

Simple Harmonic Motion

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1 Oscillation and Vibration

1.1 Oscillation

- Oscillation is the repetitive variation, typically in time, of some measure about a central value (often a point of equilibrium) or between two or more different states.
- The term vibration is precisely used to describe mechanical oscillation.
- Familiar examples of oscillation include a swinging pendulum and alternating current.

1.2 Vibration

- Vibration is a mechanical phenomenon whereby oscillations occur about an equilibrium point.
- The word comes from Latin vibrationem ("shaking, brandishing").
- The oscillations may be periodic, such as the motion of a pendulum—or random, such as the movement of a tire on a gravel road.

1.3 Differences between Oscillation and Vibration

- Oscillation is the definite displacement of a body in terms of distance or time, whereas vibration is the movement brought about in a body due to oscillation.
- Oscillation takes place in physical, biological systems, and often in our society, but vibrations is associated with mechanical systems only.
- Oscillation is about a single body, whereas vibration is the result of collective oscillation of atoms in the body.
- All vibrations are oscillations, but not all oscillations are vibrations.

2 Simple Harmonic Motion

2.1 Definition

Definition 2.1.1: Simple Harmonic Motion

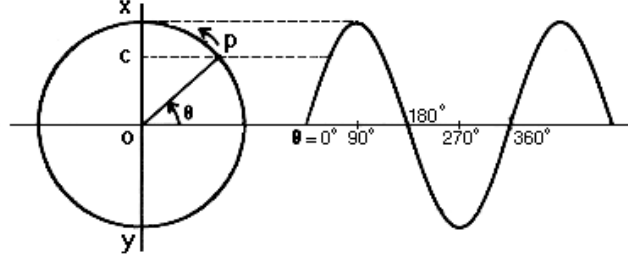
Simple harmonic motion is a type of periodic motion where the restoring force is directly proportional to the displacement and acts in the direction opposite to that of displacement.

A particle is said to execute SHM when it will

- (a) Trace and retrace the same path over and over again.
- (b) Change direction at a regular interval of time.
- (c) Move along a straight line.
- (d) Have acceleration proportional to its displacement from the mean position.

A particle which satisfies the condition (a) only is said to execute **periodic motion**. A particle which satisfies condition (a) and (b) is said to execute **vibratory motion**.

Let P be a particle moving on the circumference of a circle of radius r with a uniform velocity v . Let angular velocity be $\omega = v/r$.



Displacement of the particle from the mean position is given by $y = r \sin \omega t$

So, velocity of the particle is given by $v = \frac{dy}{dt} = \omega r \cos \omega t$

And acceleration of the particle is given by $a = \frac{dv}{dt} = -\omega^2 r \sin \omega t = -\omega^2 y$

Angle	Position of vibrating particle	Displacement $y = r \sin \omega t$	Velocity $\frac{dy}{dt} = \omega r \cos \omega t$	Acceleration $-\omega^2 r \sin \omega t = -\omega^2 y$
0	O	0	ωr	0
$\pi/2$	X	r	0	$-\omega^2 r$
π	O	0	$-\omega r$	0
$3\pi/2$	Y	$-r$	0	$\omega^2 r$
2π	O	0	ωr	0

2.2 Differential Equation of SHM

Let y be the displacement of the particle from the mean position at time t , r be the amplitude, and α be the epoch of the vibrating particle

$$y = r \sin (\omega t + \alpha) \quad (1)$$

$$\frac{dy}{dt} = r\omega \cos (\omega t + \alpha) \quad (2)$$

$$\frac{d^2y}{dt^2} = -r\omega^2 \sin (\omega t + \alpha) \quad (3)$$

Hence the differential equation of SHM is

$$\frac{d^2y}{dt^2} + \omega^2 y = 0 \quad (4)$$

2.3 Solution of the Differential Equation of SHM

$$\frac{d^2y}{dt^2} + \omega^2 y = 0 \quad (5)$$

Here,

$$\frac{d^2y}{dt^2} = \frac{dv}{dt} = \frac{dv}{dy} \frac{dy}{dt} = v \frac{dv}{dy}$$

$$\begin{aligned}
v \, dv + \omega^2 y \, dy &= 0 \\
\int v \, dv + \omega^2 \int y \, dy &= 0 \\
\frac{v^2}{2} + \frac{\omega^2 y^2}{2} &= C' \\
\left(\frac{dy}{dt} \right)^2 + \omega^2 y^2 &= C^2
\end{aligned}$$

At maximum displacement, $y = r$ and $\frac{dy}{dt} = 0$
So, $C^2 = \omega^2 r^2$

$$\begin{aligned}
\left(\frac{dy}{dt} \right)^2 &= \omega^2 (r^2 - y^2) \\
\frac{dy}{dt} &= \omega \sqrt{r^2 - y^2} \\
\int \frac{1}{\sqrt{r^2 - y^2}} \, dy &= \int \omega \, dt \\
\sin^{-1} \frac{y}{r} &= \omega t + \alpha \\
\boxed{y = r \sin (\omega t + \alpha)} & \quad (6)
\end{aligned}$$

By expanding equation (6), we get

$$y = r \sin \omega t \cos \alpha + r \cos \omega t \sin \alpha \quad (7)$$

If $y = 0$ at $t = 0$, then $\alpha = 0$

$$y = r \sin \omega t \quad (8)$$

If $y = r$ at $t = 0$, then $\alpha = \pi/2$

$$y = r \cos \omega t \quad (9)$$

Hence, the general solution of the differential equation of SHM is

$$\boxed{y = A \sin \omega t + B \cos \omega t} \quad (10)$$

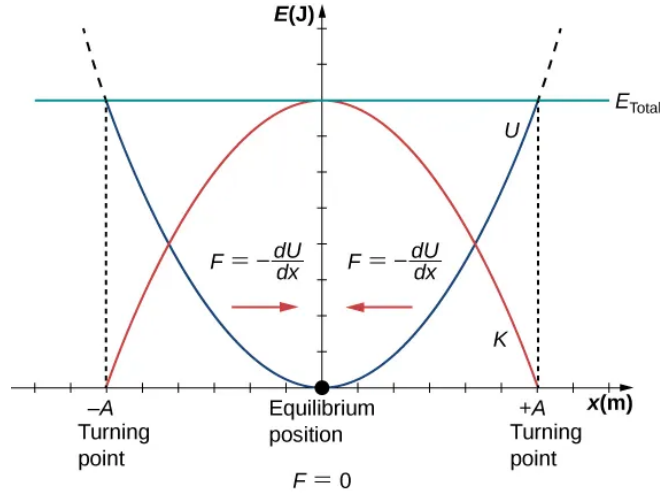
3 Energy in SHM

3.1 Total Energy of a Vibrating Particle

$$\begin{aligned}
\text{Kinetic Energy} &= \frac{1}{2} m \left(\frac{dy}{dt} \right)^2 \\
&= \frac{1}{2} m \omega^2 r^2 \cos^2 (\omega t + \alpha) \\
\text{Potential Energy} &= \frac{1}{2} k y^2 \\
&= \frac{1}{2} m \omega^2 r^2 \sin^2 (\omega t + \alpha)
\end{aligned}$$

Thus, the total energy of the vibrating particle is

$$\boxed{E = \frac{1}{2} k r^2 = \frac{1}{2} m \omega^2 r^2} \quad (11)$$



3.2 Average Kinetic Energy

Kinetic energy of the particle is given by

$$K = \frac{1}{2}m \left(\frac{dy}{dt} \right)^2 = \frac{1}{2}m\omega^2 r^2 \cos^2(\omega t + \alpha) \quad (12)$$

Hence, average kinetic energy is

$$\begin{aligned}
 \overline{K} &= \frac{1}{T} \int_0^T K dt \\
 &= \frac{1}{T} \int_0^T \frac{1}{2}m\omega^2 r^2 \cos^2(\omega t + \alpha) dt \\
 &= \frac{1}{2}m\omega^2 r^2 \frac{1}{T} \int_0^T \cos^2(\omega t + \alpha) dt \\
 &= \frac{1}{2}m\omega^2 r^2 \frac{1}{T} \int_0^T \frac{1 + \cos 2(\omega t + \alpha)}{2} dt \\
 &= \frac{1}{2}m\omega^2 r^2 \frac{1}{T} \left[\frac{t}{2} + \frac{\sin 2(\omega t + \alpha)}{4\omega} \right]_0^T \\
 &= \frac{1}{2}m\omega^2 r^2 \frac{1}{T} \left[\frac{T}{2} + \frac{\sin 2(\omega T + \alpha)}{4\omega} - \frac{\sin 2\alpha}{4\omega} \right] \\
 &= \frac{1}{2}m\omega^2 r^2 \frac{1}{T} \left[\frac{T}{2} + \frac{\sin 2\alpha}{4\omega} - \frac{\sin 2\alpha}{4\omega} \right] \\
 &= \frac{1}{4}m\omega^2 r^2 \\
 \boxed{\overline{K} = \frac{1}{4}m\omega^2 r^2 = \frac{1}{4}kr^2 = \frac{1}{2}E} \quad (13)
 \end{aligned}$$

3.3 Average Potential Energy

Potential energy of the particle is given by

$$U = \frac{1}{2}ky^2 = \frac{1}{2}m\omega^2 r^2 \sin^2(\omega t + \alpha) \quad (14)$$

Hence, average potential energy is

$$\overline{U} = \frac{1}{T} \int_0^T U dt$$

$$\begin{aligned}
&= \frac{1}{T} \int_0^T \frac{1}{2} m \omega^2 r^2 \sin^2 (\omega t + \alpha) dt \\
&= \frac{1}{2} m \omega^2 r^2 \frac{1}{T} \int_0^T \sin^2 (\omega t + \alpha) dt \\
&= \frac{1}{2} m \omega^2 r^2 \frac{1}{T} \int_0^T \frac{1 - \cos 2(\omega t + \alpha)}{2} dt \\
&= \frac{1}{2} m \omega^2 r^2 \frac{1}{T} \left[\frac{t}{2} - \frac{\sin 2(\omega t + \alpha)}{4\omega} \right]_0^T \\
&= \frac{1}{2} m \omega^2 r^2 \frac{1}{T} \left[\frac{T}{2} - \frac{\sin 2(\omega T + \alpha)}{4\omega} + \frac{\sin 2\alpha}{4\omega} \right] \\
&= \frac{1}{2} m \omega^2 r^2 \frac{1}{T} \left[\frac{T}{2} + \frac{\sin 2\alpha}{4\omega} - \frac{\sin 2\alpha}{4\omega} \right] \\
&= \frac{1}{4} m \omega^2 r^2
\end{aligned}$$

$$\boxed{\bar{U} = \frac{1}{4} m \omega^2 r^2 = \frac{1}{4} k r^2 = \frac{1}{2} E} \quad (15)$$

4 Composition of Two SHMs of Same Frequency

4.1 Same Direction

Let y_1 and y_2 be the displacements of two SHM of same frequency ω , amplitude r_1 and r_2 , and phases α_1 and α_2 respectively.

$$y_1 = r_1 \sin (\omega t + \alpha_1) \quad (16)$$

$$y_2 = r_2 \sin (\omega t + \alpha_2) \quad (17)$$

If the two SHMs are in the same direction, then the resultant displacement is

$$\begin{aligned}
y &= y_1 + y_2 \\
&= r_1 \sin (\omega t + \alpha_1) + r_2 \sin (\omega t + \alpha_2) \\
&= r_1 \sin \omega t \cos \alpha_1 + r_1 \cos \omega t \sin \alpha_1 + r_2 \sin \omega t \cos \alpha_2 + r_2 \cos \omega t \sin \alpha_2 \\
\therefore y &= (r_1 \cos \alpha_1 + r_2 \cos \alpha_2) \sin \omega t + (r_1 \sin \alpha_1 + r_2 \sin \alpha_2) \cos \omega t
\end{aligned} \quad (18)$$

In equation (18), let

$$r_1 \cos \alpha_1 + r_2 \cos \alpha_2 = A \cos \varphi \quad (19)$$

$$r_1 \sin \alpha_1 + r_2 \sin \alpha_2 = A \sin \varphi \quad (20)$$

Then, equation (18) becomes

$$y = A \cos \varphi \sin \omega t + A \sin \varphi \cos \omega t \quad (21)$$

$$y = A \sin (\omega t + \varphi) \quad (22)$$

Here, A is the amplitude of the resultant SHM and φ is the phase of the resultant SHM. From equations (19) and (20), we get

$$A^2 = A^2 \sin^2 \varphi + A^2 \cos^2 \varphi$$

$$\begin{aligned}
A^2 &= r_1^2 \sin^2 \alpha_1 + r_2^2 \sin^2 \alpha_2 + 2r_1 r_2 \sin \alpha_1 \sin \alpha_2 \\
&\quad + r_1^2 \cos^2 \alpha_1 + r_2^2 \cos^2 \alpha_2 + 2r_1 r_2 \cos \alpha_1 \cos \alpha_2 \\
A^2 &= r_1^2 + r_2^2 + 2r_1 r_2 (\sin \alpha_1 \sin \alpha_2 + \cos \alpha_1 \cos \alpha_2) \\
A^2 &= r_1^2 + r_2^2 + 2r_1 r_2 \cos (\alpha_1 - \alpha_2)
\end{aligned}$$

$$A = \sqrt{r_1^2 + r_2^2 + 2r_1 r_2 \cos (\alpha_1 - \alpha_2)} \quad (23)$$

And,

$$\varphi = \tan^{-1} \frac{r_1 \sin \alpha_1 + r_2 \sin \alpha_2}{r_1 \cos \alpha_1 + r_2 \cos \alpha_2} \quad (24)$$

4.1.1 Special Cases

(I) Same phase : $\alpha_1 = \alpha_2$

In this case, equation (21) becomes

$$A = \sqrt{r_1^2 + r_2^2 + 2r_1 r_2 \cos 0} = \sqrt{(r_1 + r_2)^2} = r_1 + r_2$$

$$A = r_1 + r_2 \quad (25)$$

And,

$$\varphi = \tan^{-1} \frac{r_1 \sin \alpha + r_2 \sin \alpha}{r_1 \cos \alpha + r_2 \cos \alpha} = \tan^{-1} \left(\frac{r_1 + r_2}{r_1 + r_2} \tan \alpha \right) = \tan^{-1} (\tan \alpha)$$

$$\varphi = \alpha \quad (26)$$

(II) Opposite phase : $\alpha_1 - \alpha_2 = (2n + 1)\pi$, **where** $n = 0, 1, 2, \dots$

In this case, equation (21) becomes

$$A = \sqrt{r_1^2 + r_2^2 + 2r_1 r_2 \cos (\alpha_2 + \pi - \alpha_2)} = \sqrt{(r_1 - r_2)^2} = r_1 - r_2$$

$$A = r_1 - r_2 \quad (27)$$

And,

$$\varphi = \tan^{-1} \frac{r_1 \sin (\alpha + \pi) + r_2 \sin \alpha}{r_1 \cos (\alpha + \pi) + r_2 \cos \alpha} = \tan^{-1} \left(\frac{r_1 \sin \alpha - r_2 \sin \alpha}{r_1 \cos \alpha - r_2 \cos \alpha} \right) = \tan^{-1} (-\tan \alpha)$$

$$\varphi = \alpha + \pi \quad (28)$$

4.2 Right Angle

Let x and y be the displacements of two SHM of same frequency ω , amplitude a and b respectively, and phase difference α , acting at right angle to each other.

$$x = a \sin (\omega t + \alpha) \quad (29)$$

$$y = b \sin (\omega t) \quad (30)$$

Or,

$$\frac{x}{a} = \sin (\omega t + \alpha) \quad (31)$$

$$\frac{y}{b} = \sin(\omega t) \quad (32)$$

Thus, we get

$$\begin{aligned} \frac{x}{a} &= \sin \omega t \cos \alpha + \cos \omega t \sin \alpha \\ \frac{x}{a} &= \frac{y}{b} \sqrt{1 - \sin^2 \alpha} + \sqrt{1 - \frac{y^2}{b^2}} \sin \alpha \\ \frac{x}{a} - \frac{y}{b} \sqrt{1 - \sin^2 \alpha} &= \sqrt{1 - \frac{y^2}{b^2}} \sin \alpha \\ \frac{x^2}{a^2} + \frac{y^2}{b^2} (1 - \sin^2 \alpha) - \frac{2xy}{ab} \sqrt{1 - \sin^2 \alpha} &= \left(1 - \frac{y^2}{b^2}\right) \sin^2 \alpha \\ \boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab} \sqrt{1 - \sin^2 \alpha} = \sin^2 \alpha} & \quad (33) \end{aligned}$$

Equation (33) represents the general equation of the resultant SHM of the two perpendicular SHMs.

4.2.1 Special Cases

(I) If $\alpha = 0$ or 2π

$$\cos \alpha = 1, \quad \sin \alpha = 0$$

Then equation (33) becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab} = 0$$

Or,

$$\frac{x}{a} = \frac{y}{b}$$

This represents a straight line passing through the origin.

(II) If $\alpha = \pi$

$$\cos \alpha = -1, \quad \sin \alpha = 0$$

Then equation (33) becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{2xy}{ab} = 0$$

Or,

$$\frac{x}{a} = -\frac{y}{b}$$

This represents a straight line with negative slope passing through the origin.

(III) If $\alpha = \pi/2$ or, $3\pi/2$

$$\cos \alpha = 0, \quad \sin \alpha = 1$$

Then equation (33) becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

This represents an ellipse.

(IV) If $\alpha = \pi/2$ or, $3\pi/2$, and $a = b$

$$\cos \alpha = 0, \quad \sin \alpha = 1$$

Then equation (33) becomes

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$$

Or,

$$x^2 + y^2 = a^2$$

This represents a circle of radius a .

(V) If $\alpha = \pi/4$ or, $7\pi/4$

$$\cos \alpha = \frac{1}{\sqrt{2}}, \quad \sin \alpha = \frac{1}{\sqrt{2}}$$

Then equation (33) becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab} \frac{1}{\sqrt{2}} = \frac{1}{2}$$

Or,

$$\left(\frac{x}{a} - \frac{y}{b} \right)^2 = \frac{1}{2}$$

This represents an oblique ellipse.

(VI) If $\alpha = 3\pi/4$ or, $5\pi/4$, and $a = b$

$$\cos \alpha = -\frac{1}{\sqrt{2}}, \quad \sin \alpha = \frac{1}{\sqrt{2}}$$

Then equation (33) becomes

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{2xy}{a^2} \frac{1}{\sqrt{2}} = \frac{1}{2}$$

Or,

$$\left(\frac{x}{a} + \frac{y}{a} \right)^2 = \frac{1}{2}$$

This represents an oblique ellipse (mirrored).

