Laplace Transform

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1 Definition, Existence, and Basic Properties of the Laplace Transform

1.1 Definition and Existence

> **Definition 1.1.1: Laplace Transform**
>
> Let F be a real-valued function of the real variable t, defined for $t > 0$. Let s be a variable that we shall assume to be real, and consider the function f defined by
>
> $$f(s) = \int_0^\infty e^{-st} F(t)\, dt \tag{1}$$
>
> for all values of s for which this integral exists. The function f defined by the integral (1) is called the Laplace Transform of the function F. We shall denote the Laplace transform of F by $\mathscr{L}\{F(t)\}$.
>
> Thus the Laplace transform of a function f is given by
>
> $$\mathscr{L}\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\, dt = \lim_{R \to \infty} \int_0^R e^{-st} F(t)\, dt \tag{2}$$

Some ways to write Laplace transforms:
$$\mathscr{L} F(t) = f(s) = \int_0^\infty e^{-st} F(t)\, dt$$
$$\mathscr{L} G(t) = g(s)$$
$$\mathscr{L} u(t) = \tilde{u}(s)$$

$F(t)$	$\mathscr{L}\{F(t)\} = f(s)$	$F(t)$	$\mathscr{L}\{F(t)\} = f(s)$
1	$\dfrac{1}{s}$	n	$\dfrac{n}{s}$
t	$\dfrac{1}{s^2}$	t^n	$\dfrac{n!}{s^{n+1}}$
e^{at}	$\dfrac{1}{s-a}$	e^{-at}	$\dfrac{1}{s+a}$
$\sin at$	$\dfrac{a}{s^2 + a^2}$	$\sinh at$	$\dfrac{a}{s^2 - a^2}$
$\cos at$	$\dfrac{s}{s^2 + a^2}$	$\cosh at$	$\dfrac{s}{s^2 - a^2}$

Table 1: Functions and their Laplace Transform

Proofs :

Let $F(t) = n$, for $t > 0$
Then

$$\mathscr{L}\{n\} = \int_0^\infty e^{-st} \cdot n \, dt$$

$$= n \frac{-e^{st}}{s} \Big|_0^\infty$$

$$= \frac{n}{s} \quad \square$$

Let $F(t) = t$, for $t > 0$
Then

$$\mathscr{L}\{t\} = \int_0^\infty e^{-st} \cdot t \, dt$$

$$= -t \frac{e^{-st}}{s} + \int_0^\infty \frac{e^{-st}}{s} \, dt$$

$$= -e^{-st} \frac{t}{s} \Big|_0^\infty - e^{-st} \frac{1}{s^2} \Big|_0^\infty$$

$$= \frac{1}{s^2} \quad \square$$

Let $F(t) = t^n$, for $t > 0$
Then

$$\mathscr{L}\{t^n\} = \int_0^\infty e^{-st} t^n \, dt$$

$$= -t^n \frac{e^{st}}{s} + \int_0^\infty n t^{n-1} \frac{e^{-st}}{s} \, dt$$

$$= -n t^{n-1} \frac{e^{-st}}{s^2} \Big|_0^\infty + \int_0^\infty n(n-1) t^{n-2} \frac{e^{-st}}{s^2} \, dt$$

$$= -n(n-1) t^{n-2} \left(\frac{e^{-st}}{s^3} \right) \Big|_0^\infty + \int_0^\infty n(n-1)(n-2) t^{n-3} \frac{e^{-st}}{s^3} \, dt$$

$$= \cdots$$

$$= n! t^{n-n} \frac{e^{-st}}{s^{n+1}} \Big|_0^\infty + \int_0^\infty n(n-1) \cdots (n-n) \frac{e^{-st}}{s^{n+1}} \, dt$$

$$= \frac{n!}{s^{n+1}} \quad \square$$

Let $F(t) = e^{at}$, for $t > 0$
Then

$$\mathscr{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at} \, dt$$

$$= \int_0^\infty e^{(a-s)t} \, dt$$

$$= \frac{e^{(a-s)t}}{a-s} \Big|_0^\infty$$

$$= \frac{1}{s-a} \quad \square$$

Let $F(t) = e^{-at}$, for $t > 0$
Then

$$\mathscr{L}\{e^{-at}\} = \int_0^\infty e^{-st} e^{-at} \, dt$$

$$= \int_0^\infty e^{-(a+s)t} \, dx$$

$$= \frac{e^{-(a+s)t}}{s+a} \Big|_0^\infty$$

$$= \frac{1}{s+a} \quad \square$$

Let $F(t) = \sin at$, for $t > 0$

Then

$$\mathcal{L}\{\sin at\} = \int_0^\infty e^{-st \sin at}\, dt$$

$$= -\frac{e^{-st}}{s^2 + a^2}(s \sin at + a \cos at)\Big|_0^\infty$$

$$= \frac{a}{s^2 + a^2} \quad \square$$

Let $F(t) = \cos at$, for $t > 0$

Then

$$\mathcal{L}\{\cos at\} = \int_0^\infty e^{-st} \cos at\, dt$$

$$= \frac{e^{-st}}{s^2 + a^2}(-s \cos at + a \sin at)\Big|_0^\infty$$

$$= \frac{s}{s^2 + a^2} \quad \square$$

Theorem 1.1.2: Hypothesis: *Let F be a real function that has the following properties:*

1. *F is a piecewise continuous in every finite closed interval $0 \le t \le a$ $(b > 0)$.*

2. *F is of exponential order, i.e, there exists α, $M > 0$, and $t_0 > 0$ such that*

$$e^{-\alpha t}|F(t)| < M \text{ for } t > t_0$$

Conclusion: *The Laplace transform of F exists for $s > \alpha$.*

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt$$

1.2   Basic Properties of the Laplace Transform

Theorem 1.2.1 (The Linear Property):
Let F_1 and F_2 be functions whose Laplace transform exist, and let c_1 and c_2 be constants. Then
$$\mathcal{L}\{c_1 F_1(t) + c_2 F_2(t)\} = c_1 \mathcal{L}\{F_1(t)\} + c_2 \mathcal{L}\{F_2(t)\}$$

Example 1.1:
$$\mathcal{L}\{4t^2 - 3\cos 2t + 5e^{-t}\}$$

$$\mathcal{L}\{4t^2 - 3\cos 2t + 5e^{-t}\} = 4\mathcal{L}\{t^2\} - 3\mathcal{L}\{\cos 2t\} + 5\mathcal{L}\{e^{-t}\}$$

$$= 4 \cdot \frac{2!}{s^3} - 3 \cdot \frac{s}{s^2 + 4} + 5 \cdot \frac{1}{s + 1}$$

$$= \frac{8}{s^3} - \frac{3s}{s^2 + 4} + \frac{5}{s + 1}$$

Example 1.2:
$$\mathcal{L}\{e^{-t}\cos 2t\}$$

Since $\mathcal{L}\{\cos 2t\} = \dfrac{s}{s^2+4}$, we have

$$\mathcal{L}\{e^{-t}\cos 2t = \frac{s+1}{(s+1)^2+4} = \frac{s+1}{s^2+2s+5}\}$$

Theorem 1.2.3 (Second translation or Shifting property):
If
$$\mathcal{L}\{F(t)\} = f(s)$$
and
$$G(t) = \begin{cases} F(t-a) & for\ t > a \\ 0 & for\ t < a \end{cases}$$
then
$$\mathcal{L}\{G(t)\} = e^{-as}f(s)$$

Example 1.3: Find $\mathcal{L}\{G(t)\}$ where $G(t) = \begin{cases} \cos t - \frac{2\pi}{3} & \textbf{for } t > \frac{2\pi}{3} \\ 0 & \textbf{for } t < \frac{2\pi}{3} \end{cases}$

$$\mathcal{L}\{G(t)\} = e^{-\frac{2\pi}{3}s} \cdot \mathcal{L}\{\cos t\}$$
$$= e^{-\frac{2\pi}{3}s} \cdot \frac{s}{s^2+1}$$
$$= \frac{se^{-\frac{2\pi}{3}s}}{s^2+1}$$

Theorem 1.2.4 (Change of scale property):
If
$$\mathcal{L}\{F(t)\} = f(s)$$
then
$$\mathcal{L}\{F(at)\} = \frac{1}{a}f\left(\frac{s}{a}\right)$$

Example 1.4: Evaluate $\mathcal{L}\{\sin 3t\}$

$$\mathcal{L}\{\sin 3t\} = \frac{1}{3} \cdot \frac{1}{\left(\frac{s}{3}\right) + 1}$$

$$= \frac{1}{3} \cdot \frac{3^2}{s^2 + 3^2}$$

$$= \frac{3}{s^2 + 9}$$

***Theorem 1.2.5* (Multiplication by t^n):**
If
$$\mathcal{L}\{F(t)\} = f(s)$$
then
$$\mathcal{L}\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s)$$

Example 1.5: Find $\mathcal{L}\{t^2 \cos at\}$

$$\mathcal{L}\{t^2 \cos at\} = (-1)^2 \cdot \frac{d^2}{dx^2} \left(\frac{s}{s^2 + a^2} \right)$$

$$= \frac{d \left[\frac{s^2 + a^2 - 2s^2}{(s^2 + a^2)^2} \right]}{ds}$$

$$= \frac{d \left[\frac{a^2 - s^2}{(s^2 + a^2)^2} \right]}{ds}$$

$$= \frac{(s^2 + a^2)^2(-2s) - (-s^2 + a^2)2(s^2 + a^2) \cdot 2s}{(s^2 + a^2)^4}$$

$$= \frac{2s^3 - 6a^2 s}{(s^2 + a^2)^3}$$

***Theorem 1.2.6* (Division by t):**
If
$$\mathcal{L}\{F(t)\} = f(s)$$
then
$$\mathcal{L}\left\{ \frac{F(t)}{t} \right\} = \int_s^\infty f(x)\, dx$$

Example 1.6: Find $\mathcal{L}\left\{ \frac{\sin t}{t} \right\}$

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{s^2 + 1}\, du$$

$$= \tan^{-1} u \Big|_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s$$

$$= \tan^{-1}\frac{1}{s}$$

Theorem 1.2.7 **(Laplace transform of Integral):**

If
$$\mathcal{L}\{F(t)\} = f(s)$$

then
$$\mathcal{L}\left\{\int_0^t F(x)\, dx\right\} = \frac{1}{s} f(s)$$

Example 1.7: Evaluate $\mathcal{L}\{\int_0^t \frac{\sin u}{u}\, du\}$

$$\mathcal{L}\left\{\frac{\sin u}{u}\right\} = \tan^{-1}\frac{1}{s}$$

$$\mathcal{L}\left\{\int_0^t \frac{\sin u}{u}\, du\right\} = \frac{f(s)}{s} = \frac{1}{s}\tan^{-1}\frac{1}{s}$$

Example 1.8: Evaluate $\mathcal{L}\{\int_0^t \frac{\sin u}{u}\, du\}$

Let $F(t) = \frac{\sin t}{t}$

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{u^2 + 1}\, du$$

$$= \tan^{-1} u \Big|_s^\infty$$

$$= \tan^{-1}\frac{1}{s}$$

$$\therefore \mathcal{L}\left\{\int_0^t \frac{\sin u}{u}\, du\right\} = \frac{1}{s}\cdot \tan^{-1}\frac{1}{s}$$

Example 1.9: Evaluate $\mathcal{L}\{\int_0^t \sin 2u\, du\}$

$$\mathcal{L}\{\sin 2t\} = \frac{2}{s^2 + 4}$$

$$\therefore \mathscr{L}\left\{\int_0^t \sin 2u \, du\right\} = \frac{2}{s^3 + 4s}$$

Theorem 1.2.8 (Periodic functions):

If

$$\mathscr{L}\{F(t)\} = f(s)$$

then

$$\mathscr{L}\{F(t)\} = \frac{1}{1 - e^{-Ts}} \int_0^T e^{-st} F(t) \, dt$$

where T is the period of $F(t)$.

Example 1.10: Find $\mathscr{L}\{F(t)\}$ for

$$F(t) = \begin{cases} \sin t & \textbf{for } 0 < t < \pi \\ 0 & \textbf{for } \pi < t < 2\pi \end{cases}$$

$$\mathscr{L}\{F(t)\} = \frac{1}{1 - e^{-2\pi s}} \int_0^{2\pi} e^{st} \sin t \, dt$$

$$= \frac{1}{1 - e^{-2\pi s}} \int_0^{\pi} e^{-st} \sin t \, dt$$

Example 1.11: Find $\mathscr{L}\{F(t)\}$ for

$$F(t) = \begin{cases} t & \textbf{for } 0 < t < 1 \\ 0 & \textbf{for } 1 < t < 2 \end{cases}$$

$$\mathscr{L}\{F(t)\} = \frac{1}{1 - e^{-2s}} \int_0^1 t \, dt$$

$$= \frac{t^2 \Big|_0^1}{2 - 2e^{-2s}}$$

$$= \frac{1}{2 - 2e^{-2s}}$$

Theorem 1.2.9 (Laplace transform of derivatives):
If
$$\mathcal{L}\{F(t)\} = f(s)$$
then
$$\mathcal{L}\{F'(t)\} = sf(s) - F(0)$$
$$\mathcal{L}\{F''(t)\} = s^2 F(s) - sF(0) - F'(0)$$
$$\mathcal{L}\{F'''(t) = s^3 f(s) - s^2 F(0) - sF'(0) - F''(0)\}$$
$$\boxed{\mathcal{L}\{F^{(n)}(t)\} = s^n f(s) - s^{n-1} F(0) - s^{n-2} F'(0) - \cdots - F^{(n-1)}(0)}$$

$$\boxed{\mathcal{L}\{F^{(n)}(t)\} = s^n f(s) - \sum_{i=n-1}^{0} \sum_{j=0}^{n-1} s^i F^{(j)}(0)}$$

Example 1.12