Regardless of how we construct a sequence  $(z_n)$  converging to zero, the limit is always  $\lim h(z_n) = 0$ .

This observation goes to the heart of what we want functional limits to entail. To assert that

$$\lim_{x \rightarrow c} h(x) = L$$

should imply that

$$h(z_n) \rightarrow L \text{ for all sequences } (z_n) \rightarrow c$$

To this point, we've been discussing continuity of a function at a particular point in its domain. This is a significant departure from thinking of continuous functions as curves that can be drawn without lifting the pen from the paper, and it leads to some fascinating questions.

In 1875, K. J. Thomae discovered the function

$$t(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1/n & \text{if } x = m/n \in \mathbb{Q} \setminus \{0\} \text{ is in lowest terms with } n > 0 \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

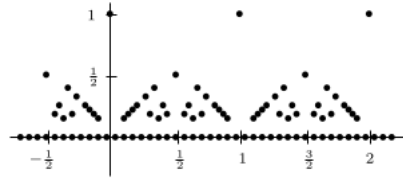


Figure 11: Thomae's Function,  $t(x)$

If  $c \in \mathbb{Q}$ , then  $t(c) > 0$ . Because  $\mathbb{Q}'$  is dense in  $\mathbb{R}$ , we can find a sequence  $(y_n)$  in  $\mathbb{Q}'$  converging to  $c$ . The result is that

$$\lim t(y_n) = 0 \neq t(c)$$

and Thomae's function fails to be continuous at any rational point.

Let's try this argument on some irrational point like  $c = \sqrt{2}$ . All irrational values get mapped to 0 by function  $t$ , so let's consider a sequence  $(x_n)$  of rational numbers instead that converges to  $\sqrt{2}$ . So, the sequence of rational approximations for  $\sqrt{2}$  might be

$$\left(1, \frac{14}{10}, \frac{141}{100}, \frac{1414}{1000}, \frac{14142}{10000}, \frac{141421}{100000}, \dots\right)$$

In this case, the sequence  $t(x_n)$  begins,

$$\left(1, \frac{1}{5}, \frac{1}{100}, \frac{1}{500}, \frac{1}{5000}, \frac{1}{100000}, \dots\right)$$

The denominator of these fractions are getting larger and fast approaching  $0 = t(\sqrt{2})$ . This always happens. The closer a rational number is chosen to a fixed irrational number, the larger its denominator must necessarily be. Consequently, Thomae's function has the bizarre property of being continuous at every irrational point and discontinuous at every rational point on  $\mathbb{R}$ .

Can there be examples of functions with the opposite property? If we're given some set  $A \subseteq \mathbb{R}$ , is it always possible to find a function that is continuous only on the set  $A^c$ ? In each of our

examples, the functions were defined to have erratic oscillations around points in the domain. What about we restrict our attention to somewhat less volatile functions like a *monotonic* (A function which is either always increasing or always decreasing on a given domain) function? What might we be able to say about the set of discontinuities of a monotonic function on  $\mathbb{R}$ ?

## 2.1 Continuity

### Definition 2.1.1: Continuity

A function  $f(x)$  is said to be continuous at  $x = a$  if,

- (i)  $\lim_{x \rightarrow a^-} f(x)$  exists
- (ii)  $\lim_{x \rightarrow a^+} f(x)$  exists
- (iii)  $f(x)$  is finite
- (iv)  $\lim_{x \rightarrow a^-} f(x) = f(a) = \lim_{x \rightarrow a^+} f(x)$

### Definition 2.1.2: Cauchy's Definition of Continuity

A function  $f : A \rightarrow \mathbb{R}$  is continuous at a point  $c \in A$  if

$$\forall \epsilon > 0, \exists \delta > 0 : \forall x \in A, |x - c| < \delta \implies |f(x) - f(c)| < \epsilon$$

**Theorem 2.1.3 (Properties of Continuous Functions):** Let  $f(x)$  and  $g(x)$  be two functions of  $x$  each being continuous at  $x = a$ . For  $\lim_{x \rightarrow a} f(x) = l$  and  $\lim_{x \rightarrow a} g(x) = m$ , the following statements are true:

- (i)  $\lim_{x \rightarrow a} \{f(x) \pm g(x)\} = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x) = f(a) \pm g(a)$
- (ii)  $\lim_{x \rightarrow a} \{f(x)g(x)\} = \lim_{x \rightarrow a} f(x) \lim_{x \rightarrow a} g(x) = f(a)g(a)$
- (iii)  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} = \frac{f(a)}{g(a)}$
- (iv) If  $f(x)$  is continuous in a closed interval, it's bounded in that interval
- (v) A function which is continuous in a closed interval attains at least once its least upper and greatest lower bounds.
- (vi) A continuous function which has opposite sign at two points meets its domain vanishes at least once between these points.
- (vii) A continuous function  $f(x)$  in the interval  $(a, b)$  assumes at least once every value between  $f(a)$  and  $f(b)$ , it being supposed that  $f(a) \neq f(b)$ .
- (viii) The converse of this theorem is not true i.e a function  $f(x)$  which takes all values between  $f(a)$  and  $f(b)$  is not necessarily continuous in the interval  $(a, b)$ .

## 2.2 Discontinuity

### Definition 2.2.1: Discontinuous Function

A function  $f(x)$  is said to be discontinuous for  $x = a$  if  $f(x)$  does not satisfy at least one of the conditions of continuity in Definition 2.1.1.

#### Discontinuities are:

- (i) Ordinary discontinuity / discontinuity of 1st kind
- (ii) Ordinary discontinuity / discontinuity of 2nd kind
- (iii) Removal discontinuity
- (iv) Mixed discontinuity
- (v) Oscillatory discontinuity
- (vi) Infinite discontinuity

**A. Discontinuity of the first kind:** If the function  $f(x)$  has finite limit, but

$$\lim_{h \rightarrow 0} f(a + h) \neq \lim_{h \rightarrow 0} f(a - h) \neq f(a)$$

the function is said to have ordinary discontinuity or discontinuity of the first kind at  $x = a$ .

**B. Discontinuity of the second kind:** If both the limits of  $f(x)$ , i.e

$$\lim_{h \rightarrow 0} f(a - h) \text{ and } \lim_{h \rightarrow 0} f(a + h)$$

do not exist for  $x = a$  then  $f(x)$  has a discontinuity of the second kind at  $x = a$ .

**C. Mixed discontinuity:** If one of the limits of  $f(x)$  exists, then the discontinuity of the function  $f(x)$  at  $x = a$  is called a mixed discontinuity. That is, if

$$\lim_{h \rightarrow 0} f(a + h) = f(a) \text{ but } \lim_{h \rightarrow 0} f(a - h) \neq f(a)$$

then  $f(x)$  is continuous on the right but it has an ordinary discontinuity on the left for  $x = a$ . In the same way, if

$$\lim_{h \rightarrow 0} f(a - h) = f(a) \text{ but } \lim_{h \rightarrow 0} f(a + h) \neq f(a)$$

then  $f(x)$  is continuous on the right at  $x = a$ , and discontinuous on the left.

**D. Removal discontinuity:** If

$$\lim_{h \rightarrow 0} f(a + h) = \lim_{h \rightarrow 0} f(a - h) \neq f(a)$$

then the function  $f(x)$  is said to have a removal discontinuity for  $x = a$ .

**E. Infinite discontinuity:** If both the limits of  $f(x)$  are infinite, the function  $f(x)$  has an infinite discontinuity. That is, if

$$\lim_{h \rightarrow 0^-} f(x) = \pm\infty \text{ or, } \lim_{h \rightarrow 0^+} f(x) = \pm\infty$$



then  $f(x)$  is said to have an infinite discontinuity at  $x = a$ .

**F. Oscillatory discontinuity:** A function  $f(x)$  having a discontinuity at a point  $x = a$  may oscillate finitely or doesn't tend to a finite limit or to  $\pm\infty$  as  $x \rightarrow \infty$ . In such a case,  $f(x)$  has an Oscillatory discontinuity at  $x = a$ .

**Example 2.1:** Examine the continuity of the function  $f(x)$  at  $x = \frac{3}{2}$  where

$$f(x) \begin{cases} 3 - 2x, & \text{if } 0 \leq x < \frac{3}{2} \\ -3 - 2x, & \text{if } x \geq \frac{3}{2} \end{cases}$$

$$\text{LHL } \lim_{x \rightarrow \frac{3}{2}^-} f(x) = \lim_{x \rightarrow \frac{3}{2}^-} \left\{ 3 - 2 \left( \frac{3}{2} \right) \right\} = 0$$

$$\text{RHL } \lim_{x \rightarrow \frac{3}{2}^+} f(x) = \lim_{x \rightarrow \frac{3}{2}^+} \left\{ -3 - 2 \left( \frac{3}{2} \right) \right\} = -6$$

$$f\left(\frac{3}{2}\right) = -3 - 2\left(\frac{3}{2}\right) = -6$$

$$\therefore \text{RHL} = f\left(\frac{3}{2}\right) \neq \text{LHL}$$

Thus  $f(x)$  has a mixed discontinuity and it's discontinuous on the left.