# Laplace Transform

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# Contents

# 1 Laplace Transform

## 1.1 Definition and Existence

> **Definition 1.1.1: Laplace Transform**
>
> Let $F$ be a real-valued function of the real variable $t$, defined for $t > 0$. Let $s$ be a variable that we shall assume to be real, and consider the function $f$ defined by
>
> $$f(s) = \int_0^\infty e^{-st} F(t)\, dt \tag{1}$$
>
> for all values of $s$ for which this integral exists. The function $f$ defined by the integral (1) is called the Laplace Transform of the function $F$. We shall denote the Laplace transform of $F$ by $\mathscr{L}\{F(t)\}$.
>
> Thus the Laplace transform of a function $f$ is given by
>
> $$\mathscr{L}\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\, dt = \lim_{R \to \infty} \int_0^R e^{-st} F(t)\, dt \tag{2}$$

Some ways to write Laplace transforms:

$$\mathscr{L}F(t) = f(s) = \int_0^\infty e^{-st} F(t)\, dt$$
$$\mathscr{L}G(t) = g(s)$$
$$\mathscr{L}u(t) = \tilde{u}(s)$$

> **Theorem 1.1.2: Hypothesis:** *Let $F$ be a real function that has the following properties:*
>
> *1. $F$ is a piecewise continuous in every finite closed interval $0 \le t \le a$ ($b > 0$).*
>
> *2. $F$ is of exponential order, i.e, there exists $\alpha$, $M > 0$, and $t_0 > 0$ such that*
>
> $$e^{-\alpha t}|F(t)| < M \text{ for } t > t_0$$
>
> ***Conclusion:*** *The Laplace transform of $F$ exists for $s > \alpha$.*
>
> $$\mathscr{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt$$

## 1.2 Some Functions and Their Laplace Transforms

| $F(t)$ | $\mathscr{L}\{F(t)\} = f(s)$ | $F(t)$ | $\mathscr{L}\{F(t)\} = f(s)$ |
|--------|------------------------------|--------|------------------------------|
| $1$ | $\dfrac{1}{s}$ | $n$ | $\dfrac{n}{s}$ |
| $t$ | $\dfrac{1}{s^2}$ | $t^n$ | $\dfrac{n!}{s^{n+1}}$ |
| $e^{at}$ | $\dfrac{1}{s-a}$ | $e^{-at}$ | $\dfrac{1}{s+a}$ |
| $\sin at$ | $\dfrac{a}{s^2+a^2}$ | $\sinh at$ | $\dfrac{a}{s^2-a^2}$ |
| $\cos at$ | $\dfrac{s}{s^2+a^2}$ | $\cosh at$ | $\dfrac{s}{s^2-a^2}$ |

Table 1: Functions and their Laplace Transform

## Proofs :

$$\mathscr{L}\{n\} = \frac{n}{s}$$

$$\mathscr{L}\{t\} = \frac{1}{s^2}$$

Let $F(t) = t$, for $t > 0$
Then

Let $F(t) = n$, for $t > 0$
Then

$$\mathscr{L}\{t\} = \int_0^\infty e^{-st} \cdot t \, dt$$

$$\mathscr{L}\{n\} = \int_0^\infty e^{-st} \cdot n \, dt$$

$$= -t\frac{e^{-st}}{s} + \int_0^\infty \frac{e^{-st}}{s} \, dt$$

$$= n\frac{-e^{st}}{s}\bigg|_0^\infty$$

$$= -e^{-st}\frac{t}{s}\bigg|_0^\infty - e^{-st}\frac{1}{s^2}\bigg|_0^\infty$$

$$= \frac{n}{s} \quad \square$$

$$= \frac{1}{s^2} \quad \square$$

$$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

Let $F(t) = t^n$, for $t > 0$
Then

$$\mathscr{L}\{t^n\} = \int_0^\infty e^{-st}t^n \, dt$$

$$= -t^n\frac{e^{st}}{s} + \int_0^\infty nt^{n-1}\frac{e^{-st}}{s} \, dt$$

$$= -nt^{n-1}\frac{e^{-st}}{s^2}\bigg|_0^\infty + \int_0^\infty n(n-1)t^{n-2}\frac{e^{-st}}{s^2} \, dt$$

$$= -n(n-1)t^{n-2}\left(\frac{e^{-st}}{s^3}\right)\bigg|_0^\infty + \int_0^\infty n(n-1)(n-2)t^{n-3}\frac{e^{-st}}{s^3} \, dt$$

$$= \cdots$$

$$= n!t^{n-n}\frac{e^{-st}}{s^{n+1}}\bigg|_0^\infty + \int_0^\infty n(n-1)\cdots(n-n)\frac{e^{-st}}{s^{n+1}} \, dt$$

$$= \frac{n!}{s^{n+1}} \quad \square$$

Let $F(t) = e^{at}$, for $t > 0$
Then

Let $F(t) = e^{-at}$, for $t > 0$
Then

$$\mathscr{L}\{e^{at}\} = \int_0^\infty e^{-st}e^{at} \, dt$$

$$\mathscr{L}\{e^{-at}\} = \int_0^\infty e^{-st}e^{-at} \, dt$$

$$= \int_0^\infty e^{(a-s)t} \, dt$$

$$= \int_0^\infty e^{-(a+s)t} \, dx$$

$$= \frac{e^{(a-s)t}}{a-s}\bigg|_0^\infty$$

$$= \frac{e^{-(a+s)t}}{s+a}\bigg|_0^\infty$$

$$= \frac{1}{s-a} \quad \square$$

$$= \frac{1}{s+a} \quad \square$$

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Let $F(t) = \sin at$, for $t > 0$
Then

$$\mathscr{L}\{\sin at\} = \int_0^\infty e^{-st\sin at}\, dt$$

$$= -\frac{e^{-st}}{s^2 + a^2}\left(s\sin at + a\cos at\right)\Big|_0^\infty$$

$$= \frac{a}{s^2 + a^2} \quad \square$$

Let $F(t) = \cos at$, for $t > 0$
Then

$$\mathscr{L}\{\cos at\} = \int_0^\infty e^{-st}\cos at\, dt$$

$$= \frac{e^{-st}}{s^2 + a^2}\left(-s\cos at + a\sin at\right)\Big|_0^\infty$$

$$= \frac{s}{s^2 + a^2} \quad \square$$

# 2    Basic Properties of the Laplace Transform

## 2.1    Linearity Property

> **_Theorem_ (The Linearity Property):**
> Let $F_1$ and $F_2$ be functions whose Laplace transform exist, and let $c_1$ and $c_2$ be constants.
> Then
> $$\mathscr{L}\{c_1 F_1(t) + c_2 F_2(t)\} = c_1\mathscr{L}\{F_1(t)\} + c_2\mathscr{L}\{F_2(t)\}$$

**Proof :**
Let $F(t) = c_1 F_1(t) + c_2 F_2(t)$, for $t > 0$
Then

$$\mathscr{L}\{c_1 F_1(t) + c_2 F_2(t)\} = \int_0^\infty e^{-st}\left[c_1 F_1(t) + c_2 F_2(t)\right] dt$$

$$= c_1\int_0^\infty e^{-st} F_1(t)\, dt + c_2\int_0^\infty e^{-st} F_2(t)\, dt$$

$$= c_1\mathscr{L}\{F_1(t)\} + c_2\mathscr{L}\{F_2(t)\} \quad \square$$

**Example 2.1:**

$$\mathscr{L}\{4t^2 - 3\cos 2t + 5e^{-t}\}$$

$$\mathscr{L}\{4t^2 - 3\cos 2t + 5e^{-t}\} = 4\mathscr{L}\{t^2\} - 3\mathscr{L}\{\cos 2t\} + 5\mathscr{L}\{e^{-t}\}$$

$$= 4\cdot\frac{2!}{s^3} - 3\cdot\frac{s}{s^2 + 4} + 5\cdot\frac{1}{s + 1}$$

$$= \frac{8}{s^3} - \frac{3s}{s^2 + 4} + \frac{5}{s + 1}$$

**Example 2.2: Find** $\mathscr{L}\{F(w)\}$**, when** $F(t) = \begin{cases} 5 & \textbf{for } 0 < t < 3 \\ 0 & \textbf{for } t > 3 \end{cases}$

$$\mathscr{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt = \int_0^3 e^{-st}\cdot 5\, dt + \int_3^\infty 0\, dt$$

$$= \int_0^3 e^{-st}\cdot 5\, dt$$

$$= \frac{5e^{-st}}{s}\Big|_0^3$$

$$= \frac{5}{s}\left(1 - e^{-3s}\right)$$

**Example 2.3: Find** $\mathscr{L}\{F(t)\}$**, when** $F(t = \begin{cases} (t - 1)^2 & \textbf{for } t > 1 \\ 0 & \textbf{for } 0 < t < 1 \end{cases})$

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$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt = \int_0^1 0\, dt + \int_1^\infty e^{-st}(t-1)^2\, dt$$

$$= \int_1^\infty e^{-st}(t^2 - 2t + 1)\, dt$$

$$= -t^2 \frac{e^{-st}}{s}\Big|_1^\infty - 2\int_1^\infty t \cdot \frac{e^{-st}}{s}\, dt - 2\int_1^\infty t \cdot e^{-st}\, dt - \frac{e^{-st}}{s}$$

$$= \frac{e^{-s}}{s} + 2\left[-\frac{t}{s}\left(\frac{e^{-st}}{s}\right)\Big|_1^\infty + \int_1^\infty \frac{e^{-st}}{s^2}\, dt + 2t\frac{e^{-st}}{s}\Big|_1^\infty - \int_1^\infty \frac{e^{-st}}{s}\, dt\right] + \frac{e^{-st}}{s}$$

$$= \frac{e^{-s}}{s} - 2\frac{e^{-s}}{s^2} + 2\frac{e^{-s}}{s^2} - 2\frac{e^{-st}}{s} + 2\frac{e^{-s}}{s^2} + \frac{e^{-st}}{s}$$

$$= 2\frac{e^{-s}}{s^3}$$

## 2.2 First Translation Property

> **Theorem 2.2.1 (First Translation of Shifting Property):**
> *If*
> $$\mathcal{L}\{F(t)\} = f(s)$$
> *then*
> $$\mathcal{L}\{e^{at} F(t)\} = f(s - a)$$

**Proof:**
Let $G(t) = e^{at} F(t)$, for $t > 0$
Then

$$\mathcal{L}\{e^{at} F(t)\} = \int_0^\infty e^{-st} e^{at} F(t)\, dt$$

$$= \int_0^\infty e^{-(s-a)t} F(t)\, dt$$

$$= \mathcal{L}\{F(t)\}\Big|_{s-a}$$

$$= f(s - a) \quad \square$$

**Example 2.4:**
$$\mathcal{L}\{e^{-t} \cos 2t\}$$

Since $\mathcal{L}\{\cos 2t\} = \dfrac{s}{s^2 + 4}$, we have

$$\mathcal{L}\{e^{-t} \cos 2t\} = \frac{s+1}{(s+1)^2 + 4} = \frac{s+1}{s^2 + 2s + 5}\}$$

**Example 2.5: Evaluate** $\mathcal{L}\{e^{-2t}(3\cos 6t - 5\sin 6t)\}$
Now,

$$f(s) = \mathcal{L}\{3\cos 6t - 5\sin 6t\}$$

$$= \frac{3s}{s^2 + 36} - \frac{30}{s^2 + 36}$$

$$= \frac{3s - 30}{s^2 + 36}$$

$$\therefore \mathcal{L}\{e^{-2t}(3\cos 6t - 5\sin 6t)\} = f(s + 2)$$

$$= \frac{3(s+2) - 30}{(s+2)^2 + 36}$$

$$= \frac{3s - 24}{s^2 + 4s + 40}$$

## 2.3   Second Translation Property

> **Theorem 2.3.1 (Second Translation or Shifting Property):**
> *If*
> $$\mathscr{L}\{F(t)\} = f(s)$$
> *and*
> $$G(t) = \begin{cases} F(t-a) & \text{for } t > a \\ 0 & \text{for } t < a \end{cases}$$
> *then*
> $$\mathscr{L}\{G(t)\} = e^{-as} f(s)$$

**<u>Proof:</u>**

Let $G(t) = \begin{cases} F(t-a) & \text{for } t > a \\ 0 & \text{for } t < a \end{cases}$

Then

$$\begin{aligned}
\mathscr{L}\{G(t)\} &= \int_0^\infty e^{-st} F(t-a)\,dt \\
&= \int_a^\infty e^{-st} F(t-a)\,dt \\
&= \int_0^\infty e^{-s(u+a)} F(u)\,du \quad \text{where } u = t - a \\
&= e^{-as} \int_0^\infty e^{-su} F(u)\,du \\
&= e^{-as} \mathscr{L}\{F(t)\} \\
&= e^{-as} f(s) \quad \square
\end{aligned}$$

---

**Example 2.6:**  Find $\mathscr{L}\{G(t)\}$ **where** $G(t) = \begin{cases} \cos\left(t - \frac{2\pi}{3}\right) & \text{for } t > \frac{2\pi}{3} \\ 0 & \text{for } t < \frac{2\pi}{3} \end{cases}$

$$\begin{aligned}
\mathscr{L}\{G(t)\} &= e^{-\frac{2\pi}{3}s} \cdot \mathscr{L}\{\cos t\} \\
&= e^{-\frac{2\pi}{3}s} \cdot \frac{s}{s^2 + 1} \\
&= \frac{s\, e^{-\frac{2\pi}{3}s}}{s^2 + 1}
\end{aligned}$$

---

**Example 2.7: Find** $\mathscr{L}\{F(t)\}$**, if** $F(t) = \begin{cases} (t-1)^2 & \text{for } t > 1 \\ 0 & \text{for } t < 0 \end{cases}$

Let
$$G(t) = t^2$$
$$\therefore \mathscr{L}\{G(t)\} = \frac{2!}{s^3}$$

Now,
$$F(t) = \begin{cases} G(t-1) & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases}$$

$$\therefore \mathscr{L}\{F(t)\} = \frac{e^{-s} \cdot 2!}{s^3} = \frac{2e^{-s}}{s^3}$$

## 2.4   Change of Scale Property

> **Theorem 2.4.1 (Change of Scale Property):**
> If
> $$\mathcal{L}\{F(t)\} = f(s)$$
> then
> $$\mathcal{L}\{F(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right)$$

**Proof:**
Let $G(t) = F(at)$, for $t > 0$
Then

$$
\begin{aligned}
\mathcal{L}\{G(t)\} &= \int_0^\infty e^{-st} F(at)\, dt \\
&= \int_0^\infty e^{-\frac{s}{a}u} F(u)\, d(u/a) \quad \text{where } u = at \\
&= \frac{1}{a} \int_0^\infty e^{-\frac{s}{a}u} F(u)\, du \\
&= \frac{1}{a}\mathcal{L}\{F(t)\} \\
&= \frac{1}{a} f\left(\frac{s}{a}\right) \quad \square
\end{aligned}
$$

**Example 2.8: Evaluate** $\mathcal{L}\{\sin 3t\}$

$$
\begin{aligned}
\mathcal{L}\{\sin 3t\} &= \frac{1}{3} \cdot \frac{1}{\left(\frac{s}{3}\right) + 1} \\
&= \frac{1}{3} \cdot \frac{3^2}{s^2 + 3^2} \\
&= \frac{3}{s^2 + 9}
\end{aligned}
$$

**Example 2.9: If** $\mathcal{L}\{\frac{\sin t}{t} = \tan^{-1}\frac{1}{s} = f(s)\}$**, then evaluate** $\mathcal{L}\{\frac{\sin at}{t}\}$

$$
\begin{aligned}
\mathcal{L}\{\frac{\sin at}{t}\} &= \mathcal{L}\left\{a \cdot \frac{\sin at}{t}\right\} \\
&= a \cdot \mathcal{L}\left\{\frac{\sin at}{at}\right\} \\
&= a \cdot \frac{1}{a} f\left(\frac{s}{a}\right) \\
&= \tan^{-1}\frac{a}{s}
\end{aligned}
$$

## 2.5   Multiplication by $t$

> **Theorem 2.5.1 (Multiplication by $t^n$):**
> If
> $$\mathcal{L}\{F(t)\} = f(s)$$
> then
> $$\mathcal{L}\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s)$$

**Proof:**
Let $G(t) = t^n F(t)$, for $t > 0$
Then

$$\mathcal{L}\{G(t)\} = \int_0^\infty e^{-st} t^n F(t) \, dt$$

$$= (-1)^n \int_0^\infty e^{-st} \frac{d^n}{ds^n} F(t) \, dt$$

$$= (-1)^n \frac{d^n}{ds^n} \int_0^\infty e^{-st} F(t) \, dt$$

$$= (-1)^n \frac{d^n}{ds^n} f(s) \quad \square$$

**Alternative Proof:**

We have

$$f(s) = \int_0^\infty e^{-st} F(t) \, dt$$

Then by Leibnitz's rule for differentiating under the integral sign, we have

$$\frac{df}{ds} = f'(s) = \frac{d}{ds} \int_0^\infty e^{-st} F(t) \, dt$$

$$= \int_0^\infty \frac{d}{ds} \left( e^{-st} F(t) \right) dt$$

$$= \int_0^\infty e^{-st} \left( -t e^{-st} F(t) + F'(t) \right) dt$$

$$= - \int_0^\infty e^{-st} \{t F(t)\} \, dt$$

$$= -\mathcal{L}\{t F(t)\}$$

$$\therefore \mathcal{L}\{t F(t)\} = -\frac{d}{ds} f(s) = f'(s)$$

This proves the theorem for $n = 1$.

To establish the theorem in general, we use mathematical induction. Suppose that the theorem is true for $n = k$, i.e

$$\int_0^\infty e^{-st} \{t^k F(t)\} \, dt = (-1)^k f^{(k)}(s) \tag{2.5.1}$$

Then

$$\frac{d}{ds} \left[ \int_0^\infty e^{-st} \{t^k F(t)\} \, dt \right] = (-1)^k f^{(k+1)}(s)$$

Or by Leibnitz's rule,

$$- \int_0^\infty \frac{d}{ds} \left( e^{-st} \{t^{k+1} F(t)\} \right) dt = (-1)^{(k)} f^{(k+1)}(s)$$

i.e

$$\int_0^\infty e^{-st} \{t^{k+1} F(t)\} \, dt = (-1)^{(k+1)} f^{(k+1)}(s) \tag{2.5.2}$$

If follows that if (2.5.1) is true for $n = k$, then (2.5.2) is true for $n = k + 1$. Since (2.5.1) is true for $n = 1$, it follows that (2.5.1) is true for all positive integers $n$. $\quad \square$

**Example 2.10:** Find $\mathcal{L}\{t^2 \cos at\}$

$$\mathcal{L}\{t^2 \cos at\} = (-1)^2 \cdot \frac{d^2}{dx^2} \left( \frac{s}{s^2 + a^2} \right)$$

$$= \frac{d}{ds} \left[ \frac{s^2 + a^2 - 2s^2}{(s^2 + a^2)^2} \right]$$

$$= \frac{d}{ds} \left[ \frac{a^2 - s^2}{(s^2 + a^2)^2} \right]$$

$$= \frac{(s^2 + a^2)^2(-2s) - (-s^2 + a^2)2(s^2 + a^2) \cdot 2s}{(s^2 + a^2)^4}$$

$$= \frac{2s^3 - 6a^2 s}{(s^2 + a^2)^3}$$

## 2.6 Division by $t$

> **Theorem 2.6.1 (Division by $t$):**
> If
> $$\mathcal{L}\{F(t)\} = f(s)$$
> then
> $$\mathcal{L}\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(x)\, dx$$

**Proof:**

Let $G(t) = \dfrac{F(t)}{t}$, for $t > 0$. Then $F(t) = tG(t)$. Taking the Laplace Transform of both sides, we get

$$\mathcal{L}\{F(t)\} = \mathcal{L}\{tG(t)\}$$
$$f(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}\{G(t)\} = -\frac{\mathrm{d}}{\mathrm{d}s}g(s)$$

Then integrating, we have

$$g(s) = -\int_\infty^s f(u)\, du$$
$$\mathcal{L}\{G(t)\} = \int_s^\infty f(u)\, du$$
$$\mathcal{L}\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(u)\, du \quad \square$$

> **Example 2.11: Find $\mathcal{L}\left\{\frac{\sin t}{t}\right\}$**
>
> $$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{s^2+1}\, du$$
> $$= \left. \tan^{-1} u \right|_s^\infty$$
> $$= \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s$$
> $$= \tan^{-1}\frac{1}{s}$$

## 2.7 Laplace Transform of Integral

> **Theorem 2.7.1 (Laplace transform of Integral):**
> If
> $$\mathcal{L}\{F(t)\} = f(s)$$
> then
> $$\mathcal{L}\left\{\int_0^t F(x)\, dx\right\} = \frac{1}{s}f(s)$$

**Proof:**

Let $G(t) = \displaystyle\int_0^t F(x)dx$, for $t > 0$. Then $G'(t) = F(t)$ and $G(0) = 0$. Taking the Laplace Transform of both sides, we have

$$\mathcal{L}\{G'(t)\} = \mathcal{L}\{F(t)\}$$
$$s\mathcal{L}\{G(t)\} - G(0) = f(s)$$
$$s\mathcal{L}\{G(t)\} = f(s)$$
$$\mathcal{L}\{G(t)\} = \frac{f(s)}{s}$$
$$\mathcal{L}\left\{\int_0^t F(u)\, du\right\} = \frac{f(s)}{s} \quad \square$$

**Example 2.12:** Evaluate $\mathcal{L}\{\int_0^t \frac{\sin u}{u} \, du\}$

$$\mathcal{L}\left\{\frac{\sin u}{u}\right\} = \tan^{-1}\frac{1}{s}$$

$$\mathcal{L}\left\{\int_0^t \frac{\sin u}{u} \, du\right\} = \frac{f(s)}{s} = \frac{1}{s}\tan^{-1}\frac{1}{s}$$

**Example 2.13:** Evaluate $\mathcal{L}\{\int_0^t \frac{\sin u}{u} \, du\}$

Let $F(t) = \frac{\sin t}{t}$

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{u^2+1} \, du$$

$$= \tan^{-1} u \Big|_s^\infty$$

$$= \tan^{-1}\frac{1}{s}$$

$$\therefore \mathcal{L}\left\{\int_0^t \frac{\sin u}{u} \, du\right\} = \frac{1}{s}\cdot\tan^{-1}\frac{1}{s}$$

**Example 2.14:** Evaluate $\mathcal{L}\{\int_0^t \sin 2u \, du\}$

$$\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4}$$

$$\therefore \mathcal{L}\left\{\int_0^t \sin 2u \, du\right\} = \frac{2}{s^3+4s}$$

## 2.8   Laplace Transform of Periodic Functions

> *Theorem 2.8.1* **(Periodic functions):**
> *If*
> $$\mathcal{L}\{F(t)\} = f(s)$$
> *then*
> $$\mathcal{L}\{F(t)\} = \frac{1}{1-e^{-Ts}}\int_0^T e^{-st}F(t)\,dt$$
> *where $T$ is the period of $F(t)$.*

**Proof:**
Let $F(t)$ has period $T$. Then $F(t) = F(t+T)$ for all $t$. Then

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st}F(t)\,dt$$

$$= \int_0^T e^{-st}F(t)\,dt + \int_T^{2T} e^{-st}F(t)\,dt + \int_{2T}^{3T} e^{-st}F(t)\,dt + \cdots$$

$$= \int_0^T e^{-st}F(t)\,dt + \int_0^T e^{-s(t+T)}F(t+T)\,dt + \int_0^T e^{-s(t+2T)}F(t+2T)\,dt + \cdots$$

$$= \int_0^T e^{-st}F(t)\,dt + e^{-sT}\int_0^T e^{-st}F(t)\,dt + e^{-2sT}\int_0^T e^{-st}F(t)\,dt + \cdots$$

$$= \left[1 + e^{-sT} + e^{-2sT} + \cdots\right]\int_0^T e^{-st}F(t)\,dt$$

$$= \frac{1}{1-e^{-sT}}\int_0^T e^{-st}F(t)\,dt \quad \square$$

**Example 2.15:** Find $\mathscr{L}\{F(t)\}$ for

$$F(t) = \begin{cases} \sin t & \textbf{for } 0 < t < \pi \\ 0 & \textbf{for } \pi < t < 2\pi \end{cases}$$

$$\mathscr{L}\{F(t)\} = \frac{1}{1 - e^{-2\pi s}} \int_0^{2\pi} e^{st} \sin t \, dt$$

$$= \frac{1}{1 - e^{-2\pi s}} \int_0^{\pi} e^{-st} \sin t \, dt$$

**Example 2.16:** Find $\mathscr{L}\{F(t)\}$ for

$$F(t) = \begin{cases} t & \textbf{for } 0 < t < 1 \\ 0 & \textbf{for } 1 < t < 2 \end{cases}$$

$$\mathscr{L}\{F(t)\} = \frac{1}{1 - e^{-2s}} \int_0^1 t \, dt$$

$$= \frac{t^2 \Big|_0^1}{2 - 2e^{-2s}}$$

$$= \frac{1}{2 - 2e^{-2s}}$$

## 2.9   Laplace Transform of Derivatives

**Theorem 2.9.1 (Laplace transform of derivatives):**
*If*

$$\mathscr{L}\{F(t)\} = f(s)$$

*then*

$$\mathscr{L}\{F'(t)\} = sf(s) - F(0)$$

$$\mathscr{L}\{F''(t)\} = s^2 F(s) - sF(0) - F'(0)$$

$$\mathscr{L}\{F'''(t)\} = s^3 f(s) - s^2 F(0) - sF'(0) - F''(0)\}$$

$$\mathscr{L}\{F^{(n)}(t)\} = s^n f(s) - s^{n-1}F(0) - s^{n-2}F'(0) - \cdots - F^{(n-1)}(0)$$

$$\mathscr{L}\{F^{(n)}(t)\} = s^n f(s) - \sum_{i=n-1}^{0} \sum_{j=0}^{n-1} s^i F^{(j)}(0)$$

**Proof:**
Using integration by parts, we have

$$\mathscr{L}\{F'(t)\} = \int_0^\infty e^{-st} F'(t) \, dt$$

$$= e^{-st} F(t) \Big|_0^\infty + s \int_0^\infty e^{-st} F(t) \, dt$$

$$= -F(0) + s \int_0^\infty e^{-st} F(t) \, dt$$

$$= sf(s) - F(0) \quad \square$$

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Similarly,

$$\mathcal{L}\{F''(t)\} = s\mathcal{L}\{F'(t)\} - F'(0)$$
$$= s\left[sf(s) - F(0)\right] - F'(0)$$
$$= s^2 f(s) - sF(0) - F'(0)$$

Thus using mathematical induction, we get

$$\mathcal{L}\{F^{(n)}(t)\} = s\mathcal{L}\{F^{(n-1)}(t)\} - F^{(n-1)}(0)$$
$$= s\left[s\mathcal{L}\{F^{(n-2)}(t)\} - F^{(n-2)}(0)\right] - F^{(n-1)}(0)$$
$$= s^2\mathcal{L}\{F^{(n-2)}(t)\} - sF^{(n-2)}(0) - F^{(n-1)}(0)$$
$$= s^2\left[s\mathcal{L}\{F^{(n-3)}(t)\} - F^{(n-3)}(0)\right] - sF^{(n-2)}(0) - F^{(n-1)}(0)$$
$$= s^3\mathcal{L}\{F^{(n-3)}(t)\} - s^2 F^{(n-3)}(0) - sF^{(n-2)}(0) - F^{(n-1)}(0)$$
$$= \cdots$$
$$= s^n\mathcal{L}\{F(t)\} - s^{n-1}F(0) - s^{n-2}F'(0) - \cdots - F^{(n-1)}(0)$$
$$= s^n f(s) - \sum_{i=n-1}^{0}\sum_{j=0}^{n-1} s^i F^{(j)}(0) \quad \square$$

# 3 Inverse Laplace Transform

## 3.1 Definition and Existence

> **Definition 3.1.1: Inverse Laplace Transform**
>
> If the Laplace Transform of a function $F(t)$ is $f(s)$, i.e
>
> $$\mathcal{L}\{F(t)\} = f(s)$$
>
> then the Inverse Laplace Transform is defined as
>
> $$\mathcal{L}^{-1}\{f(s)\} = F(t)$$

## 3.2 Some Functions and their Inverse Laplace Transforms

| $f(s)$ | $\mathcal{L}^{-1}\{f(s)\} = F(t)$ | $f(s)$ | $\mathcal{L}^{-1}\{f(s)\} = F(t)$ |
|:---:|:---:|:---:|:---:|
| $\dfrac{1}{s}$ | $1$ | $\dfrac{n}{s}$ | $n$ |
| $\dfrac{1}{s^2}$ | $t$ | $\dfrac{1}{s^n}$ | $\dfrac{t^{n-1}}{(n-1)!}$ |
| $\dfrac{1}{s-a}$ | $e^{at}$ | $\dfrac{1}{s+a}$ | $e^{-at}$ |
| $\dfrac{1}{s^2+a^2}$ | $\dfrac{\sin at}{a}$ | $\dfrac{1}{s^2-a^2}$ | $\dfrac{\sinh at}{a}$ |
| $\dfrac{s}{s^2+a^2}$ | $\cos at$ | $\dfrac{s}{s^2-a^2}$ | $\cosh at$ |

Table 2: Functions and their Inverse Laplace Transform

# 4 Basic Properties of Inverse Laplace Transform

## 4.1 Linearity Property

> **Theorem 4.1.1 (Linearity Property):**
> If
> $$\mathcal{L}^{-1}\{f(s)\} = F(t) \quad and \quad \mathcal{L}^{-1}\{g(s)\} = G(t)$$
> Then,
> $$\mathcal{L}^{-1}\{c_1 f(s) + c_2 g(s)\} = c_1\mathcal{L}^{-1}\{f(s)\} + c_2\mathcal{L}^{-1}\{g(s)\}$$

**Example 4.1:** Evaluate $\mathcal{L}^{-1}\left\{\frac{4}{s-2} - \frac{3s}{s^2+16} + \frac{s}{s^2+4}\right\}$

$$\mathcal{L}^{-1}\left\{\frac{4}{s-2} - \frac{3s}{s^2+16} + \frac{s}{s^2+4}\right\} = 4\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - 3\mathcal{L}^{-1}\left\{\frac{s}{s^2+4^2}\right\} + 5\mathcal{L}^{-1}\left\{\frac{1}{s^2+2^2}\right\}$$

$$= 4e^{2t} - 3\cos 4t + \frac{5}{2}\sin 2t$$

## 4.2 First Translation or Shifting Property

> **Theorem 4.2.1 (First Translation or Shifting Property):**
> If
> $$\mathscr{L}^{-1}\{f(s)\} = F(t)$$
> then
> $$\mathscr{L}^{-1}\{f(s-a)\} = e^{at}F(t)$$

**Example 4.2: Evaluate** $\mathscr{L}^{-1}\left\{\dfrac{6s-4}{s^2-4s+20}\right\}$

$$\mathscr{L}^{-1}\left\{\frac{6s-4}{s^2-4s+20}\right\} = \mathscr{L}^{-1}\left\{\frac{6(s-2)+8}{(s-2)^2+16}\right\}$$

$$= 6\mathscr{L}^{-1}\left\{\frac{(s-2)}{(s-2)^2+16}\right\} + 2\mathscr{L}^{-1}\left\{\frac{4}{(s-2)^2+16}\right\}$$

$$= 6e^{2t}\cos 4t + 2e^{2t}\sin 4t$$

## 4.3 Second Translation or Shifting Property

> **Theorem 4.3.1 (Second Translation or Shifting Property):**
> If
> $$\mathscr{L}^{-1}\{f(s)\} = F(t)$$
> then
> $$\mathscr{L}^{-1}\{e^{-as}f(s)\} = \begin{cases} F(t-a) & \text{for } t > a \\ 0 & \text{for } t < a \end{cases}$$

**Example 4.3: Evaluate** $\dfrac{e^{-5t}}{(s-2)^4}$

Here
$$\mathscr{L}^{-1}\left\{\frac{1}{(s-2)^4}\right\} = e^{2t}\frac{t^3}{3!} = \frac{e^{2t}}{6}t^3$$

$$\therefore \mathscr{L}^{-1}\left\{\frac{e^{-5s}}{(s-2)^4}\right\} = \frac{e^{2(t-5)}}{6}(t-5)^3, \text{ when } t > 5$$

$$\mathscr{L}^{-1}\left\{\frac{e^{-5s}}{(s-2)^4}\right\} = 0, \text{ when } t < 5$$

## 4.4 Inverse Laplace Transform of Derivatives

> **Theorem 4.4.1 (Inverse Laplace Transform of Derivatives):**
> If
> $$\mathscr{L}^{-1}\{f(s)\} = F(t)$$
> then
> $$\mathscr{L}^{-1}\{f^{(}n)(s)\} = \mathscr{L}^{-1}\left\{\frac{\mathrm{d}^n}{\mathrm{d}s^n}f(s)\right\} = (-1)^n t^n F(t)$$

**Example 4.4: Evaluate** $\mathscr{L}^{-1}\left\{\ln\left(1+\dfrac{1}{s^2}\right)\right\}$

Here

$$f(s) = \ln\left(1 + \frac{1}{s^2}\right) = \mathscr{L}\{F(t)\}$$

$$f'(s) = \frac{-\frac{2}{s^3}}{1 + \frac{1}{s^2}}$$

$$= -2\left\{\frac{1}{s(s^2+1)}\right\}$$

$$= -2\left(\frac{1}{s - \frac{s}{s^2+1}}\right)$$

$$\mathscr{L}^{-1}\{f'(s)\} = -2(1 - \cos t)$$
$$tF(t) = -2(1 - \cos t)$$
$$F(t) = \frac{2(1 - \cos t)}{t}$$
$$\therefore \mathscr{L}^{-1}\left\{\ln\left(1 + \frac{1}{s^2}\right)\right\} = \frac{2(1 - \cos t)}{t}$$

## 4.5   Inverse Laplace Transform of Integrals

*Theorem 4.5.1* (**Inverse Laplace Transform of Integrals**)*:*
*If*
$$\mathscr{L}^{-1}\{f(s)\} = F(t)$$
*then*
$$\mathscr{L}^{-1}\left\{\int_s^\infty f(u)\, du\right\} = \frac{F(t)}{t}$$

**Example 4.5: Find** $\mathscr{L}^{-1}\left\{\int_s^\infty \left(\frac{1}{u} - \frac{1}{u+1}\right) du\right\}$

Let
$$f(s) = \frac{1}{s} - \frac{1}{s+1}$$

$$\mathscr{L}^{-1}\{f(s)\} = \mathscr{L}^{-1}\left\{\frac{1}{s - \frac{1}{s+1}}\right\}$$
$$= 1 - e^{-t} = F(t)$$
$$\therefore \mathscr{L}^{-1}\left\{\int_s^\infty f(u)\, du\right\} = \frac{F(t)}{t}$$
$$= \frac{1 - e^{-t}}{t}$$

## 4.6   Division by s

*Theorem 4.6.1* (**Division by** $s$)*:*
*If*
$$\mathscr{L}^{-1}\{f(s)\} = F(t)$$
*then*
$$\mathscr{L}^{-1}\left\{\frac{f(s)}{s}\right\} = \int_0^t F(u)\, du$$

**Example 4.6: Evaluate** $\mathscr{L}^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\}$

Let
$$f(s) = \frac{1}{s^2 + 1}$$

$$\therefore \mathcal{L}^{-1}\{f(s)\} = \sin t$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \int_0^t \sin u \, du \qquad\qquad = 1 - \cos t$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = \int_0^t (1 - \cos u) \, du \qquad\qquad = t - \sin t$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\} = \int_0^t (u - \sin u) \, du \qquad\qquad = \frac{t^2}{2} + \cos t - 1$$

**Alternative approach:**
Let
$$f(s) = \frac{1}{s^4 + s^2} = \frac{1}{s^2} - \frac{1}{s^2 + 1}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2} - \frac{1}{s^2+1}\right\}$$

$$= t - \sin t$$

$$= F(t)$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\} = \int_0^t F(u) \, du$$

$$= \int_0^t (u - \sin u) \, du$$

$$= \frac{t^2}{2} + \cos t - 1$$

## 4.7   Inverse Laplace Transform by Partial Fraction

**Example 4.7: Find** $\mathcal{L}^{-1}\left\{\dfrac{5^2 - 15s - 11}{(s+1)(s-2)^2}\right\}$

$$\frac{5^2 - 15s - 11}{(s+1)(s-2)^2} = \frac{-\frac{1}{3}}{ss+1} - \frac{7}{(s-2)^3} + \frac{4}{(s-2)^2} + \frac{\frac{1}{3}}{s-2}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{5^2 - 15s - 11}{(s+1)(s-2)^2}\right\} = -\frac{1}{e}e^{-t} - 7\frac{t^2}{2}e^{2t} + 4te^{2t} + \frac{1}{3}e^{2t}$$

## 4.8   The Convolution Theorem

*Theorem 4.8.1* **(The Convolution Theorem):**
*If*
$$\mathcal{L}^{-1}\{f(s)\} = F(t) \quad and \quad \mathcal{L}^{-1}\{g(s)\} = G(t)$$
*then*
$$\mathcal{L}^{-1}\{f(s)g(s)\} = \int_0^t F(u)G(t-u) \, du = F * G$$

$$G * F = \int_0^t G(u)F(t-u) \, du = F * G$$

**Example 4.8: Find** $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2(s^2+1)^2}\right\}$

Here,
$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t = F(t)$$

$$\mathscr{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = e^{-t}t = G(t)$$

$$\therefore \mathscr{L}^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\} = G * F$$

$$= \int_0^t ue^{-u}(t-u)\,du$$

$$= \int_0^t (ut - u^2)e^{-u}\,du$$

$$= -(ut - u^2)\cdot e^{-u}\Big|_0^t + \int_0^t (t - 2u)e^{-u}\,du$$

**Example 4.9:** Find $\mathscr{L}^{-1}\left\{\dfrac{s}{(s^2+a^2)^2}\right\}$

Let

$$f(s) = \frac{s}{s^2+a^2} \qquad \text{and} \qquad g(s) = \frac{1}{s^2+a^2}$$

$$\mathscr{L}^{-1}\{f(s)\} = \cos at \qquad \text{and} \qquad \mathscr{L}^{-1}\{g(s)\} = \frac{\sin at}{a}$$

$$F * G = \frac{1}{a}\int_0^t \cos au \cdot \sin au(t-u)\,du$$

$$= \frac{1}{a}\int_0^t \cos au(\sin at \cos au - \cos at \sin au)\,du$$

$$= \frac{1}{a}\int_0^t \sin at \cos^2 au\,du - \frac{1}{a}\int_0^t \sin au \cos au\,du$$

$$= \frac{1}{2a}\sin at \int_0^t (\cos 2au + 1)\,du - \frac{1}{2a}\cos at \int_0^t \sin 2au\,du$$

$$= \frac{1}{2a}\left[\frac{1}{2}\sin 2au + u\right]_0^t - \frac{1}{2a}\left[\frac{1}{2}\cos 2au\right]_0^t$$

$$= -\frac{1}{2a}\sin at \cdot \sin 2at - \frac{1}{2a}\cos at \cdot \frac{1}{2}(\cos 2at - 1) + \frac{1}{2a}\sin at \cdot t$$

$$= \frac{1}{4a}\left[\cos at - \cos(at - 2at)\right] + \frac{t\sin at}{2a}$$

$$= \frac{1}{4a}\left[\cos at - \cos at\right] + \frac{t\sin at}{2a}$$

$$= \frac{t\sin at}{2a}$$