

# MATH-281

## Complex Variables

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# 1 Complex Numbers

## 1.1 Definition

### Definition 1.1.1: Complex Numbers

A complex number is a number that can be expressed in the form  $a + bi$ , where  $a$  and  $b$  are real numbers, and  $i$  is a solution of the equation  $x^2 = -1$ , or simply,  $i = \sqrt{-1}$ . Because no real number satisfies this equation,  $i$  is called an imaginary number. For the complex number  $a + bi$ ,  $a$  is called the real part, and  $b$  is called the imaginary part.

- The set of all complex numbers is denoted by  $\mathbb{C}$ .
- The set of all real numbers is denoted by  $\mathbb{R}$ .

### Definition 1.1.2: Modulus and Amplitude

Let  $z = a + bi$  be a complex number. The modulus of  $z$  is the non-negative real number  $|z| = \sqrt{a^2 + b^2}$ . The amplitude of  $z$  is the angle  $\theta$  such that  $\cos(\theta) = \frac{a}{|z|}$  and  $\sin(\theta) = \frac{b}{|z|}$ .

If the polar form of the point  $(a, b)$  be  $(r, \theta)$ , then  $a = r \cos \theta$  and  $b = r \sin \theta$ .

$$r = |z| = \sqrt{a^2 + b^2} \quad \text{and} \quad \theta = \arctan\left(\frac{b}{a}\right) \quad (1.1.1)$$

Here,  $r$  is the modulus of  $z$  and  $\theta$  is the amplitude of  $z$ .

In symbols, we write

$$r = \text{mod}(z) = |a + ib| \quad \text{and} \quad \theta = \arg(z) = \tan^{-1}\left(\frac{b}{a}\right) \quad (1.1.2)$$

---

## 1.2 De Moivre's Theorem

**Theorem 1.2.1 (De Moivre's Theorem):** Let  $z = r(\cos \theta + i \sin \theta)$  be a complex number. Then, for any positive integer  $n$ ,

$$z^n = r^n(\cos n\theta + i \sin n\theta) \quad (1.2.1)$$

**Proof:**

**Case 1:**  $n \in \mathbb{Z}_+$

We have,

$$\begin{aligned} z_1 z_2 \dots z_n &= (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2) \dots (\cos \theta_n + i \sin \theta_n) \\ &= \{\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)\} \dots (\cos \theta_n + i \sin \theta_n) \\ &= \cos(\theta_1 + \theta_2 + \dots + \theta_n) + i \sin(\theta_1 + \theta_2 + \dots + \theta_n) \end{aligned}$$

Hence, we get

$$z^n = (\cos n\theta + i \sin n\theta) \quad \square$$

**Case 2:**  $n \in \mathbb{Z}_-$

Let  $n = -m$ . We have,

$$\begin{aligned} z^n &= (\cos \theta + i \sin \theta)^{-m} \\ &= \frac{1}{(\cos \theta + i \sin \theta)^m} \\ &= \frac{1}{\cos m\theta + i \sin m\theta} \\ &= \frac{\cos m\theta - i \sin m\theta}{\cos^2 m\theta + \sin^2 m\theta} \\ &= \cos m\theta - i \sin m\theta \\ &= \cos n\theta + i \sin n\theta \end{aligned}$$

Hence, we get

$$z^n = (\cos n\theta + i \sin n\theta) \quad \square$$

**Case 3:**  $n \in \mathbb{Q}$ , i.e.  $n = \frac{p}{q}$ , where  $p, q \in \mathbb{Z}$  and  $q \neq 0$ .

Now,

$$\begin{aligned} \left( \cos \frac{p}{q}\theta + i \sin \frac{p}{q}\theta \right)^q &= \cos \left( q \cdot \frac{p}{q}\theta \right) + i \sin \left( q \cdot \frac{p}{q}\theta \right) \\ &= \cos p\theta + i \sin p\theta \\ &= (\cos \theta + i \sin \theta)^p \end{aligned}$$

Taking the  $q^{th}$  root of both sides, we get

$$\cos \frac{p}{q}\theta + i \sin \frac{p}{q}\theta = (\cos \theta + i \sin \theta)^{\frac{p}{q}} \quad \square$$

**Note:-**

**Some Important Results:**

- (i)  $1 = e^{i2n\pi} = \cos 2n\pi + i \sin 2n\pi$
- (ii)  $-1 = \cos \pi + i \sin \pi = e^{i\pi}$
- (iii)  $i = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} = e^{i\frac{\pi}{2}}$
- (iv)  $-i = \cos \frac{\pi}{2} - i \sin \frac{\pi}{2} = e^{-i\frac{\pi}{2}}$

## 2 Analytic Functions

### 2.1 Definitions

#### Definition 2.1.1: Complex variable

A **complex variable** is a variable that can take on complex values. A complex variable is usually denoted by  $z$ .

If  $x$  and  $y$  are real variables, then  $z = x + iy$  is a complex variable, where  $i$  is the imaginary unit.

#### Definition 2.1.2: Complex Function

A **complex function** is a function that takes complex variables as input and returns complex values. A complex function is usually denoted by  $f(z)$ .

If  $z = x + iy$  and  $w = u + iv$  are complex variables, then  $f(z) = u(x, y) + iv(x, y)$  is a complex function, where  $u(x, y)$  and  $v(x, y)$  are real functions.

#### Definition 2.1.3: Single-valued Function

A **single-valued function** is a function that returns a unique value for each input.

A complex function  $f(z)$  is single-valued if and only if  $f(z_1) = f(z_2)$  implies  $z_1 = z_2$ . In other words, if  $z_1 \neq z_2$ , then  $f(z_1) \neq f(z_2)$ .

$$\forall z_1, z_2 \in \mathbb{C} \quad \text{s.t.} \quad z_1 \neq z_2 \quad \text{implies} \quad f(z_1) \neq f(z_2)$$

#### Definition 2.1.4: Multiple-valued Function

A **multiple-valued function** is a function that returns multiple values for each input.

A complex function  $f(z)$  is multiple-valued if and only if  $f(z_1) = f(z_2)$  for some  $z_1 \neq z_2$ .

$$\exists z_1, z_2 \in \mathbb{C} \quad \text{s.t.} \quad z_1 \neq z_2 \quad \text{and} \quad f(z_1) = f(z_2)$$

#### Definition 2.1.5: Derivative

The **derivative** of a complex function  $f(z)$  is defined as

$$f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

where  $\Delta z$  is a complex number.

If the limit exists, then  $f(z)$  is said to be **differentiable** at  $z$ . If  $f(z)$  is differentiable at every point in a region  $R$ , then  $f(z)$  is said to be **analytic** in  $R$ .

### Definition 2.1.6: Analytic Function

A complex function  $f(z)$  is **analytic** in a region  $R$  if it is differentiable at every point in  $R$ .

If  $f(z)$  is analytic in a region  $R$ , then  $f(z)$  is also said to be **regular** or **holomorphic** in  $R$ .

## 2.2 Necessary Conditions for Analyticity

Let  $f(z) = u(x, y) + iv(x, y)$  be an analytic function in a region  $R$ .

That means,  $f(z)$  is differentiable at every point in  $R$ .

$$\Rightarrow f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \text{ exists at every point in } R.$$

Now, let  $z = x + iy$  and  $\Delta z = \Delta x + i\Delta y$ .

$$z + \Delta z = (x + \Delta x) + i(y + \Delta y)$$

Then,

$$\begin{aligned} f'(z) &= \lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0}} \frac{f(x + \Delta x + i(y + \Delta y)) - f(x + iy)}{\Delta x + i\Delta y} \\ &= \lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0}} \frac{u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y) - u(x, y) - iv(x, y)}{\Delta x + i\Delta y} \\ &= \lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0}} \frac{u(x + \Delta x, y + \Delta y) - u(x, y)}{\Delta x + i\Delta y} + i \lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0}} \frac{v(x + \Delta x, y + \Delta y) - v(x, y)}{\Delta x + i\Delta y} \end{aligned}$$

Along the real axis,  $\Delta y = 0$ . Hence, the limit is

$$\begin{aligned} f'(z) &= \lim_{\Delta x \rightarrow 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + i \lim_{\Delta x \rightarrow 0} \frac{v(x + \Delta x, y) - v(x, y)}{\Delta x} \\ f'(z) &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \end{aligned} \tag{2.2.1}$$

Along the imaginary axis,  $\Delta x = 0$ . Hence, the limit is

$$\begin{aligned} f'(z) &= \lim_{\Delta y \rightarrow 0} \frac{u(x, y + \Delta y) - u(x, y)}{i\Delta y} + i \lim_{\Delta y \rightarrow 0} \frac{v(x, y + \Delta y) - v(x, y)}{i\Delta y} \\ f'(z) &= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \end{aligned} \tag{2.2.2}$$

## 2.3 Cauchy-Riemann Equations

Since  $f'(z)$  exists, (2.2.4) and (2.2.5) must be equal.

$$\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} \quad (2.3.1)$$

Comparing real and imaginary parts,

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \quad \text{and} \quad \boxed{\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}} \quad (2.3.2)$$

These are called the **Cauchy-Riemann equations**.

---

## 2.4 Cauchy-Riemann Equations in Polar Form

Let

$$z = re^{i\theta}$$

$$f(z) = u(r, \theta) + iv(r, \theta)$$

Then

$$f(re^{i\theta}) = u(r, \theta) + iv(r, \theta) \quad (2.4.1)$$

Differentiating (2.4.1) with respect to  $r$ , we get

$$e^{i\theta} f'(re^{i\theta}) = \frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r} \quad (2.4.2)$$

Differentiating (2.4.1) with respect to  $\theta$ , we get

$$ire^{i\theta} f'(re^{i\theta}) = \frac{\partial u}{\partial \theta} + i \frac{\partial v}{\partial \theta} \quad (2.4.3)$$

Now, from (2.4.2) and (2.4.3),

$$\begin{aligned} \frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r} &= \frac{1}{ir} \left( \frac{\partial u}{\partial \theta} + i \frac{\partial v}{\partial \theta} \right) \\ \frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r} &= \frac{1}{ir} \frac{\partial u}{\partial \theta} + \frac{1}{r} \frac{\partial v}{\partial \theta} \\ \frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r} &= -\frac{i}{r} \frac{\partial u}{\partial \theta} + \frac{1}{r} \frac{\partial v}{\partial \theta} \end{aligned}$$

Equating the real and imaginary parts, we get

$$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}} \quad \text{and} \quad \boxed{\frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}} \quad (2.4.4)$$

These are the **Cauchy-Riemann equations in polar form**.

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## 3 Harmonic Function

### 3.1 Laplace's Equation

#### Definition 3.1.1: Laplace's Equation

An equation of the form

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \text{or} \quad \nabla^2 \phi = 0 \quad (3.1.1)$$

is called **Laplace's equation** (in two dimensions).

Here,  $\nabla^2$  is the Laplacian operator.

### 3.2 Harmonic Function

#### Definition 3.2.1: Harmonic Function

A function  $\phi(x, y)$  is called **harmonic** if it satisfies Laplace's equation

$$\nabla^2 \phi = 0 \quad (3.2.1)$$

where  $\nabla^2$  is the Laplacian operator.

**Theorem 3.2.2:** If  $f(z) = u + iv$  is an analytic function, then  $u$  and  $v$  are both harmonic functions.

#### Proof:

Since  $f(z)$  is analytic, it satisfies the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad (3.2.2)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad (3.2.3)$$

Differentiating (3.2.2) w.r.t.  $x$  and (3.2.3) w.r.t.  $y$ , we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad (3.2.4)$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \quad (3.2.5)$$

Adding (3.2.4) and (3.2.5), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad (3.2.6)$$

Similarly,

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \quad (3.2.7)$$

Hence, both  $u$  and  $v$  are harmonic functions.  $\square$

### Definition 3.2.3: Conjugate Harmonic Function

Any two functions  $\phi$  and  $\psi$  such that  $f(z) = \phi + i\psi$  is analytic, are called **Conjugate Harmonic Functions**.

## 3.3 Velocity Potential

Consider a two-dimensional flow of an incompressible fluid. The velocity of the fluid at a point  $(x, y)$  is given by the vector

$$\mathbf{v} = v_x \hat{\mathbf{i}} + v_y \hat{\mathbf{j}} \quad (3.3.1)$$

Here,  $v$  is called the stream function.

The **velocity potential**  $\phi(x, y)$  is defined as the scalar function such that

$$\mathbf{V} = \nabla\phi = \left( \hat{\mathbf{i}} \frac{\partial}{\partial x} + \hat{\mathbf{j}} \frac{\partial}{\partial y} \right) \phi = \hat{\mathbf{i}} \frac{\partial\phi}{\partial x} + \hat{\mathbf{j}} \frac{\partial\phi}{\partial y} \quad (3.3.2)$$

Comparing (3.3.1) and (3.3.2), we get

$$v_x = \frac{\partial\phi}{\partial x} \quad \text{and} \quad v_y = \frac{\partial\phi}{\partial y} \quad (3.3.3)$$

The scalar function  $\phi(x, y)$  gives the velocity components.  
Since the fluid is incompressible,

$$\begin{aligned} \nabla v &= 0 \\ \left( \hat{\mathbf{i}} \frac{\partial}{\partial x} + \hat{\mathbf{j}} \frac{\partial}{\partial y} \right) (\hat{\mathbf{i}} v_x + \hat{\mathbf{j}} v_y) &= 0 \\ \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} &= 0 \end{aligned}$$

Putting the values of  $v_x$  and  $v_y$  from (3.3.3),

$$\begin{aligned} \frac{\partial}{\partial x} \left( \frac{\partial\phi}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{\partial\phi}{\partial y} \right) &= 0 \\ \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} &= 0 \end{aligned}$$

This is Laplace's equation. Hence, the velocity potential  $\phi(x, y)$  is a harmonic function and is a real part of the analytic function

$$f(z) = \phi + i\psi$$

## 3.4 Method for Finding Conjugate Harmonic Function

### A Method 1: Real or Imaginary Part of an Analytic Function is Given

If  $f(z) = u + iv$  and  $u$  is known



We know that

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

Using C-R equations,

$$\begin{aligned} dv &= -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \\ v &= -\int \frac{\partial u}{\partial y}dx + \int \frac{\partial u}{\partial x}dy + C \end{aligned}$$

Since  $u$  is known,  $v$  can be found using the above method.

If  $v$  is known, then  $u$  can be found using the same method.

## B Method 2: Milne's Method/ Milne Thomson Method

By this method,  $f(z)$  is directly constructed without finding  $v$ .

Since

$$\begin{aligned} z &= x + iy \quad \text{and} \quad \bar{z} = x - iy \\ x &= \frac{z + \bar{z}}{2} \quad \text{and} \quad y = \frac{z - \bar{z}}{2i} \end{aligned}$$

Thus,

$$\boxed{f(z) = u\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right) + iv\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right)} \quad (3.4.1)$$

### Case 1: $u$ is given

Let  $f(z) = u + iv$  be an analytic function and  $u$  is given.

Then,

$$\frac{\partial u}{\partial x} = u_1(x, y) \quad \text{and} \quad \frac{\partial u}{\partial y} = u_2(x, y)$$

By Milne's method, we get

$$f'(z) = u_1(z, 0) - iu_2(z, 0) \quad (3.4.2)$$

Integrating (3.4.1) w.r.t.  $z$ , we get

$$f(z) = \int [u_1(z, 0) - iu_2(z, 0)] dz + C_1 \quad (3.4.3)$$

### Case 2: $v$ is given

If  $v$  is given, then

$$\frac{\partial v}{\partial y} = v_1(x, y) \quad \text{and} \quad \frac{\partial v}{\partial x} = v_2(x, y)$$

By Milne's method, we get

$$f'(z) = v_1(z, 0) + iv_2(z, 0) \quad (3.4.4)$$

Integrating (3.4.3) w.r.t.  $z$ , we get

$$f(z) = \int [v_1(z, 0) + iv_2(z, 0)] dz + C_2 \quad (3.4.5)$$

### 3.5 Complex Potential Function

#### Definition 3.5.1: Complex Potential Function

The analytic function

$$W = \phi(x, y) + i\psi(x, y)$$

is called the **Complex Potential Function**.

The real part  $\phi(x, y)$  represents the velocity potential function, and the imaginary part  $\psi(x, y)$  represents the stream function.

**Example 3.1:** If  $W = \phi + i\psi$  represents the complex potential for an electric field, and  $\psi = 3x^2y - y^3$ , then find  $\phi$ .

Given,

$$\psi = 3x^2y - y^3$$

Hence,

$$\begin{aligned}\frac{\partial \psi}{\partial y} &= 3x^2 - 3y^2 \\ \frac{\partial \phi}{\partial x} &= 6xy\end{aligned}$$

By Milne's method, we have

$$\begin{aligned}W'(z) &= \psi_1(z, 0) + i\psi_2(z, 0) \\ &= 3z^2 + i \cdot 0 \\ &= 3z^2\end{aligned}$$

Integrating  $W'(z)$  w.r.t.  $z$ , we get

$$\begin{aligned}W(z) &= \int 3z^2 dz + C \\ \phi + i\psi &= z^3 + c_1 + ic_2 \\ \phi + i\psi &= x^3 - 3xy^2 + c_1 + i(3x^2y - y^3 + c_2)\end{aligned}$$

Comparing real and imaginary parts, we get the required potential function

$$\boxed{\phi = x^3 - 3xy^2 + c_1}$$

## 4 Complex Integration

### 4.1 Definitions

#### Definition 4.1.1: Simply Connected Region

A connected region is said to be a **Simply Connected** region if all the interior points of a closed curve  $C$  drawn in the region  $D$  are the points of the region  $D$ .

#### Definition 4.1.2: Multi-Connected Region

A **Multi-connected** region is bounded by more than one curve. A multi-connected region can be divided into simply connected regions.

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### 4.2 Complex Line Integrals

#### Definition 4.2.1: Complex Line Integral

The **Complex Line Integral** of a function  $f(z)$  along a curve  $C$  is defined as

$$\oint_C f(z) dz = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(z_k^*) \Delta z_k \quad (4.2.1)$$

where  $z_k^*$  is a point on the curve  $C$  and  $\Delta z_k$  is the length of the curve  $C$ .

If  $z = x + iy$  and  $f(z) = u(x, y) + iv(x, y)$ , then

$$dz = dx + i dy$$

and

$$f(z) dz = (u dx - v dy) + i(u dy + v dx)$$

Hence, the complex line integral can be written as

$$\oint_C f(z) dz = \int_C (u dx - v dy) + i \int_C (u dy + v dx) \quad (4.2.2)$$

---

### 4.3 Cauchy's Integral Theorem

#### **Theorem 4.3.1 (Cauchy's Integral Theorem):**

If  $f(z)$  is analytic and its derivative  $f'(z)$  is continuous at all points inside and on a simple closed curve  $C$ , then

$$\oint_C f(z) dz = 0 \quad (4.3.1)$$

**Proof:**

Let the region enclosed by the curve  $C$  be  $D$ , and let

$$\begin{aligned} f(z) &= u(x, y) + iv(x, y) \\ z &= x + iy \\ dz &= dx + i dy \end{aligned}$$

Now,

$$\begin{aligned} \oint_C f(z) dz &= \oint_C (u + iv) (dx + idy) \\ &= \oint_C (u dx - v dy) + i \oint_C (u dy + v dx) \\ &= \oint_C u dx - v dy + i \oint_C u dy + v dx \\ &= \iint_R \left( -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx dy + i \iint_R \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx dy \end{aligned}$$

[By Green's Theorem]

Since  $f(z)$  is analytic, the Cauchy-Riemann equations hold, i.e.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Thus, we get

$$\oint_C f(z) dz = 0 \quad \square$$

## 4.4 Cauchy's Integral Formula

### **Theorem 4.4.1 (Cauchy's Integral Formula):**

If  $f(z)$  is analytic inside and on a simple closed curve  $C$ , and if  $a$  is a point inside the curve  $C$ , then

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - a} dz \quad (4.4.1)$$

### **Proof:**

Let  $f(z)$  be analytic inside and on a simple closed curve  $C$ , and let  $a$  be a point inside the curve  $C$ . Then, by Cauchy's Integral Theorem, we have

$$\oint_C f(z) dz = 0$$

Now, consider the function

$$g(z) = \frac{f(z)}{z - a}$$

This function is analytic inside and on the curve  $C$ , except at the point  $z = a$ . Thus, by Cauchy's Integral Theorem, we have

$$\oint_C g(z) dz = 0$$

Now, we have

$$\begin{aligned}
\oint_C g(z) dz &= \oint_C \frac{f(z)}{z-a} dz \\
&= \oint_C \frac{f(z)}{z-a} dz - \oint_C \frac{f(a)}{z-a} dz + \oint_C \frac{f(a)}{z-a} dz \\
&= \oint_C \frac{f(z) - f(a)}{z-a} dz + f(a) \oint_C \frac{1}{z-a} dz
\end{aligned}$$

For any point on the curve  $C_1$ , we have

$$z - a = re^{i\theta} \quad \text{and} \quad dz = ire^{i\theta} d\theta$$

$$\begin{aligned}
\oint_{C_1} \frac{f(z) - f(a)}{z-a} dz &= \int_{C_1} \frac{f(z) - f(a)}{z-a} dz \\
&= \int_0^{2\pi} \frac{f(re^{i\theta}) - f(a)}{re^{i\theta}} ire^{i\theta} d\theta \\
&= \int_0^{2\pi} i [f(re^{i\theta}) - f(a)] d\theta \\
&= 0 \\
\int_{C_1} \frac{1}{z-a} dz &= \int_0^{2\pi} \frac{ire^{i\theta}}{re^{i\theta}} d\theta \\
&= \int_0^{2\pi} i d\theta \\
&= 2\pi i
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\oint_C g(z) dz &= \oint_C \frac{f(z) - f(a)}{z-a} dz + f(a) \oint_C \frac{1}{z-a} dz \\
&= 0 + f(a) \cdot 2\pi i \\
&= 2\pi i f(a)
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\oint_C g(z) dz &= 2\pi i f(a) \\
\oint_C \frac{f(z)}{z-a} dz &= 2\pi i f(a) \\
f(a) &= \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz \quad \square
\end{aligned}$$

**Theorem 4.4.2 (Cauchy Integral Formula for the Derivative of an Analytic Function):**

If  $f(z)$  is analytic inside and on a simple closed curve  $C$ , and if  $a$  is a point inside the curve  $C$ , then

$$f'(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^2} dz \quad (4.4.2)$$

$$f''(a) = \frac{2!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^3} dz \quad (4.4.3)$$

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz \quad (4.4.4)$$

**Proof:**

The proof of these formulas can be obtained by differentiating the Cauchy Integral Formula and using the Cauchy Integral Formula for  $f(a)$ .

We know

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a} dz \quad (4.4.5)$$

Differentiating both sides with respect to  $a$ , we get

$$f'(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^2} dz \quad (4.4.6)$$

Differentiating again, we get

$$f''(a) = \frac{2!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^3} dz \quad (4.4.7)$$

Continuing this process, we get

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz \quad \square \quad (4.4.8)$$

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## 4.5 Cauchy's Extended Theorem

**Theorem 4.5.1 (Cauchy's Extended Theorem):**

If  $f(x)$  is analytic within and on the boundary of a region bounded by two closed curves  $C_1$  and  $C_2$ , then

$$\oint_{C_1} f(z) dz = \oint_{C_2} f(z) dz \quad (4.5.1)$$