

Laplace Transform

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1 Laplace Transform

1.1 Definition and Existence

Definition 1.1.1: Laplace Transform

Let F be a real-valued function of the real variable t , defined for $t > 0$. Let s be a variable that we shall assume to be real, and consider the function f defined by

$$f(s) = \int_0^{\infty} e^{-st} F(t) dt \quad (1)$$

for all values of s for which this integral exists. The function f defined by the integral (1) is called the Laplace Transform of the function F . We shall denote the Laplace transform of F by $\mathcal{L}\{F(t)\}$.

Thus the Laplace transform of a function f is given by

$$\mathcal{L}\{F(t)\} = f(s) = \int_0^{\infty} e^{-st} F(t) dt = \lim_{R \rightarrow \infty} \int_0^R e^{-st} F(t) dt \quad (2)$$

Some ways to write Laplace transforms:

$$\mathcal{L}F(t) = f(s) = \int_0^{\infty} e^{-st} F(t) dt$$

$$\mathcal{L}G(t) = g(s)$$

$$\mathcal{L}u(t) = \tilde{u}(s)$$

Theorem 1.1.2: Hypothesis: Let F be a real function that has the following properties:

1. F is a piecewise continuous in every finite closed interval $0 \leq t \leq a$ ($a > 0$).
2. F is of exponential order, i.e, there exists α , $M > 0$, and $t_0 > 0$ such that

$$e^{-\alpha t} |F(t)| < M \text{ for } t > t_0$$

Conclusion: The Laplace transform of F exists for $s > \alpha$.

$$\mathcal{L}\{F(t)\} = \int_0^{\infty} e^{-st} F(t) dt$$

1.2 Some Functions and Their Laplace Transforms

| $F(t)$ | $\mathcal{L}\{F(t)\} = f(s)$ | $F(t)$ | $\mathcal{L}\{F(t)\} = f(s)$ |
|-----------|------------------------------|------------|------------------------------|
| 1 | $\frac{1}{s}$ | n | $\frac{n}{s}$ |
| t | $\frac{1}{s^2}$ | t^n | $\frac{n!}{s^{n+1}}$ |
| e^{at} | $\frac{1}{s-a}$ | e^{-at} | $\frac{1}{s+a}$ |
| $\sin at$ | $\frac{a}{s^2 + a^2}$ | $\sinh at$ | $\frac{a}{s^2 - a^2}$ |
| $\cos at$ | $\frac{s}{s^2 + a^2}$ | $\cosh at$ | $\frac{s}{s^2 - a^2}$ |

Table 1: Functions and their Laplace Transform

Proofs :

$$\mathcal{L}\{n\} = \frac{n}{s}$$

$$\mathcal{L}\{t\} = \frac{1}{s^2}$$

Let $F(t) = n$, for $t > 0$

Then

$$\begin{aligned}\mathcal{L}\{n\} &= \int_0^\infty e^{-st} \cdot n \, dt \\ &= n \left. \frac{-e^{-st}}{s} \right|_0^\infty \\ &= \frac{n}{s} \quad \square\end{aligned}$$

Let $F(t) = t$, for $t > 0$

Then

$$\begin{aligned}\mathcal{L}\{t\} &= \int_0^\infty e^{-st} \cdot t \, dt \\ &= -t \frac{e^{-st}}{s} + \int_0^\infty \frac{e^{-st}}{s} \, dt \\ &= -e^{-st} \frac{t}{s} \Big|_0^\infty - e^{-st} \frac{1}{s^2} \Big|_0^\infty \\ &= \frac{1}{s^2} \quad \square\end{aligned}$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

Let $F(t) = t^n$, for $t > 0$

Then

$$\begin{aligned}\mathcal{L}\{t^n\} &= \int_0^\infty e^{-st} t^n \, dt \\ &= -t^n \frac{e^{-st}}{s} + \int_0^\infty n t^{n-1} \frac{e^{-st}}{s} \, dt \\ &= -n t^{n-1} \frac{e^{-st}}{s^2} \Big|_0^\infty + \int_0^\infty n(n-1) t^{n-2} \frac{e^{-st}}{s^2} \, dt \\ &= -n(n-1) t^{n-2} \left(\frac{e^{-st}}{s^3} \right) \Big|_0^\infty + \int_0^\infty n(n-1)(n-2) t^{n-3} \frac{e^{-st}}{s^3} \, dt \\ &= \dots \\ &= n! t^{n-n} \frac{e^{-st}}{s^{n+1}} \Big|_0^\infty + \int_0^\infty n(n-1) \dots (n-n) \frac{e^{-st}}{s^{n+1}} \, dt \\ &= \frac{n!}{s^{n+1}} \quad \square\end{aligned}$$

Let $F(t) = e^{at}$, for $t > 0$

Then

$$\begin{aligned}\mathcal{L}\{e^{at}\} &= \int_0^\infty e^{-st} e^{at} \, dt \\ &= \int_0^\infty e^{(a-s)t} \, dt \\ &= \left. \frac{e^{(a-s)t}}{a-s} \right|_0^\infty \\ &= \frac{1}{s-a} \quad \square\end{aligned}$$

Let $F(t) = e^{-at}$, for $t > 0$

Then

$$\begin{aligned}\mathcal{L}\{e^{-at}\} &= \int_0^\infty e^{-st} e^{-at} \, dt \\ &= \int_0^\infty e^{-(a+s)t} \, dt \\ &= \left. \frac{e^{-(a+s)t}}{s+a} \right|_0^\infty \\ &= \frac{1}{s+a} \quad \square\end{aligned}$$

Let $F(t) = \sin at$, for $t > 0$
Then

$$\begin{aligned}\mathcal{L}\{\sin at\} &= \int_0^\infty e^{-st} \sin at \, dt \\ &= -\frac{e^{-st}}{s^2 + a^2} (s \sin at + a \cos at) \Big|_0^\infty \\ &= \frac{a}{s^2 + a^2} \quad \square\end{aligned}$$

Let $F(t) = \cos at$, for $t > 0$
Then

$$\begin{aligned}\mathcal{L}\{\cos at\} &= \int_0^\infty e^{-st} \cos at \, dt \\ &= \frac{e^{-st}}{s^2 + a^2} (-s \cos at + a \sin at) \Big|_0^\infty \\ &= \frac{s}{s^2 + a^2} \quad \square\end{aligned}$$

2 Basic Properties of the Laplace Transform

2.1 Linearity Property

Theorem (The Linearity Property):

Let F_1 and F_2 be functions whose Laplace transform exist, and let c_1 and c_2 be constants.
Then

$$\mathcal{L}\{c_1 F_1(t) + c_2 F_2(t)\} = c_1 \mathcal{L}\{F_1(t)\} + c_2 \mathcal{L}\{F_2(t)\}$$

Proof :

Let $F(t) = c_1 F_1(t) + c_2 F_2(t)$, for $t > 0$

Then

$$\begin{aligned}\mathcal{L}\{c_1 F_1(t) + c_2 F_2(t)\} &= \int_0^\infty e^{-st} [c_1 F_1(t) + c_2 F_2(t)] \, dt \\ &= c_1 \int_0^\infty e^{-st} F_1(t) \, dt + c_2 \int_0^\infty e^{-st} F_2(t) \, dt \\ &= c_1 \mathcal{L}\{F_1(t)\} + c_2 \mathcal{L}\{F_2(t)\} \quad \square\end{aligned}$$

Example 2.1:

$$\mathcal{L}\{4t^2 - 3 \cos 2t + 5e^{-t}\}$$

$$\begin{aligned}\mathcal{L}\{4t^2 - 3 \cos 2t + 5e^{-t}\} &= 4\mathcal{L}\{t^2\} - 3\mathcal{L}\{\cos 2t\} + 5\mathcal{L}\{e^{-t}\} \\ &= 4 \cdot \frac{2!}{s^3} - 3 \cdot \frac{s}{s^2 + 4} + 5 \cdot \frac{1}{s + 1} \\ &= \frac{8}{s^3} - \frac{3s}{s^2 + 4} + \frac{5}{s + 1}\end{aligned}$$

Example 2.2: Find $\mathcal{L}\{F(t)\}$, when $F(t) = \begin{cases} 5 & \text{for } 0 < t < 3 \\ 0 & \text{for } t > 3 \end{cases}$

$$\begin{aligned}\mathcal{L}\{F(t)\} &= \int_0^\infty e^{-st} F(t) \, dt = \int_0^3 e^{-st} \cdot 5 \, dt + \int_3^\infty 0 \, dt \\ &= \int_0^3 e^{-st} \cdot 5 \, dt \\ &= \frac{5e^{-st}}{s} \Big|_0^3 \\ &= \frac{5}{s} (1 - e^{-3s})\end{aligned}$$

Example 2.3: Find $\mathcal{L}\{F(t)\}$, when $F(t) = \begin{cases} (t-1)^2 & \text{for } t > 1 \\ 0 & \text{for } 0 < t < 1 \end{cases}$

$$\begin{aligned}
\mathcal{L}\{F(t)\} &= \int_0^\infty e^{-st} F(t) dt = \int_0^1 0 dt + \int_1^\infty e^{-st} (t-1)^2 dt \\
&= \int_1^\infty e^{-st} (t^2 - 2t + 1) dt \\
&= -t^2 \frac{e^{-st}}{s} \Big|_1^\infty - 2 \int_1^\infty t \cdot \frac{e^{-st}}{s} dt - 2 \int_1^\infty t \cdot e^{-st} dt - \frac{e^{-st}}{s} \\
&= \frac{e^{-s}}{s} + 2 \left[-\frac{t}{s} \left(\frac{e^{-st}}{s} \right) \Big|_1^\infty + \int_1^\infty \frac{e^{-st}}{s^2} dt + 2t \frac{e^{-st}}{s} \Big|_1^\infty - \int_1^\infty \frac{e^{-st}}{s} dt \right] + \frac{e^{-st}}{s} \\
&= \frac{e^{-s}}{s} - 2 \frac{e^{-s}}{s^2} + 2 \frac{e^{-s}}{s^2} - 2 \frac{e^{-st}}{s} + 2 \frac{e^{-s}}{s^2} + \frac{e^{-st}}{s} \\
&= 2 \frac{e^{-s}}{s^3}
\end{aligned}$$

2.2 First Translation Property

Theorem 2.2.1 (First Translation or Shifting Property):

If

$$\mathcal{L}\{F(t)\} = f(s)$$

then

$$\mathcal{L}\{e^{at}F(t)\} = f(s-a)$$

Proof:

Let $G(t) = e^{at}F(t)$, for $t > 0$

Then

$$\begin{aligned}
\mathcal{L}\{e^{at}F(t)\} &= \int_0^\infty e^{-st} e^{at} F(t) dt \\
&= \int_0^\infty e^{-(s-a)t} F(t) dt \\
&= \mathcal{L}\{F(t)\} \Big|_{s-a} \\
&= f(s-a) \quad \square
\end{aligned}$$

Example 2.4:

$$\mathcal{L}\{e^{-t} \cos 2t\}$$

Since $\mathcal{L}\{\cos 2t\} = \frac{s}{s^2 + 4}$, we have

$$\mathcal{L}\{e^{-t} \cos 2t\} = \frac{s+1}{(s+1)^2 + 4} = \frac{s+1}{s^2 + 2s + 5}$$

Example 2.5: Evaluate $\mathcal{L}\{e^{-2t}(3 \cos 6t - 5 \sin 6t)\}$

Now,

$$\begin{aligned}
f(s) &= \mathcal{L}\{3 \cos 6t - 5 \sin 6t\} \\
&= \frac{3s}{s^2 + 36} - \frac{30}{s^2 + 36} \\
&= \frac{3s - 30}{s^2 + 36}
\end{aligned}$$

$$\begin{aligned}
\therefore \mathcal{L}\{e^{-2t}(3 \cos 6t - 5 \sin 6t)\} &= f(s+2) \\
&= \frac{3(s+2) - 30}{(s+2)^2 + 36} \\
&= \frac{3s - 24}{s^2 + 4s + 40}
\end{aligned}$$

2.3 Second Translation Property

Theorem 2.3.1 (Second Translation or Shifting Property):

If

$$\mathcal{L}\{F(t)\} = f(s)$$

and

$$G(t) = \begin{cases} F(t-a) & \text{for } t > a \\ 0 & \text{for } t < a \end{cases}$$

then

$$\mathcal{L}\{G(t)\} = e^{-as}f(s)$$

Proof:

$$\text{Let } G(t) = \begin{cases} F(t-a) & \text{for } t > a \\ 0 & \text{for } t < a \end{cases}$$

Then

$$\begin{aligned} \mathcal{L}\{G(t)\} &= \int_0^{\infty} e^{-st} F(t-a) dt \\ &= \int_a^{\infty} e^{-st} F(t-a) dt \\ &= \int_0^{\infty} e^{-s(u+a)} F(u) du \quad \text{where } u = t-a \\ &= e^{-as} \int_0^{\infty} e^{-su} F(u) du \\ &= e^{-as} \mathcal{L}\{F(t)\} \\ &= e^{-as} f(s) \quad \square \end{aligned}$$

Example 2.6: Find $\mathcal{L}\{G(t)\}$ where $G(t) = \begin{cases} \cos\left(t - \frac{2\pi}{3}\right) & \text{for } t > \frac{2\pi}{3} \\ 0 & \text{for } t < \frac{2\pi}{3} \end{cases}$

$$\begin{aligned} \mathcal{L}\{G(t)\} &= e^{-\frac{2\pi}{3}s} \cdot \mathcal{L}\{\cos t\} \\ &= e^{-\frac{2\pi}{3}s} \cdot \frac{s}{s^2 + 1} \\ &= \frac{se^{-\frac{2\pi}{3}s}}{s^2 + 1} \end{aligned}$$

Example 2.7: Find $\mathcal{L}\{F(t)\}$, if $F(t) = \begin{cases} (t-1)^2 & \text{for } t > 1 \\ 0 & \text{for } t < 1 \end{cases}$

Let

$$G(t) = t^2$$

$$\therefore \mathcal{L}\{G(t)\} = \frac{2!}{s^3}$$

Now,

$$F(t) = \begin{cases} G(t-1) & \text{for } t > 1 \\ 0 & \text{for } t < 1 \end{cases}$$

$$\therefore \mathcal{L}\{F(t)\} = \frac{e^{-s} \cdot 2!}{s^3} = \frac{2e^{-s}}{s^3}$$

2.4 Change of Scale Property

Theorem 2.4.1 (Change of Scale Property):

If

$$\mathcal{L}\{F(t)\} = f(s)$$

then

$$\mathcal{L}\{F(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right)$$

Proof:

Let $G(t) = F(at)$, for $t > 0$

Then

$$\begin{aligned} \mathcal{L}\{G(t)\} &= \int_0^\infty e^{-st} F(at) dt \\ &= \int_0^\infty e^{-\frac{s}{a}u} F(u) d(u/a) \quad \text{where } u = at \\ &= \frac{1}{a} \int_0^\infty e^{-\frac{s}{a}u} F(u) du \\ &= \frac{1}{a} \mathcal{L}\{F(t)\} \\ &= \frac{1}{a} f\left(\frac{s}{a}\right) \quad \square \end{aligned}$$

Example 2.8: Evaluate $\mathcal{L}\{\sin 3t\}$

$$\begin{aligned} \mathcal{L}\{\sin 3t\} &= \frac{1}{3} \cdot \frac{1}{\left(\frac{s}{3}\right)^2 + 1} \\ &= \frac{1}{3} \cdot \frac{3^2}{s^2 + 3^2} \\ &= \frac{3}{s^2 + 9} \end{aligned}$$

Example 2.9: If $\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \tan^{-1} \frac{1}{s} = f(s)$, then evaluate $\mathcal{L}\left\{\frac{\sin at}{t}\right\}$

$$\begin{aligned} \mathcal{L}\left\{\frac{\sin at}{t}\right\} &= \mathcal{L}\left\{a \cdot \frac{\sin at}{t}\right\} \\ &= a \cdot \mathcal{L}\left\{\frac{\sin at}{at}\right\} \\ &= a \cdot \frac{1}{a} f\left(\frac{s}{a}\right) \\ &= \tan^{-1} \frac{a}{s} \end{aligned}$$

2.5 Multiplication by t

Theorem 2.5.1 (Multiplication by t^n):

If

$$\mathcal{L}\{F(t)\} = f(s)$$

then

$$\mathcal{L}\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s)$$

Proof:

Let $G(t) = t^n F(t)$, for $t > 0$

Then

$$\begin{aligned}
\mathcal{L}\{G(t)\} &= \int_0^\infty e^{-st} t^n F(t) dt \\
&= (-1)^n \int_0^\infty e^{-st} \frac{d^n}{ds^n} F(t) dt \\
&= (-1)^n \frac{d^n}{ds^n} \int_0^\infty e^{-st} F(t) dt \\
&= (-1)^n \frac{d^n}{ds^n} f(s) \quad \square
\end{aligned}$$

Alternative Proof:

We have

$$f(s) = \int_0^\infty e^{-st} F(t) dt$$

Then by Leibnitz's rule for differentiating under the integral sign, we have

$$\begin{aligned}
\frac{df}{ds} = f'(s) &= \frac{d}{ds} \int_0^\infty e^{-st} F(t) dt \\
&= \int_0^\infty \frac{d}{ds} (e^{-st} F(t)) dt \\
&= \int_0^\infty e^{-st} (-te^{-st} F(t) + F'(t)) dt \\
&= - \int_0^\infty e^{-st} \{tF(t)\} dt \\
&= -\mathcal{L}\{tF(t)\}
\end{aligned}$$

$$\therefore \mathcal{L}\{tF(t)\} = -\frac{d}{ds} f(s) = f'(s)$$

This proves the theorem for $n = 1$.

To establish the theorem in general, we use mathematical induction. Suppose that the theorem is true for $n = k$, i.e

$$\int_0^\infty e^{-st} \{t^k F(t)\} dt = (-1)^k f^{(k)}(s) \quad (2.5.1)$$

Then

$$\frac{d}{ds} \left[\int_0^\infty e^{-st} \{t^k F(t)\} dt \right] = (-1)^k f^{(k+1)}(s)$$

Or by Leibnitz's rule,

$$- \int_0^\infty \frac{d}{ds} (e^{-st} \{t^{k+1} F(t)\}) dt = (-1)^{(k)} f^{(k+1)}(s)$$

i.e

$$\int_0^\infty e^{-st} \{t^{k+1} F(t)\} dt = (-1)^{(k+1)} f^{(k+1)}(s) \quad (2.5.2)$$

It follows that if (2.5.1) is true for $n = k$, then (2.5.2) is true for $n = k + 1$. Since (2.5.1) is true for $n = 1$, it follows that (2.5.1) is true for all positive integers n . \square

Example 2.10: Find $\mathcal{L}\{t^2 \cos at\}$

$$\begin{aligned}
\mathcal{L}\{t^2 \cos at\} &= (-1)^2 \cdot \frac{d^2}{dx^2} \left(\frac{s}{s^2 + a^2} \right) \\
&= \frac{d}{ds} \left[\frac{s^2 + a^2 - 2s^2}{(s^2 + a^2)^2} \right] \\
&= \frac{d}{ds} \left[\frac{a^2 - s^2}{(s^2 + a^2)^2} \right] \\
&= \frac{(s^2 + a^2)^2 (-2s) - (-s^2 + a^2) 2(s^2 + a^2) \cdot 2s}{(s^2 + a^2)^4} \\
&= \frac{2s^3 - 6a^2s}{(s^2 + a^2)^3}
\end{aligned}$$

2.6 Division by t

Theorem 2.6.1 (Division by t):

If

$$\mathcal{L}\{F(t)\} = f(s)$$

then

$$\mathcal{L}\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(x) dx$$

Proof:

Let $G(t) = \frac{F(t)}{t}$, for $t > 0$. Then $F(t) = tG(t)$. Taking the Laplace Transform of both sides, we get

$$\begin{aligned}\mathcal{L}\{F(t)\} &= \mathcal{L}\{tG(t)\} \\ f(s) &= -\frac{d}{ds}\mathcal{L}\{G(t)\} = -\frac{d}{ds}g(s)\end{aligned}$$

Then integrating, we have

$$\begin{aligned}g(s) &= -\int_s^\infty f(u) du \\ \mathcal{L}\{G(t)\} &= \int_s^\infty f(u) du \\ \mathcal{L}\left\{\frac{F(t)}{t}\right\} &= \int_s^\infty f(u) du \quad \square\end{aligned}$$

Example 2.11: Find $\mathcal{L}\left\{\frac{\sin t}{t}\right\}$

$$\begin{aligned}\mathcal{L}\left\{\frac{\sin t}{t}\right\} &= \int_s^\infty \frac{1}{s^2 + 1} du \\ &= \tan^{-1} u \Big|_s^\infty \\ &= \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s \\ &= \tan^{-1} \frac{1}{s}\end{aligned}$$

2.7 Laplace Transform of Integral

Theorem 2.7.1 (Laplace transform of Integral):

If

$$\mathcal{L}\{F(t)\} = f(s)$$

then

$$\mathcal{L}\left\{\int_0^t F(x) dx\right\} = \frac{1}{s}f(s)$$

Proof:

Let $G(t) = \int_0^t F(x)dx$, for $t > 0$. Then $G'(t) = F(t)$ and $G(0) = 0$. Taking the Laplace Transform of both sides, we have

$$\begin{aligned}\mathcal{L}\{G'(t)\} &= \mathcal{L}\{F(t)\} \\ s\mathcal{L}\{G(t)\} - G(0) &= f(s) \\ s\mathcal{L}\{G(t)\} &= f(s) \\ \mathcal{L}\{G(t)\} &= \frac{f(s)}{s} \\ \mathcal{L}\left\{\int_0^t F(u) du\right\} &= \frac{f(s)}{s} \quad \square\end{aligned}$$

Example 2.12: Evaluate $\mathcal{L}\left\{\int_0^t \frac{\sin u}{u} du\right\}$

$$\begin{aligned}\mathcal{L}\left\{\frac{\sin u}{u}\right\} &= \tan^{-1} \frac{1}{s} \\ \mathcal{L}\left\{\int_0^t \frac{\sin u}{u} du\right\} &= \frac{f(s)}{s} = \frac{1}{s} \tan^{-1} \frac{1}{s}\end{aligned}$$

Example 2.13: Evaluate $\mathcal{L}\left\{\int_0^t \frac{\sin u}{u} du\right\}$

Let $F(t) = \frac{\sin t}{t}$

$$\begin{aligned}\mathcal{L}\left\{\frac{\sin t}{t}\right\} &= \int_s^\infty \frac{1}{u^2 + 1} du \\ &= \tan^{-1} u \Big|_s^\infty \\ &= \tan^{-1} \frac{1}{s} \\ \therefore \mathcal{L}\left\{\int_0^t \frac{\sin u}{u} du\right\} &= \frac{1}{s} \cdot \tan^{-1} \frac{1}{s}\end{aligned}$$

Example 2.14: Evaluate $\mathcal{L}\left\{\int_0^t \sin 2u du\right\}$

$$\begin{aligned}\mathcal{L}\{\sin 2t\} &= \frac{2}{s^2 + 4} \\ \therefore \mathcal{L}\left\{\int_0^t \sin 2u du\right\} &= \frac{2}{s^3 + 4s}\end{aligned}$$

2.8 Laplace Transform of Periodic Functions

Theorem 2.8.1 (Periodic functions):

If

$$\mathcal{L}\{F(t)\} = f(s)$$

then

$$\mathcal{L}\{F(t)\} = \frac{1}{1 - e^{-Ts}} \int_0^T e^{-st} F(t) dt$$

where T is the period of $F(t)$.

Proof:

Let $F(t)$ has period T . Then $F(t) = F(t + T)$ for all t . Then

$$\begin{aligned}\mathcal{L}\{F(t)\} &= \int_0^\infty e^{-st} F(t) dt \\ &= \int_0^T e^{-st} F(t) dt + \int_T^{2T} e^{-st} F(t) dt + \int_{2T}^{3T} e^{-st} F(t) dt + \dots \\ &= \int_0^T e^{-st} F(t) dt + \int_0^T e^{-s(t+T)} F(t+T) dt + \int_0^T e^{-s(t+2T)} F(t+2T) dt + \dots \\ &= \int_0^T e^{-st} F(t) dt + e^{-sT} \int_0^T e^{-st} F(t) dt + e^{-2sT} \int_0^T e^{-st} F(t) dt + \dots \\ &= \left[1 + e^{-sT} + e^{-2sT} + \dots\right] \int_0^T e^{-st} F(t) dt \\ &= \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} F(t) dt \quad \square\end{aligned}$$

Note:-

Sum of an infinite series $1 + r + r^2 + \dots = \frac{1}{1-r}$ for $|r| < 1$.

Example 2.15: Find $\mathcal{L}\{F(t)\}$ for

$$F(t) = \begin{cases} \sin t & \text{for } 0 < t < \pi \\ 0 & \text{for } \pi < t < 2\pi \end{cases}$$

$$\begin{aligned} \mathcal{L}\{F(t)\} &= \frac{1}{1 - e^{-2\pi s}} \int_0^{2\pi} e^{st} \sin t \, dt \\ &= \frac{1}{1 - e^{-2\pi s}} \int_0^{\pi} e^{-st} \sin t \, dt \end{aligned}$$

Example 2.16: Find $\mathcal{L}\{F(t)\}$ for

$$F(t) = \begin{cases} t & \text{for } 0 < t < 1 \\ 0 & \text{for } 1 < t < 2 \end{cases}$$

$$\begin{aligned} \mathcal{L}\{F(t)\} &= \frac{1}{1 - e^{-2s}} \int_0^1 t \, dt \\ &= \frac{t^2 \Big|_0^1}{2 - 2e^{-2s}} \\ &= \frac{1}{2 - 2e^{-2s}} \end{aligned}$$

2.9 Laplace Transform of Derivatives

Theorem 2.9.1 (Laplace transform of derivatives):

If

$$\mathcal{L}\{F(t)\} = f(s)$$

then

$$\mathcal{L}\{F'(t)\} = sf(s) - F(0)$$

$$\mathcal{L}\{F''(t)\} = s^2F(s) - sF(0) - F'(0)$$

$$\mathcal{L}\{F'''(t)\} = s^3f(s) - s^2F(0) - sF'(0) - F''(0)$$

$$\mathcal{L}\{F^{(n)}(t)\} = s^n f(s) - s^{n-1}F(0) - s^{n-2}F'(0) - \dots - F^{(n-1)}(0)$$

$$\mathcal{L}\{F^{(n)}(t)\} = s^n f(s) - \sum_{i=n-1}^0 \sum_{j=0}^{n-1} s^i F^{(j)}(0)$$

Proof:

Using integration by parts, we have

$$\begin{aligned} \mathcal{L}\{F'(t)\} &= \int_0^{\infty} e^{-st} F'(t) \, dt \\ &= e^{-st} F(t) \Big|_0^{\infty} + s \int_0^{\infty} e^{-st} F(t) \, dt \\ &= -F(0) + s \int_0^{\infty} e^{-st} F(t) \, dt \\ &= sf(s) - F(0) \quad \square \end{aligned}$$

Similarly,

$$\begin{aligned}
\mathcal{L}\{F''(t)\} &= s\mathcal{L}\{F'(t)\} - F'(0) \\
&= s[sf(s) - F(0)] - F'(0) \\
&= s^2f(s) - sF(0) - F'(0)
\end{aligned}$$

Thus using mathematical induction, we get

$$\begin{aligned}
\mathcal{L}\{F^{(n)}(t)\} &= s\mathcal{L}\{F^{(n-1)}(t)\} - F^{(n-1)}(0) \\
&= s[s\mathcal{L}\{F^{(n-2)}(t)\} - F^{(n-2)}(0)] - F^{(n-1)}(0) \\
&= s^2\mathcal{L}\{F^{(n-2)}(t)\} - sF^{(n-2)}(0) - F^{(n-1)}(0) \\
&= s^2[s\mathcal{L}\{F^{(n-3)}(t)\} - F^{(n-3)}(0)] - sF^{(n-2)}(0) - F^{(n-1)}(0) \\
&= s^3\mathcal{L}\{F^{(n-3)}(t)\} - s^2F^{(n-3)}(0) - sF^{(n-2)}(0) - F^{(n-1)}(0) \\
&= \dots \\
&= s^n\mathcal{L}\{F(t)\} - s^{n-1}F(0) - s^{n-2}F'(0) - \dots - F^{(n-1)}(0) \\
&= s^n f(s) - \sum_{i=n-1}^0 \sum_{j=0}^{n-1} s^i F^{(j)}(0) \quad \square
\end{aligned}$$

3 Inverse Laplace Transform

3.1 Definition and Existence

Definition 3.1.1: Inverse Laplace Transform

If the Laplace Transform of a function $F(t)$ is $f(s)$, i.e

$$\mathcal{L}\{F(t)\} = f(s)$$

then the Inverse Laplace Transform is defined as

$$\mathcal{L}^{-1}\{f(s)\} = F(t)$$

3.2 Some Functions and their Inverse Laplace Transforms

| $f(s)$ | $\mathcal{L}^{-1}\{f(s)\} = F(t)$ | $f(s)$ | $\mathcal{L}^{-1}\{f(s)\} = F(t)$ |
|---------------------|-----------------------------------|---------------------|-----------------------------------|
| $\frac{1}{s}$ | 1 | $\frac{n}{s}$ | n |
| $\frac{1}{s^2}$ | t | $\frac{1}{s^n}$ | $\frac{t^{n-1}}{(n-1)!}$ |
| $\frac{1}{s-a}$ | e^{at} | $\frac{1}{s+a}$ | e^{-at} |
| $\frac{1}{s^2+a^2}$ | $\frac{\sin at}{a}$ | $\frac{1}{s^2-a^2}$ | $\frac{\sinh at}{a}$ |
| $\frac{s}{s^2+a^2}$ | $\cos at$ | $\frac{s}{s^2-a^2}$ | $\cosh at$ |

Table 2: Functions and their Inverse Laplace Transform

4 Basic Properties of Inverse Laplace Transform

4.1 Linearity Property

Theorem 4.1.1 (Linearity Property):

If

$$\mathcal{L}^{-1}\{f(s)\} = F(t) \quad \text{and} \quad \mathcal{L}^{-1}\{g(s)\} = G(t)$$

Then,

$$\mathcal{L}^{-1}\{c_1f(s) + c_2g(s)\} = c_1\mathcal{L}^{-1}\{f(s)\} + c_2\mathcal{L}^{-1}\{g(s)\}$$

Example 4.1: Evaluate $\mathcal{L}^{-1}\left\{\frac{4}{s-2} - \frac{3s}{s^2+16} + \frac{5}{s^2+4}\right\}$

$$\begin{aligned} \mathcal{L}^{-1}\left\{\frac{4}{s-2} - \frac{3s}{s^2+16} + \frac{5}{s^2+4}\right\} &= 4\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} - 3\mathcal{L}^{-1}\left\{\frac{s}{s^2+4^2}\right\} + 5\mathcal{L}^{-1}\left\{\frac{1}{s^2+2^2}\right\} \\ &= 4e^{2t} - 3\cos 4t + \frac{5}{2}\sin 2t \end{aligned}$$

4.2 First Translation or Shifting Property

Theorem 4.2.1 (First Translation or Shifting Property):

If

$$\mathcal{L}^{-1}\{f(s)\} = F(t)$$

then

$$\mathcal{L}^{-1}\{f(s-a)\} = e^{at}F(t)$$

Example 4.2: Evaluate $\mathcal{L}^{-1}\left\{\frac{6s-4}{s^2-4s+20}\right\}$

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{6s-4}{s^2-4s+20}\right\} &= \mathcal{L}^{-1}\left\{\frac{6(s-2)+8}{(s-2)^2+16}\right\} \\ &= 6\mathcal{L}^{-1}\left\{\frac{(s-2)}{(s-2)^2+16}\right\} + 2\mathcal{L}^{-1}\left\{\frac{4}{(s-2)^2+16}\right\} \\ &= 6e^{2t}\cos 4t + 2e^{2t}\sin 4t\end{aligned}$$

4.3 Second Translation or Shifting Property

Theorem 4.3.1 (Second Translation or Shifting Property):

If

$$\mathcal{L}^{-1}\{f(s)\} = F(t)$$

then

$$\mathcal{L}^{-1}\{e^{-as}f(s)\} = \begin{cases} F(t-a) & \text{for } t > a \\ 0 & \text{for } t < a \end{cases}$$

Example 4.3: Evaluate $\frac{e^{-5s}}{(s-2)^4}$

Here

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^4}\right\} &= e^{2t}\frac{t^3}{3!} = \frac{e^{2t}}{6}t^3 \\ \therefore \mathcal{L}^{-1}\left\{\frac{e^{-5s}}{(s-2)^4}\right\} &= \frac{e^{2(t-5)}}{6}(t-5)^3, \text{ when } t > 5 \\ \mathcal{L}^{-1}\left\{\frac{e^{-5s}}{(s-2)^4}\right\} &= 0, \text{ when } t < 5\end{aligned}$$

4.4 Inverse Laplace Transform of Derivatives

Theorem 4.4.1 (Inverse Laplace Transform of Derivatives):

If

$$\mathcal{L}^{-1}\{f(s)\} = F(t)$$

then

$$\mathcal{L}^{-1}\{f^{(n)}(s)\} = \mathcal{L}^{-1}\left\{\frac{d^n}{ds^n}f(s)\right\} = (-1)^n t^n F(t)$$

Example 4.4: Evaluate $\mathcal{L}^{-1}\left\{\ln\left(1+\frac{1}{s^2}\right)\right\}$

Here

$$\begin{aligned}
 f(s) &= \ln \left(1 + \frac{1}{s^2} \right) = \mathcal{L}\{F(t)\} \\
 f'(s) &= \frac{-\frac{2}{s^3}}{1 + \frac{1}{s^2}} \\
 &= -2 \left\{ \frac{1}{s(s^2 + 1)} \right\} \\
 &= -2 \left(\frac{1}{s} - \frac{s}{s^2 + 1} \right) \\
 \mathcal{L}^{-1}\{f'(s)\} &= -2(1 - \cos t) \\
 -tF(t) &= -2(1 - \cos t) \\
 F(t) &= \frac{2(1 - \cos t)}{t} \\
 \therefore \mathcal{L}^{-1} \left\{ \ln \left(1 + \frac{1}{s^2} \right) \right\} &= \frac{2(1 - \cos t)}{t}
 \end{aligned}$$

4.5 Inverse Laplace Transform of Integrals

Theorem 4.5.1 (Inverse Laplace Transform of Integrals):

If

$$\mathcal{L}^{-1}\{f(s)\} = F(t)$$

then

$$\mathcal{L}^{-1} \left\{ \int_s^\infty f(u) du \right\} = \frac{F(t)}{t}$$

Example 4.5: Find $\mathcal{L}^{-1} \left\{ \int_s^\infty \left(\frac{1}{u} - \frac{1}{u+1} \right) du \right\}$

Let

$$f(s) = \frac{1}{s} - \frac{1}{s+1}$$

$$\begin{aligned}
 \mathcal{L}^{-1}\{f(s)\} &= \mathcal{L}^{-1} \left\{ \frac{1}{s} - \frac{1}{s+1} \right\} \\
 &= 1 - e^{-t} = F(t)
 \end{aligned}$$

$$\begin{aligned}
 \therefore \mathcal{L}^{-1} \left\{ \int_s^\infty f(u) du \right\} &= \frac{F(t)}{t} \\
 &= \frac{1 - e^{-t}}{t}
 \end{aligned}$$

4.6 Division by s

Theorem 4.6.1 (Division by s):

If

$$\mathcal{L}^{-1}\{f(s)\} = F(t)$$

then

$$\mathcal{L}^{-1} \left\{ \frac{f(s)}{s} \right\} = \int_0^t F(u) du$$

Example 4.6: Evaluate $\mathcal{L}^{-1} \left\{ \frac{1}{s^3(s^2 + 1)} \right\}$

Let

$$f(s) = \frac{1}{s^2 + 1}$$

$$\begin{aligned}\therefore \mathcal{L}^{-1}\{f(s)\} &= \sin t \\ \therefore \mathcal{L}^{-1}\left\{\frac{1}{s(s^2 + 1)}\right\} &= \int_0^t \sin u \, du = 1 - \cos t \\ \therefore \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2 + 1)}\right\} &= \int_0^t (1 - \cos u) \, du = t - \sin t \\ \therefore \mathcal{L}^{-1}\left\{\frac{1}{s^3(s^2 + 1)}\right\} &= \int_0^t (u - \sin u) \, du = \frac{t^2}{2} + \cos t - 1\end{aligned}$$

Alternative approach:

Let

$$f(s) = \frac{1}{s^4 + s^2} = \frac{1}{s^2} - \frac{1}{s^2 + 1}$$

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2 + 1)}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s^2} - \frac{1}{s^2 + 1}\right\} \\ &= t - \sin t \\ &= F(t) \\ \therefore \mathcal{L}^{-1}\left\{\frac{1}{s^3(s^2 + 1)}\right\} &= \int_0^t F(u) \, du \\ &= \int_0^t (u - \sin u) \, du \\ &= \frac{t^2}{2} + \cos t - 1\end{aligned}$$

4.7 Inverse Laplace Transform by Partial Fraction

Example 4.7: Find $\mathcal{L}^{-1}\left\{\frac{5^2 - 15s - 11}{(s + 1)(s - 2)^2}\right\}$

$$\begin{aligned}\frac{5^2 - 15s - 11}{(s + 1)(s - 2)^2} &= \frac{-\frac{1}{3}}{ss + 1} - \frac{7}{(s - 2)^3} + \frac{4}{(s - 2)^2} + \frac{\frac{1}{3}}{s - 2} \\ \therefore \mathcal{L}^{-1}\left\{\frac{5^2 - 15s - 11}{(s + 1)(s - 2)^2}\right\} &= -\frac{1}{e}e^{-t} - 7\frac{t^2}{2}e^{2t} + 4te^{2t} + \frac{1}{3}e^{2t}\end{aligned}$$

4.8 The Convolution Theorem

Theorem 4.8.1 (The Convolution Theorem):

If

$$\mathcal{L}^{-1}\{f(s)\} = F(t) \quad \text{and} \quad \mathcal{L}^{-1}\{g(s)\} = G(t)$$

then

$$\mathcal{L}^{-1}\{f(s)g(s)\} = \int_0^t F(u)G(t - u) \, du = F * G$$

$$G * F = \int_0^t G(u)F(t - u) \, du = F * G$$

Example 4.8: Find $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2 + 1)^2}\right\}$

Here,

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t = F(t)$$

$$\mathcal{L}^{-1} \left\{ \frac{1}{(s+1)^2} \right\} = e^{-t}t = G(t)$$

$$\begin{aligned} \therefore \mathcal{L}^{-1} \left\{ \frac{1}{s^2(s+1)^2} \right\} &= G * F \\ &= \int_0^t u e^{-u}(t-u) \, du \\ &= \int_0^t (ut - u^2) e^{-u} \, du \\ &= -(ut - u^2) \cdot e^{-u} \Big|_0^t + \int_0^t (t - 2u) e^{-u} \, du \end{aligned}$$

Example 4.9: Find $\mathcal{L}^{-1} \left\{ \frac{s}{(s^2 + a^2)^2} \right\}$

Let

$$\begin{aligned} f(s) &= \frac{s}{s^2 + a^2} & \text{and} & & g(s) &= \frac{1}{s^2 + a^2} \\ \mathcal{L}^{-1}\{f(s)\} &= \cos at & \text{and} & & \mathcal{L}^{-1}\{g(s)\} &= \frac{\sin at}{a} \end{aligned}$$

$$\begin{aligned} F * G &= \frac{1}{a} \int_0^t \cos au \cdot \sin au(t-u) \, du \\ &= \frac{1}{a} \int_0^t \cos au (\sin at \cos au - \cos at \sin au) \, du \\ &= \frac{1}{a} \int_0^t \sin at \cos^2 au \, du - \frac{1}{a} \int_0^t \sin au \cos au \, du \\ &= \frac{1}{2a} \sin at \int_0^t (\cos 2au + 1) \, du - \frac{1}{2a} \cos at \int_0^t \sin 2au \, du \\ &= \frac{1}{2a} \left[-\frac{1}{2} \sin 2au + u \right]_0^t - \frac{1}{2a} \left[\frac{1}{2} \cos 2au \right]_0^t \\ &= -\frac{1}{2a} \sin at \cdot \sin 2at - \frac{1}{2a} \cos at \cdot \frac{1}{2} (\cos 2at - 1) + \frac{1}{2a} \sin at \cdot t \\ &= \frac{1}{4a} [\cos at - \cos (at - 2at)] + \frac{t \sin at}{2a} \\ &= \frac{1}{4a} [\cos at - \cos at] + \frac{t \sin at}{2a} \\ &= \frac{t \sin at}{2a} \end{aligned}$$