Math-183
Differential Equations

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Contents

1 Differential Equations and Their Solutions

1.1 Classification of Differential Equations

Definition 1.1.1: Differential Equation

Differential equation is an equation involving derivatives of one or more dependent variables with respect to one or more independent variables.

Definition 1.1.2: Ordinary Differential Equation

A differential equation involving ordinary derivatives of one or more dependent variables with respect to a single independent variable is called an ordinary differential equation.

Example 1.1.1: Ordinary Differential Equations:

$$\frac{dy}{dx} + xy\left(\frac{d}{dx}\right)^2 = 0 \tag{1.1.1}$$

$$\frac{d^4x}{dt^4} + 5\frac{d^2x}{dt^2} + 3x = \sin t \tag{1.1.2}$$

Definition 1.1.3: Partial Differential Equation

A differential equation involving partial derivatives of one or more dependent variables with respect to more than one independent variables is called an partial differential equation.

Example 1.1.2: Partial Differential Equations:

$$\frac{\partial v}{\partial s} + \frac{\partial v}{\partial t} = v \tag{1.1.3}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \tag{1.1.4}$$

Definition 1.1.4: Order and Degree of Differential Equations

Order of DE: The order of the highest ordered derivative involved in a differential equation is called the order of the differential equation.

Degree of DE: The power of the highest order derivative involved in a differential equation is called the degree of the differential equation.

Definition 1.1.5: Linearity of Differential Equations

If the dependent variable and its various derivatives occur to the first degree only, the DE is a linear DE. Otherwise it's a non-linear DE.

$$a_0(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_1(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + a_{n-1}(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_n(x)y = b(x)$$

Linear DE can also be classified as linear with *constant* and *variable* coefficients.

Example 1.1.3: Ordinary Differential Equations: Orders, Degree, Linearity

$$\begin{aligned}
\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} - 3\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 3\frac{\mathrm{d}y}{\mathrm{d}x} - 6y = \sin x &\quad \text{3rd ord 1st deg Lin}\\
\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + y = 0 &\quad \text{2nd ord 1st deg Non-Lin}\\
y = x\frac{\mathrm{d}y}{\mathrm{d}x} + \sqrt{1 + \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}} &\quad \text{2nd ord 1st deg Non-Lin}\\
\frac{\mathrm{d}^4 x}{\mathrm{d}t^4} + t^2\frac{\mathrm{d}^3 x}{\mathrm{d}t^3} + \frac{\mathrm{d}y}{\mathrm{d}x} = \sin t &\quad \text{4th ord 1st deg Lin}\\
\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 5\frac{\mathrm{d}y}{\mathrm{d}x} + 6y^2 = 0 &\quad \text{2nd ord 1st deg Non-Lin}\\
\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 5\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3 + 6y = 0 &\quad \text{2nd ord 1st deg Non-Lin}\\
\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 5y\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = 0 &\quad \text{2nd ord 1st deg Lin}
\end{aligned}$$

1.2 Solutions

A Nature of Solutions

An nth-order Differential Equation:

$$F\left[x, y, \frac{\mathrm{d}y}{\mathrm{d}x}, \cdots, \frac{\mathrm{d}^n y}{\mathrm{d}x^n}\right] = 0 \tag{1.2.1}$$

Definition 1.2.1: Explicit solution

f is an explicit solution of (1.2.1) if

$$\forall x \in I, F\left[x, f(x), f'(x), \cdots, f^{(n)}(x)\right] = 0$$

where I is a real interval.

Definition 1.2.2: Implicit solution

$g(x, y) = 0$ is an implicit solution if this relation defines at least one real function $f(x)$ on an interval I such that f is an explicit solution of (1.2.1)

Example 1.2.1: Explicit and Implicit Solutions

$$x^2 + y^2 - 25 = 0 \quad : \quad \text{Implicit solution}$$
$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$x + y\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \quad : \quad \text{Differential Equation}$$
$$y = \pm\sqrt{25 - x^2}\ ;\ -5 \le x \le 5 \quad : \quad \text{Explicit solution}$$

B Methods of Solution

The study of a Differential Equation consists of 3 phases:

1. Formulation of DE from the given physical situation.
2. Solutions of DE, evaluating the arbitrary constants from the given condition.
3. Physical interpretation of the solution.

Example 1.2.2: Show that the function $f(x) = e^x + 2x^2 + 6x + 7$ **is a solution to the DE** $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 4x^2$

$$f(x) = e^x + 2x^2 + 6x + 7$$
$$f'(x) = e^x + 4x + 6$$
$$f''(x) = e^x + 4$$

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y &= (e^x + 4) - 3(e^x + 4x + 6) + 2(e^x + 2x^2 + 6x + 7)\\ &= 0 \cdot e^x + 0 \cdot x + (4 - 18 + 14) + 4x^2\\ &= 4x^2\end{aligned}$$

$\square$

Example 1.2.3: Show that the function $f(x) = \dfrac{1}{1+x^2}$ **is a solution to the DE** $(1 + x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0$

$$f(x) = \frac{1}{1+x^2}$$
$$(1 + x^2)f(x) = 1$$
$$(1 + x^2)f'(x) + 2xf(x) = 0$$
$$(1 + x^2)f''(x) + 2xf'(x) + 2xf'(x) + 2f(x) = 0$$
$$(1 + x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0$$

$\square$

Example 1.2.4: Show that the function $y = (2x^2 + 2e^{3x} + 3)e^{-2x}$ satisfies the DE

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 6e^x + 4xe^{-2x}$$

$$y = (2x^2 + 2e^{3x} + 3)e^{-2x}$$
$$y_1 = (4x + 6e^{3x})e^{-2x} - (2x^2 + 2e^{3x} + 3)2e^{-2x}$$
$$y_1 = 4xe^{-2x} + 6e^x - 2y$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 6e^x + 4e^{-2x}$$

□

1.3 Initial-Value and Boundary-Value Problems, and Existence of Solutions

A Initial-value Problems and Boundary-value Problems

One of the most frequently encountered type of problems in Differential Equations involves both a DE and one or more supplementary conditions which the solution of the given DE must satisfy.

Definition 1.3.1: IVP and BVP

Consider the first-order DE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$$

where f is a continuous function of x and y in some domain D of the xy plane; and let (x_0, y_0) be a point of D. The **initial-value problem** associated with the DE is to find a solution ϕ of the DE, defined on some real interval containing x_0, and satisfying the initial condition

$$\phi(x_0) = y_0$$

In the customary abbreviated notation, this initial-value problem may be written

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$$

$$y(x_0) = y_0$$

If the conditions relate to two different x values (the extreme or boundary values), the proble is called a **Two-Point Boundary-Value Problem** or simply a **Boundary-Value Problem (BVP)**.

Example 1.3.1: Find the solution of the DE $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$ such that $\forall x \in I, f'(x) = 2x$ and $f(1) = 4$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$$

$$\int \frac{\mathrm{d}y}{\mathrm{d}x}\, dx = \int 2x\, dx$$

$$y = x^2 + c$$

Substituting $y = 4$ and $x = 1$,

$$4 = 1 + c \text{ or } c = 3$$

$$\therefore \text{ Solution: } y^2 = x + 3$$

□

Example 1.3.2: $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$, $y(3) = 4$

$$x + y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\int x\, dx + \int y\frac{\mathrm{d}y}{\mathrm{d}x}\, dx = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} = c'$$

$$x^2 + y^2 = c$$

Substituting $x = 3$ and $y = 4$,

$$16^2 + 3^2 = c \text{ or } c = 25$$

$$\therefore \text{ Solution: } x^2 + y^2 - 25 = 0$$

B Existence of Solutions

Not all initial-value and boundary-value problems have solutions. For example,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y = 0$$

$$y(0) = 1 \;\; , \;\; y(\pi) = 5$$

has no solutions! Thus arises the question of *existence* of solutions. We can say, every initial-value problem that satisfies definition (1.3.1) has *at least one* solution. However, there arises another question. Can a problem have more than one solution?
Let's consider the initial-value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^{1/3} \;\; ; \;\; y(0) = 0$$

One may verify that the functions f_1 and f_2 defined, respectively, by

$$\forall x \in \mathbb{R}, \;\; f_1(x) = 0$$

and

$$f_2(x) = (\frac{2}{3}x)^{3/2}, \;\; x \geq 0; \;\; f_2(x) = 0, \;\; x \leq 0$$

are both solutions of this initial-value problem. In fact, this problem has infinitely many solutions. Hence, we can state that the initial-value problem need not have a *unique* solution. In order to ensure uniqeness, some additional requiremnt must certainly be imposed.

Theorem 1.3.1 **(Basic Existence and Uniqueness Theorem)*:***
Hypothesis: *Consider the differential equation*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) \tag{1.3.1}$$

where

- *The function f is a continuous function of x and y in some domain D of the xy plane, and*
- *The partial derivative $\frac{\partial f}{\partial y}$ is also a continuous function of x and y in D; and let (x_0, y_0) be a point in D.*

Conclusion: *There exists a unique solution ϕ of the differential equation (1.3.1), defined on some interval $|x - x_0| \leq h$, where h is sufficiently small, that satisfies the condition*

$$\phi(x_0) = y_0$$

Example 1.3.3: Show that

$$y = 4e^{2x} + 2e^{-3x}$$

is a solution of the initial-value problem

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} - 6y = 0$$
$$y(0) = 6$$
$$y'(0) = 2$$

Is $y = 2e^{2x} + 4e^{-3x}$ also a solution of this problem? Explain why or why not.

$$\begin{aligned} y &= 4e^{2x} + 2e^{-3x} \\ y_1 &= 8e^{2x} - 6e^{-3x} \\ y_2 &= 16e^{2x} + 18e^{-3x} \end{aligned}$$

$$\begin{aligned} y_2 + y_1 - 6y &= (16e^{2x} + 18e^{-3x}) + (8e^{2x} - 6e^{-3x}) - 6(4e^{2x} + 2e^{-3x}) \\ &= 0 \cdot e^{2x} + 0 \cdot e^{-3x} \\ &= 0 \end{aligned}$$

The solution also satisfies $y(0) = 6$ and $y'(0) = 2$

Now, for $y = 2e^{2x} + 4e^{-3x}$,

$$y_1 = 4e^{2x} - 12e^{-3x} \ \ ; \ \ y_2 = 8e^{2x} + 36e^{-3x}$$

$$\begin{aligned} y_2 + y_1 - 6y &= (8e^{2x} + 36e^{-3x}) + (4e^{2x} - 12e^{-3x}) - 6(2e^{2x} + 4e^{-3x}) \\ &= 0 \cdot e^{2x} + 0 \cdot e^{-3x} \\ &= 0 \end{aligned}$$

However, in this case,

$$y(0) = 6 \ \ ; \ \ y'(0) = -8$$

As we can see, this solution doesn't satisfy the initial-value problem. Hence $y = 2e^{2x} + 4e^{-3x}$ is not a solution of this problem.

Example 1.3.4: Given that every solution of

$$x^3 \frac{\mathrm{d}^3 y}{\mathrm{d}x^3} - 3x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 6x \frac{\mathrm{d}y}{\mathrm{d}x} - 6y = 0$$

may be written in the form $y = c_1 x + c_2 x^2 + c_3 x^3$ **for some choice of the arbitrary constants** c_1**,** c_2**, and** c_3**, solve the initial-value problem consisting of the above DE plus the three conditions**

$$y(2) = 0 \ \ , \ \ y'(2) = 2 \ \ , \ \ y''(2) = 6$$

$$y = c_1 x + c_2 x^2 + c_3 x^3$$

$$y(2) = 0 \text{ or, } 8c_3 + 4c_2 + 2c_1 = 0 \tag{1.3.2}$$

$$y' = c_1 + 2c_2 x + 3c_3 x^2$$

$$y'(2) = 2 \text{ or, } 12c_3 + 4c_2 + c_1 = 2 \tag{1.3.3}$$

$$y'' = 0 + 2c_2 + 6c_3 x$$

$$y''(2) = 6 \text{ or, } 12c_3 + 2c_2 + 0c_1 = 6 \tag{1.3.4}$$

Solving (1.3.1), (1.3.2), and (1.3.3) we get,

$$c_1 = 2 \ \ , \ \ c_2 = -3 \ \ , \ \ c_3 = 1$$

$$\therefore \text{ Solution: } y = 2x - 3x^2 + x^3$$

2 First Order Equations for Which Exact Solutions Are Obtainable

2.1 Exact Differential Equations and Integrating Factors

A Standard Forms of First-Order Differential Equations

The first-order differential equations may be expressed in either the **Derivative Form**

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) \tag{2.1.1}$$

or the **Differential Form**

$$M(x, y)\, dx + N(x, y)\, dy = 0 \tag{2.1.2}$$

Example 2.1.1: Standard Forms

The equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2 + y^2}{x - y}$$

is the form (2.1.1). It may be written as

$$(x^2 + y^2)\, dx + (y - x)\, dy = 0$$

which is of the form (2.1.2).
Again, the equation

$$(\sin x + y)\, dx + (x + 3y)\, dy = 0$$

is of the form (2.1.2), which can also be written as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\sin x + y}{x + 3y}$$

B Exact Differential Equations

Definition 2.1.1: Exact Differential

Let F be a function of two real variables such that F has continuous first partial derivatives in a domain D. The total differential dF of the function F is defined by the formula

$$dF(x, y) = \frac{\partial F(x, y)}{\partial x}\, dx + \frac{\partial F(x, y)}{\partial y}\, dy$$

for all $(x, y) \in D$.

Comparing $dF(x, y)$ with the form (2.1.2), we get

$$\frac{\partial F(x, y)}{\partial x} = M(x, y) \quad \textit{and} \quad \frac{\partial F(x, y)}{\partial y} = N(x, y)$$

Example 2.1.2

Let F be a function

$$F(x, y) = xy^2 + 2x^3 y$$

for all real (x, y). Then

$$\frac{\partial F(x,y)}{\partial x} = y^2 + 6x^2y, \quad \frac{\partial F(x,y)}{\partial y} = 2xy + 2x^3$$

and the total differential dF is defined by

$$dF(x,y) = (y^2 + 6x^2y)\,dx + (2xy + 2x^3)\,dy$$

for all real (x, y)

Definition 2.1.2: Exact Differential Equation

The expression

$$M(x,y)\,dx + N(x,y)\,dy \tag{2.1.3}$$

is called an exact differential in a domain D if there exists a function F of two real variables such that this expression equals the total differential $dF(x, y)$ for all $(x, y) \in D$. That is, expression (2.1.3) is an exact differential in D if there exists a function F such that

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) \quad \textit{and} \quad \frac{\partial F(x,y)}{\partial y} = N(x,y)$$

for all $(x, y) \in D$.
If $M(x, y)\,dx + N(x, y)\,dy$ is an exact differential, then the differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is called an **Exact Differential Equation**.

Theorem 2.1.1* (Exact Differential Equation)*:

1. *If the DE $M(x, y)\,dx + N(x, y)\,dy = 0$ is exact in D, then*

$$\forall (x,y) \in D, \quad \frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

2. *Conversely, if*

$$\forall (x,y) \in D, \quad \frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

then the DE is exact in D.

Proof (1):

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) \quad , \quad \frac{\partial F(x,y)}{\partial y} = N(x,y)$$

$$\frac{\partial^2 F(x,y)}{\partial x \partial y} = \frac{\partial M(x,y)}{\partial y} \quad , \quad \frac{\partial^2 F(x,y)}{\partial x \partial y} = \frac{\partial N(x,y)}{\partial x}$$

$$\because \frac{\partial^2 F(x,y)}{\partial y \partial x} = \frac{\partial^2 F(x,y)}{\partial x \partial y}$$

$$\therefore \frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x} \quad \square$$

C The Solution of Exact Differential Equations

***Theorem 2.1.2* (Solution of Exact DE):**
If $M(x,y)\,dx + N(x,y)\,dy = 0$ is exact in domain D, then

$$\forall (x,y) \in D, \exists F(x,y) : \frac{\partial F(x,y)}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial F(x,y)}{\partial y} = N(x,y)$$

Then the equation may be written

$$\frac{\partial F(x,y)}{\partial x}\,dx + \frac{\partial F(x,y)}{\partial y}\,dy = 0$$

or simply,

$$dF(x,y) = 0$$

Here, $F(x,y) = c$ is a one-parameter family of solutions of this DE, where c is an arbitrary constant.

Example 2.1.3: Solve the equation

$$(3x^2 + 4xy)\,dx + (2x^2 + 2y)\,dy = 0$$

Standard Method:

$$\begin{aligned}
\frac{\partial F(x,y)}{\partial x} &= M(x,y) = 3x^2 + 4xy \\
F(x,y) &= \int (3x^2 + 4xy)\,\partial x + \phi(y) \\
&= x^3 + 2x^2y + \phi(y)
\end{aligned}$$

Again,

$$\begin{aligned}
\frac{\partial F(x,y)}{\partial y} = 2x^2 + \frac{\partial \phi(y)}{\partial y} &= 2x^2 + 2y \\
\frac{\mathrm{d}\phi(y)}{\mathrm{d}y} &= 2y \\
\int \frac{\mathrm{d}\phi(y)}{\mathrm{d}y}\,dy &= \int 2y\,dy \\
\phi(y) &= y^2 + c_0
\end{aligned}$$

Thus, we get

$$F(x,y) = x^3 + 2x^2y + y^2 + c_0$$

Hence, a one-parameter family of the solution is $F(x,y) = c_1$ or,

$$x^3 + 2x^2y + y^2 + c_0 = c_1$$

$$\boxed{x^3 + 2x^2y + y^2 = c}$$

Method of Grouping:

$$\begin{aligned}(3x^2+4xy)\,dx+(2x^2+2y)\,dy&=0\\3x^2\,dx+(4xy\,dx+2x^2\,dy)+2y\,dy&=0\\d(x^3)+d(2x^2y)+d(y^2)&=d(c)\\\boxed{x^3+2x^2y+y^2=c}\end{aligned}$$

Example 2.1.4: Solve the initial-value problem

$$(2x\cos y+3x^2y)\,dx+(x^3-x^2\sin y-y)\,dy=0\ \ ;\ \ y(0)=2$$

$$\begin{aligned}(2x\cos y\,dx-x^2\sin y\,dy)+(3x^2y\,dx+x^3\,dy)-y\,dy&=0\\d(x^2\cos y)+d(x^3y)+d(\frac{y^2}{2})&=d(c_1)\\2x^2\cos y+x^3y+y^2&=c\end{aligned}$$

Substituting $x=0$ and $y=2$,

$$2^2=c$$

Hence, the solution is:

$$2x^2\cos y+x^3y+y^2=4$$

D Integrating Factors

Definition 2.1.3: Integrating Factor (IF)

If the DE

$$M(x,y)\,dx+N(x,y)\,dy=0 \tag{2.1.4}$$

is not exact in a domain D but the DE

$$\mu(x,y)M(x,y)\,dx+\mu(x,y)N(x,y)\,dy=0 \tag{2.1.5}$$

is exact in D, then $\mu(x,y)$ is called an **Integrating Factor** of the DE.

Example 2.1.5: Integrating factor

Consider the DE

$$(3y+4xy^2)\,dx+(2x+3x^2y)\,dy=0 \tag{2.1.6}$$

This equation is of the form (2.1.4), where

$$\begin{aligned}M(x,y)&=3y+4xy^2, & N(x,y)&=2x+3x^2y\\\frac{\partial M(x,y)}{\partial y}&=3+8xy, & \frac{\partial N(x,y)}{\partial x}&=2+6xy\end{aligned}$$

Since

$$\frac{\partial M(x,y)}{\partial y}\neq\frac{\partial N(x,y)}{\partial x}$$

except for (x,y) such that $2xy+1=0$, Equation (2.1.4) is not exact in any rectangular domain D.

Let $\mu(x, y) = x^2 y$. Then the corresponding DE of the form (2.1.5) is

$$(3x^2y^2 + 4x^3y^3)\,dx + (2x^3y + 3x^4y^2)\,dy = 0$$

This equation is exact in every rectangular domain D, since

$$\frac{\partial[\mu(x, y)M(x, y)]}{\partial y} = 6x^2y + 12x^3y^2 = \frac{\partial[\mu(x, y)N(x, y)]}{\partial x}$$

For all real (x, y). Hence, $\mu(x, y) = x^2 y$ is an integrating factor of Equation (2.1.6).