Application of partial Differential Equations

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In practical problems, the following types of equations are generally used :

(*i*) *Wave equation* : $$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Example 3. *Obtain the solution of the wave equation*

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

using the method of separation of variables.

Solution. $$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

Let $y = XT$ where X is a function of x only and T is a function of t only.

$$\frac{\partial y}{\partial t} = X \frac{dT}{dt} \qquad \text{and} \qquad \frac{\partial y}{\partial x} = T \frac{dX}{dx}$$

Since T and X are functions of a single variable only.

$$\frac{\partial^2 y}{\partial t^2} = X \frac{d^2 T}{dt^2} \qquad \text{and} \qquad \frac{\partial^2 y}{\partial x^2} = T \frac{d^2 X}{dx^2}$$

Substituting these values in the given equation, we get

$$X \frac{d^2 T}{d t^2} = c^2 T \frac{d^2 X}{d x^2}$$

By separating the variables, we get

$$\frac{\frac{d^2 T}{dt^2}}{c^2 T} = \frac{\frac{d^2 X}{dx^2}}{X} = k \qquad \text{(say)}.$$

(Each side is constant, since the variables x and y are independent).

$$\therefore \qquad \frac{d^2T}{dt^2} - k\,c^2T = 0 \qquad \text{and} \qquad \frac{d^2X}{dx^2} - kX = 0$$

Auxiliary equations are

$$m^2 - k\,c^2 = 0 \quad \text{or} \quad m = \pm c\sqrt{k} \qquad \text{and} \quad m^2 - k = 0 \quad \text{or} \quad m = \pm\sqrt{k}$$

Case 1. If k > 0.

$$T = C_1\, e^{c\sqrt{k}\,t} + C_2\, e^{-c\sqrt{k}\,t}$$

$$X = C_3\, e^{\sqrt{k}\,x} + C_4\, e^{-\sqrt{k}\,x}$$

Case 2. If k < 0.

$$T = C_5 \cos c\sqrt{k}\,t + C_6 \sin c\sqrt{k}\,t$$

$$X = C_7 \cos \sqrt{k}\,x + C_8 \sin \sqrt{k}\,x$$

Case 3. If k = 0.

$$T = C_9\, t + C_{10}$$

$$X = C_{11} x + C_{12}$$

These are the three cases depending upon the particular problems. Here we are dealing with wave motion ($k < 0$).

$$y = TX$$

$$y = (C_5 \cos c\sqrt{k}\,t + C_6 \sin c\sqrt{k}\,t) \times (C_7 \cos \sqrt{k}\,x + c_8 \sin \sqrt{k}\,x) \qquad \textbf{Ans.}$$

Example 4. *Find the solution of the wave equation*

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

such that $y = P_0 \cos pt$, *(*P_0 *is a constant) when* $x = l$ *and* $y = 0$ *when* $x = 0$.

(A.M.I.E., Winter 1996)

Solution. $$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \qquad ...(1)$$

Its solution is as given in Example 3 on page 710.

$$y = (c_1 \cos c\sqrt{k}t + c_2 \sin c\sqrt{k}t)(c_3 \cos \sqrt{k}x + c_4 \sin \sqrt{k}x) \qquad ...(2)$$

Put $y = 0$, when $x = 0$

$$0 = (c_1 \cos c\sqrt{k}t + c_2 \sin c\sqrt{k}t)\, c_3 \qquad \Rightarrow c_3 = 0$$

(2) is reduced to

$$y = (c_1 \cos c\sqrt{k}t + c_2 \sin c\sqrt{k}t)\, c_4 \sin \sqrt{k}x$$

$$y = c_1 c_4 \cos c\sqrt{k}t \sin \sqrt{k}x + c_2 c_4 \sin c\sqrt{k}t \sin \sqrt{k}\, x \qquad ...(3)$$

Put $y = P_0 \cos pt$ when $x = l$

$$P_0 \cos pt = c_1 c_4 \cos c\sqrt{k}t \sin \sqrt{k}l + c_2 c_4 \sin c\sqrt{k}\, t \sin \sqrt{k}\, l$$

Equating the coefficients of sin and cos on both sides

$$P_o = c_1 c_4 \sin\sqrt{k}\, l, \quad \Rightarrow \quad c_1 c_4 = \frac{P_o}{\sin\sqrt{k}\, l}$$

$$0 = c_2 c_4 \sin\sqrt{k}\, l \quad \Rightarrow \quad c_2 = 0$$

And $p = c\sqrt{k}$ $\quad \Rightarrow \quad \frac{p}{c} = \sqrt{k}$

(3) becomes $\quad y = \frac{P_o}{\sin\sqrt{k}\, l} \cos pt \sin \frac{p}{c} x$

$$y = \frac{P_o}{\sin \frac{p}{c} l} \cos pt \sin \frac{p}{c} x$$ **Ans.**

Example 5. *A string is stretched and fastened to two points l apart. Motion is started by displacing the string in the form* $y = a \sin \frac{\pi x}{l}$ *from which it is released at a time* $t = 0$*. Show that the displacement of any point at a distance x from one end at time t is given by*

$$y(x, t) = a \sin\left(\frac{\pi x}{l}\right) \cos\left(\frac{\pi c t}{l}\right)$$ *(A.M.I.E.T.E., Winter 2003, A.M.I.E., Winter 2001)*

Solution. The vibration of the string is given by :

$$\frac{\partial^2 y}{\partial t^2} = C^2 \frac{\partial^2 y}{\partial x^2} \qquad ...(1)$$

As the end points of the string are fixed, for all time,

$$y(0, t) = 0 \qquad ...(2)$$

and

$$y(l, t) = 0 \qquad ...(3)$$

Since the initial transverse velocity of any point of the string is zero, therefore,

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \qquad ...(4)$$

Also

$$y(x, 0) = a \sin \frac{\pi x}{l} \qquad ...(5)$$

Now we have to solve (1), subject to the above boundary conditions. Since the vibration of the string is periodic, thereore, the solution of (1) is of the form

$$y(x,t) = (C_1 \cos px + C_2 \sin px)(C_3 \cos Cpt + C_4 \sin Cpt) \qquad \text{...(6)}$$

By (2) $\quad y(0,t) = C_1 (C_3 \cos Cpt + C_4 \sin Cpt) = 0$

For this to be true for all time, $C_1 = 0$.

Hence $\quad y(x,t) = C_2 \sin px (C_3 \cos Cpt + C_4 \sin Cpt) \qquad \text{...(7)}$

and $\quad \dfrac{\partial y}{\partial t} = C_2 \sin px [C_3(-Cp \sin Cpt) + C_4 (Cp \cos Cpt)]$

$\therefore$ By (4) $\quad \left(\dfrac{\partial y}{\partial t}\right)_{t=0} = C_2 \sin px (C_4 Cp) = 0$

Whence $\quad C_2 C_4 C p = 0$

If $C_2 = 0$, (7) will lead to the trivial solution $y(x, t) = 0$.

$\therefore$ the only possibility is that $C_4 = 0$

Thus (7) becomes

$$y(x,t) = C_2 C_3 \sin px \cos Cpt \qquad \text{... (8)}$$

If $x = l$ then $y = 0$, $0 = C_2 C_3 \sin pl \cos Cpt$, for all t.

Since C_2 and $C_3 \neq 0$, we have $\sin pl = 0 \;\therefore\; pl = n\pi$

i.e. $\quad p = \dfrac{n\pi}{l}$, where n is an integer.

Hence (8) reduces to

$$y(x,t) = C_2 C_3 \sin \frac{n\pi x}{l} \cos \frac{n\pi Ct}{l} \qquad \text{... (9)}$$

Finally imposing the last condition (5), we have

$$y(x, 0) = C_2 C_3 \sin \frac{n \pi x}{l} = a \sin \frac{\pi x}{l}$$

which will be satisfied by taking $C_2 C_3 = a$ and $n = 1$

Hence the required solution is

$$y(x, t) = a \sin \frac{\pi x}{l} \cos \frac{\pi C t}{l}$$ **Proved**