

# Fourier Analysis

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# 1 Introduction

## 1.1 Periodic Functions

### Definition 1.1.1: Periodic Functions

A function  $f(x)$  is said to be have a *period*  $P$  or to be *periodic* with period  $P$  if for all  $x$ ,  $f(x + P) = f(x)$  where  $P$  is a positive constant. The least value of  $P > 0$  is called the *least period* or simply the *period* of  $f(x)$ .

### Example 1.1: Some examples of periodic functions

1.  $\sin x$  has periods  $2\pi, 4\pi, 6\pi, \dots$  and  $-\pi, -3\pi, -5\pi, \dots$  and hence the least period is  $2\pi$ .
2.  $\cos x$  has the least period  $2\pi$ .
3.  $\tan x$  has the least period  $\pi$ .

Some other examples:

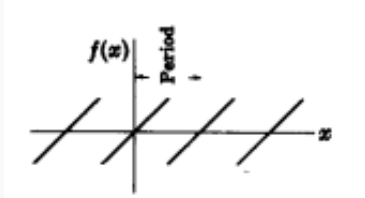


Figure 1.1.1

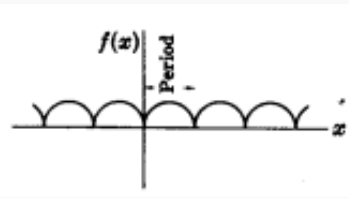


Figure 1.1.2

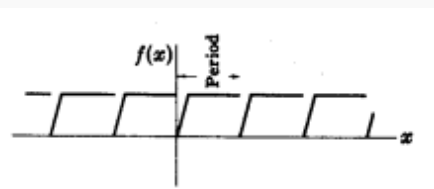


Figure 1.1.3

## 1.2 Piecewise Continuous Functions

### Definition 1.2.1: Piecewise Continuous Functions

A function  $f(x)$  is said to be *piecewise continuous* in the interval  $[a, b]$  if  $f(x)$  is continuous in the interval  $(a, b)$  and has a finite number of finite discontinuities in the interval  $[a, b]$ .

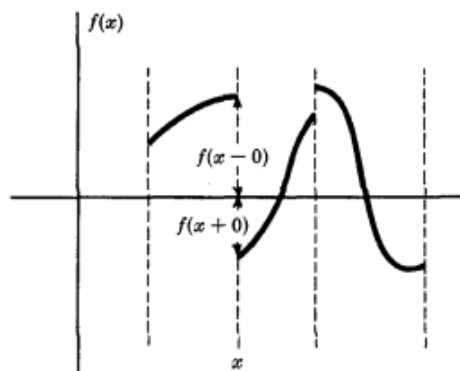


Figure 1.2.1

The right-hand limit of  $f(x)$  is often denoted by  $\lim_{\epsilon \rightarrow 0} f(x + \epsilon) = f(x + 0)$ , where  $\epsilon > 0$ .

Similarly, the left-hand limit of  $f(x)$  is denoted by  $\lim_{\epsilon \rightarrow 0} f(x - \epsilon) = f(x - 0)$ , where  $\epsilon > 0$ . The values of  $f(x + 0)$  and  $f(x - 0)$  at the point  $x$  in (1.2.1) are as indicated.

## 2 Fourier Expansion

### 2.1 Definition

#### Definition 2.1.1: Fourier Expansion

Let  $f(x)$  be defined in the interval  $(-L, L)$  and determined outside of this interval by  $f(x+2L) = f(x)$ , i.e. assume that  $f(x)$  has the period  $2L$ . The *Fourier series* or *Fourier expansion* corresponding to  $f(x)$  is defined to be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos \left( \frac{n\pi x}{L} \right) + b_n \sin \left( \frac{n\pi x}{L} \right) \right] \quad (2.1.1)$$

where the *Fourier coefficients*  $a_n$  and  $b_n$  are given by

$$\begin{cases} a_0 = \frac{1}{L} \int_{-L}^L f(x) dx \\ a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \left( \frac{n\pi x}{L} \right) dx \\ b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \left( \frac{n\pi x}{L} \right) dx \end{cases} \quad n = 0, 1, 2, \dots \quad (2.1.2)$$

### 2.2 Some pre-derivations

$$\begin{aligned} I &= \int_{-L}^L \sin^2 \frac{n\pi x}{L} dx \\ &= \sin \frac{n\pi x}{L} \cdot \frac{L}{n\pi} (\cos n\pi - \cos n\pi) + \frac{n\pi}{L} \cdot \frac{L}{n\pi} \int_{-L}^L \cos \frac{n\pi x}{L} \cos \frac{n\pi x}{L} dx \\ &= \int_{-L}^L \cos^2 \frac{n\pi x}{L} dx \\ &= \frac{1}{2} \int_{-L}^L \left( \cos \frac{2n\pi x}{L} + 1 \right) dx \\ &= \frac{1}{2} \int_{-L}^L \cos \frac{2n\pi x}{L} dx + \frac{1}{2} \int_{-L}^L dx \\ &= 0 + \frac{1}{2} \cdot 2L \\ &= L \end{aligned}$$

$$\begin{aligned}
I_1 &= \int_{-L}^L \cos \frac{m\pi x}{L} \cos \frac{n\pi x}{L} dx \quad [m \neq 0] \\
&= \cos \frac{m\pi x}{L} \int_{-L}^L \cos \frac{n\pi x}{L} dx + \frac{m}{n} \int_{-L}^L \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L} dx \\
&= \frac{L}{n\pi} \cos \frac{m\pi x}{L} (\sin n\pi + \sin n\pi) + \frac{m}{n} I_2 \\
&= 0 + \frac{m}{n} \left[ \int_{-L}^L \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L} dx \right] \\
&= \frac{m}{n} \left[ \sin \frac{m\pi x}{L} \int_{-L}^L \sin \frac{n\pi x}{L} dx + \frac{m}{n} \int_{-L}^L \cos \frac{m\pi x}{L} \cos \frac{n\pi x}{L} dx \right] \\
&= \frac{m}{n} \left[ \frac{L}{n\pi} \sin \frac{m\pi x}{L} (-\cos n\pi + \cos n\pi) + \frac{m}{n} \right] \\
&= 0 + \frac{m^2}{n^2} I_1 \\
I_1 &= 0 = I_2
\end{aligned}$$

To summarize, we have

$$\int_{-L}^L \sin^2 \frac{n\pi x}{L} dx = \int_{-L}^L \cos^2 \frac{n\pi x}{L} dx = L \quad (2.2.1)$$

$$\int_{-L}^L \cos mx dx = \int_{-L}^L \sin mx dx = 0 \quad (2.2.2)$$

$$\int_{-L}^L \sin \frac{n\pi x}{L} \cos \frac{n\pi x}{L} dx = 0 \quad (2.2.3)$$

$$\int_{-L}^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx = \int_{-L}^L \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} dx = 0 \quad [m \neq n] \quad (2.2.4)$$

### 2.3 Derivation of $a_0$

Taking integral on both sides of (1) from  $-L$  to  $L$ , we get

$$\begin{aligned}
\int_{-L}^L f(x) dx &= \frac{a_0}{2} \int_{-L}^L dx + \sum_{n=1}^{\infty} \int_{-L}^L \left[ a_n \cos \frac{m\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right] dx \\
&= \frac{a_0}{2} \cdot 2L \quad [\text{All the other terms are 0 according to equation (4)}]
\end{aligned}$$

$$a_0 = \frac{1}{L} \int_{-L}^L f(x) dx$$

### 2.4 Derivation of $a_n$

Multiplying both sides of (1) by  $\cos \frac{m\pi x}{L}$  and integrating from  $-L$  to  $L$ , we get

$$\begin{aligned}
\int_{-L}^L f(x) \cos \frac{m\pi x}{L} dx &= \frac{a_0}{2} \int_{-L}^L \cos \frac{m\pi x}{L} dx \\
&\quad + \sum_{n=1}^{\infty} \int_{-L}^L \left[ a_n \cos \frac{m\pi x}{L} \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \cos \frac{m\pi x}{L} \right] dx \\
&= a_n \int_{-L}^L \cos^2 \frac{m\pi x}{L} dx \\
&= a_n \cdot L \quad [\text{All the other terms are 0 according to equation (2)}]
\end{aligned}$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{m\pi x}{L} dx$$

## 2.5 Derivation of $b_n$

Multiplying both sides of (1) by  $\sin \frac{m\pi x}{L}$  and integrating from  $-L$  to  $L$ , we get

$$\begin{aligned}
\int_{-L}^L f(x) \sin \frac{m\pi x}{L} dx &= \frac{a_0}{2} \int_{-L}^L \sin \frac{m\pi x}{L} dx \\
&\quad + \sum_{n=1}^{\infty} \int_{-L}^L \left[ a_n \sin \frac{m\pi x}{L} \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \right] dx \\
&= a_n \int_{-L}^L \sin^2 \frac{m\pi x}{L} dx \\
&= a_n \cdot L \quad [\text{All the other terms are 0 according to equation (2)}]
\end{aligned}$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{m\pi x}{L} dx$$

## 2.6 Examples

**Example 2.1:** Obtain the fourier series for  $f(x) = x - x^2$  in the interval  $(-\pi, \pi)$  and hence evaluate

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (x - x^2) dx = \frac{1}{\pi} \left[ \frac{x^2}{2} - \frac{x^3}{3} \right]_{-\pi}^{\pi} = -\frac{2\pi^2}{3}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (x - x^2) \cos nx dx \\
&= \frac{1}{\pi} \left[ \int_{-\pi}^{\pi} x \cos nx dx - \int_{-\pi}^{\pi} x^2 \cos nx dx \right] \\
&= -\frac{2}{\pi} \int_0^{\pi} x^2 \cos nx dx \\
&= -\frac{2}{\pi} \left[ \frac{x^2}{n} \sin nx - \frac{2}{n} \int x \sin nx dx \right]_0^{\pi} \\
&= \frac{4}{n\pi} \left[ -\frac{x}{n} \cos nx + \frac{1}{n^2} \sin nx \right]_0^{\pi} \\
&= -\frac{4}{n^2} (-1)^n
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} (x - x^2) \sin nx dx \\
&= \frac{1}{\pi} \left[ \int_{-\pi}^{\pi} x \sin nx dx - \int_{-\pi}^{\pi} x^2 \sin nx dx \right] \\
&= \frac{2}{\pi} \int_0^{\pi} x \sin nx dx \\
&= \frac{2}{\pi} \left[ -\frac{x}{n} \cos nx + \frac{1}{n^2} \sin nx \right]_0^{\pi} \\
&= -\frac{2}{n} (-1)^n
\end{aligned}$$

$$\therefore f(x) = -\frac{\pi^2}{3} - \sum_{n=1}^{\infty} \left( \frac{4}{n^2} (-1)^n \cos nx + \frac{2}{n} (-1)^n \sin nx \right)$$

For  $x = 0$ , we get

$$\begin{aligned}
0 &= -\frac{\pi^2}{3} - \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n \\
\frac{\pi^2}{12} &= -\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}
\end{aligned}$$

$$\boxed{\therefore 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}}$$

**Example 2.2:** Find a fourier series to represent the function  $f(x) = e^x$  for  $-\pi < x < \pi$  and hence derive a series for  $\frac{\pi}{\sinh \pi}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x dx = \frac{1}{\pi} e^x \Big|_{-\pi}^{\pi} = \frac{e^{\pi} - e^{-\pi}}{\pi} = \frac{2 \sinh x}{\pi}$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \cos nx \, dx \\ &= \frac{1}{\pi} \left[ e^x \frac{\sin nx}{n} - \frac{1}{n} \int e^x \sin nx \, dx \right]_{-\pi}^{\pi} \\ &= -\frac{1}{n\pi} \left[ -e^x \frac{\cos nx}{n} + \frac{1}{n} \int e^x \cos nx \, dx \right]_{-\pi}^{\pi} \end{aligned}$$

$$\begin{aligned} a_n \left( 1 + \frac{1}{n^2} \right) &= \frac{(-1)^n}{n^2 \pi} (e^{\pi} - e^{-\pi}) \\ a_n &= \frac{(-1)^n}{n^2 \pi} (e^{\pi} - e^{-\pi}) \left( 1 + \frac{1}{n^2} \right)^{-1} \\ a_n &= 2 \frac{(-1)^n}{n^2 \pi} \sinh x \left( 1 + \frac{1}{n^2} \right)^{-1} \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \sin nx \, dx \\ &= \frac{1}{\pi} \left[ -e^x \frac{\cos nx}{n} + \frac{1}{n} \int e^x \cos nx \, dx \right]_{-\pi}^{\pi} \\ &= \frac{1}{\pi} \left[ -e^x \frac{\cos nx}{n} + \frac{1}{n} \left\{ e^x \frac{\sin nx}{n} - \frac{1}{n} \int e^x \sin nx \, dx \right\} \right]_{-\pi}^{\pi} \\ &= \frac{1}{\pi} \left[ -e^x \frac{\cos nx}{n} - \frac{1}{n^2} \int e^x \sin nx \, dx \right] \\ b_n &= -\frac{(-1)^n}{n\pi} (e^{\pi} - e^{-\pi}) \left( 1 + \frac{1}{n^2} \right)^{-1} \\ b_n &= -2 \frac{(-1)^n}{n\pi} \sinh x \left( 1 + \frac{1}{n^2} \right)^{-1} \end{aligned}$$

$$f(x) = e^x = 2 \frac{\sinh x}{\pi} \left[ \frac{1}{2} + \sum_{n=1}^{\infty} \left( 1 + \frac{1}{n^2} \right)^{-1} \frac{(-1)^n}{n} \left( \frac{1}{n} \cos nx - \sin nx \right) \right]$$

For  $x = 0$ , we get

$$\begin{aligned} \frac{\pi}{\sinh x} &= 1 + 2 \sum_{n=1}^{\infty} \left( 1 + \frac{1}{n^2} \right)^{-1} \frac{(-1)^n}{n^2} \\ &= 1 + 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + 1} \end{aligned}$$

$$\frac{\pi}{\sinh x} = 1 + 2 \left( -\frac{1}{2} + \frac{1}{5} - \frac{1}{10} + \cdots \right)$$

## 3 Fourier Integral

### 3.1 Definition

#### Definition 3.1.1: Fourier Integral

The Fourier integral of a function  $f$  defined on the interval  $(-\infty, \infty)$  is given by

$$f(x) = \frac{1}{\pi} \int_0^\infty [A(\alpha) \cos \alpha x + B(\alpha) \sin \alpha x] d\alpha \quad (3.1.1)$$

where the coefficients  $A(\alpha)$  and  $B(\alpha)$  are given by

$$\begin{cases} A(\alpha) = \int_{-\infty}^\infty f(x) \cos \alpha x dx \\ B(\alpha) = \int_{-\infty}^\infty f(x) \sin \alpha x dx \end{cases} \quad (3.1.2)$$

The Fourier integral can also be written in the form

$$f(x) = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty f(t) \cos \lambda(t-x) dt d\lambda \quad (3.1.3)$$

where  $\lambda = \frac{n\pi}{L}$

Fourier series were used to represent a function  $f$  defined on the finite interval  $(-L, L)$  or  $(0, L)$ . It converged to  $f$  and to its periodic extension. In this sense, Fourier series is associated with periodic functions.

Fourier integral represents a certain type of non-periodic functions that are defined on  $(-\infty, \infty)$  or  $(0, \infty)$ .

### 3.2 Derivation

Let a function  $f$  be defined on  $(-L, L)$ . The Fourier series of the function is then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^\infty \left( a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \quad (3.2.1)$$

where the coefficients are given by

$$\begin{aligned} a_0 &= \frac{1}{L} \int_{-L}^L f(t) dt \\ a_n &= \frac{1}{L} \int_{-L}^L f(t) \cos \frac{n\pi t}{L} dt \\ b_n &= \frac{1}{L} \int_{-L}^L f(t) \sin \frac{n\pi t}{L} dt \end{aligned}$$

Now, let  $a_n = \frac{n\pi}{L}$ ,

then  $\Delta\alpha = \alpha_{n+1} - \alpha_n = \frac{\pi}{L}$



So, we get

$$\Delta f(x) = \frac{1}{2\pi} \left( \int_{-L}^L f(t) dt \right) \Delta\alpha + \frac{1}{\pi} \sum_{n=1}^{\infty} \left[ \left( \int_{-L}^L f(t) \cos \alpha_n t dt \right) \cos \alpha_n x + \left( \int_{-L}^L f(t) \sin \alpha_n t dt \right) \sin \alpha_n x \right] \Delta\alpha \quad (3.2.2)$$

We now expand the interval  $(-L, L)$  by taking  $L \rightarrow \infty$ , which implies that  $\Delta\alpha \rightarrow 0$ . Consequently, we get

$$\lim_{\Delta\alpha \rightarrow 0} \sum_{n=1}^{\infty} F(\alpha_n) \Delta\alpha \rightarrow \int_0^{\infty} F(\alpha) d\alpha \quad (3.2.3)$$

Thus, the limit of the first term in the Fourier series  $\int_{-L}^L f(t) dt$  vanishes, and the limit of the sum becomes

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \left[ \left( \int_{-\infty}^{\infty} f(t) \cos \alpha t dt \right) \cos \alpha x + \left( \int_{-\infty}^{\infty} f(t) \sin \alpha t dt \right) \sin \alpha x \right] d\alpha \quad (3.2.4)$$

This is the Fourier integral of  $f$  on the interval  $(-\infty, \infty)$ .

### 3.3 Alternative Derivation

Substituting the values of  $a_0$ ,  $a_n$  and  $b_n$  in (7), we get

$$\begin{aligned} f(x) &= \frac{1}{2L} \int_{-L}^L f(t) dt + \frac{1}{L} \sum_{n=1}^{\infty} \left[ \int_{-L}^L f(t) \cos \frac{n\pi t}{L} \cos \frac{n\pi x}{L} dt + \int_{-L}^L f(t) \sin \frac{n\pi t}{L} \sin \frac{n\pi x}{L} dt \right] \\ &= \frac{1}{2L} \int_{-L}^L f(t) dt + \frac{1}{L} \sum_{n=1}^{\infty} \left[ \int_{-L}^L f(t) \left\{ \cos \frac{n\pi t}{L} \cos \frac{n\pi x}{L} + \sin \frac{n\pi t}{L} \sin \frac{n\pi x}{L} \right\} dt \right] \\ f(x) &= \frac{1}{2L} \int_{-L}^L f(t) dt + \frac{1}{L} \sum_{n=1}^{\infty} \int_{-L}^L f(t) \cos \frac{n\pi}{L} (t - x) dt \end{aligned} \quad (3.3.1)$$

Now, if we assume that  $\int_{-\infty}^{\infty} |f(x)| dx$  converges, the first term on the right side of (11) approaches 0 as  $L \rightarrow \infty$ , since

$$\left| \frac{1}{2L} \int_{-L}^L f(t) dt \right| \leq \frac{1}{2L} \int_{-\infty}^{\infty} |f(t)| dt$$

The second term on the right side of (11) approaches

$$\begin{aligned} &\lim_{L \rightarrow \infty} \frac{1}{L} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} f(t) \cos \frac{n\pi}{L} (t - x) dt \\ &= \lim_{\delta\lambda \rightarrow 0} \frac{1}{\pi} \sum_{n=1}^{\infty} \delta\lambda \int_{-\infty}^{\infty} f(t) \cos n\delta\lambda (t - x) dt \end{aligned}$$

where  $\lambda = \frac{n\pi}{L}$  which implies that  $\delta\lambda = \frac{\pi}{L}$ .

We know,

$$\lim_{\delta\lambda \rightarrow 0} \sum_{n=1}^{\infty} \delta\lambda F(\lambda_n) = \int_0^{\infty} F(\lambda) d\lambda$$

Thus we get

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(t) \cos \lambda(t-x) dt d\lambda \quad (3.3.2)$$

This is another form of the Fourier integral.

### 3.4 Fourier Sine and Cosine Integrals

We can rewrite (7) as

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \cos \lambda x \int_{-\infty}^{\infty} f(t) \cos \lambda t dt d\lambda + \frac{1}{\pi} \int_0^{\infty} \sin \lambda x \int_{-\infty}^{\infty} f(t) \sin \lambda t dt d\lambda \quad (3.4.1)$$

If  $f(x)$  is an odd function,  $f(t) \cos \lambda t$  is also an odd function while  $f(t) \sin \lambda t$  is even. Then the first term on the right side of (3.4.1) vanishes and we get

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \sin \lambda x \int_0^{\infty} f(t) \sin \lambda t dt d\lambda \quad (3.4.2)$$

This is known as the Fourier sine integral.

Similarly, if  $f(x)$  is an even function, (3.4.1) becomes

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \cos \lambda x \int_0^{\infty} f(t) \cos \lambda t dt d\lambda \quad (3.4.3)$$

This is known as the Fourier cosine integral.

### 3.5 Complex Form of Fourier Integral

Equation (3.1.3) can be written as

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \cos \lambda(t-x) dt d\lambda \quad (3.5.1)$$

because  $\cos \lambda(t-x)$  is an even function of  $\lambda$ . Also, since  $\sin \lambda(t-x)$  is an odd function of  $\lambda$ , we have

$$0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \sin \lambda(t-x) dt d\lambda \quad (3.5.2)$$

Now, multiplying (3.5.2) by  $i$  and adding it to (3.5.1), we get

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) e^{i\lambda(t-x)} dt d\lambda \quad (3.5.3)$$

This is the complex form of the Fourier integral.