Complex Numbers

1. **Definition:** A number of the form $a+ib$ where a and b are real numbers and $i=\sqrt{-1}$, is called a complex number.

If $z=a+ib$, then a is called the real part of z and b is called the imaginary part of z.

2. **Modulus and Amplitude:**

If the polar coordinates of the point (a,b) be (r,θ), then $a=r\cos\theta$ and $b=r\sin\theta$

$\therefore$ $r=\sqrt{a^2+b^2}$ and $\theta=\tan^{-1}\left(\frac{b}{a}\right)$

The number r is called the modulus or absolute value and θ is called the amplitude or argument of the complex number $z=a+ib$.

In symbols, we write

$$r=\text{mod}\,z=|z|=|a+ib|=\sqrt{a^2+b^2}$$

$$\theta=\text{amp}\,z=\arg z=\tan^{-1}\left(\frac{b}{a}\right)$$

Now, $z=a+ib=r(\cos\theta+i\sin\theta)=re^{i\theta}$

3. State and Prove De Moivre's theorem.

Statement: For all rational values of n,

$$(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta.$$

Proof: Case-1: When n is a positive integer.

We have, $(\cos\theta_1+i\sin\theta_1)(\cos\theta_2+i\sin\theta_2)$

$$=\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2+i(\sin\theta_1\cos\theta_2+\cos\theta_1\sin\theta_2)$$

$$=\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)$$

Similarly, $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)$

$$= \{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\}(\cos\theta_3 + i\sin\theta_3)$$

$$= \cos(\theta_1+\theta_2+\theta_3) + i\sin(\theta_1+\theta_2+\theta_3)$$

Proceeding in this way, the product of the n factors

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \cdots\cdots (\cos\theta_n + i\sin\theta_n)$$

$$= \cos(\theta_1+\theta_2+\cdots+\theta_n) + i\sin(\theta_1+\theta_2+\cdots+\theta_n)$$

If we put $\theta_1 = \theta_2 = \cdots = \theta_n = \theta$, then we have

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Case-2: When n is a negative integer.

Let $n = -m$, where m is a positive integer.

$$(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m}$$

$$= \frac{1}{(\cos\theta + i\sin\theta)^m}$$

$$= \frac{1}{\cos m\theta + i\sin m\theta} \quad [\text{by Case-I}]$$

$$= \frac{1 \times (\cos m\theta - i\sin m\theta)}{(\cos m\theta + i\sin m\theta)(\cos m\theta - i\sin m\theta)}$$

$$= \frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta}$$

$$= \cos m\theta - i\sin m\theta$$

$$= \cos(-n)\theta - i\sin(-n)\theta \quad [\because m = -n]$$

$$= \cos n\theta + i\sin n\theta$$

Case-3: When n is a fraction, +ve or -ve.

Let $n = \frac{p}{q}$, where q is a positive integer and p is any integer, +ve or -ve.

Now, $\left(\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta\right)^q = \cos\left(q\cdot\frac{p}{q}\theta\right) + i\sin\left(q\cdot\frac{p}{q}\theta\right)$

[by case-1]

$= \cos p\theta + i\sin p\theta$

$= (\cos\theta + i\sin\theta)^p$, since p is any integer.

Taking qth root, we get

$\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta$ = one of the values of $(\cos\theta + i\sin\theta)^{\frac{p}{q}}$

or, $\cos n\theta + i\sin n\theta$ = one of the values of $(\cos\theta + i\sin\theta)^n$

Thus, De Moivre's theorem is completely established for all rational values of n.

4. If $x_r = \cos\frac{\pi}{2^r} + i\sin\frac{\pi}{2^r}$, prove that

$$x_1 x_2 x_3 \cdots\cdots \infty = -1$$

<u>Solution:</u> LHS $= x_1 x_2 x_3 \cdots\cdots \infty$

$= \left(\cos\frac{\pi}{2^1} + i\sin\frac{\pi}{2^1}\right)\left(\cos\frac{\pi}{2^2} + i\sin\frac{\pi}{2^2}\right)\left(\cos\frac{\pi}{2^3} + i\sin\frac{\pi}{2^3}\right)\cdots\infty$

$= \cos\left(\frac{\pi}{2^1} + \frac{\pi}{2^2} + \frac{\pi}{2^3} + \cdots\infty\right) + i\sin\left(\frac{\pi}{2^1} + \frac{\pi}{2^2} + \frac{\pi}{2^3} + \cdots\infty\right)$

$= \cos\left(\frac{\pi/2}{1-1/2}\right) + i\sin\left(\frac{\pi/2}{1-1/2}\right)$ $\left[\because S_\infty = \frac{a}{1-r}\right]$

$= \cos\pi + i\sin\pi$

$= -1 + i\cdot 0$

$= -1$

$=$ R.H.S

5. If $x_r = \cos\frac{\pi}{3^r} + i\sin\frac{\pi}{3^r}$, show that

$$x_1 x_2 x_3 \cdots\cdots \infty = i$$

<u>Solution:</u> LHS $= x_1 x_2 x_3 \cdots\cdots \infty$

$= \left(\cos\frac{\pi}{3^1} + i\sin\frac{\pi}{3^1}\right)\left(\cos\frac{\pi}{3^2} + i\sin\frac{\pi}{3^2}\right)\cdots\cdots\infty$

$= \cos\left(\frac{\pi}{3^1} + \frac{\pi}{3^2} + \frac{\pi}{3^3} + \cdots\infty\right) + i\sin\left(\frac{\pi}{3^1} + \frac{\pi}{3^2} + \frac{\pi}{3^3} + \cdots\infty\right)$

$\therefore$ LHS $= \cos\left(\frac{\pi/3}{1-1/3}\right) + i\sin\left(\frac{\pi/3}{1-1/3}\right)$

$= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

$= 0 + i\cdot 1$

$= i$

$=$ R.H.S

6. If n be a positive integer, prove that

(i) $(1+i)^n + (1-i)^n = 2^{\frac{n}{2}+1}\cos\frac{n\pi}{4}$

(ii) $(\sqrt{3}+i)^n + (\sqrt{3}-i)^n = 2^{n+1}\cos\frac{n\pi}{6}$

Solution: (i) Let $1 = r\cos\theta \cdots (1)$

$1 = r\sin\theta \cdots (2)$

Squaring (1) and (2), then adding

$2 = r^2(\cos^2\theta + \sin^2\theta)$

$= r^2\cdot 1$

$\therefore r = \sqrt{2}$

Dividing (2) by (1), we get $\tan\theta = 1$

or, $\tan\theta = \tan\frac{\pi}{4}$

$\therefore \theta = \frac{\pi}{4}$

Now, $1+i = 1+i\cdot 1$

$= r\cos\theta + ir\sin\theta$

$= r(\cos\theta + i\sin\theta)$

$= \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$\therefore (1+i)^n = \left\{\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right\}^n$

$= 2^{n/2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^n$

$= 2^{n/2}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right) \cdots\cdots (3)$

Similarly, $(1-i)^n = 2^{n/2}\left(\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right) \cdots\cdots (4)$

Adding (3) and (4), we get

$(1+i)^n + (1-i)^n = 2\cdot 2^{n/2}\cos\frac{n\pi}{4}$

$= 2^{n/2+1}\cos\frac{n\pi}{4}$

(ii) Let $\sqrt{3} = r\cos\theta$ ... (1)

$1 = r\sin\theta$ .... (2)

Squaring (1) and (2), then adding

$$3+1 = r^2(\cos^2\theta + \sin^2\theta)$$

or, $4 = r^2 \cdot 1$

$\therefore r = 2$

Dividing (2) by (1), $\tan\theta = \frac{1}{\sqrt{3}}$

or, $\tan\theta = \tan\frac{\pi}{6}$

$\therefore \theta = \frac{\pi}{6}$

Now, $\sqrt{3}+i = \sqrt{3}+i\cdot 1$

$= r\cos\theta + i\, r\sin\theta$

$= r(\cos\theta + i\sin\theta)$

$= 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$\therefore (\sqrt{3}+i)^n = \left\{2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right\}^n$

$= 2^n\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^n$

$= 2^n\left(\cos\frac{n\pi}{6} + i\sin\frac{n\pi}{6}\right)$ ...(3)

Similarly, $(\sqrt{3}-i)^n = 2^n\left(\cos\frac{n\pi}{6} - i\sin\frac{n\pi}{6}\right)$ --(4)

Adding (3) and (4), we get

$(\sqrt{3}+i)^n + (\sqrt{3}-i)^n = 2\cdot 2^n\cos\frac{n\pi}{6}$

$= 2^{n+1}\cos\frac{n\pi}{6}$

7. Determine the locus represented by

(i) $|z-2| = 3$ (ii) $|z-2| = |z+4|$ (iii) $|z-3|+|z+3| = 10$

Solution: (i) $|z-2| = 3$

or, $|x+iy-2| = 3$

or, $|(x-2)+iy| = 3$

or $\sqrt{(x-2)^2+y^2} = 3$

or, $(x-2)^2 + y^2 = 3^2$ which represents a circle of radius 3 and centre (2,0).

(ii) $|z-2| = |z+4|$

or, $|x+iy-2| = |x+iy+4|$

or, $(x-2)^2 + y^2 = (x+4)^2 + y^2$

or, $x = -1$ which represents a straight line.

(iii) $|z-3| + |z+3| = 10$

or, $\sqrt{(x-3)^2+y^2} + \sqrt{(x+3)^2+y^2} = 10$

or, $\sqrt{(x-3)^2+y^2} = 10 - \sqrt{(x+3)^2+y^2}$

or, $(x-3)^2 + y^2 = 100 - 20\sqrt{(x+3)^2+y^2} + (x+3)^2 + y^2$

or, $12x + 100 = 20\sqrt{(x+3)^2+y^2}$

or, $3x + 25 = 5\sqrt{(x+3)^2+y^2}$

or, $9x^2 + 150x + 625 = 25(x^2+6x+9+y^2)$

or, $16x^2 + 25y^2 = 400$

or, $\frac{x^2}{25} + \frac{y^2}{16} = 1$

or, $\frac{x^2}{5^2} + \frac{y^2}{4^2} = 1$ which represents an ellipse.

8. If $x + \frac{1}{x} = 2\cos\theta$, show that $x^n + \frac{1}{x^n} = 2\cos n\theta$

<u>Solution:</u> We have, $x + \frac{1}{x} = 2\cos\theta$

or, $x^2 - 2x\cos\theta + 1 = 0$

$\therefore x = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$

$= \cos\theta \pm i\sin\theta$

Take +ve sign only, $x = \cos\theta + i\sin\theta$

Now LHS $= x^n + \frac{1}{x^n}$

$= (\cos\theta + i\sin\theta)^n + (\cos\theta + i\sin\theta)^{-n}$

$= \cos n\theta + i\sin n\theta + \cos(-n)\theta + i\sin(-n)\theta$ [By De Moivre's theorem]

$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$= 2\cos n\theta$

$=$ RHS

9. If $x = \cos\theta + i\sin\theta$ and $1+\sqrt{1-a^2} = na$, prove that

$$1 + a\cos\theta = \frac{a}{2n}(1+nx)\left(1+\frac{n}{x}\right)$$

Solution: Given, $x = \cos\theta + i\sin\theta$

$$\therefore \frac{1}{x} = \frac{1}{\cos\theta + i\sin\theta}$$

$$= (\cos\theta + i\sin\theta)^{-1}$$

$$= \cos\theta - i\sin\theta$$

$$\therefore x + \frac{1}{x} = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta$$

$$= 2\cos\theta$$

Also, $1+\sqrt{1-a^2} = na$

or, $\sqrt{1-a^2} = na - 1$

or, $1 - a^2 = n^2a^2 - 2na + 1$

or, $n^2a^2 + a^2 = 2na$

or, $a^2(1+n^2) = 2na$

or, $\dfrac{a^2(1+n^2)}{2na} = 1$

$\therefore \dfrac{a(1+n^2)}{2n} = 1$

Now, LHS $= 1 + a\cos\theta$

$$= \frac{a(1+n^2)}{2n} + a \cdot \frac{x+\frac{1}{x}}{2}$$

$$= \frac{a}{2n}\left\{(1+n^2) + n\left(x+\frac{1}{x}\right)\right\}$$

$$= \frac{a}{2n}\left(1+n^2 + nx + \frac{n}{x}\right)$$

$$= \frac{a}{2n}\left\{1\cdot(1+nx) + \frac{n}{x}(1+nx)\right\}$$

$$= \frac{a}{2n}(1+nx)\left(1+\frac{n}{x}\right)$$

$=$ R.H.S

Important results: (i) $1 = e^{i2n\pi} = \cos 2n\pi + i\sin 2n\pi$

(ii) $-1 = \cos\pi + i\sin\pi = e^{i\pi}$

(iii) $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\pi/2}$

(iv) $-i = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2} = e^{-i\pi/2}$

10. If α, β be the roots of $x^2-2x+4=0$, prove that $\alpha^n+\beta^n = 2^{n+1}\cos\frac{n\pi}{3}$

Solution: Given, $x^2-2x+4=0$

$$\therefore x = \frac{-(-2)\pm\sqrt{(-2)^2-4\cdot 1\cdot 4}}{2\cdot 1}$$
$$= \frac{2\pm\sqrt{-12}}{2}$$
$$= 1\pm\sqrt{-3}$$
$$= 1\pm i\sqrt{3}$$

$\therefore$ $\alpha = 1+i\sqrt{3}$
$$= 2\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$$
$$= 2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$$

and $\beta = 1-i\sqrt{3}$
$$= 2\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)$$
$$= 2\left(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}\right)$$

$$\therefore \alpha^n = 2^n\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^n$$
$$= 2^n\left(\cos\frac{n\pi}{3}+i\sin\frac{n\pi}{3}\right)\cdots(1)$$

and $\beta^n = 2^n\left(\cos\frac{n\pi}{3}-i\sin\frac{n\pi}{3}\right)\cdots(2)$

Adding (1) and (2) we get
$$\alpha^n+\beta^n = 2^n\cdot 2\cos\frac{n\pi}{3}$$
$$= 2^{n+1}\cos\frac{n\pi}{3}$$

11. Show that $\sin(\ln i^i) = -1$

Solution: $i = e^{\frac{i\pi}{2}}\cdot e^{i2n\pi}$ $\quad [\because 1 = e^{i2n\pi}]$
$$= e^{\frac{i\pi}{2}(4n+1)}$$
$$\therefore i^i = e^{-(4n+1)\frac{\pi}{2}}$$
$$\therefore \ln i^i = -(4n+1)\frac{\pi}{2}$$

Now, $\sin(\ln i^i) = \sin\left\{-(4n+1)\frac{\pi}{2}\right\}$
$$= -\sin\left(2n\pi+\frac{\pi}{2}\right)$$
$$= -\sin\frac{\pi}{2}$$
$$= -1$$

12. If $x=\cos\alpha+i\sin\alpha$, $y=\cos\beta+i\sin\beta$, $z=\cos\gamma+i\sin\gamma$ and $x+y+z=0$, then prove that

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=0.$$

Solution: We have, $x+y+z=0$

$$\text{or, } \cos\alpha+i\sin\alpha+\cos\beta+i\sin\beta+\cos\gamma+i\sin\gamma=0$$

$$\text{or, } (\cos\alpha+\cos\beta+\cos\gamma)+i(\sin\alpha+\sin\beta+\sin\gamma)=0$$

Equating real and imaginary parts, we get

$$\cos\alpha+\cos\beta+\cos\gamma=0 \quad \cdots(1)$$

and $\sin\alpha+\sin\beta+\sin\gamma=0 \quad \cdots(2)$

Now LHS $=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$

$$=(\cos\alpha+i\sin\alpha)^{-1}+(\cos\beta+i\sin\beta)^{-1}+(\cos\gamma+i\sin\gamma)^{-1}$$

$$=\cos\alpha-i\sin\alpha+\cos\beta-i\sin\beta+\cos\gamma-i\sin\gamma$$

$$=(\cos\alpha+\cos\beta+\cos\gamma)-i(\sin\alpha+\sin\beta+\sin\gamma)$$

$$=0-i.0$$

$$=0$$

13. Using De Moivre's theorem, solve the equations:

(i) $x^9=1$ (ii) $(x+1)^5+(x-1)^5=0$ (iii) $x^4+x^2+1=0$

Solution: (i) $x^9=1$

$$\text{or, } x^9=\cos 2n\pi+i\sin 2n\pi \quad [\because 1=e^{i2n\pi}=\cos 2n\pi+i\sin 2n\pi]$$

$$\text{or, } x=(\cos 2n\pi+i\sin 2n\pi)^{\frac{1}{9}}$$

$$\text{or, } x=\cos\frac{2n\pi}{9}+i\sin\frac{2n\pi}{9}$$

Putting $n=0,1,2,3,4,5,6,7,8$, the required solutions are

$\cos 0+i\sin 0$, $\cos\frac{2\pi}{9}+i\sin\frac{2\pi}{9}$, $\cos\frac{4\pi}{9}+i\sin\frac{4\pi}{9}$,
$\cos\frac{6\pi}{9}+i\sin\frac{6\pi}{9}$, $\cos\frac{8\pi}{9}+i\sin\frac{8\pi}{9}$, $\cos\frac{10\pi}{9}+i\sin\frac{10\pi}{9}$,
$\cos\frac{12\pi}{9}+i\sin\frac{12\pi}{9}$, $\cos\frac{14\pi}{9}+i\sin\frac{14\pi}{9}$, $\cos\frac{16\pi}{9}+i\sin\frac{16\pi}{9}$

(ii) $(x+1)^5+(x-1)^5=0$

or, $(x+1)^5=-(x-1)^5$

or, $\left(\frac{x+1}{x-1}\right)^5=-1$

or, $\frac{x+1}{x-1}=(-1)^{\frac{1}{5}}$

or, $\frac{x+1}{x-1}=\left(e^{i\pi}\cdot e^{i2n\pi}\right)^{\frac{1}{5}}$

or, $\frac{x+1}{x-1}=\left\{e^{i(2n\pi+\pi)}\right\}^{\frac{1}{5}}$

or, $\frac{x+1}{x-1}=\left\{\cos(2n\pi+\pi)+i\sin(2n\pi+\pi)\right\}^{\frac{1}{5}}$

or, $\frac{x+1}{x-1}=\cos\left(\frac{2n\pi+\pi}{5}\right)+i\sin\left(\frac{2n\pi+\pi}{5}\right)$

or, $\frac{x+1}{x-1}=\frac{\cos\theta+i\sin\theta}{1}$ where $\theta=\frac{2n\pi+\pi}{5}$

Using Componendo and dividendo, we get

$$\frac{x+1+x-1}{x+1-x+1}=\frac{\cos\theta+i\sin\theta+1}{\cos\theta+i\sin\theta-1}$$

or, $\frac{2x}{2}=-\frac{2\cos^2\theta/2+i\cdot 2\sin\theta/2\cos\theta/2}{2\sin^2\theta/2-i\cdot 2\sin\theta/2\cos\theta/2}$

or, $x=-\frac{2\cos\theta/2(\cos\theta/2+i\sin\theta/2)}{2\sin\theta/2(\sin\theta/2-i\cos\theta/2)}$

or, $x=-i\cot\theta/2\left[\frac{\cos\theta/2+i\sin\theta/2}{i\sin\theta/2+\cos\theta/2}\right]$

or, $x=-i\cot\theta/2$

$\therefore x=-i\cot(2n+1)\frac{\pi}{10}$.

Putting $n=0,1,2,3,4$, the required solutions are

$-i\cot\frac{\pi}{10}$, $-i\cot\frac{3\pi}{10}$, $-i\cot\frac{\pi}{2}$, $-i\cot\frac{7\pi}{10}$, $-i\cot\frac{9\pi}{10}$.

(iii) Given, $x^4+x^2+1=0$

or, $(x^2-1)(x^4+x^2+1)=0$ [Multiplying bothsides by (x^2-1)]

or, $(x^2)^3-(1)^3=0$

or, $x^6-1=0$ or, $x^6=1$

or, $x^6=\cos 2n\pi + i\sin 2n\pi$ [$\because 1=\cos 2n\pi + i\sin 2n\pi$]

or, $x=(\cos 2n\pi + i\sin 2n\pi)^{\frac{1}{6}}$

or, $x=\cos\frac{2n\pi}{6}+i\sin\frac{2n\pi}{6}$

$\therefore x=\cos\frac{n\pi}{3}+i\sin\frac{n\pi}{3}$ .. (1)

Putting $n=0,1,2,3,4,5$ in (1), we get

$x=1$, $\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$, $\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$, -1, $\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}$, $\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}$

Among these values $x=\pm 1$ will be omitted as we have multiplied the equation by x^2-1. Hence the four roots of the given equation are

$x=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$, $\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$, $\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}$, $\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}$.

14. Find all the values of (i) $(1+i)^{\frac{1}{5}}$ (ii) $(-i)^{\frac{1}{6}}$

Solution: Let us put $1=r\cos\theta$ and $1=r\sin\theta$

$\therefore r=\sqrt{2}$ and $\theta=\frac{\pi}{4}$

Now, $(1+i)^{\frac{1}{5}}=(1+i.1)^{\frac{1}{5}}$

$=(r\cos\theta+ir\sin\theta)^{\frac{1}{5}}$

$=\{r(\cos\theta+i\sin\theta)\}^{\frac{1}{5}}$

$=\{r.e^{i\theta}.e^{i2n\pi}\}^{\frac{1}{5}}$

$\therefore \quad (1+i)^{\frac{1}{5}} = \left\{ r.\, e^{i(2n\pi+\theta)} \right\}^{\frac{1}{5}}$

$= \left\{ \sqrt{2}\, e^{i\left(2n\pi+\frac{\pi}{4}\right)} \right\}^{\frac{1}{5}}$

$= 2^{\frac{1}{10}} \left\{ \cos\left(2n\pi+\frac{\pi}{4}\right) + i\sin\left(2n\pi+\frac{\pi}{4}\right) \right\}^{\frac{1}{5}}$

$= 2^{\frac{1}{10}} \left\{ \cos(8n+1)\frac{\pi}{20} + i\sin(8n+1)\frac{\pi}{20} \right\}$

Putting $n = 0, 1, 2, 3, 4$, the required values are

$2^{\frac{1}{10}}\left(\cos\frac{\pi}{20} + i\sin\frac{\pi}{20}\right)$, $2^{\frac{1}{10}}\left(\cos\frac{9\pi}{20} + i\sin\frac{9\pi}{20}\right)$,

$2^{\frac{1}{10}}\left(\cos\frac{17\pi}{20} + i\sin\frac{17\pi}{20}\right)$, $2^{\frac{1}{10}}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$,

$2^{\frac{1}{10}}\left(\cos\frac{33\pi}{20} + i\sin\frac{33\pi}{20}\right)$

(ii) $\quad (-i)^{\frac{1}{6}} = \left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right)^{\frac{1}{6}}$

$= \left(e^{-i\frac{\pi}{2}}\right)^{\frac{1}{6}}$

$= \left(e^{-\frac{i\pi}{2}} \cdot e^{i2n\pi}\right)^{\frac{1}{6}} \quad \left[\because 1 = e^{i2n\pi}\right]$

$= \left\{ e^{i(4n-1)\frac{\pi}{2}} \right\}^{\frac{1}{6}}$

$= \left\{ \cos(4n-1)\frac{\pi}{2} + i\sin(4n-1)\frac{\pi}{2} \right\}^{\frac{1}{6}}$

$= \cos(4n-1)\frac{\pi}{12} + i\sin(4n-1)\frac{\pi}{12}$

Putting $n = 0, 1, 2, 3, 4, 5$, the required values are

$\cos\frac{\pi}{12} - i\sin\frac{\pi}{12}$, $\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$, $\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}$,

$\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}$, $\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}$, $\cos\frac{19\pi}{12} + i\sin\frac{19\pi}{12}$