

Complex Numbers

1. Definition: A number of the form $a+ib$ where a and b are real numbers and $i=\sqrt{-1}$, is called a complex number.

If $z = a+ib$, then a is called the real part of z and b is called the imaginary part of z .

2. Modulus and Amplitude:

If the polar coordinates of the point (a,b) be (r,θ) , then $a = r\cos\theta$ and $b = r\sin\theta$

$$\therefore |z| = \sqrt{a^2+b^2} \text{ and } \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

The number $|z|$ is called the modulus or absolute value and θ is called the amplitude or argument of the complex number $z = a+ib$.

In symbols, we write

$$|z| = \sqrt{a^2+b^2}$$

$$\theta = \arg z = \tan^{-1}\left(\frac{b}{a}\right)$$

$$\text{Now, } z = a+ib = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

3. State and Prove De Moivre's theorem.

Statement: For all rational values of n ,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Proof: Case-1: When n is a positive integer.

We have, $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$$\begin{aligned} &= \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2 + i(\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2) \\ &= \cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2) \end{aligned}$$

$$\begin{aligned} \text{Similarly, } & (\cos\alpha_1 + i\sin\alpha_1)(\cos\alpha_2 + i\sin\alpha_2)(\cos\alpha_3 + i\sin\alpha_3) \\ &= \{\cos(\alpha_1 + \alpha_2) + i\sin(\alpha_1 + \alpha_2)\}(\cos\alpha_3 + i\sin\alpha_3) \\ &= \cos(\alpha_1 + \alpha_2 + \alpha_3) + i\sin(\alpha_1 + \alpha_2 + \alpha_3) \end{aligned}$$

Proceeding in this way, the product of the n factors

$$\begin{aligned} & (\cos\alpha_1 + i\sin\alpha_1)(\cos\alpha_2 + i\sin\alpha_2) \dots (\cos\alpha_n + i\sin\alpha_n) \\ &= \cos(\alpha_1 + \alpha_2 + \dots + \alpha_n) + i\sin(\alpha_1 + \alpha_2 + \dots + \alpha_n) \end{aligned}$$

If we put $\alpha_1 = \alpha_2 = \dots = \alpha_n = \alpha$, then we have

$$(\cos\alpha + i\sin\alpha)^n = \cos n\alpha + i\sin n\alpha$$

Case-2: When n is a negative integer.

Let $n = -m$, where m is a positive integer.

$$(\cos\alpha + i\sin\alpha)^n = (\cos\alpha + i\sin\alpha)^{-m} = \frac{1}{(\cos\alpha + i\sin\alpha)^m}$$

$$\text{Left side} = \frac{1}{\cos m\alpha + i\sin m\alpha} \quad [\text{by Case-1}]$$

$$\text{Right side} = \frac{1}{(\cos m\alpha - i\sin m\alpha)} = \frac{1}{(\cos m\alpha + i\sin m\alpha)(\cos m\alpha - i\sin m\alpha)}$$

$$\text{Left side} = \frac{\cos m\alpha - i\sin m\alpha}{\cos^2 m\alpha + \sin^2 m\alpha}$$

$$\text{Right side} = \cos m\alpha - i\sin m\alpha$$

$$\therefore \text{Left side} = \cos(-m)\alpha - i\sin(-m)\alpha \quad [:: m = -n]$$

$$\therefore \text{Left side} = \cos m\alpha + i\sin m\alpha$$

Case-3: When n is a fraction, +ve or -ve.

Let $n = \frac{p}{q}$, where q is a positive integer and p is any integer, +ve or -ve.

$$\begin{aligned}
 \text{Now, } (\cos \frac{p}{q}\theta + i \sin \frac{p}{q}\theta)^q &= \cos(q \cdot \frac{p}{q}\theta) + i \sin(q \cdot \frac{p}{q}\theta) \\
 &= \cos p\theta + i \sin p\theta \\
 &= (\cos \theta + i \sin \theta)^p, \text{ since } p \text{ is any integer.}
 \end{aligned}$$

Taking q -th root, we get—

$$\begin{aligned}
 \cos \frac{p}{q}\theta + i \sin \frac{p}{q}\theta &= \text{one of the values of } (\cos \theta + i \sin \theta)^{\frac{p}{q}} \\
 \text{or, } \cos n\theta + i \sin n\theta &= \text{one of the values of } (\cos \theta + i \sin \theta)^n
 \end{aligned}$$

Thus, De Moivre's theorem is completely established for all rational values of n .

4. If $x_n = \cos \frac{\pi}{2^n} + i \sin \frac{\pi}{2^n}$, prove that—

$$x_1 x_2 x_3 \dots \infty = -1$$

$$\begin{aligned}
 \text{Solution: L.H.S.} &= x_1 x_2 x_3 \dots \infty \\
 &= \left(\cos \frac{\pi}{2^1} + i \sin \frac{\pi}{2^1} \right) \left(\cos \frac{\pi}{2^2} + i \sin \frac{\pi}{2^2} \right) \left(\cos \frac{\pi}{2^3} + i \sin \frac{\pi}{2^3} \right) \dots \infty \\
 &= \cos \left(\frac{\pi}{2^1} + \frac{\pi}{2^2} + \frac{\pi}{2^3} + \dots \infty \right) + i \sin \left(\frac{\pi}{2^1} + \frac{\pi}{2^2} + \frac{\pi}{2^3} + \dots \infty \right) \\
 &= \cos \left(-\frac{\pi}{1-\frac{1}{2}} \right) + i \sin \left(-\frac{\pi}{1-\frac{1}{2}} \right) \quad \left[: \frac{1}{1-\frac{1}{2}} = \frac{2}{1} \right] \\
 &= \cos \pi + i \sin \pi \\
 &= -1 + i \cdot 0 \\
 &= -1 \\
 &= \text{R.H.S}
 \end{aligned}$$

5. If $x_n = \cos \frac{\pi}{3^n} + i \sin \frac{\pi}{3^n}$, show that—

$$x_1 x_2 x_3 \dots \infty = i$$

Solution: L.H.S. = $x_1 x_2 x_3 \dots \infty$

$$\begin{aligned}
 &= \left(\cos \frac{\pi}{3^1} + i \sin \frac{\pi}{3^1} \right) \left(\cos \frac{\pi}{3^2} + i \sin \frac{\pi}{3^2} \right) \dots \infty \\
 &= \cos \left(\frac{\pi}{3^1} + \frac{\pi}{3^2} + \frac{\pi}{3^3} + \dots \infty \right) + i \sin \left(\frac{\pi}{3^1} + \frac{\pi}{3^2} + \frac{\pi}{3^3} + \dots \infty \right)
 \end{aligned}$$

$$\begin{aligned}
 \text{L.H.S.} &= \cos\left(\frac{\sqrt{3}}{1-i_3}\right) + i \sin\left(\frac{\sqrt{3}}{1-i_3}\right) \\
 &= \cos\frac{\pi}{2} + i \sin\frac{\pi}{2} \\
 &= 0 + i \cdot 1 \\
 &= i \\
 &= \text{R.H.S.}
 \end{aligned}$$

6. If n be a positive integer, prove that—

$$(i) (1+i)^n + (1-i)^n = 2^{\frac{n}{2}+1} \cos \frac{n\pi}{4}$$

$$(ii) (\sqrt{3}+i)^n + (\sqrt{3}-i)^n = 2^{\frac{n}{2}+1} \cos \frac{n\pi}{6}$$

Solution: (i) Let $i = r \cos \theta \dots \text{--- } (1)$

$$i = r \sin \theta \dots \text{--- } (2)$$

Squaring (1) and (2), then adding

$$2 = r^2 (\cos^2 \theta + \sin^2 \theta)$$

$$\text{Dividing by } r^2, 1 = \cos^2 \theta + \sin^2 \theta \Rightarrow \theta = 45^\circ$$

$$1 + i = r \cos 45^\circ + i \sin 45^\circ$$

Dividing (2) by (1), we get $\tan \theta = 1$

$$\text{or, } \tan \theta = \tan 45^\circ$$

$$(1+i)(\cos 45^\circ + i \sin 45^\circ)(\cos 45^\circ - i \sin 45^\circ) \therefore \theta = 45^\circ$$

$$\text{Now, } 1+i = 1+i \cdot 1 + i \cdot i =$$

$$= \sqrt{2} \cos 45^\circ + i \sin 45^\circ$$

$$= \sqrt{2} (\cos 45^\circ + i \sin 45^\circ)$$

$$\therefore (1+i)^n = \left\{ \sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \right\}^n$$

$$= 2^{\frac{n}{2}} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)^n$$

$$= 2^{\frac{n}{2}} \left(\cos \frac{n\pi}{4} + i \sin \frac{n\pi}{4} \right) \dots \text{--- } (3)$$

$$\text{Similarly, } (1-i)^n = 2^{\frac{n}{2}} \left(\cos \frac{n\pi}{4} - i \sin \frac{n\pi}{4} \right) \dots \text{--- } (4)$$

Adding (3) and (4), we get—

$$\begin{aligned}
 (1+i)^n + (1-i)^n &= 2 \cdot 2^{\frac{n}{2}} \cos \frac{n\pi}{4} \\
 &= 2^{\frac{n}{2}+1} \cos \frac{n\pi}{4}
 \end{aligned}$$

$$(iii) \text{ Let } \sqrt{3} = r \cos \theta \quad \dots (1)$$

$$1 = r \sin \theta \quad \dots (2)$$

Squaring (1) and (2), then adding

$$3+1 = r^2 (\cos^2 \theta + \sin^2 \theta)$$

$$\therefore r^2 = 4 \Rightarrow r = 2$$

$$\text{Dividing (2) by (1), } \tan \theta = \frac{1}{\sqrt{3}}$$

$$\therefore \tan \theta = \tan \frac{\pi}{6}$$

$$\text{Now, } \sqrt{3} + i = r \left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right)$$

$$= r \cos \theta + i r \sin \theta$$

$$= r (\cos \theta + i \sin \theta)$$

$$= 2 \left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right)$$

$$(\sqrt{3} + i)^n = \left\{ 2 \left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right) \right\}^n$$

$$= 2^n \left(\cos \frac{n\pi}{6} + i \sin \frac{n\pi}{6} \right)^n$$

$$= 2^n \left(\cos \frac{n\pi}{6} + i \sin \frac{n\pi}{6} \right) \dots (3)$$

$$\text{Similarly, } (\sqrt{3} - i)^n = 2^n \left(\cos \frac{n\pi}{6} - i \sin \frac{n\pi}{6} \right) \dots (4)$$

Adding (3) and (4), we get

$$(\sqrt{3} + i)^n + (\sqrt{3} - i)^n = 2 \cdot 2^n \cos \frac{n\pi}{6}$$

$$= 2^{n+1} \cos \frac{n\pi}{6}$$

7. Determine the locus represented by

$$(i) |z-2| = 3 \quad (ii) |z-2| = |z+4| \quad (iii) |z-3| + |z+3| = 10$$

$$\text{Solution: (i) } |z-2| = 3$$

$$\text{or, } |x+iy-2| = 3$$

$$\text{or, } |(x-2)+iy| = 3$$

$$\text{or, } \sqrt{(x-2)^2 + y^2} = 3$$

$$\text{or, } (x-2)^2 + y^2 = 3^2 \text{ which}$$

represents a circle of radius 3 and centre (2,0).

$$(ii) |z-2| = |z+4|$$

$$\text{or, } |x+iy-2| = |x+iy+4|$$

$$\text{or, } (x-2)^2 + y^2 = (x+4)^2 + y^2$$

or, $x = -1$ which represents a straight line.

$$(iii) |z-3| + |z+3| = 10$$

$$\text{or, } \sqrt{(x-3)^2 + y^2} + \sqrt{(x+3)^2 + y^2} = 10$$

$$\text{or, } \sqrt{(x-3)^2 + y^2} = 10 - \sqrt{(x+3)^2 + y^2}$$

$$\text{or, } (x-3)^2 + y^2 = 100 - 20\sqrt{(x+3)^2 + y^2} + (x+3)^2 + y^2$$

$$\text{or, } 12x + 100 = 20\sqrt{(x+3)^2 + y^2}$$

$$\text{or, } 3x + 25 = 5\sqrt{(x+3)^2 + y^2}$$

$$\text{or, } 9x^2 + 150x + 625 = 25(x^2 + 6x + 9 + y^2)$$

$$\text{or, } 16x^2 + 25y^2 = 400$$

$$\text{or, } \frac{x^2}{25} + \frac{y^2}{16} = 1$$

$$\text{or, } \frac{x^2}{5^2} + \frac{y^2}{4^2} = 1 \text{ which represents an ellipse.}$$

8. If $x + \frac{1}{x} = 260^\circ$, show that $x^n + \frac{1}{x^n} = 260n^\circ$

Solution: We have, $x + \frac{1}{x} = 260^\circ$

$$\text{or, } x^2 - 2x60^\circ + 1 = 0$$

$$\therefore x = \frac{260^\circ \pm \sqrt{460^2 - 4}}{2} \\ = 60^\circ \pm i\sin 60^\circ$$

Take +ve sign only, $x = 60^\circ + i\sin 60^\circ$

$$\text{Now LHS} = x^n + \frac{1}{x^n}$$

$$= (60^\circ + i\sin 60^\circ)^n + (60^\circ + i\sin 60^\circ)^{-n}$$

$$= \cos n\theta + i\sin n\theta + 60^\circ(-n)\theta + i\sin(-n)\theta \quad [\text{By De Moivre's theory}]$$

$$= \cos n\theta + i\sin n\theta + 60^\circ n\theta - i\sin n\theta$$

$$= \text{RHS}$$

9. If $x = \cos\theta + i\sin\theta$ and $1 + \sqrt{1-a^2} = na$, prove that

$$1 + a\cos\theta = \frac{a}{2n}(1+nx)(1+\frac{n}{a})$$

Solution: Given, $x = \cos\theta + i\sin\theta$

$$\therefore \frac{1}{x} = \frac{1}{\cos\theta + i\sin\theta}$$

$$= (\cos\theta + i\sin\theta)^{-1}$$

$$= \cos\theta - i\sin\theta$$

$$\therefore x + \frac{1}{x} = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta$$

$$= 2\cos\theta$$

Also, $1 + \sqrt{1-a^2} = na$

or, $\sqrt{1-a^2} = na - 1$

or, $1-a^2 = n^2a^2 - 2na + 1$

or, $n^2a^2 + a^2 = 2na$

or, $a^2(1+n^2) = 2na$

or, $\frac{a^2(1+n^2)}{2na} = 1$

$$\therefore \frac{a(1+n^2)}{2n} = 1$$

Now, L.H.S. = $1 + a\cos\theta$

$$= \frac{a(1+n^2)}{2n} + a \cdot \frac{x + \frac{1}{x}}{2}$$

$$= \frac{a}{2n} \left\{ (1+n^2) + n(x + \frac{1}{x}) \right\}$$

$$= \frac{a}{2n} \left(1 + n^2 + nx + \frac{n}{x} \right)$$

$$= \frac{a}{2n} \left\{ 1(1+nx) + \frac{n}{x}(1+nx) \right\}$$

$$= \frac{a}{2n} (1+nx)(1+\frac{n}{x})$$

$$= R.H.S$$

Important results: (i) $1 = e^{i2n\pi} = \cos 2n\pi + i\sin 2n\pi$

$$(ii) -1 = \cos\pi + i\sin\pi = e^{i\pi}$$

$$(iii) i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\frac{\pi}{2}}$$

$$(iv) -i = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2} = e^{-i\frac{\pi}{2}}$$

10. If α, β be the roots of $x^2 - 2x + 4 = 0$, then
 that $\alpha^n + \beta^n = 2^{n+1} \cos \frac{n\pi}{3}$
Solution: Given, $x^2 - 2x + 4 = 0$

$$\therefore x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot 4}}{2 \cdot 1}$$

$$= \frac{2 \pm \sqrt{-12}}{2}$$

$$= 1 \pm \sqrt{-3}$$

$$= 1 \pm i\sqrt{3}$$

$$\therefore \alpha = 1 + i\sqrt{3}$$

$$= 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$

$$= 2\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)$$

$$\text{and } \beta = 1 - i\sqrt{3}$$

$$= 2\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$

$$= 2\left(\cos \frac{4\pi}{3} - i \sin \frac{4\pi}{3}\right)$$

$$\therefore \alpha^n = 2^n \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)^n$$

$$= 2^n \left(\cos \frac{n\pi}{3} + i \sin \frac{n\pi}{3}\right) \dots (1)$$

$$\text{and } \beta^n = 2^n \left(\cos \frac{n\pi}{3} - i \sin \frac{n\pi}{3}\right) \dots (2)$$

Adding (1) and (2) we get

$$\alpha^n + \beta^n = 2^n \cdot 2 \cos \frac{n\pi}{3}$$

$$= 2^{n+1} \cos \frac{n\pi}{3}$$

11. Show that $\sin(\ln i^i) = -1$

$$\text{Solution: } i = e^{\frac{i\pi}{2}} \cdot e^{i2n\pi}$$

$$\therefore i = e^{i2n\pi}$$

$$\therefore i^i = e^{-(4n+1)\frac{\pi}{2}}$$

$$\therefore \ln i^i = -(4n+1)\frac{\pi}{2}$$

$$\text{Now, } \sin(\ln i^i) = \sin\left\{-(4n+1)\frac{\pi}{2}\right\}$$

$$= -\sin\left(2n\pi + \frac{\pi}{2}\right)$$

$$= -\sin \frac{\pi}{2}$$

$$= -1$$

12. If $x = \cos\alpha + i\sin\alpha$, $y = \cos\beta + i\sin\beta$, $z = \cos\gamma + i\sin\gamma$ and $x+y+z=0$, then prove that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0.$$

Solution: We have, $x+y+z=0$

$$\text{or, } \cos\alpha + i\sin\alpha + \cos\beta + i\sin\beta + \cos\gamma + i\sin\gamma = 0$$

Equating real and imaginary parts, we get

$$\cos\alpha + \cos\beta + \cos\gamma = 0 \quad \dots \text{(1)}$$

$$\text{and } \sin\alpha + \sin\beta + \sin\gamma = 0 \quad \dots \text{(2)}$$

$$\text{Now } LHS = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

$$\begin{aligned} &= (\cos\alpha + i\sin\alpha)^{-1} + (\cos\beta + i\sin\beta)^{-1} + (\cos\gamma + i\sin\gamma)^{-1} \\ &= \cos\alpha - i\sin\alpha + \cos\beta - i\sin\beta + \cos\gamma - i\sin\gamma \\ &= (\cos\alpha + \cos\beta + \cos\gamma) - i(\sin\alpha + \sin\beta + \sin\gamma) \\ &= 0 - i \cdot 0 \\ &= 0 \end{aligned}$$

13. Using De Moivre's theorem, solve the equations:

$$(i) x^9 = 1 \quad (ii) (x+1)^5 + (x-1)^5 = 0 \quad (iii) x^4 + x^2 + 1 = 0$$

Solution: (i) $x^9 = 1$

$$\text{or, } x^9 = \cos 2n\pi + i\sin 2n\pi \quad [\because 1 = e^{i2n\pi} = \cos 2n\pi + i\sin 2n\pi]$$

$$\text{or, } x = \left(\cos 2n\pi + i\sin 2n\pi\right)^{\frac{1}{9}}$$

$$\text{or, } x = \cos \frac{2n\pi}{9} + i\sin \frac{2n\pi}{9}$$

Putting $n=0, 1, 2, 3, 4, 5, 6, 7, 8$, the required solutions are

$$\cos 0 + i\sin 0, \cos \frac{2\pi}{9} + i\sin \frac{2\pi}{9}, \cos \frac{4\pi}{9} + i\sin \frac{4\pi}{9},$$

$$\cos \frac{6\pi}{9} + i\sin \frac{6\pi}{9}, \cos \frac{8\pi}{9} + i\sin \frac{8\pi}{9}, \cos \frac{10\pi}{9} + i\sin \frac{10\pi}{9},$$

$$\cos \frac{12\pi}{9} + i\sin \frac{12\pi}{9}, \cos \frac{14\pi}{9} + i\sin \frac{14\pi}{9}, \cos \frac{16\pi}{9} + i\sin \frac{16\pi}{9}$$

$$(ii) (x+1)^5 + (x-1)^5 = 0$$

$$\text{or, } (x+1)^5 = - (x-1)^5$$

$$\text{or, } \left(\frac{x+1}{x-1}\right)^5 = -1$$

$$\text{or, } \frac{x+1}{x-1} = (-1)^{\frac{1}{5}}$$

$$\text{or, } \frac{x+1}{x-1} = \left(e^{in\pi} \cdot e^{i2n\pi}\right)^{\frac{1}{5}}$$

$$\text{or, } \frac{x+1}{x-1} = \left\{ e^{i(2n\pi+n)}\right\}^{\frac{1}{5}}$$

$$\text{or, } \frac{x+1}{x-1} = \left\{ \cos(2n\pi+n) + i\sin(2n\pi+n)\right\}^{\frac{1}{5}}$$

$$\text{or, } \frac{x+1}{x-1} = \cos\left(\frac{2n\pi+n}{5}\right) + i\sin\left(\frac{2n\pi+n}{5}\right)$$

$$\text{or, } \frac{x+1}{x-1} = \frac{\cos\theta + i\sin\theta}{1} \text{ where } \theta = \frac{2n\pi+n}{5}$$

Using componendo and dividendo, we get-

$$\frac{x+1+x-1}{x+1-x+1} = \frac{\cos\theta + i\sin\theta + 1}{\cos\theta + i\sin\theta - 1}$$

$$\text{or, } \frac{2x}{2} = - \frac{2\cos^2\theta_2 + i \cdot 2\sin\theta_2 \cos\theta_2}{2\sin^2\theta_2 - i \cdot 2\sin\theta_2 \cos\theta_2}$$

$$\text{or, } x = - \frac{2\cos\theta_2(\cos\theta_2 + i\sin\theta_2)}{2\sin\theta_2(\sin\theta_2 - i\cos\theta_2)}$$

$$\text{or, } x = -i\cot\theta_2 \left[\frac{\cos\theta_2 + i\sin\theta_2}{i\sin\theta_2 + \cos\theta_2} \right]$$

$$\text{or, } x = -i\cot\theta_2$$

$$\therefore x = -i\cot\left(\frac{(2n+1)\pi}{10}\right).$$

Putting $n=0, 1, 2, 3, 4$, the required solutions are

$$-i\cot\frac{\pi}{10}, -i\cot\frac{3\pi}{10}, -i\cot\frac{\pi}{2}, -i\cot\frac{7\pi}{10}, -i\cot\frac{9\pi}{10}.$$

(iii) Given, $x^4 + x^2 + 1 = 0$

or, $(x^2 - 1)(x^4 + x^2 + 1) = 0$ [Multiplying both sides by $(x^2 - 1)$]

or, $(x^2)^3 - (1)^3 = 0$

or, $x^6 - 1 = 0$ or, $x^6 = 1$

or, $x^6 = \cos 2n\pi + i \sin 2n\pi$ [$\because 1 = \cos 0 + i \sin 0$]

or, $x = (\cos 2n\pi + i \sin 2n\pi)^{\frac{1}{6}}$

or, $x = \cos \frac{2n\pi}{6} + i \sin \frac{2n\pi}{6}$

$$\therefore x = \cos \frac{n\pi}{3} + i \sin \frac{n\pi}{3} \dots (1)$$

Putting $n = 0, 1, 2, 3, 4, 5$ in (1), we get

$$x = 1, \cos \frac{\pi}{3} + i \sin \frac{\pi}{3}, \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}, -1,$$

$$\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}, \cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3}$$

Among these values $x = \pm 1$ will be omitted as we have multiplied the equation by $x^2 - 1$. Hence the four roots of the given equation are

$$x = \cos \frac{\pi}{3} + i \sin \frac{\pi}{3}, \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}, \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3},$$

$$\cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3}$$

14. find all the values of (i) $(1+i)^{\frac{1}{5}}$ (ii) $(-i)^{\frac{1}{6}}$

Solution: let us put $1 = r \cos \theta$ and $i = r \sin \theta$

$$\therefore r = \sqrt{2} \text{ and } \theta = \frac{\pi}{4}$$

Now, $(1+i)^{\frac{1}{5}} = (1+i \cdot 1)^{\frac{1}{5}}$

$$= (r \cos \theta + i r \sin \theta)^{\frac{1}{5}}$$

$$= \left\{ r(\cos \theta + i \sin \theta) \right\}^{\frac{1}{5}}$$

$$= \left\{ r \cdot e^{i\theta}, e^{i2\pi n} \right\}^{\frac{1}{5}}$$

$$\begin{aligned}
 (1+i)^{\frac{1}{5}} &= \left\{ r e^{i(2n\pi + \theta)} \right\}^{\frac{1}{5}} \\
 &= \left\{ \sqrt{2} e^{i(2n\pi + \frac{\pi}{4})} \right\}^{\frac{1}{5}} \\
 &= 2^{\frac{1}{10}} \left\{ \cos(2n\pi + \frac{\pi}{4}) + i \sin(2n\pi + \frac{\pi}{4}) \right\}^{\frac{1}{5}}
 \end{aligned}$$

Putting $n=0, 1, 2, 3, 4$, the required values are

$$\begin{aligned}
 &2^{\frac{1}{10}} \left(\cos \frac{\pi}{20} + i \sin \frac{\pi}{20} \right), 2^{\frac{1}{10}} \left(\cos \frac{9\pi}{20} + i \sin \frac{9\pi}{20} \right), \\
 &2^{\frac{1}{10}} \left(\cos \frac{17\pi}{20} + i \sin \frac{17\pi}{20} \right), 2^{\frac{1}{10}} \left(\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4} \right), \\
 &2^{\frac{1}{10}} \left(\cos \frac{33\pi}{20} + i \sin \frac{33\pi}{20} \right)
 \end{aligned}$$

$$\begin{aligned}
 (\text{ii}) \quad (-i)^{\frac{1}{6}} &= \left(\cos \frac{\pi}{2} - i \sin \frac{\pi}{2} \right)^{\frac{1}{6}} \\
 &= \left(e^{-i\frac{\pi}{2}} \right)^{\frac{1}{6}} \\
 &= \left(e^{-i\frac{\pi}{2}} \cdot e^{i2n\pi} \right)^{\frac{1}{6}} \quad [\because 1 = e^{i2n\pi}] \\
 &= \left\{ e^{i(4n-1)\frac{\pi}{2}} \right\}^{\frac{1}{6}} \\
 &= \left\{ \cos(4n-1)\frac{\pi}{2} + i \sin(4n-1)\frac{\pi}{2} \right\}^{\frac{1}{6}} \\
 &= \cos(4n-1)\frac{\pi}{12} + i \sin(4n-1)\frac{\pi}{12}
 \end{aligned}$$

Putting $n=0, 1, 2, 3, 4, 5$, the required values are

$$\begin{aligned}
 &\cos \frac{\pi}{12} - i \sin \frac{\pi}{12}, \cos \frac{\pi}{4} + i \sin \frac{\pi}{4}, \cos \frac{7\pi}{12} + i \sin \frac{7\pi}{12}, \\
 &\cos \frac{11\pi}{12} + i \sin \frac{11\pi}{12}, \cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4}, \cos \frac{19\pi}{12} + i \sin \frac{19\pi}{12}
 \end{aligned}$$

Analytic functions-1

Complex variable: A symbol, such as z , which can stand for any one of a set of complex numbers is called a complex variable.

If x and y are real variables, then $z=x+iy$ is called a complex variable.

Function: If to each value which a complex variable z may assume there corresponds one or more values of a complex variable w , then w is called a function of z , written $w=f(z)$.

Single-valued function: If for each value of z there corresponds only one value of w , then w is called a single-valued function of z or that $f(z)$ is single-valued.

Multiple-valued function: If for each value of z there corresponds more than one value of w , then w is called a multiple-valued or many-valued function of z or that $f(z)$ is multiple-valued.

Derivative: If $f(z)$ is a single-valued function defined in a region R of the z -plane, the derivative of $f(z)$ is defined as

$$f'(z) = \frac{df(z)}{dz} = \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

provided the limit exists.

1. Define an analytic function. Find the necessary conditions for a function $f(z) = u + iv$ to be analytic in a region R .

Solution:

Analytic function: A single-valued function $f(z)$ which is differentiable at every point of a region R , is called as an analytic function of z in R .

An analytic function can also be called as regular function or holomorphic function.

Necessary Conditions for a function $f(z) = u + iv$ to be analytic:

Let $f(z) = u(x, y) + iv(x, y)$ be an analytic function in a region R .

$\therefore f(z)$ is differentiable in R .

$$\Rightarrow f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \text{ exists in } R.$$

$$\text{Let } z = x + iy$$

$$\therefore \Delta z = \Delta x + i \Delta y$$

$$\therefore z + \Delta z = (x + iy) + (\Delta x + i \Delta y)$$

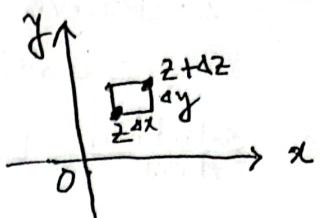
$$\therefore z + \Delta z = (x + \Delta x) + i(y + \Delta y)$$

$$\therefore f(z + \Delta z) = u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y)$$

If $\Delta z \rightarrow 0$ or, $\Delta x + i \Delta y \rightarrow 0$, then $\Delta x \rightarrow 0$, $\Delta y \rightarrow 0$.

$$\therefore f'(z) = \lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0}} \frac{[u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y)] - [u(x, y) + iv(x, y)]}{\Delta x + i \Delta y}$$

$$= \lim_{\substack{\Delta x \rightarrow 0 \\ \Delta y \rightarrow 0}} \frac{[u(x + \Delta x, y + \Delta y) - u(x, y)] + i[v(x + \Delta x, y + \Delta y) - v(x, y)]}{\Delta x + i \Delta y}$$



Along x -axis, $\Delta y = 0$, then the limit is

$$\begin{aligned}
 f'(z) &= \lim_{\Delta x \rightarrow 0} \frac{[u(x+\Delta x, y) - u(x, y)] + i[v(x+\Delta x, y) - v(x, y)]}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{u(x+\Delta x, y) - u(x, y)}{\Delta x} + i \lim_{\Delta x \rightarrow 0} \frac{v(x+\Delta x, y) - v(x, y)}{\Delta x} \\
 &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \quad \dots \dots \text{(1)} \\
 &\quad (\text{By definition})
 \end{aligned}$$

Along y -axis, $\Delta x = 0$, then the limit is

$$\begin{aligned}
 f'(z) &= \lim_{\Delta y \rightarrow 0} \frac{[u(x, y+\Delta y) - u(x, y)] + i[v(x, y+\Delta y) - v(x, y)]}{i \Delta y} \\
 &= \lim_{i \Delta y \rightarrow 0} \frac{u(x, y+\Delta y) - u(x, y)}{i \Delta y} + \lim_{i \Delta y \rightarrow 0} \frac{v(x, y+\Delta y) - v(x, y)}{i \Delta y} \\
 &= \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \\
 &= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad \dots \dots \text{(2)} \\
 &\quad (\text{By definition})
 \end{aligned}$$

since $f'(z)$ exists, so (1) and (2) must be equal.

$$\text{i.e. } \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Comparing real and imaginary parts, we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \text{or} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\text{i.e. } u_x = v_y \quad \text{and} \quad u_y = -v_x$$

These two equations are called Cauchy-Riemann (C-R) equations.

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2. Derive the polar form of Cauchy-Riemann (C-R) equations.

Solution: Let $z = r e^{i\theta}$ and $f(z) = u(r, \theta) + i v(r, \theta)$

[By using polar form]

$$\text{Then } u(r, \theta) + i v(r, \theta) = f(re^{i\theta}) \dots (1)$$

Differentiating (1) partially wrt r , we get

$$\frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r} = f'(re^{i\theta}) \cdot e^{i\theta} \dots (2)$$

Differentiating (1) partially wrt θ , we get

$$\frac{\partial u}{\partial \theta} + i \frac{\partial v}{\partial \theta} = f'(re^{i\theta}) \cdot i re^{i\theta} \dots (3)$$

Now from (2) and (3), we get

$$\frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r} = \frac{1}{ir} \left(\frac{\partial u}{\partial \theta} + i \frac{\partial v}{\partial \theta} \right)$$

$$\text{or, } \frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r} = \frac{1}{ir} \frac{\partial u}{\partial \theta} + \frac{1}{r} \frac{\partial v}{\partial \theta}$$

$$\text{or, } \frac{\partial u}{\partial r} + i \frac{\partial v}{\partial r} = -\frac{i}{r} \frac{\partial u}{\partial \theta} + \frac{1}{r} \frac{\partial v}{\partial \theta}$$

Equating the real and imaginary parts, we get

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta} \text{ and } \frac{\partial v}{\partial r} = -\frac{i}{r} \frac{\partial u}{\partial \theta}$$

The above equations are called Cauchy-Riemann (C-R) equations in Polar form.

Laplace's equation: An equation of the form

$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ or $\nabla^2 \phi = 0$, is called a Laplace's equation (in two dimension).

Harmonic function: Any function having continuous second order partial derivatives which satisfies the Laplace's equation is called harmonic function.

Conjugate harmonic functions: Any two harmonic functions u and v such that $f(z) = u+iv$ is analytic, are called conjugate harmonic functions.

3. Examine whether the following functions are analytic or not:

- (i) $e^x (\cos y - i \sin y)$
- (ii) $\frac{1}{z}$ ($z \neq 0$)
- (iii) \bar{z}
- (iv) $2xy + i(x^2 - y^2)$

Solution: (i) let $f(z) = e^x (\cos y - i \sin y)$

$$\text{or, } u+iv = e^x \cos y - i e^x \sin y$$

Equating real and imaginary parts, we get—

$$u = e^x \cos y, v = -e^x \sin y$$

$$\therefore \frac{\partial u}{\partial x} = -e^x \cos y, \quad \frac{\partial u}{\partial y} = -e^x \sin y$$

$$\frac{\partial v}{\partial x} = -e^x \sin y, \quad \frac{\partial v}{\partial y} = -e^x \cos y$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

i.e. C-R equations are satisfied

$\therefore f(z) = e^x (\cos y - i \sin y)$ is analytic.

(ii) let $f(z) = \frac{1}{z}$ ($z \neq 0$)

$$= \frac{1}{x+iy}$$

$$= \frac{x-iy}{x^2+y^2}$$

or, $u+iv = \frac{x}{x^2+y^2} - i \frac{y}{x^2+y^2}$

Equating real and imaginary parts, we get—

$$u = \frac{x}{x^2+y^2}, v = -\frac{y}{x^2+y^2}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{(x^2+y^2) \cdot 1 - x \cdot 2x}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(x^2+y^2) \cdot 0 - x \cdot 2y}{(x^2+y^2)^2}$$

$$\therefore \text{Im } f'(z) \neq 0 \Rightarrow u = -\frac{-2xy}{(x^2+y^2)^2} \quad (i)$$

$$\frac{\partial v}{\partial x} = -\frac{(x^2+y^2) \cdot 0 - y \cdot 2x}{(x^2+y^2)^2}$$

$$= \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial y} = -\frac{(x^2+y^2) \cdot 1 - y \cdot 2y}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

i.e., C-R equations are satisfied

$\therefore f(z) = \frac{1}{z}$ is analytic except at $z=0$.

(iii) Let $f(z) = \bar{z}$

or, $u+iv = x-iy$ [$\because z=x+iy \therefore \bar{z}=x-iy$]

$$\therefore u = x, v = -y$$

$$\therefore \frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial y} = 0 \quad | \quad \frac{\partial v}{\partial x} = 0, \frac{\partial v}{\partial y} = -1$$

$$\text{since } \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$$

\Rightarrow C-R equations are not satisfied.

$\therefore f(z) = \bar{z}$ is not analytic.

(iv) let $f(z) = 2xy + i(x^2 - y^2)$

$$\text{or, } u + iv = 2xy + i(x^2 - y^2)$$

$$\therefore u = 2xy, v = x^2 - y^2$$

$$\therefore \frac{\partial u}{\partial x} = 2y, \frac{\partial u}{\partial y} = 2x$$

$$\frac{\partial v}{\partial x} = 2x, \frac{\partial v}{\partial y} = -2y$$

$$\text{since } \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$$

\Rightarrow C-R equations are not satisfied.

$\therefore f(z) = 2xy + i(x^2 - y^2)$ is not analytic.

Construction of Conjugate harmonic functions/ an analytic function whose real or imaginary part is given.

Method-1: Suppose u is given.

Then $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ are known.

By total differentiation,

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$= -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \quad [\because \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}]$$

$$\text{integrating it, } v = \int \left(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \right) + c_1$$

since v is known we can construct $f(z) = u + iv$.

similarly, if v is given we can find $\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$

$$\text{Then } du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$= \frac{\partial v}{\partial y} dx - \frac{\partial v}{\partial x} dy$$

$$\Rightarrow u = \int \left(\frac{\partial v}{\partial y} dx - \frac{\partial v}{\partial x} dy \right) + c_2$$

$\therefore f(z) = u + iv$

Method-2: Milne's method / Milne Thomson method

Let $f(z) = u + iv$ is to be constructed

(i) Suppose the real part 'u' is given.

Then $\frac{\partial u}{\partial x} (= u_1(x,y))$ and $\frac{\partial u}{\partial y} (= u_2(x,y))$ are known.

Then by Milne's method we have,

$$f'(z) = u_1(z,0) - iu_2(z,0)$$

Integrating it wrt z, we get

$$f(z) = \int [u_1(z,0) - iu_2(z,0)] dz + c_1$$

(ii) Similarly suppose the imaginary part 'v' is given.

Then $\frac{\partial v}{\partial y} (= v_1(x,y))$ and $\frac{\partial v}{\partial x} (= v_2(x,y))$ are known.

Then by Milne's method we have,

$$f'(z) = v_1(z,0) + iv_2(z,0)$$

Integrating it wrt z, we get

$$f(z) = \int [v_1(z,0) + iv_2(z,0)] dz + c_2$$

which is the analytic function.

4. Show that $u = 3x^2y + 2x - y^3 - 2y^2$ is harmonic function. Also find its conjugate harmonic.

Solution: Given, $u = 3x^2y + 2x - y^3 - 2y^2$

$$\therefore \frac{\partial u}{\partial x} = 6xy + 4x, \frac{\partial^2 u}{\partial x^2} = 6y + 4 \quad \text{... (1)}$$

$$\frac{\partial u}{\partial y} = 3x^2 - 3y^2 - 4y, \frac{\partial^2 u}{\partial y^2} = -6y - 4 \quad \text{... (2)}$$

Adding (1) and (2), we get—

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\text{or, } \nabla^2 u = 0$$

i.e. u satisfies Laplace's equation.

So, u is a harmonic function.

Similarly let v be the conjugate harmonic function.

Then we have, $dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$

$\therefore (1) - (2)$ (that is, equating $\frac{\partial v}{\partial x}$ to $\frac{\partial u}{\partial y}$) $\Rightarrow \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$

$$\therefore \text{L.H.S. of (1) - (2)} = -(3x^2 - 3y^2 - 4y)dx + (6xy + 4x)dy$$

$$(1) - (2) \Rightarrow \frac{\partial v}{\partial x} = d(3xy^2 + 4xy - x^3)$$

Integrating it, we get—

$$\therefore v = 3xy^2 + 4xy - x^3 + C$$

5. In a two dimensional flow, the stream function is $\psi = \tan^{-1}(y/x)$. Find the velocity potential ϕ .

Solution: Given that $\psi = \tan^{-1}(y/x)$

$$\therefore \frac{\partial \psi}{\partial x} = \frac{1}{1+(y/x)^2} \cdot y \cdot (-\frac{1}{x^2})$$

$$= \frac{-y}{x^2+y^2}$$

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$$\begin{aligned}\frac{\partial \Psi}{\partial y} &= \frac{1}{1+y^2} \cdot \frac{1}{x} \\ &= \frac{x}{x^2+y^2}\end{aligned}$$

Then we have,

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy \quad [\because w = \phi + i\Psi]$$

$$= \frac{\partial \Psi}{\partial y} dx - \frac{\partial \Psi}{\partial x} dy \quad \therefore \frac{\partial \phi}{\partial x} = \frac{\partial \Psi}{\partial y}, \frac{\partial \phi}{\partial y} = -\frac{\partial \Psi}{\partial x}$$

$$= \frac{x}{x^2+y^2} dx + \frac{y}{x^2+y^2} dy$$

integrating it, we get

$$\phi = \frac{1}{2} \ln(x^2+y^2) + \frac{1}{2} \ln(x^2+y^2) + C$$

$\phi = \ln(x^2+y^2) + C$ is the required velocity potential.

b. (a) Prove that the function $u = 2x(1-y)$ is harmonic.

- (b) Find a function v such that $f(z) = u+iv$ is analytic [ie. find the conjugate function of u] (c) Express $f(z)$ in terms of z .

Solution: (a) Given that $u = 2x(1-y) \dots (1)$

$$\frac{\partial u}{\partial x} = 2 - 2y$$

$$\frac{\partial u}{\partial y} = u_1(x, y), \text{ say}$$

$$\frac{\partial u}{\partial x^2} = 0 \quad \dots (2)$$

$$\frac{\partial u}{\partial y} = -2x$$

$$= u_2(x, y), \text{ say}$$

$$\frac{\partial u}{\partial y^2} = 0 \quad \dots (3)$$

Adding (2) and (3) we get

$$\frac{\partial u}{\partial x^2} + \frac{\partial u}{\partial y^2} = 0$$

Since u satisfies Laplace's equation, so u is a harmonic function.

(c) By Milne's method we have

$$\begin{aligned} f'(z) &= u_1(z, 0) - iu_2(z, 0) \\ &= 2 - i \cdot (-2z) \\ &= 2 + 2z \end{aligned}$$

Integrating it, we get—

$$f(z) = 2z + i \cdot \frac{z^2}{2} + c, \text{ where } c$$

is the complex const.

$\therefore f(z) = 2z + i \cdot \frac{z^2}{2} + c$ is the required function.

(b) From (c) we have,

$$\begin{aligned} f(z) &= 2z + i \cdot \frac{z^2}{2} + c \\ &= 2(x+iy) + i(x+iy)^2 + 4+iC_2 \\ \text{or, } u+iv &= 2(x+iy) + i(x^2 + i2xy - y^2) + 4+iC_2 \\ &= 2x - 2xy + i(2y + x^2 - y^2) + 4+iC_2 \end{aligned}$$

Equating the imaginary parts, we get

$v = 2y + x^2 - y^2 + C_2$ is the required conjugate harmonic.

7. Show that the function $f(z) = u+iv$, where

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, \quad z \neq 0$$

$$= 0, \quad z=0$$

satisfies the Cauchy-Riemann conditions at $z=0$. Is the function analytic at $z=0$? Justify your answer.

Solution: Given, $f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$

$$\text{or, } u+iv = \frac{x^3-y^3+i(x^3+y^3)}{x^2+y^2}$$

$$\therefore u(x, y) = \frac{x^3-y^3}{x^2+y^2}, \quad v(x, y) = \frac{x^3+y^3}{x^2+y^2}$$

Also, $u(0,0) = 0, v(0,0) = 0$ [$A + B = 0, f(0) = 0$]

At the origin ($i.e.$ at $z=0$)

$$\begin{aligned}\frac{\partial u}{\partial x} &= \lim_{h \rightarrow 0} \frac{u(0+h,0) - u(0,0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{h^3/h^2}{h} \\ &= 1\end{aligned}$$

$$\begin{aligned}\frac{\partial u}{\partial y} &= \lim_{k \rightarrow 0} \frac{u(0,0+k) - u(0,0)}{k} \\ &= \lim_{k \rightarrow 0} \frac{-k^3/k^2}{k} \\ &= -1\end{aligned}$$

$$\begin{aligned}\frac{\partial v}{\partial x} &= \lim_{h \rightarrow 0} \frac{v(0+h,0) - v(0,0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{h^3/h^2}{h} \\ &= 1\end{aligned} \quad \left| \begin{array}{l} \frac{\partial v}{\partial y} = \lim_{k \rightarrow 0} \frac{v(0,0+k) - v(0,0)}{k} \\ = \lim_{k \rightarrow 0} \frac{k^3/k^2}{k} \\ = 1 \end{array} \right.$$

$$\text{Also, } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

i.e., Cauchy-Riemann Conditions are satisfied

at $z=0$.

$$\begin{aligned}\text{Now, } f'(0) &= \lim_{z \rightarrow 0} \frac{f(z) - f(0)}{z} \\ &= \lim_{z \rightarrow 0} \left[\frac{x^3 - y^3 + i(x^3 + y^3)}{x^2 + y^2} \cdot \frac{1}{x+iy} \right]\end{aligned}$$

Let $z \rightarrow 0$ along $y=x$, then

$$f'(0) = \lim_{x \rightarrow 0} \left[\frac{x^3 - x^3 + i(x^3 + x^3)}{x^2 + x^2} \cdot \frac{1}{x+ix} \right]$$

$$= \frac{2i}{2(1+i)} = \frac{i(1-i)}{1-i^2} = \frac{1}{2}(1+i)$$

Again let $z \rightarrow 0$ along $y=0$, then

$$f'(0) = \lim_{x \rightarrow 0} \frac{x^3 + ix^3}{x^2} \cdot \frac{1}{x} = 1+i$$

so we see that $f'(0)$ is not unique. Hence the function $f(z)$ is not analytic at $z=0$.

Analytic function-2

The complex potential function

The analytic function $w = \varphi(x,y) + i\psi(x,y)$ is referred to as the complex potential function. Its real part $\varphi(x,y)$ represents the velocity potential function and the imaginary part $\psi(x,y)$ represents the stream function.

Problem-1(a). If $w = \varphi + i\psi$ represents the complex potential for an electric field and $\psi = 3xy - y^3$, find the potential function φ .

Solution: Given $\psi = 3xy - y^3$

$$\begin{aligned}\frac{\partial \psi}{\partial y} &= 3x^2 - 3y^2 \\ &= \psi_1(x,y), \text{ say}\end{aligned}$$

$$\begin{aligned}\text{and } \frac{\partial \psi}{\partial x} &= 6xy \\ &= \psi_2(x,y), \text{ say}\end{aligned}$$

By Milne's method we have,

$$\begin{aligned}w'(z) &= \psi_1(z,0) + i\psi_2(z,0) \\ &= 3z^2 + i \cdot 0 \\ &= 3z^2\end{aligned}$$

Integrating w.r.t. z, we get

$$w(z) = z^3 + C$$

$$\text{or } \varphi + i\psi = (x+iy)^3 + C_1 + iC_2$$

$$\text{or, } \varphi + i\psi = x^3 + i3x^2y - 3xy^2 - iy^3 + C_1 + iC_2$$

$\therefore \varphi = x^3 - 3xy^2 + C_1$ is the required potential function.

Problem-1(i). If $\psi = \phi + i\psi$ represents the complex potential for an electric field and $\psi = x^2 - y^2 + \frac{2xy}{x^2+y^2}$, find ϕ . Ans. $\phi = -2xy + \frac{y}{x^2+y^2} + c_1$

Problem-1(ii). An incompressible fluid flowing over the xy -plane has the velocity potential

$$\phi = x^2 - y^2 + \frac{x}{x^2+y^2}$$

Examine if this is possible and find a stream function ψ .

Solution: Given, $\phi = x^2 - y^2 + \frac{x}{x^2+y^2}$

$$\therefore \frac{\partial \phi}{\partial x} = 2x + \frac{(x^2+y^2) \cdot 1 - x \cdot 2x}{(x^2+y^2)^2} = 2x + \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 \phi}{\partial x^2} = 2 + \frac{(x^2+y^2)^2 \cdot (-2x) - (y^2-x^2) \cdot 2(x^2+y^2) \cdot 2x}{(x^2+y^2)^4} = \phi_1(x, y), \text{ say}$$

$$= 2 + \frac{2x^3 - 6xy^2}{(x^2+y^2)^3}$$

$$\frac{\partial \phi}{\partial y} = -2y + \frac{(x^2+y^2)(0) - x(2y)}{(x^2+y^2)^2} = -2y - \frac{2xy}{(x^2+y^2)^2} = \phi_2(x, y), \text{ say}$$

$$\frac{\partial^2 \phi}{\partial y^2} = -2 + \frac{(x^2+y^2)^2(-2x) - (-2xy) \cdot 2(x^2+y^2) \cdot 2y}{(x^2+y^2)^4}$$

$$= -2 + \frac{-2x^3 + 6xy^2}{(x^2+y^2)^3}$$

$$\therefore \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \text{i.e., } \phi \text{ is harmonic.}$$

Hence it can be a possible form of the velocity potential function.

By Milne's method, we have

$$f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$$

$$= 2z - \frac{1}{z^2} - i \cdot 0$$

Integrating it, we get $f(z) = z^2 + \frac{1}{z} + C$

$$\text{or, } \phi + i\psi = (x+iy)^2 + \frac{1}{x+iy} + 4+iC_2$$

$$= x^2 - y^2 + 2ixy + \frac{x-iy}{x^2+y^2} + 4+iC_2$$

$\therefore \psi = 2xy - \frac{y}{x^2+y^2} + C_2$ is the required.

Problem-1(b). Prove that the real and imaginary parts of an analytic function $f(z) = u(x, y) + iv(x, y)$ satisfies the Laplace's equation.

Solution: Since $f(z) = u(x, y) + iv(x, y)$ is an analytic function, so we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \dots \text{(1)} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \dots \text{(2)}$$

Differentiate (1) partially wrt x , we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad \dots \text{(3)}$$

Differentiate (2) partially wrt y , we get

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \quad \dots \text{(4)}$$

Adding (3) and (4) we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{or, } \nabla^2 u = 0$$

$$\text{similarly } \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \quad \text{or, } \nabla^2 v = 0$$

i.e. u and v satisfy their Laplace's equations.

[Both u and v are harmonic functions]

problem-2. Show that an analytic function with constant real part is constant.

Solution: Let $f(z) = u + iv$ be an analytic function.

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \dots \text{(1)} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \dots \text{(2)}$$

Given that $u = \text{constant} = c_1$, say

$$\therefore \frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial y} = 0$$

so from (1) and (2) we get, $\frac{\partial v}{\partial x} = 0$ and $\frac{\partial v}{\partial y} = 0$

i.e. v is independent of x and y

$$\Rightarrow v = \text{constant} = c_2, \text{ say}$$

$$\therefore f(z) = u + iv = c_1 + ic_2 \text{ is a constant}$$

Problem-3(a). Show that an analytic function with constant imaginary part is constant.

Solution: Let $f(z) = u + iv$ be an analytic function.

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{--- (1)}$$

Given that $v = \text{constant} = c_1$, say

$$\therefore \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = 0$$

$$\Rightarrow \frac{\partial u}{\partial y} = 0, \quad \frac{\partial u}{\partial x} = 0 \quad [\text{by (1)}]$$

$\Rightarrow u$ is independent of x and y

$\Rightarrow u = \text{constant} = c_2$, say

$$\therefore f(z) = u + iv = c_2 + ic_1 = \text{constant}$$

Problem-3(b). Determine the analytic function whose real part is $x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$.

Solution: Given that $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$$\therefore \frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x$$

$$= u_1(x, y), \text{ say}$$

$$\frac{\partial u}{\partial y} = -6xy - 6y$$

$$= u_2(x, y), \text{ say}$$

By Milne's method, we have

$$\begin{aligned} f'(z) &= u_1(z, 0) - iu_2(z, 0) \\ &= 3z^2 + 6z - i \cdot 0 \end{aligned}$$

Integrating it, we get $f(z) = 3 \cdot \frac{z^3}{3} + 6 \cdot \frac{z^2}{2} + C$

$$= z^3 + 3z^2 + C, \text{ where}$$

C is the complex constant.

Problem-4: Show that an analytic function with constant absolute value/modulus is constant.

Solution: Let an analytic function be $f(z) = u + iv$

$$\therefore |f(z)| = \sqrt{u^2 + v^2}$$

But we are given, $|f(z)| = \text{constant} = k$, say

$$\therefore u^2 + v^2 = k^2$$

By differentiation, $uu_x + vv_x = 0$, $uv_y + vuy = 0$

Now we use $v_x = -u_y$ in the first equation and $v_y = u_x$ in the second, we get

$$uu_x - vuy = 0 \quad \text{--- (1)}$$

$$vuy + vu_x = 0 \quad \text{--- (2)}$$

Multiplying (1) by u and (2) by v , then adding and also multiplying (1) by $-v$ and (2) by u , then adding we get

$$(u^2 + v^2)u_x = 0, \quad (u^2 + v^2)v_y = 0$$

If $k^2 = u^2 + v^2 = 0$, then $u = 0 = v$, hence $f = 0$.

If $k \neq 0$, then $u_x = u_y = 0$, hence by Cauchy-Riemann equations, also $v_x = v_y = 0$.

Together, $u = \text{constant}$ and $v = \text{constant}$, hence $f = \text{constant}$.

✓ Problem-5: Test whether the function $f(z) = z^3 + z$ is analytic or not.

Solution: We have, $f(z) = z^3 + z$

$$= (x+iy)^3 + (x+iy)$$

$$= x^3 + i3x^2y - 3xy^2 - iy^3 + x + iy$$

$$\text{or, } u+iv = (x^3 - 3xy^2 + x) + i(3x^2y - y^3 + y)$$

Equating the real and imaginary parts, we get

$$u = x^3 - 3xy^2 + x$$

$$v = 3x^2y - y^3 + y$$

$$\therefore \frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 1, \quad \frac{\partial u}{\partial y} = -6xy$$

$$\frac{\partial v}{\partial x} = 6xy, \quad \frac{\partial v}{\partial y} = 3x^2 - 3y^2 + 1$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

⇒ C-R equations are satisfied.

∴ $f(z) = z^3 + z$ is analytic.

✓ Problem-6: Find the constants a, b and c if $f(z) = x + ay + i(bx + cy)$ is analytic.

Solution: Given, $f(z) = x + ay + i(bx + cy)$

$$\text{or, } u+iv = x + ay + i(bx + cy)$$

$$\therefore u = x + ay, v = bx + cy$$

$$\therefore \frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = a$$

$$\frac{\partial v}{\partial x} = b, \quad \frac{\partial v}{\partial y} = c$$

Since $f(z)$ is analytic, so Cauchy-Riemann (C-R) equations are satisfied.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\therefore f(z) = c, \quad a = -b$$

$$\Rightarrow c = 1, \quad a = -b, \quad b \text{ may be any value.}$$

Problem-7: Determine b such that $u = e^{bx} \cos 5y$ is harmonic.

Solution: Given, $u = e^{bx} \cos 5y$

$$\textcircled{1} \quad \therefore \frac{\partial u}{\partial x} = b e^{bx} \cos 5y$$

$$\frac{\partial^2 u}{\partial x^2} = b^2 e^{bx} \cos 5y$$

$$\frac{\partial u}{\partial y} = e^{bx} (-5 \sin 5y)$$

$$\frac{\partial^2 u}{\partial y^2} = -5 e^{bx} \cdot 5 \cos 5y$$

$$\textcircled{2} \quad \therefore -25 e^{bx} \cos 5y$$

$\because u$ is harmonic function, so

$$0 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\text{or, } b^2 e^{bx} \cos 5y - 25 e^{bx} \cos 5y = 0$$

$$\text{and on further dividing, or, } e^{bx} \cos 5y (b^2 - 25) = 0$$

$$\text{or, } b^2 - 25 = 0 \quad [\because e^{bx} \cos 5y \neq 0]$$

$$\therefore b = \pm 5$$

Problem-8: (a) Prove that $u = e^{-x}(x \sin y - y \cos y)$ is harmonic. (b) Find v such that $f(z) = u + iv$ is analytic.

Solution: Given, $u = e^{-x}(x \sin y - y \cos y)$

$$\therefore \frac{\partial u}{\partial x} = e^{-x} \cdot \sin y + (-e^{-x})(x \cos y - y \sin y)$$

$$\frac{\partial^2 u}{\partial x^2} = e^{-x} (\sin y - x \cos y + y \sin y)$$

$$\frac{\partial^2 u}{\partial x^2} = -e^{-x} (\sin y - x \cos y + y \sin y) + e^{-x} (-\sin y) \\ = -e^{-x} (2 \sin y - x \cos y + y \sin y) \quad \text{--- (1)}$$

$$\frac{\partial u}{\partial y} = e^{-x} (x \cos y - 1 \cdot \cos y + y \sin y)$$

$$= u_2(x, y), \text{ say}$$

$$\frac{\partial^2 u}{\partial y^2} = e^{-x} (-x \sin y + \cos y + 1 \cdot \sin y + y \cos y) \\ = e^{-x} (-x \sin y + 2 \sin y + y \cos y) \quad \text{--- (2)}$$

Adding (1) and (2), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

since u satisfies Laplace's equation,
so u is harmonic function.

and part b) By Milne's method we have,

$$f(z) = u(z, 0) - i u_2(z, 0) \\ = 0 - i(z e^{-2} - e^{-2})$$

$$\text{Integrating, } f(z) = -iz \cdot (-e^{-2}) + i \int 1 \cdot (e^{-2}) dz + i \int e^{-2} dz \\ = ie^{-2} + C$$

$$\text{or, } u + iv = i(x+iy) e^{-(x+iy)} + C$$

$$= i(x+iy) e^{-x} \cdot e^{-iy} + C$$

$$= i(x+iy) \cdot e^{-x} (\cos y - i \sin y) + C$$

$$\text{Taking real part, we get } = ix e^{-x} \cos y + x e^{-x} \sin y - y e^{-x} \cos y + iy e^{-x} \sin y + C_1 + iC_2$$

$$= (x e^{-x} \sin y - y e^{-x} \cos y + C_1) + i(x e^{-x} \cos y + y e^{-x} \sin y + C_2)$$

Equating imaginary parts, we get—

$$v = e^{-x} (x \cos y + y \sin y) + C_2$$

Problem-9. In a two dimensional flow of a fluid, the velocity potential $\varphi = x^2 - y^2$. Find the stream function ψ .

Solution: Given that $\varphi = x^2 - y^2$

$$\therefore \frac{\partial \varphi}{\partial x} = 2x$$

$$\text{and } \frac{\partial \varphi}{\partial y} = -2y$$

$$\text{and } \frac{\partial \varphi}{\partial y} = -2y$$

$$= \varphi_1(x, y), \text{ say}$$

$$= \varphi_2(x, y), \text{ say}$$

By Milne's method we have,

$$\begin{aligned} w'(z) &= \varphi_1(z, 0) - i\varphi_2(z, 0) \\ &= 2z - i \cdot 0 \\ &= 2z \end{aligned}$$

Integrating it, we get $w(z) = z^2 + C$

$$\text{or, } W(z) = z^2 + C$$

$$\text{or, } \varphi + i\psi = (x+iy)^2 + 4+iC_2 \\ = x^2 - y^2 + 4 + i(2xy + C_2)$$

Equating imaginary parts, we get

$$\psi = 2xy + C_2 \text{ is the required stream function.}$$

Problem-10. Show that xy^2 cannot be real part of an analytic function.

Solution: Given $u = xy^2$

$$\therefore \frac{\partial u}{\partial x} = y^2, \frac{\partial^2 u}{\partial x^2} = 0 \quad \dots \textcircled{1}$$

$$\frac{\partial u}{\partial y} = 2xy, \frac{\partial^2 u}{\partial y^2} = 2x \quad \dots \textcircled{2}$$

Adding (1) and (2) we get—

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2x \neq 0$$

$\therefore u$ is not harmonic function.

i.e. u cannot be a real part of an analytic function.

Complex Integration

1. state and prove Cauchy's theorem/Cauchy's integral theorem.

Statement: If $f(z)$ is analytic inside and on a simple closed curve C , then $\oint_C f(z) dz = 0$.

Proof: Let $f(z) = u(x,y) + iv(x,y)$ be analytic.

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \dots \text{(1)}$$

Since $z = x+iy$, so $dz = dx+idy$

$$\begin{aligned} \text{Now } \oint_C f(z) dz &= \oint_C (u+iv)(dx+idy) \\ &= \oint_C (u dx + iu dy + iv dx - vd y) \\ &= \oint_C (u dx - vd y) + i \oint_C (v dx + u dy) \quad \dots \text{(1)} \end{aligned}$$

By Green's theorem we have,

$$\oint_C (u dx - vd y) = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx dy$$

$$\text{and } \oint_C (v dx + u dy) = \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx dy$$

where R is the region bounded by C .

Hence (1) becomes,

$$\begin{aligned} \oint_C f(z) dz &= \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx dy + i \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx dy \\ &= \iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y} \right) dx dy + i \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x} \right) dx dy \\ &= 0 + i \cdot 0 \\ &= 0 \end{aligned}$$

2. State and prove Cauchy's integral formula.

Statement: If $f(z)$ is analytic inside and on a simple closed curve C , and 'a' is any point within C , then $\oint_C \frac{f(z)dz}{z-a} = 2\pi i f(a)$

Proof: Since $f(z)$ is analytic inside and on C , $\frac{f(z)}{z-a}$ is also analytic inside and on C , except at the point $z=a$. Hence, we draw a small circle with centre at $z=a$ and radius r_c lying entirely inside C .



Now, $\frac{f(z)}{z-a}$ is analytic in the region enclosed between C and C_1 .

Hence, by Cauchy's extended theorem,

$$\oint_C \frac{f(z)dz}{z-a} = \oint_{C_1} \frac{f(z)dz}{z-a} \quad \dots \text{(1)}$$

On C_1 , any point z is given by $z=a+re^{i\theta}$

$$dz = ire^{i\theta} d\theta$$

where θ varies from 0 to 2π .

$$\begin{aligned} \therefore \oint_{C_1} \frac{f(z)dz}{z-a} &= \int_{\theta=0}^{2\pi} \frac{f(a+re^{i\theta}) \cdot ire^{i\theta} d\theta}{re^{i\theta}} \\ &= i \int_{\theta=0}^{2\pi} f(a+re^{i\theta}) d\theta \end{aligned}$$

As $r_c \rightarrow 0$, the circle tends to a point.

Taking limit $r_c \rightarrow 0$, we get

$$\begin{aligned} \oint_{C_1} \frac{f(z)dz}{z-a} &= i \int_{\theta=0}^{2\pi} f(a) d\theta \\ &= i f(a) [\theta]_{\theta=0}^{2\pi} \\ &= 2\pi i f(a) \end{aligned}$$

So from (1), we get $\oint_C \frac{f(z)dz}{z-a} = 2\pi i f(a)$

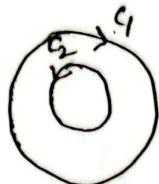
$$* \text{ In general, } \oint_C \frac{f(z) dz}{(z-a)^{n+1}} = \frac{2\pi i}{n!} f^{(n)}(a)$$

where $n=0, 1, 2, 3, \dots$

$$\text{and } f^{(0)}(a) = f(a)$$

* Cauchy's extended theorem: If $f(z)$ is analytic within and on the boundary of a region bounded by two closed curves C_1 and C_2 , then

$$\oint_{C_1} f(z) dz = \oint_{C_2} f(z) dz$$



3. Evaluate $\frac{1}{2\pi i} \oint_C \frac{e^z dz}{z-2}$ if C is (a) the circle $|z|=3$,

(b) the circle $|z|=1$.

Solution: (a) Here $f(z) = e^z$ is analytic inside and on the circle $|z|=3$ and $z=a=2$ is a point inside the given circle.

Then by using the Cauchy's integral formula, we have

$$\oint_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

$$\therefore \oint_C \frac{e^z}{z-2} dz = 2\pi i \cdot e^2$$

$$\text{or, } \frac{1}{2\pi i} \oint_C \frac{e^z}{z-2} dz = e^2$$

(b) Here $f(z) = \frac{e^z}{z-2}$ is analytic inside and on the circle $|z|=1$ and $z=2$ is a point outside the given circle.

Then by using the Cauchy's integral theorem, $\oint_C f(z) dz = 0$ we get

$$\oint_C \frac{e^z}{z-2} dz = 0 \text{ or, } \frac{1}{2\pi i} \oint_C \frac{e^z}{z-2} dz = 0$$

4. Evaluate $\oint_C \frac{\sin z^2}{z + \frac{\pi i}{2}} dz$ if C is the circle $|z|=5$.

Solution: Here $f(z) = \sin z^2$ is analytic inside and on the circle $|z|=5$ and $z=\frac{\pi i}{2}$ lies inside the given circle.

Then by using Cauchy's integral formula,

$$\oint_C \frac{f(z)}{z-a} dz = 2\pi i f(a) \text{ we get -}$$

$$\oint_C \frac{\sin z^2}{z - \left(-\frac{\pi i}{2}\right)} dz = 2\pi i f\left(-\frac{\pi i}{2}\right) \quad \left| \begin{array}{l} \therefore f\left(-\frac{\pi i}{2}\right) = \sin\left(-\frac{\pi^2}{4}\right) \\ = -(-1) \\ = 1 \end{array} \right.$$

$$\text{or, } \oint_C \frac{\sin z^2}{z + \frac{\pi i}{2}} dz = 2\pi i$$

5. Evaluate $\oint_C \frac{e^{z^2}}{z-\pi i} dz$ if C is (a) the circle $|z-1|=4$,
(b) the ellipse $|z-2|+|z+2|=6$.

Solution: (a) Here $f(z) = e^{z^2}$ is analytic inside and on the given circle $|z-1|=4$, and $z=\pi i$ is a point inside the given circle.

Then by using Cauchy's integral formula,

$$\oint_C \frac{f(z)}{z-a} dz = 2\pi i f(a) \text{ we get -}$$

$$\oint_C \frac{e^{z^2}}{z-\pi i} dz = 2\pi i f(\pi i) \\ = 2\pi i \cdot e^{3\pi i}$$

$$= 2\pi i (653\pi + i \sin 3\pi)$$

$$= 2\pi i (-1 + i \cdot 0)$$

$$= -2\pi i$$

(b) Here $f(z) = \frac{e^{z^2}}{z-\pi i}$ is analytic inside and on the ellipse C , and $z=\pi i$ lies outside the given ellipse C .

Then by using Cauchy's integral theorem,

$$\oint_C f(z) dz = 0 \text{ we get -}$$

$$\oint_C \frac{e^{z^2}}{z-\pi i} dz = 0$$

* * Locus of $|z-2|+|z+2|=6$ is $\frac{x^2}{3^2} + \frac{y^2}{(6)^2} = 1$.

Its foci $(\pm ae, 0) = (\pm 3 \cdot \frac{2}{3}, 0) = (\pm 2, 0)$, $a = \sqrt{1 - \frac{4}{9}} = \frac{2}{3}$
and length of major axis $= 2 \cdot 3 = 6$

6. Evaluate $\frac{1}{2\pi i} \oint_C \frac{\cos z}{z^2 - 1} dz$ around a rectangle with vertices at: (a) $2 \pm i, -2 \pm i$ (b) $-i, 2-i, 2+i, i$.

Solution: We have $\frac{1}{2\pi i} \oint_C \frac{\cos z}{z^2 - 1} dz = \frac{1}{4\pi i} \oint_C \left[\frac{\cos z}{z-1} - \frac{\cos z}{z+1} \right] dz$

$$= \frac{1}{4\pi i} \left[\oint_C \frac{\cos z}{z-1} dz - \oint_C \frac{\cos z}{z+1} dz \right] \quad \text{--- (1)}$$

(a) Here $f(z) = \cos z$ is analytic inside and on C , and also both points $z = \pm 1$ lie inside the rectangle $2 \pm i, -2 \pm i$.

Then by using Cauchy's integral formula, $\oint_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$, we get from (1)

$$\begin{aligned} \frac{1}{2\pi i} \oint_C \frac{\cos z}{z^2 - 1} dz &= \frac{1}{4\pi i} [2\pi i \cos(-\pi) - 2\pi i \cos(\pi)] \\ &= \frac{1}{4\pi i} [-2\pi i + 2\pi i] \\ &= 0 \end{aligned}$$

(b) Here only the point $z=1$ lies inside the rectangle $\pm i, 2 \pm i$.

Then by using the Cauchy's integral formula and also the Cauchy's integral theorem, we get from (1),

$$\begin{aligned} \frac{1}{2\pi i} \oint_C \frac{\cos z}{z^2 - 1} dz &= \frac{1}{4\pi i} [2\pi i \cos 0] \\ &= -\frac{1}{2} \end{aligned}$$

7. Show that $\frac{1}{2\pi i} \oint_C \frac{e^{zt}}{z^2 + 1} dz = \sin t$ if $t > 0$ and C is the circle $|z|=3$

Solution: We have $\frac{1}{2\pi i} \oint_C \frac{e^{zt}}{z^2 + 1} dz = \frac{1}{2\pi i} \oint_C \frac{e^{zt} dz}{(z+i)(z-i)}$

$$= \frac{1}{2\pi i} \cdot \frac{1}{2i} \left[\oint_C \frac{e^{zt}}{z-i} dz - \oint_C \frac{e^{zt}}{z+i} dz \right]$$

Here $f(z) = e^{zt}$ is analytic inside and on the given circle $|z|=3$ and $z=\pm i$ are inside C . --- (1)

Then by using Cauchy's integral formula, we get from (1)

$$\begin{aligned}
 \frac{1}{2\pi i} \oint_C \frac{e^{iz}}{z^2+1} dz &= \frac{1}{2\pi i} \cdot \frac{1}{2i} [2\pi i f(i) - 2\pi i f(-i)] \\
 &= \frac{1}{2i} [e^{it} - e^{-it}] \\
 &= \frac{1}{2i} \cdot 2i \sin t \quad [\because e^{it} - e^{-it} = 2i \sin t] \\
 &= \sin t
 \end{aligned}$$

8. Evaluate $\oint_C \frac{e^{iz}}{z^3} dz$ where C is the circle $|z|=2$.

Solution: Here $f(z) = e^{iz}$ is analytic inside and on the circle $|z|=2$ and $z=0$ is a point inside the given circle.

Then by using Cauchy's integral formula,

$$\oint_C \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(a), \text{ we get}$$

$$\begin{aligned}
 \oint_C \frac{f(z)}{(z-0)^3} dz &= \frac{2\pi i}{2!} f^{(2)}(0) & f(z) = e^{iz} \\
 \therefore \oint_C \frac{e^{iz}}{z^3} dz &= \frac{2\pi i}{2} \cdot (-1) & f'(z) = i e^{iz} \\
 &= -\pi i & f''(z) = -e^{iz} \\
 && f''(0) = -1 = f^{(2)}(0)
 \end{aligned}$$

9. Find the value of (a) $\oint_C \frac{\sin^6 z}{z - \frac{\pi i}{6}} dz$, (b) $\oint_C \frac{\sin^6 z dz}{(z - \frac{\pi i}{6})^3}$ if C is the circle $|z|=1$.

Solution: Here $f(z) = \sin^6 z$ is analytic inside and on the circle $|z|=1$ and $z=a=\frac{\pi i}{6}$ is a point inside the given circle.

(a) Then by using Cauchy's integral formula,

$$\oint_C \frac{f(z)}{z-a} dz = 2\pi i f(a) \text{ we get}$$

$$\begin{aligned}
 \oint_C \frac{\sin^6 z}{z - \frac{\pi i}{6}} dz &= 2\pi i \left(\sin \frac{\pi i}{6}\right)^6 \\
 &= 2\pi i \cdot \left(\frac{1}{2}\right)^6 \\
 &= 2\pi i \cdot \frac{1}{64} \\
 &= \frac{\pi i}{32}
 \end{aligned}$$

(b) Then by using Cauchy's integral formula,

$$\oint_C \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(a) \text{ we get}$$

$$\oint_C \frac{\sin^6 z}{(z-\frac{\pi}{6})^3} dz = \frac{2\pi i}{12!} f''(\frac{\pi}{6}) \dots \textcircled{1}$$

$$We have f(z) = \sin^6 z$$

$$\therefore f'(z) = 6 \sin^5 z \cdot \cos z$$

$$f''(z) = 30 \sin^4 z \cos^2 z + 6 \sin^5 z (-\sin z)$$

$$\begin{aligned} \therefore f''(\frac{\pi}{6}) &= f''(\frac{\pi}{6}) = 30 \cdot (\frac{1}{2})^4 \cdot (\frac{\sqrt{3}}{2})^2 - 6 \cdot (\frac{1}{2})^6 \\ &= 30 \cdot \frac{1}{16} \cdot \frac{3}{4} - 6 \cdot \frac{1}{64} \\ &= \frac{90-6}{64} \\ &= \frac{84}{64} \\ &= \frac{21}{16} \end{aligned}$$

So from (1) we get

$$\begin{aligned} \oint_C \frac{\sin^6 z}{(z-\frac{\pi}{6})^3} dz &= \frac{2\pi i}{2} \cdot \frac{21}{16} \\ &= \frac{21\pi i}{16} \end{aligned}$$

10. Evaluate $\frac{1}{2\pi i} \oint_C \frac{e^{zt}}{(z^2+1)^2} dz$ if $t > 0$ and C is the circle $|z| = 3$.

Solution: We have, $\frac{1}{(z^2+1)^2} = \frac{1}{(z+i)^2(z-i)^2}$

$$= \frac{1}{4iz} \left[\frac{1}{(z-i)^2} - \frac{1}{(z+i)^2} \right]$$

$$\therefore \oint_C \frac{e^{zt}}{(z^2+1)^2} dz = \frac{1}{4i} \left[\oint_C \frac{\frac{e^{zt}}{z-i} dz}{(z-i)^2} - \oint_C \frac{\frac{e^{zt}}{z+i} dz}{(z+i)^2} \right]$$

Hence $f(z) = \frac{e^{zt}}{z^2}$ is analytic inside on the given circle $|z|=3$ and $z=\pm i$ are inside C .

Then by using Cauchy's integral theorem,

$$\oint_C \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{L} f^{(n)}(a) \text{ we get -}$$

$$\oint_C \frac{e^z dz}{(z+1)^2} = \frac{1}{4i} \left[\frac{2\pi i}{L} f'(i) - \frac{2\pi i}{L} f'(-i) \right] \dots (1)$$

$$\text{we have, } f(z) = \frac{e^z}{z}$$

$$\therefore f'(z) = \frac{z \cdot e^{2z} - 1 \cdot e^{2z}}{z^2}$$

$$\therefore f'(i) = \frac{ie^{it} - e^{it}}{i^2}$$

$$= e^{it} - ie^{it} \quad [\because i^2 = -1]$$

$$\text{Also, } f'(-i) = \frac{-ie^{-it} - e^{-it}}{(-i)^2}$$

$$= ie^{-it} + e^{-it}$$

so from (1) we get

$$\frac{1}{2\pi i} \oint_C \frac{e^z dz}{(z+1)^2} = \frac{1}{4i} [e^{it} - ie^{it} - ie^{-it} - e^{-it}]$$

$$= \frac{1}{4i} [(e^{it} - e^{-it}) - it(e^{it} + e^{-it})]$$

$$= \frac{1}{4i} [2isint - it \cdot 2cost]$$

$$= \frac{1}{4i} \cdot 2i [sint - tcost]$$

$$= \frac{1}{2} (sint - tcost) \quad \left[\begin{array}{l} \because e^{i\theta} = \cos\theta + i\sin\theta \\ e^{-i\theta} = \cos\theta - i\sin\theta \\ \Rightarrow 2isint = e^{i\theta} + e^{-i\theta} \end{array} \right]$$

- II. Evaluate $\oint_C \frac{e^z dz}{z(1-z)^3}$ if (i) 0 lies inside C and 1 lies outside C, (ii) 1 lies inside C and 0 lies outside C, (iii) 0 and 1 lie inside C.

Solution: (i) Since 0 lies inside C and 1 lies outside C.

$$\begin{aligned} \therefore \oint_C \frac{e^z dz}{z(1-z)^3} &= \oint_C \frac{f(z)}{z-0} dz \quad \text{where } f(z) = \frac{e^z}{(1-z)^3} \\ &= 2\pi i f(0) \\ &= 2\pi i \cdot 1 \\ &= 2\pi i \end{aligned}$$

(iii) since 1 lies inside C and 0 lies outside C.

$$\therefore \oint_C \frac{e^z dz}{z(1-z)^3} = \oint_C \frac{f(z) dz}{(z-1)^3}$$

$$= \frac{2\pi i}{1^2} f''(1)$$

$$\text{Where, } f(z) = -\frac{e^z}{z}$$

$$\therefore f'(z) = \frac{e^z}{z^2} - \frac{e^z}{z}$$

$$f''(z) = -\frac{2e^z}{z^3} + \frac{e^z}{z^2} + \frac{e^z}{z^2} - \frac{e^z}{z^2}$$

$$\therefore f''(1) = -2e^1 + 2e^1 - e^1$$

$$= -e$$

$$\therefore \oint_C \frac{e^z dz}{z(1-z)^3} = \frac{2\pi i}{2} \cdot (-e)$$

$$= -\pi ie$$

(iii) Since 0 and 1 lie inside C, so we express

$\frac{1}{z(1-z)^3}$ in partial fractions.

$$\text{Let } \frac{1}{z(1-z)^3} = \frac{1}{z} + \frac{1}{1-z} + \frac{A}{(1-z)^2} + \frac{B}{(1-z)^3} \dots (1)$$

$$\Rightarrow 1 = (1-z)^3 + z(1-z)^2 + A z(1-z) + B z \dots (2)$$

Putting $z=1$ in (2), we get $1=B$

Equating the coefficients of z^2 from both sides of (2), we get

$$0 = 3 - 2 - A \quad \text{or, } A = 1$$

so from (1) we get

$$\frac{1}{z(1-z)^3} = \frac{1}{z} + \frac{1}{1-z} + \frac{1}{(1-z)^2} + \frac{1}{(1-z)^3}$$

$$\therefore \oint_C \frac{e^z dz}{z(1-z)^3} = \oint_C \frac{e^z dz}{z} + \oint_C \frac{e^z dz}{1-z} + \oint_C \frac{e^z dz}{(1-z)^2} + \oint_C \frac{e^z dz}{(1-z)^3}$$

$$= 2\pi i \cdot (e^0) - 2\pi i \cdot (e^1) + \frac{2\pi i}{1!} f'(1) - \frac{2\pi i}{2!} f''(1)$$

$$= 2\pi i - 2\pi ie + 2\pi ie - \frac{2\pi i}{2} \cdot e$$

$$= \pi i (2 - e)$$

12. What is the value of $\oint_C \frac{z^2+1}{z^2-1} dz$ if C is a circle of unit radius with centre at (i) $z=1$ and (ii) $z=-1$.

Solution: (i) If C is a circle of unit radius with centre at $z=1$, then

$$\begin{aligned}\oint_C \frac{(z^2+1)dz}{z^2-1} &= \oint_C \frac{\frac{z^2+1}{z-1}}{z+1} dz \\ &= 2\pi i f(1) \quad \text{where } f(z) = \frac{z^2+1}{z-1} \\ &= 2\pi i \cdot 1 \\ &= 2\pi i\end{aligned}$$

(ii) If C is a circle of unit radius with centre $z=-1$, then

$$\begin{aligned}\oint_C \frac{z^2+1}{z^2-1} dz &= \oint_C \frac{\frac{z^2+1}{z+1}}{z-1} dz \\ &= 2\pi i f(-1) \quad \text{where } f(z) = \frac{z^2+1}{z+1} \\ &= 2\pi i (-1) \\ &= -2\pi i\end{aligned}$$

13. Using Cauchy's integral formula, evaluate

$$\oint_C \frac{z dz}{(z-1)(z-2)} \quad \text{where } C \text{ is the circle } |z-2|=\frac{1}{2}$$

Solution: Since $z=2$ is the only point lies inside the circle $|z-2|=\frac{1}{2}$,

$$\begin{aligned}\therefore \oint_C \frac{z dz}{(z-1)(z-2)} &= \oint_C \frac{\left(\frac{z}{z-1}\right) dz}{z-2} \\ &= 2\pi i f(2) \quad \text{where } f(z) = \frac{z}{z-1} \\ &= 2\pi i \cdot 2 \\ &= 4\pi i\end{aligned}$$

14. Evaluate $\oint_C \frac{dz}{(z^2+4)^2}$, where C is the circle $|z-i|=2$

Solution: Let $F(z) = \frac{1}{(z^2+4)^2}$

\because Singular points of $f(z)$ are $z = \pm 2i$. Among this only $z = 2i$ lies inside the circle $|z-i|=2$.

$$\begin{aligned}\therefore \oint_C \frac{dz}{(z^2+4)^2} &= \oint_C \frac{\frac{1}{(z+2i)^2} \cdot \frac{d}{dz}(z+2i)^2 dz}{(z-2i)^2} \\ &= \frac{2\pi i}{11} f'(2i) \\ &= 2\pi i \cdot \frac{1}{32i} \\ &= \frac{\pi i}{16}\end{aligned}$$

$$\begin{aligned}\text{where } f(z) &= \frac{1}{(z+2i)^2} \\ \therefore f'(z) &= -\frac{2}{(z+2i)^3} \\ \therefore f(2i) &= -\frac{2}{(4i)^3} \\ &= \frac{1}{32i}\end{aligned}$$

Complex Integration

1. State and prove Cauchy's theorem/Cauchy's integral theorem.

Statement: If $f(z)$ is analytic inside and on a simple closed curve C , then $\oint_C f(z) dz = 0$.

Proof: Let $f(z) = u(x, y) + iv(x, y)$ be analytic.

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \dots \text{(1)}$$

Since $z = x+iy$, so $dz = dx+idy$

$$\begin{aligned} \text{Now } \oint_C f(z) dz &= \oint_C (u+iv)(dx+idy) \\ &= \oint_C (u dx + iudy + iv dx - v dy) \\ &= \oint_C (u dx - v dy) + i \oint_C (v dx + u dy) \quad \dots \text{(1)} \end{aligned}$$

By Green's theorem we have,

$$\oint_C (u dx - v dy) = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx dy$$

$$\text{and } \oint_C (v dx + u dy) = \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx dy$$

where R is the region bounded by C .

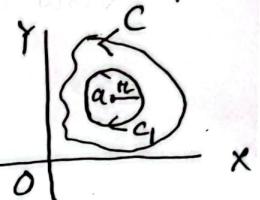
Hence (1) becomes,

$$\begin{aligned} \oint_C f(z) dz &= \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dx dy + i \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) dx dy \\ &= \iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y} \right) dx dy + i \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x} \right) dx dy \\ &= 0 + i \cdot 0 \\ &= 0 \end{aligned}$$

2. State and prove Cauchy's integral formula.

Statement: If $f(z)$ is analytic inside and on a simple closed curve C , and 'a' is any point within C , then $\oint_C \frac{f(z)dz}{z-a} = 2\pi i f(a)$

Proof: Since $f(z)$ is analytic inside and on C , $\frac{f(z)}{z-a}$ is also analytic inside and on C , except at the point $z=a$. Hence, we draw a small circle with centre at $z=a$ and radius r lying entirely inside C .



Now, $\frac{f(z)}{z-a}$ is analytic in the region enclosed between C and C_1 .

Hence, by Cauchy's extended theorem,

$$\oint_C \frac{f(z)dz}{z-a} = \oint_{C_1} \frac{f(z)dz}{z-a} \dots (1)$$

On C_1 , any point z is given by $z=a+re^{i\theta}$

$$dz = ire^{i\theta} d\theta$$

where θ varies from 0 to 2π .

$$\begin{aligned} \therefore \oint_{C_1} \frac{f(z)dz}{z-a} &= \int_{\theta=0}^{2\pi} \frac{f(a+re^{i\theta}) \cdot ire^{i\theta} d\theta}{re^{i\theta}} \\ &= i \int_{\theta=0}^{2\pi} f(a+re^{i\theta}) d\theta \end{aligned}$$

As $r \rightarrow 0$, the circle tends to a point.

Taking limit $r \rightarrow 0$, we get

$$\begin{aligned} \oint_{C_1} \frac{f(z)dz}{z-a} &= i \int_{\theta=0}^{2\pi} f(a) d\theta \\ &= i f(a) [\theta]_{\theta=0}^{2\pi} \\ &= 2\pi i f(a) \end{aligned}$$

So from (1), we get $\oint_C \frac{f(z)dz}{z-a} = 2\pi i f(a)$

* In general, $\oint_C \frac{f(z) dz}{(z-a)^{n+1}} = \frac{2\pi i}{n!} f^{(n)}(a)$

where $n=0, 1, 2, 3, \dots$

and $f^{(0)}(a) = f(a)$

* Cauchy's extended theorem: If $f(z)$ is analytic within and on the boundary of a region bounded by two closed curves C_1 and C_2 , then

$$\oint_{C_1} f(z) dz = \oint_{C_2} f(z) dz$$



3. Evaluate $\frac{1}{2\pi i} \oint_C \frac{e^z}{z-2} dz$ if C is (a) the circle $|z|=3$,

(b) the circle $|z|=1$.

Solution: (a) Here $f(z) = e^z$ is analytic inside and on the circle $|z|=3$ and $z=a=2$ is a point inside the given circle.

Then by using the Cauchy's integral formula, we have

$$\oint_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$$

$$\therefore \oint_C \frac{e^z}{z-2} dz = 2\pi i \cdot e^2$$

$$\text{or, } \frac{1}{2\pi i} \oint_C \frac{e^z}{z-2} dz = e^2$$

(b) Here $f(z) = \frac{e^z}{z-2}$ is analytic inside and on the circle $|z|=1$ and $z=2$ is a point outside the given circle.

Then by using the Cauchy's integral theorem, $\oint_C f(z) dz = 0$ we get

$$\oint_C \frac{e^z}{z-2} dz = 0 \text{ or, } \frac{1}{2\pi i} \oint_C \frac{e^z}{z-2} dz = 0$$

4. Evaluate $\oint_C \frac{\sin z^2}{z+\frac{1}{2}i} dz$ if C is the circle $|z|=5$.

Solution: Here $f(z) = \sin z^2$ is analytic inside and on the circle $|z|=5$ and $z= -\frac{1}{2}i$ lies inside the given circle.

Then by using Cauchy's integral formula,

$$\oint_C \frac{f(z)}{z-a} dz = 2\pi i f(a) \text{ we get—}$$

$$\oint_C \frac{\sin z^2}{z-\left(-\frac{1}{2}i\right)} dz = 2\pi i f\left(-\frac{1}{2}i\right) \quad \left| \begin{array}{l} \therefore f\left(-\frac{1}{2}i\right) = \sin\left(-\frac{1}{2}\right) \\ = -(-1) \\ = 1 \end{array} \right.$$

$$\text{or, } \oint_C \frac{\sin z^2}{z+\frac{1}{2}i} dz = 2\pi i$$

5. Evaluate $\oint_C \frac{e^{z^2}}{z-ni} dz$ if C is (a) the circle $|z-1|=4$,
(b) the ellipse $|z-2|+|z+2|=6$.

Solution: (a) Here $f(z) = e^{z^2}$ is analytic inside and on the given circle $|z-1|=4$, and $z=ni$ is a point inside the given circle.

Then by using Cauchy's integral formula,

$$\oint_C \frac{f(z)}{z-a} dz = 2\pi i f(a) \text{ we get—}$$

$$\begin{aligned} \oint_C \frac{e^{z^2}}{z-ni} dz &= 2\pi i f(ni) \\ &= 2\pi i \cdot e^{3\pi i} \\ &= 2\pi i (453\pi + i \sin 3\pi) \\ &= 2\pi i (-1 + i \cdot 0) \\ &= -2\pi i \end{aligned}$$

(b) Hence $f(z) = \frac{e^{z^2}}{z-ni}$ is analytic inside and on the ellipse C , and $z=ni$ lies outside the given ellipse C .

Then by using Cauchy's integral theorem,

$$\oint_C f(z) dz = 0 \text{ we get—}$$

$$\oint_C \frac{e^{z^2}}{z-ni} dz = 0$$

** Locus of $|z-2|+|z+2|=6$ is $\frac{x^2}{3^2} + \frac{y^2}{(5)^2} = 1$.

It's foci $(\pm ae, 0) = (\pm 3, \frac{2}{3}, 0) = (\pm 2, 0)$, $a = \sqrt{1-\frac{4}{9}} = \frac{2}{3}$
and length of major axis $= 2 \cdot 3 = 6$

6. Evaluate $\frac{1}{2\pi i} \oint_C \frac{\cos z}{z^2 - 1} dz$ around a rectangle with vertices at: (a) $2 \pm i, -2 \pm i$ (b) $-i, 2-i, 2+i, i$.

Solution: We have $\frac{1}{2\pi i} \oint_C \frac{\cos z}{z^2 - 1} dz = \frac{1}{4\pi i} \oint_C \left[\frac{\cos z}{z-1} - \frac{\cos z}{z+1} \right] dz$

$$= \frac{1}{4\pi i} \left[\oint_C \frac{\cos z}{z-1} dz - \oint_C \frac{\cos z}{z+1} dz \right]$$

(a) Here $f(z) = \cos z$ is analytic inside and on C , and also both points $z = \pm 1$ lie inside the rectangle $2 \pm i, -2 \pm i$. (1)

Then by using Cauchy's integral formula,

$$\oint_C \frac{f(z)}{z-a} dz = 2\pi i f(a), \text{ we get from (1)}$$

$$\begin{aligned} \frac{1}{2\pi i} \oint_C \frac{\cos z}{z-1} dz &= \frac{1}{4\pi i} [2\pi i \cos(-\pi) - 2\pi i \cos(\pi)] \\ &= \frac{1}{4\pi i} [-2\pi i + 2\pi i] \\ &= 0 \end{aligned}$$

(b) Here only the point $z=1$ lies inside the rectangle $\pm i, 2 \pm i$.

Then by using the Cauchy's integral formula and also the Cauchy's integral theorem, we get from (1),

$$\begin{aligned} \frac{1}{2\pi i} \oint_C \frac{\cos z}{z^2 - 1} dz &= \frac{1}{4\pi i} [2\pi i \cos 0] \\ &= -\frac{1}{2} \end{aligned}$$

7. Show that $\frac{1}{2\pi i} \oint_C \frac{e^{zt}}{z^2 + 1} dz = \sin t$ if $t > 0$ and C is the circle $|z|=3$

Solution: We have $\frac{1}{2\pi i} \oint_C \frac{e^{zt}}{z^2 + 1} dz = \frac{1}{2\pi i} \oint_C \frac{e^{zt}}{(z+i)(z-i)} dz$

$$= \frac{1}{2\pi i} \cdot \frac{1}{2i} \left[\oint_C \frac{e^{zt}}{z-i} dz - \oint_C \frac{e^{zt}}{z+i} dz \right]$$

Here $f(z) = e^{zt}$ is analytic inside and on the given circle $|z|=3$ and $z=\pm i$ are inside C . (1)

Then by using Cauchy's integral formula, we get from (1)

$$\begin{aligned}
 \frac{1}{2\pi i} \oint_C \frac{e^{iz}}{z^2+1} dz &= \frac{1}{2\pi i} \cdot \frac{1}{2i} [2\pi i f(i) - 2\pi i f(-i)] \\
 &= \frac{1}{2i} [e^{it} - e^{-it}] \\
 &= \frac{1}{2i} \cdot 2i \sin t \quad [\because e^{it} - e^{-it} = 2i \sin t] \\
 &= \sin t
 \end{aligned}$$

8. Evaluate $\oint_C \frac{e^{iz}}{z^3} dz$ where C is the circle $|z|=2$.

Solution: Here $f(z) = e^{iz}$ is analytic inside and on the circle $|z|=2$ and $z=0$ is a point inside the given circle.

Then by using Cauchy's integral formula,

$$\oint_C \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(a), \text{ we get}$$

$$\begin{aligned}
 \oint_C \frac{f(z)}{(z-0)^3} dz &= \frac{2\pi i}{2!} f^{(2)}(0) \quad \left| \begin{array}{l} f(z) = e^{iz} \\ f'(z) = ie^{iz} \\ f''(z) = -e^{iz} \\ f'''(z) = -1 = f^{(2)}(0) \end{array} \right. \\
 \therefore \oint_C \frac{e^{iz}}{z^3} dz &= \frac{2\pi i}{2} \cdot (-1) \\
 &= -\pi i
 \end{aligned}$$

9. Find the value of (a) $\oint_C \frac{\sin^6 z}{z-\frac{\pi}{6}} dz$, (b) $\oint_C \frac{\sin^6 z dz}{(z-\frac{\pi}{6})^3}$ if C is the circle $|z|=1$.

Solution: Here $f(z) = \sin^6 z$ is analytic inside and on the circle $|z|=1$ and $z=a=\frac{\pi}{6}$ is a point inside the given circle.

(a) Then by using Cauchy's integral formula,

$$\oint_C \frac{f(z)}{z-a} dz = 2\pi i f(a) \text{ we get}$$

$$\begin{aligned}
 \oint_C \frac{\sin^6 z}{z-\frac{\pi}{6}} dz &= 2\pi i \left(\sin \frac{\pi}{6}\right)^6 \\
 &= 2\pi i \cdot \left(\frac{1}{2}\right)^6 \\
 &= 2\pi i \cdot \frac{1}{64} \\
 &= \frac{\pi i}{32}
 \end{aligned}$$

(b) Then by using Cauchy's integral formula,

$$\oint_C \frac{f(z)}{(z-a)^m} dz = \frac{2\pi i}{m!} f^{(m)}(a) \text{ we get}$$

$$\oint_C \frac{\sin^6 z}{(z-\frac{\pi}{6})^3} dz = \frac{2\pi i}{12} f''(\frac{\pi}{6}) \dots (1)$$

$$\text{We have } f(z) = \sin^6 z$$

$$\therefore f'(z) = 6 \sin^5 z \cdot \cos z$$

$$f''(z) = 30 \sin^4 z \cos^2 z + 6 \sin^5 z (-\sin z)$$

$$\begin{aligned} \therefore f''(\frac{\pi}{6}) &= f''(\frac{\pi}{6}) = 30 \cdot (\frac{1}{2})^4 \cdot (\frac{\sqrt{3}}{2})^2 - 6 \cdot (\frac{1}{2})^6 \\ &= 30 \cdot \frac{1}{16} \cdot \frac{3}{4} - 6 \cdot \frac{1}{64} \\ &= \frac{90-6}{64} \\ &= \frac{84}{64} \\ &= \frac{21}{16} \end{aligned}$$

So from (1) we get

$$\begin{aligned} \oint_C \frac{\sin^6 z}{(z-\frac{\pi}{6})^3} dz &= \frac{2\pi i}{2} \cdot \frac{21}{16} \\ &= \frac{21\pi i}{16} \end{aligned}$$

10. Evaluate $\frac{1}{2\pi i} \oint_C \frac{e^{zt}}{(z^2+1)^2} dz$ if $t > 0$ and C is the circle $|z| = 3$.

Solution: We have, $\frac{1}{(z^2+1)^2} = \frac{1}{(z+i)^2(z-i)^2}$

$$= \frac{1}{4iz} \left[\frac{1}{(z-i)^2} - \frac{1}{(z+i)^2} \right]$$

$$\therefore \oint_C \frac{e^{zt}}{(z^2+1)^2} dz = \frac{1}{4i} \left[\oint_C \frac{\frac{e^{zt}}{z-i} dz}{(z-i)^2} - \oint_C \frac{\frac{e^{zt}}{z+i} dz}{(z+i)^2} \right]$$

Hence $f(z) = \frac{e^{zt}}{z^2}$ is analytic inside on the given circle $|z|=3$ and $z=\pm i$ are inside C .

Then by using Cauchy's integral theorem,

$$\oint_C \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{L} f^{(n)}(a) \text{ we get}$$

$$\oint_C \frac{e^{zt}}{(z+1)^2} dz = \frac{1}{4i} \left[\frac{2\pi i}{L} f'(i) - \frac{2\pi i}{L} f'(-i) \right] \quad (1)$$

$$\text{we have, } f(z) = \frac{e^{zt}}{z}$$

$$\therefore f'(z) = \frac{z \cdot e^{zt} \cdot t - 1 \cdot e^{zt}}{z^2}$$

$$\therefore f'(i) = \frac{ite^{it} - e^{it}}{i^2} \quad [\because i^2 = -1]$$

$$= e^{it} - ite^{it}$$

$$\text{Also, } f'(-i) = \frac{-ite^{-it} - e^{-it}}{(-i)^2}$$

$$= ite^{-it} + e^{-it}$$

so from (1) we get

$$\begin{aligned} \frac{1}{2\pi i} \oint_C \frac{e^{zt}}{(z+1)^2} dz &= \frac{1}{4i} [e^{it} - ite^{it} - ite^{-it} - e^{-it}] \\ &= \frac{1}{4i} [(e^{it} - e^{-it}) - it(e^{it} + e^{-it})] \\ &= \frac{1}{4i} [2isint - it \cdot 2cost] \\ &= \frac{1}{4i} \cdot 2i [sint - tcost] \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2} (sint - tcost) \quad [\because e^{i\theta} = \cos\theta + i\sin\theta \\ &\quad e^{-i\theta} = \cos\theta - i\sin\theta \\ &\quad \Rightarrow 2isint = e^{i\theta} + e^{-i\theta}] \end{aligned}$$

- II. Evaluate $\oint_C \frac{e^z dz}{z(1-z)^3}$ if (i) 0 lies inside C and 1 lies outside C, (ii) 1 lies inside C and 0 lies outside C, (iii) 0 and 1 lie inside C.

Solution: (i) since 0 lies inside C and 1 lies outside C.

$$\begin{aligned} \therefore \oint_C \frac{e^z dz}{z(1-z)^3} &= \oint_C \frac{f(z)}{z-0} dz \text{ where } f(z) = \frac{e^z}{(1-z)^3} \\ &= 2\pi i f(0) \\ &= 2\pi i \cdot 1 \\ &= 2\pi i \end{aligned}$$

(iii) since 1 lies inside C and 0 lies outside C.

$$\therefore \oint_C \frac{e^z dz}{z(1-z)^3} = \oint_C \frac{f(z) dz}{(z-1)^3}$$

$$= \frac{2\pi i}{1^2} f''(1)$$

$$\text{Where, } f(z) = -\frac{e^z}{z}$$

$$\therefore f'(z) = \frac{e^z}{z^2} - \frac{e^z}{z}$$

$$f''(z) = -\frac{2e^z}{z^3} + \frac{e^z}{z^2} + \frac{e^z}{z^2} - \frac{e^z}{z}$$

$$\therefore f''(1) = -2e^1 + 2e^1 - e^1$$

$$= -e$$

$$\therefore \oint_C \frac{e^z dz}{z(1-z)^3} = \frac{2\pi i}{2} \cdot (-e)$$

$$= -\pi ie$$

(iii) Since 0 and 1 lie inside C, so we express

$\frac{1}{z(1-z)^3}$ in partial fractions.

$$\text{Let } \frac{1}{z(1-z)^3} = \frac{1}{z} + \frac{1}{1-z} + \frac{A}{(1-z)^2} + \frac{B}{(1-z)^3} \dots (1)$$

$$\Rightarrow 1 = (1-z)^3 + z(1-z)^2 + A z(1-z) + Bz \dots (2)$$

Putting $z=1$ in (2), we get $1=B$

Equating the coefficients of z^2 from both sides of (2), we get

$$0 = 3 - 2 - A \quad \text{or, } A = 1$$

so from (1) we get

$$\frac{1}{z(1-z)^3} = \frac{1}{z} + \frac{1}{1-z} + \frac{1}{(1-z)^2} + \frac{1}{(1-z)^3}$$

$$\therefore \oint_C \frac{e^z dz}{z(1-z)^3} = \oint_C \frac{e^z dz}{z} + \oint_C \frac{e^z dz}{1-z} + \oint_C \frac{e^z dz}{(1-z)^2} + \oint_C \frac{e^z dz}{(1-z)^3}$$

$$= 2\pi i \cdot (e^0) - 2\pi i \cdot (e^1) + \frac{2\pi i}{1!} f'(1) - \frac{2\pi i}{2!} f''(1)$$

$$= 2\pi i - 2\pi ie + 2\pi ie - \frac{2\pi i}{2} \cdot e$$

$$= \pi i (2-e)$$

12. What is the value of $\oint_C \frac{z^2+1}{z^2-1} dz$ if C is a circle of unit radius with centre at (i) $z=1$ and (ii) $z=-1$.

Solution: (i) If C is a circle of unit radius with centre at $z=1$, then

$$\begin{aligned}\oint_C \frac{(z^2+1)dz}{z^2-1} &= \oint_C \frac{\frac{z^2+1}{z-1}}{z+1} dz \\ &= 2\pi i f(1) \text{ where } f(z) = \frac{z^2+1}{z-1} \\ &= 2\pi i \cdot 1 \\ &= 2\pi i\end{aligned}$$

(ii) If C is a circle of unit radius with centre $z=-1$, then

$$\begin{aligned}\oint_C \frac{z^2+1}{z^2-1} dz &= \oint_C \frac{\frac{z^2+1}{z+1}}{z-1} dz \\ &= 2\pi i f(-1) \text{ where } f(z) = \frac{z^2+1}{z+1} \\ &= 2\pi i (-1) \\ &= -2\pi i\end{aligned}$$

13. Using Cauchy's integral formula, evaluate

$$\oint_C \frac{z dz}{(z-1)(z-2)} \text{ where } C \text{ is the circle } |z-2|=\frac{1}{2}$$

Solution: Since $z=2$ is the only point lies inside the circle $|z-2|=\frac{1}{2}$,

$$\begin{aligned}\therefore \oint_C \frac{z dz}{(z-1)(z-2)} &= \oint_C \frac{\left(\frac{z}{z-1}\right) dz}{z-2} \\ &= 2\pi i f(2) \text{ where } f(z) = \frac{z}{z-1} \\ &= 2\pi i \cdot 2\end{aligned}$$

14. Evaluate $\oint_C \frac{dz}{(z^2+4)^2}$, where C is the circle $|z-i|=2$

Solution: Let $F(z) = \frac{1}{(z^2+4)^2}$

∴ Singular points of $f(z)$ are $z = \pm 2i$. Among this only $z = 2i$ lies inside the circle $|z-i|=2$.

$$\begin{aligned}\therefore \oint_C \frac{dz}{(z^2+4)^2} &= \oint_C \frac{\frac{1}{(z+2i)^2} \cdot dz}{(z-2i)^2} \quad \text{where } f(z) = \frac{1}{(z+2i)^2} \\ &= \frac{2\pi i}{11} f'(2i) \\ &= 2\pi i \cdot \frac{1}{32i} \\ &= \frac{\pi i}{16}\end{aligned}$$

$$\begin{aligned}\therefore f'(2i) &= -\frac{2}{(2+2i)^3} \\ \therefore f(2i) &= -\frac{2}{(4i)^3} \\ &= \frac{1}{32i}\end{aligned}$$