

Math-183

Differential Equations

Note taken by: Turja Roy
ID: 2108052

Contents

1	Differential Equations and Their Solutions	2
1.1	Classification of Differential Equations	2
1.2	Solutions	3
A	Nature of Solutions	3
B	Methods of Solution	4
1.3	Initial-Value and Boundary-Value Problems, and Existence of Solutions	5
A	Initial-value Problems and Boundary-value Problems	5
B	Existence of Solutions	6
2	First Order Equations for Which Exact Solutions Are Obtainable	9
2.1	Exact Differential Equations and Integrating Factors	9
A	Standard Forms of First-Order Differential Equations	9
B	Exact Differential Equations	9
C	The Solution of Exact Differential Equations	11
D	Integrating Factors	12
2.2	Separable Equations and Equations Reducible to this Form	14
A	Separable Equations	14
B	Homogeneous Equations	15
2.3	Linear Equations and Bernoulli Equations	24
A	Linear Equation	24
B	Bernoulli Equations	27
2.4	Special Integrating Factors and Transformations	29
A	Finding Integrating Factors	30
B	A Special Transformation	32
3	Explicit Methods of Solving Higher-Order Linear Differential Equations	34
3.1	Basic Theory of Linear Differential Equations	34
A	Definition and Basic Existence Theorem	34
B	The Homogeneous Equation	36

1 Differential Equations and Their Solutions

1.1 Classification of Differential Equations

Definition 1.1.1: Differential Equation

Differential equation is an equation involving derivatives of one or more dependent variables with respect to one or more independent variables.

Definition 1.1.2: Ordinary Differential Equation

A differential equation involving ordinary derivatives of one or more dependent variables with respect to a single independent variable is called an ordinary differential equation.

Example 1.1.1: Ordinary Differential Equations:

$$\frac{dy}{dx} + xy \left(\frac{d}{dx} \right)^2 = 0 \quad (1.1.1)$$

$$\frac{d^4x}{dt^4} + 5\frac{d^2x}{dt^2} + 3x = \sin t \quad (1.1.2)$$

Definition 1.1.3: Partial Differential Equation

A differential equation involving partial derivatives of one or more dependent variables with respect to more than one independent variables is called a partial differential equation.

Example 1.1.2: Partial Differential Equations:

$$\frac{\partial v}{\partial s} + \frac{\partial v}{\partial t} = v \quad (1.1.3)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad (1.1.4)$$

Definition 1.1.4: Order and Degree of Differential Equations

Order of DE: The order of the highest ordered derivative involved in a differential equation is called the order of the differential equation.

Degree of DE: The power of the highest order derivative involved in a differential equation is called the degree of the differential equation.

Definition 1.1.5: Linearity of Differential Equations

If the dependent variable and its various derivatives occur to the first degree only, the DE is a linear DE. Otherwise it's a non-linear DE.

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = b(x)$$

Linear DE can also be classified as linear with *constant* and *variable* coefficients.

Example 1.1.3: Ordinary Differential Equations: Orders, Degree, Linearity

$$\frac{d^3 y}{dx^3} - 3\frac{d^2 y}{dx^2} + 3\frac{dy}{dx} - 6y = \sin x \quad \text{3rd ord 1st deg Lin}$$

$$\frac{d^2 y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + y = 0 \quad \text{2nd ord 1st deg Non-Lin}$$

$$y = x\frac{dy}{dx} + \sqrt{1 + \frac{d^2 y}{dx^2}} \quad \text{2nd ord 1st deg Non-Lin}$$

$$\frac{d^4 x}{dt^4} + t^2\frac{d^3 x}{dt^3} + \frac{dy}{dx} = \sin t \quad \text{4th ord 1st deg Lin}$$

$$\frac{d^2 y}{dx^2} + 5\frac{dy}{dx} + 6y^2 = 0 \quad \text{2nd ord 1st deg Non-Lin}$$

$$\frac{d^2 y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 + 6y = 0 \quad \text{2nd ord 1st deg Non-Lin}$$

$$\frac{d^2 y}{dx^2} + 5y\frac{dy}{dx} + 6y = 0 \quad \text{2nd ord 1st deg Lin}$$

1.2 Solutions

A Nature of Solutions

An nth-order Differential Equation:

$$F\left[x, y, \frac{dy}{dx}, \dots, \frac{d^n y}{dx^n}\right] = 0 \quad (1.2.1)$$

Definition 1.2.1: Explicit solution

f is an explicit solution of (1.2.1) if

$$\forall x \in I, F\left[x, f(x), f'(x), \dots, f^{(n)}(x)\right] = 0$$

where I is a real interval.

Definition 1.2.2: Implicit solution

$g(x, y) = 0$ is an implicit solution if this relation defines at least one real function $f(x)$ on an interval I such that f is an explicit solution of (1.2.1)

Example 1.2.1: Explicit and Implicit Solutions

$$x^2 + y^2 - 25 = 0 \quad : \quad \text{Implicit solution}$$

$$2x + 2y \frac{dy}{dx} = 0$$

$$x + y \frac{dy}{dx} = 0 \quad : \quad \text{Differential Equation}$$

$$y = \pm \sqrt{25 - x^2} ; -5 \leq x \leq 5 \quad : \quad \text{Explicit solution}$$

B Methods of Solution

The study of a Differential Equation consists of 3 phases:

1. Formulation of DE from the given physical situation.
2. Solutions of DE, evaluating the arbitrary constants from the given condition.
3. Physical interpretation of the solution.

Example 1.2.2: Show that the function $f(x) = e^x + 2x^2 + 6x + 7$ is a solution to the DE $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 4x^2$

$$f(x) = e^x + 2x^2 + 6x + 7$$

$$f'(x) = e^x + 4x + 6$$

$$f''(x) = e^x + 4$$

$$\begin{aligned} \frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y &= (e^x + 4) - 3(e^x + 4x + 6) + 2(e^x + 2x^2 + 6x + 7) \\ &= 0 \cdot e^x + 0 \cdot x + (4 - 18 + 14) + 4x^2 \\ &= 4x^2 \end{aligned}$$

□

Example 1.2.3: Show that the function $f(x) = \frac{1}{1+x^2}$ is a solution to the DE $(1+x^2)\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = 0$

$$f(x) = \frac{1}{1+x^2}$$

$$(1+x^2)f(x) = 1$$

$$(1+x^2)f'(x) + 2xf(x) = 0$$

$$(1+x^2)f''(x) + 4xf'(x) + 2f(x) = 0$$

$$(1+x^2)\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = 0$$

□

Example 1.2.4: Show that the function $y = (2x^2 + 2e^{3x} + 3)e^{-2x}$ satisfies the DE

$$\frac{dy}{dx} + 2y = 6e^x + 4xe^{-2x}$$

$$\begin{aligned} y &= (2x^2 + 2e^{3x} + 3)e^{-2x} \\ y_1 &= (4x + 6e^{3x})e^{-2x} - (2x^2 + 2e^{3x} + 3)2e^{-2x} \\ y_1 &= 4xe^{-2x} + 6e^x - 2y \\ \frac{dy}{dx} + 2y &= 6e^x + 4e^{-2x} \end{aligned}$$

□

1.3 Initial-Value and Boundary-Value Problems, and Existence of Solutions

A Initial-value Problems and Boundary-value Problems

One of the most frequently encountered type of problems in Differential Equations involves both a DE and one or more supplementary conditions which the solution of the given DE must satisfy.

Definition 1.3.1: IVP and BVP

Consider the first-order DE

$$\frac{dy}{dx} = f(x, y)$$

where f is a continuous function of x and y in some domain D of the xy plane; and let (x_0, y_0) be a point of D . The **initial-value problem** associated with the DE is to find a solution ϕ of the DE, defined on some real interval containing x_0 , and satisfying the initial condition

$$\phi(x_0) = y_0$$

In the customary abbreviated notation, this initial-value problem may be written

$$\frac{dy}{dx} = f(x, y)$$

$$y(x_0) = y_0$$

If the conditions relate to two different x values (the extreme or boundary values), the problem is called a **Two-Point Boundary-Value Problem** or simply a **Boundary-Value Problem (BVP)**.

Example 1.3.1: Find the solution of the DE $\frac{dy}{dx} = 2x$ such that $\forall x \in I, f'(x) = 2x$ and $f(1) = 4$

$$\begin{aligned}\frac{dy}{dx} &= 2x \\ \int \frac{dy}{dx} dx &= \int 2x dx \\ y &= x^2 + c\end{aligned}$$

Substituting $y = 4$ and $x = 1$,

$$4 = 1 + c \text{ or } c = 3$$

$$\therefore \text{Solution: } y^2 = x + 3$$

□

Example 1.3.2: $\frac{dy}{dx} = -\frac{x}{y}$, $y(3) = 4$

$$\begin{aligned}x + y \frac{dy}{dx} &= 0 \\ \int x dx + \int y \frac{dy}{dx} dx &= 0 \\ \frac{x^2}{2} + \frac{y^2}{2} &= c' \\ x^2 + y^2 &= c\end{aligned}$$

Substituting $x = 3$ and $y = 4$,

$$16 + 9 = c \text{ or } c = 25$$

$$\therefore \text{Solution: } x^2 + y^2 - 25 = 0$$

B Existence of Solutions

Not all initial-value and boundary-value problems have solutions. For example,

$$\begin{aligned}\frac{d^2y}{dx^2} + y &= 0 \\ y(0) &= 1, \quad y(\pi) = 5\end{aligned}$$

has no solutions! Thus arises the question of *existence* of solutions. We can say, every initial-value problem that satisfies definition (1.3.1) has *at least one* solution. However, there arises another question. Can a problem have more than one solution?

Let's consider the initial-value problem

$$\frac{dy}{dx} = y^{1/3}; \quad y(0) = 0$$

One may verify that the functions f_1 and f_2 defined, respectively, by

$$\forall x \in \mathbb{R}, \quad f_1(x) = 0$$

and

$$f_2(x) = \left(\frac{2}{3}x\right)^{3/2}, \quad x \geq 0; \quad f_2(x) = 0, \quad x \leq 0$$

are both solutions of this initial-value problem. In fact, this problem has infinitely many solutions. Hence, we can state that the initial-value problem need not have a *unique* solution. In order to ensure uniqueness, some additional requirement must certainly be imposed.

Theorem 1.3.1 (Basic Existence and Uniqueness Theorem):

Hypothesis: Consider the differential equation

$$\frac{dy}{dx} = f(x, y) \quad (1.3.1)$$

where

- The function f is a continuous function of x and y in some domain D of the xy plane, and
- The partial derivative $\frac{\partial f}{\partial y}$ is also a continuous function of x and y in D ; and let (x_0, y_0) be a point in D .

Conclusion: There exists a unique solution ϕ of the differential equation (1.3.1), defined on some interval $|x - x_0| \leq h$, where h is sufficiently small, that satisfies the condition

$$\phi(x_0) = y_0$$

Example 1.3.3: Show that

$$y = 4e^{2x} + 2e^{-3x}$$

is a solution of the initial-value problem

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$$

$$y(0) = 6$$

$$y'(0) = 2$$

Is $y = 2e^{2x} + 4e^{-3x}$ also a solution of this problem? Explain why or why not.

$$y = 4e^{2x} + 2e^{-3x}$$

$$y_1 = 8e^{2x} - 6e^{-3x}$$

$$y_2 = 16e^{2x} + 18e^{-3x}$$

$$\begin{aligned} y_2 + y_1 - 6y &= (16e^{2x} + 18e^{-3x}) + (8e^{2x} - 6e^{-3x}) - 6(4e^{2x} + 2e^{-3x}) \\ &= 0 \cdot e^{2x} + 0 \cdot e^{-3x} \\ &= 0 \end{aligned}$$

The solution also satisfies $y(0) = 6$ and $y'(0) = 2$

Now, for $y = 2e^{2x} + 4e^{-3x}$,

$$y_1 = 4e^{2x} - 12e^{-3x} ; \quad y_2 = 8e^{2x} + 36e^{-3x}$$

$$\begin{aligned}
y_2 + y_1 - 6y &= (8e^{2x} + 36e^{-3x}) + (4e^{2x} - 12e^{-3x}) - 6(2e^{2x} + 4e^{-3x}) \\
&= 0 \cdot e^{2x} + 0 \cdot e^{-3x} \\
&= 0
\end{aligned}$$

However, in this case,

$$y(0) = 6 \ ; \ y'(0) = -8$$

As we can see, this solution doesn't satisfy the initial-value problem. Hence $y = 2e^{2x} + 4e^{-3x}$ is not a solution of this problem.

Example 1.3.4: Given that every solution of

$$x^3 \frac{d^3 y}{dx^3} - 3x^2 \frac{d^2 y}{dx^2} + 6x \frac{dy}{dx} - 6y = 0$$

may be written in the form $y = c_1 x + c_2 x^2 + c_3 x^3$ for some choice of the arbitrary constants c_1 , c_2 , and c_3 , solve the initial-value problem consisting of the above DE plus the three conditions

$$y(2) = 0 \ , \ y'(2) = 2 \ , \ y''(2) = 6$$

$$y = c_1 x + c_2 x^2 + c_3 x^3$$

$$y(2) = 0 \text{ or, } 8c_3 + 4c_2 + 2c_1 = 0 \tag{1.3.2}$$

$$y' = c_1 + 2c_2 x + 3c_3 x^2$$

$$y'(2) = 2 \text{ or, } 12c_3 + 4c_2 + c_1 = 2 \tag{1.3.3}$$

$$y'' = 0 + 2c_2 + 6c_3 x$$

$$y''(2) = 6 \text{ or, } 12c_3 + 2c_2 + 0c_1 = 6 \tag{1.3.4}$$

Solving (1.3.1), (1.3.2), and (1.3.3) we get,

$$c_1 = 2 \ , \ c_2 = -3 \ , \ c_3 = 1$$

$$\therefore \text{ Solution: } y = 2x - 3x^2 + x^3$$

2 First Order Equations for Which Exact Solutions Are Obtainable

2.1 Exact Differential Equations and Integrating Factors

A Standard Forms of First-Order Differential Equations

The first-order differential equations may be expressed in either the **Derivative Form**

$$\frac{dy}{dx} = f(x, y) \quad (2.1.1)$$

or the **Differential Form**

$$M(x, y) dx + N(x, y) dy = 0 \quad (2.1.2)$$

Example 2.1.1: Standard Forms

The equation

$$\frac{dy}{dx} = \frac{x^2 + y^2}{x - y}$$

is the form (2.1.1). It may be written as

$$(x^2 + y^2) dx + (y - x) dy = 0$$

which is of the form (2.1.2).

Again, the equation

$$(\sin x + y) dx + (x + 3y) dy = 0$$

is of the form (2.1.2), which can also be written as

$$\frac{dy}{dx} = -\frac{\sin x + y}{x + 3y}$$

B Exact Differential Equations

Definition 2.1.1: Exact Differential

Let F be a function of two real variables such that F has continuous first partial derivatives in a domain D . The total differential dF of the function F is defined by the formula

$$dF(x, y) = \frac{\partial F(x, y)}{\partial x} dx + \frac{\partial F(x, y)}{\partial y} dy$$

for all $(x, y) \in D$.

Comparing $dF(x, y)$ with the form (2.1.2), we get

$$\frac{\partial F(x, y)}{\partial x} = M(x, y) \quad \text{and} \quad \frac{\partial F(x, y)}{\partial y} = N(x, y)$$

Example 2.1.2

Let F be a function

$$F(x, y) = xy^2 + 2x^3y$$

for all real (x, y) . Then

$$\frac{\partial F(x, y)}{\partial x} = y^2 + 6x^2y, \quad \frac{\partial F(x, y)}{\partial y} = 2xy + 2x^3$$

and the total differential dF is defined by

$$dF(x, y) = (y^2 + 6x^2y) dx + (2xy + 2x^3) dy$$

for all real (x, y)

Definition 2.1.2: Exact Differential Equation

The expression

$$M(x, y) dx + N(x, y) dy \quad (2.1.3)$$

is called an exact differential in a domain D if there exists a function F of two real variables such that this expression equals the total differential $dF(x, y)$ for all $(x, y) \in D$.

That is, expression (2.1.3) is an exact differential in D if there exists a function F such that

$$\frac{\partial F(x, y)}{\partial x} = M(x, y) \quad \text{and} \quad \frac{\partial F(x, y)}{\partial y} = N(x, y)$$

for all $(x, y) \in D$.

If $M(x, y) dx + N(x, y) dy$ is an exact differential, then the differential equation

$$M(x, y) dx + N(x, y) dy = 0$$

is called an **Exact Differential Equation**.

Theorem 2.1.1 (Exact Differential Equation):

1. If the DE $M(x, y) dx + N(x, y) dy = 0$ is exact in D , then

$$\forall (x, y) \in D, \quad \frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x}$$

2. Conversely, if

$$\forall (x, y) \in D, \quad \frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x}$$

then the DE is exact in D .

Proof (1):

$$\begin{aligned} \frac{\partial F(x, y)}{\partial x} &= M(x, y) \quad , \quad \frac{\partial F(x, y)}{\partial y} = N(x, y) \\ \frac{\partial^2 F(x, y)}{\partial x \partial y} &= \frac{\partial M(x, y)}{\partial y} \quad , \quad \frac{\partial^2 F(x, y)}{\partial x \partial y} = \frac{\partial N(x, y)}{\partial x} \\ \therefore \frac{\partial^2 F(x, y)}{\partial y \partial x} &= \frac{\partial^2 F(x, y)}{\partial x \partial y} \\ \therefore \frac{\partial M(x, y)}{\partial y} &= \frac{\partial N(x, y)}{\partial x} \quad \square \end{aligned}$$

C The Solution of Exact Differential Equations

Theorem 2.1.2 (Solution of Exact DE):

If $M(x, y) dx + N(x, y) dy = 0$ is exact in domain D , then

$$\forall (x, y) \in D, \exists F(x, y) : \frac{\partial F(x, y)}{\partial x} = M(x, y) \quad \text{and} \quad \frac{\partial F(x, y)}{\partial y} = N(x, y)$$

Then the equation may be written

$$\frac{\partial F(x, y)}{\partial x} dx + \frac{\partial F(x, y)}{\partial y} dy = 0$$

or simply,

$$dF(x, y) = 0$$

Here, $F(x, y) = c$ is a one-parameter family of solutions of this DE, where c is an arbitrary constant.

Example 2.1.3: Solve the equation

$$(3x^2 + 4xy) dx + (2x^2 + 2y) dy = 0$$

Standard Method:

$$\begin{aligned} \frac{\partial F(x, y)}{\partial x} &= M(x, y) = 3x^2 + 4xy \\ F(x, y) &= \int (3x^2 + 4xy) dx + \phi(y) \\ &= x^3 + 2x^2y + \phi(y) \end{aligned}$$

Again,

$$\begin{aligned} \frac{\partial F(x, y)}{\partial y} &= 2x^2 + \frac{\partial \phi(y)}{\partial y} = 2x^2 + 2y \\ \frac{d\phi(y)}{dy} &= 2y \\ \int \frac{d\phi(y)}{dy} dy &= \int 2y dy \\ \phi(y) &= y^2 + c_0 \end{aligned}$$

Thus, we get

$$F(x, y) = x^3 + 2x^2y + y^2 + c_0$$

Hence, a one-parameter family of the solution is $F(x, y) = c_1$ or,

$$x^3 + 2x^2y + y^2 + c_0 = c_1$$

$$\boxed{x^3 + 2x^2y + y^2 = c}$$

Method of Grouping:

$$\begin{aligned}(3x^2 + 4xy) dx + (2x^2 + 2y) dy &= 0 \\ 3x^2 dx + (4xy dx + 2x^2 dy) + 2y dy &= 0 \\ d(x^3) + d(2x^2y) + d(y^2) &= d(c) \\ \boxed{x^3 + 2x^2y + y^2} &= c\end{aligned}$$

Example 2.1.4: Solve the initial-value problem

$$(2x \cos y + 3x^2y) dx + (x^3 - x^2 \sin y - y) dy = 0 ; \quad y(0) = 2$$

$$\begin{aligned}(2x \cos y dx - x^2 \sin y dy) + (3x^2y dx + x^3 dy) - y dy &= 0 \\ d(x^2 \cos y) + d(x^3y) + d\left(\frac{y^2}{2}\right) &= d(c_1) \\ 2x^2 \cos y + x^3y + y^2 &= c\end{aligned}$$

Substituting $x = 0$ and $y = 2$,

$$2^2 = c$$

Hence, the solution is:

$$2x^2 \cos y + x^3y + y^2 = 4$$

D Integrating Factors

Definition 2.1.3: Integrating Factor (IF)

If the DE

$$M(x, y) dx + N(x, y) dy = 0 \quad (2.1.4)$$

is not exact in a domain D but the DE

$$\mu(x, y)M(x, y) dx + \mu(x, y)N(x, y) dy = 0 \quad (2.1.5)$$

is exact in D , then $\mu(x, y)$ is called an **Integrating Factor** of the DE.

Example 2.1.5: Integrating factor

Consider the DE

$$(3y + 4xy^2) dx + (2x + 3x^2y) dy = 0 \quad (2.1.6)$$

This equation is of the form (2.1.4), where

$$\begin{aligned}M(x, y) &= 3y + 4xy^2, & N(x, y) &= 2x + 3x^2y \\ \frac{\partial M(x, y)}{\partial y} &= 3 + 8xy, & \frac{\partial N(x, y)}{\partial x} &= 2 + 6xy\end{aligned}$$

Since

$$\frac{\partial M(x, y)}{\partial y} \neq \frac{\partial N(x, y)}{\partial x}$$

except for (x, y) such that $2xy + 1 = 0$, Equation (2.1.4) is not exact in any rectangular domain D .

Let $\mu(x, y) = x^2y$. Then the corresponding DE of the form (2.1.5) is

$$(3x^2y^2 + 4x^3y^3) dx + (2x^3y + 3x^4y^2) dy = 0$$

This equation is exact in every rectangular domain D , since

$$\frac{\partial[\mu(x, y)M(x, y)]}{\partial y} = 6x^2y + 12x^3y^2 = \frac{\partial[\mu(x, y)N(x, y)]}{\partial x}$$

For all real (x, y) . Hence, $\mu(x, y) = x^2y$ is an integrating factor of Equation (2.1.6).

Example 2.1.6: Determine whether or not the following equation is exact

$$\left(\frac{x}{y^2} + x\right) dx + \left(\frac{x^2}{y^3} + y\right) dy = 0$$

$$\begin{aligned}\frac{\partial M(x, y)}{\partial y} &= -\frac{x}{2y^3} \\ \frac{\partial N(x, y)}{\partial x} &= \frac{2x}{y^3}\end{aligned}$$

Here, $\frac{\partial M(x, y)}{\partial y} \neq \frac{\partial N(x, y)}{\partial x}$. Hence, the equation is not exact.

Example 2.1.7: Determine the constant A in the following equations such that the equation is exact

1. $(Ax^2y + 2y^2) dx + x^3 + 4xy dy = 0$
2. $\left(\frac{Ay}{x^3} + \frac{y}{x^2}\right) dx + \left(\frac{1}{x^2} - \frac{1}{x}\right) dy = 0$

1.

$$\begin{aligned}\frac{\partial M(x, y)}{\partial y} &= \frac{\partial(Ax^2y + 2y^2)}{\partial y} = Ax^2 + 4y \\ \frac{\partial N(x, y)}{\partial x} &= \frac{\partial(x^3 + 4xy)}{\partial x} = 3x^2 + 4y\end{aligned}$$

Equating the coefficients of x^2 , we get

$$\boxed{A = 3}$$

2.

$$\begin{aligned}\frac{\partial M(x, y)}{\partial y} &= \frac{\partial\left(\frac{Ay}{x^3} + \frac{y}{x^2}\right)}{\partial y} = \frac{A}{x^3} + \frac{1}{x^2} \\ \frac{\partial N(x, y)}{\partial x} &= \frac{\partial\left(\frac{1}{x^2} - \frac{1}{x}\right)}{\partial x} = -\frac{1}{2x^3} + \frac{1}{x^2}\end{aligned}$$

Equating the coefficients of $\frac{1}{x^3}$, we get

$$\boxed{A = -\frac{1}{2}}$$

2.2 Separable Equations and Equations Reducible to this Form

A Separable Equations

Definition 2.2.1: Separable Equations

An equations of the form

$$F(x)G(y) dx + f(x)g(y) dy = 0 \quad (2.2.1)$$

is called an equation with separable variables or simply a separable equation.

Theorem 2.2.1 (Solution of Separable Differential Equations):

In general, the separable equations are not exact, but they possess an obvious integrating factor $\frac{1}{f(x)G(y)}$

Thus the equation (2.2.1) becomes

$$\frac{F(x)}{f(x)} dx + \frac{g(y)}{G(y)} dy = 0 \quad (2.2.2)$$

which is exact, because

$$\frac{\partial}{\partial y} \left(\frac{F(x)}{f(x)} \right) = 0 = \frac{\partial}{\partial x} \left(\frac{g(y)}{G(y)} \right)$$

We can write the equation (2.2.2) as

$$M(x) dx + N(y) dy = 0$$

where $M(x) = \frac{F(x)}{f(x)}$ and $N(y) = \frac{g(y)}{G(y)}$

A one-parameter family solution to the DE is

$$\int M(x) dx + \int N(y) dy = c \quad (2.2.3)$$

Example 2.2.1: Solve the equation

$$(x - 4)y^4 dx - x^3(y^2 - 3) dy = 0$$

The equation is separable; dividing by x^3y^4 we obtain

$$\frac{x - 4}{x^3} dx - \frac{y^2 - 3}{y^4} dy = 0$$
$$\int (x^{-2} - 4x^{-3}) dx - \int (y^{-2} - 3y^{-4}) dy = 0$$

$$\boxed{-\frac{1}{x} + \frac{2}{x^2} + \frac{1}{y} - \frac{1}{y^3} = c}$$

The DE in derivative form:

$$\frac{dy}{dx} = \frac{(x - 4)y^4}{x^3(y^2 - 3)}$$

Here, $y = 0$ is a solution which was lost in the separation process.

Example 2.2.2: Solve the initial-value problem that consists of the DE

$$x \sin y \, dx + (x^2 + 1) \cos y \, dy = 0$$

and the initial condition $y(1) = \frac{\pi}{2}$

$$\begin{aligned}\frac{x}{x^2 + 1} \, dx + \frac{\cos y}{\sin y} \, dy &= 0 \\ \frac{1}{2} \int \frac{d(x^2 + 1)}{x^2 + 1} + \int \cot y \, dy &= 0 \\ \frac{1}{2} \ln |x^2 + 1| + \ln |\sin y| &= \ln |c_1| \\ \ln |(x^2 + 1) \sin^2 y| &= \ln |c| \\ \therefore (x^2 + 1) \sin^2 y &= c\end{aligned}$$

Applying the initial condition, we get

$$2 \sin^2 \frac{\pi}{2} = c \text{ or, } c = 2$$

Thus, the solution is

$$(x^2 + 1) \sin^2 y = 2$$

B Homogeneous Equations

Definition 2.2.2: Homogeneous Equations

The first-order differential equation

$$M(x, y) \, dx + N(x, y) \, dy = 0$$

is said to be homogeneous if, when written in the derivative form

$$\frac{dy}{dx} = f(x, y)$$

there exists a function g such that $f(x, y)$ can be expressed in the form $g(y/x)$

Example 2.2.3

The DE

$$x^2 - 3y^2 \, dx + 2xy \, dy = 0$$

is homogeneous. To see this, we first write the derivative form of the equation:

$$\frac{dy}{dx} = \frac{3y^2 - x^2}{2xy} = \frac{3y}{2x} - \frac{x}{2y}$$

We see that the DE can be written as

$$\frac{dy}{dx} = \frac{3}{2} \left(\frac{y}{x} \right) - \frac{1}{2} \left(\frac{1}{y/x} \right)$$

in which the right side of the equation is of the form $g(y/x)$ for a certain function g .

Example 2.2.4: The equation

$$(y + \sqrt{x^2 + y^2}) dx - x dy = 0$$

is homogeneous.

Derivative form:

$$\begin{aligned} \frac{dy}{dx} &= \frac{y + \sqrt{x^2 + y^2}}{x} \\ &= \frac{y}{x} + \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2}} \\ &= \frac{y}{x} + \sqrt{1 + \left(\frac{y}{x}\right)^2} = g\left(\frac{y}{x}\right) \end{aligned}$$

Definition 2.2.3: Homogeneous Equation of degree n

A function F is called homogeneous of degree n if

$$F(tx, ty) = t^n F(x, y)$$

This means that if the tx and ty are substituted for x and y respectively in $F(x, y)$, and if t^n is then factored out, the other factor that remains is the original expression $F(x, y)$ itself.

For example, the function given by $F(x, y) = x^2 + y^2$ is homogeneous of degree 2, since

$$F(tx, ty) = (tx)^2 + (ty)^2 = t^2(x^2 + y^2) = t^2 F(x, y)$$

Now, suppose both $M(x, y)$ and $N(x, y)$ in the DE

$$M(x, y) dx + N(x, y) dy = 0$$

are homogeneous of the same degree n . Since $M(tx, ty) = t^n M(x, y)$, for $t = \frac{1}{x}$, we have

$$\begin{aligned} M\left(1, \frac{y}{x}\right) &= \left(\frac{1}{x}\right)^n M(x, y) \\ M(x, y) &= \left(\frac{1}{x}\right)^{-n} M\left(1, \frac{y}{x}\right) \end{aligned}$$

Similarly,

$$N(x, y) = \left(\frac{1}{x}\right)^{-n} N\left(1, \frac{y}{x}\right)$$

Now, writing the DE in derivative form, we get

$$\begin{aligned} \frac{dy}{dx} &= -\frac{M(x, y)}{N(x, y)} \\ &= -\frac{\left(\frac{1}{x}\right)^{-n} M\left(1, \frac{y}{x}\right)}{\left(\frac{1}{x}\right)^{-n} N\left(1, \frac{y}{x}\right)} \\ &= -\frac{M\left(1, \frac{y}{x}\right)}{N\left(1, \frac{y}{x}\right)} \\ &= g\left(\frac{y}{x}\right) \end{aligned}$$

Note:-

If $M(x, y)$ and $N(x, y)$ in

$$M(x, y) dx + N(x, y) dy = 0$$

are both homogeneous functions of the same degree n , then the differential equation is a homogeneous differential equation.

Theorem 2.2.2:

If

$$M(x, y) dx + N(x, y) dy = 0 \quad (2.2.4)$$

is a homogeneous equation, then the change of variables $y = vx$ transforms the equation into a separable equation in the variables v and x .

Proof:

Since $M(x, y) dx + N(x, y) dy = 0$ is homogeneous, it may be written in the form

$$\frac{dy}{dx} = g\left(\frac{y}{x}\right)$$

Let $y = vx$. Then

$$\frac{dy}{dx} = v + x \frac{dv}{dx}$$

and the initial equation becomes

$$v + x \frac{dv}{dx} = g(v)$$

or,

$$[v - g(v)] dx + x dv = 0$$

This equation is separable. Separating the variables we obtain

$$\frac{dv}{v - g(v)} + \frac{dx}{x} = 0 \quad \square \quad (2.2.5)$$

Theorem 2.2.3 (Solution of a Homogeneous Differential Equation): To solve a DE of the form (2.2.4), we let $y = vx$ and transform the homogeneous equation into a separable equation of the form (2.2.5). From this, we have

$$\int \frac{dv}{v - g(v)} + \int \frac{dx}{x} = c$$

where c is an arbitrary constant. Letting $F(v)$ denote

$$\int \frac{dv}{v - g(v)}$$

and returning to the original dependent variable y , the solution takes the form

$$F\left(\frac{y}{x}\right) + \ln|x| = c$$

Example 2.2.5: Solve the equation

$$(x^2 - 3y^2) dx + 2xy dy = 0$$

Derivative form:

$$\frac{dy}{dx} = -\frac{x}{2y} + \frac{3y}{2x}$$

Letting $y = vx$, we get

$$\begin{aligned} v + x \frac{dv}{dx} &= -\frac{1}{2v} + \frac{3v}{2} \\ x \frac{dv}{dx} &= \frac{v^2 - 1}{2v} \\ \frac{2v dv}{v^2 - 1} &= \frac{dx}{x} \end{aligned}$$

Integrating, we find

$$\begin{aligned} \ln |v^2 - 1| &= \ln |x| + \ln |c| \\ |v^2 - 1| &= |cx| \\ \left| \frac{y^2}{x^2} - 1 \right| &= |cx| \\ |y^2 - x^2| &= x^2 |cx| \end{aligned}$$

For $y \geq x \geq 0$, it can be written as

$$\boxed{y^2 - x^2 = cx^3}$$

Example 2.2.6: Solve the initial-value problem

$$(y + \sqrt{x^2 + y^2}) dx - x dy = 0, y(1) = 0$$

Derivative form:

$$\frac{dy}{dx} = \frac{y}{x} + \sqrt{1 + \left(\frac{y}{x}\right)^2}$$

Letting $y = vx$, we get

$$\begin{aligned} v + x \frac{dv}{dx} &= v + \sqrt{1 + v^2} \\ \frac{dv}{\sqrt{1 + v^2}} &= \frac{dx}{x} \\ \ln |v + \sqrt{1 + v^2}| &= \ln |x| + \ln |c| \\ v + \sqrt{v^2 + 1} &= cx \\ \frac{y}{x} + \frac{1}{x} \sqrt{y^2 + x^2} &= cx \\ y + \sqrt{x^2 + y^2} &= cx^2 \end{aligned}$$

Applying the initial condition, we get

$$0 + \sqrt{1} = c \cdot 1 \text{ or, } c = 1$$

Hence, the solution:

$$\boxed{y + \sqrt{x^2 + y^2} = x^2} \text{ or, } \boxed{y = \frac{1}{2}(x^2 - 1)}$$

Exercise 2.1: Solve the following differential equations

1. $(xy + 2x + y + 2) dx + (x^2 + 2x) dy = 0$
2. $(2x \cos y + 3x^2y) dx + (x^3 - x^2 - y) dy = 0, y(0) = 2$
3. $(e^v + 1) \cos u du + e^v(\sin u + 1) dv = 0$
4. $(x + 4)(y^2 + 1) dx + y(x^2 + 3x + 2) dy = 0$
5. $(2xy + 3y^2) dx - (2xy + x^2) dy = 0$
6. $(x + y) dx - x dy = 0$
7. $v^3 du + (u^3 - uv^2) dv = 0$
8. $(x \tan \frac{y}{x} + y) dx - x dy = 0$
9. $(2s^2 + 2st + t^2) ds + (s^2 + 2st - t^2) dt = 0$
10. $(x^3 + y^2\sqrt{x^2 + y^2}) dx - xy\sqrt{x^2 + y^2} dy = 0$
11. $(\sqrt{x+y} + \sqrt{x-y}) dx + (\sqrt{x-y} - \sqrt{x+y}) dy = 0$
12. $(3x + 8)(y^2 + 4) dx - 4y(x^2 + 5x + 6) dy = 0, y(1) = 2$
13. $(2x^2 + 2xy + y^2) dx + (x^2 + 2xy) dy = 0$

1.

$$\begin{aligned} (xy + 2x + y + 2) dx + (x^2 + 2x) dy &= 0 \\ (x + 1)(y + 2) dx + x(x + 2) dy &= 0 \\ \int \frac{x + 1}{x(x + 2)} dx + \int \frac{dy}{y + 2} &= 0 \\ \frac{1}{2} \ln |x^2 + 2x| + \ln |y + 2| &= \ln |c_1| \\ \ln |(x^2 + 2x)(y + 2)^2| &= \ln |c| \\ \boxed{(x^2 + 2x)(y + 2)^2 = c} \end{aligned}$$

2.

$$\begin{aligned} (2x \cos y + 3x^2y) dx + (x^3 - x^2 \sin^2 y - y) dy &= 0 \\ F(x, y) &= \int (2x \cos y + 3x^2y) \partial x + \phi(y) \\ &= x^2 \cos y + x^3y + \phi(y) \end{aligned}$$

Now,

$$\frac{\partial F(x, y)}{\partial y} = x^3 - x^2 \sin y - y = x^3 - x^2 \sin y + \frac{d}{dx} \phi(y)$$

$$\therefore \phi(y) = - \int y \, dy = -\frac{y^2}{2} + c_0$$

$$\begin{aligned} F(x, y) &= x^2 \cos y + x^3 y - \frac{y^2}{2} + c_0 = c_1 \\ 2x^2 \cos y + 2x^3 y - y^2 &= c \end{aligned}$$

Applying initial value,

$$c = -4$$

$$\boxed{2x^2 \cos y + 2x^3 y - y^2 + 4 = 0}$$

3.

$$\begin{aligned} (e^v + 1) \cos u \, du + e^v (\sin u + 1) \, dv &= 0 \\ (e^v \cos u \, du + e^v \sin u \, dv) + \cos u \, du + e^v \, dv &= 0 \\ d(e^v \sin u) + d(\sin u) + de^v &= d(c) \end{aligned}$$

$$\boxed{\sin u + e^v (\sin u + 1) = c}$$

4.

$$\begin{aligned} (x+4)(y^2+1) \, dx + y(x^2+3x+2) \, dy &= 0 \\ \int \frac{x+4}{x^2+3x+2} \, dx + \int \frac{y}{y^2+1} \, dy &= 0 \\ \int \frac{x+4}{(x+2)(x+1)} \, dx + \frac{1}{2} \int \frac{2y}{y^2+1} \, dy &= 0 \\ \int \frac{3}{x+1} \, dx - \int \frac{2}{x+2} \, dx + \frac{1}{2} \ln |y^2+1| & \\ 3 \ln |x+1| - 2 \ln |x+2| + \frac{1}{2} \ln |y^2+1| &= \ln |c_1| \end{aligned}$$

$$\ln \left| \frac{(x+1)^6}{(x+2)^4} \cdot (y^2+1) \right| = \ln |c|$$

$$\boxed{(x+6)^6(y^2+1) = c(x+2)^4}$$

5.

$$(2xy + 3y^2) dx - (2xy + x^2) dy = 0$$

$$\frac{dy}{dx} = \frac{2xy + 3y^2}{2xy + x^2} = \frac{2\left(\frac{y}{x}\right) + 3\left(\frac{y}{x}\right)^2}{2\left(\frac{y}{x}\right) + 1}$$

Letting $y = vx$, we get

$$v + x \frac{dv}{dx} = \frac{2 + 3v^2}{2v + 1}$$

$$x \frac{dv}{dx} = \frac{v^2 + v}{2v + 1}$$

$$\int \frac{2v + 1}{v^2 + v} dv = \int \frac{dx}{x}$$

$$\int \frac{d(v^2 + v)}{v^2 + v} = \int \frac{dx}{x}$$

$$\ln |v^2 + v| = \ln |cx|$$

$$\frac{y^2}{x^2} + \frac{y}{x} = cx$$

$$\boxed{y^2 + xy = cx^3}$$

6.

$$(x + y) dx - x dy = 0$$

$$\frac{dy}{dx} = \frac{x + y}{x} = 1 + \frac{y}{x}$$

Letting $y = vx$, we get

$$v + x \frac{dv}{dx} = 1 + v$$

$$\int dv = \int \frac{dx}{x}$$

$$\frac{y}{x} = \ln |cx|$$

$$\boxed{cx = e^{y/x}}$$

7.

$$v^3 du + (u^3 - uv^2) dv = 0$$

$$\frac{du}{dv} = \frac{uv^2 - u^3}{v^3} = \frac{u}{v} - \left(\frac{u}{v}\right)^3$$

Letting $u = wv$, we get

$$w + v \frac{dw}{dv} = w - w^3$$

$$- \int \frac{dw}{w^3} = \int \frac{dv}{v}$$

$$\frac{1}{2w^2} = \ln |v| + c_1$$

$$\boxed{v^2 = u^2(\ln v^2 + c)}$$

8.

$$\left(x \tan \frac{y}{x} + y\right) dx - x dy = 0$$

$$\frac{dy}{dx} = \tan \frac{y}{x} + \frac{y}{x}$$

Letting $y = vx$, we get

$$v + x \frac{dv}{dx} = \tan v + v$$

$$\int \frac{dv}{\tan v} = \int \frac{dx}{x}$$

$$\ln |\sin v| = \ln |cx|$$

$$\boxed{\sin \frac{y}{x} = cx}$$

9.

$$(2s^2 + 2st + t^2) ds + (s^2 + 2st - t^2) dt = 0$$

$$\begin{aligned} \frac{ds}{dt} &= \frac{t^2 - 2st - s^2}{2s^2 + 2st + t^2} \\ &= \frac{1 - 2\left(\frac{s}{t}\right) - \left(\frac{s}{t}\right)^2}{2\left(\frac{s}{t}\right)^2 + 2\left(\frac{s}{t}\right) + 1} \end{aligned}$$

Letting $s = vt$, we get

$$\begin{aligned}
 v + t \frac{dv}{dt} &= \frac{1 - 2v - v^2}{2v^2 + 2v + 1} \\
 t \frac{dv}{dt} &= -\frac{2v^3 + 3v^2 + 3v - 1}{2v^2 + 2v + 1} \\
 -\int \frac{2v^2 + 2v + 1}{2v^3 + 3v^2 + 3v - 1} dv &= \int \frac{dt}{t} \\
 -\frac{1}{3} \int \frac{d(2v^3 + 3v^2 + 3v - 1)}{2v^3 + 3v^2 + 3v - 1} &= \ln |t| + \ln |c_1| \\
 \boxed{2v^3 + 3v^2 + 3v - 1} &= \frac{c}{t^3}
 \end{aligned}$$

10.

$$\begin{aligned}
 (x^3 + y^2 \sqrt{x^2 + y^2}) dx - xy \sqrt{x^2 + y^2} dy &= 0 \\
 \frac{dy}{dx} = \frac{x^3 + y^2 \sqrt{x^2 + y^2}}{xy \sqrt{x^2 + y^2}} &= \frac{1 + \left(\frac{y}{x}\right)^2 \sqrt{1 + \left(\frac{y}{x}\right)^2}}{\frac{y}{x} \sqrt{1 + \left(\frac{y}{x}\right)^2}}
 \end{aligned}$$

Letting $y = vx$,

$$\begin{aligned}
 v + x \frac{dv}{dx} &= \frac{1 + v^2 \sqrt{1 + v^2}}{v \sqrt{1 + v^2}} \\
 x \frac{dv}{dx} &= \frac{1}{v \sqrt{1 + v^2}} \\
 \int v \sqrt{1 + v^2} dv &= \int \frac{dx}{x} \\
 \frac{1}{2} \int \sqrt{1 + v^2} d(1 + v^2) &= \ln |c_1 x| \\
 (1 + v^2)^{3/2} &= 3 \ln |c_1 x| \\
 \left(1 + \frac{y^2}{x^2}\right) \sqrt{1 + \frac{y^2}{x^2}} &= \ln |cx^3| \\
 \boxed{(x^2 + y^2) \sqrt{x^2 + y^2}} &= x^3 \ln |cx^3|
 \end{aligned}$$

11.

$$\begin{aligned}
 (\sqrt{x+y} + \sqrt{x-y}) dx + (\sqrt{x-y} - \sqrt{x+y}) dy &= 0 \\
 \frac{dy}{dx} &= \frac{\sqrt{\frac{x}{y} + 1} - \sqrt{x-y} - 1}{\sqrt{\frac{x}{y} + 1} + \sqrt{\frac{x}{y} - 1}}
 \end{aligned}$$

Letting $x = vy$,

$$\begin{aligned}
 v + y \frac{dv}{dy} &= \frac{\sqrt{v+1} - \sqrt{v-1}}{\sqrt{v+1} + \sqrt{v-1}} \\
 &= \frac{v+1 + v-1 - 2\sqrt{v^2-1}}{v+1 - v+1} \\
 v + y \frac{dv}{dy} &= v - \sqrt{v^2-1} \\
 \int \frac{dv}{\sqrt{v^2-1}} &= - \int \frac{dy}{y} \\
 \ln |v + \sqrt{v^2-1}| &= \ln \left| \frac{c}{y} \right| \\
 \frac{x}{y} + \sqrt{\frac{x^2}{y^2} - 1} &= \frac{c}{y}
 \end{aligned}$$

$$\boxed{x + \sqrt{x^2 - y^2} = c}$$

12.

$$\begin{aligned}
 (3x+8)(y^2+4) dx + 4y(x^2+5x+6) dy &= 0 \\
 \frac{3x+8}{x^2+5x+6} dx - \frac{4y}{y^2+4} dy &= 0
 \end{aligned}$$

Here,

$$\begin{aligned}
 \frac{3x+8}{(x+3)(x+2)} &= \frac{1}{x+3} + \frac{2}{x+2} \\
 \therefore \int \frac{dx}{x+3} + 2 \int \frac{dx}{x+2} - 2 \ln |y^2+4| &= c_2 \\
 \ln |x+3| + 2 \ln |x+2| &= 2 \ln |c_1(y^2+4)| \\
 (x+3)(x+2)^2 &= c(y^2+4)
 \end{aligned}$$

Applying initial value,

$$c = \frac{9}{16}$$

$$\boxed{16(x+3)(x+2)^2 = 9(y^2+4)}$$

13.

$$\begin{aligned}
 (2x^2 + 2xy + y^2) dx + (x^2 + 2xy) dy &= 0 \\
 (2xy dx + x^2 dy) + (y^2 dx + 2xy dy) + 2x^2 dx &= 0 \\
 d(x^2y) + d(xy^2) + d\left(\frac{2}{3}x^3\right) &= d(c_1) \\
 x^2y + xy^2 + \frac{2}{3}x^3 &= c_1 \\
 \boxed{2x^3 + 3x^2y + 3xy^2} &= c
 \end{aligned}$$

Exercise 2.2: Show that the homogeneous equation

$$(Ax^2 + Bxy + Cy^2) dx + (Dx^2 + Exy + Fy^2) dy = 0$$

is exact if and only if $B = 2D$ and $E = 2C$.

The equation is exact if and only if

$$\frac{\partial(Ax^2 + Bxy + Cy^2)}{\partial y} = \frac{\partial(Dx^2 + Exy + Fy^2)}{\partial x}$$

$$Bx + 2Cy = 2Dx + Ey$$

Equating the coefficients of x , $B = 2D$

Equating the coefficients of y , $E = 2C$

2.3 Linear Equations and Bernoulli Equations**A Linear Equation****Definition 2.3.1: Linear Equation**

A first-order ordinary differential equation is linear in the dependent variable y and independent variable x if it is, or can be, written in the form

$$\frac{dy}{dx} + P(x)y = Q(x) \quad (2.3.1)$$

Equation (2.3.1) can also be written as

$$[P(x)y - Q(x)] dx + dy = 0 \quad (2.3.2)$$

Here,

$$\frac{\partial}{\partial y} M(x, y) = P(x, y) \quad \text{and} \quad \frac{\partial}{\partial x} N(x, y) = 0$$

The equation is not exact. So we multiply both sides of (2.3.2) by an integrating factor:

$$[\mu(x)P(x)y - \mu(x)Q(x)] dx + \mu(x) dy = 0$$

Now,

$$\frac{\partial}{\partial y} [\mu(x)M(x, y)] = \frac{\partial}{\partial x} [\mu(x)N(x, y)]$$

$$\frac{\partial}{\partial y} [\mu(x)P(x)y - \mu(x)Q(x)] = \frac{\partial}{\partial x} [\mu(x)]$$

$$\mu P(x) = \frac{d}{dx} \mu$$

$$\int P(x) dx = \int \frac{d\mu}{\mu}$$

$$\ln |\mu| = \int P(x) dx$$

$$\boxed{\mu = e^{\int P(x) dx}}$$

Theorem 2.3.1 (Solution of Linear Differential Equation):

The linear differential equation

$$\frac{dy}{dx} + P(x)y = Q(x)$$

has an integrating factor of the form

$$\mu = e^{\int P(x) dx} \quad (2.3.3)$$

A one-parameter family of solution of this equation is

$$\mu y = \int \mu Q(x) dx + c$$

or

$$y e^{\int P(x) dx} = \int e^{\int P(x) dx} Q(x) dx + c \quad (2.3.4)$$

That is,

$$y = e^{-\int P(x) dx} \left[\int e^{\int P(x) dx} Q(x) dx + c \right] \quad (2.3.5)$$

Example 2.3.1: Solve the Linear Differential Equation

$$\frac{dy}{dx} + \left(\frac{2x+1}{x} \right) y = e^{-2x}$$

$$P(x) = 2 + \frac{1}{x}$$

$$\therefore \text{IF} = e^{\int P(x) dx} = e^{2x + \ln x} = x e^{2x}$$

Now,

$$x e^{2x} \frac{dy}{dx} + e^{2x} (2x+1) y = x$$

$$\frac{d}{dx} (x e^{2x} y) = x$$

$$x e^{2x} y = \frac{x^2}{2} + c_1$$

$$y = \frac{1}{2} x e^{-2x} + \frac{c}{x} e^{-2x}$$

Example 2.3.2: Solve the initial value problem

$$(x^2 + 1) \frac{dy}{dx} + 4xy = x, \quad y(2) = 1$$

$$\frac{dy}{dx} + \left(\frac{4x}{x^2 + 1} \right) y = \frac{x}{x^2 + 1}$$

$$\therefore \text{IF} = \exp \left(2 \int \frac{2x}{x^2 + 1} dx \right) = \exp (2 \ln |x^2 + 1|) = (x^2 + 1)^2$$

Therefore, the solution is

$$\begin{aligned}(x^2 + 1)^2 y &= \int (x^2 + 1)^2 \cdot \frac{x}{x^2 + 1} dx + c_1 \\&= \int (x^3 + x) dx + c_1 \\&= \frac{x^4}{4} + \frac{x^2}{2} + c\end{aligned}$$

Applying initial value,

$$c = 19$$

$$(x^2 + 1)^2 y = \frac{x^4}{4} + \frac{x^2}{2} + 19$$

Example 2.3.3: Solve the linear DE

$$y^2 dx + (3xy - 1) dy = 0$$

$$\begin{aligned}\frac{dx}{dy} &= -\frac{3x - 1}{y^2} = -\frac{3}{y}x + \frac{1}{y^2} \\ \therefore \frac{dx}{dy} + \frac{3}{y}x &= \frac{1}{y^2} \\ \therefore \text{IF} &= e^{\int \frac{3}{y} dy} = y^3\end{aligned}$$

Now,

$$\begin{aligned}y^3 \frac{dx}{dy} + 3xy^2 &= y \\ y^3 dx + (3xy^2 - y) dy &= 0 \\ (y^3 dx + 3xy^2 dy) - y dy &= 0 \\ d(xy^3) - d\left(\frac{y^2}{2}\right) &= d(c_1) \\ 2xy^3 - y^2 &= c\end{aligned}$$

$$x = \frac{1}{2y} + \frac{c}{y^3}$$

B Bernoulli Equations

Definition 2.3.2: Bernoulli Equations

An equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

is called a Bernoulli Equation.

If $n = 0$ or $n = 1$, the equation is simply a linear DE. However, in general case in which $n \neq 0$ or $n \neq 1$, we must proceed in a different manner.

Theorem 2.3.2 (Transformation of Bernoulli Equation to Linear Equation):

Suppose $n \neq 0$ or $n \neq 1$. Then the transformation

$$v = y^{1-n}$$

reduces the Bernoulli Equation to a linear equation in v .

Proof:

$$\begin{aligned}\frac{dy}{dx} + P(x)y &= Q(x)y^n \\ y^{-n} \frac{dy}{dx} + P(x)y^{1-n} &= Q(x)\end{aligned}$$

Substituting $v = y^{1-n}$,

$$\begin{aligned}\frac{dv}{dx} &= (1-n)y^{-n} \frac{dy}{dx} \\ \frac{1}{1-n} \frac{dv}{dx} + P(x)v &= Q(x) \\ \frac{dv}{dx} + (1-n)P(x)v &= (1-n)Q(x)\end{aligned}$$

Letting $P_1(x) = (1-n)P(x)$ and $Q_1(x) = (1-n)Q(x)$ we get,

$$\frac{dv}{dx} + P_1(x)v = Q_1(x) \quad \square$$

Example 2.3.4:

$$\frac{dy}{dx} + y = xy^3$$

$$y^{-3} \frac{dy}{dx} + y^{-2} = x$$

Letting $v = y^{-2}$,

$$\begin{aligned}\frac{dv}{dx} &= -2y^{-3} \frac{dy}{dx} \\ -\frac{1}{2} \frac{dv}{dx} + v &= x \\ \frac{dv}{dx} - 2v &= x\end{aligned}$$

$$\therefore \text{IF} = e^{-\int 2 dx} = e^{-2x}$$

$$\begin{aligned}e^{-2x}v &= \int e^{-2x}(-2x) dx + c \\ e^{-2x} \frac{1}{y^2} &= -2 \int x e^{-2x} dx + c \\ &= x e^{-2x} + \frac{1}{2} e^{-2x} + c\end{aligned}$$

$$\frac{1}{y^2} = x + \frac{1}{2} + c e^{2x}$$

Exercise 2.3: Solve the Differential Equations

1. $x^4 \frac{dy}{dx} + 2x^3 y = 1$
2. $\frac{dy}{dx} + 3y = 3x^2 e^{-3x}$
3. $(x^2 + x - 2) \frac{dy}{dx} + 3(x + 1)y = x - 1$
4. $y dx + (xy^2 + x - y) dy = 0$
5. $\cos \theta dr + (r \sin \theta - \cos^4 \theta) d\theta = 0$
6. $(y \sin 2x - \cos x) dx + (1 + \sin^2 x) dy = 0$
7. $x \frac{dy}{dx} + y = -2x^6 y^4$
8. $\frac{dy}{dx} - \frac{y}{x} = -\frac{y^2}{x}$
9. $e^x [y - 3(e^x + 1)^2] dx + (e^x + 1) dy = 0$
10. $\frac{dy}{dx} + \frac{y}{2x} = \frac{x}{y^3}$

Handwritten solutions on notebook. No time for updating in L^AT_EXnow. If anyone is interested, contact; I can provide handwritten solutions.

Exercise 2.4: Consider the equation

$$a \frac{dy}{dx} + by = ke^{-\lambda x}$$

where a , b , and k are positive constants and λ is a non-negative constant.

(a) Solve this equation.

(b) Show that if $\lambda = 0$, every solution approaches $\frac{k}{b}$ as $x \rightarrow \infty$, but if $\lambda > 0$ every solution approaches 0 as $x \rightarrow \infty$.

Handwritten solutions on notebook. No time for updating in L^AT_EXnow. If anyone is interested, contact; I can provide handwritten solutions.

Exercise 2.5:

(a) Prove that if f and g are two different solutions of

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{A}$$

then $f - g$ is a solution of the equation

$$\frac{dy}{dx} + P(x)y = 0$$

(b) Thus show that if f and g are two different solutions of Equation (A) and c is an arbitrary constant, then

$$c(f - g) + f$$

is a one-parameter family of solutions of (A).

Handwritten solutions on notebook. No time for updating in L^AT_EXnow. If anyone is interested, contact; I can provide handwritten solutions.

2.4 Special Integrating Factors and Transformations

The five basic types of differential equations we've encountered so far:

- Exact \rightarrow Direct solution
- Separable \rightarrow Integrating Factor \rightarrow Exact DE
- Homogeneous \rightarrow Integrating Factor \rightarrow Exact DE
- Linear \rightarrow Appropriate Transformation \rightarrow Separable DE
- Bernoulli \rightarrow Appropriate Transformation \rightarrow Linear DE

How to solve a DE that is not of one of the five types?

1. Either multiply by proper IF \rightarrow Exact DE
2. Or, appropriate transformation \rightarrow One of the five basic forms.

A Finding Integrating Factors

Separable equations always possess integrating factors that can be determined by immediate inspection. However, some non-separable equations also possess such integrating factors that can be determined.

Suppose a non-exact DE

$$M(x, y) dx + N(x, y) dy = 0 \quad (2.4.1)$$

has an IF $\mu(x, y)$. Then the equation is

$$\mu(x, y)M(x, y) dx + \mu(x, y)N(x, y) dy = 0 \quad (2.4.2)$$

is exact. Now, we can say the equation (2.4.2) is exact if and only if

$$\begin{aligned} \frac{\partial}{\partial y} [\mu(x, y)M(x, y)] &= \frac{\partial}{\partial x} [\mu(x, y)N(x, y)] \\ M(x, y)\frac{\partial\mu(x, y)}{\partial y} + \mu(x, y)\frac{\partial M(x, y)}{\partial y} &= N(x, y)\frac{\partial\mu(x, y)}{\partial x} + \mu(x, y)\frac{\partial N(x, y)}{\partial x} \\ N(x, y)\frac{\partial\mu}{\partial x} - M(x, y)\frac{\partial\mu}{\partial y} &= \left[\frac{\partial M(x, y)}{\partial y} - \frac{\partial N(x, y)}{\partial x} \right] \mu \end{aligned} \quad (2.4.3)$$

Equation (2.4.3) is a PDE for the general IF μ , and we're in no position to attempt to solve such an equation. Let's attempt to determine IF of certain special types instead.

If M and N are functions of x and y , but the IF μ depends only upon x , then equation (2.4.3) reduces to

$$N(x, y) \frac{d\mu(x)}{dx} = \mu(x) \frac{\partial M(x, y)}{\partial y} - \mu(x) \frac{\partial N(x, y)}{\partial x}$$

or,

$$\frac{d\mu(x)}{\mu(x)} = \frac{1}{N(x, y)} \left[\frac{\partial M(x, y)}{\partial y} - \frac{\partial N(x, y)}{\partial x} \right] dx \quad (2.4.4)$$

Here, if

$$\frac{1}{N(x, y)} \left[\frac{\partial M(x, y)}{\partial y} - \frac{\partial N(x, y)}{\partial x} \right]$$

depends upon x only, equation (2.4.4) is a separable ordinary equation in the single independent variable x and the single dependent variable μ . In this case, we may integrate to obtain the IF

$$\mu(x) = \exp \left\{ \int \frac{1}{N(x, y)} \left[\frac{\partial M(x, y)}{\partial y} - \frac{\partial N(x, y)}{\partial x} \right] dx \right\}$$

Likewise, if

$$\frac{1}{M(x, y)} \left[\frac{\partial N(x, y)}{\partial x} - \frac{\partial M(x, y)}{\partial y} \right]$$

depends upon y only, then we may obtain an IF that depends only on y .

Theorem 2.4.1 (Integrating Factors):

Consider the differential equation

$$M(x, y) dx + N(x, y) dy = 0 \quad (2.4.5)$$

If

$$\frac{1}{N(x, y)} \left[\frac{\partial M(x, y)}{\partial y} - \frac{\partial N(x, y)}{\partial x} \right] \quad (2.4.6)$$

depends upon x only, then IF

$$\mu(x) = \exp \left\{ \int \frac{1}{N(x, y)} \left[\frac{\partial M(x, y)}{\partial y} - \frac{\partial N(x, y)}{\partial x} \right] dx \right\} \quad (2.4.7)$$

And if

$$\frac{1}{M(x, y)} \left[\frac{\partial N(x, y)}{\partial x} - \frac{\partial M(x, y)}{\partial y} \right] \quad (2.4.8)$$

depends upon y only, then IF

$$\mu(y) = \exp \left\{ \int \frac{1}{M(x, y)} \left[\frac{\partial N(x, y)}{\partial x} - \frac{\partial M(x, y)}{\partial y} \right] dy \right\} \quad (2.4.9)$$

Example 2.4.1:

$$(2x^2 + y) dx + (x^2y - x) dy = 0$$

This equation is not any of the five basic types of differential equations. We can apply Theorem 2.4.1 in this case. Here, $M(x, y) = 2x^2 + y$ and $N(x, y) = x^2y - x$, and the equation (2.4.6) becomes

$$\frac{1}{x^2y - x} [1 - (2xy - 1)] = \frac{2(1 - xy)}{x(xy - y)} = -\frac{2}{x}$$

This depends upon x only, so

$$\text{IF} = \exp \left(- \int \frac{2}{x} dx \right) = \exp(-2 \ln |x|) = \frac{1}{x^2}$$

Thus we obtain the equation

$$\left(2 + \frac{y}{x^2} \right) dx + \left(y - \frac{1}{x} \right) dy = 0$$

This equation is exact, and the solution is

$$2x + \frac{y^2}{2} - \frac{y}{x} = c$$

B A Special Transformation

Theorem 2.4.2 (A Special Transformation):

Consider the equation

$$(a_1x + b_1y + c_1) dx + (a_2x + b_2y + c_2) dy = 0 \quad (2.4.10)$$

where $a_1, b_1, c_1, a_2, b_2, c_2$ are constants.

Case 1: If $\frac{a_2}{a_1} \neq \frac{b_2}{b_1}$, then the transformation

$$\begin{aligned} x &= X + h \\ y &= Y + k \end{aligned}$$

where (h, k) is the solution of the system

$$\begin{aligned} a_1h + b_1k + c_1 &= 0 \\ a_2h + b_2k + c_2 &= 0 \end{aligned}$$

reduces the equation (2.4.10) to the Homogeneous Equation

$$(a_1X + b_1Y) dX + (a_2X + b_2Y) dY = 0 \quad (2.4.11)$$

Case 2: If $\frac{a_2}{a_1} = \frac{b_2}{b_1}$, then the transformation

$$z = a_1x + b_1y$$

reduces the equation (2.4.10) to a separable equation in the variables x and z .

Example 2.4.2:

$$(x - 2y + 1) dx + (4x - 3y - 6) dy = 0$$

Here,

$$\frac{a_2}{a_1} = 4 \neq \frac{3}{2} = \frac{b_2}{b_1}$$

Therefore, we make the transformation

$$\begin{aligned} x &= X + h \\ y &= Y + k \end{aligned}$$

where (h, k) is the solution of the system

$$\begin{aligned} h - 2k + 1 &= 0 \\ 4h - 3k - 6 &= 0 \end{aligned}$$

The solution of the system is $(3, 2)$, and so the transformation is

$$\begin{aligned} x &= X + 3 \\ y &= Y + 2 \end{aligned}$$

This reduces the given equation to the homogeneous equation

$$(X - 2Y) dX + (4X - 3Y) dY = 0$$

$$\frac{dY}{dX} = \frac{1 - 2(Y/X)}{3(Y/X) - 4}$$

Letting $Y = vX$,

$$\begin{aligned} v + X \frac{dv}{dX} &= \frac{1 - 2v}{3v - 4} \\ \int \frac{3v - 4}{3v^2 - 2v - 1} dv &= - \int \frac{dX}{X} \\ \frac{1}{2} \int \frac{d(3v^2 - 2v - 1)}{3v^2 - 2v - 1} - 3 \int \frac{dv}{3v^2 - 2v - 1} &= - \int \frac{dX}{X} \\ \frac{1}{2} \int \frac{d(3v^2 - 2v - 1)}{3v^2 - 2v - 1} - 3 \int \left[\frac{1}{4} \int \frac{dv}{v - 1} - \frac{3}{4} \int \frac{dv}{3v + 1} \right] &= - \int \frac{dX}{X} \\ \frac{1}{2} \ln |3v^2 - 2v - 1| - \frac{3}{4} \ln |v - 1| + \frac{9}{4} \ln |3v + 1| + \ln |X| &= \ln |c_1| \\ \ln \left| X^4 \cdot \frac{(v - 1)^2 (3v + 1)^{11}}{(v - 1)^3} \right| &= \ln |c| \\ |3Y + X|^{11} &= X^6 c |Y - X| \end{aligned}$$

$$\boxed{|x + 3y - 9|^{11} = c(x - 3)^6 |y - x + 1|}$$

Example 2.4.3:

$$(x + 2y + 3) dx + (2x + 4y - 1) dy = 0$$

Here,

$$\frac{a_2}{a_1} = 2 = \frac{b_2}{b_1}$$

Therefore, we apply the transformation

$$z = x + 2y$$

$$\therefore (z + 3) dx + (2z - 1) \left(\frac{dz - dx}{2} \right) = 0$$

$$7 dx + (2z - 1) dz = 0$$

$$7x + z^2 - z = c$$

$$7x + x^2 + 4y^2 + 4xy - x - 2y = c$$

$$\boxed{x^2 + 4xy + 4y^2 + 6x - 2y = c}$$

3 Explicit Methods of Solving Higher-Order Linear Differential Equations

3.1 Basic Theory of Linear Differential Equations

A Definition and Basic Existence Theorem

Definition 3.1.1: Linear ODE and Homogeneous DE of Order n

A **linear ordinary differential equation of order n** in the dependent variable y and the independent variable x is an equation that is in, or can be expressed in the form

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = F(x) \quad (3.1.1)$$

where a_0 is not identically zero. In the equation, a_0, a_1, \dots, a_n and F are continuous real functions on a real interval $a \leq x \leq b$ and that $a_0(x) \neq 0$ for any x on $a \leq x \leq b$. The $F(x)$ is called the nonhomogeneous term. If F is identically zero, Equation (3.1.1) reduces to

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = 0 \quad (3.1.2)$$

Equation (3.1.2) is a **homogeneous differential equation of order n** .

Example 3.1.1

The equation

$$y'' + 3xy' + x^3y = e^x$$

is a linear ordinary differential equation.

The equation

$$y''' + xy'' + 3x^2y' - 5y = \sin x$$

is a linear ODE of third order.

Theorem 3.1.1 (Basic Existence Theorem):

Hypothesis:

1. Consider the n th-order linear differential equation

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = F(x) \quad (3.1.3)$$

where a_0, a_1, \dots, a_n and F are real functions on a real interval $a \leq x \leq b$ and $a_0(x) \neq 0$ for any x on $a \leq x \leq b$.

2. Let x_0 be any point of the interval $a \leq x \leq b$, and let c_0, c_1, \dots, c_{n-1} be n arbitrary real constants.

Conclusion: There exists a unique solution f of (3.1.3) such that

$$f(x_0) = c_0, f'(x_0) = c_1, \dots, f^{(n-1)}(x_0) = c_{n-1}$$

and this solution is defined over the entire interval $a \leq x \leq b$.

Example 3.1.2: Consider the initial-value problem

$$2y''' + xy'' + 3x^2y' - 5y = \sin x$$

$$y(4) = 3$$

$$y'(4) = 5$$

$$y''(4) = -\frac{7}{2}$$

Here we have a third-order problem. The coefficients $2, x, 3x^2$, and -5 , as well as the nonhomogeneous term $\sin x$, are all continuous for all $x \in (-\infty, \infty)$. The point $x_0 = 4$ certainly belongs to this interval; the real numbers c_0, c_1 , and c_2 in this problem are $3, 5$, and $-\frac{7}{2}$ respectively. Theorem 3.1.1 assures us that this problem also has a unique solution which is defined for all $x \in (-\infty, \infty)$.

Corollary 3.1.2:

Hypothesis: Let f be a solution of the n th-order homogeneous linear DE

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = 0 \quad (3.1.4)$$

such that

$$f(x_0) = 0, f'(x_0) = 0, \dots, f^{(n-1)}(x_0) = 0,$$

where x_0 is a point of the interval $a \leq x \leq b$ in which the coefficients a_0, a_1, \dots, a_n are all continuous and $a_0(x) \neq 0$.

Conclusion: Then $f(x) = 0$ for all x on $a \leq x \leq b$.

Example 3.1.3

The unique solution of f of the third-order homogeneous equation

$$y''' + 2y'' + 4xy' + x^2y = 0$$

which is such that

$$f(2) = f'(2) = f''(2) = 0$$

is the trivial solution f such that $f(x) = 0$ for all x .

B The Homogeneous Equation

We now consider the fundamental results concerning the homogeneous equation

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = 0 \quad (3.1.5)$$

Theorem 3.1.3 (Basic Theorem on Linear Homogeneous Differential Equations):

Hypothesis: Let f_1, f_2, \dots, f_m be any m solutions of the homogeneous linear differential equation (3.1.5).

Conclusion: Then

$$c_1f_1 + c_2f_2 + \cdots + c_mf_m$$

is also a solution of (3.1.5), where c_1, c_2, \dots, c_m are m arbitrary constants.

In other words: Any linear combination of solutions of the homogeneous linear differential equation (3.1.5) is also a solution of (3.1.5).

Definition 3.1.2: Linear Combination

If f_1, f_2, \dots, f_m are m given functions, and c_1, c_2, \dots, c_m are m constants, then the expression

$$c_1f_1 + c_2f_2 + \cdots + c_mf_m$$

is called a linear combination of f_1, f_2, \dots, f_m .

Example 3.1.4

e^x, e^{-x}, e^{2x} are solutions of

$$y''' - 2y'' - y' + 2y = 0$$

Theorem 3.1.3 states that the linear combination $c_1e^x + c_2e^{-x} + c_3e^{2x}$ is also a solution for any constants c_1, c_2, c_3 . For example, the particular linear combination

$$2e^x - 3e^{-x} + \frac{2}{3}e^{2x}$$

is a solution.

Definition 3.1.3: Linear Dependence

The n functions f_1, f_2, \dots, f_n are called *linearly dependent* on $a \leq x \leq b$ if there exist constants c_1, c_2, \dots, c_n , not all zero, such that

$$c_1f_1(x) + c_2f_2(x) + \cdots + c_nf_n(x) = 0$$

for all x such that $a \leq x \leq b$.