Solution of differential Equations with Constant Coefficient

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S. No.	*Corresponding part of C.F.*	*Nature of roots of auxiliary equation (A.E)*
1.	(*i*) One real root m_1	$c_1e^{m_1x}$
	(*ii*) Two real and different roots m_1, m_2	$c_1e^{m_1x} + c_2e^{m_2x}$
	(*iii*) Three real and different roots m_1, m_2, m_3	$c_1e^{m_1x} + c_2e^{m_2x} + c_3e^{m_3x}$
2.	(*i*) Two real and equal roots m_1, m_1	$(c_1 + c_2x)\, e^{m_1x}$
	(*ii*) Three real and equal roots m_1, m_1, m_1	$(c_1 + c_2x + c_3x^2)\, e^{m_1x}$
3.	(*i*) One pair of complex roots $\alpha \pm i\beta$	$e^{\alpha x}\,(c_1 \cos \beta x + c_2 \sin \beta x)$ or $c_1e^{\alpha x} \cos (\beta x + c_2)$ or $c_1\, e^{\alpha x} \sin (\beta x + c_2)$
	(*ii*) Two pairs of complex and equal roots $\alpha \pm i\beta, \alpha \pm i\beta$	$e^{\alpha x}\,[(c_1 + c_2x) \cos \beta x + (c_3 + c_4x) \sin \beta x]$
4.	(*i*) One pair of surd roots $\alpha \pm \sqrt{\beta}$	$e^{\alpha x}\,(c_1 \cosh x\sqrt{\beta} + c_2 \sinh x\sqrt{\beta})$ or $c_1\, e^{\alpha x} \cosh (x\sqrt{\beta} + c_2)$ or $c_1e^{\alpha x} \sinh (x\sqrt{\beta} + c_2)$
	(*ii*) Two pairs of surd and equal roots $\alpha \pm \sqrt{\beta}, \alpha \pm \sqrt{\beta}$	$e^{\alpha x}\,[(c_1 + c_2x) \cosh x\sqrt{\beta} + (c_3 + c_4x) \sinh x\sqrt{\beta}]$

Ex. 2. *Solve* $(D^3 + 3D^2 + 3D + 1)\, y = 0$ **[Delhi Maths. (G) 1994]**

Sol. The auxiliary equation is $D^3 + 3D^2 + 3D + 1 = 0$ or $(D + 1)^3 = 0 \Rightarrow -1, -1, -1.$

$\therefore$ The required solution is $y = (c_1 + c_2x + c_3x^2)\, e^{-x}$, c_1, c_2, c_3 being arbitrary constants.

Ex. 3 *Solve* $(d^4y/dx^4) - (d^3y/dx^3) - 9(d^2y/dx^2) - 11(dy/dx) - 4y = 0.$ **[Delhi Maths. (G) 1997]**

Sol. Let $D = d/dx$. Then the given equation can be written as

$(D^4 - D^3 - 9\, D^2 - 11\, D - 4)\, y = 0$ or $(D + 1)^3\, (D - 4) = 0$ so that $D = 4, -1, -1, -1.$

$\therefore$ The required solution is $y = c_1e^{4x} + (c_2 + c_3x + c_4x^2)\, e^{-x}$, c_1, c_2, c_3, c_4 being arbitrary constants.

Ex. 4. *Solve* **(a)** $(D^4 - 5D^2 + 4)\, y = 0$

(b) $(D^4 + 2D^3 - 3D^2 - 4D + 4)\, y = 0$

(c) $(D^3 - 3D^2 + 2D)\, y = 0$

Sol. (a) Here auxiliary equation is $D^4 - 5D^2 + 4 = 0$

or $(D^2 - 4)\, (D^2 - 1) = 0$ or $D^2 = 4$ or 1 so that $D = 2, -2, 1, -1.$

$\therefore$ The required general solution is $y = c_1e^{2x} + c_2e^{-2x} + c_3e^x + c_4e^{-x}$,

c_1, c_2, c_3, c_4 being arbitrary constants

(b) Here auxiliary equation is $D^4 + 2D^3 - 3D^2 - 4D + 4 = 0$ or $(D - 1)\, (D^3 + 3D^2 - 4) = 0$

or $(D - 1)\, \{(D - 1)\, (D^2 + 4D + 4)\}$ or $(D - 1)^2\, (D + 2)^2 = 0$ so that $D = 1, 1, -2, -2.$

$\therefore$ The required solution is $y = (c_1 + c_2x)\, e^x + (c_3 + c_4x)^{-2x}$, c_1, c_2, c_3 being arbitrary constants.

(*c*) Here the auxiliary equation is $D^3 - 3D^2 + 2D = 0$ or $D(D^2 - 3D + 2) = 0$

or $D(D-1)(D-2) = 0$ so that $D = 0, 1, 2.$

Hence the required solution is $y = c_1 e^{0x} + c_2 e^x + c_3 e^{2x}$

or $y = c_1 + c_2 e^x + c_3 e^{2x}$, c_1, c_2, c_3 being arbitrary constants

Ex. 5. *Solve* $(D^3 - 8)\,y = 0.$

Sol. (*a*) Here auxiliary equation is $D^3 - 8 = 0$ or $(D-2)(D^2 + 2D + 4) = 0$ so that

$D = 2$ or $D = \{-2 \pm (4-16)^{1/2}\}/2$ or $D = 2, -1 \pm i\sqrt{3}$.

$\therefore$ The required solution is $y = c_1 e^{2x} + e^{-x}\{c_2 \cos(x\sqrt{3}) + c_3 \sin(x\sqrt{3})\}$,

c_1, c_2, c_3 being arbitrary constants

Ex. 6. *Solve* (*i*) $d^4 y/dx^4 + m^4 y = 0$

(*ii*) $d^4 y/dx^4 + y = 0$ **[(I.A.S.Prel 2001; Agra 2006]**

Sol. (*i*) Let $D \equiv d/dx$. Then, the given equation can be rewritten as $(D^4 + m^4)\,y = 0$

Its auxiliary equation is $D^4 + m^4 = 0$ or $(D^2 + m^2)^2 - (\sqrt{2}D\,m)^2 = 0$

or $(D^2 + m^2 + \sqrt{2}Dm)(D^2 + m^2 - \sqrt{2}Dm) = 0 \Rightarrow D^2 + m^2 + \sqrt{2}\,Dm = 0$ or $D^2 + m^2 - \sqrt{2}\,Dm = 0$

$\therefore$ $D = \{-\sqrt{2}m \pm (2m^2 - 4m^2)^{1/2}\}/2 = -(m/\sqrt{2}) \pm i\,(m/\sqrt{2})$,

and $D = \{\sqrt{2}m \pm (2m^2 - 4m^2)^{1/2}\}/2 = m/\sqrt{2} \pm i\,(m/\sqrt{2})$

Hence the required general solution is $y = e^{-(mx/\sqrt{2})}\,\{c_1 \cos(mx/\sqrt{2}) + c_2 \sin(mx/\sqrt{2}\}$

$+ e^{mx/\sqrt{2}}\left\{c_3 \cos(mx/\sqrt{2}) + c_4 \sin(mx/\sqrt{2})\right\}$, c_1, c_2, c_3, c_4 being arbitrary constants.

(*ii*) This is a particular case of part (*i*). Here $m = 1$. Solution is

$y = e^{-(x/\sqrt{2})}\,\{c_1 \cos(x/\sqrt{2}) + c_2 \sin(x/\sqrt{2})\} + e^{x/\sqrt{2}}\,\{c_3 \cos(x/\sqrt{2}) + c_4 \sin(x/\sqrt{2})\}.$

...

Ex. 15 (*b*). *Find the solution of* $(d^2i/dt^2) + (R/L)(di/dt) + (1/LC)\, i = 0$, *where* $R^2C = 4L$ *and* R, C, L *are constants.*

Sol. Let $D \equiv d/dt$. Then the given equation can be written as $[(D^2 + (R/L)\, D + (1/LC)]\, i = 0$.

Here the auxiliary equation is $$D^2 + (R/L)\, D + (1/LC) = 0$$

so that $$D = [-(R/L) \pm \{(R^2/L^2) - (4/LC)\}^{1/2}]/2 = -(R/2L), \text{ as } R^2C = 4L$$

Thus, $D = -(R/2L)$ (twice). Hence the required general solution is

$$y = (c_1 + c_2 t)\, e^{-t\,(R/2L)},\ c_1,\ c_2 \text{ being arbitrary constants.}$$

H.W. 7-12

Ex. 1. *Solve the following differential equations :*

(*a*) $(D^2 - 3D + 2)\, y = e^{3x}$. **[I.A.S. (Preliminary) 1993, Meerut 1994]**

(*b*) $(4D^2 + 12D + 9)\, y = 144\, e^{-3x}$. **[Rohilkhand 1992, 93]**

(*c*) $[D^2 + 2pD + (p^2 + q^2)]\, y = e^{ax}$.

(*d*) $D^2\, (D + 1)^2\, (D^2 + D + 1)^2\, y = e^x$

Sol. (*a*) Here the auxiliary equation is $D^2 - 3D + 2 = 0$ so that $D = 1, 2$

$\therefore$ C.F. $= c_1 e^x + c_2 e^{2x}$, c_1, c_2 being arbitrary constants.

and
$$\text{P.I.} = \frac{1}{D^2 - 3D + 2} e^{3x} = \frac{1}{3^3 - (3 \times 3) + 2} e^{3x} = \frac{1}{2} e^{3x}$$

$\therefore$ The required general solution is $y = c_1 e^x + c_2 e^{2x} + (1/2)\, e^{3x}$.

(*b*) Here the A.E. is $(2D + 3)^2 = 0$ so that $D = -3/2, -3/2$

$\therefore$ C.F. $= (c_1 + c_2 x)\, e^{-3x/2}$, c_1, c_2 being arbitrary constants.

$$\text{and P.I.} = \frac{1}{4D^2 + 12D + 9} 144 e^{-3x} = 144 \frac{1}{(2D + 3)^2} e^{-3x} = \frac{144}{(-6 + 3)^2} e^{-3x} = 16 e^{-3x}$$

Hence the required solution is $y = (c_1 + c_2 x)\, e^{-3x/2} + 16\, e^{-3x}$.

(*c*) Here the auxiliary equation is $D^2 + 2pD + (p^2 + q^2) = 0$

Solving
$$D = \frac{-2p \pm \sqrt{4p^2 - 4(p^2 + q^2)}}{2} = -p \neq iq \text{ (complex roots)}$$

$\therefore$ C.F. $= e^{-px}(c_1 \cos qx + c_2 \sin qx)$, c_1, c_2 being arbitrary constants

and $$\text{P.I.} = \frac{1}{D^2 + 2pD + (p^2 + q^2)} e^{ax} = \frac{1}{a^2 + 2pa + p^2 + q^2} e^{ax} = \frac{e^{ax}}{(p+a)^2 + q^2}$$

$\therefore$ Required solution is $$y = e^{-p}(c_1 \cos qx + c_2 \sin qx) + e^{ax}/\{(p+a)^2 + q^2\}$$

(*d*) Here the auxiliary equation is $$D^2 (D+1)^2 (D^2 + D + 1)^2 = 0$$

Solving, we get $D = 0, \quad 0, \quad -1, \quad -1, \quad -(1/2) \pm i(\sqrt{3}/2), \quad -(1/2) \pm i(\sqrt{3}/2).$

Ex. 2. *Solve the following differential equations :*

(a) $(4D^2 - 12D + 9)\, y = 144\, e^{3x/2}$.

(b) $(D^2 + 4D + 4)\, y = e^{2x} - e^{-2x}$ or $(D^2 + 4D + 4)\, y = 2 \sinh 2x$.

Sol. (a) Here the auxiliary equation is $4D^2 - 12D + 9 = 0$.

or $(2D - 3)^2 = 0$ so that $D = 3/2, 3/2$.

$\therefore$ C.F. $= (c_1 + c_2 x)\, e^{3x/2}$, c_1, c_2 being arbitrary constants.

and
$$\text{P.I.} = \frac{1}{(4D^2 - 12D + 9)} 144\, e^{3x/2} = 144 \frac{1}{(2D-3)} e^{3x/2} = \frac{144}{4} \frac{1}{\{D-(3/2)\}^2} e^{3x/2}$$

$$= 36 \cdot \frac{x^2}{2!} e^{3x/2}, \text{ as } \frac{1}{(D-a)^n} e^{ax} = \frac{x^n}{n!} e^{ax}$$

$\therefore$ Solution is $y = (c_1 + c_2 x)\, e^{3x/2} + 18x^2\, e^{3x/2}$. c_1, c_2, being arbitrary constants.

(b) Given equation is $(D^2 + 4D + 4)\, y = 2 \sinh 2x$.

or $(D + 2)^2\, y = e^{2x} - e^{-2x}$, as $\sinh 2x = (e^{2x} - e^{-2x})/2$.

Here the auxiliary equation is $(D + 2)^2 = 0$ so that $D = -2, -2$

$\therefore$ C.F. $= (c_1 + c_2 x)\, e^{-2x}$, c_1, c_2 being arbitrary constants.

and
$$\text{P.I.} = \frac{1}{(D+2)^2}(e^{2x} - e^{-2x}) = \frac{1}{(D+2)^2} e^{2x} - \frac{1}{(D+2)^2} e^{-2x}$$

$$= \frac{1}{(2+2)^2} e^{2x} - \frac{x^2}{2!} e^{-2x} = \frac{1}{16} e^{2x} - \frac{x^2}{2} e^{-2x}.$$

Hence the required solution is $y = (c_1 + c_2 x)\, e^{-2x} + (1/16) \times e^{2x} - (x^2/2) \times e^{-2x}$

H.W: 3,4,6,8

Ex. 1. *Solve the following differential equations*

(a) $(D^2 + 1)\, y = \cos 2x$

(b) $(D^2 + 9)\, y = \cos 4x$.

Sol. (*a*) Here the auxiliary eqution is $\quad D^2 + 1 = 0 \quad$ so that $\quad D = \pm\, i$,

$\therefore$ C.F. $= c_1 \cos x + c_2 \sin x$, c_1, c_2 being arbitrary constants.

Now, $\quad$ P.I. $= \dfrac{1}{D^2+1}\cos 2x = \dfrac{1}{-2^2+1}\cos 2x = -\dfrac{1}{3}\cos 2x.$

The required general solution is $\quad y = c_1 \cos x + c_2 \sin x - (1/3) \cos 2x.$

(*b*) Try yourself. **Ans.** $y = c_1 \cos 3x + c_2 \sin 3x - (1/7) \cos 4x.$

Ex. 2. (*a*) *Solve* $(D^2 - 3D + 2)\, y = \sin 3x$. **[Delhi Maths (G) 1996]**

(*b*) $(D^2 - 4D + 4)\, y = \sin 2x$. **[G.N.D.U. (Amritsar) 2010]**

Sol. (***a***) Here auxiliary equation $\quad D^2 - 3D + 2 = 0 \quad$ gives $\quad D = 1, 2.$

C.F. $= c_1 e^x + c_2 e^{2x}$, c_1, c_2 being arbitrary constants.

and $\quad$ P.I. $= \dfrac{1}{D^2 - 3D + 2}\sin 3x = \dfrac{1}{-3^2 - 3D + 2}\sin 3x$

$$= -\frac{1}{3D+7}\sin 3x = -(3D-7)\frac{1}{(3D-7)(3D+7)}\sin 3x$$

$$= -(3D-7)\frac{1}{9D^2-49}\sin 3x = -(3D-7)\frac{1}{9(-3^2)-49}\sin 3x$$

$$= (1/130) \times (3D-7)\sin 3x = (1/130) \times (9\cos 3x - 7\sin 3x).$$

$\therefore$ Solution is $\quad y = c_1 e^x + c_2 e^{2x} + (1/130) \times (9\cos 3x - 7\sin 3x).$

Ex. 4. *Solve the following differential equations:*

(a) $(D^3 + a^2 D)\, y = \sin ax$ **[I.A.S. Prel. 2006, Rajsthan 2010, Purvanchal 1999]**

(b) $(D^3 + 9D)\, y = \sin 3x.$

(c) $(d^3x/dy^3) + b^2(dx/dy) = \sin by.$

Sol. *(a)* Here auxiliary equation is $D^3 + a^2 D = 0$ so that $D = 0, 0 \pm ia.$

$\therefore$ C.F. $= c_1 e^{0.x} + e^{0.x}(c_2 \cos ax + c_3 \sin ax) = c_1 + c_2 \cos ax + c_3 \sin ax,$

where c_1, c_2 and c_3 arbitrary constants.

$$\text{P.I.} = \frac{1}{D^3+a^2D}\sin ax = \frac{1}{D^2+a^2}\frac{1}{D}\sin ax = \frac{1}{D^2+a^2}\left(-\frac{1}{a}\cos ax\right)$$

$$= -\frac{1}{a}\left[\text{Real part of } \frac{1}{D^2+a^2}(\cos ax + i\sin ax)\right]$$

$$= -\frac{1}{a}\left[\text{Real part of } \frac{1}{D^2+a^2}e^{iax}\right], \text{ by Euler's theorem}$$

$$= -\frac{1}{a}\left[\text{Real part of } \left(\frac{x}{2a}\sin ax - \frac{ix}{2a}\text{as } ax\right)\right]$$

$$\left[\text{As in Ex. 3. } (a), \text{ prove that } \frac{1}{D^2+a^2}e^{iax} = \frac{x}{2a}\sin ax - \frac{ix}{2a}\cos ax\right]$$

$$= -(1/a)(x/2a)\sin ax = -(x/2a^2)\sin ax.$$

Hence the general solution is $y = c_1 + c_2 \cos ax + c_3 \sin ax - (x/2a^2)\sin ax.$

(b) Do as in part *(a)* **Ans.** $y = c_1 + c_2 \cos 3x + c_3 \sin 3x - (x/18)\sin 3x.$

(c) Proceed as in part *(a)*. Here y is independent and x is dependent variable

Ans. $x = c_1 + c_2 \cos by + c_3 \sin by - (y/2b^2)\sin by$

H.W: 5,6,8,11

Ex. 1. *Solve* $(D^4 - D^2)\, y = 2.$ **[Agra 2005]**

Sol. Here auxiliary equation is $D^4 - D^2 = 0$ or $D^2(D^2 - 1) = 0 \Rightarrow D = 0, 0, 1, -1$

$\therefore$ C.F. $= (c_1 + c_2 x)\, e^{0.x} + c_3 e^x + c_4 e^{-x} = c_1 + c_2 x + c_3 e^x + c_4 e^{-x}$,

where c_1, c_2, c_3 and c_4 are arbitrary constants.

$$\text{P.I.} = \frac{1}{D^4 - D^2}\, 2 = -\frac{2}{D^2}\frac{1}{(1-D^2)}\, 1 = -\frac{2}{D^2}(1 - D^2)^{-1} \cdot 1$$

$$= -\frac{2}{D^2}(1 + D^2 + D^4 + \ldots) \cdot 1 = -\frac{2}{D^2}\, 1 = -\frac{2}{D}\, x = (-2) \times \frac{x^2}{2} = -x^2$$

$\therefore$ Solution is $\quad y = c_1 + c_2 x + c_3 e^x + c_4 e^{-x} - x^2$

Ex. 2. *Find the particular integral of* $(D^2 + D)\, y = x^2 + 2x + 4.$ **[I.A.S. Prel. 1994]**

Sol. The required particular integral

$$= \frac{1}{D^2 + D}(x^2 + 2x + 4) = \frac{1}{D(1+D)}(x^2 + 2x + 4) = \frac{1}{\mathrm{D}}(1 + \mathrm{D})^{-1}(x^2 + 2x + 4)$$

$= (1/D)\,(1 - D + D^2 - D^3 + \ldots)\,(x^2 + 2x + 4)$

$= (1/D)\,[(x^2 + 2x + 4) - D\,(x^2 + 2x + 4) + D^2\,(x^2 + 2x + 4)]$

$= (1/D)\,[x^2 + 2x + 4 - (2x + 2) + 2] = (1/D)\,(x^2 + 4) = x^3/3 + 4x.$

Ex. 4. *Solve* $(D^3 + 8)\, y = x^4 + 2x + 1.$ **[Delhi Maths Hons. 1998]**

Sol. Here the auxiliary equation is $D^3 + 2^3 = 0$ or $(D + 2)\,(D^2 - 2D + 4) = 0$

so that $D = -2,\ \{2 \pm (4 - 16)^{1/2}\}/2 = -2,\ 1 \pm i\sqrt{3}$.

$\therefore$ C.F. $= c_1 e^{-2x} + e^{x}\,(c_2 \cos x\sqrt{3} + c_3 \sin x\sqrt{3})$, c_1, c_2, c_3 being arbitrary constans

$$\therefore \quad \text{P.I.} = \frac{1}{D^3+8}(x^4+2x+1) = \frac{1}{8(1+D^3/8)}(x^4+2x+1)$$

$$= \frac{1}{8}\left(1+\frac{D^3}{8}\right)^{-1}(x^4+2x+1) = \frac{1}{8}\left(1-\frac{D^3}{8}+\ldots\right)(x^4+2x+1)$$

$$= (1/8) \times [\,(x^4 + 2x + 1) - (1/8) \times D^3(x^4 + 2x + 1)]$$

$$= (1/8) \times [(x^4 + 2x + 1) - (1/8)\,(24x)] = (1/8) \times (x^4 + 2x + 1 - 3x) = (x^4 - x + 1)/8.$$

$\therefore$ Required solution is $y = c_1 e^{-2x} + e^{x}\,(c_2 \cos x\sqrt{3} + c_3 \sin x\sqrt{3}) + (x^4 - x + 1)/8.$

Ex. 5. *Solve (a)* $(D^2 + 2D + 2)\, y = x^2.$

(b) $(D^2 - 4D + 4)\, y = x^2.$

Sol. (*a*) Here the auxiliary equation is $D^2 + 2D + 2 = 0$ so that $D = -1 \pm i.$

$\therefore$ C.F. $= e^{-x}\,(c_1 \cos x + c_2 \sin x)$, c_1, c_2 being arbitrary constants

$$\text{and} \quad \text{P.I.} = \frac{1}{D^2+2D+2}x^2 = \frac{1}{2[1+(D+D^2/2)]}x^2 = \frac{1}{2}\{1+(D+D^2/2)\}^{-1}$$

$$= (1/2)\,\{1 - (D + D^2/2) + (D + D^2/2)^2 - \ldots\}x^2 = (1/2)\,\{1 - (D + D^2/2) + D^2 + \ldots\}x^2$$

$$= (1/2)\,\{1 - D + D^2/2 + \ldots\}x^2 = (1/2)\,(x^2 - 2x + 1)$$

$\therefore$ The required solution is $y = e^{-x}\,(c_1 \cos x + c_2 \sin x) + (x^2 - 2x + 1)/2.$

(*b*) Try yourself. **Ans.** $y = (c_1 + c_2 x)\, e^{2x} + (2x^2 + 4x + 3)/4.$

H.W: 5,7,11

Ex. 1. *Solve* $(D^2 - 2D + 1)\, y = x^2 e^{3x}$.

[Purvanchal 2007, Delhi 1993, 2005, 08; Agra 2003]

Sol. The auxiliary equation of the given equation $D^2 - 2D + 1 = 0 \Rightarrow D = 1, 1.$

$\therefore$ C.F. $= (c_1 + c_2 x)\, e^x$, c_1, c_2 being arbitrary constants.

$$\text{P.I.} = \frac{1}{D^2 - 2D + 1} x^2 e^{3x} = \frac{1}{(D-1)^2} x^2 e^{3x} = e^{3x} \frac{1}{(D+3-1)^2} x^2$$

$$= e^{3x} \frac{1}{(D+2)^2} x^2 = e^{3x} \frac{1}{4(1 + D/2)^2} x^2 = \frac{1}{4} e^{3x} \left(1 + \frac{D}{2}\right)^{-2} x^2$$

$$= \frac{1}{4} e^{3x} \left[1 - \frac{D}{2} + \frac{(-2)(-3)}{2!} \frac{D^2}{4} + \ldots\right] x^2 = \frac{1}{4} e^{3x} \left(1 - \frac{D}{2} + \frac{3}{4} D^2 + \ldots\right) x^2$$

Ex. 2. *Solve (a)* $(D^2 - 2D + 1)\, y = x^2 e^x$

(b) $(D^2 - 6D + 9)\, y = x^2\, e^{3x}$ **[G.N.D.U. Amritsar 2010]**

Sol. (*a*) Here the auxiliary equation $D^2 - 2D + 1 = 0 \Rightarrow D = 1, 1.$

$\therefore$ C.F. $= (c_1 + c_2 x)\, e^x$, c_1, c_2 being arbitrary constants.

and $$\text{P.I.} = \frac{1}{D^2 - 2D + 1} e^x x^2 = \frac{1}{(D-1)^2} e^x x^2 = e^x \frac{1}{(D+1-1)^2} x^2$$

$$= e^x \frac{1}{D^2} x^2 = e^x \frac{1}{D}\int x^2 dx = e^x \frac{1}{D}\frac{x^3}{3} = e^x \int \frac{x^3}{3} dx = \frac{e^x}{3}\cdot\frac{x^4}{4}.$$

Hence the required solution is $y = (c_1 + c_2 x)\, e^x + (1/12) \times x^4\, e^x.$

(*b*) Try yourself. **Ans.** $y = (c_1 + c_2 x)\, e^{3x} + (x^4/12) \times e^{3x}$

Ex. 3. *Find the particular solution of* $(D - 1)^2 y = e^x \sec^2 x \tan x.$ **[Kuvempa 2005]**

Sol. $$\text{P.I.} = \frac{1}{(D-1)^2} e^x \sec^2 x \tan x = e^x \frac{1}{(D+1-1)^2} \tan x \sec^2 x$$

$$= e^x \frac{1}{D}\int \tan x \sec^2 x\, dx = e^x \frac{1}{D}\left(\frac{\tan^2 x}{2}\right) = \frac{e^x}{2}\int \tan^2 x\, dx$$

$$= \frac{e^x}{2}\int (\sec^2 x - 1)\, dx = \frac{e^x}{2}(\tan x - x)$$

Ex. 4. *Solve* $(D - a)^2\, y = e^{ax} f'(x)$ **[Agra 2006]**

Sol. Here auxiliary equation of the given equation is $(D - a)^2 = 0$ so that $D = a, a.$

$\therefore$ C.F. $= (c_1 + c_2 x)\, e^{ax}$, c_1 and c_2 being arbitrary constants

$$\text{P.I.} = \frac{1}{(D-a)^2} e^{ax} f'(x) = e^{ax} \frac{1}{(D+a-a)^2} f'(x) = e^{ax} \frac{1}{D}\frac{1}{D} f'(x) = e^{ax} \frac{1}{D} f(x) = e^{ax}\int f(x)\, dx$$

$\therefore$ The required solution is $y = (c_1 + c_2 x)\, e^{ax} + e^{ax}\int f(x)\, dx$

H.W: 5,6,8,12,16

Ex. 10. *solve* $(D^5 - D)\, y = 12\, e^x + 8 \sin x - 2x.$

Sol. A.E. $D(D^4 - 1) = 0 \;\Rightarrow\; D(D^2 - 1)(D^2 + 1) = 0 \;\Rightarrow\; D = 0, 1, -1, \pm i.$

$\therefore$ C.F. $= c_1 + c_2 e^x + c_3 e^{-x} + c_4 \cos x + c_5 \sin x$, $c_1, \ldots, c_5$ being arbitrary constants

Now P.I. corresponding to $12e^x$

$$= 12 \frac{1}{(D-1)D(D+1)(D^2+1)} e^x = 12 \frac{1}{(D-1)\,1\cdot(1+1)(1+1)} e^x = 3 \frac{1}{D-1} e^x = 3 \cdot \frac{x}{1!} e^x = 3xe^x.$$

P.I. corresponding to $8 \sin x$ is

$$= 8 \frac{1}{(D^2+1)\, D\, (D^2-1)} \sin x = 8 \frac{1}{(D^2+1)\, D\, (-1^2-1)} \sin x = -4 \frac{1}{(D^2+1)} \left[\frac{1}{D} \sin x \right]$$

$$= 4 \frac{1}{D^2+1} \cos x = 4 \left(\frac{x}{2 \times 1} \sin x \right) = 2x \sin x \qquad \left[\because \frac{1}{D^2+a^2} \cos ax = \frac{x}{2a} \sin x \right]$$

P.I. corresponding to $(-2x)$ is

$$= -2 \frac{1}{D(D^2-1)(D^2+1)} x = 2 \frac{1}{D(1-D^2)(1+D^2)} x = 2 \frac{1}{D} (1-D^2)^{-1} (1+D^2)^{-1} x$$

$$= 2 \frac{1}{D} (1 + D^2 + \ldots)(1 - D^2 + \ldots)\, x = 2 \frac{1}{D} (1 + D^2 - D^2 + \ldots)\, x = 2 \frac{1}{D} x = 2 \left(\frac{x^2}{2} \right) = x^2.$$

$\therefore$ Solution is $\quad y = c_1 + c_2 e^x + c_3 e^{-x} + c_4 \cos x + c_5 \sin x + 3\, xe^x + 2x \sin x + x^2.$