Signals and Systems
Lecture # 1 Topic: Signals

Text Books

- Signals and Systems
  - Alan V. Oppenheim, Alan S. Willsky, S. Hamid Nawab, Prentice Hall Ltd. Second edition
- Signals and Systems
  - Simon Haykin, Barry Van Veen, John Wiley & Sons Inc.
- Signals and Systems
  - M. J. Roberts, McGraw-Hill Edition
- Analysis of Linear Systems
  - David K. Cheng, Narosa Publishing House

What is a signal?

- A signal is a varying quantity whose value can be measured and which conveys information
- Example: temperature
  - It can vary over time
  - We can measure it using a thermometer
  - It conveys information: knowing the temperature outside will inform our decision as to which clothes to wear
- In the digital signal processing system, the signal is represented as a sequence of numbers either on a computer or in digital hardware
- Example:
  - we could store the temperature at various times of the day as a sequence of numbers in an array on a computer: each reading might be a temperature reading in Celsius

Examples of Signals

- Signals are functions of one or more variables (independent variables) that carry/convey information.
- For example:
  - Electrical signals - voltages and currents in a circuit, power
  - Thermal signals – temperature
  - Light signals – light intensity,
  - Mechanical signals – force, torque, pressure
  - Acoustic signals ---audio or speech signals (analog or digital) such as human voice, a dog's bark, bird's song,
  - Video signals ---intensity variations in an image (e.g. a CAT scan, MRI data)
  - Biological signals ---sequence of bases in a gene
  - Human Body Signal – ECG, EMG, EOG; Brain signal - EEG signal
  - Stock market data

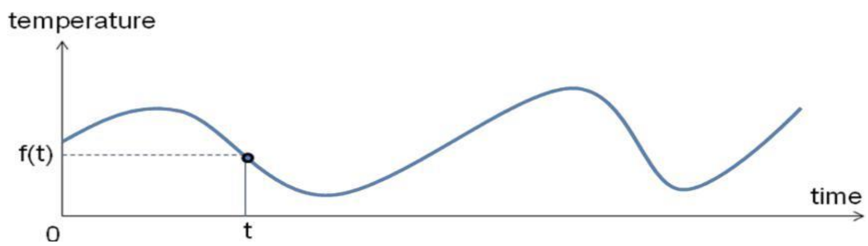
The domain of a Signal

- The domain is a very widely used term in DSP.
  - For instance, a signal that uses time as the independent variable (i.e., the parameter on the horizontal axis), is said to be in the time domain
  - Another common signal in DSP uses frequency as the independent variable, resulting in the term, frequency domain.

– Likewise, signals that use distance as the independent parameter are said to be in the spatial domain (distance is a measure of space). The type of parameter on the horizontal axis is the domain of the signal; it's that simple. What if the x-axis is labeled with something very generic, such as sample number? Authors commonly refer to these signals as being in the time domain. This is because sampling at equal intervals of time is the most common way of obtaining signals, and they don't have anything more specific to call it.

Domain and range of a signal

- Temperature is a function of single real-valued variable, T=f(t)
- We say that ***domain*** of the signal is one-dimensional



- Some signals are functions of more than one variable
- *Example*: black and white photograph – can be regarded as a signal
  - Brightness u of a point on the photograph is a function of two variables, the x and y coordinates of the point on the photograph
  - The domain of the signal is two-dimensional (fig. 2)
  - The range of the signal is one-dimensional

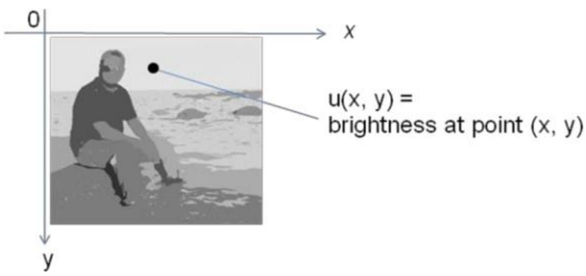


Fig. 2

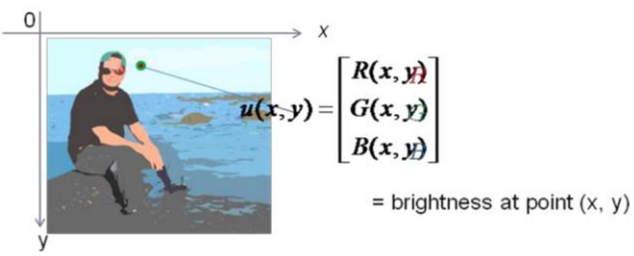


Fig. 3

- *Example*: color photograph – can be regarded as a signal
  - The color of a point on the photograph can be expressed as three real-numbers, especially giving the amount of red (R), green (G), and blue (B) that go to make up the color
  - In this case, we could say that the ***range*** of the signal is three-dimensional and the ***domain*** is two-dimensional (Fig. 3)

- *Example*: color movie – can be regarded as a signal
  - The color of a point on the screen with given x and y coordinates can be expressed as three real numbers for the amounts of red (R), green (G), and blue (B) that go to make up the color
  - The picture also changes with time, which adds an extra dimension to the domain of the signal
  - In total, we, therefore, have three dimensions in the domain (x, y, and time) and three in the range (red, green, and blue)

The dependent variable of a signal

- The dependent variable of a signal
  - Examples: Specific: voltage, light intensity, sound pressure, or an infinite number of other parameters
  - Generic label: amplitude
- Another name of the Dependent variable
  - vertical axis
  - y-axis,
  - range, and
  - ordinate

Independent variable of a signal

- The independent variable of a signal
  - Most common parameter: Time
  - Distance
  - x
  - Number
  - Generic name: sample number
- Another name of the independent variable
  - horizontal axis
  - x-axis,
  - the domain, and
  - the abscissa

Dependent and independent variable of a signal

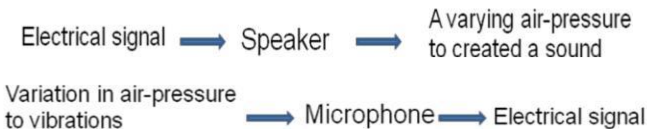
- Dependent variables
  - Can be continuous
    - Speech, current, voltage, temperature, audio
  - Can be discrete
    - DNA base sequence
    - Digital image pixels
  - Can be 1-D (speech signal), 2-D (image signal), ••• N-D
- Independent variable
  - Single (1-D) independent variable which we call" time”.
  - Continuous-Time (CT) signals: x(t), t—continuous values
  - Discrete-Time (DT) signals: x[n], n—integer values only

Relation between variables

- The two parameters that form a signal are generally not interchangeable.
- The parameter on the y-axis (the dependent variable) is said to be a function of the parameter on the x-axis (the independent variable).
- In other words, the independent variable describes how or when each sample is taken, while the dependent variable is the actual measurement.
- Given a specific value on the x-axis, we can always find the corresponding value on the y-axis, but usually not the other way around.

Converting a signal from one form to another

- A device that converts a signal from one form to another is called a transducer
- Example: Loudspeaker, microphone



- Variation of air-pressure is a real-valued function of time

What is Noise? How can noise be represented?

- Noise is a random signal, is like a signal in that it is a time-varying physical phenomenon, but unlike a signal, it usually does not carry useful information and is almost considered undesirable

- Noise Representation: Noise can be represented as a function of time like a signal

  The function of time t : $n(t) = a(t)[\cos(2\pi ft + \Theta(t))]$;

  a(t) and *Θ(t)* are time-variant amplitude and phase of the noise; $\Theta(t) \in [-\pi, \pi]$

What do you mean by signal processing task? Give some examples.

- Modify the amount of bass and treble in an audio signal
- Analyze the image to determine what object s are present in it
- Compute seasonally adjusted temperature values
- Make a photograph sharper and increase its contrast
- Measure the pitch of a musical instrument
- Compression of data for transmitting and store

Analog vs. digital signal/data

- Data can be analog or digital. The term analog data refers to information that is continuous; digital data refers to information that has discrete states. Analog data take on continuous values. Digital data take on discrete values.
- Analog and Digital data
  - Data can be analog or digital.
  - Analog data are continuous and take continuous values.
  - Digital data have discrete states and take discrete values.
- Analog and Digital Signal
  - Signals can be analog or digital.
  - Analog signals can have an infinite number of values in a range.
  - Digital signals can have only a limited number of values

Sample questions

- **Q1**. In this study, we used outside temperature as an example of a signal, considering it only as a function of time. A weather forecast, however, will talk about temperature not only as a function of time but as a function of location in the country: how many dimensions does the domain of this signal have? An aircraft pilot whose hobby is signal processing thinks of the outside temperature as a signal. How many dimensions would you imagine he thinks its domain has?
- **Q2**. Consider a black-and-white movie as a signal. How many dimensions do its domain and range have?

Signals and Systems
Lecture #2 Topic: Classification of Signals

- Continuous-time signal vs. discrete time signal
- Analog vs. Digital signal
- Periodic vs. aperiodic
- Causal vs. non-causal
- Even vs. odd
- Deterministic vs. random
- Right-handed vs. left-handed
- Finite vs. infinite length (duration) signal
- Energy signal vs. power signal

(i) Continuous-Time vs. Discrete-Time

As the names suggest, this classification is determined by whether or not the time axis (x-axis) is *discrete* (countable) or *continuous* (Figure 1) . A continuous-time signal will contain a value for all real numbers along the time axis. In contrast to this, a discrete-time signal is often created by using the sampling theorem to sample a continuous signal, so it will only have values at equally spaced intervals along the time axis.

CT signal: $x(t) = a\cos(wt + \theta)$

DT signal: $x[n] = x(nT) = \{x(0), x(T), \ldots, x(N-1)T\}, \qquad n = 0, \pm 1, \pm 2, \ldots$

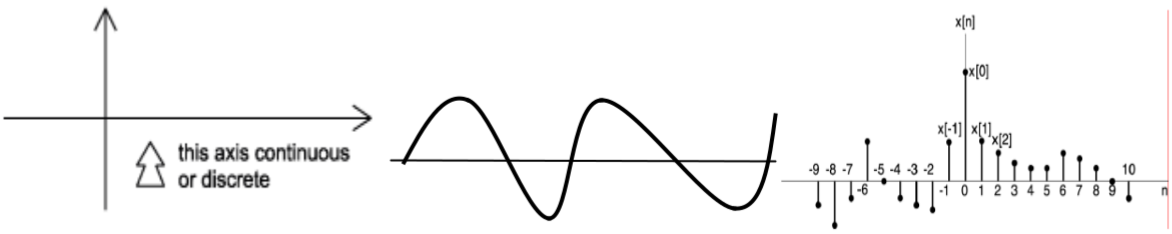


Figure 1

Examples:

- Continuous time signals (**t->time, x->signal, x(t) -> CT signal**)
  – Most of the signals in the physical world are CT signals—E.g. voltage & current, pressure, temperature, velocity, etc.
- Discrete time signals in nature: (n->**time, x->signal, x[n] -> DT signal)**
  – DNA base sequence
  – Population of the nth generation of certain species

(ii) Analog vs. Digital

The difference between *analog* and *digital* is similar to the difference between continuous-time and discrete-time. In this case, however, the difference is with respect to the value of the function (y-axis) (Figure 2). Analog corresponds to a continuous y-axis, while digital corresponds to a discrete y-axis.

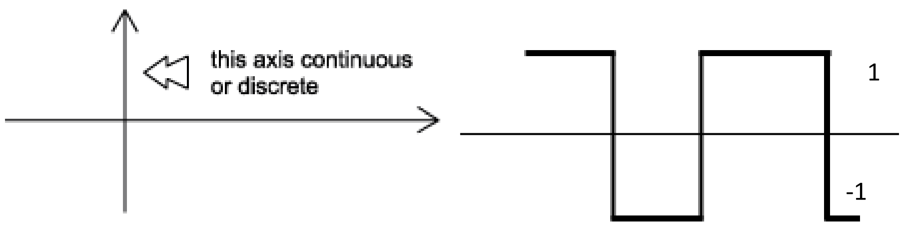


Figure 2

Examples:

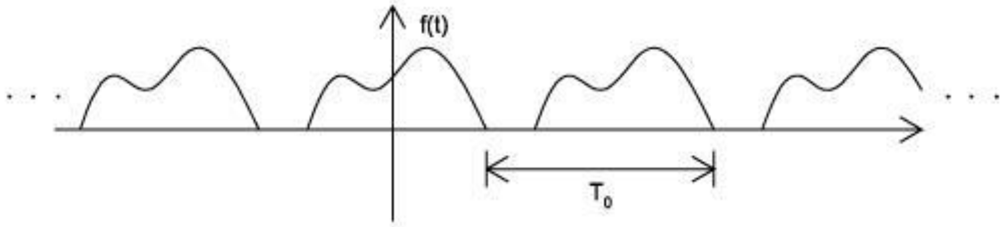
- Analog: signals in nature, such as a speech signal
- Digital: binary sequence

(iii) Periodic vs. Aperiodic

Periodic signals repeat with some *period* T, while aperiodic, or nonperiodic, signals do not (Figure 3). We can define a periodic function through the following mathematical expression, where t can be any number and T is a positive constant: $f(t) = f(T+t)$ for all t (1)

The *fundamental period* of our function, $f(t)$, is the smallest value of T that the still allows Equation 1 to be true.

(a) A periodic signal with period T_0



(b) An aperiodic signal

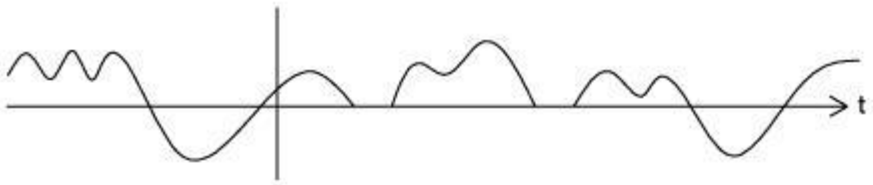


Figure 3

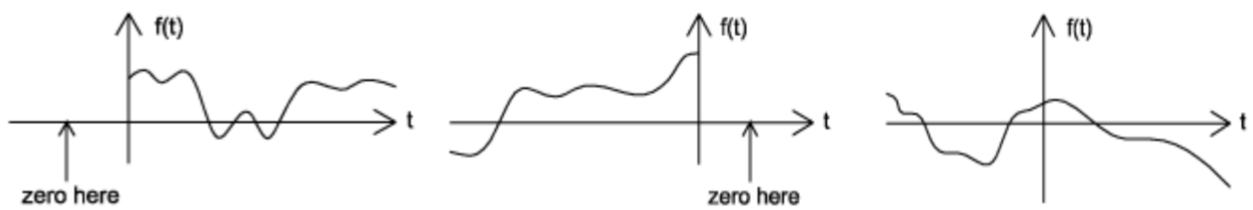
A discrete-time signal x[n] is said to be periodic if it satisfies the following condition:

$$x[n] = x[n+T], \text{ for all integer n}$$

The fundamental angular frequency or fundamental frequency of x[n] is defined as $\Omega = \dfrac{2\pi}{N}$ in radians

(iv) Causal vs. Anticausal vs. Noncausal // Causal vs. non-causal

Causal signals are signals that are zero for all negative time, while *anticausal* are signals that are zero for all positive time. *Noncausal* signals are signals that have nonzero values in both positive and negative time (Figure 4). A signal $x[n]$ is *causal* if $x[n] = 0$ for all $n < 0$. It is *anti-causal* if $x[n] = 0$ for all $n > 0$.



(a) A causal signal **(b) An anticausal signal** **(c) A noncausal signal**

Figure 4

(v) Even vs. Odd

An *even signal* is any signal f such that $f(t) = f(-t)$. Even signals can be easily spotted as they are *symmetric* around the vertical axis. An *odd signal*, on the other hand, is a signal f such that $f(t) = -(f(-t))$ (Figure 5).

(a) An even signal **(b) An odd signal**

Figure 5

Using the definitions of even and odd signals, we can show that any signal can be written as a combination of an even and odd signal. That is, every signal has an odd-even decomposition. To demonstrate this, we have to look no further than a single equation.

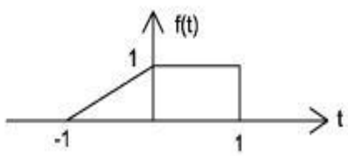
$$f(t)=1/2\,[f(t)+f(-t)] + 1/2\,[f(t) - f(-t)] \qquad (2)$$

By multiplying and adding this expression out, it can be shown to be true. Also, it can be shown that $f(t) + f(-t)$ fulfills the requirement of an even function, while $f(t) - f(-t)$ fulfills the requirement of an odd function.

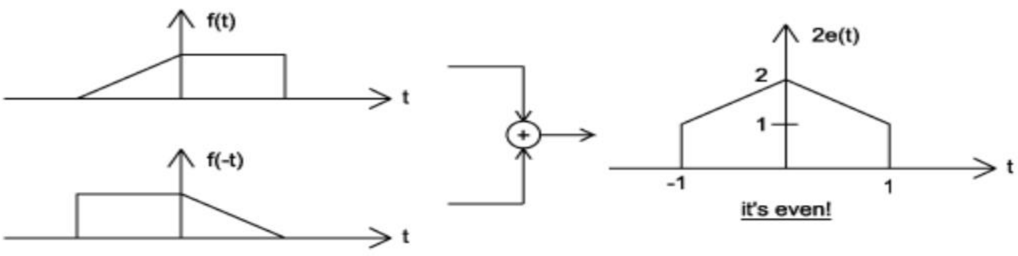
A function whose even part is zero is odd and a function whose odd part is zero is even.

Even function: $x_e(t) = \dfrac{x(t) + x(-t)}{2}$, Odd function: $x_o(t) = \dfrac{x(t) - x(-t)}{2}$

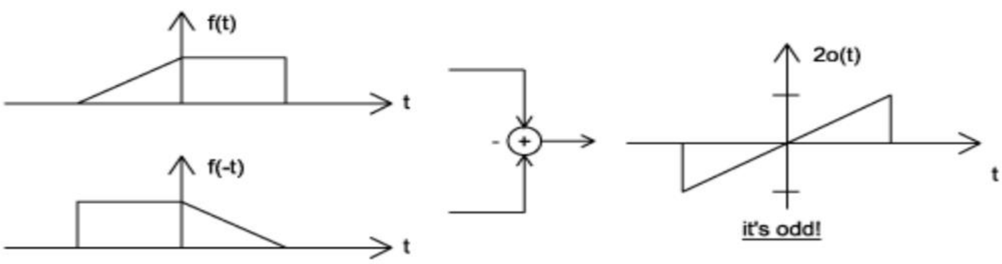
Example 1(a) The signal we will decompose using odd-even decomposition



(b) Even part: *e*(*t*) = ½ (*f*(*t*) +*f*(−*t*))



(c) Odd part: *o*(*t*) = ½ (*f*(*t*) −*f*(−*t*))



(d) Check: *e*(*t*) +*o*(*t*) =*f*(*t*)

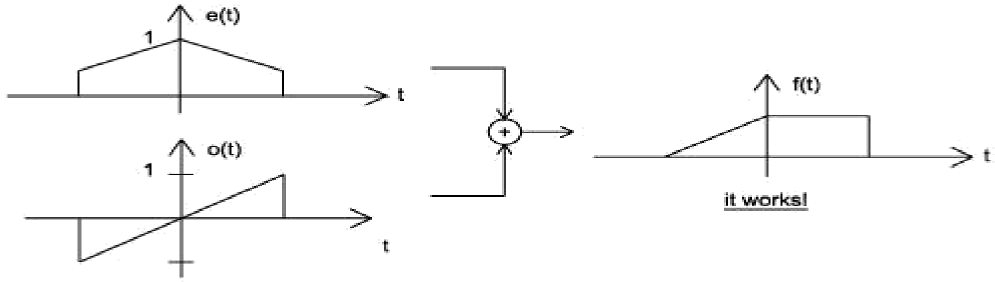
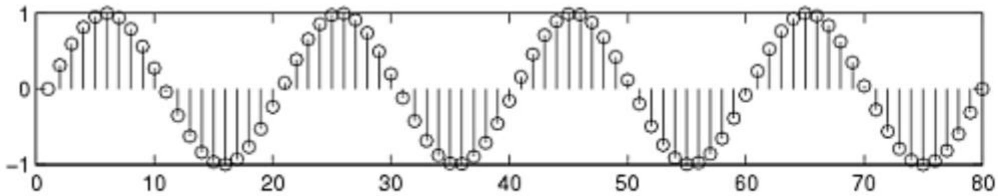


Figure 6

(vi) Deterministic vs. Random

A *deterministic signal* is a signal in which each value of the signal is fixed and can be determined by a mathematical expression, rule, or table. Because of this, the future values of the signal can be calculated from past values with complete confidence. On the other hand, a *random signal* has a lot of uncertainty about its behavior. The future values of a random signal cannot be accurately predicted and can usually only be guessed based on the averages of sets of signals (Figure 7).

(a) Deterministic Signal



(b) Random Signal

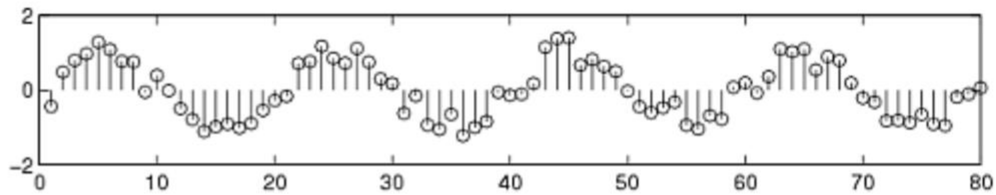
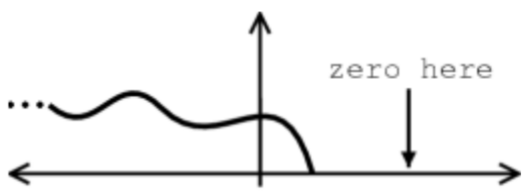


Figure 7

(vii) Right-handed vs. Left-Handed

A *right-handed* signal and *left-handed* signal are those signals whose value is zero between a given variable and positive or negative infinity. Mathematically speaking, a right-handed signal is defined as any signal where $f(t)=0$ for t<t_1<∞, and a left-handed signal is defined as any signal where $f(t)=0$ for $t>t_1>-\infty$. See (Figure 8) for an example. Both figures "begin" at t_1 and then extends to positive or negative infinity with mainly nonzero values.

(a) Right-handed signal



(b) Left-handed signal

Figure 8

(viii) Finite vs. Infinite Length (duration)

As the name applies, signals can be characterized as to whether they have a finite or infinite length set of values. Most finite length signals are used when dealing with discrete-time signals or a given sequence of values. Mathematically speaking, $f(t)$ is a *finite-length signal* if it is *nonzero* over a finite interval $t_1 < f(t) < t_2$ where $t_1 > -\infty$ and $t_2 < \infty$. An example can be seen in Figure 9. Similarly, an *infinite-length signal*, $f(t)$, is defined as nonzero overall real numbers: $\infty \leq f(t) \leq -\infty$.

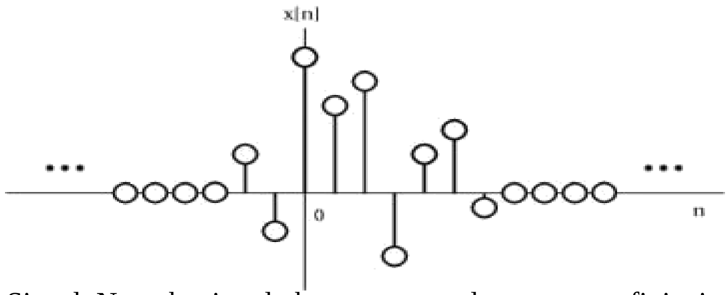


Figure 9: Finite-Length Signal. Note that it only has nonzero values on a set, finite interval.

First, we divide all signals into two classes: those that are of finite duration and those that are of infinite duration. A signal $x[n]$ is of *finite duration* if there exists two integers $\infty < N1 <= N2 < \infty$, such that $x[n] = 0$ only for $N1 <= n <= N2$. Otherwise, it is of *infinite duration*.

(ix) Energy and power signals

- Energy Signals: an energy signal is a signal with finite energy and zero average power ($0 \leq E < \infty$, $P = 0$)
- Power Signals: a power signal is a signal with infinite energy but finite average power ($0 < P < \infty$, $E \to \infty$).

In general, power is given by $p(t) = x^2(t)$, since $p(t) = v^2(t)/R$ or $p(t) = i^2(t)R$

The total energy of the continuous time signal x(t) as $E = \lim_{T \to \infty} \int_{-T/2}^{T/2} x^2(t)dt = \int_{-\infty}^{\infty} x^2(t)dt$ and its average power as $P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)dt$

The average power of a periodic signal x(t) of fundamental period T is given by $P = \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)dt$

The square root of the average power P is called the root-mean-square (RMS) value of the signal x(t).

In case of discrete-time signal x[n], the total energy of x[n] is defined as $E = \sum_{n=-\infty}^{\infty} x^2[n]$ and its average power is defined as $P = \lim_{N \to \infty} \frac{1}{2N} \sum_{n=-N}^{N} x^2[n]$.

The average power of a periodic signal x[n] with fundamental period N is given by $P = \frac{1}{N} \sum_{n=0}^{N-1} x^2[n]$

General rule: A signal cannot be both an energy and power signal.

- A signal may be neither energy nor power signal.
- All periodic signals are power signals (but not all non–periodic signals are energy signals).
- Any signal f that has limited amplitude ($|f| < \infty$) and is time limited ($f = 0$ for $|t| > t0$) is an energy signal.
- The square root of the average power of a power signal is called the RMS value.
- Periodic and random signals are power signals. Signals that are both deterministic and non-periodic are energy signals.

Q1. Evaluate *E* and *P* and determine the type of the signal $a(t) = 3\sin(2\pi t), -\infty < t < \infty$ Solution: It is a power signal

$$E_a = \int_{-\infty}^{\infty} |a(t)|^2 dt = \int_{-\infty}^{\infty} |3\sin(2\pi t)|^2 dt$$

$$= 9\int_{-\infty}^{\infty} \frac{1}{2}[1-\cos(4\pi t)]dt$$

$$= 9\int_{-\infty}^{\infty} \frac{1}{2}dt - 9\int_{-\infty}^{\infty} \cos(4\pi t)dt$$

$$= \infty \quad \text{J}$$

$$P_a = \frac{1}{1}\int_0^1 |a(t)|^2 dt = \int_0^1 |3\sin(2\pi t)|^2 dt$$

$$= 9\int_0^1 \frac{1}{2}[1-\cos(4\pi t)]dt$$

$$= 9\int_0^1 \frac{1}{2}dt - 9\int_0^1 \cos(4\pi t)dt$$

$$= \frac{9}{2} - \left[\frac{9}{4\pi}\sin(4\pi t)\right]_0^1$$

$$= \frac{9}{2} \text{ W}$$

Q2. Evaluate *E* and *P* and determine the type of the signal $b(t) = 5e^{-2|t|}, \; -\infty < t < \infty$

Solution: It is an energy signal

$$E_b = \int_{-\infty}^{\infty} |b(t)|^2 \, dt = \int_{-\infty}^{\infty} \left|5e^{-2|t|}\right|^2 dt$$

$$= 25\int_{-\infty}^{0} e^{4t} \, dt + 25\int_{0}^{\infty} e^{-4t} \, dt$$

$$= \frac{25}{4}\left[e^{4t}\right]_{-\infty}^{0} + \frac{25}{4}\left[e^{-4t}\right]_{0}^{\infty}$$

$$= \frac{25}{4} + \frac{25}{4} = \frac{50}{4} \quad \text{J}$$

$$P_b = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} |b(t)|^2 \, dt = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} \left|5e^{-2|t|}\right|^2 dt$$

$$= 25\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{0} e^{4t} \, dt + 25\lim_{T\to\infty}\frac{1}{T}\int_{0}^{T/2} e^{-4t} \, dt$$

$$= \frac{25}{4}\lim_{T\to\infty}\frac{1}{T}\left[e^{4t}\right]_{-T/2}^{0} + \frac{25}{4}\lim_{T\to\infty}\frac{1}{T}\left[e^{-4t}\right]_{0}^{T/2}$$

$$= \frac{25}{4}\lim_{T\to\infty}\frac{1}{T}\left[\quad {}^{-2T}\right] + \frac{25}{4}\lim_{T\to\infty}\frac{1}{T}\left[\quad {}^{-2T}\right]$$

$$=0+0=0$$

Combination of Odd and even function

Function type	Sum	Difference	Product	Quotient
Both even	Even	Even	Even	Even
Both odd	Odd	Odd	Even	Even
Even and odd	Neither	Neither	Odd	Odd

Signals and Systems
Lecture #3 Topic: Basic Operations on Signals

- Operations performed on dependent variables
    - Amplitude scaling
    - Addition
    - Multiplication
    - Differentiation
    - Integration
- Operations performed on the independent variables
    - Time scaling
    - Reflection
    - Time shifting

(1) Operations performed on dependent variables

(i) Amplitude or Magnitude Scaling: Example – electronic amplifier

CT signal: $y(t) = cx(t)$, where c is the scaling factor ; voltage = resistor*current

DT signal: $y[n] = cx[n]$

(ii) Addition : Example - audio mixer, which combines music and voice signals

CT signal: $y(t) = x_1(t) + x_2(t)$

DT signal: $y[n] = x_1[n] + x_2[n]$

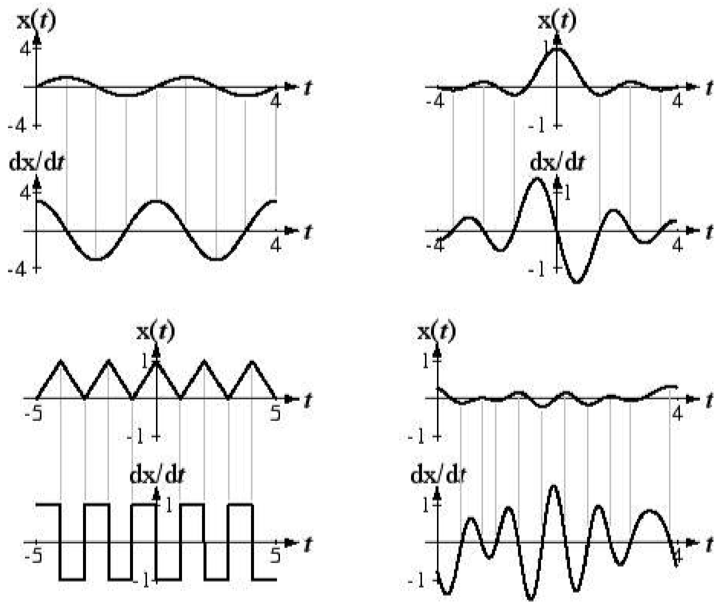
(iii) Multiplication: Example – AM radio signal

CT signal: $y(t) = x_1(t)x_2(t)$; x_1(t) = audio signal + a dc component, x_2(t)=sinusoidal signal (carrier)

DT signal: $y[n] = x_1[n]x_2[n]$

(iv) Differentiation: Inductor performs differentiation

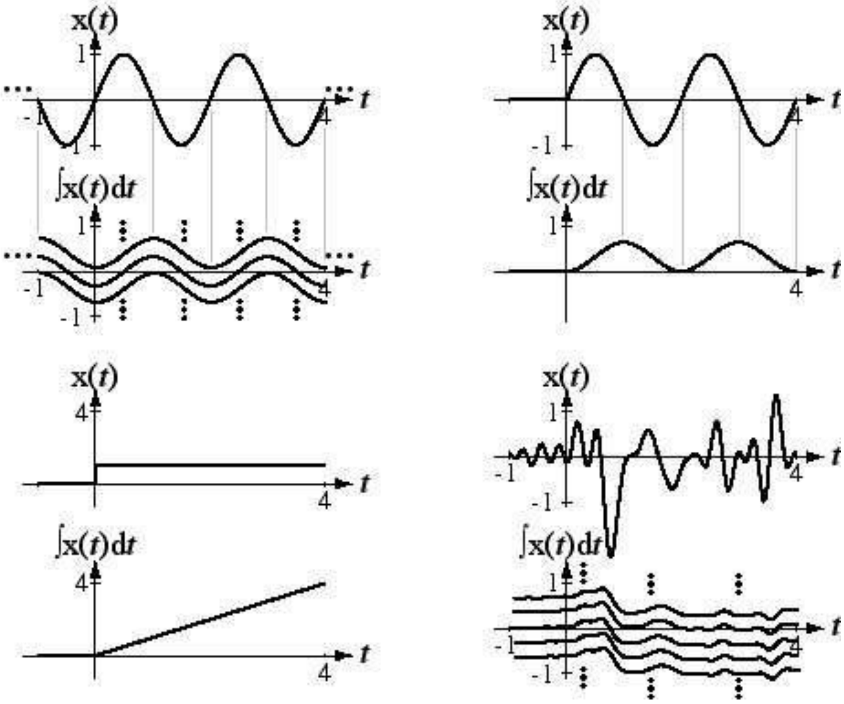
CT signal: $y(t) = \frac{d}{dt}x(t)$, $v(t) = L\frac{d}{dt}i(t)$



(v) Integration: A capacitor performs integration

CT signal: $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$, where τ is the integration variable

Voltage develop across the capacitor, $v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau$



(2) Operations performed on independent variables

(i) Time scaling: output is obtained by scaling of the independent variable

CT signal: $y(t) = x(at)$

DT signal: $y[n] = x[kn]$, k>0

If a>1, the signal y(t) is a compressed version of x(t).
If 0<a<1, the signal y(t) is an extended (stretched) version of x(t)

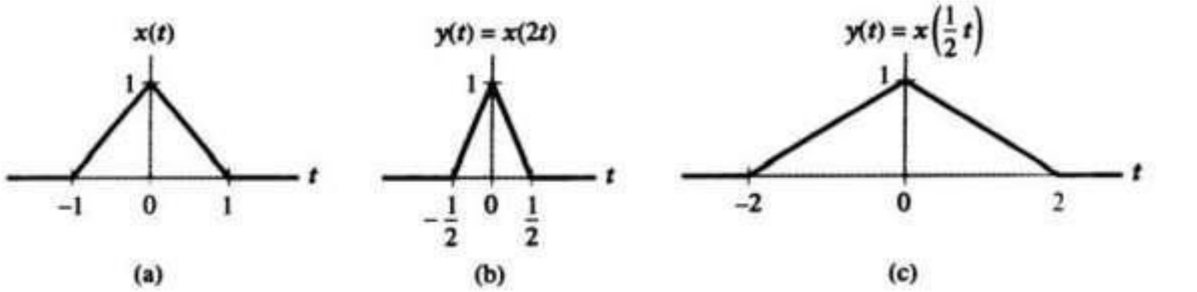


FIGURE 1.20 Time-scaling operation: (a) continuous-time signal $x(t)$, (b) version of $x(t)$ compressed by a factor of 2, and (c) version of $x(t)$ expanded by a factor of 2.

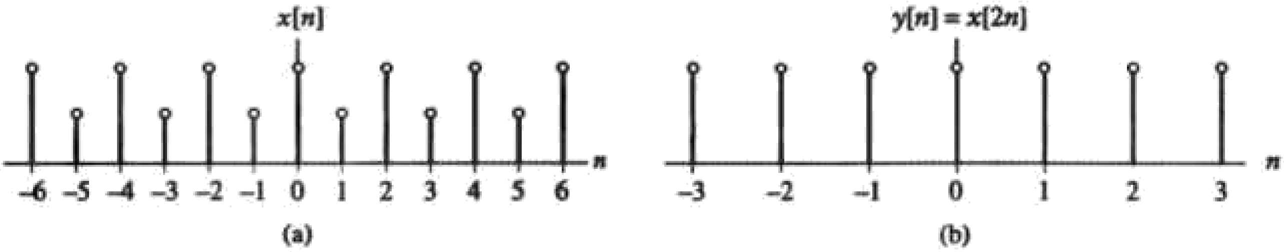


FIGURE 1.21 Effect of time scaling on a discrete-time signal: (a) discrete-time signal $x[n]$ and (b) version of $x[n]$ compressed by a factor of 2, with some values of the original $x[n]$ lost as a result of the compression.

(ii) Reflection:

CT signal: $y(t) = x(-t)$

DT signal: $y[n] = x[-n]$

The signal y(t) represents a reflected version of x(t) about the amplitude axis

Two cases:

Even signals: $x(-t) = x(t)$ for all t; an even signal is same as the reflected version

Even signals: $x(-t) = -x(t)$ for all t; an odd signal is the negative of its reflected version

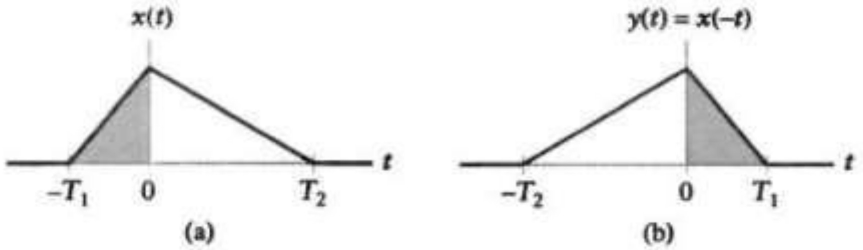


FIGURE 1.22 Operation of reflection: (a) continuous-time signal $x(t)$ and (b) reflected version of $x(t)$ about the origin.

Here, $x(t) = 0$ for $t < -T_1$ and $t > T_2$

$y(t) = 0$ for $t > T_1$ and $t < -T_2$

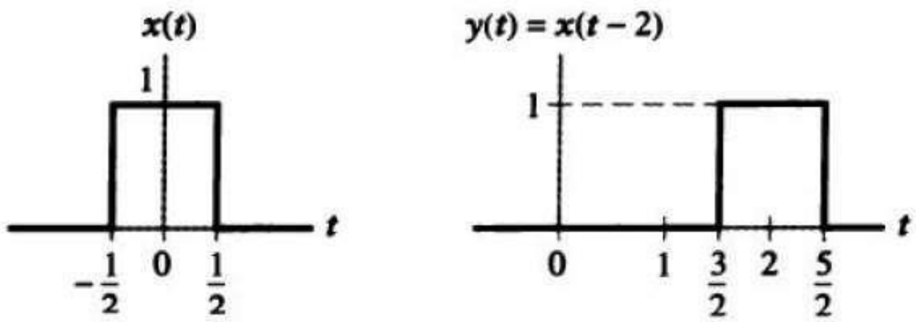
(vii) Time-shifting: Time-shifted version of $x(t)$ is defined by

CT signal: $y(t) = x(t - t_0)$, where t_0 is the time shift

If $t_0 > 0$, waveform representing $x(t)$ is shifted intact to the right, relative to the time axis

If $t_0 < 0$, it is shifted to the left

DT signal: $y[n] = x[n - m]$, where the shift m must be positive or negative integer

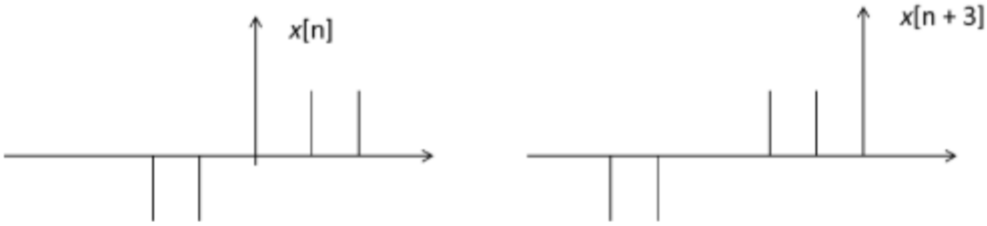


▶ Problem 1.13 The discrete-time signal

$$x[n] = \begin{cases} 1, & n = 1,2 \\ -1, & n = -1,-2 \\ 0, & n = 0 \text{ and } |n| > 2 \end{cases}.$$

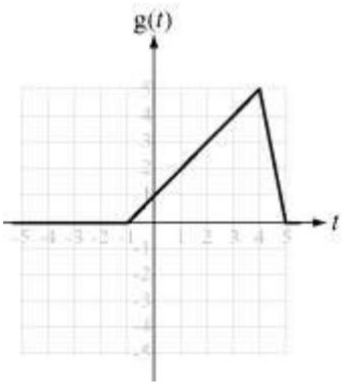
Find the time-shifted signal $y[n] = x[n+3]$.

Answer: After drawing figure, it can be easily shown that

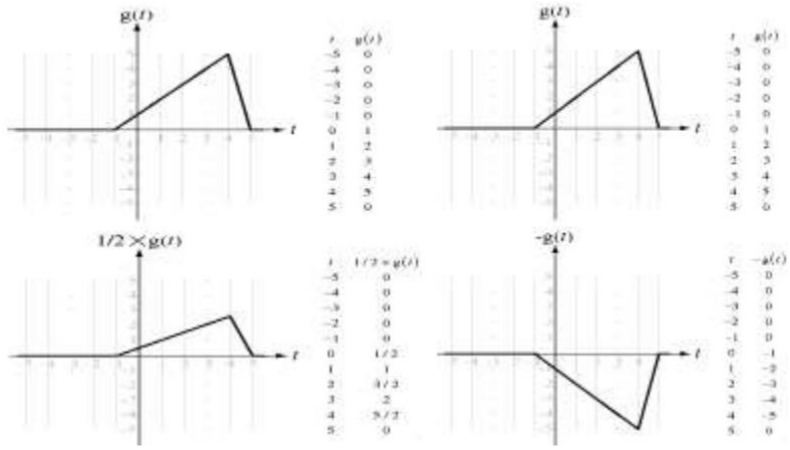


$$y[n] = \begin{cases} 1, & n = -1,-2 \\ -1, & n = -4,-5 \\ 0, & n = -3, n < -5, \text{and } n > -1 \end{cases}.$$

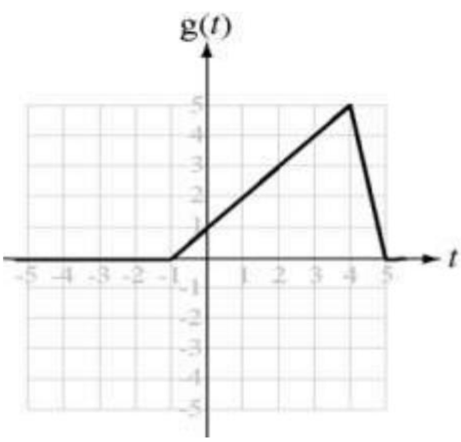
Example: Amplitude scaling - Let g(t)=0 |t|>5



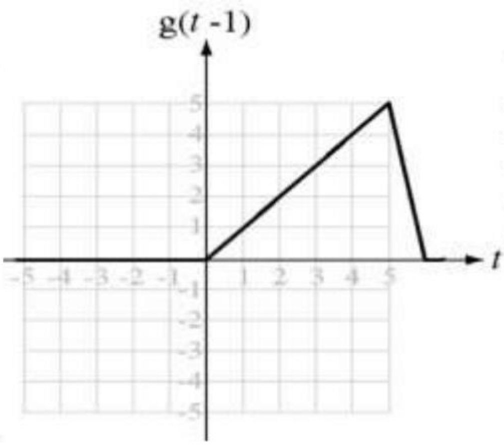
t	$g(t)$
−5	0
−4	0
−3	0
−2	0
−1	0
0	1
1	2
2	3
3	4
4	5
5	0



Example: Time-shifting: $t \rightarrow t{-}t_0$: Shifting the function to the right or left by t_0

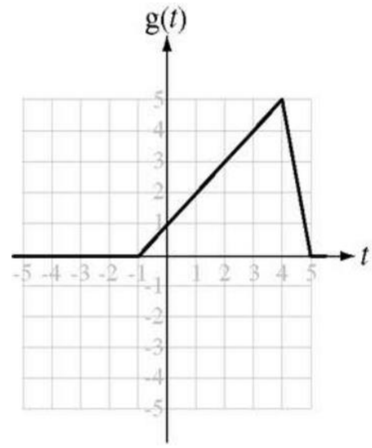


t	$g(t)$
−5	0
−4	0
−3	0
−2	0
−1	0
0	1
1	2
2	3
3	4
4	5
5	0

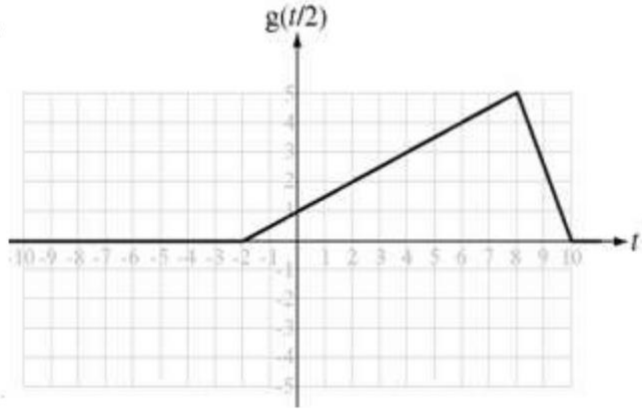


t	$t-1$	$g(t-1)$
−5	−6	0
−4	−5	0
−3	−4	0
−2	−3	0
−1	−2	0
0	−1	0
1	0	1
2	1	2
3	2	3
4	3	4
5	4	5

Example: Time-scaling (i) Expands the function horizontally by a factor of |a|

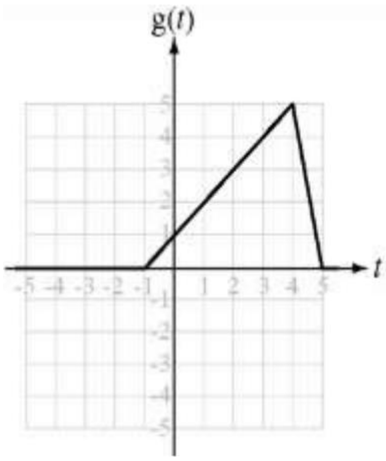


t	$g(t)$
−5	0
−4	0
−3	0
−2	0
−1	0
0	1
1	2
2	3
3	4
4	5
5	0

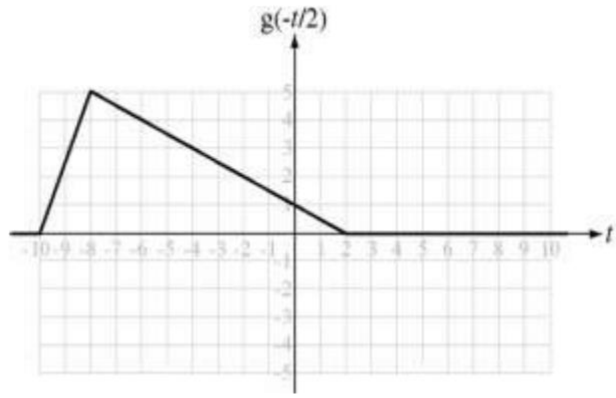


t	$t/2$	$g(t/2)$
−10	−5	0
−8	−4	0
−6	−3	0
−4	−2	0
−2	−1	0
0	0	1
2	1	2
4	2	3
6	3	4
8	4	5
10	5	0

Example: Time-scaling (ii) If a < 0, the function is also time inverted. The time inversion means flipping the curve 180^0 with the g axis as the rotation axis of the flip.



t	$g(t)$
−5	0
−4	0
−3	0
−2	0
−1	0
0	1
1	2
2	3
3	4
4	5
5	0



t	$-t/2$	$g(-t/2)$
−10	5	0
−8	4	5
−6	3	4
−4	2	3
−2	1	2
0	0	1
2	−1	0
4	−2	0
6	−3	0
8	−4	0
10	−5	0

Multiple transformations: $g(t) \to Ag\left(\frac{t-t_0}{a}\right)$**: Amplitude scaling, time scaling and time shifting can be applied simultaneously. How?**

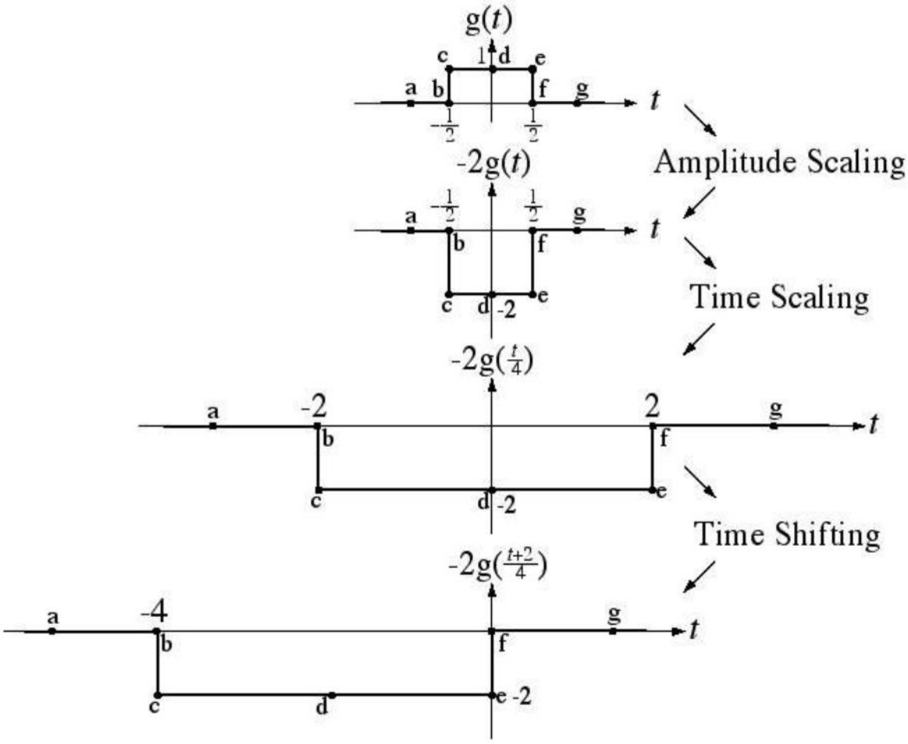
Answer: A multiple transformation can be done in steps

$$g(t) \xrightarrow{\text{amplitude scaling}, A} Ag(t) \xrightarrow{t\to t/a} Ag\left(\frac{t}{a}\right) \xrightarrow{t\to t-t_0} Ag\left(\frac{t-t_0}{a}\right)$$

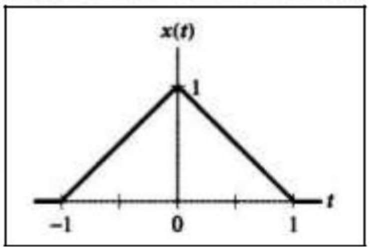
The order of the changes is important. For example, if we exchange the order of the time-scaling and time-shifting operations, we get:

$$g(t) \xrightarrow{\text{amplitude scaling}, A} Ag(t) \xrightarrow{t\to t-t_0} Ag(t-t_0) \xrightarrow{t\to t/a} Ag\left(\frac{t}{a}-t_0\right) \neq Ag\left(\frac{t-t_0}{a}\right)$$

#A sequence of amplitude scaling, time scaling and time shifting: $g(t) \to Ag\left(\frac{t-t_0}{a}\right)$



▶ **Problem 1.14** A triangular pulse signal $x(t)$ is depicted in Fig. 1.26. Sketch each of the following signals derived from $x(t)$:



(a) $x(3t)$

(b) $x(3t + 2)$

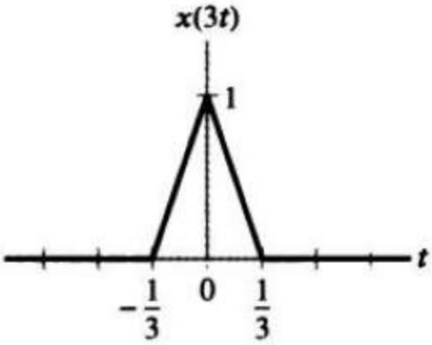
(c) $x(-2t - 1)$

(d) $x(2(t + 2))$

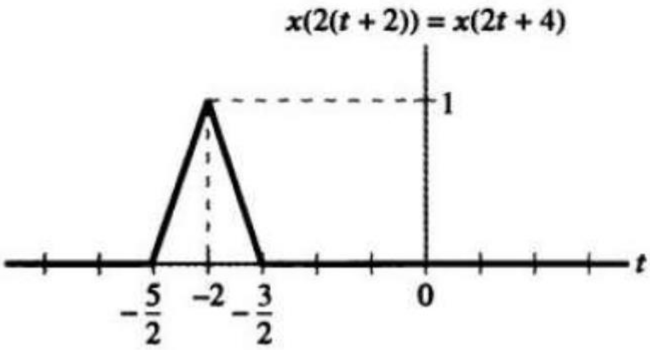
(e) $x(2(t - 2))$

(f) $x(3t) + x(3t + 2)$

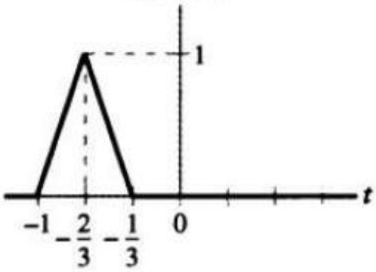
Answer:



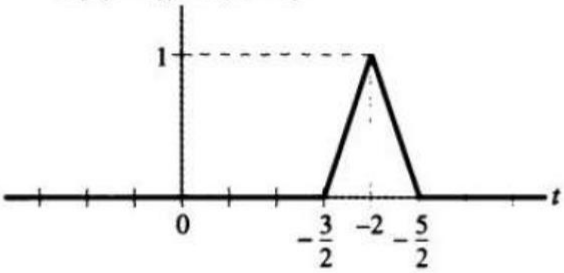
(a)



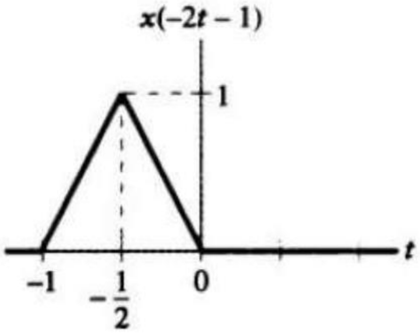
(d)



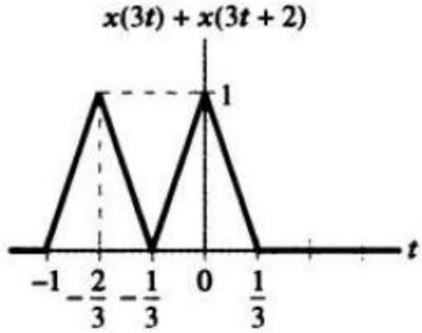
(b)



(e)



(c)



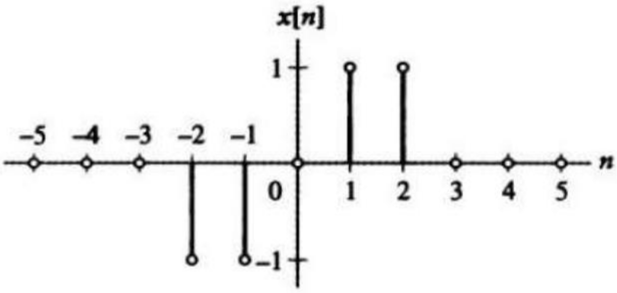
(f)

EXAMPLE 1.6 PRECEDENCE RULE FOR DISCRETE-TIME SIGNAL A discrete-time signal is defined by

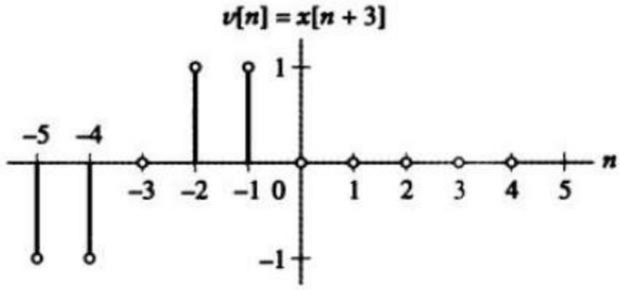
$$x[n] = \begin{cases} 1, & n = 1, 2 \\ -1, & n = -1, -2 \\ 0, & n = 0 \text{ and } |n| > 2 \end{cases}.$$

Find $y[n] = x[2n + 3]$.

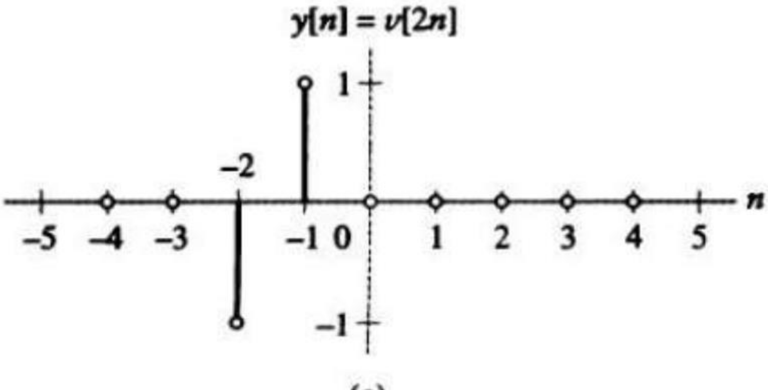
Answer:



(a)



(b)



(c)

Signals and Systems
Lecture # 4 Topic: Elementary Signals

- Impulse function
- Step function
- Exponential signals
- Sinusoidal signals
- Exponentially damped sinusoidal signals
- Ramp function
- Pulse function
- Square or Rectangular signal

(i) Impulse function: Another name ***Dirac Delta function***

The discrete-time version of the impulse-function, commonly denoted by δ[n] is defined as follows:

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

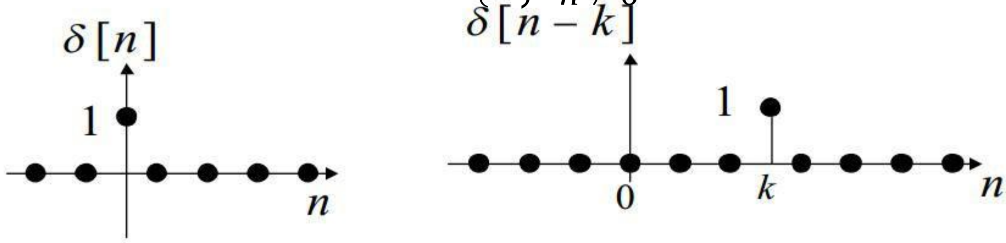
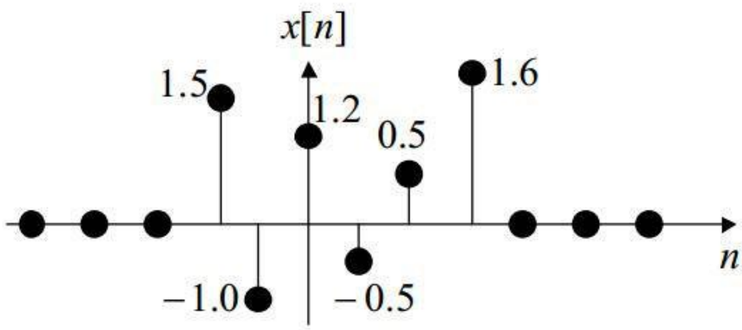


Figure: An impulse function δ[n] and a shifted impulse δ[n – k]

Any discrete time signal can be expanded into the superposition of elementary shifted impulses, each one representing each of the samples. This is expressed as

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

where each term $x[k]\delta[n-k]$ in the summation expresses the n-th sample of the sequence.



Example: the sequence shown in above figure can be expanded as

x[n] = 1.5 δ [n + 2] - 1.0 δ [n + 1] + 1.2 δ [n] - 0.5 δ [n - 1] + 0.5 δ [n - 2] + 1.6 δ [n - 3]

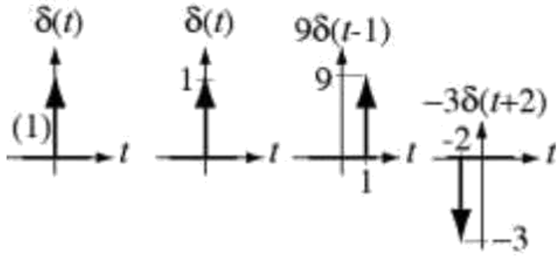
The continuous-time version of the step-function, commonly denoted by δ(t) is defined by the following pairs of relations:

$\delta(t) = 0, \textit{ for } t \neq 0$; Impulse δ(t) is zero except at the origin

and $\int_{-\infty}^{\infty} \delta(t)dt = 1$; Total area under the unit impulse is infinity

Graphical representation of the impulse:

The area under an impulse is called its strength or weight. It is represented graphically by a vertical arrow. Its strength is either written beside it or is represented by its length. An impulse with strength of one is called a unit impulse.



(ii) Step function: a battery or a dc source

The discrete-time version of the step-function, commonly denoted by u[n] is defined as follows:

$$u[n] = \begin{cases} k, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

The continuous-time version of the step-function, commonly denoted by u(t) is defined as follows:

$$u(t) = \begin{cases} k, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Unit step function:

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}, \qquad u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

It can be show that

$$u[n] = \sum_{k=-\infty}^{\infty} \delta[k] \qquad\qquad u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$$

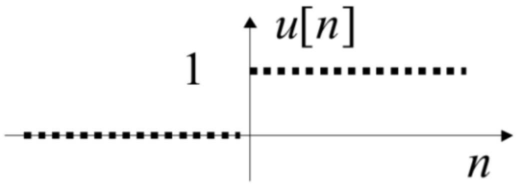


Figure: Unit step function

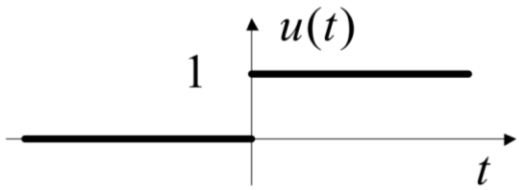


Figure: Unit step function

(iii) Exponential signals: Example – Lossy capacitor

CT signal: $x(t) = Be^{at}$, where B and a are real parameters, B=amplitude at time t=0

Two cases of exponential signal:

Decaying exponential: $a < 0$
Growing exponential: $a > 0$

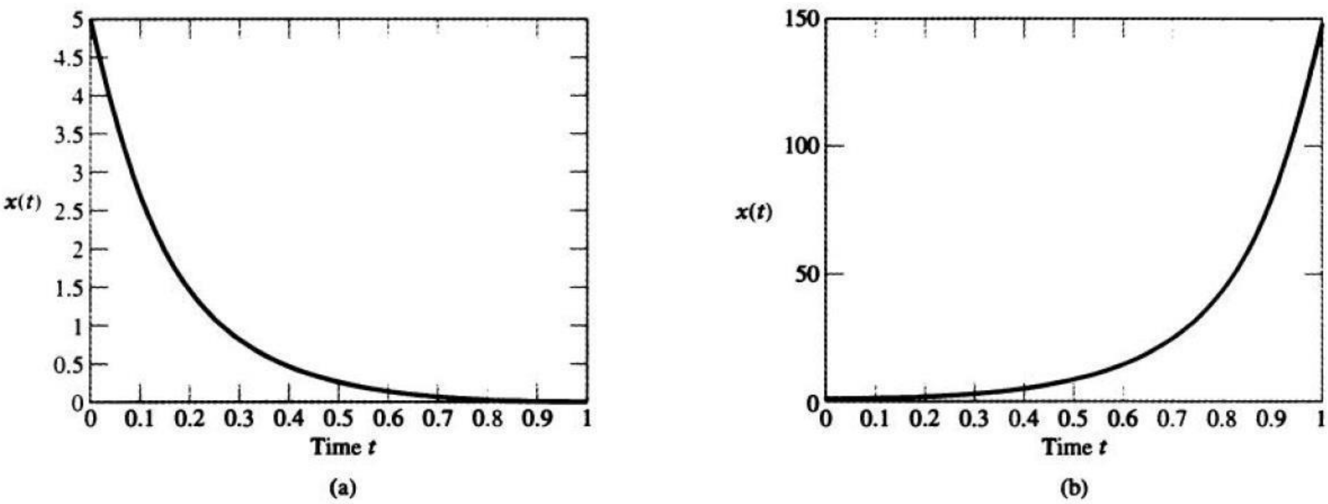


FIGURE 1.28 (a) Decaying exponential form of continuous-time signal. (b) Growing exponential form of continuous-time signal.

DT signal: $x[n] = Br^{n}$, where r=e$^{\alpha}$

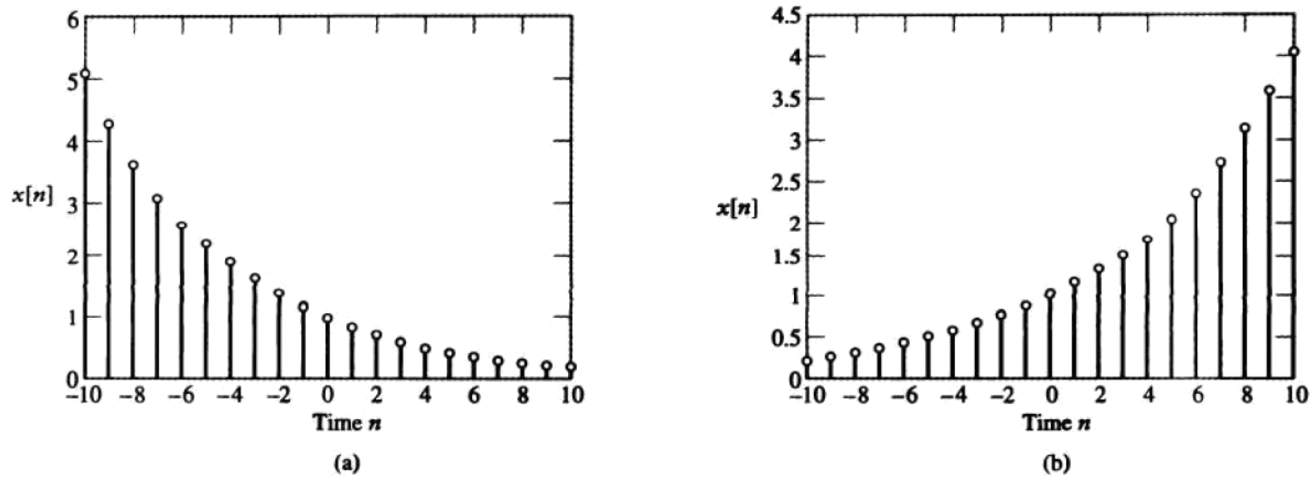


FIGURE 1.30 (a) Decaying exponential form of discrete-time signal. (b) Growing exponential form of discrete-time signal.

(iv) Sinusoidal Signal: Example – an AC voltage, sinusoidal voltage

CT signal: $x(t) = A\cos(wt + \phi)$, A=amplitude, w=frequency in rad/sec, and φ=phase angle. A sinusoidal signal is an example of periodic signal, the period is $T = \frac{2\pi}{\omega}$

We can easily show that this is a periodic signal $x(t) = x(t + T)$

Proof:

$$x(t + T) = A\cos(w(t + T) + \phi) = A\cos(wt + wT + \phi)$$
$$= A\cos(wt + 2\pi + \phi) = A\cos(wt + \phi) = x(t)$$

DT signal: $x[n] = A\cos(\Omega n + \phi)$

The period of a periodic discrete-time signal is measured in samples. The angular frequency is given by, $\Omega = \frac{2\pi m}{N}$ rad/cycle, where m is an integer.

It can be easily shown that $x[n] = x[n + N]$

Proof:

$$x[n + N] = A\cos(\Omega(n + N) + \phi) = A\cos(\Omega n + \Omega N + \phi)$$
$$= A\cos(\Omega n + \frac{2\pi m}{N} N + \phi) = A\cos(\Omega n + 2\pi m + \phi) = A\cos(\Omega n + \phi) = x[n]$$

Qs: For a sinusoidal signal, prove that $x[n + N] = x[n]$

Qs: For a sinusoidal signal, prove that $x(t + T) = x(t)$

Qs. Consider the following sinusoidal signals: Determine whether each x[n] is period, and if it is, find its fundamental period.

(a) $x[n] = 5\sin[2n]$ - non-periodic

(b) $x[n] = 5\cos[0.2\pi n]$ - periodic, fundamental period=10

(c) $x[n] = 5\cos[6\pi n]$ - periodic, fundamental period=1

(d) $x[n] = 5\sin[6\pi n / 35]$ - periodic, fundamental period=35

***Relation between sinusoidal and complex exponential signals

(v) Exponentially damped sinusoidal signals

CT signal: $x(t) = Ae^{-at}\sin(wt + \phi), \quad a > 0$;

DT signal: $x[n] = Br^{n}\sin[\Omega n + \phi)$; r must lie in the range 0 < |r| < 1

(vi) Ramp function:

The discrete-time version of the ramp-function, commonly denoted by *r*[n] is defined as follows:

$$r[n] = \begin{cases} n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$ or equivalently, $u[n] = nu[n]$

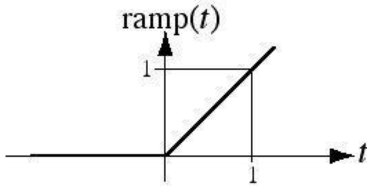
The continuous-time version of the step-function, commonly denoted by *r*(t) is defined as follows:

$$r(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

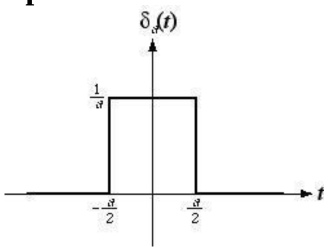
Unit-ramp function:

- The unit ramp function is the integral of the unit step function.
- It is called the unit ramp function because for positive t, its slope is one amplitude unit per time.

$$\text{ramp}(t) = \begin{cases} t & , \ t > 0 \\ 0 & , \ t \leq 0 \end{cases} = \int_{-\infty}^{t} u(\lambda)\, d\lambda = t\,u(t)$$



(vii) Rectangular pulse function:

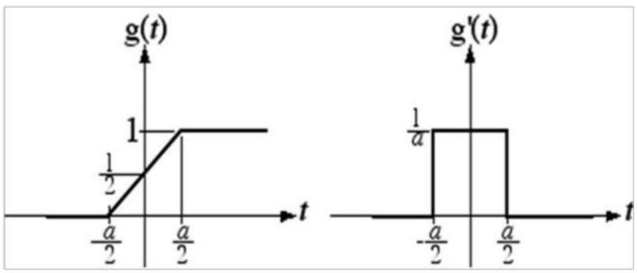


$$\delta_a(t) = \begin{cases} 1/a & , |t| < a/2 \\ 0 & , |t| > a/2 \end{cases}$$

(viii) Unit-step and unit impulse function:

The unit step is the integral of the unit impulse and the unit impulse is the generalized derivative of the unit step

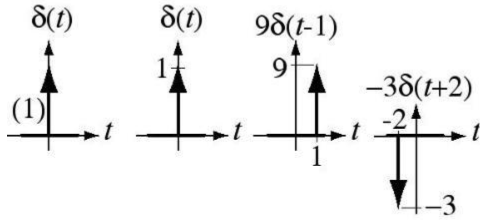
As a approaches zero, $g(t)$ approaches a unit step and $g'(t)$ approaches a unit impulse



Functions that approach unit step and unit impulse

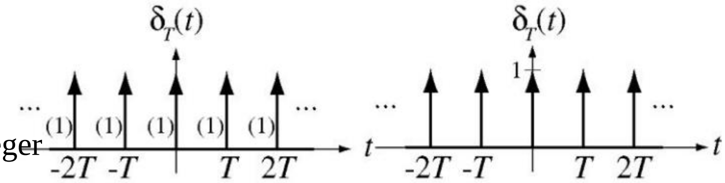
Graphical representation of the impulse:

The area under an impulse is called its strength or weight. It is represented graphically by a vertical arrow. Its strength is either written beside it or is represented by its length. An impulse with strength of one is called a unit impulse.



Unit periodic impulse: The unit periodic impulse/impulse train is defined by

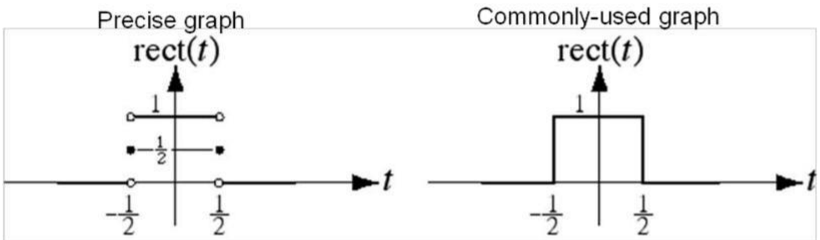
$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT), \quad n \text{ an integer}$$



The periodic impulse is a sum of infinitely many uniformly-spaced impulses

Unit Rectangular function:

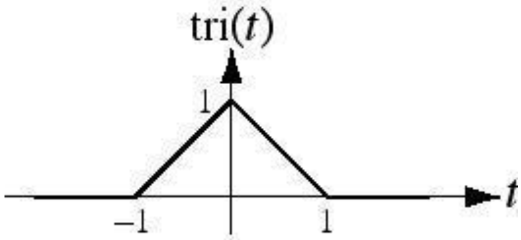
$$\mathrm{rect}(t) = \begin{cases} 1, & |t| < 1/2 \\ 1/2, & |t| = 1/2 \\ 0, & |t| > 1/2 \end{cases} = \mathrm{u}(t+1/2) - \mathrm{u}(t-1/2)$$



The signal "turned on" at time t = -1/2 and "turned back off" at time t = +1/2.

Unit triangular function:

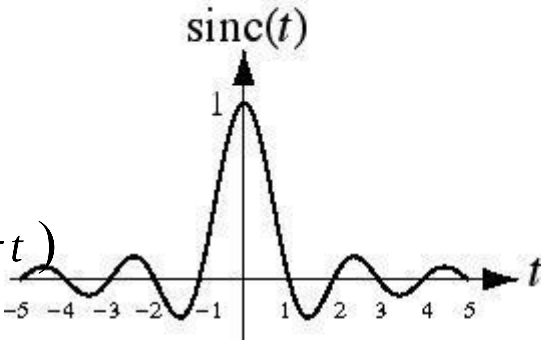
$$\mathrm{tri}(t) = \begin{cases} 1 - |t|, & |t| < 1 \\ 0, & |t| \geq 1 \end{cases}$$



The unit triangle is related to the unit rectangle through an operation called convolution. It is called a unit triangle because its height and area are both one (but its base width is not).

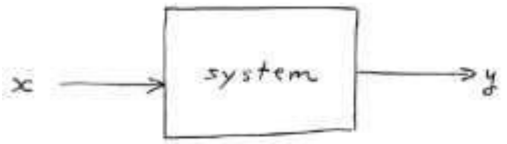
Unit Sinc function: The unit sinc function is related to the unit rectangle function through the Fourier transform.

$$\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$

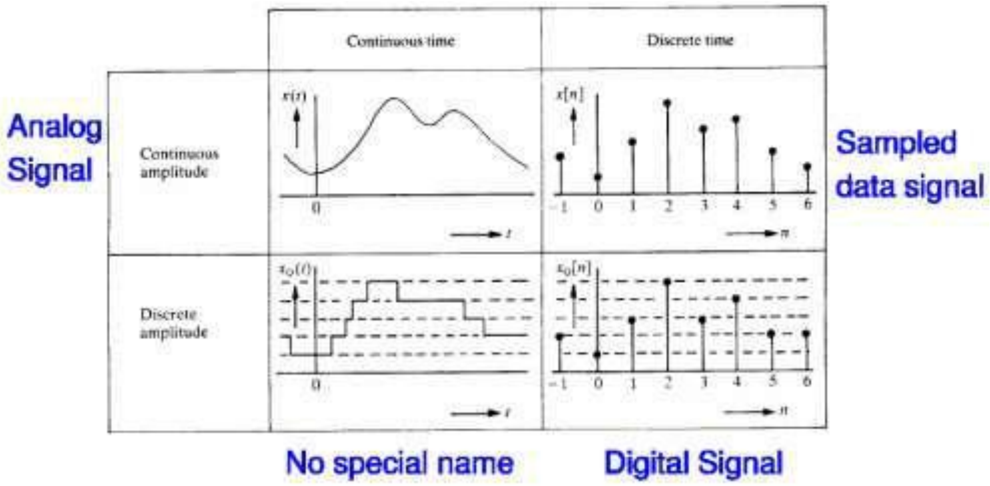


Signals and Systems
Lecture #5 Topic: Systems and Linear Systems
Additional Study: Analysis of Linear Systems by D. K. Cheng, Chapter 1 & 2

Systems: A system is an operation that transforms input signal *x* into output signal *y.*

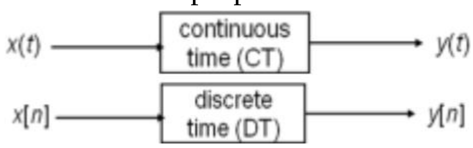


Types of Signals:



Linear systems:

- A system takes a signal as an input and transforms it into another signal
- Linear systems play a crucial role in most areas of science
  - Closed-form solutions often exist
  - Theoretical analysis is considerably simplified
  - Non-linear systems can often be regarded as linear, for small perturbations, so-called linearization
- For the remainder of the lecture/course, we're primarily going to be considering Linear, Time-Invariant Systems (LTI) and consider their properties



Examples of Simple Systems:

To get some idea of typical systems (and their properties), consider the electrical circuit example:

$\frac{dv_c(t)}{dt} + \frac{1}{RC} v_c(t) = \frac{1}{RC} v_s(t)$ which is **first order**, **CT differential** equation.

Examples of the **first order**, **DT difference** equations: $y[n] = x[n] + 1.01 y[n-1]$
where *y* is the monthly bank balance, and *x* is monthly net deposit.

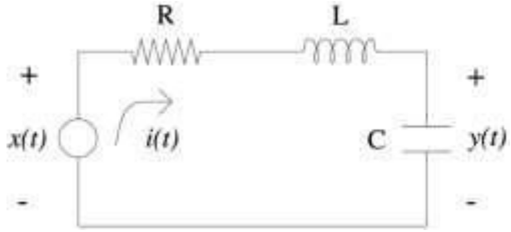
$v[n] - \frac{RC}{RC+k} v[n-1] = \frac{k}{RC+k} f[n]$ which represents a discretized version of the electrical circuit

Example of the second-order system includes: $a\frac{d^2 y(t)}{dt^2} + b\frac{dy(t)}{dt} + cy(t) = x(t)$

System described by **order** and **parameters** (*a*, *b*, *c*)

Some Examples of Systems

Electrical system: An RLC circuit



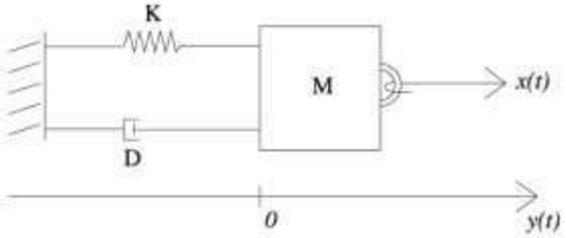
$$Ri(t) + L\frac{di(t)}{dt} + y(t) = x(t)$$

$$i(t) = C\frac{dy(t)}{dt}$$

$$\Downarrow$$

$$LC\frac{d^2y(t)}{dt^2} + RC\frac{dy(t)}{dt} + y(t) = x(t)$$

Mechanical System:



$$M\frac{d^2y(t)}{dt^2} = x(t) - Ky(t) - D\frac{dy(t)}{dt}$$

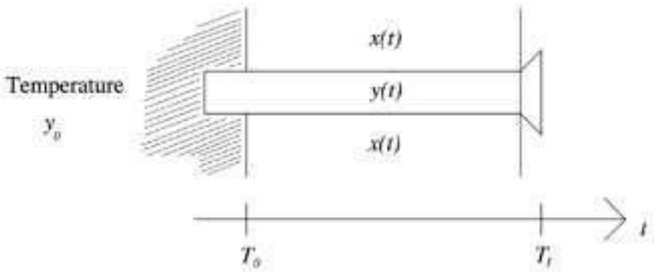
$$\Downarrow$$

$$M\frac{d^2y(t)}{dt^2} + D\frac{dy(t)}{dt} + Ky(t) = x(t)$$

Here, x(t) – applied force, K -spring constant, D- damping constant, y(t) – displacement from rest

Observation: Very different physical systems may be modeled mathematically in very similar ways.

Thermal system: Cooling Fin in steady-state



Here, t= distance along rod, y(t) = Fin temperature as a function of time, x(t) surrounding temperature along the Fin

$$\frac{d^2y(t)}{dt^2} = k[y(t) - x(t)]$$

$$y(T_0) = y_0$$

$$\frac{dy}{dt}(T_1) = 0$$

Observation:

- Independent variable can be something other than time, such as space.
- Such systems may, more naturally, have boundary conditions, rather than "initial" conditions.

Financial system:

Fluctuations in the price of zero-coupon bonds

$t = 0$ Time of purchase at price, y_0

$t = T$ Time of maturity at value y_T

$y(t)$ = Values of bond at time t

$x(t)$= Influence of external factors on fluctuations in bond price

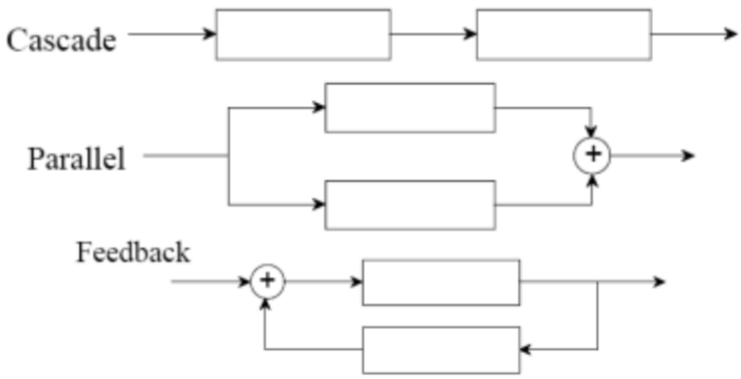
$$\frac{d^2y(t)}{dt^2} = f\left(y(t), \frac{dy(t)}{dt}, x_1(t), x_2(t), \ldots, x_N(t), t\right)$$
$$y(0) = y_0$$
$$y(T) = y_T$$

Observation: Even if the independent variable is time, there are interesting and important systems which have boundary conditions.

Dynamics of an aircraft or space vehicle
An algorithm for analyzing financial and economic factors to predict bond prices
An algorithm for post-flight analysis of a space launch
An edge detection algorithm for medical images

Systems Interconnections:

- An important concept is that of interconnecting systems
  - To build more complex systems by interconnecting simpler subsystems
  - To modify the response of a system
- Signal flow (Block) diagram



System Linearity: Specifically, a linear system must satisfy the two properties:

1 Additive: the response to $x_1(t)+x_2(t)$ is $y_1(t) + y_2(t)$

2 Scaling: the response to $ax_1(t)$ is $ay_1(t)$ where $a \in C$

Combined: $ax_1(t)+bx_2(t) \rightarrow ay_1(t) + by_2(t)$

E.g. Linear $y(t) = 3*x(t)$ why?

Non-linear $y(t) = 3*x(t)+2$, $y(t) = 3*x^2(t)$ why?

(equivalent definition for DT systems)

System Properties: Why?

- Important practical/physical implications
- They provide us with insight and structure that we can exploit both to analyze and understand systems more deeply.

1.8.1 Stability:

A system is said to be bounded input bounded output (BIBO) stable if and only if every bounded input results in a bounded output. The output of such a system does not diverge if the input does not diverge.

The operator H is BIBO stable if the output signal y(t) satisfies the condition $|y(t)| \le M_y < \infty$ for all t

Whenever the input x(t) signals satisfy the condition

$$|x(t)| \le M_x < \infty \quad \text{for all } t$$

Both M_x and M_y represent finite positive numbers

Example 1.13: show that the moving-average system described by the input-output relation $y[n] = \frac{1}{3}\left(x[n] + x[n-1] + x[n-2]\right)$ is BIBO stable.

Answer: Assume, $|x[n]| \le M_X < \infty$ for all n

Using the input-output relation, $y[n] = \frac{1}{3}\left(x[n] + x[n-1] + x[n-2]\right)$

We may write,

$$
\begin{aligned}
|y[n]| &= \frac{1}{3}\left|x[n] + x[n-1] + x[n-2]\right| \\
&\le \frac{1}{3}\left(|x[n]| + |x[n-1]| + |x[n-2]|\right) \\
&\le \frac{1}{3}\left(M_X + M_X + M_X\right) \\
&= M_x
\end{aligned}
$$

Hence, |y[n]| is always less than the maximum value of |x[n]| for all value of *n*, which shows that the moving-average system is stable.

Example 1.14: Consider a discrete-time system whose input-output relation is defined by $y[n] = r^n x[n]$, where r>1. Show that the system is unstable.

Answer: Assume input signal x[n] satisfies the condition, $|x[n]| \le M_X < \infty$ for all n We then find that,

$$|y[n]| = |r^n x[n]| = |r^n| . |x[n]|$$

With r > 1, r^n → diverges for increasing n

Therefore, bounded input signal does not guarantee a bounded output signal, so the system is unstable.

1.8.2 Memory

A system is said to process memory if its output signal depends on past values of the input signal.

- A system is said to be memoryless if its output for each value of the independent variable at a given time is dependent on the output at only that same time (no system dynamics)

$$y[n] = (2x[n] - x^2[n])^2$$

- e.g. a resistor is a memoryless CT system where *x*(*t*) is current and *y*(*t*) is the voltage

$$x(t) = \frac{1}{R} y(t) \leftrightarrow i(t) = \frac{1}{R} v(t)$$

- An inductor has memory since the current x(t) flowing through it is related to the applied voltage v(t) as follows:

$$x(t) = \frac{1}{L}\int_{-\infty}^{t} y(\tau)d\tau \leftrightarrow i(t) = \frac{1}{L}\int_{-\infty}^{t} v(\tau)d\tau$$

  The memory of an inductor extends into the infinite past.
- The moving average system describe by the input-output relation $y[n] = \frac{1}{3}\left(x[n] + x[n-1] + x[n-2]\right)$ has memory, since the values of output signal y[n] at time n depends on the present and two past values of the input signal x[n].
- A DT system with memory is an accumulator (integrator) $y[n] = \sum_{k=-\infty}^{n} x[k]$ and a delay $y[n] = x[n-1]$

- Roughly speaking, a memory corresponds to a mechanism in the system that retains information about input values other than the current time.

$$y[n] = \sum_{k=-\infty}^{n-1} x[k] + x[n] = y[n-1] + x[n]$$

1.8.3 Causality:

- A system is said to be causal if the present value of output signal depends only on the present or past values of the input signal. In contrast, the output signal of a non-causal system depends on one or more future values of the input signal.
- All real-time physical systems are causal because time only moves forward. The effect occurs after cause. (Imagine if you own a noncausal system whose output depends on tomorrow's stock price.)
- Causality does not apply to spatially varying signals. (We can move both left and right, up and down.)
- Causality does not apply to systems processing recorded signals, e.g. taped sports games vs. live broadcast.

For example, the moving-average system described by $y[n] = \frac{1}{3}\left(x[n] + x[n-1] + x[n-2]\right)$ is causal.

By contrast, the moving-average system described by $y[n] = \frac{1}{3}\left(x[n+1] + x[n] + x[n-1]\right)$ is noncausal, since the output signal *y[n]* depends on a future value of the input signal, namely, x[n + 1].

Mathematically (in CT): A system $x(t) \to y(t)$ is causal if

When $x_1(t) \to y_1(t)$ $\quad x_2(t) \to y_2(t)$
And $x_1(t) = x_2(t)$ for all $t \le t_o$

State whether the systems are Causal or Non-causal

$y(t) = x^2(t-1)$: y(5) depends on x(4) → Causal

$y(t) = x(t+1)$: y(5) depends on future → Noncausal

$y[n] = x[-n]$: y[5]=x[-5] ok, but y[-5]=x[5] depend on future → Noncausal

$y[n] = \left(\frac{1}{2}\right)^{n+1} x^3[n-1]$: y[5] depend on x[4] → Causal

1.8.4 Invertibility: A system is said to be invertible if the input of the system can be recovered from the system output

Let the operator H represent a continuous-time system, with the input signal x(t) producing the output signal y(t).

$$\xrightarrow{x(t)} H \xrightarrow{y(t)} H^{-1} \xrightarrow{x(t)}$$

The output of the second system is defined by

$$\begin{aligned} H^{-1}\{y(t)\} &= H^{-1}\{H\{x(t)\}\} \\ &= H^{-1}H\{x(t)\} \end{aligned}$$

▶ **Problem 1.32** An inductor is described by the input–output relation

$$y(t) = \frac{1}{L}\int_{-\infty}^{t} x(\tau)\, d\tau.$$

Find the operation representing the inverse system.

Answer: $x(t) = L\frac{d}{dt}y(t)$

EXAMPLE 1.16 NON-INVERTIBLE SYSTEM Show that a square-law system described by the input–output relation

$$y(t) = x^2(t)$$

is not invertible.

Solution: Note that the square-law system violates a necessary condition for invertibility, namely, that distinct inputs must produce distinct outputs. Specifically, the distinct inputs $x(t)$ and $-x(t)$ produce the same output $y(t)$. Accordingly, the square-law system is not invertible. ■

1.8. 5 Time-invariance:

A system is said to be *time invariant* if a time delay or time advance of the input signal leads to an identical time shift in the output signal. This implies that a time-invariant system responds identically no matter when the input signal is applied.

Informally, a system is time-invariant (TI) if its behavior does not depend on what time it is.

Mathematically (in DT): A system $x[n] \rightarrow y[n]$ is TI if for any input $x[n]$ and anytime shift n_0,

If $x[n] \rightarrow y[n]$, then $x[n - n_0] \rightarrow y[n - n_0]$.

Similarly for a CT time-invariant system,

If $x(t) \rightarrow y(t)$, then $x(t - t_0) \rightarrow y(t - t_0)$

Fact: If the input to a TI System is periodic, then the output is periodic with the same period.

Proof: Suppose, x(t + T) = x(t)
and x(t) →y(t)
Then by TI, x(t + T) → y(t + T)
These are the same input, so these must be the same output.
i.e., y(t) = y(t + T).

Check whether the system is TI or time-varying?

$y(t) = x^2(t+1)$: Time-Invariant (TI)

$y[n] = \left(\frac{1}{2}\right)^{n+1} x^3[n-1]$: Time-varying

1.8.6 Linearity:

A system is said to be ***linear*** in terms of the system input (excitation) $x(t)$ and the system output (response) $y(t)$ if it satisfies the following two properties of superposition and homogeneity:

1. *Superposition.* Consider a system that is initially at rest. Let the system be subjected to an input $x(t) = x_1(t)$, producing an output $y(t) = y_1(t)$. Suppose next that the same system is subjected to a different input $x(t) = x_2(t)$, producing a corresponding output $y(t) = y_2(t)$. Then for the system to be linear, it is necessary that the composite input $x(t) = x_1(t) + x_2(t)$ produce the corresponding output $y(t) = y_1(t) + y_2(t)$. What we have described here is a statement of the ***principle of superposition*** in its simplest form.
2. *Homogeneity.* Consider again a system that is initially at rest, and suppose an input $x(t)$ results in an output $y(t)$. Then the system is said to exhibit the property of homogeneity if, whenever the input $x(t)$ is scaled by a constant factor a, the output $y(t)$ is scaled by exactly the same constant factor a.

When a system violates either the principle of superposition or the property of homogeneity, the system is said to be ***nonlinear***.

Let, H represents a continuous-time system, Let signal applied to the system input be defined by the weighted sum

$$x(t) = \sum_{i=1}^{N} a_i x_i(t)$$

Where, $x_1(t)$, $x_2(t)$, …, $x_N(t)$ denotes a set of input signals, and a_1, a_2, …, a_N denote the corresponding weighting factors. The resulting output signal is written as –

$$y(t) = H\{x(t)\} = H\left\{\sum_{i=1}^{N} a_i x_i(t)\right\}$$

If the system is linear, we may express the output signal y(t) of the system as

$$y(t) = \sum_{i=1}^{N} a_i y_i(t) \text{ ,}$$

where $y_i(t)$ is the output of the system in response to the input $x_i(t)$ acting alone.

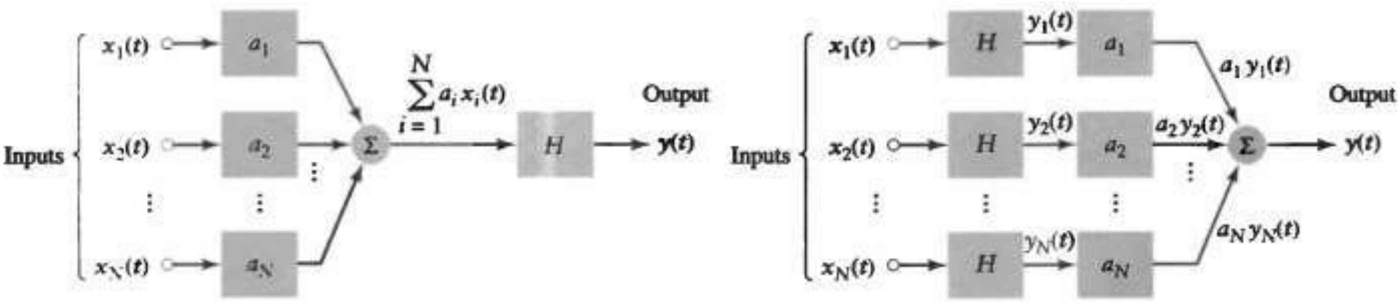


FIGURE 1.56 The linearity property of a system. (a) The combined operation of amplitude scaling and summation precedes the operator H for multiple inputs. (b) The operator H precedes amplitude scaling for each input; the resulting outputs are summed to produce the overall output y(*t*). If these two configurations produce the same output *y(t}*, the operator H is linear.

Examples of Linear and non-linear system:

Examples of ***Linear*** Systems

Wave propagation such as sound and electromagnetic waves
Electrical circuits composed of resistors, capacitors, and inductors
Electronic circuits, such **as** amplifiers and filters
Mechanical motion from the interaction of masses, springs, and dashpots (dampeners)
Systems described by differential equations such as resistor-capacitor-inductor networks
Multiplication by a Constant, that **is,** amplification **or** attenuation of the signal
Signal changes, such as echoes, resonances, and image blurring
The unity system where the output is always equal to the input
The null system where the output is always equal to the zero, regardless of the input
Differentiation and integration, and the analogous operations *of first* ***difference*** and ***running sum*** for discrete signals
Small perturbations in an otherwise nonlinear system: for instance, a small signal being amplified by a properly biased transistor
Convolution, a mathematical operation where each value in the output is expressed as the sum of values in the input multiplied by a set of weighing coefficients
Recursion, a technique similar to convolution, except previously calculated values in the output are used in addition to values from the input

Examples of ***Nonlinear*** Systems

Systems that do not have static linearity: for instance, the voltage and power in a resistor: $P = V^2/R$, the radiant energy emission of a hot object depending on its temperature: $R = kT^4$, the intensity of light transmitted through a thickness of translucent material: $I = e^{\beta t}$.

Systems that do not have sinusoidal fidelity, such as electronics circuits for: peak detection, squaring, sine wave ***to*** square wave conversion, frequency doubling, etc.
Common electronic distortion, such as clipping, crossover distortion and slewing
Multiplication of one signal by another signal, such as in amplitude modulation and automatic gain controls
Hysteresis phenomena, such as magnetic **flux** density versus magnetic intensity in iron, or mechanical stress versus strain in vulcanized rubber
Saturation, such as electronic amplifiers and transformers driven **too** hard
Systems with a threshold: for example, digital logic gates, **or** seismic vibrations that are strong enough to pulverize the intervening rock

Formally, linear systems are defined by the properties **of *homogeneity, additivity,*** and ***shift invariance.*** Informally, most scientists and engineers think lof linear systems in terms of ***static linearity*** and ***sinusoidal fidelity.***

Linearity: A CT system is linear if it has the superposition property:

If $x_1(t) \rightarrow y_1(t)$ and $x_2(t) \rightarrow y_2(t)$

then $ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t)$

$y[n] = x^2[n]$ Nonlinear, TI, Causal

$y(t) = x(2t)$ Linear, not TI, Noncausal

Can you find systems with other combinations ?
- e.g. Linear, TI, Noncausal
  Linear, not TI, Causal

EXAMPLE 1.19 LINEAR DISCRETE-TIME SYSTEM Consider a discrete-time system described by the input–output relation

$$y[n] = nx[n].$$

Show that this system is linear.

Solution: Let the input signal $x[n]$ be expressed as the weighted sum

$$x[n] = \sum_{i=1}^{N} a_i x_i[n].$$

We may then express the resulting output signal of the system as

$$\begin{aligned} y[n] &= n \sum_{i=1}^{N} a_i x_i[n] \\ &= \sum_{i=1}^{N} a_i n x_i[n] \\ &= \sum_{i=1}^{N} a_i y_i[n], \end{aligned}$$

where

$$y_i[n] = nx_i[n]$$

is the output due to each input acting independently. We thus see that the given system satisfies both superposition and homogeneity and is therefore linear. ■

Self-Study: Please see the example 1.20 pp. 65 (Signals and Systems, Symon Haykin)

PROBLEMS

1.1 Find the even and odd components of each of the following signals:

(a) $x(t) = \cos(t) + \sin(t) + \sin(t)\cos(t)$

(b) $x(t) = 1 + t + 3t^2 + 5t^3 + 9t^4$

(c) $x(t) = 1 + t\cos(t) + t^2\sin(t) + t^3\sin(t)\cos(t)$

(d) $x(t) = (1 + t^3)\cos^3(10t)$

1.2 Determine whether the following signals are periodic. If they are periodic, find the fundamental period.

(a) $x(t) = (\cos(2\pi t))^2$

(b) $x(t) = \Sigma_{k=-5}^{5} w(t - 2k)$ for $w(t)$ depicted in Fig. P1.2b.

(c) $x(t) = \Sigma_{k=-\infty}^{\infty} w(t - 3k)$ for $w(t)$ depicted in Fig. P1.2b.

(d) $x[n] = (-1)^n$

(e) $x[n] = (-1)^{n^2}$

(f) $x[n]$ depicted in Fig. P1.2f.

(g) $x(t)$ depicted in Fig. P1.2g.

(h) $x[n] = \cos(2n)$

(i) $x[n] = \cos(2\pi n)$

1.3 The sinusoidal signal

$$x(t) = 3\cos(200t + \pi/6)$$

is passed through a square-law device defined by the input–output relation

$$y(t) = x^2(t)$$

Using the trigonometric identity

$$\cos^2\theta = \tfrac{1}{2}(\cos 2\theta + 1)$$

show that the output $y(t)$ consists of a dc component and a sinusoidal component.

(a) Specify the dc component.

(b) Specify the amplitude and fundamental frequency of the sinusoidal component in the output $y(t)$.

1.4 Categorize each of the following signals as an energy or power signal, and find the energy or power of the signal.

(a) $x(t) = \begin{cases} t, & 0 \le t \le 1 \\ 2 - t, & 1 \le t \le 2 \\ 0, & \text{otherwise} \end{cases}$

(b) $x[n] = \begin{cases} n, & 0 \le n \le 5 \\ 10 - n, & 5 \le n \le 10 \\ 0, & \text{otherwise} \end{cases}$

(c) $x(t) = 5\cos(\pi t) + \sin(5\pi t), \quad -\infty < t < \infty$

(d) $x(t) = \begin{cases} 5\cos(\pi t), & -1 \le t \le 1 \\ 0, & \text{otherwise} \end{cases}$

(e) $x(t) = \begin{cases} 5\cos(\pi t), & -0.5 \le t \le 0.5 \\ 0, & \text{otherwise} \end{cases}$

(f) $x[n] = \begin{cases} \sin(\pi/2\ n), & -4 \le n \le 4 \\ 0, & \text{otherwise} \end{cases}$

(g) $x[n] = \begin{cases} \cos(\pi n), & -4 \le n \le 4 \\ 0, & \text{otherwise} \end{cases}$

(h) $x[n] = \begin{cases} \cos(\pi n), & n \ge 0 \\ 0, & \text{otherwise} \end{cases}$

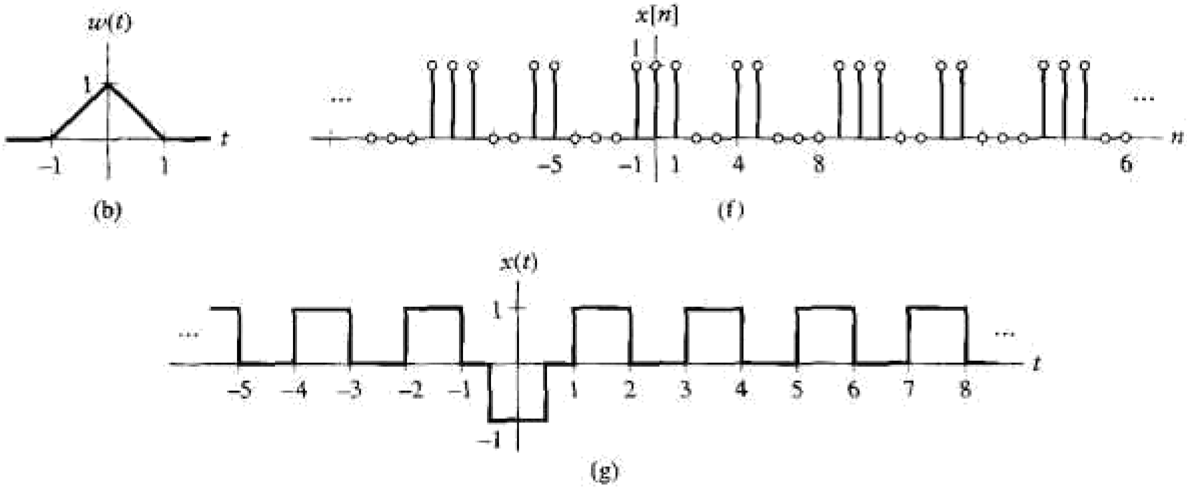


FIGURE P1.2

1.5 Consider the sinusoidal signal

$$x(t) = A\cos(\omega t + \phi)$$

Determine the average power of $x(t)$.

1.6 The angular frequency Ω of the sinusoidal signal

$$x[n] = A\cos(\Omega n + \phi)$$

satisfies the condition for $x[n]$ to be periodic. Determine the average power of $x[n]$.

1.7 The raised-cosine pulse $x(t)$ shown in Fig. P1.7 is defined as

$$x(t) = \begin{cases} \frac{1}{2}[\cos(\omega t) + 1], & -\pi/\omega \le t \le \pi/\omega \\ 0, & \text{otherwise} \end{cases}$$

Determine the total energy of $x(t)$.

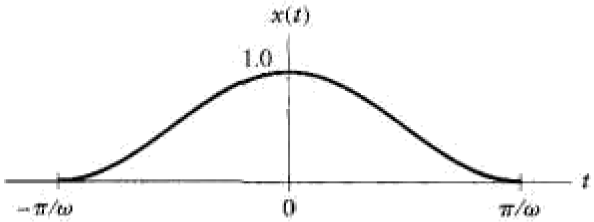


FIGURE P1.7

1.8 The trapezoidal pulse $x(t)$ shown in Fig. P1.8 is defined by

$$x(t) = \begin{cases} 5 - t, & 4 \le t \le 5 \\ 1, & -4 \le t \le 4 \\ t + 5 & -5 \le t \le -4 \\ 0, & \text{otherwise} \end{cases}$$

Determine the total energy of $x(t)$.

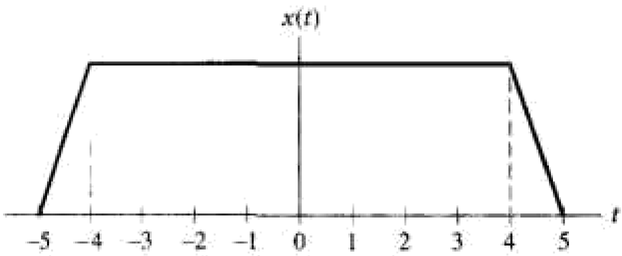


FIGURE P1.8

1.9 The trapezoidal pulse $x(t)$ of Fig. P1.8 is applied to a differentiator, defined by

$$y(t) = \frac{d}{dt}x(t)$$

(a) Determine the resulting output $y(t)$ of the differentiator.

(b) Determine the total energy of $y(t)$.

1.10 A rectangular pulse $x(t)$ is defined by

$$x(t) = \begin{cases} A, & 0 \le t \le T \\ 0, & \text{otherwise} \end{cases}$$

The pulse $x(t)$ is applied to an integrator defined by

$$y(t) = \int_0^t x(\tau)\,d\tau$$

Find the total energy of the output $y(t)$.

1.11 The trapezoidal pulse $x(t)$ of Fig. P1.8 is time scaled, producing

$$y(t) = x(at)$$

Sketch $y(t)$ for (a) $a = 5$ and (b) $a = 0.2$.

1.12 A triangular pulse signal $x(t)$ is depicted in Fig. P1.12. Sketch each of the following signals derived from $x(t)$:

(a) $x(3t)$

(b) $x(3t + 2)$

(c) $x(-2t - 1)$

(d) $x(2(t + 2))$

(e) $x(2(t - 2))$

(f) $x(3t) + x(3t + 2)$

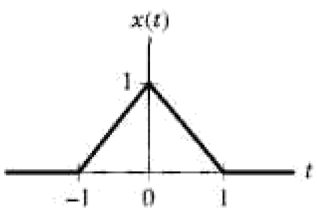


FIGURE P1.12

1.13 Sketch the trapezoidal pulse $y(t)$ that is related to that of Fig. P1.8 as follows:

$$y(t) = x(10t - 5)$$

1.14 Let $x(t)$ and $y(t)$ be given in Figs. P1.14(a) and (b), respectively. Carefully sketch the following signals:

(a) $x(t)y(t - 1)$

(b) $x(t - 1)y(-t)$

(c) $x(t + 1)y(t - 2)$

(d) $x(t)y(-1 - t)$

(e) $x(t)y(2 - t)$

(f) $x(2t)y(\frac{1}{2}t + 1)$

(g) $x(4 - t)y(t)$

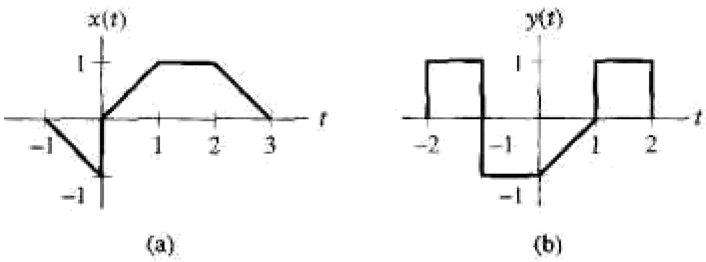


FIGURE P1.14

1.15 Figure P1.15(a) shows a staircase-like signal $x(t)$ that may be viewed as the superposition of four rectangular pulses. Starting with the rectangular pulse $g(t)$ shown in Fig. P1.15(b), construct this waveform, and express $x(t)$ in terms of $g(t)$.

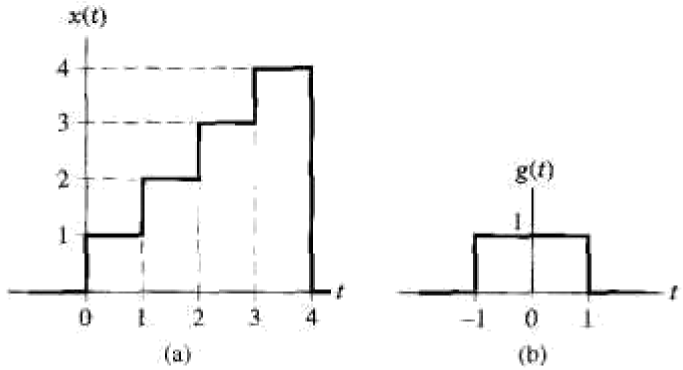


FIGURE P1.15

1.16 Sketch the waveforms of the following signals:

(a) $x(t) = u(t) - u(t - 2)$

(b) $x(t) = u(t + 1) - 2u(t) + u(t - 1)$

(c) $x(t) = -u(t + 3) + 2u(t + 1) - 2u(t - 1) + u(t - 3)$

(d) $y(t) = r(t + 1) - r(t) + r(t - 2)$

(e) $y(t) = r(t + 2) - r(t + 1) - r(t - 1) + r(t - 2)$

1.17 Figure P1.17(a) shows a pulse $x(t)$ that may be viewed as the superposition of three rectangular pulses. Starting with the rectangular pulse $g(t)$ of Fig. P1.17(b), construct this waveform, and express $x(t)$ in terms of $g(t)$.

1.18 Let $x[n]$ and $y[n]$ be given in Figs. P1.18(a) and (b), respectively. Carefully sketch the following signals:

(a) $x[2n]$

(b) $x[3n - 1]$

(c) $y[1 - n]$

(d) $y[2 - 2n]$

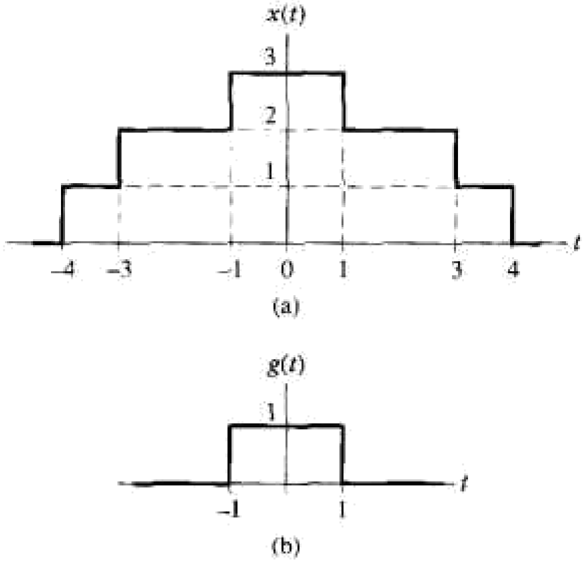


FIGURE P1.17

(e) $x[n - 2] + y[n + 2]$

(f) $x[2n] + y[n - 4]$

(g) $x[n + 2]y[n - 2]$

(h) $x[3 - n]y[n]$

(i) $x[-n]y[-n]$

(j) $x[n]y[-2 - n]$

(k) $x[n + 2]y[6 - n]$

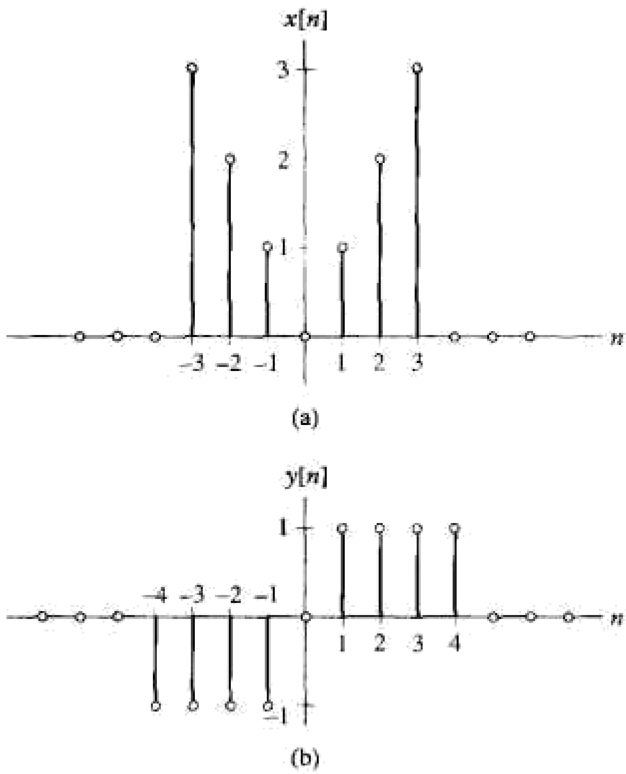


FIGURE P1.18

1.19 Consider the sinusoidal signal

$$x[n] = 10 \cos\left(\frac{4\pi}{31} n + \frac{\pi}{5}\right)$$

Determine the fundamental period of $x(n)$.

1.20 *The sinusoidal signal $x[n]$ has fundamental period* $N = 10$ samples. Determine the smallest angular frequency Ω for which $x[n]$ is periodic.

1.21 Determine whether the following signals are periodic. If they are periodic, find the fundamental period.

(a) $x[n] = \cos(\frac{8}{15}\pi n)$

(b) $x[n] = \cos(\frac{7}{15}\pi n)$

(c) $x(t) = \cos(2t) + \sin(3t)$

(d) $x(t) = \sum_{k=-\infty}^{\infty}(-1)^k \delta(t - 2k)$

(e) $x[n] = \sum_{k=-\infty}^{\infty}\{\delta[n - 3k] + \delta[n - k^2]\}$

(f) $x(t) = \cos(t)u(t)$

(g) $x(t) = v(t) + v(-t)$, where $v(t) = \cos(t)u(t)$

(h) $x(t) = v(t) + v(-t)$, where $v(t) = \sin(t)u(t)$

(i) $x[n] = \cos(\frac{1}{5}\pi n)\sin(\frac{1}{3}\pi n)$

1.22 A complex sinusoidal signal $x(t)$ has the following components:

$$\mathrm{Re}\{x(t)\} = x_R(t) = A\cos(\omega t + \phi)$$
$$\mathrm{Im}\{x(t)\} = x_I(t) = A\sin(\omega t + \phi)$$

The amplitude of $x(t)$ is defined by the square root of $x_R^2(t) + x_I^2(t)$. Show that this amplitude equals A, independent of the phase angle ϕ.

1.23 Consider the complex-valued exponential signal

$$x(t) = Ae^{\alpha t + j\omega t}, \qquad \alpha > 0$$

Evaluate the real and imaginary components of $x(t)$.

1.24 Consider the continuous-time signal

$$x(t) = \begin{cases} t/T + 0.5, & -T/2 \le t \le T/2 \\ 1, & t \ge T/2 \\ 0, & t < -T/2 \end{cases}$$

which is applied to a differentiator. Show that the output of the differentiator approaches the unit impulse $\delta(t)$ as T approaches zero.

1.25 In this problem, we explore what happens when a unit impulse is applied to a differentiator. Consider a triangular pulse $x(t)$ of duration T and amplitude $1/2T$, as depicted in Fig. P1.25. The area under the pulse is unity. Hence as the duration T approaches zero, the triangular pulse approaches a unit impulse.

(a) Suppose the triangular pulse $x(t)$ is applied to a differentiator. Determine the output $y(t)$ of the differentiator.

(b) What happens to the differentiator output $y(t)$ as T approaches zero? Use the definition of a unit impulse $\delta(t)$ to express your answer.

(c) What is the total area under the differentiator output $y(t)$ for all T? Justify your answer.

Based on your findings in parts (a) to (c), describe in succinct terms the result of differentiating a unit impulse.

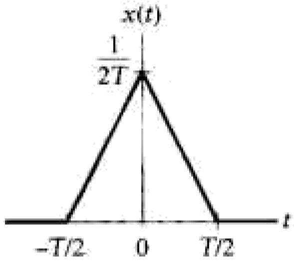


FIGURE P1.25

1.26 The derivative of impulse function $\delta(t)$ is referred to as a *doublet*. It is denoted by $\delta'(t)$. Show that $\delta'(t)$ satisfies the sifting property

$$\int_{-\infty}^{\infty} \delta'(t - t_0)\, f(t)\, dt = f'(t_0)$$

where

$$f'(t_0) = \left.\frac{d}{dt} f(t)\right|_{t=t_0}$$

Assume that the function $f(t)$ has a continuous derivative at time $t = t_0$.

1.27 A system consists of several subsystems connected as shown in Fig. P1.27. Find the operator H relating $x(t)$ to $y(t)$ for the subsystem operators given by:

$H_1: y_1(t) = x_1(t)x_1(t-1)$
$H_2: y_2(t) = |x_2(t)|$
$H_3: y_3(t) = 1 + 2x_3(t)$
$H_4: y_4(t) = \cos(x_4(t))$

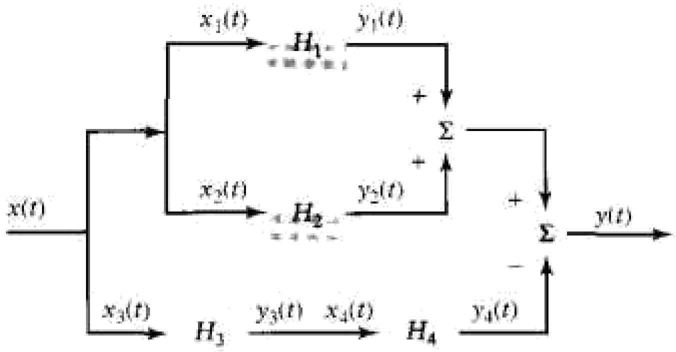


FIGURE P1.27

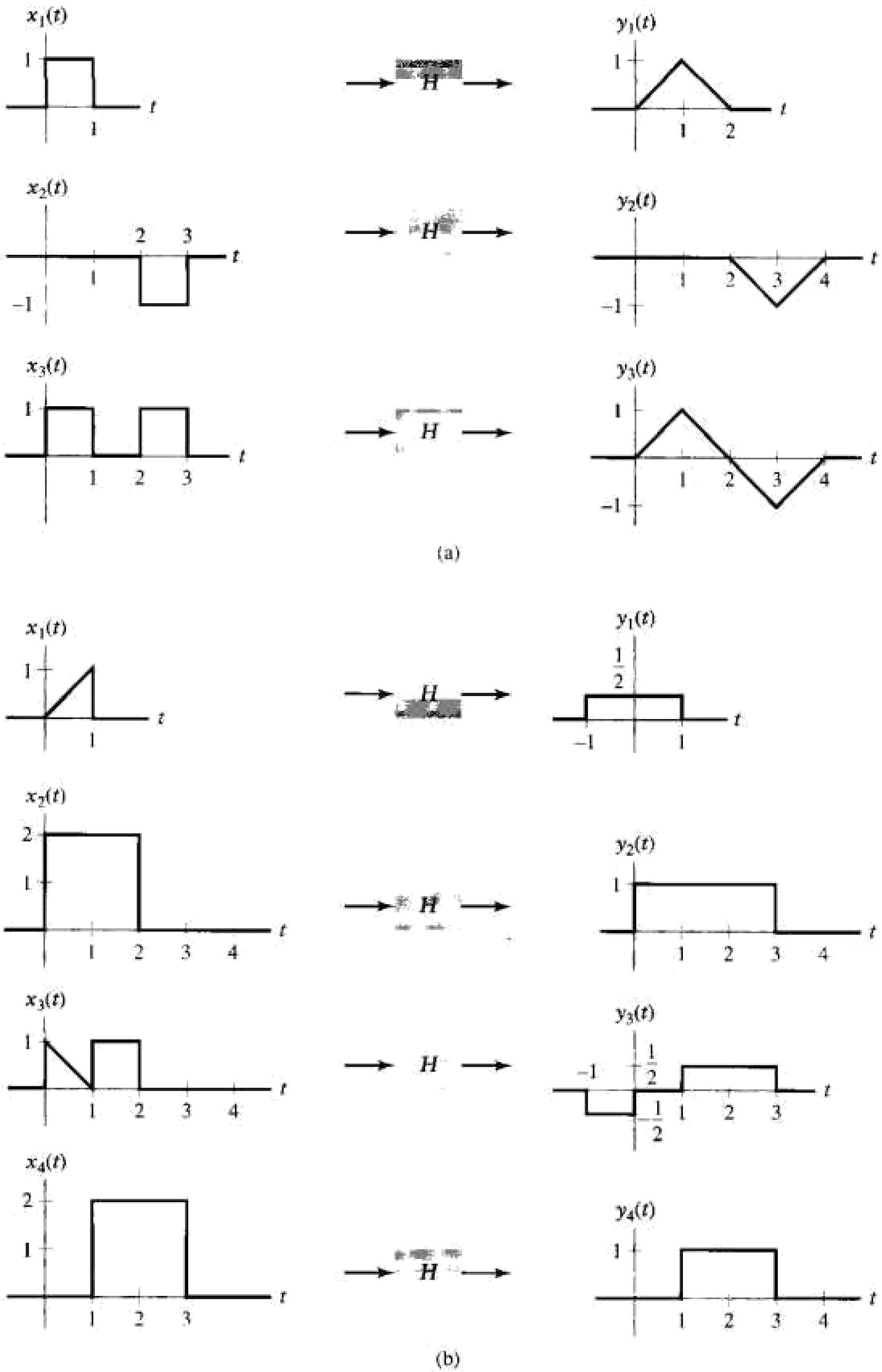


FIGURE P1.39

1.28 The systems given below have input $x(t)$ or $x[n]$ and output $y(t)$ or $y[n]$, respectively. Determine whether each of them is (i) memoryless, (ii) stable, (iii) causal, (iv) linear, and (v) time invariant.

(a) $y(t) = \cos(x(t))$

(b) $y[n] = 2x[n]u[n]$

(c) $y[n] = \log_{10}(|x[n]|)$

(d) $y(t) = \int_{-\infty}^{t/2} x(\tau)\,d\tau$

(e) $y[n] = \sum_{k=-\infty}^{n} x[k+2]$

(f) $y(t) = \dfrac{d}{dt}x(t)$

(g) $y[n] = \cos(2\pi x[n+1]) + x[n]$

(h) $y(t) = \dfrac{d}{dt}\{e^{-t}x(t)\}$

(i) $y(t) = x(2-t)$

(j) $y[n] = x[n]\sum_{k=-\infty}^{\infty}\delta[n-2k]$

(k) $y(t) = x(t/2)$

(l) $y[n] = 2x[2^n]$

1.29 The output of a discrete-time system is related to its input $x[n]$ as follows:

$$y[n] = a_0x[n] + a_1x[n-1] + a_2x[n-2] + a_3x[n-3]$$

Let the operator S^k denote a system that shifts the input $x[n]$ by k time units to produce $x[n-k]$. Formulate the operator H for the system relating $y[n]$ to $x[n]$. Hence develop a block diagram representation for H, using (a) cascade implementation and (b) parallel implementation.

1.30 Show that the system described in Problem 1.29 is BIBO stable for all a_0, a_1, a_2, and a_3.

1.31 How far does the memory of the discrete-time system described in Problem 1.29 extend into the past?

1.32 Is it possible for a noncausal system to possess memory? Justify your answer.

1.33 The output signal $y[n]$ of a discrete-time system is related to its input signal $x[n]$ as follows:

$$y[n] = x[n] + x[n-1] + x[n-2]$$

Let the operator S denote a system that shifts its input by one time unit.

(a) Formulate the operator H for the system relating $y[n]$ to $x[n]$.

(b) The operator H^{-1} denotes a discrete-time system that is the inverse of this system. How is H^{-1} defined?

1.34 Show that the discrete-time system described in Problem 1.29 is time invariant, independent of the coefficients a_0, a_1, a_2, and a_3.

1.35 Is it possible for a time-variant system to be linear? Justify your answer.

1.36 Show that an Nth power-law device defined by the input–output relation

$$y(t) = x^N(t), \quad N \text{ integer and } N \neq 0, 1$$

is nonlinear.

1.37 A linear time-invariant system may be causal or noncausal. Give an example for each one of these two possibilities.

1.38 Figure 1.50 shows two equivalent system configurations on condition that the system operator H is linear. Which of these two configurations is simpler to implement? Justify your answer.

1.39 A system H has its input–output pairs given. Determine whether the system could be memoryless, causal, linear, and time invariant for (a) signals depicted in Fig. P1.39(a) and (b) signals depicted in Fig. P1.39(b). For all cases, justify your answers.

1.40 A linear system H has the input–output pairs depicted in Fig. P1.40(a). Determine the following and explain your answers:

(a) Is this system causal?

(b) Is this system time invariant?

(c) Is this system memoryless?

(d) Find the output for the input depicted in Fig. P1.40(b).

1.41 A discrete-time system is both linear and time invariant. Suppose the output due to an input $x[n] = \delta[n]$ is given in Fig. P1.41(a).

(a) Find the output due to an input $x[n] = \delta[n-1]$.

(b) Find the output due to an input $x[n] = 2\delta[n] - \delta[n-2]$.

(c) Find the output due to the input depicted in Fig. P1.41(b).

► Computer Experiments

1.42 Write a set of MATLAB commands for approximating the following continuous-time periodic waveforms:

(a) Square wave of amplitude 5 volts, fundamental frequency 20 Hz, and duty cycle 0.6.

Signals and Systems
Lecture #7 Topic: Problems and Solutions – Haykin (chapter-1)

Problem 1: Find the even and odd components of each of the following signals

Problem 2:
(a) Periodic: Fundamental period = 0.5s
(b) Nonperiodic
(c) Periodic: Fundamental period = 3s
(d) Periodic: Fundamental period = 2 samples
(e) Nonperiodic
(f) Periodic: Fundamental period = 10 samples
(g) Nonperiodic
(h) Nonperiodic
(i) Periodic: Fundamental period = 1 sample

Problem 1.3:
Answer: Here, $x(t) = 3\cos(200t + \pi/6)$

$$y(t) = \left(3\cos(200t + \pi/6)\right)^2 = 9\cos^2(200t + \pi/6) = \frac{9}{2}\left[1 + \cos\left(400t + \frac{\pi}{3}\right)\right]$$

(a) DC component = 9/2

(b) Sinusoidal Component = $\frac{9}{2}\cos\left(400t + \frac{\pi}{3}\right)$

Amplitude =9/2, Fundamental Frequency = 400/(2π)=200/π Hz

Problem 1.6:
Answer: Let N denote the fundamental period of $x[N]$, which is defined by $N=2\pi/\Omega$
The average power of $x[n]$ is, therefore,

$$P = \frac{1}{N}\sum_{n=0}^{N-1} x^2(n) = \frac{1}{N}\sum_{n=0}^{N-1} A^2\cos^2\left(\frac{2\pi n}{N} + \phi\right) = \frac{A^2}{N}\sum_{n=0}^{N-1}\cos^2\left(\frac{2\pi n}{N} + \phi\right)$$

Problem 1.7:
Answer: The energy of the raised cosine pulse x(t) is

$$E = \int_{-\pi/w}^{\pi/w} \frac{1}{4}\left(\cos(wt) + 1\right)^2 dt = \frac{1}{2}\int_0^{\pi/w}\left(\cos^2(wt) + 2\cos wt + 1\right)dt$$

$$= \frac{1}{2}\int_0^{\pi/w}\left(\frac{1}{2}\cos 2wt + \frac{1}{2} + 2\cos wt + 1\right)dt = \frac{1}{2}\left(\frac{3}{2}\right)\left(\frac{\pi}{3}\right) = \frac{3\pi}{4w}$$

Problem 1.8:
Answer: The signal $x(t)$ is even; its total energy is therefore

$$E = 2\int_0^5 x^2(t)dt = 2\int_0^4 1^2\, dt + \int_4^5 (5-t)dt = 2\left[t\right]_{t=0}^{4} + 2\left[-\frac{1}{3}(5-t)^3\right]_{t=4}^{5} = 8+2/3= 26/3$$

Problem 1.9:

Answer: (a) The differentiator output is

$$y(t) = \begin{cases} 1 & for \quad -5 < t < -4 \\ -1 & for \quad 4 < t < 5 \\ 1 & \text{otherwise} \end{cases}$$

(b) The total energy is

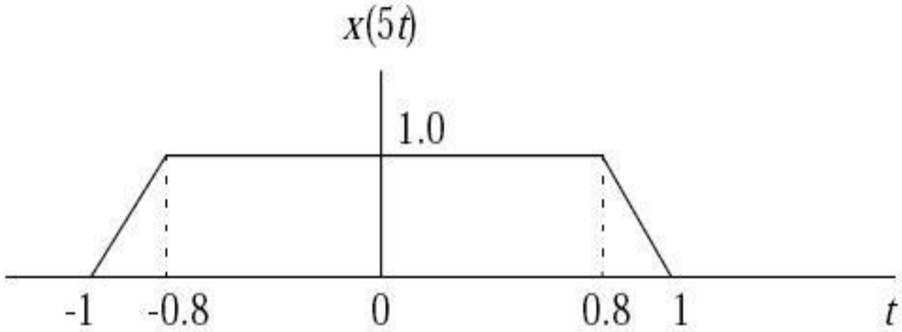
$$E = \int_{-5}^{-4} (1)^2 \, dt + \int_{4}^{5} (-1)^2 \, dt = 1 + 1 = 2$$

Problem 1.10:

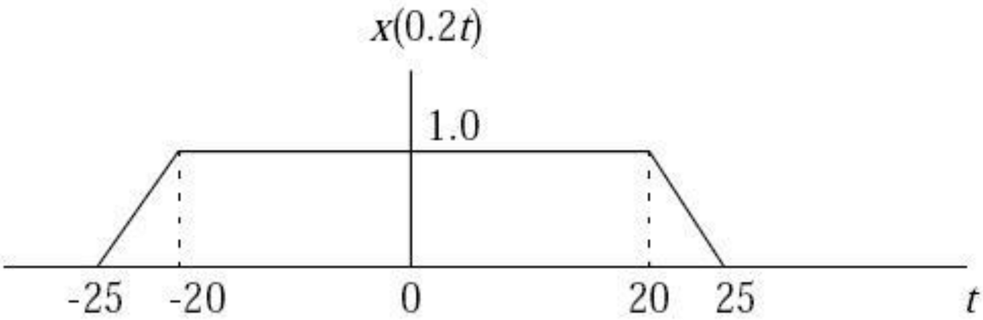
Answer: The output of the integrator is $y(t) = A\int_0^t \tau d\tau = At \quad for \quad 0 \le t \le T$

Hence the energy of y(t) is $E = \int_0^T (At)^3 \, dt = \frac{A^2T^3}{3}$

Answer: Problem 1.11:

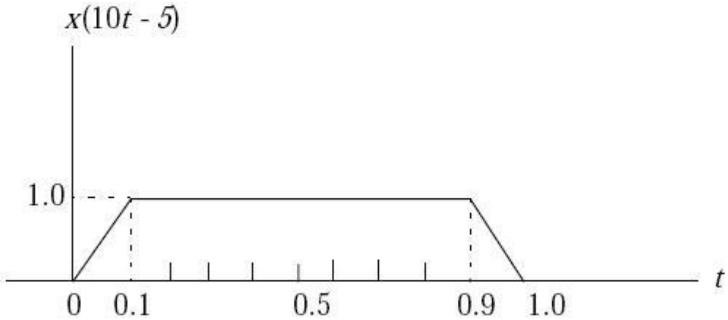


(a) y(t) for a=5

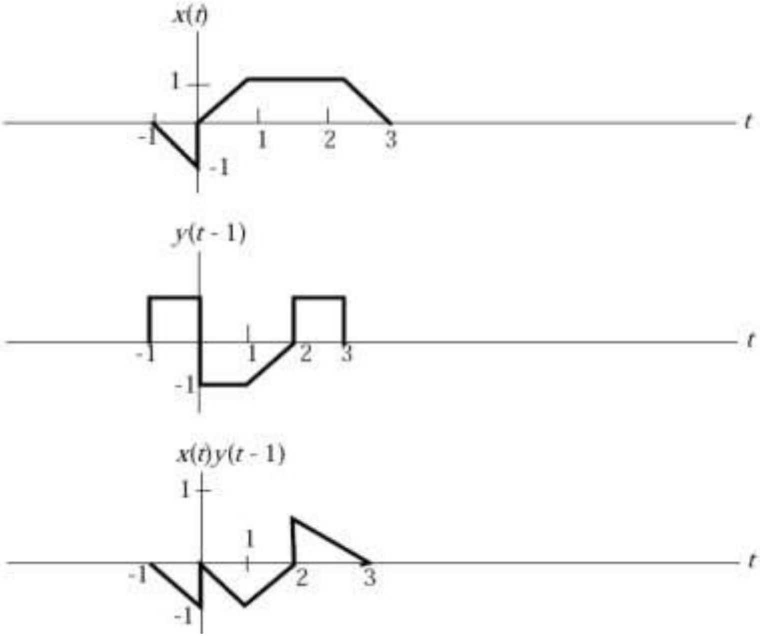


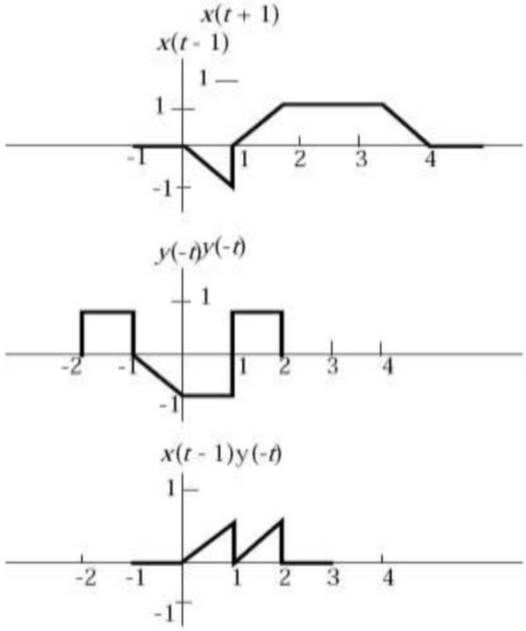
(b) y(t) for a=0.2

Problem 1.13:



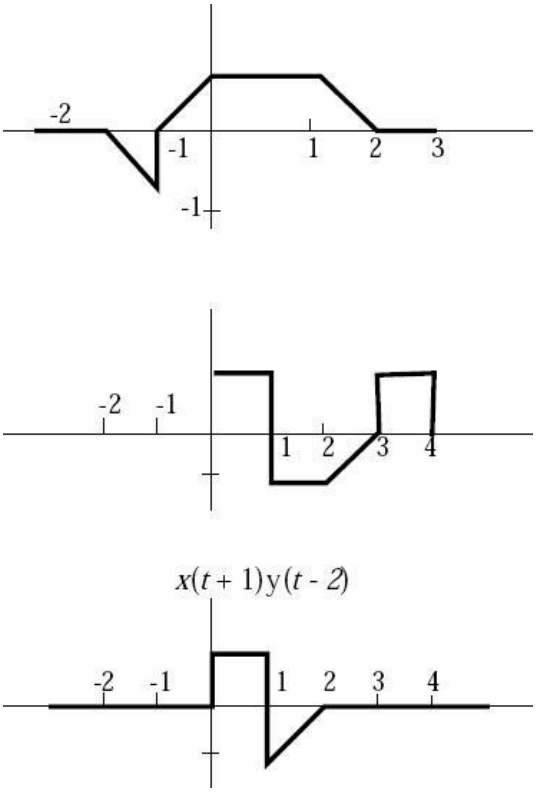
Problem 1.14:

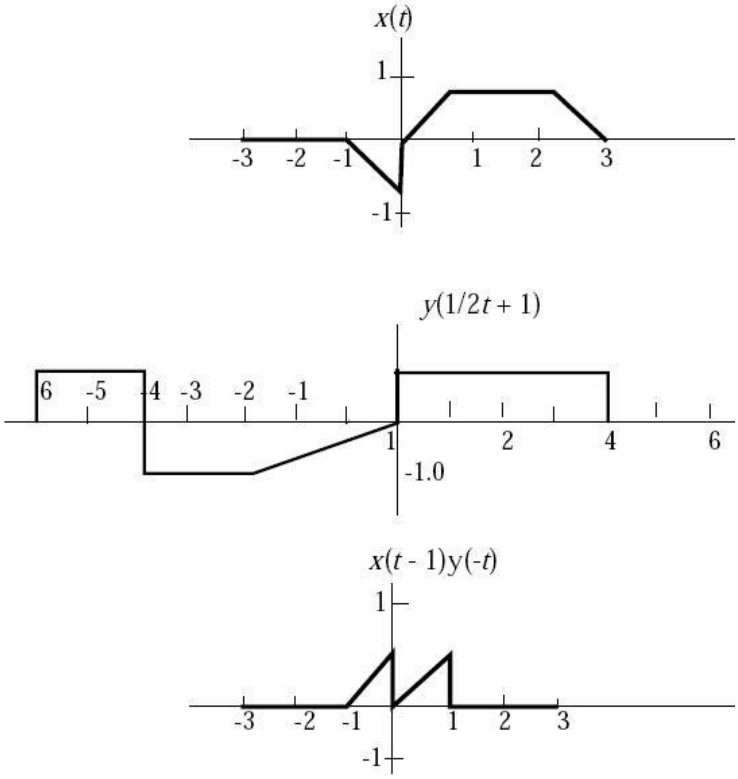




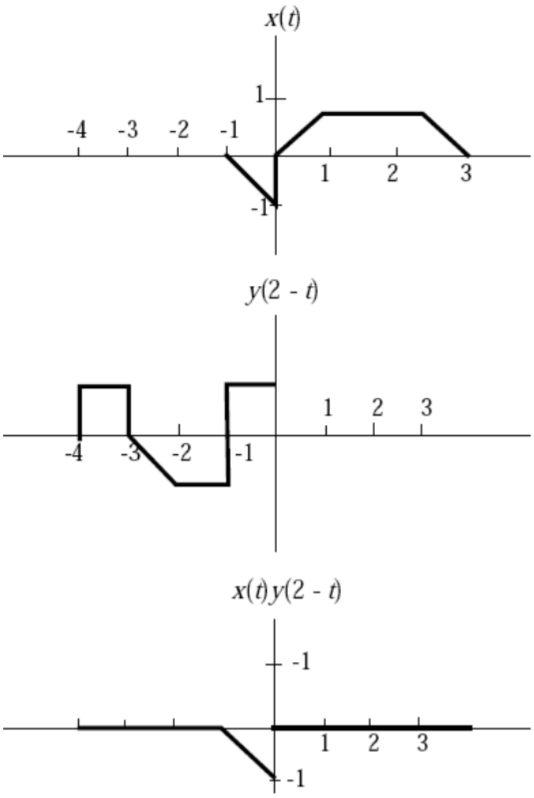
(a) **(b)**

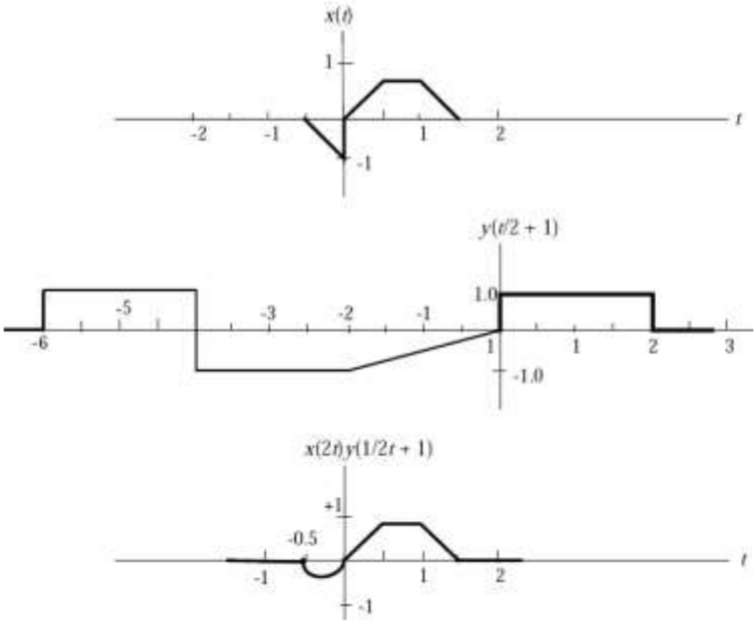
1.14 (c) Answer: **1.14(d) answer**



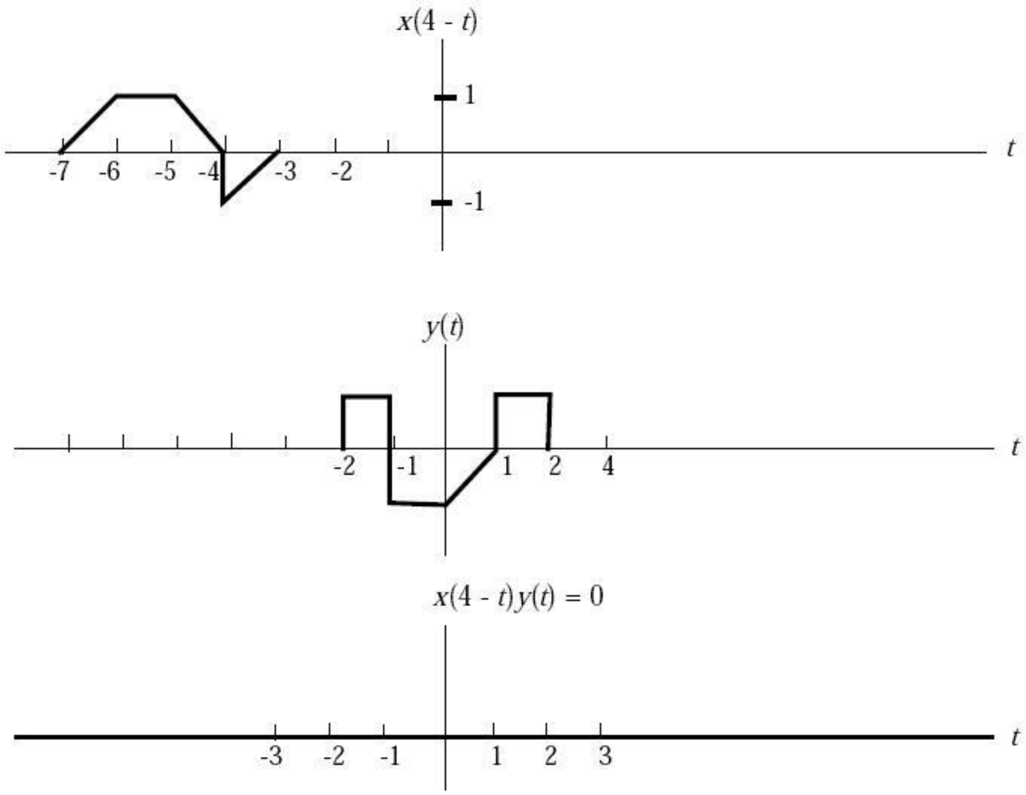


1.14 (e) Answer: **1.14(f) answer**

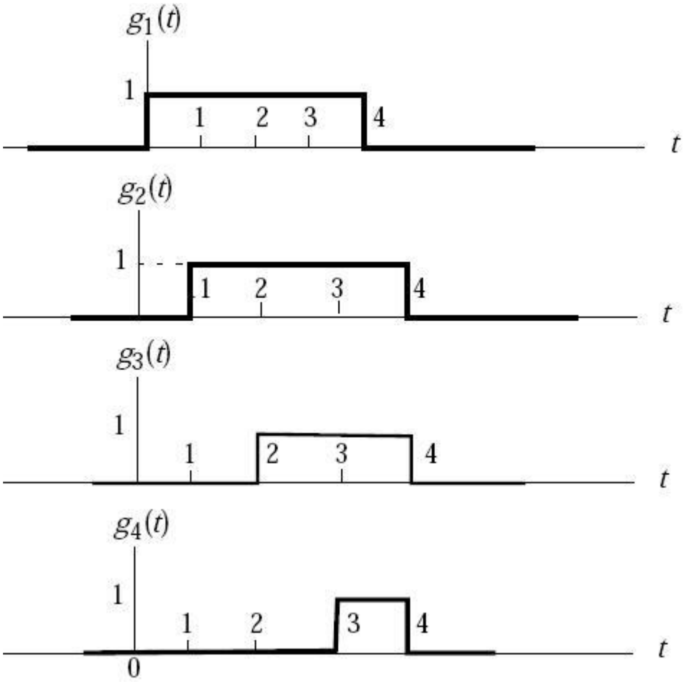




1.14 (g) Answer:



1.15 Answer: We may represent $x(t)$ as the superposition of 4 rectangular pulses as follows:



To generate $g1(t)$ from the prescribed $g(t)$, we let $g_1(t) = g(at - b)$ where a and b are to be determined. The width of pulse $g(t)$ is 2, whereas the width of pulse $g_1(t)$ is 4. We, therefore, need to expand $g(t)$ by a factor of 2, which, in turn, requires that we choose

$$a = \frac{1}{2}$$

The mid-point of $g(t)$ is at $t = 0$, whereas the mid-point of $g_1(t)$ is at $t = 2$. Hence, we must choose b to satisfy the condition

$$a(2) - b = 0$$

$$\Rightarrow b = 2a = 2 \times \frac{1}{2} = 1$$

Hence, $g_1(t) = g(\frac{1}{2}t - 1)$

In a similar way, we can determine

$$g_2(t) = g\left(\frac{2}{3}t - \frac{5}{3}\right)$$

$$g_3(t) = g(t - 3)$$

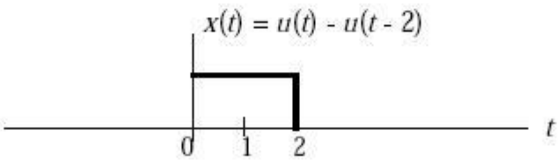
$$g_4(t) = g(2t - 7)$$

Accordingly, we may express the staircase signal $x(t)$ in terms of the rectangular pulse $g(t)$ as follows:

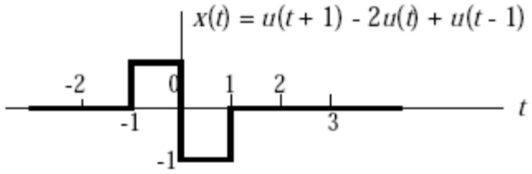
$$x(t) = g\left(\frac{1}{2}t - 1\right) + g\left(\frac{2}{3}t - \frac{5}{3}\right) + g(t - 3) + g(2t - 7)$$

1.16 Answer:

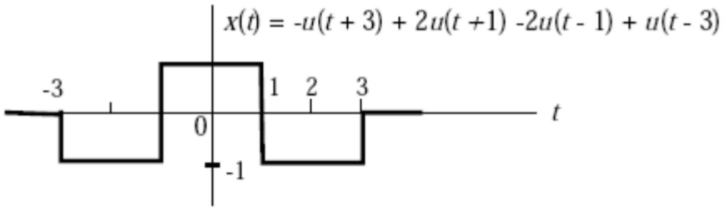
(a)



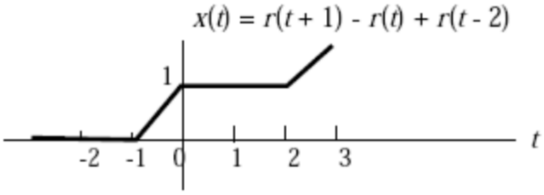
(b)



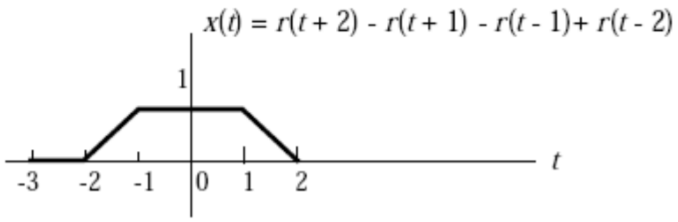
(c)



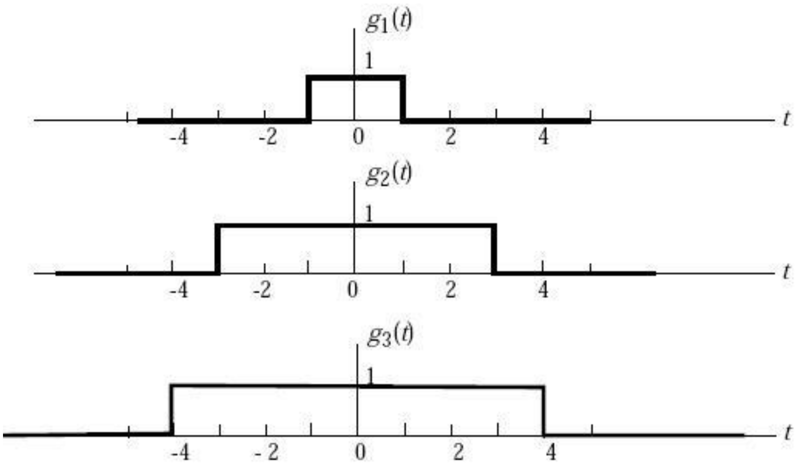
(d)



(e)



1.17 Answer: We may generate $x(t)$ as the superposition of 3 rectangular pulses as follows:



All three pulses, $g_1(t)$, $g_2(t)$, and $g_3(t)$, are symmetrically positioned around the origin:

1. $g_1(t)$ is exactly the same as $g(t)$.
2. $g_2(t)$ is an expanded version of $g(t)$ by a factor of 3.
3. $g_3(t)$ is an expanded version of $g(t)$ by a factor of 4.

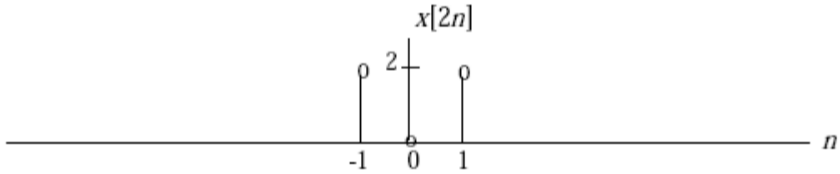
Hence, it follows that

$$g_1(t) = g(t), \qquad g_2(t) = g\left(\frac{1}{3}t\right), \qquad g_3(t) = g\left(\frac{1}{4}t\right)$$

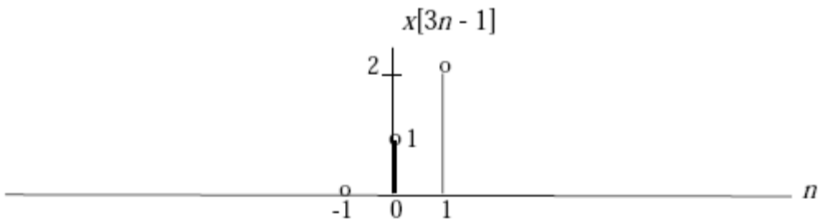
Therefore, $x(t) = g(t) + g\left(\frac{1}{3}t\right) + g\left(\frac{1}{4}t\right)$

1.18 Answer:

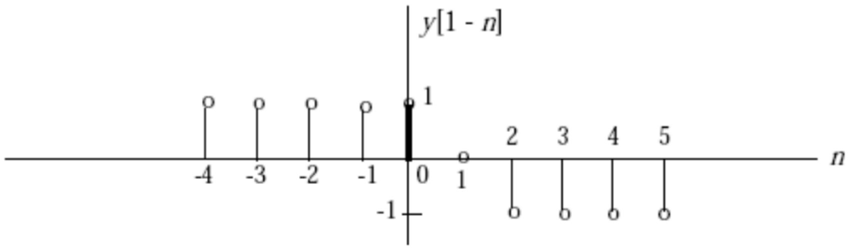
(a)



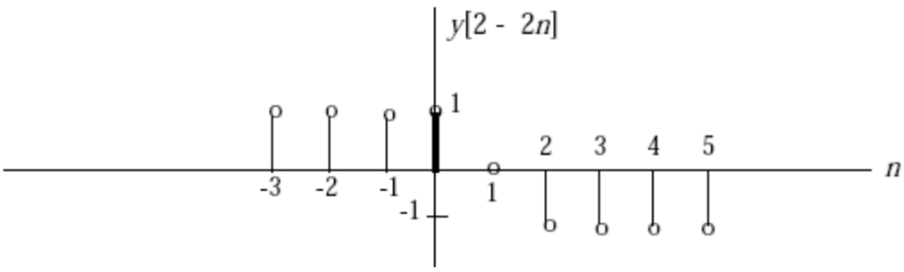
(b)



(c)



(d)



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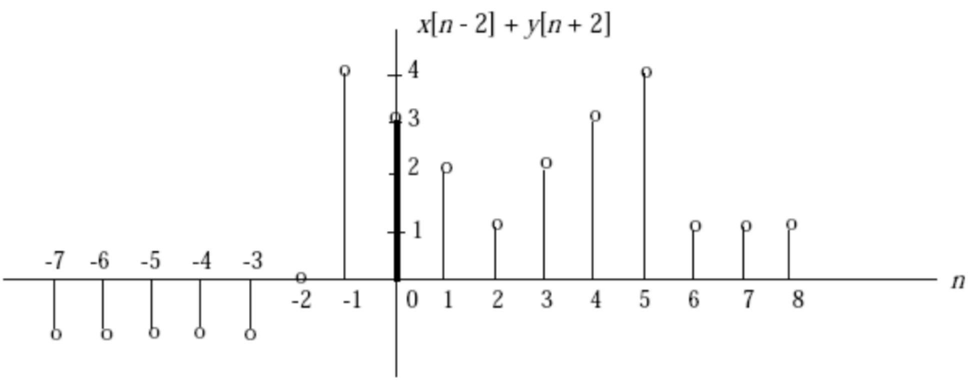

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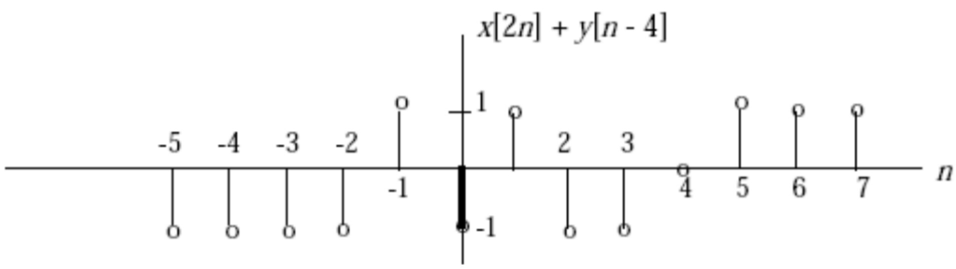

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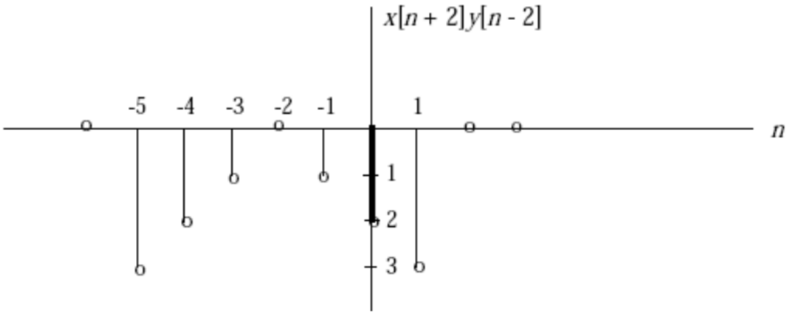

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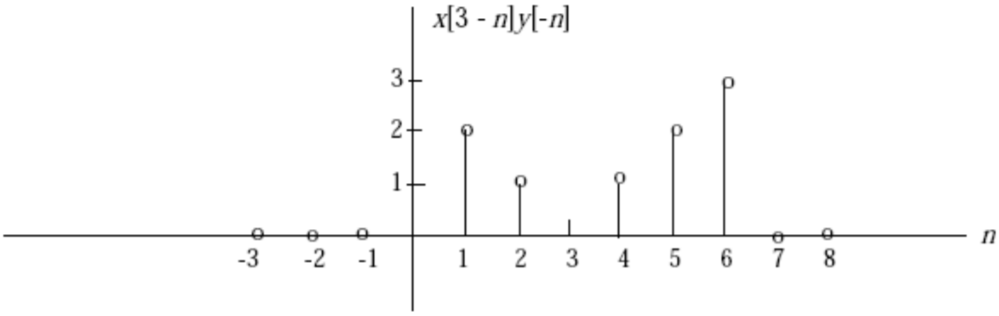
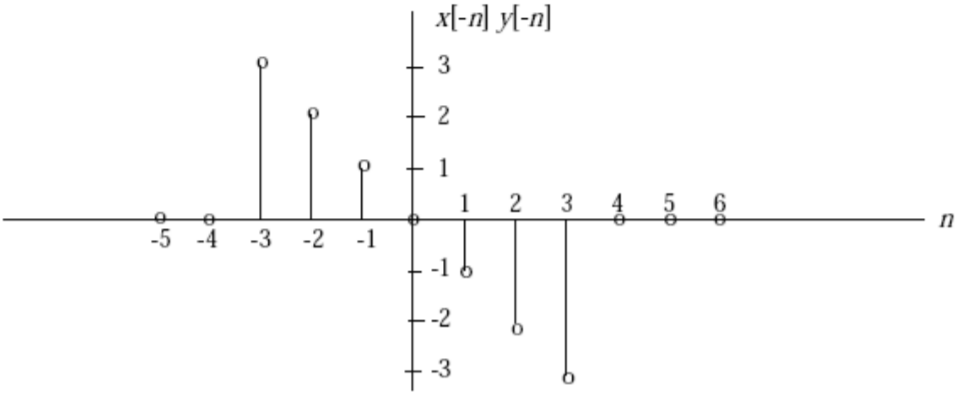


(i)



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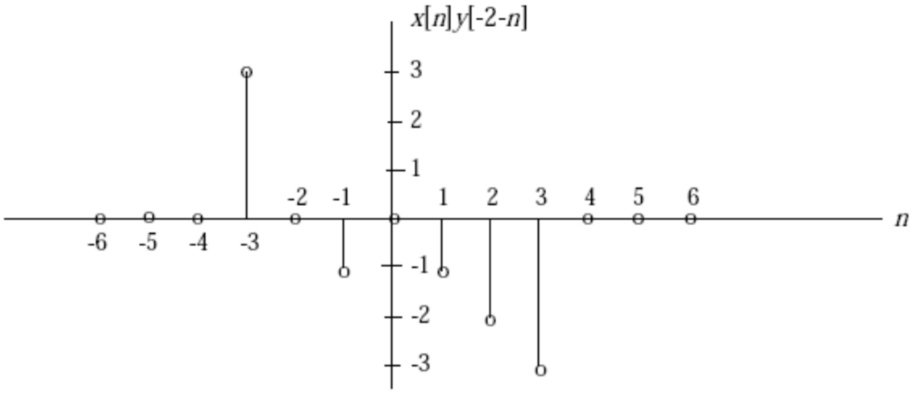

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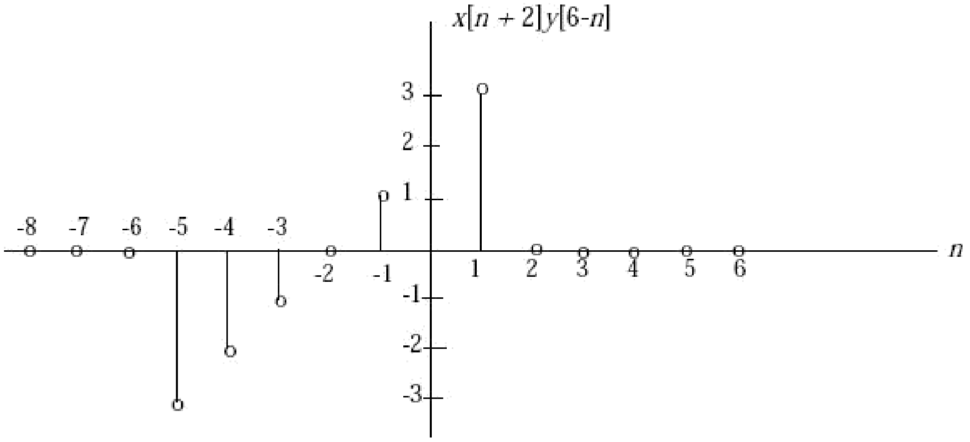


Problem 1.20

Answer:

The fundamental period of the sinusoidal signal $x[n]$ is $N = 10$.

Hence the angular frequency of $x[n]$ is

$$\Omega = \frac{2\pi m}{N}, \text{ where } m\text{: integer}$$

The smallest value of Ω is attained with $m = 1$. Hence, $\Omega = \frac{2\pi}{10} = \frac{\pi}{5}$ radians/cycle

Problem 1.21

Answers:

(a) Periodic, Fundamental period = 15 samples
(b) Periodic, Fundamental period = 30 samples
(c) Nonperiodic
(d) Periodic, Fundamental period = 2 samples
(e) Nonperiodic
(f) Nonperiodic
(g) Periodic, Fundamental period = 2π seconds
(h) Nonperiodic
(i) Periodic, Fundamental period = 15 samples

Problem 1.22

Answers:

The amplitude of complex signal $x(t)$ is defined by

$$\begin{aligned}\sqrt{x_R^2(t) + x_I^2(t)} &= \sqrt{A^2\cos^2(\omega t + \phi) + A^2\sin^2(\omega t + \phi)} \\ &= A\sqrt{\cos^2(\omega t + \phi) + \sin^2(\omega t + \phi)} \\ &= A\end{aligned}$$

Problem 1.23

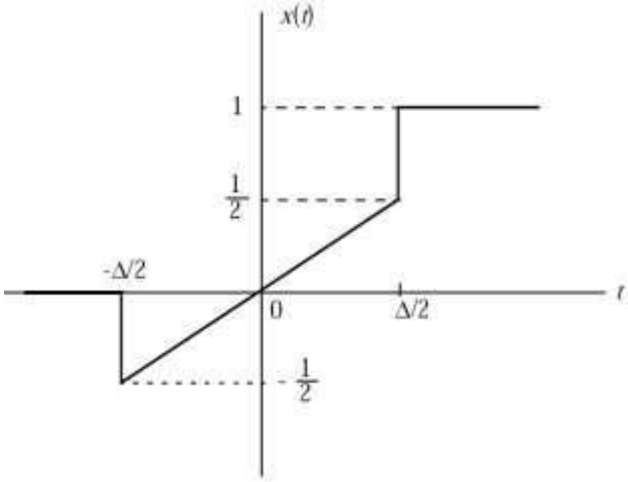
Answer: real part of x(t) is $\mathrm{Re}\{x(t)\} = Ae^{\alpha t}\cos(wt)$

The imaginary part of x(t) is $\mathrm{Im}\{x(t)\} = Ae^{\alpha t}\sin(wt)$

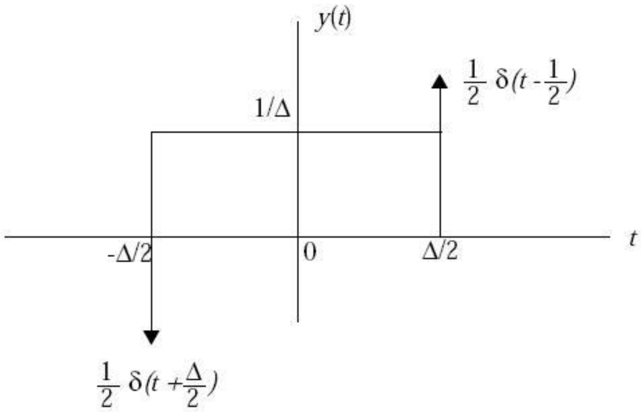
Problem 1.24
Answer: The continuous-time signal is given

$$x(t) = \begin{cases} t/T + 0.5, & -T/2 \le t \le T/2 \\ 1, & t \ge T/2 \\ 0, & t < -T/20 \end{cases},$$

The waveform of x(t) is as follows: Here, T=Δ



The output of a differentiator in response to $x(t)$ has the corresponding waveform:



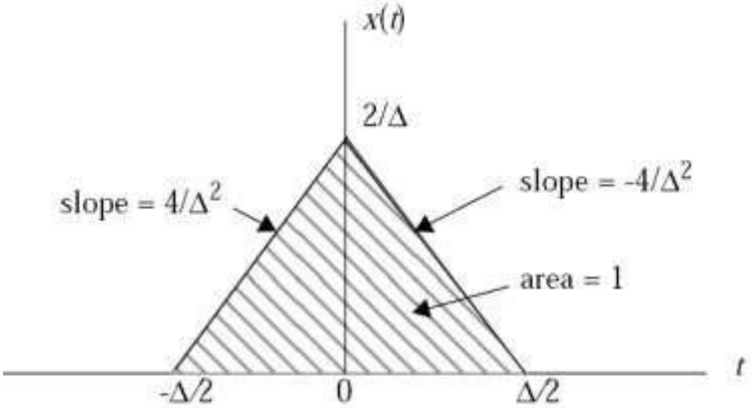
$y(t)$ consists of the following components:
1. A rectangular pulse of duration Δ and amplitude 1/Δ centered on the origin; the area under this pulse is unity.
2. An impulse of strength 1/2 at t = Δ/2.
3. An impulse of strength -1/2 at t = -Δ/2.

As the duration Δ is permitted to approach zero, the impulses (1/2)δ(t-Δ/2) and -(1/2)δ(t+ Δ/2) coincide and therefore cancel each other. At the same time, the rectangular pulse of unit area (i.e., component 1) approaches a unit impulse at t = 0. We may thus state that in the limit:

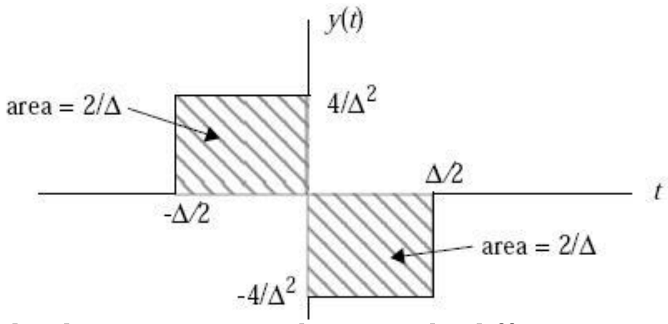
$$\lim_{\Delta\to 0} y(t) = \lim_{\Delta\to 0} \frac{d}{dt} x(t) = \delta(t)$$

Problem 1.25

Answer: The triangular pulse of total duration Δ and unit area are given, which is symmetrical about the origin: (Let T=Δ)



(a) Applying $x(t)$ to a differentiator, we get an output $y(t)$ depicted as follows:



(b) As the triangular pulse duration Δ approaches zero, the differentiator output approaches the combination of two impulse functions described as follows:
- An impulse of positive infinite strength at $t = 0$-.
- An impulse of negative infinite strength at $t = 0$+.

(c) The total area under the differentiator output $y(t)$ is equal to $(2/\Delta) + (-2/\Delta) = 0$.
In light of the results presented in parts (a), (b), and (c) of this problem, we may now make the following statement:
When the unit impulse $\delta(t)$ is differentiated with respect to time t, the resulting output consists of a pair of impulses located at $t = 0$- and $t = 0$+, whose respective strengths are $+\infty$ and $-\infty$.

Problem 1.27

Answer: From Fig. P1.27 we observe the following:

$$x_1(t) = x_2(t) = x_3(t) = x(t)$$

$$x_4(t) = y_3(t)$$

Hence, we may write

$$y_1(t) = x(t)x(t-1) \quad (1)$$

$$y_2(t) = |x(t)| \quad (2)$$

$$y_4(t) = \cos(y_3(t)) = \cos(1 + 2x(t)) \quad (3)$$

The overall system output is

$$y(t) = y_1(t) + y_2(t) - y_4(t) \quad (4)$$

Substituting Eqs. (1) to (3) into (4):

$$y(t) = x(t)x(t-1) + |x(t)| - \cos(1 + 2x(t)) \quad (5)$$

Equation (5) describes the operator H that defines the output $y(t)$ in terms of the input $x(t)$.

Problem 1.28
Answer:

	Memoryless	Stable	Causal	Linear	Time-invariant
(a)	✓	✓	✓	x	✓
(b)	✓	✓	✓	✓	✓
(c)	✓	✓	✓	x	✓
(d)	x	✓	✓	✓	✓
(e)	x	✓	x	✓	✓
(f)	x	✓	✓	✓	✓
(g)	✓	✓	x	x	✓
(h)	x	✓	✓	✓	✓
(i)	x	✓	x	✓	✓
(j)	✓	✓	✓	✓	✓
(k)	✓	✓	✓	✓	✓
(l)	✓	✓	✓	x	✓

Problem 1.29
Answer:

We are given

$$y[n] = a_0x[n] + a_1x[n-1] + a_2x[n-2] + a_3x[n-3]$$

Let

$$S^k\{x(n)\} = x(n-k)$$

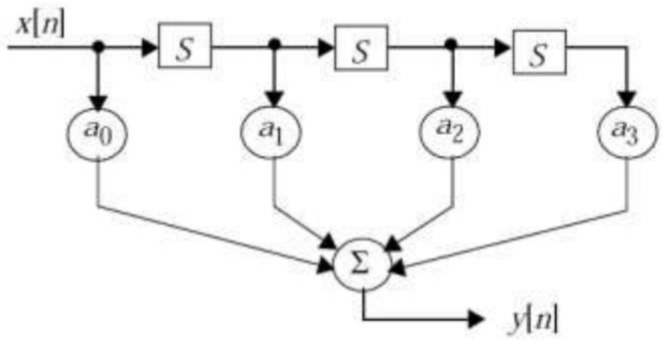
We may then rewrite Eq. (1) in the equivalent form

$$\begin{aligned} y[n] &= a_0x[n] + a_1S^1\{x[n]\} + a_2S^2\{x[n]\} + a_3S^3\{x[n]\} \\ &= (a_0 + a_1S^1 + a_2S^2 + a_3S^3)\{x[n]\} \\ &= H\{x[n]\} \end{aligned}$$

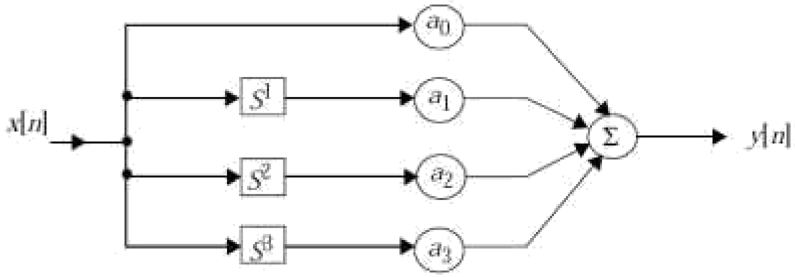
where

$$H = a_0 + a_1S^1 + a_2S^2 + a_3S^3$$

(a) Cascade implementation of operator H:



(b) Parallel implementation of operator H:



Problem 1.30
Answer:

Using the given input-output relation:

$$y[n] = a_0 x[n] + a_1 x[n-1] + a_2 x[n-2] + a_3 x[n-3]$$

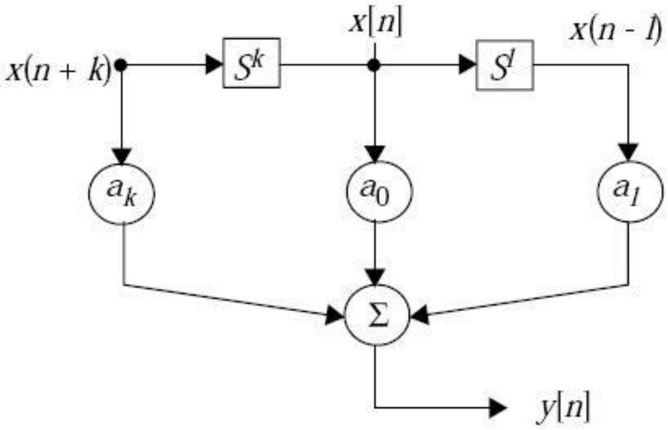
we may write

$$\begin{aligned} |y[n]| &= |a_0 x[n] + a_1 x[n-1] + a_2 x[n-2] + a_3 x[n-3]| \\ &\le |a_0 x[n]| + |a_1 x[n-1]| + |a_2 x[n-2]| + |a_3 x[n-3]| \\ &\le |a_0| M_x + |a_1| M_x + |a_2| M_x + |a_3| M_x \\ &= (|a_0| + |a_1| + |a_2| + |a_3|) M_x \end{aligned}$$

where $M_X = | x(n) |$. Hence, provided that M_X is finite, the absolute value of the output will always be finite. This assumes that the coefficients $a0$, $a1$, $a2$, $a3$ have finite values of their own. It follows therefore that the system described by the operator H of Problem 1.29 is stable.

Problem 1.31
Answer: The memory of the discrete-time described in Problem 1.29 extends 3 time units into the past.

Problem 1.32
Answer: It is indeed possible for a noncausal system to possess memory. Consider, for example, the system illustrated below:



That is, with $S^l\{x[n]\} = x[n-l]$, we have the input-output relation

$$y[n] = a_0\, x[n] + a_k\, x[n+k] + a_l\, x[n-l]$$

This system is noncausal by virtue of the term $a_k x[n+k]$. The system has memory by virtue of the term $a_l x[n-l]$.

CHAPTER 1

CHARACTERISTICS OF A LINEAR SYSTEM

1-1 Introduction. The study of linear systems is important for two reasons: (1) a great majority of engineering situations are linear, at least within specified ranges; and (2) exact solutions of the behavior of linear systems can usually be found by standard techniques. Except for a very few special types, there are no standard methods for analyzing nonlinear systems. The practical ways of solving nonlinear problems involve graphical or experimental approaches. Approximations are often necessary, and each situation usually requires special handling. The present state of the art is such that there is neither a standard technique which can be used to solve nonlinear problems exactly, nor is there any assurance that a good solution can be obtained at all for a given nonlinear system. Hence, we are indeed fortunate that a great majority of engineering problems are linear and can be solved. However, we must realize that not all physical systems are linear without restrictions.

We are all familiar with the Ohm's law that governs the relation between the voltage across and the current through a resistor. It is a *linear* relationship because the voltage across a resistor is (linearly) proportional to the current through it. But even for this simple situation, the linear relationship does not apply under all conditions. For instance, as the current in a resistor is greatly increased, the value of its resistance will increase due to heat developed in the resistor, the amount of increase being dependent upon the magnitude of the current; and it is no longer correct to say that the voltage across the resistor bears a linear relationship to the current through it. The same can be said about Hooke's law, which states that the stress is (linearly) proportional to the strain of a spring. But this linear relationship breaks down when the stress on the spring is too great. When the stress exceeds the elastic limit of the material of which the spring is made, stress and strain are no longer linearly related. The actual relationship is much more complicated than the Hooke's law situation. We are therefore forewarned that restrictions always exist for linear physical situations; saturation, breakdown, or material changes will ultimately set in and destroy linearity. Under ordinary circumstances, however, physical conditions in many engineering problems stay well within the restrictions, and the linear relationship holds.

Ohm's law and Hooke's law describe only special linear systems. There exist systems that are much more complicated and so cannot be conveniently described by simple voltage-current or stress-strain relationships.

Other more universal criteria are necessary to establish that a system is linear. Linear systems are characterized by certain definite properties which make them simpler to describe physically and easier to solve mathematically. In the following sections, we shall examine the characteristics of a linear system from both a physical and a mathematical viewpoint.

1–2 Linear system from a physical viewpoint. An engineer's interest in a physical situation is very frequently the determination of the response of a system to a given excitation. Both the excitation and the response may be any physically measurable quantity, depending upon the particular problem. Figure 1–1 depicts such a situation. Suppose that an excitation function $e_1(t)$, which varies with time in a specified manner, produces a response function $w_1(t)$, and that a second excitation function $e_2(t)$ produces a second response function $w_2(t)$.

FIG. 1–1. A physical situation.

Symbolically, we may write

$$e_1(t) \rightarrow w_1(t), \tag{1-1}$$

$$e_2(t) \rightarrow w_2(t). \tag{1-2}$$

Then, for a linear system,

$$e_1(t) + e_2(t) \rightarrow w_1(t) + w_2(t). \tag{1-3}$$

Relation (1–3), in conjunction with (1–1) and (1–2), states that a superposition of excitation functions results in a response which is the superposition of the individual response functions. Hence, from a physical point of view, we may say that *a necessary condition for a system to be linear is that the principle of superposition applies.* We note in passing that the different excitations do not have to be applied on the same part of the system.

The validity of the principle of superposition means that the presence of one excitation does not affect the responses due to other excitations; there are no interactions among responses of different excitations within a linear system. To analyze the combined effect of a number of excitations on a linear system, we can start with the analysis of the effect of each individual excitation as if the other excitations were not present, and then combine (add, or superpose) the results.

If there are n identical excitations applied to the same part of the system, that is, if

$$e_1(t) = e_2(t) = \cdots = e_n(t), \tag{1–4}$$

then, for a linear system,

$$\sum_{k=1}^{n} e_k(t) = ne_1(t) \to \sum_{k=1}^{n} w_k(t) = nw_1(t). \tag{1–5}$$

Comparing relation (1–5) with (1–1), we see that n appears as a scale factor (a magnitude change). Hence, *a characteristic of linear systems is that the magnitude scale factor is preserved.* This characteristic is sometimes referred to as the property of *homogeneity.*

At this point the reader must be warned that although the "derivation" of (1–5) from (1–3) seemed flawless, there are situations in which we cannot automatically assume the property of homogeneity (1–5) when the principle of superposition (1–3) holds. This may be illustrated by the following example. Let Fig. 1–2 represent a *nonlinear* system in which the filters 1 and 2 separate the input signal or excitation into two nonoverlapping spectral bands. Then if the spectrum of $e_1(t)$ falls entirely inside the passband of filter 1 and that of $e_2(t)$ falls entirely inside the passband of filter 2, relation (1–3) would be satisfied and yet the system remains nonlinear. Here, then, we have a situation where relation (1–3) does not imply relation (1–5). It is for this reason that the properties of superposition and homogeneity should be regarded as two *separate* requirements for a linear system. *A system is linear if and only if both* (1–3) *and* (1–5) *are satisfied.*

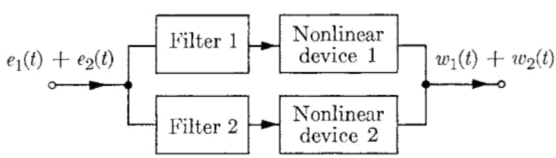


FIG. 1–2. A nonlinear system.

There is another physical aspect that characterizes a linear system with constant parameters. If the excitation function $e(t)$ applied to such a system is an alternating function of time with frequency f, then the steady-state response $w(t)$, after the initial transient has died out, appearing in any part of the system will also be alternating with the frequency f. We were aware of this fact when we solved a-c circuit problems. When a 60-cycle source is applied to a network of *fixed, linear* elements R, L, and C, the voltages and currents in all parts of the network will also be of

60-cycle frequency; no frequencies other than that of the source can exist in the network after transients have died out. In other words, *stationary (non-time-varying) linear systems create no new frequencies.* The qualification of *stationarity* implies that if

$$e(t) \rightarrow w(t) \tag{1-6}$$

then

$$e(t - \tau) \rightarrow w(t - \tau), \tag{1-7}$$

where τ is an arbitrary time delay. This qualification is to exclude situations with variable system parameters. A familiar example for such a situation is the carbon microphone circuit, in which a sinusoidal variation of the resistance in an R-L circuit will produce currents of harmonic frequencies. Another example is a linear radar system in which a moving target will cause a so-called Doppler frequency shift.

We occasionally hear the use of the terms "linear oscillators," "linear modulators," and "linear detectors." These are unfortunate choices of words. Oscillators are generators of definite frequencies, in which the only sources are d-c (zero frequency). Linear systems with constant parameters cannot do this. It is also evident that some sort of nonlinear process is there to limit the oscillation amplitude. Modulators inherently involve conversion of frequencies and are not linear devices. The term "linear detectors" is rather commonly used for large-signal detectors where the detected output follows the envelope of the modulated carrier at the input. But large-signal detectors operate under class C conditions and are basically nonlinear. They are sometimes called "linear detectors" perhaps to emphasize their difference from small-signal or square-law detectors.

The physical viewpoints that have been discussed in this section will become clearer and can all be proved after we have examined the characteristics of a linear system from a mathematical point of view. This will be done in the next section.

1–3 Linear system from a mathematical viewpoint. In mathematical language we can define linear systems as systems whose behavior is governed by linear equations, whether linear algebraic equations, linear difference equations, or linear differential equations. Let us be more specific with a typical linear differential equation, since we shall be dealing with differential equations throughout this book:

$$\frac{d^2w}{dt^2} + a_1\frac{dw}{dt} + a_0w = e(t). \tag{1-8}$$

In Eq. (1-8), t is used as the independent variable,* e is the excitation function, and w is the response function. Coefficients a_1 and a_0 are system parameters determined entirely by the number, type, and arrangement of the elements in the system; they may or may not be functions of the independent variable t. Since there are no partial derivatives (there is only one independent variable) in Eq. (1-8), and the highest order of the derivative is 2, Eq. (1-8) is an ordinary differential equation of the second order.† Equation (1-8) is a *linear* ordinary differential equation of the second order because neither the dependent variable w nor any of its derivatives is raised to a power greater than one and because none of its terms contains a product of two or more derivatives of the dependent variable or a product of the dependent variable and one of its derivatives.

The validity of the principle of superposition here can be verified as follows. We assume that the excitations $e_1(t)$ and $e_2(t)$ give rise to responses $w_1(t)$ and $w_2(t)$ respectively, as before. Hence

$$\frac{d^2w_1}{dt^2} + a_1\frac{dw_1}{dt} + a_0w_1 = e_1, \tag{1-9}$$

$$\frac{d^2w_2}{dt^2} + a_1\frac{dw_2}{dt} + a_0w_2 = e_2. \tag{1-10}$$

Adding Eqs. (1-9) and (1-10), we have directly

$$\frac{d^2}{dt^2}(w_1 + w_2) + a_1\frac{d}{dt}(w_1 + w_2) + a_0(w_1 + w_2) = (e_1 + e_2). \tag{1-11}$$

---

* Although the symbol t is used here, the independent variable does *not* have to be time. It is just a mathematical symbol. What it is in a physical system depends upon the problem; it may be time, distance, angle, or some other physical quantity.

† The *degree* of a differential equation is the same as the power of the highest derivative in the equation. Hence Eq. (1-8) is of the first degree; an equation like

$$\left(\frac{d^2w}{dt^2}\right)^3 - 3\frac{d^2w}{dt^2}\frac{dw}{dt} + \left(\frac{dw}{dt}\right)^4 = 0$$

is of the third degree; and an equation like

$$w + t\frac{dw}{dt} = \sqrt{\frac{dw}{dt}},$$

which can be reduced to

$$\left(w + t\frac{dw}{dt}\right)^2 = \frac{dw}{dt}, \qquad \text{or} \qquad t^2\left(\frac{dw}{dt}\right)^2 + (2wt - 1)\frac{dw}{dt} + w^2 = 0,$$

is of the second degree.

Equation (1–11) states that the response of the system to an excitation $e_1(t) + e_2(t)$ is equal to the sum of the responses to the individual excitations, $w_1(t) + w_2(t)$. Note that *the principle of superposition applies and the system is linear even when the coefficients a_1 and a_0 are functions of the independent variable t.* The property of homogeneity (preservation of the magnitude scale factor) can also be easily verified.

The reader can satisfy himself in proving that the principle of superposition applies to *none* of the following equations:

$$3\frac{d^2y}{dx^2} + y\frac{dy}{dx} + 2y = 5x^2, \tag{1–12}$$

$$\frac{du}{d\theta} + u + u^2 = \sin^3\theta, \tag{1–13}$$

$$t\left(\frac{d^2v}{dt^2}\right)^2 + 5\frac{dv}{dt} + t^2v = \epsilon^{-t}. \tag{1–14}$$

Equation (1–12) is nonlinear because the second term, $y(dy/dx)$ is a product of the dependent variable and its derivative; Eq. (1–13) is nonlinear because the third term, u^2, is a second power of the dependent variable; and Eq. (1–14) is nonlinear because the first term $t(d^2v/dt^2)^2$ contains a second power of a derivative of the dependent variable. *The existence of powers or other nonlinear functions of the independent variable does not make an equation nonlinear.*

1–4 General properties of linear differential equations. There exist certain properties which are characteristic of all linear differential equations. We shall discuss these general properties here without referring to any particular physical situation but with a view toward understanding the nature of linear systems better. We shall make no attempt to solve the equations in this section.

An ordinary linear differential equation of an arbitrary order n may be written as

$$a_n(t)\frac{d^nw}{dt^n} + a_{n-1}(t)\frac{d^{n-1}w}{dt^{n-1}} + \cdots + a_1(t)\frac{dw}{dt} + a_0(t)w = e(t), \tag{1–15}$$

where the coefficients $a_n(t), a_{n-1}(t), \ldots, a_1(t), a_0(t)$ and the right member of the equation, $e(t)$, are given functions of the independent variable t, determined by the system and the excitation function respectively. The equation is said to be *homogeneous* if $e(t) = 0$, and *nonhomogeneous* if $e(t) \neq 0$.

It is convenient to employ an abbreviated symbol for the long left side of the equation. Thus, if $e(t) = 0$, we represent the homogeneous linear differential equation as follows:

$$\left[a_n(t)\frac{d^n}{dt^n} + a_{n-1}(t)\frac{d^{n-1}}{dt^{n-1}} + \cdots + a_1(t)\frac{d}{dt} + a_0(t)\right]w(t) = 0.$$

Using the abbreviation

$$L = a_n(t)\frac{d^n}{dt^n} + a_{n-1}(t)\frac{d^{n-1}}{dt^{n-1}} + \cdots + a_1(t)\frac{d}{dt} + a_0(t), \tag{1-16}$$

we write

$$L[w] = 0, \tag{1-17}$$

where L can be regarded as an operator, operating on the dependent variable w.

(A) Since multiplying the dependent variable w by a constant multiplies each term in the equation by the same constant, we have

$$L[cw] = cL[w] \tag{1-18}$$

and

$$L[cw] = 0, \qquad \text{if} \qquad L[w] = 0. \tag{1-19}$$

Relations (1-18) and (1-19) state that if $w(t)$ is a solution of the homogeneous equation $L[w] = 0$, then so also is $cw(t)$.

(B) Since replacing w by $w_1 + w_2$ replaces each term by the sum of two similar terms, one in w_1 and one in w_2, we have

$$L[w_1 + w_2] = L[w_1] + L[w_2] \tag{1-20}$$

and

$$L[w_1 + w_2] = 0, \qquad \text{if} \qquad L[w_1] = 0 \qquad \text{and} \qquad L[w_2] = 0. \tag{1-21}$$

Relations (1-20) and (1-21) state that if $w_1(t)$ and $w_2(t)$ are solutions of the homogeneous equation $L[w] = 0$, then so also is $w_1(t) + w_2(t)$.

By combining the results in (A) and (B), we see that *if $w_1(t), w_2(t), \ldots, w_n(t)$ are solutions of the homogeneous linear differential equation $L[w] = 0$, then so also is a linear combination of them: $c_1w_1(t) + c_2w_2(t) + \cdots + c_nw_n(t)$, where the c's are arbitrary constants.*

(C) The solution $w_c(t) = c_1w_1(t) + c_2w_2(t) + \cdots + c_nw_n(t)$ with n (the order of the original differential equation) arbitrary constants is a general solution of the homogeneous equation (1-17) provided the n individual solutions $w_1(t), w_2(t), \ldots, w_n(t)$ are linearly independent. The solutions are linearly independent if none of them can be expressed as a linear combination of the others.* This general solution of the homogeneous equation is called the *complementary function* of the nonhomogeneous equation (1–15).

(D) If $w_p(t)$ is any particular solution of the nonhomogeneous equation such that

$$L[w_p] = e(t), \tag{1–23}$$

then the sum of this particular solution (called a *particular integral*) and the complementary function

$$\begin{aligned} w(t) &= w_c(t) + w_p(t) \\ &= c_1w_1(t) + c_2w_2(t) + \cdots + c_nw_n(t) + w_p(t) \end{aligned} \tag{1–24}$$

is the *general* or *complete* solution of the nonhomogeneous equation (1–15). In other words, any solution whatsoever of Eq. (1–15) can be written as a combination of the complementary function and a particular integral as

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* The n solutions $w_1, w_2, \ldots, w_n$ are *linearly dependent* if constants $b_1, b_2, \ldots, b_n$ (which are not all zero) can be found such that

$$b_1w_1 + b_2w_2 + \cdots + b_nw_n = 0. \tag{1–22}$$

Hence $w_1 = \epsilon^{-(1+j2)t}$, $w_2 = \epsilon^{-(1-j2)t}$, and $w_3 = \epsilon^{-t}\sin(2t - \pi/4)$ are linearly dependent because

$$\begin{aligned} w_3 &= \epsilon^{-t}(\cos\pi/4\sin 2t - \sin\pi/4\cos 2t) = \frac{\epsilon^{-t}}{\sqrt{2}}(\sin 2t - \cos 2t) \\ &= \frac{\epsilon^{-t}}{\sqrt{2}}\left[\frac{1}{2j}(\epsilon^{j2t} - \epsilon^{-j2t}) - \frac{1}{2}(\epsilon^{j2t} + \epsilon^{-j2t})\right] \\ &= -\frac{\epsilon^{-t}}{2\sqrt{2}}[(1+j)\epsilon^{j2t} + (1-j)\epsilon^{-j2t}] \\ &= -\frac{1}{2\sqrt{2}}[(1+j)w_2 + (1-j)w_1] \end{aligned}$$

or

$$(1-j)w_1 + (1+j)w_2 + 2\sqrt{2}\,w_3 = 0.$$

Compared with Eq. (1–22), we have $b_1 = (1-j)$, $b_2 = (1+j)$, and $b_3 = 2\sqrt{2}$.

There is an elegant method for testing whether a set of n solutions are linearly independent. They are linearly independent if their *Wronskian* does not vanish. The Wronskian of n solutions $w_1(t), w_2(t), \ldots, w_n(t)$ is the determinant formed by these functions and their first $n-1$ derivatives. A detailed discussion of this method is beyond the scope of this book. Interested readers are referred to E. L. Ince, *Ordinary Differential Equations*, Sec. 5.2, Dover Publications, 1944.

Eq. (1–24). For, if w is any solution of Eq. (1–15) and w_p is a particular integral, then, from Eq. (1–20), we can write

$$\begin{aligned} L[w - w_p] &= L[w] - L[w_p] \\ &= e(t) - e(t) = 0. \end{aligned}$$

Hence $w - w_p = w_c$ is a solution of the homogeneous equation (1–17), which, by property (C), must be expressible as

$$w - w_p = c_1 w_1(t) + c_2 w_2(t) + \cdots + c_n w_n(t).$$

Transferring w_p to the right side, we obtain the solution in the form of (1–24).

(E) The n arbitrary constants $c_1, c_2, \ldots, c_n$ in the complete solution (1–24) are determined by n known values* of the response function or its derivatives for specific values of the independent variable.

Remarks (A) through (E) above apply to general linear differential equations of an arbitrary order. If all the coefficients $a_n, a_{n-1}, \ldots, a_1$, and a_0 are constants, we have a linear differential equation with constant coefficients. Linear differential equations with constant coefficients are of extreme importance because they characterize a large number of physical and engineering situations. They are of such a nature that transformation methods can be applied with advantage. They will receive our prime attention throughout this book.

1–5 Illustrative examples. A few examples are given below to illustrate the properties of linear differential equations and their solutions.

Example 1–1. Verify that the function

$$y = c_1 \sin x + c_2 \cos x - \tfrac{1}{2} x \cos x$$

is a general solution of the linear differential equation

$$\frac{d^2 y}{dx^2} + y = \sin x. \tag{1–25}$$

Solution. Let us examine the complementary function and the particular integral separately: $y = y_c + y_p$.

---

* These are commonly referred to as *initial conditions*, but this term is inappropriate when the independent variable is not time. Even when the independent variable is time, this term does not always apply because final conditions or conditions given at any t are just as useful as initial conditions in determining the arbitrary constants.

Complementary function:

$$y_c = c_1 \sin x + c_2 \cos x. \tag{1-26}$$

Differentiating twice, we obtain

$$\frac{d^2 y_c}{dx^2} = -c_1 \sin x - c_2 \cos x. \tag{1-27}$$

It is obvious that

$$\frac{d^2 y_c}{dx^2} + y_c = 0, \tag{1-28}$$

which is the homogeneous equation of Eq. (1-25). Since $\sin x$ and $\cos x$ are linearly independent of each other, it follows that y_c in (1-26) with two arbitrary constants is the complementary function of Eq. (1-25).

*Particular integral:**

$$y_p = -\tfrac{1}{2} x \cos x. \tag{1-29}$$

Differentiating twice, we obtain

$$\frac{dy_p}{dx} = -\tfrac{1}{2} (\cos x - x \sin x), \tag{1-30}$$

$$\frac{d^2 y_p}{dx^2} = \tfrac{1}{2}(x \cos x + 2 \sin x). \tag{1-31}$$

Adding (1-29) and (1-31), we see that y_p satisfies Eq. (1-25) and is a particular integral. Therefore

$$y = y_c + y_p = c_1 \sin x + c_2 \cos x - \tfrac{1}{2} x \cos x \tag{1-32}$$

is a general solution of the linear differential equation (1-25).

Example 1-2. Find the differential equation which has a general solution of the form

$$w = c_1 \epsilon^{-t} + c_2 \epsilon^{-2t} + 3. \tag{1-33}$$

---

* The particular-integral part of the general solution to a differential equation is generally considered as the solution with all arbitrary constants set to equal zero. This agreement will prevent ambiguity. For instance,

$$y = 2 \sin x + 3 \cos x - \tfrac{1}{2} x \cos x$$

is also a particular solution of Eq. (1-25), as can be proved readily by direct substitution. But when we refer to the particular integral, we agree that it is

$$y_p = y\big|_{c_1=0,\ c_2=0} = -\tfrac{1}{2} x \cos x$$

Solution. Since there are two arbitrary constants, we expect it to be the solution of a second-order differential equation. Differentiating it with respect to t twice, we have

$$\frac{dw}{dt} = -c_1\epsilon^{-t} - 2c_2\epsilon^{-2t} \tag{1–34}$$

and

$$\frac{d^2w}{dt^2} = c_1\epsilon^{-t} + 4c_2\epsilon^{-2t}. \tag{1–35}$$

The two constants c_1 and c_2 can be eliminated from the three expressions above by substitution. We can proceed as follows:

$$(1\text{–}34) + (1\text{–}33):\qquad \frac{dw}{dt} + w = -c_2\epsilon^{-2t} + 3, \tag{1–36}$$

$$(1\text{–}35) - (1\text{–}33):\qquad \frac{d^2w}{dt^2} - w = 3c_2\epsilon^{-2t} - 3, \tag{1–37}$$

$$3(1\text{–}36) + (1\text{–}37):\qquad \frac{d^2w}{dt^2} + 3\frac{dw}{dt} + 2w = 6, \tag{1–38}$$

which is the required differential equation.

EXAMPLE 1–3. Assume in the above example that the values of w and its first derivative are known at $t = 0$.

$$\text{At } t = 0: \qquad w = 0 \qquad \text{and} \qquad \frac{dw}{dt} = 5.$$

Determine the arbitrary constants c_1 and c_2 in the general solution.

Solution. Substitute the known values into Eqs. (1–33) and (1–34):

$$t = 0, \qquad 0 = c_1 + c_2 + 3,$$
$$t = 0, \qquad 5 = -c_1 - 2c_2.$$

Solving for c_1 and c_2, we have

$$c_1 = -1, \qquad c_2 = -2.$$

Hence the complete solution of Eq. (1–38) under the given conditions is

$$w = -\epsilon^{-t} - 2\epsilon^{-2t} + 3.$$

PROBLEMS

1–1. Prove by mathematical reasoning the statement in Section 1–2 that stationary linear systems create no new frequencies.

1–2. Determine which of the following differential equations are linear and which are nonlinear:

(a) $4\dfrac{d^2w}{dx^2} = w\dfrac{dw}{dx}$ (b) $x^2\dfrac{d^3y}{dx^3} - \epsilon^{-x}\dfrac{dy}{dx} + 2y = \sin x$

(c) $\dfrac{1}{r}\dfrac{d^2r}{dt^2} + \dfrac{1}{r}\dfrac{dr}{dt} + 1 = 0$ (d) $\dfrac{du}{d\theta} = \sqrt{\theta}$

(e) $\dfrac{d^2w}{dt^2} + \dfrac{1}{w}\dfrac{dw}{dt} - 3 = 0$

(f) $\dfrac{d^2e}{dt^2} + \dfrac{1}{t}\dfrac{de}{dt} + \left(1 - \dfrac{n^2}{t^2}\right)e = 0$, n is a constant

(g) $(1 - x^2)\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + n(n+1)y = 0$, n is a constant

1–3. Verify that $i = c_1 \sinh 2t + c_2 \cosh 2t$ is a general solution of the equation $(d^2i/dt^2) - 4i = 0$.

1–4. Suppose it is found that the functions $3 \cos 2t$, $4 \cos (2t + 1)$, and $5 \sin 2(t - \frac{1}{3})$ all satisfy the following homogeneous equation:

$$\frac{d^2w}{dt^2} + a_1\frac{dw}{dt} + a_0w = 0.$$

(a) Determine the general solution of the given equation. (b) Determine the coefficients a_1 and a_0.

1–5. Verify that $y = c_1x + c_2x^2 + \frac{1}{2}x^3$ is a general solution of the equation

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = x^3.$$

1–6. Find the differential equation which has a general solution of the form $e = E \cos (377t + \theta)$, where E and θ are arbitrary constants.

1–7. (a) Find the differential equation which has a general solution of the form

$$y = c_1\epsilon^{-x} + c_2x\epsilon^{-x} + 2 \sin x.$$

(b) If at $x = 0$, $y = \frac{1}{2}$ and $dy/dx = 2$, determine the constants c_1 and c_2.

1–8. Determine which of the following sets of functions are linearly independent for all x, and which are linearly dependent. Give your reasons.

(a) $\epsilon^x, \epsilon^{-x}$ (b) $\epsilon^x, \epsilon^{3x}, \epsilon^{2x} \cosh x$

(c) $3x, 1 - \dfrac{x}{2}, 2(1 + x)$ (d) $3x, 1 + x, x^2$

(e) $\sin x, \cos x, \sin 2x$

(f) $\epsilon^{(-\alpha+j\beta)x}, \epsilon^{(-\alpha-j\beta)x}, \epsilon^{-\alpha x} \cos (\beta x + \Phi)$

1–9. (a) Show that the particular integral of the following linear differential equation can only be of the form $K\epsilon^{kt}$:

$$\frac{d^2w}{dt^2} + A\,\frac{dw}{dt} + Bw = C\epsilon^{kt},$$

where A, B, C, and k are given constants, from which the constant K is determined.

(b) From the result of part (a), show further that the particular integral of the following linear differential equation can only be of the form $K_1 \sin \omega t + K_2 \cos \omega t$:

$$\frac{d^2w}{dt^2} + A\,\frac{dw}{dt} + Bw = C \sin \omega t.$$

CHAPTER 2

CLASSICAL SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

2–1 Introduction. In this chapter we shall review some of the classical methods of solving linear ordinary differential equations. These methods are referred to as classical methods because they do not involve the transformation of functions and operations, the use of which has become popular only relatively recently. Although one of the primary purposes of this book is to introduce the Laplace transform method of solving linear differential and integro-differential equations and to show how, in many circumstances, it is simpler and more convenient to use than classical methods, we must not minimize the importance of understanding the classical methods. First of all, there are definite limitations to the applicability of transform methods. For example, the Laplace transform method cannot conveniently be used to solve linear differential equations with variable coefficients even of the first order, while the classical approach can yield solutions to many such equations of practical importance. If the known conditions of a problem are specified at values of the independent variable other than zero, the Laplace transform method becomes cumbersome to use even when the physical situation can be described by differential equations with constant coefficients. On the other hand, the application of classical methods is not modified by the way in which the known conditions are specified. Secondly, the separation of the general solution to a differential equation into a complementary function and a particular integral in the classical approach helps the understanding of the general nature of system response. As we shall see more clearly later on, the complementary function, being the solution of the homogeneous equation (with no excitation), represents the transient response of the system and depends entirely on the type, size, and arrangement of system elements, whereas the particular integral represents the steady-state response of the system and depends not only on the system itself but also on the excitation. For many types of excitation which we encounter in engineering both the complementary-function and the particular-integral parts of the response can be obtained quite readily without having to go through the formal, and sometimes rather tedious, transformation procedures.

In this chapter we shall first discuss classical methods of solving general linear differential equations of the first order. Since no general formulas are available for the solution of equations with variable coefficients of order greater than one, we shall devote the rest of the chapter to solutions of higher-order and simultaneous linear differential equations with constant coefficients. We need not be unduly dismayed by these restrictions because, fortunately, most engineering systems can be described or approximated within useful ranges by equations with constant coefficients. We shall not attempt to present an exhaustive account of all classical methods. Only a few important techniques will be discussed.

2-2 Linear equations of the first order. A general linear differential equation of the first order can be written in the following form:*

$$\frac{dw}{dt} + a(t)w = e(t). \tag{2-1}$$

Here we have used t as the independent variable. What it represents in a physical system is immaterial to the mathematical problem at hand. To solve this equation, we note that the presence of dw/dt and w in the two terms on the left side of Eq. (2-1) strongly suggests that it may be possible to arrange these two terms as the derivative of a product containing w as a factor, such as (wu), where u is as yet an unknown function of t. We know that if we could express the left side of Eq. (2-1) as an exact derivative of (wu), then w could be found by integrating the right side of the equation and dividing the result by u. But first we have to determine the function u. Now

$$\frac{d}{dt}(wu) = u\frac{dw}{dt} + \frac{du}{dt}w. \tag{2-2}$$

Comparing the right side of Eq. (2-2) with the two terms on the left side of Eq. (2-1), we are led to try multiplying the entire equation by the unknown function, u, of t, which yields

$$u\frac{dw}{dt} + ua(t)w = ue(t). \tag{2-3}$$

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* The fact that the coefficient of dw/dt is unity in Eq. (2-1) represents no loss in generality, for if we have

$$a_1(t)\frac{dw}{dt} + a_0(t)w = e_1(t)$$

we can always divide the entire equation by the coefficient $a_1(t)$ and write

$$\frac{dw}{dt} + \frac{a_0(t)}{a_1(t)}w = \frac{e_1(t)}{a_1(t)}.$$

Equation (2-1) is obtained if we let

$$\frac{a_0(t)}{a_1(t)} = a(t) \qquad \text{and} \qquad \frac{e_1(t)}{a_1(t)} = e(t).$$

Hence for the left side of Eq. (2-3) to be an exact derivative of the product (wu), we require

$$\frac{du}{dt} = ua(t)$$

or, separating the variables,

$$\frac{du}{u} = a(t)\,dt.$$

Integrating, we obtain

$$\ln u = \int a(t)\,dt$$

or

$$u = \epsilon^{\int a(t)\,dt}. \tag{2-4}$$*

Since the coefficient $a(t)$ is a given function, u can be found from Eq. (2-4) as a function of t. Substituting back into Eq. (2-3), we write

$$\left[\frac{dw}{dt} + a(t)w\right]\epsilon^{\int a(t)\,dt} = e(t)\epsilon^{\int a(t)\,dt},$$

$$\frac{d}{dt}\left[w\epsilon^{\int a(t)\,dt}\right] = e(t)\epsilon^{\int a(t)\,dt}.$$

Integrating both sides and rearranging terms, we have finally

$$w = \epsilon^{-\int a(t)\,dt}\left[\int e(t)\epsilon^{\int a(t)\,dt}\,dt + c\right], \tag{2-5}$$

which is the required solution for Eq. (2-1). The function u as given in Eq. (2-4), introduced to make the left side of the original equation an exact derivative (so that it can be integrated), is called an *integrating factor*.

Recalling the general properties of the solutions of a linear differential equation from Section 1-4, we note that the solution in Eq. (2-5) can actually be separated into two parts:

$$w = w_c + w_p, \tag{2-6}$$

where

$$w_c = \frac{c}{u}, \qquad u = \epsilon^{\int a(t)\,dt} \tag{2-7}$$

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* The reader need not be concerned with the integration constant which accompanies the indefinite integral of $a(t)$ for, as will be apparent from Eq. (2-5), the integration constant will be cancelled, and has no effect on the final solution of the differential equation.

is the complementary function which satisfies the homogeneous equation with $e(t) = 0$, and

$$w_p = \frac{1}{u}\int ue(t)\,dt, \qquad u = \epsilon^{\int a(t)\,dt} \tag{2–8}$$

is the particular integral which depends upon the excitation function $e(t)$.

Example 2–1. Solve the following linear first-order differential equation:

$$\frac{dw}{dt} - w = \epsilon^t.$$

Solution. Here the integrating factor is

$$u = \epsilon^{\int(-1)\,dt} = \epsilon^{-t}.$$

Complementary function:

$$w_c = \frac{c}{u} = c\epsilon^t.$$

Particular integral:

$$w_p = \frac{1}{u}\int ue(t)\,dt = \epsilon^t\int \epsilon^{-t}\cdot\epsilon^t\,dt = t\epsilon^t.$$

Hence the general solution is

$$w = w_c + w_p = \epsilon^t(c + t).$$

Example 2–2. Solve the following linear first-order differential equation:

$$(\cos x)\frac{dy}{dx} + (\sin x)\,y = 1.$$

Solution. Dividing the given equation by $\cos x$, we have

$$\frac{dy}{dx} + (\tan x)\,y = \sec x.$$

Integrating factor:

$$u = \epsilon^{\int \tan x\,dx} = \epsilon^{\ln \sec x} = \sec x.^*$$

Complementary function:

$$y_c = \frac{c}{u} = c\cos x$$

Particular integral:

$$y_p = \frac{1}{u}\int u\,(\sec x)\,dx = \cos x\int \sec^2 x\,dx = \sin x.$$

Hence the general solution is

$$y = y_c + y_p = c\cos x + \sin x.$$

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* The relation $\epsilon^{\ln z} = z$ holds for any z. This can be seen quite easily by taking the natural logarithm of both sides.

2–3 Higher-order linear equations with constant coefficients. As we have indicated in Section 2–1, no general formulas are available for the solution of linear differential equations with variable coefficients of order greater than one. Hence we are unable to extend the general method of the preceding section to equations of a higher order. Fortunately, a great many engineering systems can be described by equations with constant coefficients, and general methods of solving higher-order linear differential equations with constant coefficients do exist. In this section we shall present a method which makes use of an operator notation and which can be used to solve almost all problems of practical importance.

We first introduce a symbol D to denote the differentiating operation with respect to the independent variable. If t is used as the independent variable, then symbolically we have

$$D \equiv \frac{d}{dt}\cdot \tag{2–9}$$

Note that D is not simply a function; it is an *operator*. When D is placed to the left of a function, it implies that this function is to be differentiated once with respect to the independent variable t. *D must have something to operate on.* Hence

$$Dw = \frac{dw}{dt} \tag{2–10}$$

and

$$D^n w = \frac{d^n w}{dt^n}\cdot \tag{2–11}$$

But *wD and wD^n mean nothing at all!* Although D is not an algebraic quantity, an operation which contains more than one D can be rearranged or regrouped in accordance with the fundamental laws of algebra so long as the relative positions of the operators and the functions to be operated on are not interchanged. Thus

(A) For addition:

The commutative law: $$(D^m + D^n)w = D^m w + D^n w. \tag{2–12}$$

The associative law: $$[D^l + (D^m + D^n)]w = [(D^l + D^m) + D^n]w. \tag{2–13}$$

(B) For multiplication:

The commutative law: $$(D^m \cdot D^n)w = (D^n \cdot D^m)w = D^{m+n}w. \tag{2–14}$$

The associative law: $$D^l(D^m \cdot D^n)w = (D^l \cdot D^m)D^n w. \tag{2–15}$$

The distributive law: $$D^l(D^m + D^n)w = (D^{l+m} + D^{l+n})w. \tag{2–16}$$

Before we apply the operator notation to an equation of an arbitrary order, let us apply it to a first-order equation with constant coefficients:

$$\frac{dw}{dt} + aw = e(t). \tag{2-17}$$

Equation (2–17) is the same as Eq. (2–1) except that now the coefficient is a constant (independent of t). With operator notation (2–10), we write Eq. (2–17) as

$$(D + a)w = e(t). \tag{2-18}$$

We must always remember that *equations containing D, such as Eq.* (2–18), *are not algebraic equations.* Now, what is the general solution of Eq. (2–18)? We have the answer to this question because we have just obtained the solution in the preceding section. We know that the appropriate integrating factor is

$$u = \epsilon^{\int a\,dt} = \epsilon^{at} \tag{2-19}$$

and that the general solution consists of two parts:

(1) *Complementary function* [which is the general solution of the homogeneous equation $(D + a)w_c = 0$]:

$$w_c = \frac{c}{u} = c\epsilon^{-at}. \tag{2-20}$$

(2) *Particular integral* [which is a particular solution of the nonhomogeneous equation $(D + a)w_p = e(t)$]:

$$w_p = \frac{1}{D + a}\,e(t). \tag{2-21}$$

By virtue of Eq. (2–8),

$$w_p = \frac{1}{u}\int ue(t)\,dt = \epsilon^{-at}\int \epsilon^{at}e(t)\,dt. \tag{2-22}$$

Thus, the operation of $1/(D + a)$ on a function $e(t)$ in Eq. (2–21) can be considered as a shorthand way of writing the right side of Eq. (2–22).

The general solution of Eq. (2–18) is then

$$w = w_c + w_p = \epsilon^{-at}\left[\int \epsilon^{at}e(t)\,dt + c\right]\cdot \tag{2-23}$$

Next, consider an equation of the second order:

$$a_2\frac{d^2w}{dt^2} + a_1\frac{dw}{dt} + a_0w = e_1(t), \tag{2-24}$$

where a_2, a_1, a_0 are constants and $e_1(t)$ is an arbitrary excitation function.

Dividing Eq. (2-24) by a_2 and writing it in operator notation, we have

$$(D^2 + b_1 D + b_0)w = \frac{1}{a_2} e_1(t) = e(t), \tag{2-25}$$

where b_1 is written for a_1/a_2 and b_0 for a_0/a_2. To find the general solution of Eq. (2-25), we consider the complementary function and the particular integral separately.

(1) *Complementary function*, w_c. The homogeneous equation to be solved is

$$(D^2 + b_1 D + b_0)w_c = 0, \tag{2-26}$$

which can be factored to give

$$(D - s_1)(D - s_2)w_c = 0, \tag{2-27}$$

where*

$$s_1 = -\tfrac{1}{2}(b_1 + \sqrt{b_1^2 - 4b_0}), \tag{2-28}$$

$$s_2 = -\tfrac{1}{2}(b_1 - \sqrt{b_1^2 - 4b_0}). \tag{2-29}$$

The solution of either the equation

$$(D - s_1)w_{c_1} = 0 \tag{2-30}$$

or the equation

$$(D - s_2)w_{c_2} = 0 \tag{2-31}$$

will then be a solution of Eq. (2-27). Equations (2-30) and (2-31) are both first-order homogeneous linear equations and their solutions (similar to Eq. 2-20) are, respectively,

$$w_{c_1} = c_1 \epsilon^{s_1 t} \tag{2-32}$$

and

$$w_{c_2} = c_2 \epsilon^{s_2 t}. \tag{2-33}$$

Hence the complementary function of Eq. (2-25) is

$$w_c = w_{c_1} + w_{c_2} = c_1 \epsilon^{s_1 t} + c_2 \epsilon^{s_2 t}, \tag{2-34}$$

which correctly contains two arbitrary constants c_1 and c_2.

Another way of arriving at Eq. (2-34) is to start with a trial solution of the type ϵ^{st} for the homogeneous equation (2-26). The reason for choosing an exponential solution of this type is that all derivatives of the exponential function contain the function itself, which can then be factored out,

* We assume here that $s_1 \neq s_2$. The case of multiple roots will be discussed in Section 2-5.

leaving a purely algebraic equation. Thus, substituting ϵ^{st} for w_c in Eq. (2–26), we have

$$(s^2 + b_1 s + b_0)\epsilon^{st} = 0. \tag{2–35}$$

Inasmuch as ϵ^{st} does not vanish, Eq. (2–35) can be satisfied only if

$$s^2 + b_1 s + b_0 = 0. \tag{2–36}$$

Equation (2–36) is called the *characteristic* or *auxiliary equation* of either Eq. (2–25) or Eq. (2–26). It is an algebraic equation obtained by replacing D by s in the terms within the parentheses on the left side of Eq. (2–25) or Eq. (2–26). Equation (2–36) has two roots, namely,

$$s = s_1 \qquad \text{and} \qquad s = s_2$$

if we use the notations in Eqs. (2–28) and (2–29). This tells us that there exist two independent solutions, $\epsilon^{s_1 t}$ and $\epsilon^{s_2 t}$ for the homogeneous equation (2–26). By property (C) stated in Section 1–4, the complementary function is then of the form shown in Eq. (2–34).

(2) *Particular integral*, w_p. The nonhomogeneous equation to be solved is

$$(D^2 + b_1 D + b_0)w_p = e(t). \tag{2–37}$$

We factor the operators as we did in Eq. (2–27) to give

$$(D - s_1)(D - s_2)w_p = e(t), \tag{2–38}$$

where s_1 and s_2 take the values given in Eqs. (2–28) and (2–29) respectively. The solution of Eq. (2–38) can be written in operator form as follows:

$$w_p = \frac{1}{(D - s_1)(D - s_2)}\, e(t). \tag{2–39}$$

We emphasize here again that Eq. (2–39) is not an algebraic equation and its right side does not represent a simple division. We can determine w_p from Eq. (2–39) by the *method of successive integrations*.

We write the two factors that appear on the right side of Eq. (2–39) separately and carry out the operations successively, one at a time. Thus,

$$w_p = \frac{1}{D - s_1}\left[\frac{1}{D - s_2}\, e(t)\right]\cdot \tag{2–40}$$

From Eqs. (2–21) and (2–22) we can write the result of the operation contained in the brackets of Eq. (2–40) immediately:

$$\frac{1}{D - s_2}\, e(t) = \epsilon^{s_2 t}\int \epsilon^{-s_2 t} e(t)\, dt.$$

Substituting this back into Eq. (2–40) and performing the remaining operation, we have

$$\begin{aligned} w_p &= \frac{1}{D - s_1}\left[\epsilon^{s_2 t}\int \epsilon^{-s_2 t} e(t)\, dt\right] \\ &= \epsilon^{s_1 t}\int \epsilon^{-s_1 t}\left[\epsilon^{s_2 t}\int \epsilon^{-s_2 t} e(t)\, dt\right] dt \\ &= \epsilon^{s_1 t}\int \epsilon^{(s_2 - s_1)t}\int \epsilon^{-s_2 t} e(t)\, (dt)^2. \end{aligned} \tag{2–41}$$

The general solution of the original second-order equation (2–24) is then the sum of the complementary function as given in Eq. (2–34) and the particular integral as given in Eq. (2–41).

In general, if the roots of the characteristic or auxiliary equation of an nth-order differential equation are $s_1, s_2, s_3, \ldots, s_n$, then

$$\begin{aligned} w_p &= \frac{1}{(D - s_1)(D - s_2)(D - s_3)\cdots(D - s_n)}\, e(t) \\ &= \epsilon^{s_1 t}\int \epsilon^{(s_2 - s_1)t}\int \epsilon^{(s_3 - s_2)t}\int\cdots\int \epsilon^{(s_n - s_{n-1})t}\int \epsilon^{-s_n t} e(t)\, (dt)^n. \end{aligned} \tag{2–42}$$

EXAMPLE 2–3. Find the solution for the following second-order linear differential equation:

$$2\frac{d^2 w}{dt^2} + 3\frac{dw}{dt} + w = 10\epsilon^{-3t}, \tag{2–43}$$

for which $w = 1$ and $dw/dt = 0$ at $t = 0$.

Solution. First let us divide the given equation by 2 and rewrite it in operator notation as

$$(D^2 + \tfrac{3}{2}D + \tfrac{1}{2})w = 5\epsilon^{-3t}. \tag{2–44}$$

Factoring the left side, we have

$$(D + 1)(D + \tfrac{1}{2})w = 5\epsilon^{-3t}. \tag{2–45}$$

(1) *Complementary function.* Since $s_1 = -1$, $s_2 = -\frac{1}{2}$, we can write the complementary function directly from Eq. (2–34):

$$w_c = c_1\epsilon^{-t} + c_2\epsilon^{-t/2}. \tag{2–46}$$

(2) *Particular integral.* Direct substitution of the values of s_1, s_2, and $e(t)$ into Eq. (2–41) yields

$$w_p = \epsilon^{-t}\int \epsilon^{t/2}\int \epsilon^{t/2}(5\epsilon^{-3t})\, (dt)^2 = \epsilon^{-3t}. \tag{2–47}$$

Note that the exponential form of the particular integral is the same as that of the excitation function. This could have been predicted from Problem 1–9. The correctness of the particular integral can always be checked by substituting it back into the given differential equation. The general solution of the given equation is

$$w = w_c + w_p = c_1\epsilon^{-t} + c_2\epsilon^{-t/2} + \epsilon^{-3t}. \tag{2–48}$$

Now we apply the two given initial conditions to determine the constants c_1 and c_2:

$$\text{At } t = 0: \qquad w = c_1 + c_2 + 1 = 1,$$

$$\frac{dw}{dt} = -c_1 - \tfrac{1}{2}c_2 - 3 = 0,$$

from which we find

$$c_1 = -6 \qquad \text{and} \qquad c_2 = 6.$$

Hence the desired solution is

$$w = -6\epsilon^{-t} + 6\epsilon^{-t/2} + \epsilon^{-3t}. \tag{2–49}$$

EXAMPLE 2–4. Find the general solution for the following third-order linear differential equation:

$$\frac{d^3y}{dx^3} + 4\,\frac{d^2y}{dx^2} + 5\,\frac{dy}{dx} = x. \tag{2–50}$$

Solution. Writing Eq. (2–50) in operator notation, we have

$$(D^3 + 4D^2 + 5D)y = x. \tag{2–51}$$

The characteristic or auxiliary equation is obtained by substituting s for D in the operator part of the homogeneous equation:

$$s^3 + 4s^2 + 5s = 0. \tag{2–52}$$

This third-degree algebraic equation has three roots and can be factored as

$$(s - s_1)(s - s_2)(s - s_3) = 0, \tag{2–53}$$

where

$$s_1 = 0, \qquad s_2 = -(2 + j), \qquad s_3 = -(2 - j) = s_2^*. \tag{2–54}$$

Note that the roots of Eq. (2–52) or Eq. (2–53) are 0, s_2, and s_3, and that s_2 and s_3 are complex conjugates of each other. *It is a property of algebraic equations with real coefficients that complex roots always occur in conjugate pairs.*

(1) *Complementary function.* With the roots of the auxiliary equation known, we can write the complementary function in the standard exponential form

$$
\begin{aligned}
y_c &= c_1\epsilon^{s_1x} + c_2\epsilon^{s_2x} + c_3\epsilon^{s_3x} \\
&= c_1 + c_2\epsilon^{-(2+j)x} + c_3\epsilon^{-(2-j)x}.
\end{aligned} \tag{2-55}
$$

But Eq. (2–55) can be put in a better form by combining the terms with conjugate imaginary exponents:

$$y_c = c_1 + \epsilon^{-2x}(c_2\epsilon^{-jx} + c_3\epsilon^{jx}). \tag{2-56}$$

Recalling the relationship

$$\epsilon^{\pm jx} = \cos x \pm j\sin x, \tag{2-57}$$

we write Eq. (2–56) as

$$y_c = c_1 + \epsilon^{-2x}[(c_2 + c_3)\cos x + j(c_3 - c_2)\sin x]. \tag{2-58}$$

For the sake of simplicity, we call

$$A = c_2 + c_3, \tag{2-59}$$

$$B = j(c_3 - c_2). \tag{2-60}$$

Hence an equivalent form of the complementary function is

$$y_c = c_1 + \epsilon^{-2x}(A\cos x + B\sin x), \tag{2-61}$$

which contains the three arbitrary constants c_1, A, and B.

At this point we can make a generalization for the form of the complementary function $y_c(x)$: *each distinct* real root s_k (including zero) of the characteristic or auxiliary equation gives rise to a $c_k\epsilon^{s_kx}$ term, and each distinct pair of complex conjugate roots $-\alpha \pm j\beta$ (including purely imaginary conjugate roots when $\alpha = 0$) yields two terms which can be written as* $\epsilon^{-\alpha x}(A\cos\beta x + B\sin\beta x)$.

(2) *Particular integral.*

$$y_p = \frac{1}{D(D+2+j)(D+2-j)}\,x. \tag{2-62}$$

---

* A root of an equation is said to be *distinct* when it is not equal to any other root of the equation. Special treatment is needed when the characteristic or auxiliary equation of a differential equation has multiple or repeated roots (see Section 2–5).

Applying Eq. (2-42), we have

$$y_p = \int \epsilon^{-(2+j)x} \int \epsilon^{j2x} \int \epsilon^{(2-j)x} x\,(dx)^3 = \frac{x}{5}\left(\frac{x}{2} - \frac{4}{5}\right). \tag{2-63}$$

That this particular integral is a solution of the given equation (2-50) or (2-51) can be readily checked by direct substitution. The general solution is the sum of the complementary function y_c, as found in Eq. (2-61), and this particular integral y_p:

$$y = y_c + y_p = c_1 + \epsilon^{-2x}(A \cos x + B \sin x) + \frac{x}{5}\left(\frac{x}{2} - \frac{4}{5}\right). \tag{2-64}$$

The preceding discussions pertaining to second-order and third-order equations can be readily extended to linear differential equations with *constant* coefficients of an arbitrary order n:

$$a_n \frac{d^n w}{dt^n} + a_{n-1} \frac{d^{n-1} w}{dt^{n-1}} + \cdots + a_1 \frac{dw}{dt} + a_0 w = e_1(t), \tag{2-65}$$

where $a_n \neq 0$. Dividing the entire equation by a_n and writing it in operator notation, we have

$$(D^n + b_{n-1}D^{n-1} + \cdots + b_1 D + b_0)w = \frac{1}{a_n} e_1(t) = e(t), \tag{2-66}$$

where b_{n-1} is written for $a_{n-1}/a_n, \ldots, b_1$ for a_1/a_n, and b_0 for a_0/a_n.

The general solution of Eq. (2-65) or (2-66) can be found systematically in the following manner:

(1) We form the characteristic or auxiliary equation by replacing D's by s's in the operator part of the given differential equation and setting it equal to zero:

$$s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0 = 0. \tag{2-67}$$

(2) We then determine the n roots of the nth-degree algebraic equation (2-67),* and call these roots $s_1, s_2, s_3, \ldots, s_n$.

(3) If the n roots are all *distinct* (unequal), then the complementary function of Eq. (2-65) or (2-66) is

$$w_c = c_1\epsilon^{s_1 t} + c_2\epsilon^{s_2 t} + \cdots + c_n\epsilon^{s_n t} = \sum_{k=1}^{n} c_k \epsilon^{s_k t}. \tag{2-68}$$

---

* When n is large, it may be difficult to determine the roots of Eq. (2-67). No formulas are available for finding the exact roots of algebraic equations of a degree higher than the fourth. In such cases, we must resort to graphical or numerical methods for determining the roots approximately. Appendix A discusses numerical solution of algebraic equations.

(4) The particular integral can be found by the method of successive integrations from an integral of the nth order:

$$w_p = \frac{1}{(D - s_1)(D - s_2) \cdots (D - s_n)} e(t)$$
$$= \epsilon^{s_1 t} \int \epsilon^{(s_2 - s_1)t} \int \epsilon^{(s_3 - s_2)t} \int \cdots \int \epsilon^{(s_n - s_{n-1})t} \int \epsilon^{-s_n t} e(t)\,(dt)^n. \quad (2\text{–}69)$$

(5) The general solution of the given equation is the sum of the complementary function w_c in (2–68) and the particular integral w_p in (2–69).

The expression for the complementary function, Eq. (2–68), must be modified in accordance with Section 2–5 when the characteristic or auxiliary equation has multiple or repeated roots. The above procedure also assumes that no terms in w_p duplicate any of the exponential terms in w_c. If this assumption does not hold, the general solution should be modified as though multiple roots existed (see Example 2–9).

2–4 Method of undetermined coefficients. In the preceding section we have described the method of successive integrations for determining a particular integral. Other methods are available for determining particular integrals, among which the method of undetermined coefficients is particularly simple and easy to apply. This method has an advantage over others in that it involves only differentiations and no integrations are needed. As we well know, differentiations are always easier than integrations. The method of undetermined coefficients does not apply to all types of the excitation function $e(t)$, but it is useful when $e(t)$ is composed of functions of the following types:

A constant, K.
A power of the independent variable, t^k (k a positive integer).
An exponential function, $\epsilon^{\gamma t}$.
A cosine function, $\cos \gamma t$.
A sine function, $\sin \gamma t$.

It so happens that a majority of excitation functions we encounter in practice can be represented in terms of the functions listed above. Let us consider these functions separately. For simplicity we shall describe this method with a second-order equation:

$$\frac{d^2 w}{dt^2} + b_1 \frac{dw}{dt} + b_0 w = e(t). \quad (2\text{–}70)$$

However, the following development will hold for linear differential equations with constant coefficients of any order.

I. $e(t) = K$ (*a constant*). The differential equation (2–70) becomes

$$(D^2 + b_1 D + b_0)w = K. \tag{2–71}$$

It is obvious that the particular integral is

$$w_p = A \text{ (a constant)}. \tag{2–72}$$

Since $D^2 A = 0$ and $DA = 0$, we can determine A immediately by substituting it into Eq. (2–71):

$$b_0 A = K, \qquad w_p = A = \frac{K}{b_0}. \tag{2–73}$$

II. $e(t) = t^k$ (*k a positive integer*).*

$$(D^2 + b_1 D + b_0)w_p = t^k. \tag{2–74}$$

Here it is reasonable to assume that w_p will be a polynomial in t, because a positive integral power of t such as t^k can be obtained as the derivative of only another power of t. Moreover, if $b_0 \neq 0$, there can be no terms in w_p that are of a power higher than k. Let us then assume that

$$w_p = A_0 + A_1 t + A_2 t^2 + \cdots + A_k t^k. \tag{2–75}$$

Hence

$$b_0 w_p = b_0 A_0 + b_0 A_1 t + b_0 A_2 t^2 + \cdots + b_0 A_k t^k, \tag{2–76}$$

$$b_1 D w_p = b_1 A_1 + 2b_1 A_2 t + 3b_1 A_3 t^2 + \cdots + k b_1 A_k t^{k-1}, \tag{2–77}$$

$$D^2 w_p = 2A_2 + 6A_3 t + \cdots + k(k-1) A_k t^{k-2}. \tag{2–78}$$

Adding Eqs. (2–76), (2–77), and (2–78), and equating the result to t^k, we have

$$\begin{aligned}(D^2 + b_1 D + b_0)w_p = {} & (b_0 A_0 + b_1 A_1 + 2A_2) \\ & + (b_0 A_1 + 2b_1 A_2 + 6A_3)t + \cdots \\ & + (b_0 A_{k-1} + k b_1 A_k)t^{k-1} + b_0 A_k t^k = t^k.\end{aligned}$$

Equating the coefficients of the like powers in t, we obtain $(k + 1)$ equations:

$$\begin{aligned} b_0 A_0 \quad + \quad b_1 A_1 + 2A_2 &= 0, \\ b_0 A_1 \quad + 2b_1 A_2 + 6A_3 &= 0, \\ \vdots \qquad\qquad & \\ b_0 A_{k-1} + k b_1 A_k \qquad &= 0, \\ b_0 A_k \qquad &= 1, \end{aligned}$$

* We note that case I, where $e(t)$ equals a constant, is a special case with $k = 0$.

from which the $(k+1)$ coefficients, $A_0, A_1, A_2, \ldots, A_{k-1}$, and A_k in the assumed solution (2-75) can be determined.

EXAMPLE 2-5. Find the general solution of the equation

$$\frac{d^2w}{dt^2} - w = 2t^2. \tag{2-79}$$

Solution. Here we have

$$(D^2 - 1)w = 2t^2,$$
$$(D+1)(D-1)w = 2t^2.$$

Complementary function:

$$w_c = c_1\epsilon^{-t} + c_2\epsilon^t. \tag{2-80}$$

Particular integral: Since the excitation function has a second power in t, we assume w_p to be a polynomial in t of the second degree:

$$w_p = A_0 + A_1t + A_2t^2, \tag{2-81}$$

$$D^2w_p = 2A_2. \tag{2-82}$$

Subtracting Eq. (2-81) from Eq. (2-82) and equating the difference to $2t^2$, we have

$$(D^2 - 1)w_p = (2A_2 - A_0) - A_1t - A_2t^2 = 2t^2.$$

We can now write three simultaneous algebraic equations relating the coefficients:

$$2A_2 - A_0 = 0, \qquad -A_1 = 0, \qquad -A_2 = 2. \tag{2-83}$$

Hence $A_0 = -4$, $A_1 = 0$, $A_2 = -2$, and

$$w_p = -2(2 + t^2). \tag{2-84}$$

The general solution is obtained by adding the results in Eqs. (2-80) and (2-84):

$$w = w_c + w_p = c_1\epsilon^{-t} + c_2\epsilon^t - 2(2 + t^2). \tag{2-85}$$

Note that we did not need to evaluate any integrals in obtaining the solution.

EXAMPLE 2-6. Find the general solution of the equation

$$\frac{d^3w}{dt^3} - \frac{dw}{dt} = 2t^2. \tag{2-86}$$

Solution. Writing the given equation in operator notation, we have

$$D(D^2 - 1)w = 2t^2. \tag{2–87}$$

The characteristic equation has three distinct roots:

$$s_1 = 0, \qquad s_2 = +1, \qquad s_3 = -1.$$

Hence the complementary function is

$$w_c = c_1 + c_2\epsilon^t + c_3\epsilon^{-t}. \tag{2–88}$$

We note the similarity between the given equation and Eq. (2–79) in Example 2–5. Although the excitation function now is $2t^2$, as before, an assumed w_p in accordance with Eq. (2–81),

$$w_p = A_0 + A_1t + A_2t^2,$$

will not work, because $D^3w_p = 0$ and

$$D(D^2 - 1)w_p = -(A_1 + 2A_2t) = 2t^2$$

cannot be satisfied for any constant A_0, A_1, and A_2.

We can look at the given equation (2–86) or (2–87) as

$$(D^2 - 1)w = \int 2t^2\,dt = \tfrac{2}{3}t^3 + K. \tag{2–89}$$

The right side of Eq. (2–89) is a polynomial in t of the third degree ($k = 3$). For a w_p to satisfy Eq. (2–89) we should assume

$$w_p = A_0 + A_1t + A_2t^2 + A_3t^3. \tag{2–90}$$

Substituting the above w_p in the given equation, we have

$$D(D^2 - 1)w_p = 6A_3 - (A_1 + 2A_2t + 3A_3t^2) = 2t^2.$$

Three simultaneous equations are obtained by equating the corresponding coefficients:

$$6A_3 - A_1 = 0, \qquad -2A_2 = 0, \qquad -3A_3 = 2,$$

from which we find

$$A_1 = -4, \qquad A_2 = 0, \qquad A_3 = -\tfrac{2}{3}.$$

The value of A_0 is immaterial [since it will eventually be absorbed in the arbitrary constant c_1 in the complementary function w_c in Eq. (2–88)].

We set $A_0 = 0$. We have finally

$$w_p = -4t - \tfrac{2}{3}t^3, \tag{2-91}$$

and the complete solution is

$$w = w_c + w_p = c_1 + c_2\epsilon^t + c_3\epsilon^{-t} - 2(2 + \tfrac{1}{3}t^2)t. \tag{2-92}$$

We see that the particular integral in Eq. (2-91) is the integral of that in Eq. (2-84) found in Example 2-5. This enables us to make the following general statement: *when the characteristic equation of a differential equation has a zero root, the assumed form of the particular integral should be the integral of that which would normally be used in the method of undetermined coefficients.* This statement can be extended to include cases where the characteristic equation possesses multiple zero roots.

III. $e(t) = \epsilon^{\gamma t}$.

$$(D^2 + b_1 D + b_0)w_p = \epsilon^{\gamma t}. \tag{2-93}$$

Since the derivatives of an exponential function $\epsilon^{\gamma t}$ will all contain the same function $\epsilon^{\gamma t}$, it is certainly reasonable to choose

$$w_p = A\epsilon^{\gamma t}. \tag{2-94)*$$

Substituting w_p in Eq. (2-93) and cancelling the factor $\epsilon^{\gamma t}$ on both sides, we have

$$A(\gamma^2 + b_1\gamma + b_0) = 1.$$

Hence

$$A = \frac{1}{\gamma^2 + b_1\gamma + b_0} \tag{2-95}$$

and

$$w_p = \frac{\epsilon^{\gamma t}}{\gamma^2 + b_1\gamma + b_0}. \tag{2-96}$$

This method can be applied to find the particular integral for Eq. (2-43) or Eq. (2-44) in Example 2-3, where the given equation was

$$(D^2 + \tfrac{3}{2}D + \tfrac{1}{2})w = 5\epsilon^{-3t}.$$

Here, in view of Eq. (2-96), we obtain directly, without integration,

$$w_p = \frac{5\epsilon^{-3t}}{(-3)^2 + \frac{3}{2}(-3) + \frac{1}{2}} = \epsilon^{-3t},$$

as in Eq. (2-47).

---

* We note here that case I is also a special case of III with $\gamma = 0$.

IV. $e(t) = \cos \gamma t$ or $e(t) = \sin \gamma t$.

It is convenient to examine the cosine and sine functions in terms of their exponential components, since we now know how to handle an excitation function of the exponential type:

$$\cos \gamma t = \frac{1}{2} (\epsilon^{j\gamma t} + \epsilon^{-j\gamma t}), \tag{2-97}$$

$$\sin \gamma t = \frac{1}{2j} (\epsilon^{j\gamma t} - \epsilon^{-j\gamma t}). \tag{2-98}$$

For every type of exponential excitation function we must assume that an exponential function of the same type exists in the particular integral. Thus, for either $\cos \gamma t$ or $\sin \gamma t$, we choose

$$w_p = B_1\epsilon^{j\gamma t} + B_2\epsilon^{-j\gamma t}. \tag{2-99}$$

Equation (2–99) can be put in another form which avoids the combination and manipulation of exponential functions and complex coefficients:

$$\begin{aligned} w_p &= B_1 (\cos \gamma t + j \sin \gamma t) + B_2 (\cos \gamma t - j \sin \gamma t) \\ &= (B_1 + B_2) \cos \gamma t + j (B_1 - B_2) \sin \gamma t \\ &= A_1 \cos \gamma t + A_2 \sin \gamma t, \end{aligned} \tag{2-100}$$

where A_1 has been written for $(B_1 + B_2)$ and A_2 for $j(B_1 - B_2)$. Let us illustrate this with an example.

EXAMPLE 2–7. Find the general solution of the equation

$$\frac{d^2w}{dt^2} - w = 3 \sin \omega t. \tag{2-101}$$

Solution. We note that the left side of Eq. (2–101) is the same as that of Eq. (2–79) in Example 2–5. Therefore these two equations have the same characteristic equation and hence also the same complementary function.

$$\begin{aligned} w_c &= c_1\epsilon^{-t} + c_2\epsilon^{t} \\ &= c_1 (\cosh t - \sinh t) + c_2 (\cosh t + \sinh t) \\ &= C_1 \cosh t + C_2 \sinh t, \end{aligned} \tag{2-102}$$

where C_1 has been written for $(c_1 + c_2)$ and C_2 for $(c_2 - c_1)$. To find the particular integral, we first choose, as suggested in Eq. (2–99),

$$w_p = B_1\epsilon^{j\omega t} + B_2\epsilon^{-j\omega t}. \tag{2-103}$$

Substituting w_p for w in Eq. (2–101), we have

$$-B_1(\omega^2 + 1)\epsilon^{j\omega t} - B_2(\omega^2 + 1)\epsilon^{-j\omega t} = \frac{3}{2j}(\epsilon^{j\omega t} - \epsilon^{-j\omega t}).$$

Equating the coefficients on both sides of the equation for like exponential terms, we determine the two constants B_1 and B_2 directly:

$$B_1 = -\frac{3}{2j(\omega^2 + 1)}, \tag{2–104}$$

$$B_2 = +\frac{3}{2j(\omega^2 + 1)}. \tag{2–105}$$

Hence, from Eqs. (2–103), (2–104), and (2–105),

$$\begin{aligned} w_p &= -\frac{3}{2j(\omega^2 + 1)}(\epsilon^{j\omega t} - \epsilon^{-j\omega t}) \\ &= -\frac{3}{\omega^2 + 1}\sin \omega t, \end{aligned} \tag{2–106}$$

and the general solution of Eq. (2–101) is

$$w = w_c + w_p = C_1 \cosh t + C_2 \sinh t - \frac{3}{\omega^2 + 1}\sin \omega t. \tag{2–107}$$

If we had chosen to use a trigonometric form instead of Eq. (2–103), we would write, according to Eq. (2–100),*

$$w_p = A_1 \cos \omega t + A_2 \sin \omega t. \tag{2–108}$$

Substituting back into Eq. (2–101), we now have

$$-A_1(\omega^2 + 1)\cos \omega t - A_2(\omega^2 + 1)\sin \omega t = 3 \sin \omega t.$$

Equating the coefficients of cosine and sine terms on both sides, we find

$$A_1 = 0, \tag{2–109}$$

$$A_2 = -\frac{3}{\omega^2 + 1}, \tag{2–110}$$

and

$$w_p = -\frac{3}{\omega^2 + 1}\sin \omega t,$$

as before.

---

* Note that even though the excitation function is a sine function we must include *both* the cosine *and* the sine terms in the trial solution. Sometimes the cosine term will finally drop out because $A_1 = 0$, as happens to be true in this example; but this is a special case. In general, this will not be true, and there is no easy way to tell from the given equation whether A_1 will be zero.

To recapitulate, the method of undetermined coefficients affords us a simple way of finding the particular integrals of linear differential equations with constant coefficients when the excitation function is composed of the sum or the product of functions of certain special types. These special types of functions and their corresponding forms of particular integrals are listed in Table 2-1.

TABLE 2-1

METHOD OF UNDETERMINED COEFFICIENTS*

Excitation function	Form of particular integral
K (constant)	A (constant)
Kt^k (k a positive integer)	$A_0 + A_1t + A_2t^2 + \cdots + A_kt^k$
$K\epsilon^{\gamma t}$	$A\epsilon^{\gamma t}$
$K \cos \gamma t$	$A_1 \cos \gamma t + A_2 \sin \gamma t$
$K \sin \gamma t$	$A_1 \cos \gamma t + A_2 \sin \gamma t$

* Modifications are necessary (1) when the characteristic equation has a zero root (see Example 2-6), and (2) when the excitation function contains a term which appears in the complementary function (see Example 2-9).

If the excitation function is the sum or the product of two or more terms of the types listed in Table 2-1, then the particular integral will also be the sum or the product of two or more forms corresponding to the given excitation functions. Thus for

$$e(t) = Kt^3\epsilon^{\gamma t} \sin kt \tag{2-111}$$

the correct form for the particular integral would be

$$\begin{aligned} w_p = {} & (A_0 + A_1t + A_2t^2 + A_3t^3)\epsilon^{\gamma t} \cos kt \\ & + (B_0 + B_1t + B_2t^2 + B_3t^3)\epsilon^{\gamma t} \sin kt. \end{aligned} \tag{2-112}$$

In general, when the excitation function is of such a type that the method of undetermined coefficients is applicable, then this method is the simplest way of finding the particular-integral part of the solution.

2-5 Equations with multiple roots. Suppose now that we wish to solve the following homogeneous differential equation of the second order:

$$\frac{d^2w}{dt^2} + b_1\frac{dw}{dt} + b_0w = 0, \tag{2-113}$$

where the coefficients b_1 and b_0 are related in the following way:

$$b_1^2 = 4b_0. \tag{2-114}$$

Let us first write Eq. (2–113) in operator notation:

$$(D^2 + b_1 D + b_0)w = 0. \tag{2-115}$$

Equation (2–115) can be factored to give

$$(D - s_1)(D - s_1)w = 0 \tag{2-116}$$

because, by virtue of Eq. (2–114), the characteristic equation has a double root, that is, the two roots are not distinct:

$$s_1 = s_2 = -\frac{b_1}{2}. \tag{2-117}$$

If we tried to follow the method outlined in Section 2–3 without modification, we would obtain the following solution:

$$w = c_1\epsilon^{s_1 t} + c_2\epsilon^{s_1 t}. \tag{2-118}$$

But the two terms in Eq. (2–118) can be combined:

$$w = (c_1 + c_2)\epsilon^{s_1 t} = c\epsilon^{s_1 t}. \tag{2-119}$$

The solution then actually contains only one arbitrary constant, and hence cannot be the general solution of the second-order equation (2–113).

Going back to Eq. (2–116), let us first set

$$(D - s_1)w = r \tag{2-120}$$

and write Eq. (2–116) as

$$(D - s_1)r = 0, \tag{2-121}$$

which has as its solution

$$r = c_2\epsilon^{s_1 t}. \tag{2-122}$$

Substituting this value of r in Eq. (2–120), we have

$$(D - s_1)w = c_2\epsilon^{s_1 t}. \tag{2-123}$$

Equation (2–123) is a nonhomogeneous first-order linear differential equation, and its solution is composed of two parts, namely,

a complementary function:

$$w_c = c_1\epsilon^{s_1 t}$$

and a particular integral:

$$w_p = \epsilon^{s_1 t}\int \epsilon^{-s_1 t}(c_2\epsilon^{s_1 t})\,dt = c_2 t\epsilon^{s_1 t}.$$

Hence the general solution of Eq. (2-123) or Eq. (2-113) is

$$w = w_c + w_p = (c_1 + c_2 t)\epsilon^{s_1 t}, \tag{2-124}$$

which now correctly contains two arbitrary constants.

By extending the above procedure, we can easily establish that if the characteristic equation possesses a root s of multiplicity k, then the solution corresponding to that root is

$$w_s = \epsilon^{st}(c_1 + c_2 t + c_3 t^2 + \cdots + c_k t^{k-1}). \tag{2-125}$$

Of course, if the characteristic equation has other roots, they should be added to w_s to form the complementary function.

Example 2-8. Find the general solution of the following differential equation:

$$\frac{d^3 w}{dt^3} - 3\frac{dw}{dt} - 2w = \epsilon^{j\omega t}. \tag{2-126}$$

Solution. We first write Eq. (2-126) in operator form:

$$(D^3 - 3D - 2)w = \epsilon^{j\omega t}. \tag{2-127}$$

The characteristic equation is $(s^3 - 3s - 2) = 0$, or

$$(s + 1)^2(s - 2) = 0. \tag{2-128}$$

Hence it has a double root -1 and a single root $+2$.

(1) *Complementary function.*

$$w_c = (c_1 + c_2 t)\epsilon^{-t} + c_3\epsilon^{2t}. \tag{2-129}$$

(2) *Particular integral.* Let us use the method of successive integration. Specializing Eq. (2-69) for

$$s_1 = s_2 = -1 \qquad \text{and} \qquad s_3 = 2,$$

we have

$$\begin{aligned} w_p &= \epsilon^{s_1 t}\iint \epsilon^{(s_3 - s_1)t}\int \epsilon^{-s_3 t}(\epsilon^{j\omega t})\,(dt)^3 \\ &= \epsilon^{-t}\iint \epsilon^{3t}\int \epsilon^{-2t}(\epsilon^{j\omega t})\,(dt)^3 \\ &= -\frac{1}{2 + j\omega(3 + \omega^2)}\,\epsilon^{j\omega t}. \end{aligned} \tag{2-130}$$

We could have determined the particular integral by the method of undetermined coefficients as described in the preceding section without integrations. We start by assuming the form of the particular integral (see Table 2–1) to be

$$w_p = A\epsilon^{j\omega t}$$

and substituting it directly in Eq. (2–126) for w:

$$A[(j\omega)^3 - 3(j\omega) - 2]\epsilon^{j\omega t} = \epsilon^{j\omega t}.$$

Hence

$$A = -\frac{1}{2 + j\omega(3 + \omega^2)},$$

which, after being combined with the $\epsilon^{j\omega t}$ factor, checks with the answer found in Eq. (2–130).

The general solution of Eq. (2–126) is then

$$w = w_c + w_p = (c_1 + c_2 t)\epsilon^{-t} + c_3\epsilon^{2t} - \frac{\epsilon^{j\omega t}}{2 + j\omega(3 + \omega^2)}. \tag{2–131}$$

Having solved Eq. (2–126), we have in fact also solved the following two equations:

$$\frac{d^3w}{dt^3} - 3\frac{dw}{dt} - 2w = \cos\omega t \tag{2–132}$$

and

$$\frac{d^3w}{dt^3} - 3\frac{dw}{dt} - 2w = \sin\omega t. \tag{2–133}$$

The reason is twofold: (1) Since the characteristic equations for Eqs. (2–132) and (2–133) are the same as that for Eq. (2–126), the complementary function in both cases would remain the same as that given in Eq. (2–129). (2) Since $\cos\omega t = \mathrm{Re}\,(\epsilon^{j\omega t})$ and $\sin\omega t = \mathrm{Im}\,(\epsilon^{j\omega t})$,* then, by virtue of the linear property of the equations,

Particular integral for Eq. (2–132) is

$$\mathrm{Re}\left[-\frac{\epsilon^{j\omega t}}{2 + j\omega(3 + \omega^2)}\right] = -\frac{2\cos\omega t + \omega(3 + \omega^2)\sin\omega t}{4 + \omega^2(3 + \omega^2)^2}, \tag{2–134}$$

Particular integral for Eq. (2–133) is

$$\mathrm{Im}\left[-\frac{\epsilon^{j\omega t}}{2 + j\omega(3 + \omega^2)}\right] = -\frac{2\sin\omega t - \omega(3 + \omega^2)\cos\omega t}{4 + \omega^2(3 + \omega^2)^2}. \tag{2–135}$$

---

* Re $(\epsilon^{j\omega t})$ stands for "the real part of" $\epsilon^{j\omega t}$. Im $(\epsilon^{j\omega t})$ stands for "the imaginary part of" $\epsilon^{j\omega t}$.

EXAMPLE 2–9. Find the general solution of the following differential equation:

$$\frac{d^2w}{dt^2} - w = \epsilon^{-t}. \tag{2–136}$$

Solution. The given differential equation in operator form is

$$(D^2 - 1)w = \epsilon^{-t}. \tag{2–137}$$

The roots of the characteristic equation $s^2 - 1 = 0$ are

$$s_1 = +1 \qquad \text{and} \qquad s_2 = -1.$$

The complementary function is then

$$w_c = c_1\epsilon^t + c_2\epsilon^{-t}. \tag{2–138}$$

If, for the particular integral, we followed Table 2–1 without modification and assumed

$$w_p = A\epsilon^{-t}, \tag{2–139}$$

then we would find

$$(D^2 - 1)w_p = (D^2 - 1)A\epsilon^{-t} = 0,$$

because w_p in Eq. (2–139) is of the same form as the $c_2\epsilon^{-t}$ term in w_c. The expression in Eq. (2–139) then does not satisfy the given equation and cannot be a particular integral.

We should treat this case, where the excitation function contains a term which appears in the complementary function, in the same way as if a double root for the characteristic equation existed. Thus, assume

$$w_p = At\epsilon^{-t}. \tag{2–140}$$

Substituting this back into the given equation, we find that it is satisfied when

$$A = -\tfrac{1}{2}. \tag{2–141}$$

Hence

$$w_p = -\tfrac{1}{2}t\epsilon^{-t}, \tag{2–142}$$

and the general solution of Eq. (2–136) or (2–137) is

$$w = w_c + w_p = c_1\epsilon^t + (c_2 - \tfrac{1}{2}t)\epsilon^{-t}. \tag{2–143}$$

2–6 Simultaneous differential equations. In engineering we sometimes have to solve simultaneous linear differential equations with constant coefficients that contain one independent variable but several dependent variables. A typical problem of this type in electrical engineering would be the determination of loop currents in an electrical network of several interconnected loops in response to some given source of excitation. The most straightforward way of solving simultaneous equations is to reduce the given system of equations to one with a single dependent variable, which can then be solved with methods we already know. We shall illustrate the procedure with an example.

EXAMPLE 2–10. Find the general solution of the following pair of simultaneous equations:

$$\frac{dx}{dt} + x - 2y = \epsilon^{-t}, \tag{2–144a}$$

$$6x - 3\frac{dy}{dt} + 3y = -t. \tag{2–144b}$$

Solution. We shall try to solve this problem in systematic steps.

Step 1. Obtain an equation with one dependent variable only, say x. It is convenient to use operator notation and write D for d/dt.

$$(D+1)x - 2y = \epsilon^{-t} \tag{2–145a}$$

$$6x - 3(D-1)y = -t. \tag{2–145b}$$

To eliminate y and its derivatives from Eqs. (2–145), we operate on Eq. (2–145a) with the operator $3(D-1)$ and multiply Eq. (2–145b) by -2:

$$3(D^2-1)x - 6(D-1)y = 3(D-1)\epsilon^{-t} = -6\epsilon^{-t}, \tag{2–146a}$$

$$-12x + 6(D-1)y = 2t. \tag{2–146b}$$

Adding Eqs. (2–146a) and (2–146b), we obtain a second-order equation in x:

$$3(D^2-5)x = 2t - 6\epsilon^{-t},$$

or

$$(D^2-5)x = \tfrac{2}{3}t - 2\epsilon^{-t}. \tag{2–147}$$

Step 2. We next solve this equation for x. The characteristic equation for Eq. (2–147) is

$$s^2 - 5 = 0,$$

which has two distinct roots: $s_1 = +\sqrt{5}$ and $s_2 = -\sqrt{5}$. Hence the complementary function is

$$x_c = c_1\epsilon^{\sqrt{5}t} + c_2\epsilon^{-\sqrt{5}t}. \tag{2–148}$$

Next we assume, according to Table 2–1, that the particular integral of Eq. (2–147) is of the form

$$x_p = A_0 + A_1t + A_2\epsilon^{-t}. \tag{2–149}$$

Substituting x_p in Eq. (2-147) for x and carrying out the differentiations, we have

$$-5A_0 - 5A_1t - 4A_2\epsilon^{-t} = \tfrac{2}{3}t - 2\epsilon^{-t}.$$

This will hold if

$$-5A_0 = 0, \qquad -5A_1 = \tfrac{2}{3}, \qquad -4A_2 = -2,$$

or

$$A_0 = 0, \qquad A_1 = -\tfrac{2}{15}, \qquad A_2 = \tfrac{1}{2}.$$

Hence

$$x_p = -\tfrac{2}{15}t + \tfrac{1}{2}\epsilon^{-t}$$

and

$$x = x_c + x_p = c_1\epsilon^{\sqrt{5}t} + c_2\epsilon^{-\sqrt{5}t} - \tfrac{2}{15}t + \tfrac{1}{2}\epsilon^{-t}. \qquad (2\text{-}150)$$

Step 3. We now solve for y. Of course, we could solve for y from the given Eqs. (2-144) in the same way as x has been solved. If we did this, we would obtain the following equation for y:

$$(D^2 - 5)y = \tfrac{1}{3}(1 + t) + 2\epsilon^{-t}. \qquad (2\text{-}151)$$

But it is unnecessary to solve this second-order differential equation. Moreover, if we proceeded to solve Eq. (2-151) for y, two additional arbitrary constants would result which would have to be expressed in terms of the c_1 and c_2 in Eq. (2-150) through the original Eqs. (2-144). We can solve for y much more simply by noting from Eq. (2-144a) that

$$\begin{aligned} y &= \frac{1}{2}\left(\frac{dx}{dt} + x - \epsilon^{-t}\right) \\ &= \frac{c_1}{2}(\sqrt{5} + 1)\epsilon^{\sqrt{5}t} - \frac{c_2}{2}(\sqrt{5} - 1)\epsilon^{-\sqrt{5}t} - \frac{1}{15}(1 + t) - \frac{1}{2}\epsilon^{-t}. \end{aligned} \qquad (2\text{-}152)$$

Equations (2-150) and (2-152) constitute the general solution for the pair of simultaneous differential equations (2-144).

We are now ready to make several important observations from a pair of general simultaneous linear differential equations with constant coefficients. We consider two equations as follows:

$$f_1(D)x + g_1(D)y = e_1(t), \qquad (2\text{-}153a)$$

$$f_2(D)x + g_2(D)y = e_2(t), \qquad (2\text{-}153b)$$

where $f_1(D)$, $f_2(D)$, $g_1(D)$, and $g_2(D)$ are operators with constant coefficients. To obtain an equation for the dependent variable x, we operate on Eq. (2-153a) with $g_2(D)$ and on Eq. (2-153b) with $g_1(D)$, and then subtract:

$$[g_2(D)f_1(D) - g_1(D)f_2(D)]x = g_2(D)e_1(t) - g_1(D)e_2(t). \qquad (2\text{-}154)$$

Similarly, to obtain an equation for the dependent variable y, we operate on Eq. (2–153a) with $f_2(D)$ and on Eq. (2–153b) with $f_1(D)$, and then subtract:

$$[f_1(D)g_2(D) - f_2(D)g_1(D)]y = f_1(D)e_2(t) - f_2(D)e_1(t). \qquad (2\text{–}155)$$

We note that the operator in brackets operating on x in Eq. (2–154) is exactly the same as that operating on y in Eq. (2–155), both being equal to the expansion of the determinant formed by the operator coefficients in the original Eqs. (2–153a) and (2–153b):

$$\Delta(D) = \begin{vmatrix} f_1(D) & g_1(D) \\ f_2(D) & g_2(D) \end{vmatrix} \qquad (2\text{–}156)$$

By extending this development to simultaneous equations with more than two dependent variables, we are able to draw the following general conclusions about the solutions of simultaneous linear differential equations with constant coefficients:

1. The characteristic equations for all dependent variables are the same: $\Delta(s) = 0$. Consequently, the complementary functions of the solutions for all dependent variables have the same number and types of terms.

2. The *total* number of independent arbitrary constants in the general solution is the same as the order of the resulting differential equation in one dependent variable only, after all the other dependent variables and their derivatives have been eliminated. This number is equal to the highest degree of D in the expansion of the determinant $\Delta(D)$.

3. The coefficients for the terms in the complementary functions for different dependent variables are definitely related through the original equations. Hence, when the coefficients of the terms in the complementary function of any one dependent variable are chosen arbitrarily, the coefficients for the terms in the complementary functions of all other dependent variables can no longer be chosen at will.

Problems

Find the general solution of each of the following equations:

2–1. $L\frac{di}{dt} + Ri = E \sin \omega t$ (L, R, E, and ω are constants)

2–2. $x^2\frac{dy}{dx} = \epsilon^x - 2xy$

2–3. $2t^2\frac{dw}{dt} = \sin \beta t - 4tw$

2–4. $\sec \theta \frac{d\alpha}{d\theta} + \alpha = 1$

2–5. $(x + 1)\, dy = [y + (x - 1)]\, dx$

2–6. $\frac{dy}{dx} = \frac{1}{x + 3\epsilon^{2y}}$

2–7. Show that the nonlinear *Bernoulli's equation*

$$\frac{dy}{dx} + a(x)y = b(x)y^n$$

can be reduced to a linear first-order equation by the substitution $z = y^{1-n}$.

2–8. Solve the equation: $\frac{dy}{dx} + \frac{y}{x} + 3x^2y^3 = 0$

2–9. Solve the equation: $\frac{dw}{dt} + 3\frac{w}{t} = \sqrt{3w}$

2–10. Prove that $D^n(\epsilon^{mt}w) = \epsilon^{mt}(D + m)^n w$

Find the general solution of each of the following equations:

2–11. $2\frac{d^2y}{dx^2} - \frac{dy}{dx} - 10y = x^2\epsilon^{-x}$

2–12. $\frac{d^2y}{dx^2} + y = \sin x$

2–13. $\frac{d^3y}{dx^3} - 4\frac{dy}{dx} = 8$

2–14. $\frac{d^3y}{dx^3} - 8y = 3x + \epsilon^x$

2–15. $4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = \epsilon^{-x/2} + x$

2–16. $3\frac{d^2i}{dt^2} + 4\frac{di}{dt} + 2i = 20\epsilon^{-t/5} \sin (t/5)$

2–17. $\frac{d^3w}{dt^3} + 6\frac{d^2w}{dt^2} + 11\frac{dw}{dt} + 6w = t^2$

2–18. $\frac{d^3w}{dx^3} + 2\frac{d^2w}{dx^2} + \frac{dw}{dx} = 3 + \sin x$

2–19. (a) Show that a linear differential equation with variable coefficients of the following type (known as *Euler's* or *Cauchy's differential equation*):

$$a_nx^n\frac{d^ny}{dx^n} + a_{n-1}x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1x\frac{dy}{dx} + a_0y = f(x)$$

can be reduced to an equation with constant coefficients by means of the substitution $x = \epsilon^z$.

(b) Solve the equation

$$x^2 \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} + 2y = \log x$$

2–20. Find the general solution of

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = \frac{\epsilon^{-2x}}{x^2}$$

2–21. Find the general solution of

$$\frac{d^3w}{dt^3} + 3\frac{d^2w}{dt^2} + 3\frac{dw}{dt} + w = 3t\epsilon^{-t/2}$$

Find the general solutions of the following simultaneous differential equations:

2–22. $\dfrac{d^2y}{dx^2} = y - z \qquad \dfrac{d^2z}{dx^2} = z - y$

2–23. $\dfrac{d^2u}{dx^2} = 2v \qquad \dfrac{d^2v}{dx^2} = 2u$

2–24. $\dfrac{dy}{dx} + w = -1 \qquad \dfrac{dz}{dx} - 3w = 5 \qquad \dfrac{dw}{dx} + y - z = 2x - 3$

2–25. $L\dfrac{d^2i_1}{dt^2} + R\dfrac{di_1}{dt} - R\dfrac{di_2}{dt} = E\omega \cos \omega t,$

$$-R\frac{di_1}{dt} + R\frac{di_2}{dt} + \frac{1}{C}i_2 = 0,$$

where L, R, C, E, and ω are given constants, $R = \frac{1}{2}\sqrt{L/C}$, and $\omega^2 = 1/LC$.

ANSWERS TO PROBLEMS

Chapter 1

1-4. (a) $w = c_1 \cos 2t + c_2 \sin 2t$
(b) $a_0 = 4, \quad a_1 = 0$

1-6. $d^2e/dt^2 + 377^2e = 0$

1-7. (a) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = 4 \cos x$
(b) $c_1 = c_2 = \frac{1}{2}$

1-8. (a), (d), and (e) are linearly independent; (b), (c), and (f) are linearly dependent.

Chapter 2

2-1. $i = c\epsilon^{-Rt/L} + \dfrac{E}{\sqrt{R^2 + \omega^2L^2}} \sin [\omega t - \tan^{-1} (\omega L/R)]$

2-2. $y = (c + \epsilon^x)/x^2$

2-3. $w = \dfrac{1}{2t^2}\left(c - \dfrac{1}{\beta} \cos \beta t\right)$

2-4. $\alpha = c\epsilon^{-\sin \theta} + 1$

2-5. $y = (x + 1)[c + \ln (x + 1)] - 2$

2-6. $x = \epsilon^y(c + 3\epsilon^y) \qquad \text{or} \qquad y = \ln (-c \pm \sqrt{c^2 + 12x}) - 1.79$

2-7. $\dfrac{dz}{dx} + (1 - n)a(x)z = (1 - n)b(x)$

2-8. $y = \pm 1/x\sqrt{c + 6x}$

2-9. $w = (ct^{-3/2} + \sqrt{3}t/5)^2$

2-11. $y = c_1\epsilon^{5x/2} + c_2\epsilon^{-2x} - (78 - 70x + 49x^2)\epsilon^{-x}/7^3$

2-12. $y = c_1 \sin x + (c_2 - x/2) \cos x$

2-13. $y = c_1 + c_2\epsilon^{2x} + c_3\epsilon^{-2x} - 2x$

2-14. $y = c_1\epsilon^{2x} + (c_2 \cos \sqrt{3}x + c_3 \sin \sqrt{3}x)\epsilon^{-x} - 3x/8 - \epsilon^x/7$

2-15. $y = (c_1 + c_2x + x^2/8)\epsilon^{-x/2} + x - 4$

2-16. $i = (c_1 \cos \sqrt{2}t/3 + c_2 \sin \sqrt{2}t/3)\epsilon^{-2t/3}$
$\qquad + \dfrac{125}{137}(15 \sin t/5 - 7 \cos t/5)\epsilon^{-t/5}$

2-17. $w = c_1\epsilon^{-t} + c_2\epsilon^{-2t} + c_3\epsilon^{-3t} + \dfrac{1}{6}\left(\dfrac{85}{18} - \dfrac{11}{3}t + t^2\right)$

2-18. $w = c_1 + (c_2 + c_3x)\epsilon^{-x} + (3x - \frac{1}{2} \sin x)$

2-19. (b) $y = (c_1 \cos \log x + c_2 \sin \log x)/x + \frac{1}{2}(\log x - 1)$

2-20. $y = (c_1 + c_2 x - \ln x)\epsilon^{-2x}$

2-21. $w = (c_0 + c_1 t + c_2 t^2)\epsilon^{-t} + 24(t - 6)\epsilon^{-t/2}$

2-22. $y = c_1 + c_2 x + c_3 \epsilon^{\sqrt{2}x} + c_4 \epsilon^{-\sqrt{2}x}$

$z = c_1 + c_2 x - c_3 \epsilon^{\sqrt{2}x} - c_4 \epsilon^{-\sqrt{2}x}$

2-23. $u = c_1 \epsilon^{\sqrt{2}x} + c_2 \epsilon^{-\sqrt{2}x} + c_3 \cos \sqrt{2}x + c_4 \sin \sqrt{2}x$

$v = c_1 \epsilon^{\sqrt{2}x} + c_2 \epsilon^{-\sqrt{2}x} - c_3 \cos \sqrt{2}x - c_4 \sin \sqrt{2}x$

2-24. $w = c_1 \epsilon^{2x} + c_2 \epsilon^{-2x} - 2$

$$y = -\frac{c_1}{2}\epsilon^{2x} + \frac{c_2}{2}\epsilon^{-2x} + c_3 - 3 + x$$

$$z = \tfrac{3}{2}c_1 \epsilon^{2x} - \tfrac{3}{2}c_2 \epsilon^{-2x} + c_3 - x$$

2-25. $i_1 = c_0 + (c_1 + c_2 t)\epsilon^{-t/2CR} + \dfrac{E}{4R}(\sin \omega t - 4\omega CR \cos \omega t)$

$$i_2 = (4CRc_2 - c_1 - c_2 t)\epsilon^{-t/2CR} + \frac{E}{4R}\sin \omega t$$

Chapter 3

3-10. $i_1 = \dfrac{L_2}{L_1 L_2 - M^2}\displaystyle\int_0^t v_1\,dt - \dfrac{M}{L_1 L_2 - M^2}\int_0^t v_2\,dt + i_1(0)$

$$i_2 = -\frac{M}{L_1 L_2 - M^2}\int_0^t v_1\,dt + \frac{L_1}{L_1 L_2 - M^2}\int_0^t v_2\,dt + i_2(0)$$

3-11. (a) $i_m = V\sqrt{\dfrac{C}{L}}\,\epsilon^{-\alpha t_m}; \quad t_m = \dfrac{1}{b}\tanh^{-1}\left(\dfrac{b}{\alpha}\right)$

where $\alpha = \dfrac{R}{2L}$ and $b = \sqrt{\left(\dfrac{R}{2L}\right)^2 - \dfrac{1}{LC}}$

(b) $i_m = \dfrac{2V}{\epsilon R}; \quad t_m = \dfrac{2L}{R}$

(c) $i_m = V\sqrt{\dfrac{C}{L}}\,\epsilon^{-\alpha t_m}; \quad t_m = \dfrac{1}{\beta}\tan^{-1}\left(\dfrac{\beta}{\alpha}\right)$

where $\alpha = \dfrac{R}{2L}$ and $\beta = \sqrt{\dfrac{1}{LC} - \left(\dfrac{R}{2L}\right)^2}$

3-12. $i_s = V_s\sqrt{\dfrac{R_2^2 + \omega^2(L_2 + M)^2}{a^2 + b^2}}$

$$\cos\left[\omega t + \alpha + \tan^{-1}\frac{\omega a(L_2 + M) - bR_2}{aR_2 + \omega b(L_2 + M)}\right]$$

CHAPTER 4

ANALOGOUS SYSTEMS

4–1 Introduction. In the analysis of linear systems the mathematical procedure for obtaining the solutions to a given set of equations does not depend upon what physical system the equations represent. Hence if the response of one physical system to a given excitation is determined, the responses of all other systems which can be described by the same set of equations are known for the same excitation function. Systems which are governed by the same types of equations are called *analogous systems.*

Analogous systems may have entirely different physical appearances. For example, a given electrical circuit consisting of resistances, inductances, and capacitances may be analogous to a mechanical system consisting of a suitable combination of dashpots, weights, and springs; or it may be analogous to an acoustical device consisting of an appropriate arrangement of fine-mesh screens, tubes, and cavities. Dual electrical circuits that are governed by the same differential equations are a special type of analogous system. As we discussed in Section 3–6, the series R_a-L_a-C_a circuit fed by a voltage source v_a, as shown in Fig. 4–1(a), is a dual of the parallel G_b-C_b-L_b circuit fed by a current source i_b as shown in Fig. 4–1(b). The equations describing these two circuits are:

For Fig. 4–1(a),

$$L_a \frac{di_a}{dt} + R_a i_a + \frac{1}{C_a}\left[\int_0^t i_a\,dt + q_a(0)\right] = v_a. \tag{4–1}$$

For Fig. 4–1(b),

$$C_b \frac{dv_b}{dt} + G_b v_b + \frac{1}{L_b}\left[\int_0^t v_b\,dt + \phi_b(0)\right] = i_b. \tag{4–2}$$

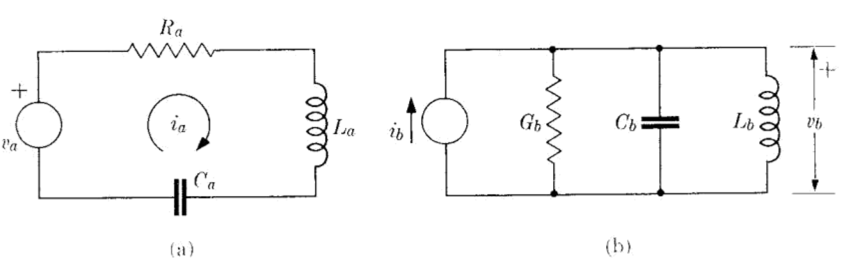

FIG. 4–1. Dual electrical circuits.

Equations (4–1) and (4–2) are identical if the following changes of symbols are made:

$$L_a \leftrightarrow C_b, \qquad R_a \leftrightarrow G_b, \qquad C_a \leftrightarrow L_b,$$

$$v_a \leftrightarrow i_b, \qquad i_a \leftrightarrow v_b, \qquad q_a \leftrightarrow \phi_b.$$

Mathematically, dual circuits are governed by the same types of equations. Physically, an electrical circuit can be changed to its dual by making appropriate conversions in accordance with Table 3–2 or by using the dot method described in Section 3–6.

When we deal with systems other than electrical, there are distinct advantages if we can reduce the systems under consideration to their analogous electrical circuits. First of all, electrical engineers have developed a set of convenient symbols for circuit elements that permits a complex system to be set down in the form of a circuit diagram from which the behavior of the system can be readily analyzed. Once the circuit diagram of the analogous electrical system is determined, it is possible to visualize and even predict system behaviors (resonances, passbands, damping coefficients, time constants, etc.) by inspection. Second, electrical circuit-theory techniques, such as the use of the impedance concept and the various network theorems, can be applied in the actual analysis of the system. Third, the ease of changing the values of electrical components, of connecting and disconnecting them in a circuit, and of measuring the voltages and currents all prove invaluable in model construction and testing.

In this chapter, we shall establish the analogy between linear mechanical and electrical systems, the ultimate objective being the ability to draw analogous electrical circuits (and solve as such) for given mechanical systems by inspection. This technique becomes even more valuable when we deal with electromechanical systems, in which electrical and mechanical phenomena are interrelated. Analogies can be extended beyond electrical and mechanical systems to acoustical, thermal, and even economic systems, but these require thorough knowledge of the parameters and system relationships in the respective fields and will not be treated here.

4–2 Linear mechanical elements. Three fundamental passive elements, or system parameters, will come into play in linear mechanical systems; they correspond to the coefficients in the expressions of three types of mechanical forces that resist motion. We must define the parameters separately for translational and for rotational systems.

A. *Translational systems.* There are three types of forces that resist motion:

1. Inertia force: Newton's second law of motion states that the inertia force is equal to mass times acceleration:

$$f_M = Ma = M\,\frac{du}{dt} = M\,\frac{d^2x}{dt^2}\,, \tag{4–3}$$

where a denotes the acceleration, u the velocity, and x the displacement. The *mass* M of a body is thus the coefficient in the force equation and is the inertia force per unit acceleration. Symbolically, we represent it by a block, as shown in Fig. 4–2(a).

2. Damping force: In linear systems we assume the damping force to be proportional to the velocity. However, this is true only in the case of viscous friction; it is, in general, not a good approximation for dry friction. Coils moving in a uniform magnetic field experience a damping force which is proportional to the velocity. We have, for linear damping,

$$f_D = Du = D\,\frac{dx}{dt}\,, \tag{4–4}$$

where D, the *damping coefficient*, is the damping force per unit velocity. Symbolically, we represent D by a dashpot, as shown in Fig. 4–2(b), signifying viscous damping between the piston and the cylinder.

3. Spring force: The restoring force of a spring is proportional to the displacement (amount of stretch or compression):

$$f_K = \frac{1}{K}\,x = \frac{1}{K}\int u\,dt = \frac{1}{K}\left[\int_0^t u\,dt + x(0)\right], \tag{4–5}$$

where K, the *compliance* of the spring, is the reciprocal of its *stiffness*. The stiffness of a spring is the restoring force per unit displacement. We represent the element by a coil spring, as shown in Fig. 4–2(c).

Note that Eqs. (4–3), (4–4), and (4–5) will hold in any self-consistent system of units.

In mechanical systems, it is convenient to define two kinds of ideal sources (active elements), corresponding to the ideal voltage and current

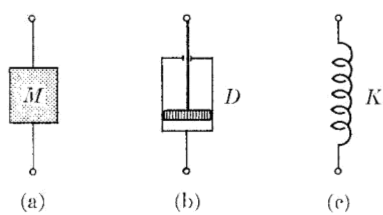

FIG. 4–2. Passive mechanical elements for translational motion.

generators in electrical systems. When the driving force $f(t)$ acting in a system is known, we can think of the system as being under the influence of a *force source*. The force developed by a force source is assumed to be independent of what is connected to it. When the driving velocity $u(t)$ at some point in a mechanical system is known and is controlled by external means, we can think of the system as being under the influence of a *velocity source*. The velocity developed by a velocity source is likewise assumed to be independent of what is connected to it.

B. *Rotational systems*. Corresponding to the three types of forces resisting translational motion, there are three types of torques resisting rotational motion. They are:

1. Inertia torque: The inertia torque, τ_I, is equal to the *moment of inertia* I_θ times the angular acceleration α:*

$$\tau_I = I_\theta \alpha = I_\theta \frac{d\Omega}{dt} = I_\theta \frac{d^2\theta}{dt^2}, \tag{4-6}$$

where Ω denotes the angular velocity and θ the angular displacement.

2. Damping torque: The damping torque, τ_D, is equal to the *rotational damping coefficient* D_θ times the angular velocity Ω in a linear system:

$$\tau_D = D_\theta \Omega = D_\theta \frac{d\theta}{dt} \cdot \tag{4-7}$$

3. Spring torque: The restoring torque, τ_K, of a spring is equal to the angular displacement θ divided by the *torsional compliance* K_θ. The reciprocal of K_θ is the *torsional stiffness* of the spring.

$$\tau_K = \frac{1}{K_\theta}\,\theta = \frac{1}{K_\theta}\int \Omega\, dt = \frac{1}{K_\theta}\left[\int_0^t \Omega\, dt + \theta(0)\right] \cdot \tag{4-8}$$

Comparing Eqs. (4-6), (4-7), and (4-8) for rotational motion to Eqs. (4-3), (4-4), and (4-5), respectively, for translational motion, we see that these two systems are entirely similar mathematically. Here then is another example of analogous systems.

Table 4-1 tabulates the analogous quantities in translational and rotational mechanical systems.

Inasmuch as all the characteristics of a rotational system can be discussed in terms of the analogous translational system, no attempt will be made to assign special symbols to rotational mechanical elements as we have done for the translational system in Fig. 4-2.

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* In rotational motion it is necessary to refer all quantities (torque, angular displacement, angular velocity, angular acceleration, moment of inertia, etc.) to an axis of rotation.

TABLE 4-1

ANALOGOUS QUANTITIES IN TRANSLATIONAL AND ROTATIONAL MECHANICAL SYSTEMS

Translational	Rotational
Force, f	Torque, τ
Acceleration, a	Angular acceleration, α
Velocity, u	Angular velocity, Ω
Displacement, x	Angular displacement, θ
Mass, M	Moment of inertia, I_θ
Damping coefficient, D	Rotational damping coefficient, D_θ
Compliance, K	Torsional compliance, K_θ

It should be realized that expressions such as "force on" and "velocity of" a mechanical element, though conventionally used, are loose expressions. This becomes clear when we recall the element symbols in Fig. 4-2, where all three elements are represented as two-terminal devices. Which terminal do we refer to when we talk about the "force on" and the "velocity of" an element? In order to be more specific, let us consider the dashpot in Fig. 4-2(b). Obviously, the two terminals (the piston and the cylinder) of the dashpot can move with different velocities. In fact, the cylinder may remain stationary while the piston is moving. Furthermore, there would be no damping force if the piston and the cylinder moved with the same velocity. Hence the significant quantity is the *relative* velocity (velocity difference) of the two terminals. In other words, element relationships hold for velocity *across* a mechanical element. Similarly, relative to mechanical elements, we must imply displacement *across* and acceleration *across* the terminals. Motion of a rigid body (mass) refers to the stationary ground. The force, on the other hand, acts *through* mechanical elements. The same comments apply to torque *through* and angular displacement, velocity, or acceleration *across* mechanical elements in rotational motion.

4-3 D'Alembert's principle. D'Alembert's principle applies the conditions of static equilibrium to problems in dynamics by considering both the externally applied driving forces and the reaction forces of mechanical elements which resist motion. It is actually a slightly modified form of Newton's second law of motion, and can be stated as follows:

For any body, the algebraic sum of externally applied forces and the forces resisting motion in any given direction is zero.

D'Alembert's principle applies for all instants of time. A positive reference direction must first be chosen. Forces acting in the reference direction are then considered as positive and those against the reference direction as negative. D'Alembert's principle is as useful in writing the equations of motion for a mechanical system as Kirchhoff's laws are in writing the circuit equations for an electrical network. Let us apply it to the system in Fig. 4–3.

In Fig. 4–3, the mass M is attached to a fixed wall through a spring with compliance K. It is assumed that the contact between the mass and the floor offers viscous damping with damping coefficient D. (The problem would be exactly the same if we assumed that the floor was frictionless but that there existed a dashpot with damping coefficient D, in addition to the spring, between the mass and the fixed wall.) In the direction toward the right:

External force: f

Resisting forces: (1) inertia force, $f_M = -M\dfrac{du}{dt}$,

(2) damping force, $f_D = -Du$,

(3) spring force, $f_K = -\dfrac{1}{K}\left[\int_0^t u\,dt + x(0)\right]$.

By d'Alembert's principle, we then have

$$f + f_M + f_D + f_K = 0, \tag{4-9}$$

or

$$M\frac{du}{dt} + Du + \frac{1}{K}\left[\int_0^t u\,dt + x(0)\right] = f, \tag{4-10}$$

which is the equation of motion for the system in Fig. 4–3.

A rotational system analogous to the translational system of Fig. 4–3 is shown in Fig. 4–4. Here a flywheel with moment of inertia I_θ is supported

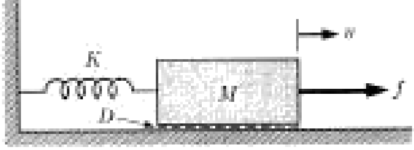

FIG. 4–3. A translational mechanical system.

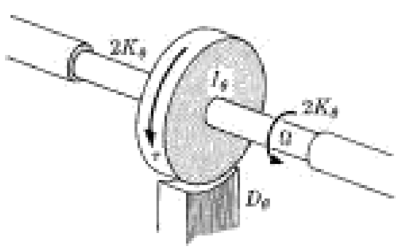

FIG. 4–4. A rotational mechanical system which is analogous to the translational system in Fig. 4–3.

on a shaft with torsional compliance K_θ (each half having a torsional compliance $2K_\theta$). The shaft is securely clamped on both ends. The brake below the flywheel is to symbolize a linear (viscous) damping with rotational damping coefficient D_θ. D'Alembert's principle modified for a rotational system can be stated as follows:

For any body, the algebraic sum of externally applied torques and the torques resisting rotation about any axis is zero.

For Fig. 4-4:

External torque: τ

Resisting torques: (1) inertia torque, $\tau_I = -I_\theta \dfrac{d\Omega}{dt}$,

(2) damping torque, $\tau_D = -D_\theta\Omega$,

(3) spring torque, $\tau_K = -\dfrac{1}{K_\theta}\left[\int_0^t \Omega\, dt + \theta(0)\right]$.

Hence

$$\tau + \tau_I + \tau_D + \tau_K = 0, \tag{4-11}$$

or

$$I_\theta \frac{d\Omega}{dt} + D_\theta\Omega + \frac{1}{K_\theta}\left[\int_0^t \Omega\, dt + \theta(0)\right] = \tau, \tag{4-12}$$

which is, of course, entirely analogous to Eq. (4-10) for the translational case.

4-4 Force-voltage analogy. Comparison of Eq. (4-10) with Eq. (4-1) immediately reveals their close similarity. They therefore represent analogous systems. In other words, the behavior of the mechanical system of Fig. 4-3 can be completely predicted by what we know about the simple series *R-L-C* electrical circuit in Fig. 4-1(a) by making appropriate conversions of physical quantities, as listed in Table 4-2. Since force, f, in the mechanical system is set to be analogous to voltage, v, in the electrical system, we designate this type of analogy as the *force-voltage* (f-v) *analogy*.*

The usefulness of the electromechanical analogy lies in our ability to draw the analogous electrical circuit directly from a given mechanical system (without first writing the mechanical equations of motion) and to solve the problem entirely as an electrical one, with known techniques. For the simple system in Fig. 4-3, it is not too difficult to see that the circuit in Fig. 4-1(a) is its force-voltage electrical analog. However, with

* It is also referred to in the literature as the *direct analogy*, or the *impedance-type analogy*.

TABLE 4–2

TABLE OF CONVERSION FOR FORCE-VOLTAGE ANALOGY

Mechanical system	Electrical system (f-v analogy)
Force, f	Voltage, v
Velocity, u	Current, i
Displacement, x	Charge, q
Mass, M	Inductance, L
Damping coefficient, D	Resistance, R
Compliance, K	Capacitance, C

a complex system, it is not so easy to visualize. The following rule for drawing f-v analogous electrical circuits from mechanical systems will prove useful:

Each junction in the mechanical system corresponds to a closed loop which consists of electrical excitation sources and passive elements analogous to the mechanical driving sources and passive elements connected to the junction. All points on a rigid mass are considered as the same junction.

For the system in Fig. 4–3, since all points on the mass are considered as the same junction, there is only one junction, to which a driving force f and three passive mechanical elements, D, M, and K, are connected. This converts into one closed loop consisting of a voltage excitation source, v, and three passive electrical circuit elements, R, L, and C, as shown in Fig. 4–1(a). Let us now consider a more complex system.

EXAMPLE 4–1. Find the equations that describe the motion of the mechanical system of Fig. 4–5.

Solution. We shall solve this problem in two different ways. In part (a) the normal approach based upon d'Alembert's principle is used; in part (b) we make use of analogy.

(a) Using d'Alembert's principle.

It is clear that the system in Fig. 4–5 is a two-coordinate system, i.e., two variables, x_1 and x_2 as shown (measured from the equilibrium position before the external force is applied), are needed to describe the system completely. First consider the forces on mass M_1:

External force: f

Resisting forces:

(1) inertia force, $$f_{M_1} = -M_1\frac{d^2x_1}{dt^2} = -M_1\frac{du_1}{dt},$$

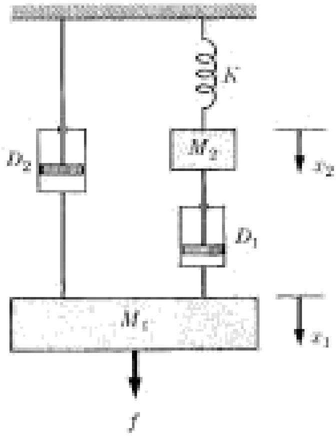

FIG. 4-5. A two-coordinate mechanical system.

(2) damping forces, $f_{D_1} = -D_1 \dfrac{d}{dt}(x_1 - x_2) = -D_1(u_1 - u_2)$,

$$f_{D_2} = -D_2 \frac{dx_1}{dt} = -D_2 u_1.$$

Application of d'Alembert's principle yields

$$M_1 \frac{du_1}{dt} + (D_1 + D_2)u_1 - D_1 u_2 = f. \tag{4-13}$$

Next consider the forces on mass M_2:

External force: 0

Resisting forces:

(1) inertia force, $f_{M_2} = -M_2 \dfrac{d^2x_2}{dt^2} = -M_2 \dfrac{du_2}{dt}$,

(2) damping force, $f_{D_1} = -D_1 \dfrac{d}{dt}(x_2 - x_1) = -D_1(u_2 - u_1)$,

(3) spring force, $f_K = -\dfrac{1}{K} x_2 = -\dfrac{1}{K}\left[\int_0^t u_2\,dt + x_2(0)\right]\cdot$

Hence

$$-D_1 u_1 + M_2 \frac{du_2}{dt} + D_1 u_2 + \frac{1}{K}\left[\int_0^t u_2\,dt + x_2(0)\right] = 0. \tag{4-14}$$

Equations (4-13) and (4-14) completely describe the motion of the system.

(b) Using f-v analogy.

Corresponding to the two coordinates x_1 and x_2, the mechanical system has two junctions. Hence, we will have two loops in the f-v analogous electrical circuit. The first loop consists of a voltage source $v\,[f]$, an inductance $L_1\,[M_1]$, and two resistances $R_1\,[D_1]$ and $R_2\,[D_2]$, and the second loop consists of an inductance $L_2\,[M_2]$, a capacitance $C\,[K]$, and a resistance $R_1\,[D_1]$, the last element being common to both loops. The analogous electrical circuit is shown in Fig. 4–6.

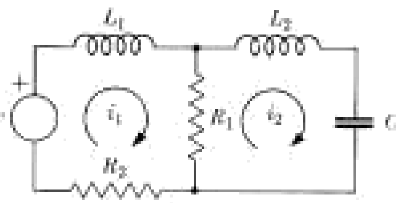

FIG. 4–6. An electrical circuit analogous to that in Fig. 4–5, based upon f-v analogy.

It is now a simple matter to write the equations for the two-loop circuit of Fig. 4–6:

$$L_1 \frac{di_1}{dt} + (R_1 + R_2)i_1 - R_1 i_2 = v, \tag{4–15}$$

$$-R_1 i_1 + L_2 \frac{di_2}{dt} + R_1 i_2 + \frac{1}{C}\left[\int_0^t i_2\,dt + q_2(0)\right] = 0. \tag{4–16}$$

Equations (4–15) and (4–16) are identical with Eqs. (4–13) and (4–14) except for the conversion of the parameters in accordance with Table 4–2, and the response of the given mechanical system can be determined by examining the behavior of the analogous electrical circuit. Note that Fig. 4–6 can be drawn directly from Fig. 4–5 by inspection; there is no need to write Eqs. (4–13) and (4–14) at all.

4–5 Force-current analogy. The force-voltage analogy described in the preceding section is based on the mathematical similarity between Eqs. (4–10) and (4–1). From the point of view of physical interpretation, it is not a natural analogy, because *forces* acting *through* mechanical elements are made to be analogous to *voltages across* the corresponding electrical elements, and *velocities across* (velocity differences between the terminals of) mechanical elements are made to be analogous to *currents through* the corresponding electrical elements. A direct consequence is that a *junction* in the mechanical system goes over to the analogous electrical circuit as a *loop.* Realizing these inherent imperfections, F. A. Firestone* advocated an analogy of the mobility type, which we shall call the force-current (f-i) analogy,† and which will be the subject matter of this section.

We arrived at the force-voltage analogy by noticing the similarity between Eq. (4–10) for the mechanical system in Fig. 4–3 and Eq. (4–1) for the electrical circuit in Fig. 4–1(a). Equation (4–10), of course, is also similar to Eq. (4–2) describing the circuit in Fig. 4–1(b), which is therefore also analogous to the mechanical system in Fig. 4–3. The corresponding electrical and mechanical quantities for this new analogy are listed in Table 4–3.

TABLE 4–3

TABLE OF CONVERSION FOR FORCE-CURRENT ANALOGY

Mechanical system	Electrical system (f-i analogy)
Force, f	Current, i
Velocity, u	Voltage, v
Displacement, x	Flux linkage, ϕ
Mass, M	Capacitance, C
Damping coefficient, D	Conductance, G
Compliance, K	Inductance, L

Since force, f, in the mechanical system is now set to be analogous to current, i, in the electrical system, we designate this type of analogy as the *force-current* (f-i) *analogy.* From the physical point of view, this analogy is more satisfactory than the force-voltage analogy, since force *through* is now analogous to current *through* and velocity *across* is now analogous to voltage *across.* Consequently, a *junction* in the mechanical system goes over to the electrical system as a *node* (junction), instead of as a loop. Furthermore, in mechanical devices we can measure velocity (or displacement) with a vibration pickup without disturbing the machine, just as we can measure voltage in electrical circuits with a voltmeter without disturbing the circuit; force and current, however, cannot be measured unless we break into the system.

The rule for drawing f-i analogous electrical circuits from mechanical systems can be stated as follows:

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* F. A. Firestone, "The Mobility Method of Computing the Vibration of Linear Mechanical and Acoustical Systems: Mechanical-Electrical Analogies," *Journal of Applied Physics,* 9, pp. 373–387; June 1938.

† It is also referred to as the *inverse analogy,* although it is in reality more direct than the force-voltage analogy.

Each junction in the mechanical system corresponds to a node (junction) which joins electrical excitation sources and passive elements analogous to the mechanical driving sources and passive elements connected to the junction. All points on a rigid mass are considered as the same junction and one terminal of the capacitance analogous to a mass is always connected to the ground.

The reason that one terminal of the capacitance analogous to a mass is always connected to the ground is that the velocity (or displacement, or acceleration) of a mass is always referred to the earth. Recognition of this fact greatly simplifies the drawing of analogous electrical circuits. Two or more masses rigidly connected go over to the analogous electrical circuit as two or more capacitances connected between the same node and the ground.

The mechanical system in Fig. 4–3 has only one junction, to which a driving force f and three passive mechanical elements, D, M, and K, are connected. This converts by f-i analogy into a single node consisting of a current source i and three passive electrical elements G, C, and L all connected to the ground (reference or datum node), as shown in Fig. 4–1(b).* It is quite clear that *the electrical circuits drawn from the f-v and f-i analogies are duals of each other*.

EXAMPLE 4–2. Draw the electrical circuit analogous to the mechanical system of Fig. 4–5, using force-current analogy.

Solution. Corresponding to the two coordinates x_1 and x_2 for the mechanical system, we will have two independent nodes in the f-i analogous electrical circuit. The first node joins a current source i $[f]$, a capacitance C_1 $[M_1]$ and two conductances G_1 $[D_1]$ and G_2 $[D_2]$; the second node joins a capacitance C_2 $[M_2]$, an inductance L $[K]$ and a conductance G_1 $[D_1]$, the last element being common to both nodes. This is shown in Fig. 4–7, which is clearly the dual of the circuit in Fig. 4–6 and can be derived from the latter by the dot method described in Section 3–6.

By applying Kirchhoff's current law, we can readily write the node equations from Fig. 4–7:

$$C_1 \frac{dv_1}{dt} + (G_1 + G_2)v_1 - G_1 v_2 = i, \tag{4–17}$$

$$-G_1 v_1 + C_2 \frac{dv_2}{dt} + G_1 v_2 + \frac{1}{L}\left[\int_0^t v_2\,dt + \phi_2(0)\right] = 0. \tag{4–18}$$

* Of course, in the actual electrical circuit, there is no difference between an element with a conductance G and an element with a resistance $R = 1/G$.

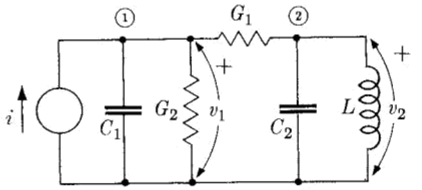


FIG. 4-7. An electrical circuit analogous to that in Fig. 4-5, based upon f-i analogy.

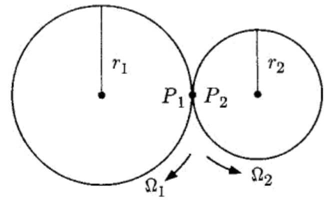


FIG. 4-8. A pair of nonslipping friction wheels.

Equations (4-17) and (4-18) can be converted to the mechanical equations of motion (4-13) and (4-14) by using the conversions in Table 4-3. However, this conversion of parameters does not have to be done until after the desired response has been determined. As a matter of fact, the circuit elements in Fig. 4-7 could have been labeled with their analogous mechanical quantities; then no conversion of parameters in the solution would be necessary.

What we have said about translational mechanical systems would obviously apply equally well to analogous rotational mechanical systems except for changes in terminology, in accord with Table 4-1.

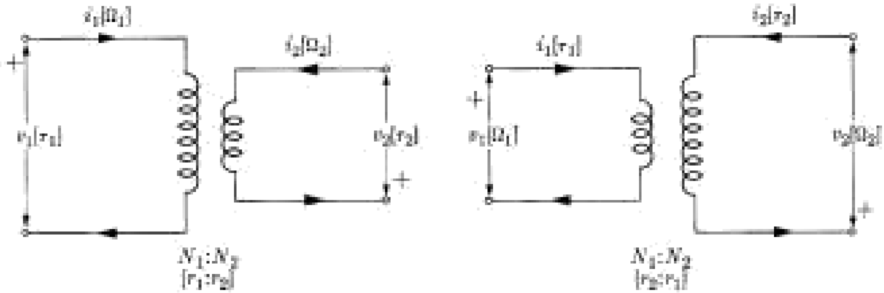
4-6 Mechanical coupling devices. Common mechanical coupling devices, such as gears, friction wheels, and levers, also have electrical analogs. Let us first consider the pair of nonslipping friction wheels shown in Fig. 4-8. (In function and in mathematical description, there is no difference between a pair of nonslipping friction wheels and a pair of positively engaged gears.)

At the point of contact, P_1 on wheel 1 and P_2 on wheel 2 must have the same linear velocity because they move together, and experience equal and opposite forces (action and reaction). Since this is a rotational system, it is convenient to use angular velocities and torques. The following relations between magnitudes hold:

$$\frac{\tau_1}{\tau_2} = \frac{r_1}{r_2}, \tag{4-19}$$

$$\frac{\Omega_1}{\Omega_2} = \frac{r_2}{r_1}. \tag{4-20}$$

Equations (4-19) and (4-20) remind us immediately of the relations that exist between the voltages and the currents in the primary and secondary windings of an ideal transformer. If $r_1:r_2$ is considered as the turns ratio $N_1:N_2$ of an ideal transformer, torque will then be analogous to voltage and angular velocity to current. This forms the basis of the f-v analogy in Fig. 4-9(a). If $r_2:r_1$ is considered as the turns ratio $N_1:N_2$, we have



(a) (b)

FIG. 4–9. Ideal transformer as electrical analog of a pair of friction wheels or meshed gears. (a) f-v analogy. (b) f-i analogy.

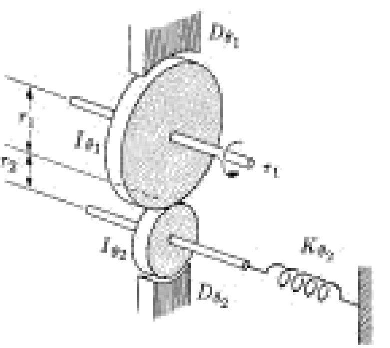


FIG. 4–10. A mechanical system with friction-wheel coupling.

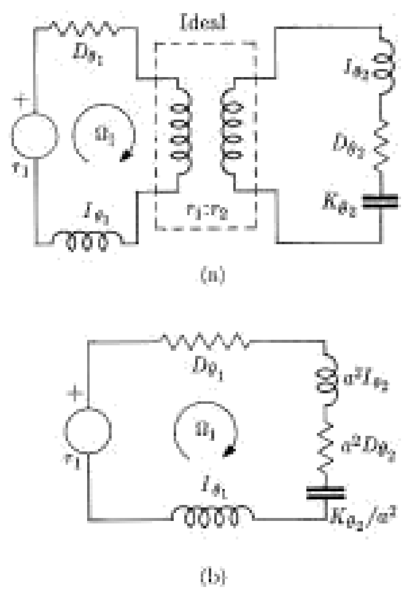


FIG. 4–11. Analogous electrical circuits for the system in Fig. 4–10 (on τ-v basis). (a) Circuit containing ideal transformer. (b) Circuit with ideal transformer removed, $a = r_1/r_2$.

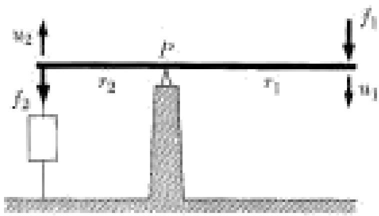


FIG. 4–12. A simple lever supported at rigid fulcrum P.

the analogous ideal transformer based on the f-i analogy in Fig. 4-9(b). The corresponding mechanical quantities are shown in brackets. The reversal of current directions and voltage polarities in the secondaries of these two figures is to show the reversal of the directions of both torque and angular velocity due to coupling; it is equivalent to putting dots on opposite ends of the primary and secondary windings of the transformer. In general, it is easy to determine the relative directions of motion of the coupled mechanical elements by inspection of the mechanical system, without elaborate notations in the electrical circuit.

In applying the electrical analogy in either Fig. 4-9(a) or Fig. 4-9(b) to mechanical systems containing friction wheels or gears, we must remember that the transformer involved is an ideal one. The voltage and current relationships in an ideal transformer are fixed by the turns ratio. Primary, secondary, and mutual inductances do not enter into the equations; in fact, in an ideal transformer they should all be infinitely large. The transformer in the analogous electrical circuit merely serves to properly transform the quantities in the secondary circuit to the primary circuit, and vice versa.

EXAMPLE 4-3. In the system shown in Fig. 4-10, a sinusoidally varying torque $\tau_1 = T_0 \sin \omega t$ is applied to wheel 1, which is engaged with wheel 2. Assume the shafts of both wheels to be inertialess and supported on frictionless bearings. Determine the steady-state angular velocity of wheel 1.

Solution. Let us use τ-v analogy in this example. The analogous electrical circuit is given in Fig. 4-11(a), where all electrical quantities have been written in terms of their analogous mechanical quantities. The ideal transformer can be removed when all the elements in the secondary circuit have been transformed to the primary circuit as shown in Fig. 4-11(b). Since the steady-state angular velocity Ω_1 is wanted, a simple application of the impedance concept yields the desired result:

$$\Omega_1 = \frac{T_0}{\sqrt{(D_{\theta_1} + a^2 D_{\theta_2})^2 + [\omega(I_{\theta_1} + a^2 I_{\theta_2}) - (a^2/\omega K_{\theta_2})]^2}} \sin(\omega t - \Psi),$$

where

$$\Psi = \tan^{-1}\left[\frac{\omega(I_{\theta_1} + a^2 I_{\theta_2}) - (a^2/\omega K_{\theta_2})}{D_{\theta_1} + a^2 D_{\theta_2}}\right].$$

The simple lever is another type of mechanical coupling device that is analogous to a transformer.* Consider the lever in Fig. 4-12, which rests

* L. L. Beranek, *Acoustics*, McGraw-Hill Book Company, New York, Chapter 3; 1954.

on a rigid fulcrum P. The lever is assumed to be massless but rigid, and its left end is connected to the ground through some mechanical element which resists motion. If a force f_1 applied to the right end makes it move with a velocity u_1, the following relations hold:

$$\frac{u_1}{u_2} = \frac{r_1}{r_2}, \tag{4-21}$$

$$\frac{f_1}{f_2} = \frac{r_2}{r_1}. \tag{4-22}$$

The similarity between these two equations and Eqs. (4–19) and (4–20) is obvious. We find that the velocities of the ends of the simple lever correspond to the torques on the gears, and the forces on the lever correspond to the angular velocities of the gears. Therefore, although the electrical analog of a simple lever is also an ideal transformer, the f-v analog for a lever will correspond to the f-i analog of a pair of meshed gears, and vice versa. This is shown in Figs. 4–13(a) and 4–13(b). The relative directions of motion of the two ends can be more readily determined from Fig. 4–12.

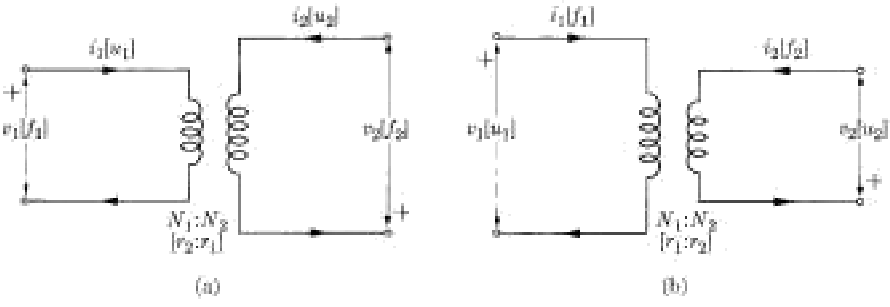


FIG. 4–13. Ideal transformer as electrical analog of a simple lever. (a) f-v analogy. (b) f-i analogy.

EXAMPLE 4–4. Find the f-i analogous electrical circuit of the mechanical system shown in Fig. 4–14. Assume that the bar is rigid but massless, and that the junctions are restricted to have vertical motion only.

Solution. Since this system has a lever-type coupling, the existence of an ideal transformer in the electrical analog is apparent. However, we do not have a simple lever here because the bar does not rest or pivot on a fixed fulcrum. The primary circuit in the electrical analog can be drawn without difficulty. By the rule of f-i analogy we know that the primary circuit has one independent node with a current source (f) and three elements (a capacitance M_1, a resistance $1/D_1$, and an inductance K_1) connected to it.

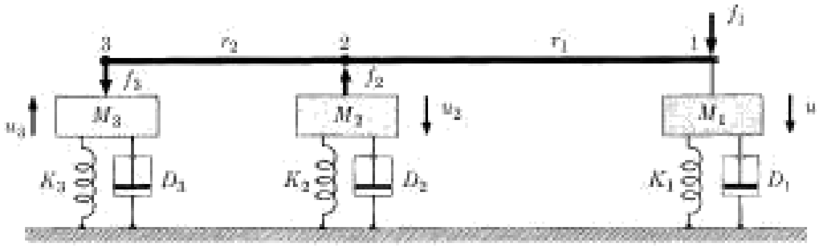


FIG. 4-14. A mechanical system with lever-type coupling.

To draw the secondary circuit, we apply the principle of superposition. First, consider junction 3 as fixed. We then have

$$\frac{u_1}{u_2} = \frac{r_1 + r_2}{r_2}, \tag{4-23}$$

$$\frac{f_1}{f_2} = \frac{r_2}{r_1 + r_2}. \tag{4-24}$$

Equations (4-23) and (4-24) indicate a primary to secondary turns ratio of $N_1 : N_2 = (r_1 + r_2):r_2$ in the f-i analogy. Next, considering junction 2 as fixed, we have

$$\frac{u_1}{u_3} = \frac{r_1}{r_2}, \tag{4-25}$$

$$\frac{f_1}{f_3} = \frac{r_2}{r_1}, \tag{4-26}$$

Equations (4-25) and (4-26) require a primary to secondary turns ratio of $N_1:N_3 = r_1:r_2$. In each of the two secondaries, three passive elements

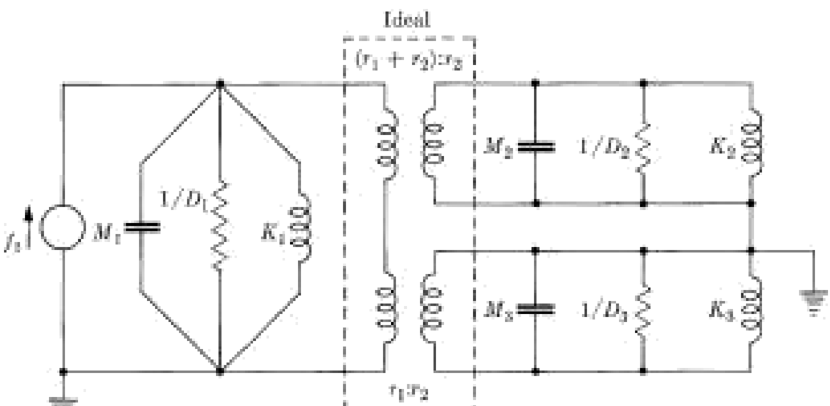


FIG. 4-15. Analogous electrical circuit for system in Fig. 4-14 (on f-i basis). Values beside resistive elements are resistances, not conductances.

(a mass, a spring, and a dashpot) meet at a common junction. This results in the f-i analogous electrical circuit of Fig. 4–15, in which all electrical quantities have been written in terms of their analogous mechanical quantities.

4–7 Electromechanical systems. Systems in which electrical and mechanical elements occur in combination and interact are called *electromechanical* systems. They are quite often referred to as *electromechanical transducers*, which convert electrical energy into mechanical energy or vice versa. Typical examples include microphones, loudspeakers, vibration pickups, and electrical machineries. We shall discuss in this section how an all-electrical analogous circuit can be drawn for an electromechanical transducer.

The basic relations can be derived from the two simple situations shown in Figs. 4–16(a) and 4–16(b), where a conductor of length l lies perpendicularly in a uniform magnetic field with flux density B. In Fig. 4–16(a), a current i through the conductor produces an upward force f on the conductor (left-hand rule):

$$f = Bli, \tag{4–27}$$

which can also be written as

$$i = \left(\frac{1}{Bl}\right) f. \tag{4–28}$$

In Fig. 4–16(b), the conductor moving with an upward velocity u will have induced in it an open-circuit voltage v with polarities as indicated (right-hand rule):

$$v = (Bl)u. \tag{4–29}$$

Equations (4–28) and (4–29) demonstrate the properties of an ideal transformer with turns ratio Bl:1; one side of the "transformer" carries electrical quantities v and i, while the other side carries mechanical quantities u and f. This is shown in Fig. 4–17. Since f and u on the mechanical side

FIG. 4–16. A conductor situated in a uniform magnetic field. (a) Current i in conductor produces force f. (b) Motion of conductor with velocity u induces voltage v.

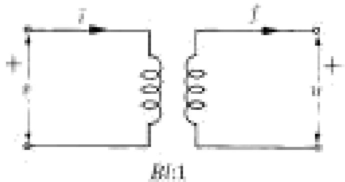


FIG. 4-17. Ideal transformer analogous to the simple electromechanical situation in Fig. 4-16.

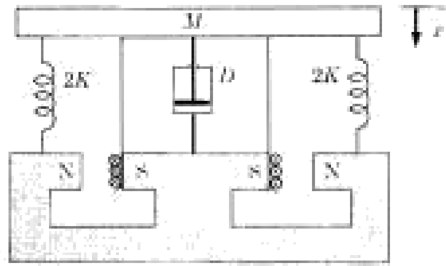


FIG. 4-18. An electromechanical system.

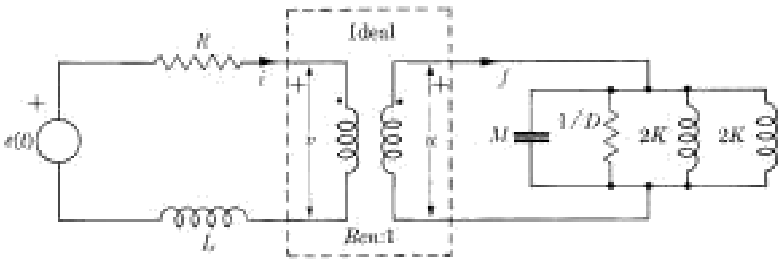


FIG. 4-19. Analogous circuit for the electromechanical system in Fig. 4-18 ($1/D$ is value of the analogous resistance).

are analogous to i and v respectively, this analogy is of the f-i type. We note that Eqs. (4-28) and (4-29) hold for all frequencies, including zero [d-c current in Fig. 4-16(a) and constant velocity in Fig. 4-16(b)]; hence we must also assume that the ideal "transformer" in Fig. 4-17 is operative at all frequencies, including zero.

EXAMPLE 4-5. The electromechanical system shown in Fig. 4-18 may represent a loudspeaker or an electromagnetic relay. The moving coil in the uniform magnetic field B has n turns of circumference c and its inductance and resistance are L and R respectively. A voltage $e(t)$ is applied to the coil. Determine the equation of motion of the mass M.

Solution. In this system we have electrical elements L and R of the coil, and mechanical elements M, D, $2K$, and $2K$. The coupling is provided by the moving coil in the magnetic field. Making use of the analogous ideal transformer in Fig. 4-17 and converting the mechanical elements into their electrical analogs on the f-i basis, we obtain the circuit of Fig. 4-19. The governing equations for the primary and the secondary are

$$L\frac{di}{dt} + Ri + v = e(t), \tag{4-30}$$

$$M\frac{du}{dt} + Du + \frac{1}{K}\left[\int_0^t u\,dt + x(0)\right] = f. \tag{4-31}$$

Kirchhoff's voltage law has been used for the primary circuit, and current law for the secondary circuit. Now we have

$$v = (Ben)u \tag{4-32}$$

and

$$f = (Ben)i. \tag{4-33}$$

Hence Eqs. (4-30) and (4-31) become

$$L\frac{di}{dt} + Ri + (Ben)u = e(t) \tag{4-34}$$

and

$$M\frac{du}{dt} + Du + \frac{1}{K}\left[\int_0^t u\,dt + x(0)\right] = (Ben)i. \tag{4-35}$$

Equations (4-34) and (4-35) are simultaneous integro-differential equations in two unknowns, i and u. The correctness of the signs of the coupling terms $(Ben)u$ and $(Ben)i$ can be checked by noting that a positive u (downward motion) would induce an increased voltage drop in the electrical circuit and that current i in the indicated direction tends to pull M down (positive u).

If it is desired to find the current i in the coil, we can refer all elements to the primary circuit and remove the transformer, as shown in Fig. 4-20(a). On the other hand, if the motion (x, or u) of the mass M is to be determined, then it is more convenient to refer all quantities to the secondary, as in Fig. 4-20(b).

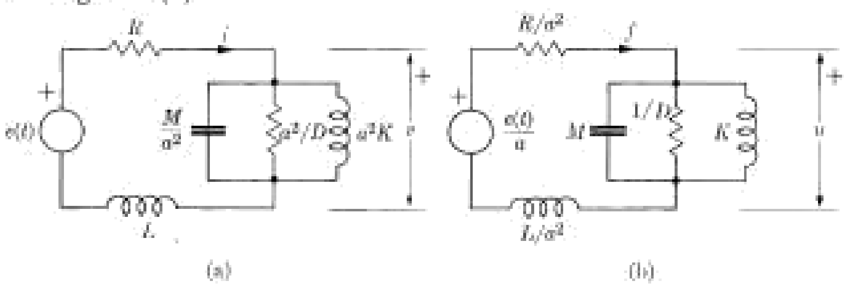


FIG. 4-20. Equivalent circuits for Fig. 4-19, $a = Ben$. (a) Referred to the primary. (b) Referred to the secondary.

Problems

For each of the mechanical systems given in problems 4–1 through 4–10, (a) draw the f-v (τ-v if system is rotational) analogous electrical circuit, (b) draw the f-i (τ-i if system is rotational) analogous electrical circuit, and (c) write the equations of motion in terms of the given mechanical quantities.

4–1.

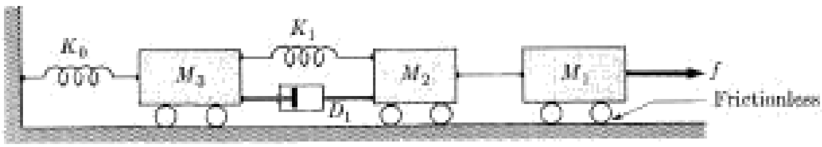


Figure 4–21

4–2.

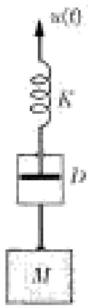


Figure 4–22

4–3.

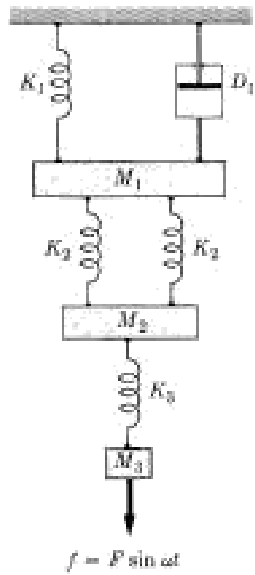


Figure 4–23

4–4.

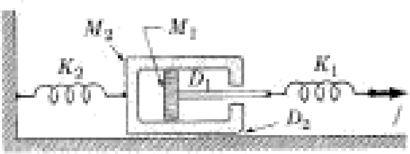


Figure 4–24

4–5.

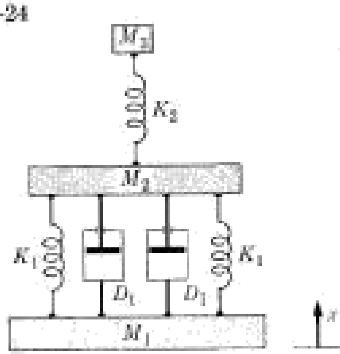


Figure 4–25

4-6.

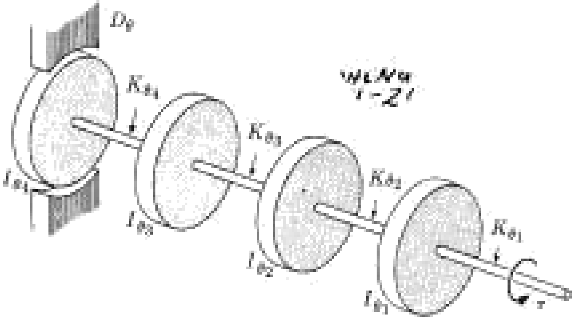


FIGURE 4-26

4-7.

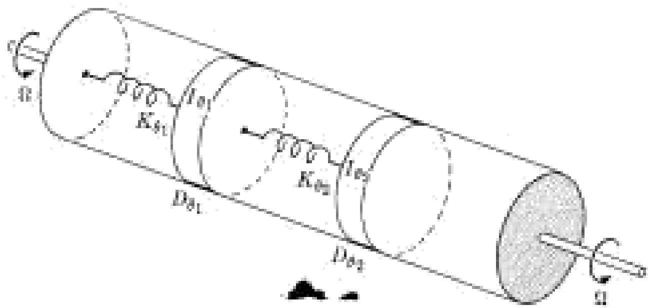


FIGURE 4-27

4-8.

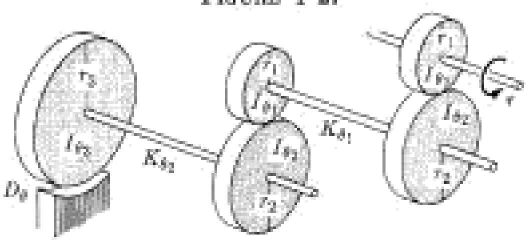


FIGURE 4-28

4-9.

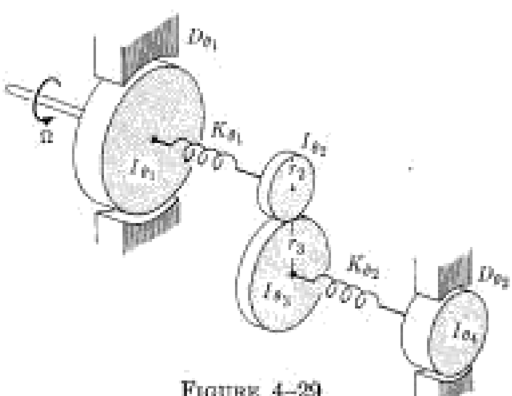


FIGURE 4-29

4-10.

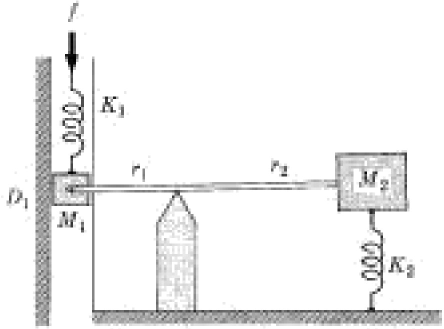


FIGURE 4-30

4-11. (a) Draw the f-v analogous mechanical system for the electrical circuit of Fig. 4-31. (b) Difficulty in drawing the f-v analogous mechanical system will arise if the circuit in Fig. 4-31 is rearranged as in Fig. 4-32. Why?

4-12. Draw the f-i analogous mechanical system for the electrical circuit of Fig. 4-33.

4-13. It has been suggested that it is possible to reduce the vibration of an electric shaver by attaching a mass and spring combination on the case, as illustrated schematically in Fig. 4-34. Comment on the merit of this suggestion and give your reasons analytically.

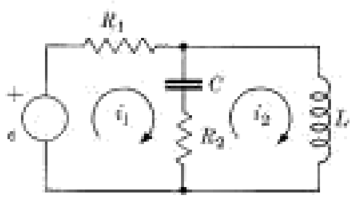


FIGURE 4-31

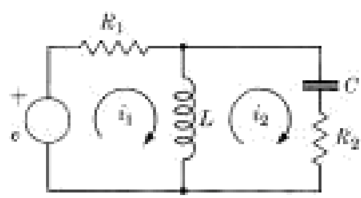


FIGURE 4-32

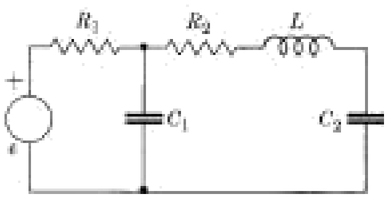


FIGURE 4-33

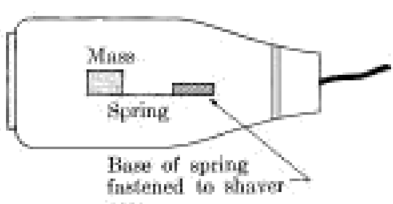


FIGURE 4-34

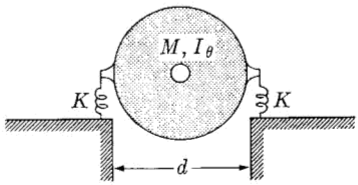


FIGURE 4–35

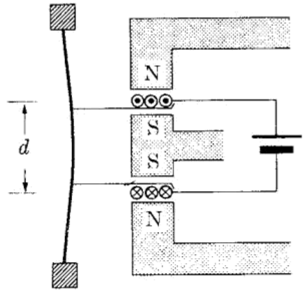


FIGURE 4–36

4–14. Figure 4–35 is a diagram of a rotary machine with mass M and moment of inertia I_θ about its shaft in the center. It is supported on two springs with identical compliance K, and translational motion is possible in the vertical direction only. Draw an analogous electrical circuit for this mechanical system with combined translational and rotational motion and express all elements in the circuit in terms of the given mechanical quantities.

4–15. Figure 4–36 is a schematic diagram of a moving-coil microphone. Sound waves impinging on the diaphragm make the coil move in the magnetic field, thereby producing a current change in the electrical circuit. Assuming that the diaphragm-coil combination has an effective mass M, a damping coefficient D, and a compliance K, and that the n-turn coil, which has an inductance L and a resistance R, moves in a uniform magnetic field of flux density B, draw an analogous circuit for this electromechanical device and write the equations that completely describe the system.

4–16. Figure 4–37 is a schematic diagram of a small d-c shunt motor. Assuming that the armature winding has an inductance L and a resistance R and that the entire rotor has a rotational damping coefficient D_θ and a moment of inertia I_θ about its axis, draw an analogous circuit for the motor after the switch is closed and write the equations that completely describe the system. The torque τ developed in a d-c motor is proportional to the armature current I and to the total flux per pole ϕ: $\tau = kI\phi$, where k is a constant for any one machine.

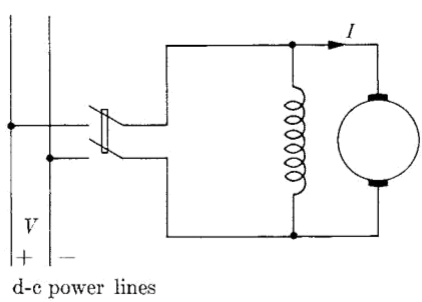


FIGURE 4–37

(b) Norton's equivalent.

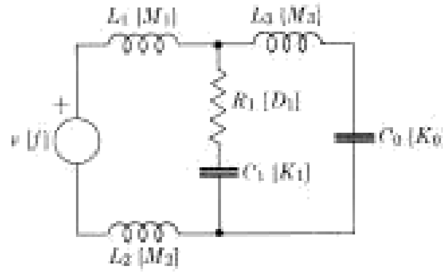
Current source: $i_0 = \dfrac{E}{R_1}(\cos\omega t - 1)$.

Passive network: parallel combination of R_1 and L

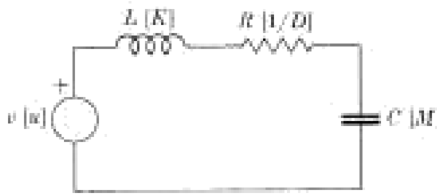
$$i_{R_2} = \frac{2E}{3R_1[1 + (R_1/3\omega L)^2]}\left(-\epsilon^{-R_1 t/3L} + \cos\omega t - \frac{R_1}{3\omega L}\sin\omega t\right)$$

Chapter 4

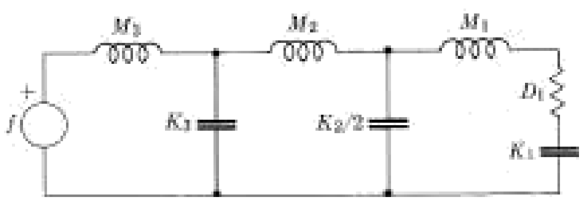
4–1. (a)



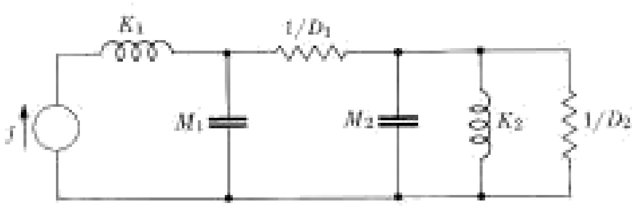
4–2. (b)



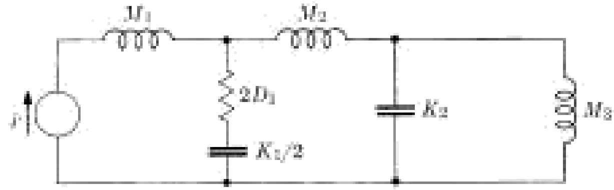
4–3. (a)



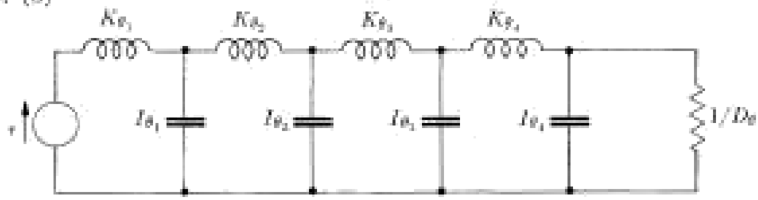
4–4. (b)



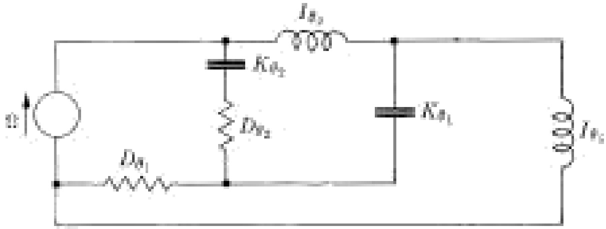
4–5. (a)



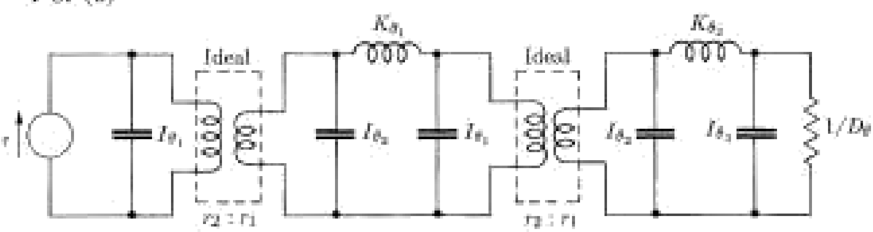
4–6. (b)



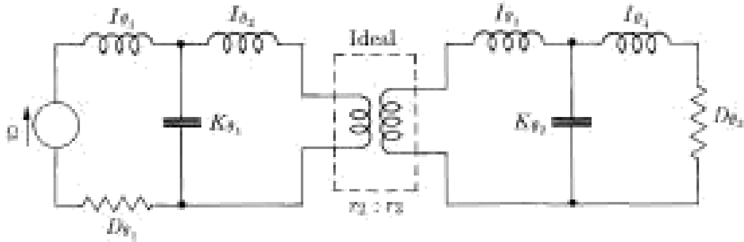
4–7. (a)



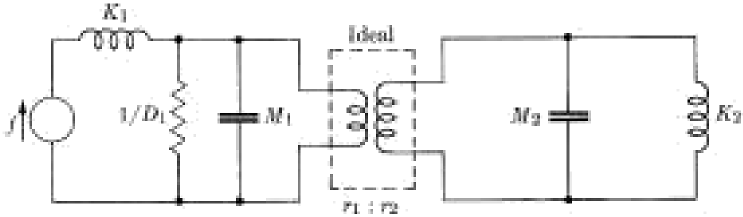
4–8. (b)



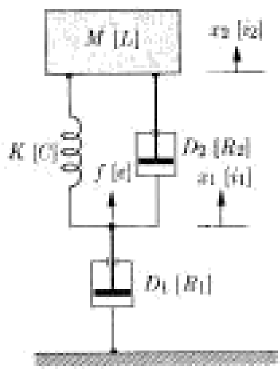
4–9. (a)



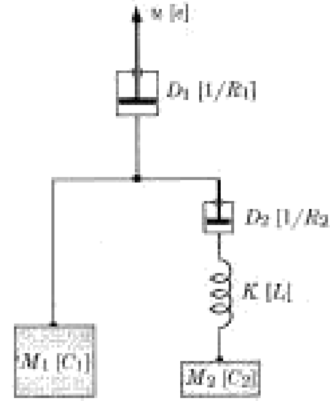
4–10. (b)



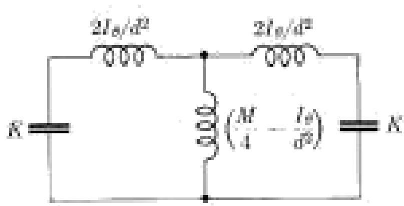
4–11. (a)



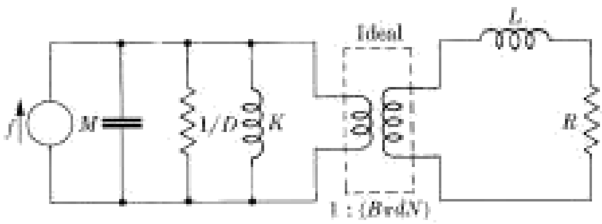
4–12. (b)



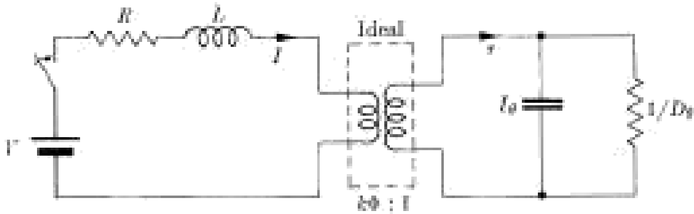
4–14.



4–15.



4-16.



Chapter 5

5-1. (a) $f(t) = \dfrac{A}{2} + \dfrac{A}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{n}\sin(2n\pi t/T)$

5-2. (a) $f(x) = \dfrac{A}{\pi}\left[1 + \dfrac{\pi}{2}\cos 2x + 2\sum_{k=1}^{\infty}\dfrac{(-1)^{k+1}}{(4k^2-1)}\cos 4kx\right]$

5-3. (a) $$f(\theta) = \frac{2A}{\pi}\left[(\theta_0 \csc\theta_0 + \cos\theta_0)\sin\theta + \sum_{n=3,5,7}\frac{2}{n^2-1}\right.$$
$$\left.\left(\cot\theta_0 \sin n\theta_0 - \frac{1}{n}\cos n\theta_0\right)\sin n\theta\right]$$

5-4. (a) $f(t) = 0.4A\left\{1 + \sum_{n=\text{odd}}\dfrac{1}{n\pi}\left[5\sin(2n\pi t/T) - \dfrac{2}{n\pi}\cos(2n\pi t/T)\right]\right\}$

5-5. (a) $f(t) = \dfrac{4A}{\pi}\sum_{k=1}^{\infty}\dfrac{2k}{(2k)^2-1}\sin(4k\pi t/T)$

5-6. (a) $f(x) = \frac{1}{2}(1-\epsilon^{-2}) + \sum_{n=1}^{\infty}\dfrac{4}{4+n^2\pi^2}\left[1+(-1)^{n+1}\epsilon^{-2}\right]\cos(n\pi x/2)$

5-7. (a) $f(x) = \dfrac{1}{3} + \dfrac{1}{2\pi^2}\sum_{n=1}^{\infty}\dfrac{1}{n^2}\left[(1+jn\pi)\epsilon^{j2n\pi x} + (1-jn\pi)\epsilon^{-j2n\pi x}\right]$

5-8. (a) $a_{2k+1} = 0, \quad b_{2k} = 0$

$$a_{2k} = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} f(\theta)\cos 2k\theta\, d\theta$$

$$b_{2k+1} = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} f(\theta)\sin(2k+1)\theta\, d\theta$$

Across and Through Variables

	Across Variable	Through Variable
Electrical	V = voltage (volts) or (V)	I = current (Amps) or (A)
Mechanical translational	v = velocity $\left(\frac{\text{m}}{\text{sec}}\right)$	F = force (newtons) or (N) or $\left(\text{Kg}\cdot\frac{\text{m}}{\text{sec}^2}\right)$
Mechanical rotational	ω = angular velocity $\left(\frac{\text{rad}}{\text{sec}}\right)$	T = torque (N·m)
Fluid	P = pressure $\left(\frac{\text{N}}{\text{m}^2}\right)$ or (Pa)	Q = flow $\left(\frac{\text{m}^3}{\text{sec}}\right)$

Elements

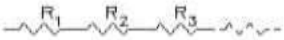
	Dissipation	Across Variable Energy Storage	Through Variable Energy Storage
Electrical	R = resistance $\left(\frac{\text{V}}{\text{A}}\right)$ or (Ω)	C = capacitance $\left(\frac{\text{A}\cdot\text{sec}}{\text{V}}\right)$ or (F)	L = inductor $\left(\frac{\text{V}\cdot\text{sec}}{\text{A}}\right)$ or (H)
Mechanical translational	B = damping $\left(\frac{\text{N}\cdot\text{sec}}{\text{m}}\right)$	M = mass (Kg) or $\left(\frac{\text{N}\cdot\text{sec}^2}{\text{m}}\right)$	k = Spring constant $\left(\frac{\text{N}}{\text{m}}\right)$
Mechanical rotational	B = damping (N·m·sec) or $\left[\frac{\text{N}\cdot\text{m}}{\left(\frac{\text{rad}}{\text{sec}}\right)}\right]$	J = moment of inertia $(\text{Kg}\cdot\text{m}^2)$ or $\left(\frac{\text{N}\cdot\text{m}^3}{\text{sec}^2}\right)$	k = Spring constant $\left(\frac{\text{N}\cdot\text{m}}{\text{rad}}\right)$
Fluid	R_f = fluid resistance $\left(\frac{\text{N}\cdot\text{sec}}{\text{m}^5}\right)$	C_f = fluid capacitance $\left(\frac{\text{m}^5}{\text{N}}\right)$	I = fluid inertia $\left(\frac{\text{Kg}}{\text{m}^4}\right)$

Basic Electric Circuit Analysis

Element	Parts like resistors, capacitors, inductors & transformers
Wires and connections	Direct the current, but do not affect voltage
Circuit	Wires and elements connected to form loops
Voltage	Measured as a difference **across** an element
Current	Flows **through** a wire or element
Kirchhoff's Current Law (KCL)	Current in = current out of all elements, wires & connections
Kirchhoff's Voltage Law (KVL)	Voltage gains = voltage "losses" around any circuit loop
Node	Connected wires and connections which all have the same voltage
Ground	Zero-reference node for all other nodal voltages
Branch	Connected wires and elements which all have the same current
Power $P = V\cdot I$	Power = Across variable x Through variable
Voltage Source 	Constant voltage regardless of current in or out
Current Source 	Constant current regardless of voltage + or -

Passive Electrical Elements

Resistors

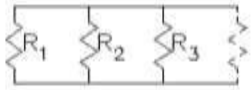


series: $R_{eq} = R_1 + R_2 + R_3 + \ldots$

Exactly the **same current** through each resistor

voltage divider: $V_{Rn} = V_{total} \cdot \dfrac{R_n}{R_1 + R_2 + R_3 + \ldots}$

parallel: $R_{eq} = \dfrac{1}{\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3} + \ldots}$



Exactly the **same voltage** across each resistor

current divider: $I_{Rn} = I_{total} \cdot \dfrac{\dfrac{1}{R_n}}{\dfrac{1}{R_1} + \dfrac{1}{R_2} + \dfrac{1}{R_3} + \ldots}$

Resistors dissipate power $P = V \cdot I = I^2 \cdot R = \dfrac{V^2}{R}$

Capacitors

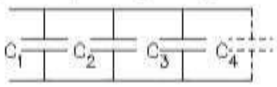
$C = \dfrac{Q}{V}$ $\text{farad} = \dfrac{\text{coul}}{\text{volt}} = \dfrac{\text{amp}\cdot\text{sec}}{\text{volt}}$ $v_C = \dfrac{1}{C} \cdot \displaystyle\int_{-\infty}^{t} i_C \, dt = \dfrac{1}{C} \cdot \int_{0}^{t} i_C \, dt + v_C(0)$ $i_C = C \cdot \dfrac{d}{dt} v_C$

initial voltage

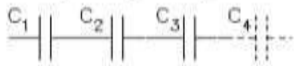
Energy stored in electric field: $E_C = \dfrac{1}{2} \cdot C \cdot V_C^2$

Capacitor voltage **cannot** change instantaneously

parallel: $C_{eq} = C_1 + C_2 + C_3 + \ldots$



series: $C_{eq} = \dfrac{1}{\dfrac{1}{C_1} + \dfrac{1}{C_2} + \dfrac{1}{C_3} + \ldots}$



Steady-state sinusoids:
Impedance: $\mathbf{Z_C} = \dfrac{1}{j \cdot \omega \cdot C} = \dfrac{-j}{\omega \cdot C}$

Current leads voltage by 90 deg

Laplace:
Impedance: $\mathbf{Z_C} = \dfrac{1}{C \cdot s}$

Inductors

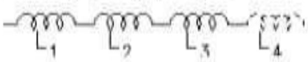
$\text{henry} = \dfrac{\text{volt}\cdot\text{sec}}{\text{amp}}$ $i_L = \dfrac{1}{L} \cdot \displaystyle\int_{-\infty}^{t} v_L \, dt = \dfrac{1}{L} \cdot \int_{0}^{t} v_L \, dt + i_L(0)$ $v_L = L \cdot \dfrac{d}{dt} i_L$

initial current

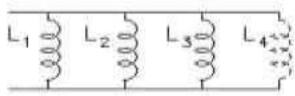
Energy stored in magnetic field: $E_L = \dfrac{1}{2} \cdot L \cdot I_L^2$

Inductor current **cannot** change instantaneously

series: $L_{eq} = L_1 + L_2 + L_3 + \ldots$



parallel: $L_{eq} = \dfrac{1}{\dfrac{1}{L_1} + \dfrac{1}{L_2} + \dfrac{1}{L_3} + \ldots}$



Steady-state sinusoids:
Impedance: $\mathbf{Z_L} = j \cdot \omega \cdot L$

Current lags voltage by 90 deg

Laplace:
Impedance: $\mathbf{Z_L} = L \cdot s$

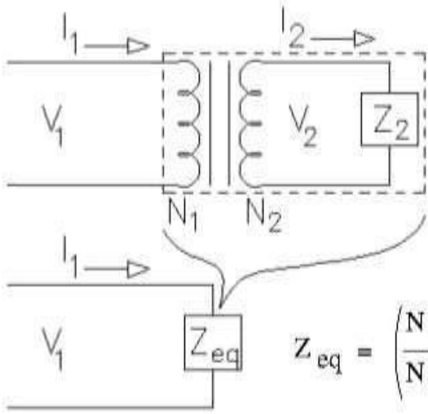
Transformers (ideal)

Ideal: $P_1 = P_2$ power in = power out

Turns ratio $= N = \dfrac{N_1}{N_2} = \dfrac{V_1}{V_2} = \dfrac{I_2}{I_1}$

Note: some books define the turns ratio as N_2/N_1

Equivalent impedance in primary: $\mathbf{Z_{eq}} = N^2 \cdot \mathbf{Z_2} = \left(\dfrac{N_1}{N_2}\right)^2 \cdot \mathbf{Z_2}$



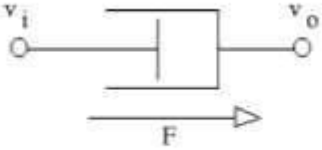
$Z_{eq} = \left(\dfrac{N_1}{N_2}\right)^2 \cdot Z_2$

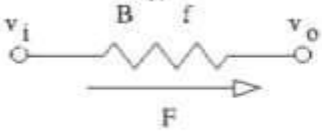
You can replace the entire transformer and load with ($\mathbf{Z_{eq}}$).
This "impedance transformation" can work across systems.

Mechanical system with linear motion (translational)

	Mechanical translational	**Electrical**	
Through Variable:	F = Force (N)	I = current (A)	Source:
Across Variable:	v = velocity $\left(\frac{m}{sec}\right)$	V = voltage (V)	Source:
$\int v\,dt$	x = displacement (m)	Source: $v = \frac{d}{dt}x$ or $s \cdot X(s)$	
$\frac{V(s)}{s}$	$X(s)$ = displacement $(m \cdot sec)$ (in freq domain)		

			Impedance
Dissipation element: power $P = v \cdot F = \frac{F^2}{B} = v^2 \cdot B$	**Damper or friction** B or f	**Resistor** $\frac{1}{B}$ or $\frac{1}{f}$	$\frac{1}{B}$ or $\frac{1}{f}$

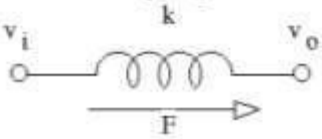




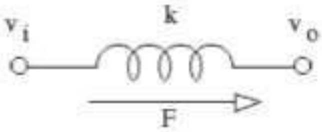
Through variable energy storage:

$$E = \frac{1}{2} \cdot \frac{1}{k} \cdot F^2 \underset{(F=kx)}{=} \frac{1}{2} \cdot k \cdot x^2$$

Spring k

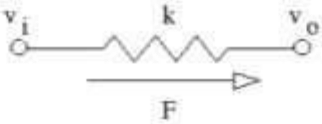


Inductor $\frac{1}{k}$



Impedance: $\frac{s}{k}$

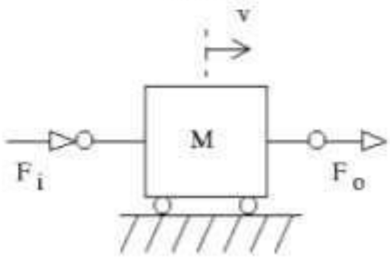
Springs are sometimes shown like this:



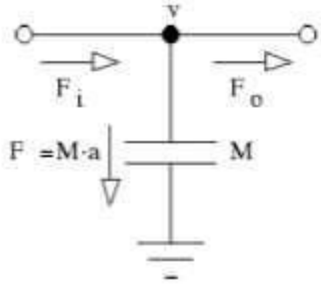
Through variable energy storage:

$$E = \frac{1}{2} \cdot M \cdot v^2$$

Mass

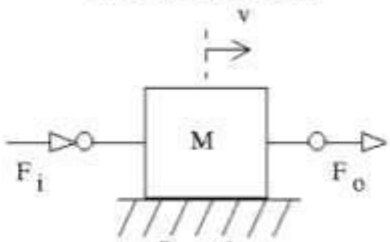


Capacitor hooked to ground



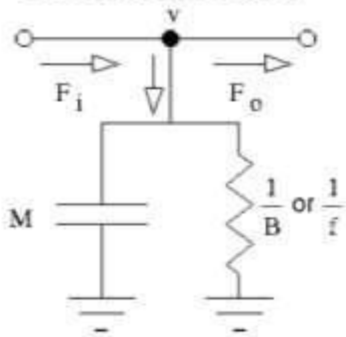
Impedance: $\frac{1}{M \cdot s}$

Mass with friction



B or f
sliding friction

Capacitor and resistor



$$\frac{1}{\frac{1}{\left(\frac{1}{M \cdot s}\right)} + \frac{1}{\left(\frac{1}{B}\right)}}$$

$$\frac{1}{M \cdot s + B}$$

Mechanical system with circular motion (rotational)

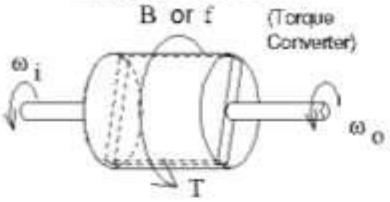
	Mechanical rotational	**Electrical**	
Through Variable:	T = Torque $(N \cdot m)$	I = current (A)	Source: 
Across Variable:	ω = angular velocity $\left(\frac{rad}{sec}\right)$	V = voltage (V)	Source: 
$\int \omega\, dt$	θ = angular displacement (rad)	Source:  $\omega = \frac{d}{dt}\theta$ or $s \cdot \theta(s)$	
$\frac{\omega(s)}{s}$	$\theta(s)$ = angular displacement $(rad \cdot sec)$ (in freq domain)		

Dissipation element:

power

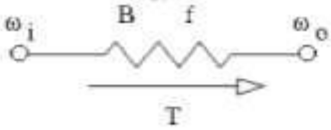
$$P = v \cdot T = \frac{T^2}{B} = \omega^2 \cdot B$$

Damper or friction



Resistor

$\frac{1}{B}$ or $\frac{1}{f}$



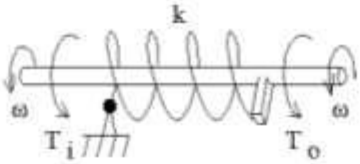
Impedance

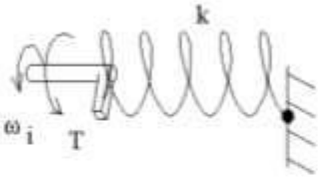
$\frac{1}{B}$ or $\frac{1}{f}$

Through variable energy storage:

$$E = \frac{1}{2} \cdot \frac{1}{k} \cdot T^2$$

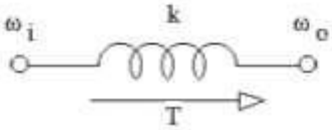
Springs



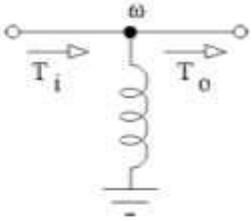


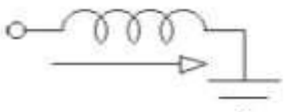
Inductor

$\frac{1}{k}$



$\frac{s}{k}$

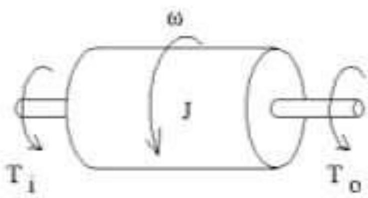




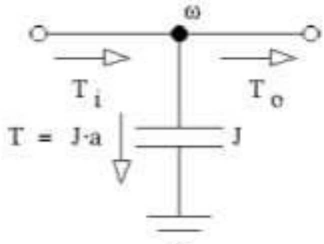
Through variable energy storage:

$$E = \frac{1}{2} \cdot J \cdot \omega^2$$

Moment of Inertia, J

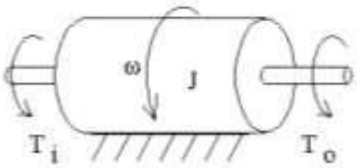


Capacitor hooked to ground



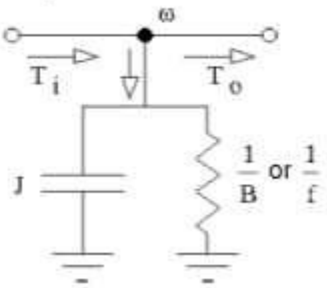
$\frac{1}{J \cdot s}$

J with friction



B or f
sliding friction

Capacitor and resistor

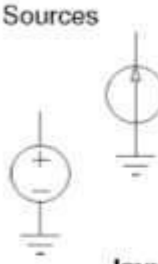


$$\frac{1}{\frac{1}{\left(\frac{1}{J \cdot s}\right)} + \frac{1}{\left(\frac{1}{B}\right)}}$$

$$\frac{1}{J \cdot s + B}$$

Fluid (hydraulic) system

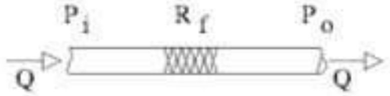
	Fluid	Electrical	Sources
Through Variable:	Q = volumetric flow rate $\left(\frac{m^3}{sec}\right)$	I = current (A)	
Across Variable:	P = Pressure $\left(\frac{N}{m^2}\right)$ or (Pa)	V = voltage (V)	



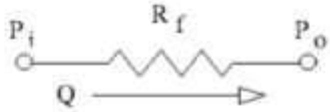
Dissipation element: power

$$P = P \cdot Q = \frac{Q^2}{R_f} = P^2 \cdot R_f$$

Fluid resistance



Resistor



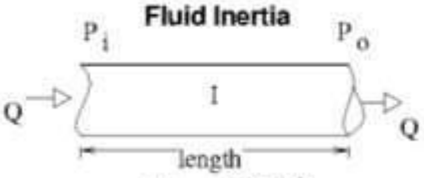
Impedance

R_f

Through variable energy storage:

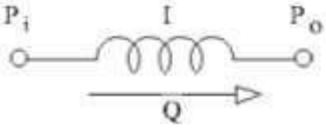
$$E = \frac{1}{2} \cdot I \cdot Q^2$$

Fluid Inertia



$$I = \frac{M}{A^2} = \frac{\rho \cdot length}{A}$$

Inductor

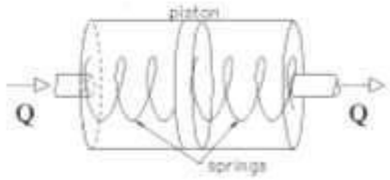


$I \cdot s$

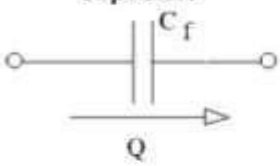
Through variable energy storage:

$$E = \frac{1}{2} \cdot C_f \cdot P^2$$

Fluid Capacitors

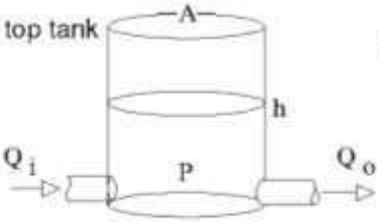


Capacitor

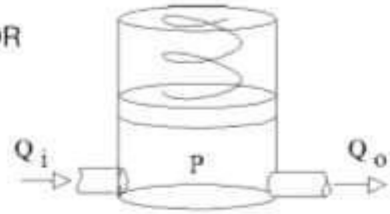


$$\frac{1}{C_f \cdot s}$$

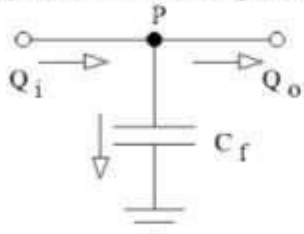
open top tank



OR



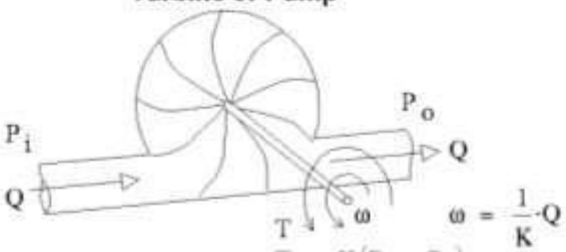
Capacitor hooked to ground



$$C_f = \frac{\Delta volume}{\Delta pressure} = \frac{\Delta h \cdot A}{\Delta h \cdot \rho \cdot g} = \frac{A}{\rho \cdot g}$$

for all capacitors — For open top tank

Turbine or Pump



$$\omega = \frac{1}{K} \cdot Q$$

$$T = K\left(P_o - P_i\right)$$

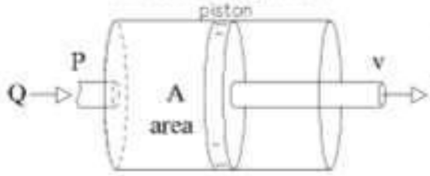
Turbines & pistons convert through variables to across variables & vice versa, so there are no good electrical analogies.

Yet you can still transform an impedance from a mechanical system into the fluid system. You,ll find that capacitors become inductors, inductors become capacitors and parallel swaps with series.

$$Z_{eq} = \frac{\Delta \cdot P}{Q} = \frac{\left(\frac{T}{K}\right)}{K \cdot \omega} = \frac{1}{K^2} \cdot \frac{T}{\omega} = \frac{1}{K^2} \cdot \frac{1}{Z_2} = \frac{1}{K^2 \cdot Z_2}$$

Piston & Cylinder

$$P = \frac{F}{A} \qquad Q = A \cdot v$$



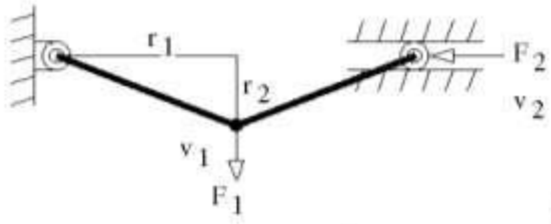
$$v = \frac{Q}{A} \qquad F = A \cdot P$$

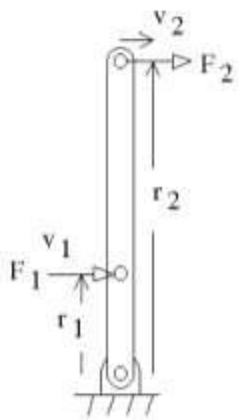
$$Z_{eq} = \frac{P}{Q} = \frac{\left(\frac{F}{A}\right)}{A \cdot v} = \frac{1}{A^2} \cdot \frac{F}{v} = \frac{1}{A^2} \cdot \frac{1}{Z_2} = \frac{1}{A^2 \cdot Z_2}$$

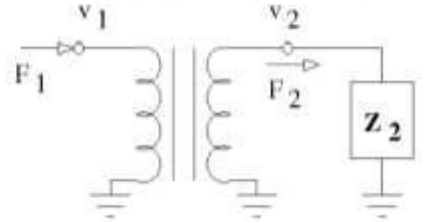
Transducers and Transformers

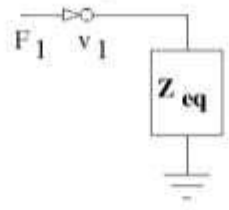
A transducer converts power from one type to another. We can model many of them with transformers. Transformers increase the through variable and correspondingly decrease the across variable or vice-versa.

Levers









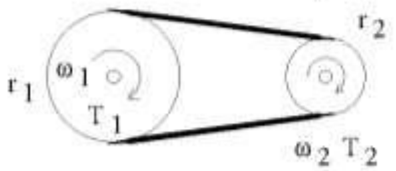
$$N = \frac{r_1}{r_2} = \frac{v_1}{v_2} = \frac{F_2}{F_1}$$

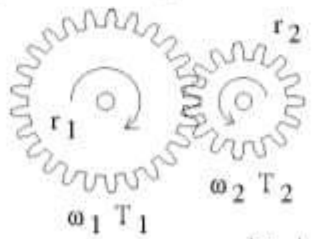
(not really this simple)

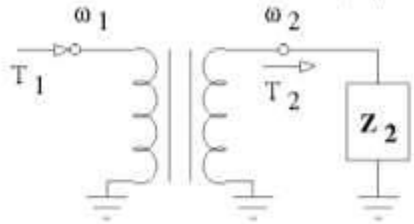
$$Z_{eq} = N^2 \cdot Z_2 = \left(\frac{r_1}{r_2}\right)^2 \cdot Z_2$$

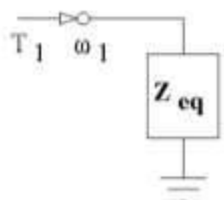
Belts, chains, & gears

r = radius of pulley or pitch radius of gears







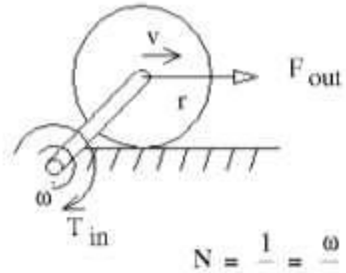


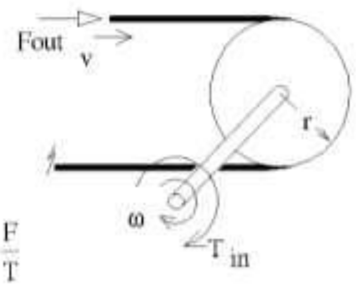
$$N = \frac{r_2}{r_1} = \frac{\omega_1}{\omega_2} = \frac{T_2}{T_1} = \text{gear tooth ratio} \left(\frac{N_2}{N_1}\right)$$

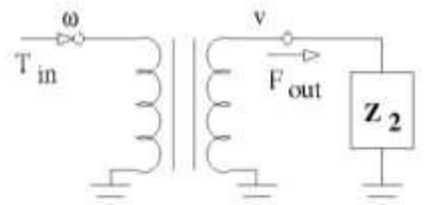
$$Z_{eq} = N^2 \cdot Z_2 = \left(\frac{r_2}{r_1}\right)^2 \cdot Z_2$$

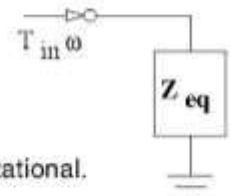
Tires, racks, & conveyors

r = radius of wheel or pitch radius of pinion gear







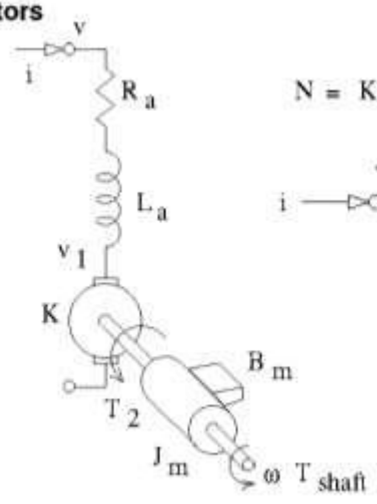


$$N = \frac{1}{r} = \frac{\omega}{v} = \frac{F}{T}$$

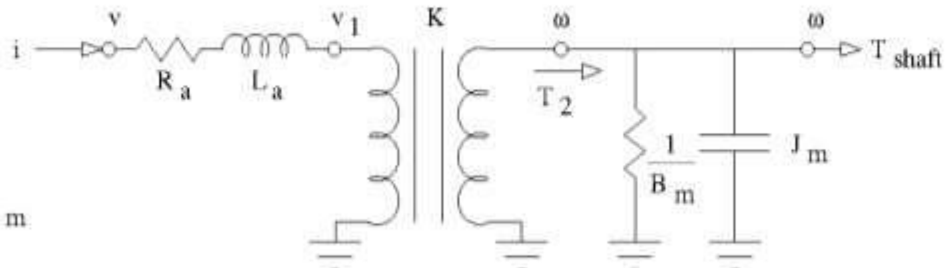
$$Z_{eq} = N^2 \cdot Z_2 = \left(\frac{1}{r}\right)^2 \cdot Z_2$$

Note: N = r if the input is linear motion and output is rotational.

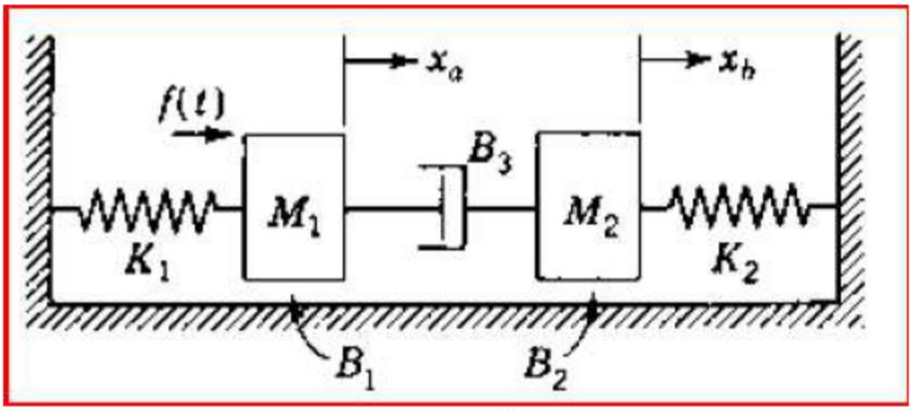
DC Motors

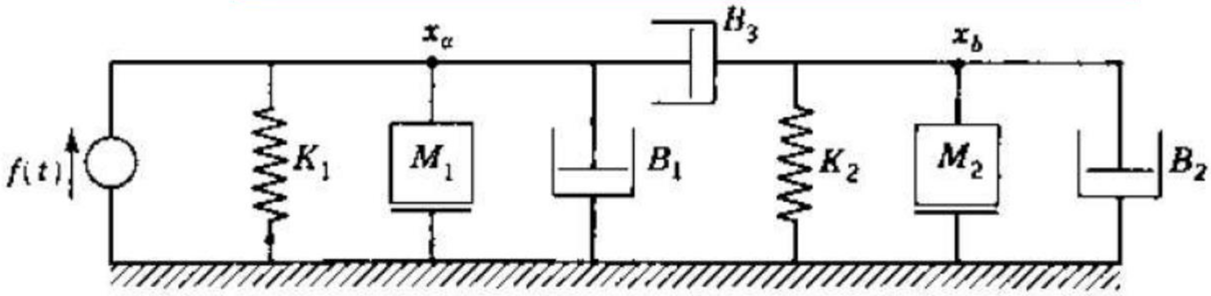


$$N = K = \frac{v_1}{\omega} = \frac{T_2}{i}$$



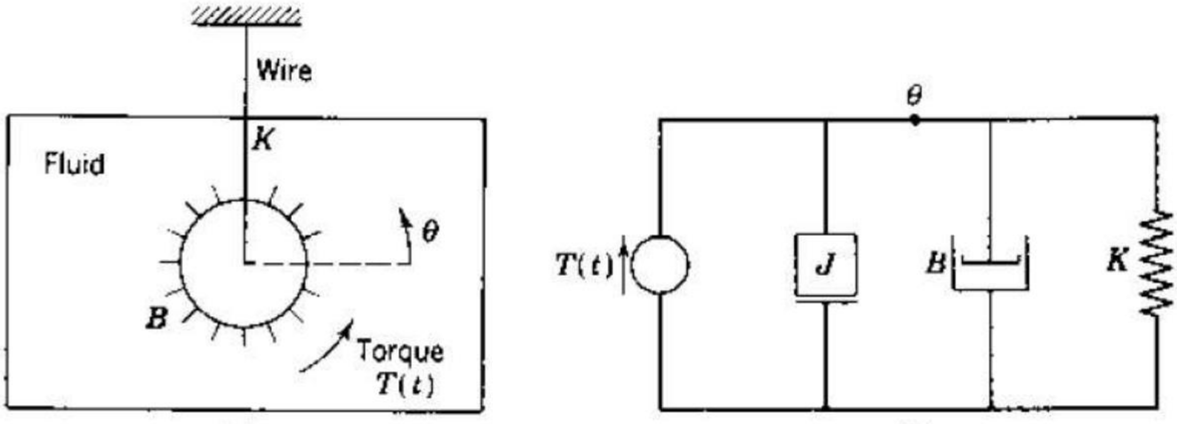
Problem:



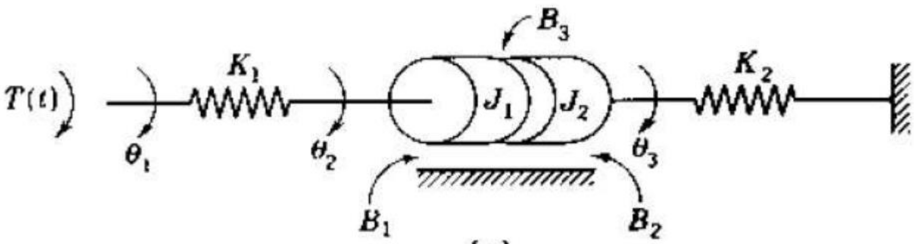


For node a: $$(M_1D^2 + B_1D + B_3D + K_1)x_a - (B_3D)x_b = f$$

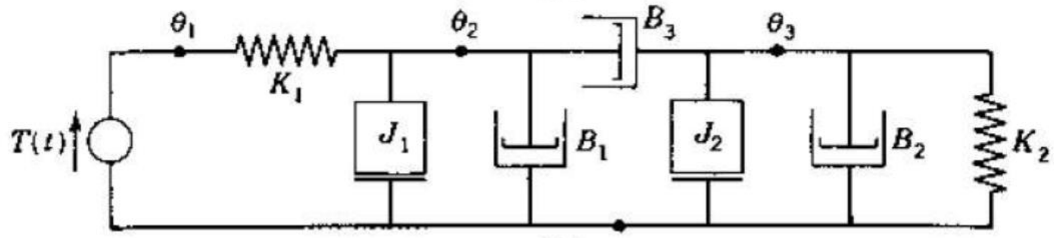
For node b: $$-(B_3D)x_a + (M_2D^2 + B_2D + B_3D + K_2)x_b = 0$$



(a) (b)



(a)



(b)

FIGURE 2.16 *(a)* Rotational system; *(b)* corresponding mechanical network.

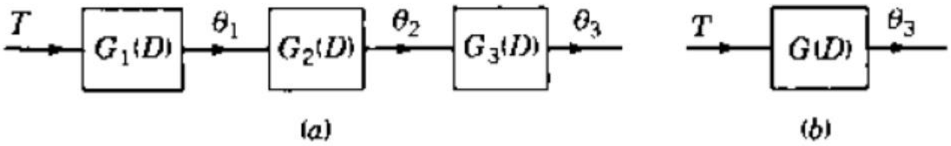


FIGURE 2.17 Detailed and overall block diagram representations of Fig. 2.16.

Node 1: $K_1\theta_1 - K_1\theta_2 = T(t)$

Node 2: $-K_1\theta_1 + [J_1D^2 + (B_1 + B_3)D + K_1]\theta_2 - (B_3D)\theta_3 = 0$

Node 3: $-(B_3D)\theta_2 + [J_2D^2 + (B_2 + B_3)D + K_2]\theta_3 = 0$

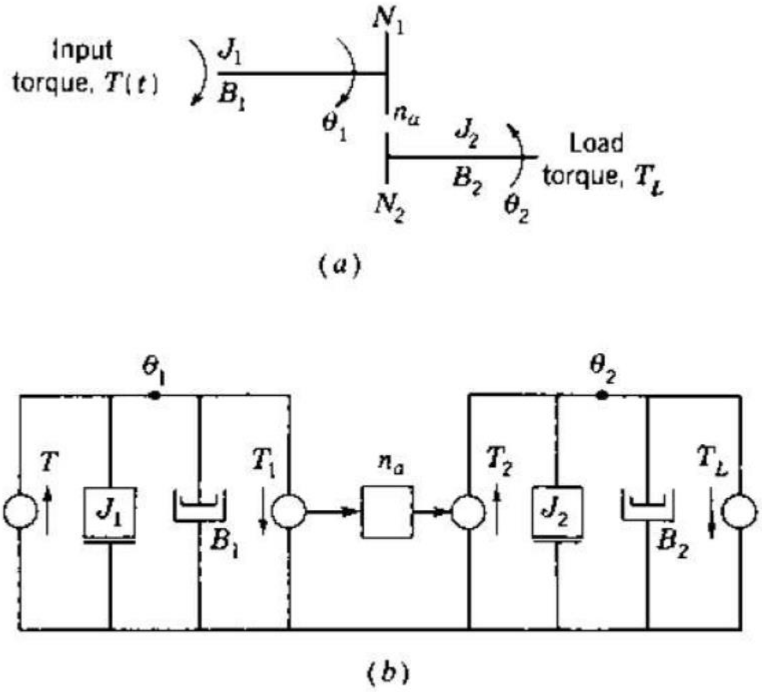


FIGURE 2.18 (*a*) Representation of a gear train; (*b*
network.

N = number of teeth on each gear
$\omega = D\theta$ = velocity of each gear
n_a = ratio of (speed of driving shaft)/(speed of driven shaft)
θ = angular position

The equations describing the system are

$$J_1D^2\theta_1 + B_1D\theta_1 + T_1 = T \qquad J_2D^2\theta_2 + B_2D\theta_2 + T_L = T_2$$
$$n_a = \frac{\omega_1}{\omega_2} = \frac{\theta_1}{\theta_2} = \frac{N_2}{N_1} \qquad \theta_2 = \frac{\theta_1}{n_a} \qquad T_2 = n_aT_1 \tag{2.79}$$

The equations can be combined to produce

$$J_1D^2\theta_1 + B_1D\theta_1 + \frac{1}{n_a}\left(J_2D^2\theta_2 + B_2D\theta_2 + T_L\right) = T \tag{2.80}$$

This equation can be expressed in terms of the torque and the input position θ_1 only:

$$\left(J_1 + \frac{J_2}{n_a^2}\right)D^2\theta_1 + \left(B_a + \frac{B_2}{n_a^2}\right)D\theta_1 + \frac{T_L}{n_a} = T \tag{2.81}$$

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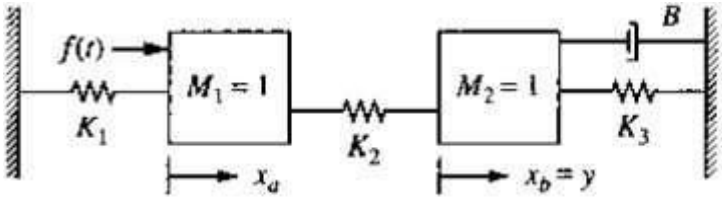
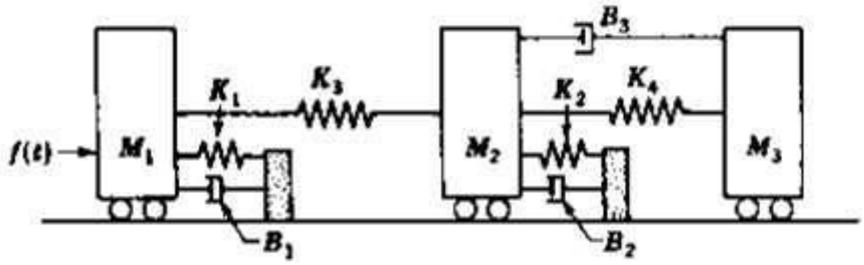
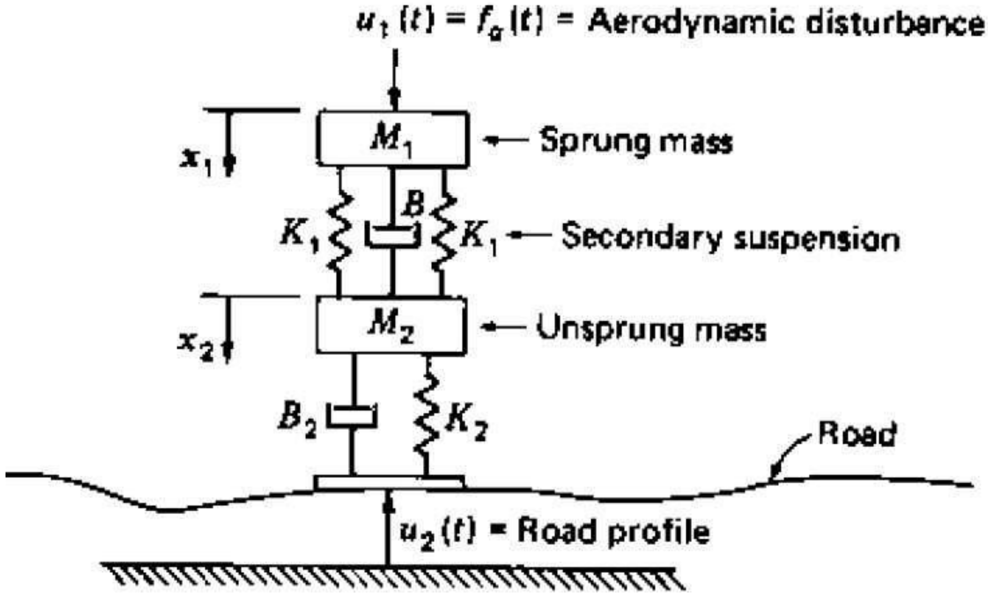
2.6 (a) Derive the differential equation relating the position y(t) and the force f(t). (b) Draw the mechanical network. (c) Determine the transfer function G(D) =y/f. (d) Identify a suitable set of independent state variables. Write the state equation in matrix form.



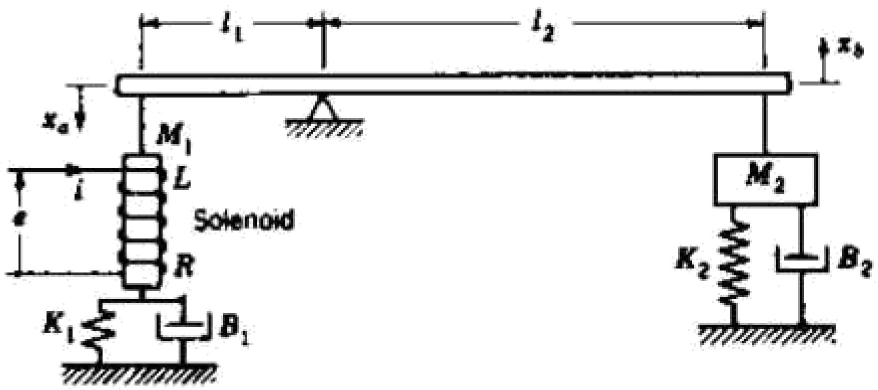
2.7 (a) Draw the mechanical network for the mechanical system shown. (b)Write the differential equations of performance. (c) Draw the analogous electric circuit in which force is analogous to current. (d) Write the state equations.



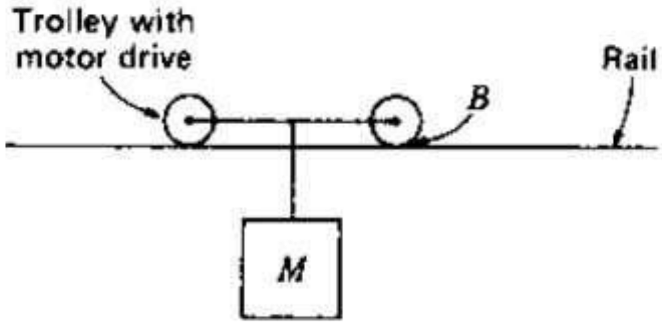
2.9. A simplified model for the vertical suspension of an automobile is shown in the figure. (a) Draw the mechanical network; (b) write the differential equations of performance; (c) derive the state equations; (d) determine the transfer function $x_1 = \hat{u}_2$, where $\hat{u}_2 = (B_2D + K_2)/u_2$ is the force exerted by the road.



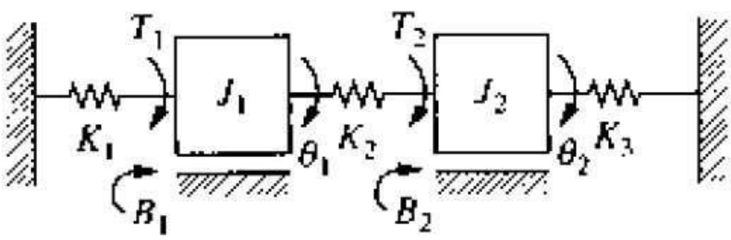
2.10. An electromagnetic actuator contains a solenoid, which produces a magnetic force proportional to the current in the coil, $f = K_i i$. The coil has resistance and inductance. (a) Write the differential equations of performance. (b)Write the state equations.



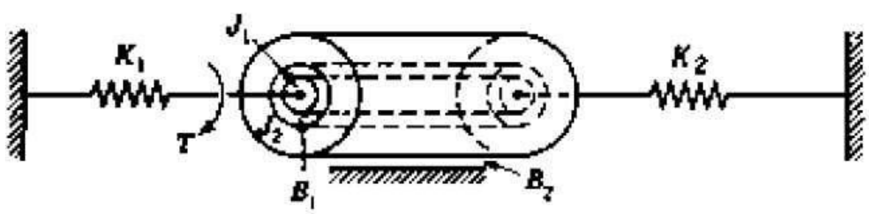
2.11. A warehouse transportation system has a motor drive moving a trolley on a rail. Write the equation of motion.



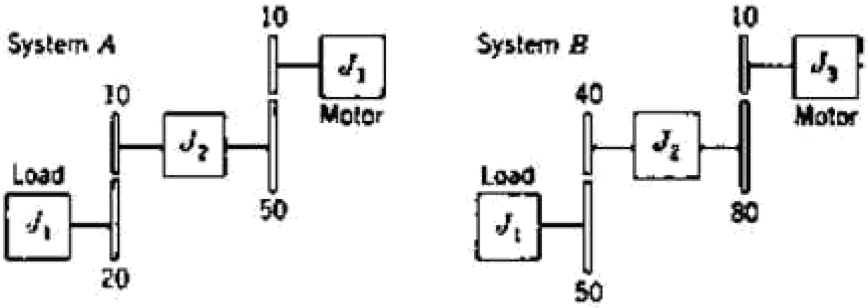
2.13. (a)Write the equations of motion for this system. (b)Using the physical energy variables, write the matrix state equation.



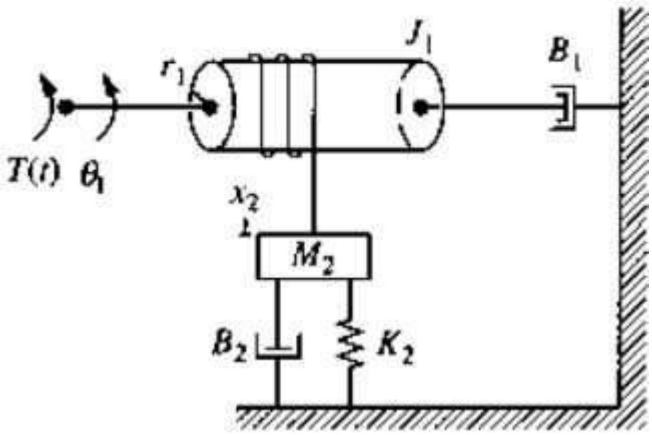
2.14. The figure represents a cylinder of inertia J_1 inside a sleeve of inertia J_2. There is viscous damping B_1 between the cylinder and the sleeve. The springs K_1 and K_2 are fastened to the inner cylinder. (a) Draw the mechanical network. (b) Write the system equations. (c) Draw the analogous circuit. (d) Write the state and output equations. The outputs are θ_1 and θ_2. (e) Determine the transfer function $G=\theta_2/T$.



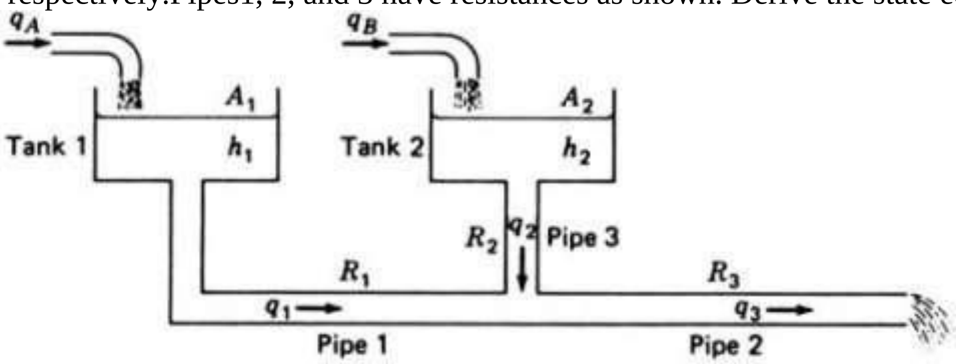
2.15. The two gear trains have an identical net reduction, have identical inertias at each stage, and are driven by the same motor. The number of teeth on each gear is indicated on the figures. At the instant of starting, the motor develops a torque T. Which system has the higher initial load acceleration? (a) Let J1=J2=J3; (b) let J1=40J2=4J3.



2.16. In the mechanical system shown, r_1 is the radius of the drum. (a) Write the necessary differential equations of performance of this system. (b) Obtain a differential equation expressing the relationship of the output x_2 in terms of the input T(t) and the corresponding transfer function. (c) Write the state equations with T(t) as the input.



2.20. The sewage system leading to a treatment plant is shown. The variables q_A and q_B are input flow rates into tanks1and 2, respectively.Pipes1, 2, and 3 have resistances as shown. Derive the state equations.



2.21. Most control systems require some type of motive power. One of the most commonly used units is the electric motor. Write the differential equations for the angular displacement of a moment of inertia, with damping, connected directly to a dc motor shaft when a voltage is suddenly applied to the armature terminals with the field separately energized.