Solution of First Order and First Degree Differential Equations

2.2 Separation of variables

If in an equation, it is possible to get all the functions of x and dx to one side and all the functions of y and dy to the other, the variables are said to be separable.

Ex. 1. (*a*) *Solve* $dy/dx = e^{x-y} + x^2 e^{-y}$

[Agra 1995; Lucknow 1998; Mysore 2004;Punjab 1994; Meerut 2009; Agra 2005]

(*b*) *Solve* $dy/dx = e^{x+y} = x^2 e^y$

Sol. (*a*) For separating variables, we re-write the given equation as

$dy/dx = e^{-y}(e^x + x^2)$ or $e^y dy = (x^2 + e^x)\, dx$.

Integrating, $e^y = x^3/3 + e^x + c$, c being an arbitrary constant.

(*b*) Do like part (*a*). **Ans.** $-e^{-y} = x^3/3 + e^x + c$.

Ex. 3. *Solve* $(dy/dx) \tan y = \sin (x + y) + \sin (x - y)$.

Sol. Using formula $\sin C + \sin D = 2 \sin \{(C + D)/2\} \cos \{(C - D)/2\}$, the given equation can be rewritten as

$$(\tan y)\,(dy/dx) = 2 \sin x \cos y \qquad \text{or} \qquad \sec y \tan y\, dy = 2 \sin x\, dx.$$

Integrating, $\quad \sec y = -2 \cos x + c$, c being an arbitrary constant.

Ex. 4. *Solve the following differential equations:*

(*i*) $\dfrac{dy}{dx} = \dfrac{\sin x + x \cos x}{y\,(2 \log y + 1)}$ (*ii*) $\dfrac{dy}{dx} = \dfrac{x\,(2 \log x + 1)}{\sin y + y \cos y}$. **[I.A.S. (Prel.) 2009]**

Sol. (*i*) Re-writing the given equation, $\quad (\sin x + x \cos x)\, dx = (2y \log y + y)\, dy$.

Integrating, $$-\cos x + \int x \cos x\, dx = 2 \int y \log y\, dy + (y^2/2) + c. \qquad \text{... (1)}$$

Now, $$\int x \cos x\, dx = x \sin x - \int \sin x\, dx, \text{ integrating by parts}$$

or $$\int x \cos x\, dx = x \sin x + \cos x. \qquad \text{... (2)}$$

Also, $$\int y \log y\, dy = (\log y) \times (y^2/2) - \int \{(1/y) \times (y^2/2)\}\, dy, \text{ integrating by parts}$$

or $$\int y \log y\, dy = (y^2/2) \times \log y - y^2/4 \qquad \text{... (3)}$$

Using (2) and (3), (1) reduces to

$$-\cos x + x \sin x + \cos x = 2\,\{(y^2/2) \times \log y - y^2/4\} + y^2/2 + c.$$

or $$x \sin x = y^2 \log y + c, \; c \text{ being an arbitrary constant.}$$

(*ii*) Proceed exactly as in part (*i*). **Ans.** $x^2 \log x = y \sin y + c$.

Ex. 6. *Solve* $y - x\,(dy/dx) = a\,(y^2 + dy/dx)$. **[Meerut 1993; Delhi Maths (G) 1994; Purvanchal 2006, Rajasthan 1995; Agra 1993; Indore 1993]**

Sol. The given equation can be re-written as

$$(a + x)\frac{dy}{dx} = y - ay \qquad \text{or} \qquad \frac{dx}{x+a} = \frac{dy}{y(1-ay)}$$

or $$\frac{dx}{x+a} = \left[\frac{a}{1-ay} + \frac{1}{y}\right] dy,$$ on resolving into partial fractions.

Integrating, $$\log(x+a) = -\log(1-ay) + \log y + \log c,$$

or $$\log(x+a) = \log\left[\frac{cy}{1-ay}\right] \qquad \text{or} \qquad x + a = \frac{cy}{1-ay}$$

or $(x+a)(1-ay) = cy$, which is the required solution.

Ex. 7. *Solve* $3e^x \tan y\, dx + (1 - e^x) \sec^2 y\, dy = 0$. **[Meerut 2008; Kanpur 1997]**

Sol. Separating the variables, we get $$\frac{3e^x}{1-e^x}dx + \frac{\sec^2 y}{\tan y}dy = 0.$$

Integrating, $-3\log(1-e^x) + \log(\tan y) = \log c$, c being an arbitrary constant.

or $$\log(\tan y) = \log(1-e^x)^3 + \log c \qquad \text{or} \qquad \tan y = c\,(1-e^x)^3.$$

Ex. 9. *Solve* $dy/dx = e^{x+y} + x^2 e^{x^3+y}$.

Sol. From given equation, we get $\qquad dy/dx = e^y\,(e^x + x^2 e^{x^3})$.

or $$e^{-y}\,dy = (e^x + x^2 e^{x^3})\,dx. \qquad \text{... (1)}$$

Integrating (1), $$\int e^{-y}\,dy = \int e^x\,dx + \int x^2 e^{x^3}\,dx$$

or $$-e^{-y} = e^x + (1/3)\int e^t\,dt + c, \text{ putting } x^3 = t$$

or $$-e^{-y} = e^x + (1/3)\,e^t + c = e^x + (1/3)\,e^{x^3} + c.$$

Ex. 10. *If* $dy/dx = e^{x+y}$ *and it is given that for* $x = 1$, $y = 1$; *find* y *when* $x = -1$.

Sol. Rewriting the given equation, we get $\qquad e^{-y}\,dy = e^x\,dx$.

Integrating it, $\qquad -e^{-y} = e^x + c. \qquad$ ... (1)

Putting $x = 1, y = 1$ in (1), $\qquad -e^{-1} = e + c \qquad$ so that $\qquad c = -e^{-1} - e$.

Hence (1) becomes $\qquad -e^{-y} = e^x - e^{-1} - e. \qquad$ ... (2)

Putting $x = -1$ in (2), we obtain $\qquad -e^{-y} = e^{-1} - e^{-1} - e \qquad$ so that $\qquad y = -1$.

Ex. 1. (a) *Solve* $dy/dx = (4x + y + 1)^2$. **[I.A.S. (Prel.) 2006]**

(b) $dy/dx = (4x + y + 1)^2$ *if* $y(0) = 1$. **[Delhi Maths (G) 2006]**

Sol. Let $4x + y + 1 = v$. ... (1)

Differentiating (1) with respect to x, we get

$4 + (dy/dx) = dv/dx$ or $dy/dx = (dv/dx) - 4$ ... (2)

Using (1) and (2), the given equation becomes

$(dv/dx) - 4 = v^2$ or $dv/dx = 4 + v^2$

Now, separating variables x and v, $dx = (dv) / (4 + v^2)$

Integrating, $x + c' = (1/2) \times \tan^{-1}(v/2)$, where c' is an arbitrary constant.

or $2x + c = \tan^{-1}(v/2)$ or $v = 2\tan(2x + c)$, where $c = 2c'$

or $4x + y + 1 = 2\tan(2x + c)$, using (1) ... (2)

(b) Putting $x = 0$, $y = 1$ in (2), we get $\tan c = 1$, so that $c = \pi/4$.

$\therefore$ Required solution is $4x + y + 1 = 2\tan(2x + \pi/4)$.

Ex. 2. *Solve* $(x + y)^2 (dy/dx) = a^2$. **[Meerut 1997; Indore 1998; I.A.S. (Prel.) 1994; Delhi Maths (G) 1997; Ravishankar 1992]**

Sol. Let $x + y = v$. ... (1)

Differentiating, $1 + (dy/dx) = dv/dx$ or $dy/dx = dv/dx - 1$ ... (2)

Using (1) and (2), the given equation becomes

$$v^2\left(\frac{dv}{dx} - 1\right) = a^2 \quad \text{or} \quad v^2\frac{dv}{dx} = a^2 + v^2$$

or $$dx = \frac{v^2}{v^2 + a^2}\,dv \quad \text{or} \quad dx = \left[1 - \frac{a^2}{a^2 + v^2}\right]dv.$$

Integrating, $x + c = v - a^2 \times (1/a) \times \tan^{-1}(v/a)$, where c is arbitrary constant

or $$x + c = x + y - a\tan^{-1}\left(\frac{x + y}{a}\right) \quad \text{or} \quad y - a\tan^{-1}\left(\frac{x + y}{a}\right) = c.$$

Ex. 3. *Solve* $dy/dx = \sec(x+y)$ **[Delhi Maths (P) 2005]**

or $\cos(x+y)\,dy = dx$. **[Kanpur 1992]**

Sol. Let $x + y = v$ so that $dy/dx = (dv/dx) - 1$.... (1)

Using (1), the given equation becomes

$$\frac{dv}{dx} - 1 = \sec v \quad \text{or} \quad \frac{dv}{dx} = 1 + \frac{1}{\cos v}$$

or $$dx = \frac{\cos v}{1+\cos v}\,dv = \frac{2\cos^2 \frac{1}{2}v - 1}{1 + 2\cos^2 \frac{1}{2}v - 1}\,dv \quad \text{or} \quad dx = (1 - \tfrac{1}{2}\sec^2 \tfrac{1}{2}v)\,dv.$$

Integrating, $x + c = v - \tan \frac{1}{2}v$ or $y - \tan \frac{1}{2}(x+y) = c$, by (1).

Ex. 4. *Solve* $dy/dx = \sin(x+y) + \cos(x+y)$. **[Guwahati 2007; Garhwal 1994]**

Sol. Let $x + y = v$ ... (1)

Differentiating (1) w.r.t 'x', $1 + \dfrac{dy}{dx} = \dfrac{dv}{dx}$ or $\dfrac{dy}{dx} = \dfrac{dv}{dx} - 1$... (2)

Using (1) and (2), the given equation becomes

$$\frac{dv}{dx} - 1 = \sin v + \cos v \quad \text{or} \quad \frac{dv}{dx} = 1 + \sin v + \cos v \text{.... (3)}$$

But $1 + \sin v + \cos v = 1 + 2\sin(v/2)\cos(v/2) + 2\cos^2(v/2) - 1 = 2\cos^2(v/2)\,[1 + \tan v/2]$.

$\therefore$ (3) reduces to $$dx = \frac{dv}{2\cos^2(v/2)\,[1+\tan(v/2)]} = \frac{\frac{1}{2}\sec^2(v/2)\,dv}{1+\tan(v/2)}.$$

Integrating, $x + c = \log[1 + \tan(v/2)]$, c being an arbitrary constant

or $x + c = \log[1 + \tan\{(x+y)/2\}]$, on using (1).

Ex. 5. *Solve* $(x+y)(dx-dy) = dx+dy$. **[Calcutta 1995]**

Sol. Re-writing the given equation, we get

$(x+y-1)\,dx = (x+y+1)\,dy$ or $\dfrac{dy}{dx} = \dfrac{x+y-1}{x+y+1}$. ... (1)

Let $x+y=v$. ... (2)

(2) $\Rightarrow$ $1+dy/dx = dv/dx$ so that $dy/dx = (dv/dx)-1$. ... (3)

Using (2) and (3), (1) becomes

$$\frac{dv}{dx}-1 = \frac{v-1}{v+1} \quad \text{or} \quad \frac{dv}{dx} = \frac{2v}{v+1} \quad \text{or} \quad 2dx = \left(1+\frac{1}{v}\right)dv.$$

$\therefore$ Integrating, $2x+c = v+\log v$ or $x-y+c = \log(x+y)$, by (2)

Ex. 6, 7.

3. $(2x+y+1)\,dx + (4x+2y-1)\,dy = 0$. **Ans.** $2y+x+\log(2x+y-1)=c$

4. $(x-y-2)\,dx-(2x-2y-3)\,dy=0$. **Ans.** $x-2y-\log(x-y-1)=c$