Differential Equations and Vector Calculus

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Introduction and formation of Differential Equations

1.1 Differential equation

Definition. An equation involving derivatives or differentials of one or more dependent variables with respect to one or more independent variables is called a *differential equation*.

For examples of differential equations we list the following:

$$dy = (x + \sin x)\, dx, \qquad \dots (1)$$

$$\frac{d^4x}{dt^4} + \frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^5 = e^t, \qquad \dots (2)$$

$$y = \sqrt{x}\,\frac{dy}{dx} + \frac{k}{dy/dx}, \qquad \dots (3)$$

$$k\,(d^2y/dx^2) = \{1 + (dy/dx)^2\}^{3/2} \qquad \dots (4)$$

$$\partial^2 v/\partial t^2 = k\,(\partial^3 v/\partial x^3)^2 \qquad \dots (5)$$

and

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2 = 0 \qquad \dots (6)$$

Note. Unless otherwise stated, y' (or y_1), y'' (or y_2), ..., $y^{(n)}$ (or y_n) will denote $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^ny}{dx^n}$ respectively. Thus, for example equation (3) may be re-written as

$$y = \sqrt{x}\, y' + k/y' \qquad \text{or} \qquad y = \sqrt{x}\, y_1 + k/y_1.$$

1.2 Ordinary differential equation

Definition. A differential equation involving derivatives with respect to a single independent variable is called an *ordinary differential equation*.

In Art. 1.1 equations (1), (2), (3) and (4) are all ordinary differential equations.

1.3 Partial differential equation

Definition. A differential equation involving partial derivatives with respect to more than one independent variables is called a *partial differential equation*.

In Art. 1.1 equations (5) and (6) are both partial differential equations.

1.4 Order of a differential equation

Definition. The order of the highest order derivative involved in a differential equation is called the *order of the differential equation*.

1.5 Degree of a differential equation

Definition. The *degree of a differential equation* is the degree of the highest derivative which occurs in it, after the differential equation has been made free from radicals and fractions as far as the derivatives are concerned. Note that the above definition of degree does not require variables x, t, u etc. to be free from radicals and fractions.

1.6 Linear and non-linear differential equations

Definition. A differential equation is called *linear* if (*i*) every dependent variable and every derivative involved occurs in the first degree only, and (*ii*) no products of dependent variables and/or derivatives occur. A differential equation which is not linear is called a *non-linear differential equation.*

In Art 1.1 equations (1) and (6) are linear and equations (2), (3), (4) and (5) are all non-linear.

1.7 Solution of a differential equation

Definition. Any relation between the dependent and independent variables, when substituted in the differential equation, reduces it to an identity is called a *solution* or *integral of the differential equation*. It should be noted that a solution of a differential equation does not involve the derivatives of the dependent variable with respect to the independent variable or variables.

For example, $y = ce^{2x}$ is a solution of $dy/dx = 2y$ because by putting $y = ce^{2x}$ and $dy/dx = 2ce^{2x}$, the given differential equation reduces to the identity $2ce^{2x} = 2ce^{2x}$. Observe that $y = ce^{2x}$ is a solution of the given differential equation for any real constant c which is called an *arbitrary constant*.

Ex. 1. *Show that* $y = (A/x) + B$ *is a solution of* $(d^2y/dx^2) + (2/x) \times (dy/dx) = 0$.

Sol. Given that $$(d^2y/dx^2) + (2/x) \times (dy/dx) = 0. \quad \text{... (1)}$$

Also given that $$y = (A/x) + B. \quad \text{... (2)}$$

Differentiating (2) w.r.t. 'x', $$dy/dx = -(A/x^2) \quad \text{... (3)}$$

Differentiating (3) w.r.t. 'x', $$d^2y/dx^2 = 2A/x^3 \quad \text{... (4)}$$

Substituting for dy/dx and d^2y/dx^2 from (3) and (4) in (1), we get

$$(2A/x^3) + (2/x) \times (-A/x^2) = 0 \qquad \text{or} \qquad 0 = 0,$$

which is true. Hence (2) is a solution of (1).

Ex. 2. *Show that* $y = a \cos(mx + b)$ *is a solution of the differential equation* $d^2y/dx^2 + m^2y = 0$.

Sol. Try yourself.

Ex. 1. *Find the differential equation of the family of curves $y = e^{mx}$, where m is an arbitrary constant.*

Sol. Given that $y = e^{mx}$. ... (1)

Differentiating (1) w.r.t. 'x', we get $dy/dx = me^{mx}$. ... (2)

Now, (1) and (2) $\Rightarrow$ $dy/dx = my$ $\Rightarrow$ $m = (1/y) \times (dy/dx)$. ... (3)

Again, from (1), $mx = \log_e y$ so that $m = (\log_e y)/x$. ... (4)

Eliminating m from (3) and (4), we get $(1/y) \times (dy/dx) = (1/x) \times \log_e y$.

Ex. 2. (*a*) *Find the differential equation of all straight lines passing through the origin.*

(*b*) *Find the differential equation of all the straight lines in the xy-plane.*

Sol. (*a*) Equation of any straight line passing through the origin is

$y = mx$, m being arbitrary constant. ... (1)

Differentiating (1) w.r.t. 'x', $dy/dx = m$. ... (2)

Eliminating m from (1) and (2), we get $y = x\,(dy/dx)$.

(*b*) We know that equation of any straight line in the xy-plane is given by

$y = mx + c$, m and c being arbitrary constants. ... (1)

Differentiating (1) w.r.t. 'x', we get $dy/dx = m$. ... (2)

Differentiating (2) w.r.t. 'x', we get $d^2y/dx^2 = 0$, ... (3)

which is the required differential equation.

Ex. 3. (*a*) *Obtain a differential equation satisfied by family of circles $x^2 + y^2 = a^2$, a being an arbitrary constant.*

(*b*) *Obtain a differential equation satisfied by the family of concentric circles.*

Sol. (*a*) Given $$x^2 + y^2 = a^2. \qquad \ldots (1)$$

Differentiating (1) w.r.t. 'x', we get $\quad 2x + 2y\,(dy/dx) = 0 \quad$ or $\quad x + y\,(dy/dx) = 0$, which is the required differential equation.

(*b*) Let the centre of the given family of concentric circles be (0, 0). Then we know that the equation of the family of concentric circles is given by $x^2 + y^2 = a^2$, *a* being arbitrary constant.

Now proceed as in part (*a*). **Ans.** $x + y\,(dy/dx) = 0$.

Ex. 8. (*a*) *Find the differential equation of the family of curves $y = e^x\,(A \cos x + B \sin x)$, where A and B are arbitrary constants.* **[G.N.D.U. Amritsar 2010]**

(*b*) *Form a differential equation of which $y = e^x\,(A \cos 2x + B \sin 2x)$ is a solution, A and B being arbitrary constants.*

Differentiating (1), $$y' = e^x\,(-A \sin x + B \cos x) + e^x\,(A \cos x + B \sin x)$$

r $$y' = e^x\,(-A \sin x + B \cos x) + y, \text{ using (1).} \qquad \ldots (2)$$

Differentiating (2) again with respect to x, we get

$$y'' = -\,e^x\,(A \cos x + B \sin x) + e^x\,(-A \sin x + B \cos x) + y'. \qquad \ldots (3)$$

Now from (2), we get $$e^x\,(-A \sin x + B \sin x) = y' - y. \qquad \ldots (4)$$

Hence, eliminating A and B from (1), (3) and (4), we get

$$y'' = -y + y' - y + y' \qquad \text{or} \qquad y'' - 2y' + 2y = 0.$$

(*b*) Proceed as in Ex. 8(*a*). **Ans.** $y'' - 2y' + 5y = 0$

Ex. 10. *Find the differential equation corresponding to the family of curves $y = c(x - c)^2$, where c is an arbitrary constant.* **[I.A.S. (Prel.) 2009; Karnatak 1995]**

Sol. Given that $$y = c(x - c)^2. \quad \ldots (1)$$

Diff. (1) w.r.t. 'x', we get $$y' = 2c(x - c). \quad \ldots (2)$$

From (1) and (2), $y'/y = 2/(x - c)$ so that $c = x - (2y/y')$. ... (3)

Putting this value of c in (2), the required equation is

$$y' = 2\{x - (2y/y')\} \times (2y/y') \quad \text{or} \quad (y')^3 = 4y(xy' - 2y).$$

Ex. 11. *Find the differential equation of all circles of radius a.* **[Nagarjuna 2003]**

Sol. The equation of all circles of radius a is given by

$$(x - h)^2 + (y - k)^2 = a^2, \quad \ldots (1)$$

where h and k, are to be taken as arbitrary constants.

Diff. (1) w.r.t. 'x', we get $$(x - h) + (y - k)y' = 0. \quad \ldots (2)$$

Diff. (2), $1 + (y')^2 + (y - k)y'' = 0$ or $y - k = -\{1 + (y')^2\}/y''$. ... (3)

Putting this value of $y - k$ in (2), we get

$$x - h = -(y - k)y' = \{1 + (y')^2\} \times (y'/y''). \quad \ldots (4)$$

Using (3) and (4), (1) gives the required equation as

$$\frac{\{1 + (y')^2\}^2 (y')^2}{(y'')^2} + \frac{\{1 + (y')^2\}^2}{(y'')^2} = a^2 \quad \text{or} \quad \{1 + (y')^2\}^3 = a^2 (y'')^2.$$