Laplace Transform

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Contents

1 Laplace Transform

1.1 Definition and Existence

Definition 1.1.1: Laplace Transform

Let F be a real-valued function of the real variable t, defined for $t > 0$. Let s be a variable that we shall assume to be real, and consider the function f defined by

$$f(s) = \int_0^\infty e^{-st} F(t)\, dt \tag{1}$$

for all values of s for which this integral exists. The function f defined by the integral (1) is called the Laplace Transform of the function F. We shall denote the Laplace transform of F by $\mathscr{L}\{F(t)\}$.
Thus the Laplace transform of a function f is given by

$$\mathscr{L}\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\, dt = \lim_{R\to\infty} \int_0^R e^{-st} F(t)\, dt \tag{2}$$

Some ways to write Laplace transforms:

$$\mathscr{L}\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\, dt$$

$$\mathscr{L}\{G(t)\} = g(s)$$

$$\mathscr{L}\{u(t)\} = \tilde{u}(s)$$

Theorem 1.1.2: Hypothesis: *Let F be a real function that has the following properties:*

1. *F is a piecewise continuous in every finite closed interval $0 \le t \le a$ $(b > 0)$.*
2. *F is of exponential order, i.e, there exists α, $M > 0$, and $t_0 > 0$ such that*

$$e^{-\alpha t}|F(t)| < M \text{ for } t > t_0$$

Conclusion: *The Laplace transform of F exists for $s > \alpha$.*

$$\mathscr{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt$$

1.2 Some Functions and Their Laplace Transforms

$F(t)$	$\mathscr{L}\{F(t)\} = f(s)$	$F(t)$	$\mathscr{L}\{F(t)\} = f(s)$
1	$\frac{1}{s}$	n	$\frac{n}{s}$
t	$\frac{1}{s^2}$	t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$	e^{-at}	$\frac{1}{s+a}$
$\sin at$	$\frac{a}{s^2+a^2}$	$\sinh at$	$\frac{a}{s^2-a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$	$\cosh at$	$\frac{s}{s^2-a^2}$

Table 1: Functions and their Laplace Transform

Proofs :

$$\mathscr{L}\{n\} = \frac{n}{s}$$

Let $F(t) = n$, for $t > 0$
Then

$$\begin{aligned}
\mathscr{L}\{n\} &= \int_0^\infty e^{-st} \cdot n \, dt \\
&= n \frac{-e^{st}}{s} \bigg|_0^\infty \\
&= \frac{n}{s} \quad \square
\end{aligned}$$

$$\mathscr{L}\{t\} = \frac{1}{s^2}$$

Let $F(t) = t$, for $t > 0$
Then

$$\begin{aligned}
\mathscr{L}\{t\} &= \int_0^\infty e^{-st} \cdot t \, dt \\
&= -t\frac{e^{-st}}{s} + \int_0^\infty \frac{e^{-st}}{s} \, dt \\
&= -e^{-st}\frac{t}{s}\bigg|_0^\infty - e^{-st}\frac{1}{s^2}\bigg|_0^\infty \\
&= \frac{1}{s^2} \quad \square
\end{aligned}$$

$$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

Let $F(t) = t^n$, for $t > 0$
Then

$$\begin{aligned}
\mathscr{L}\{t^n\} &= \int_0^\infty e^{-st} t^n \, dt \\
&= -t^n \frac{e^{st}}{s} + \int_0^\infty n t^{n-1} \frac{e^{-st}}{s} \, dt \\
&= -n t^{n-1} \frac{e^{-st}}{s^2} \bigg|_0^\infty + \int_0^\infty n(n-1) t^{n-2} \frac{e^{-st}}{s^2} \, dt \\
&= -n(n-1)t^{n-2} \left(\frac{e^{-st}}{s^3} \right) \bigg|_0^\infty + \int_0^\infty n(n-1)(n-2) t^{n-3} \frac{e^{-st}}{s^3} \, dt \\
&= \cdots \\
&= n! t^{n-n} \frac{e^{-st}}{s^{n+1}} \bigg|_0^\infty + \int_0^\infty n(n-1)\cdots(n-n) \frac{e^{-st}}{s^{n+1}} \, dt \\
&= \frac{n!}{s^{n+1}} \quad \square
\end{aligned}$$

Let $F(t) = e^{at}$, for $t > 0$
Then

$$\begin{aligned}
\mathscr{L}\{e^{at}\} &= \int_0^\infty e^{-st} e^{at} \, dt \\
&= \int_0^\infty e^{(a-s)t} \, dt \\
&= \frac{e^{(a-s)t}}{a-s} \bigg|_0^\infty \\
&= \frac{1}{s-a} \quad \square
\end{aligned}$$

Let $F(t) = e^{-at}$, for $t > 0$
Then

$$\begin{aligned}
\mathscr{L}\{e^{-at}\} &= \int_0^\infty e^{-st} e^{-at} \, dt \\
&= \int_0^\infty e^{-(a+s)t} \, dx \\
&= \frac{e^{-(a+s)t}}{s+a} \bigg|_0^\infty \\
&= \frac{1}{s+a} \quad \square
\end{aligned}$$

Let $F(t) = \sin at$, for $t > 0$
Then

$$\begin{aligned}\mathcal{L}\{\sin at\} &= \int_0^\infty e^{-st\sin at}\,dt \\ &= -\frac{e^{-st}}{s^2+a^2}\left(s\sin at + a\cos at\right)\Big|_0^\infty \\ &= \frac{a}{s^2+a^2} \quad \square\end{aligned}$$

Let $F(t) = \cos at$, for $t > 0$
Then

$$\begin{aligned}\mathcal{L}\{\cos at\} &= \int_0^\infty e^{-st}\cos at\,dt \\ &= \frac{e^{-st}}{s^2+a^2}\left(-s\cos at + a\sin at\right)\Big|_0^\infty \\ &= \frac{s}{s^2+a^2} \quad \square\end{aligned}$$

2 Basic Properties of the Laplace Transform

2.1 Linearity Property

***Theorem* (The Linearity Property):**
Let F_1 and F_2 be functions whose Laplace transform exist, and let c_1 and c_2 be constants. Then

$$\mathcal{L}\{c_1F_1(t) + c_2F_2(t)\} = c_1\mathcal{L}\{F_1(t)\} + c_2\mathcal{L}\{F_2(t)\}$$

Proof :
Let $F(t) = c_1F_1(t) + c_2F_2(t)$, for $t > 0$
Then

$$\begin{aligned}\mathcal{L}\{c_1F_1(t) + c_2F_2(t)\} &= \int_0^\infty e^{-st}\left[c_1F_1(t) + c_2F_2(t)\right]dt \\ &= c_1\int_0^\infty e^{-st}F_1(t)\,dt + c_2\int_0^\infty e^{-st}F_2(t)\,dt \\ &= c_1\mathcal{L}\{F_1(t)\} + c_2\mathcal{L}\{F_2(t)\} \quad \square\end{aligned}$$

Example 2.1:

$$\mathcal{L}\{4t^2 - 3\cos 2t + 5e^{-t}\}$$

$$\begin{aligned}\mathcal{L}\{4t^2 - 3\cos 2t + 5e^{-t}\} &= 4\mathcal{L}\{t^2\} - 3\mathcal{L}\{\cos 2t\} + 5\mathcal{L}\{e^{-t}\} \\ &= 4\cdot\frac{2!}{s^3} - 3\cdot\frac{s}{s^2+4} + 5\cdot\frac{1}{s+1} \\ &= \frac{8}{s^3} - \frac{3s}{s^2+4} + \frac{5}{s+1}\end{aligned}$$

Example 2.2: Find $\mathcal{L}\{F(w)\}$, when $F(t) = \begin{cases} 5 & \text{for } 0 < t < 3 \\ 0 & \text{for } t > 3\end{cases}$

$$\begin{aligned}\mathcal{L}\{F(t)\} &= \int_0^\infty e^{-st}F(t)\,dt = \int_0^3 e^{-st}\cdot 5\,dt + \int_3^\infty 0\,dt \\ &= \int_0^3 e^{-st}\cdot 5\,dt \\ &= \frac{5e^{-st}}{s}\Big|_0^3 \\ &= \frac{5}{s}\left(1 - e^{-3s}\right)\end{aligned}$$

Example 2.3: Find $\mathcal{L}\{F(t)\}$, when $F(t = \begin{cases} (t-1)^2 & \text{for } t > 1 \\ 0 & \text{for } 0 < t < 1\end{cases})$

$$
\begin{aligned}
\mathscr{L}\{F(t)\} &= \int_0^\infty e^{-st}F(t)\,dt = \int_0^1 0\,dt + \int_1^\infty e^{-st}(t-1)^2\,dt \\
&= \int_1^\infty e^{-st}(t^2-2t+1)\,dt \\
&= -t^2\frac{e^{-st}}{s}\bigg|_1^\infty - 2\int_1^\infty t\cdot\frac{e^{-st}}{s}\,dt - 2\int_1^\infty t\cdot e^{-st}\,dt - \frac{e^{-st}}{s} \\
&= \frac{e^{-s}}{s} + 2\left[-\frac{t}{s}\left(\frac{e^{-st}}{s}\right)\bigg|_1^\infty + \int_1^\infty \frac{e^{-st}}{s^2}\,dt + 2t\frac{e^{-st}}{s}\bigg|_1^\infty - \int_1^\infty \frac{e^{-st}}{s}\,dt\right] + \frac{e^{-st}}{s} \\
&= \frac{e^{-s}}{s} - 2\frac{e^{-s}}{s^2} + 2\frac{e^{-s}}{s^2} - 2\frac{e^{-st}}{s} + 2\frac{e^{-s}}{s^2} + \frac{e^{-st}}{s} \\
&= 2\frac{e^{-s}}{s^3}
\end{aligned}
$$

2.2 First Translation Property

Theorem 2.2.1* (First Translation of Shifting Property)*:
If

$$\mathscr{L}\{F(t)\} = f(s)$$

then

$$\mathscr{L}\{e^{at}F(t)\} = f(s-a)$$

Proof:
Let $G(t) = e^{at}F(t)$, for $t > 0$
Then

$$
\begin{aligned}
\mathscr{L}\{e^{at}F(t)\} &= \int_0^\infty e^{-st}e^{at}F(t)\,dt \\
&= \int_0^\infty e^{-(s-a)t}F(t)\,dt \\
&= \mathscr{L}\{F(t)\}\bigg|_{s-a} \\
&= f(s-a) \quad \square
\end{aligned}
$$

Example 2.4:

$$\mathscr{L}\{e^{-t}\cos 2t\}$$

Since $\mathscr{L}\{\cos 2t\} = \dfrac{s}{s^2+4}$, we have

$$\mathscr{L}\{e^{-t}\cos 2t\} = \frac{s+1}{(s+1)^2+4} = \frac{s+1}{s^2+2s+5}\}$$

Example 2.5: Evaluate $\mathscr{L}\{e^{-2t}(3\cos 6t - 5\sin 6t)\}$
Now,

$$
\begin{aligned}
f(s) &= \mathscr{L}\{3\cos 6t - 5\sin 6t\} \\
&= \frac{3s}{s^2+36} - \frac{30}{s^2+36} \\
&= \frac{3s-30}{s^2+36}
\end{aligned}
$$

$$
\begin{aligned}
\therefore \mathscr{L}\{e^{-2t}(3\cos 6t - 5\sin 6t)\} &= f(s+2) \\
&= \frac{3(s+2)-30}{(s+2)^2+36} \\
&= \frac{3s-24}{s^2+4s+40}
\end{aligned}
$$

2.3 Second Translation Property

***Theorem 2.3.1* (Second Translation or Shifting Property):**
If

$$\mathscr{L}\{F(t)\} = f(s)$$

and

$$G(t) = \begin{cases} F(t-a) & \text{for } t > a \\ 0 & \text{for } t < a \end{cases}$$

then

$$\mathscr{L}\{G(t)\} = e^{-as} f(s)$$

Proof:

Let $G(t) = \begin{cases} F(t-a) & \text{for } t > a \\ 0 & \text{for } t < a \end{cases}$

Then

$$\begin{aligned} \mathscr{L}\{G(t)\} &= \int_0^\infty e^{-st} F(t-a)\, dt \\ &= \int_a^\infty e^{-st} F(t-a)\, dt \\ &= \int_0^\infty e^{-s(u+a)} F(u)\, du \quad \text{where } u = t-a \\ &= e^{-as} \int_0^\infty e^{-su} F(u)\, du \\ &= e^{-as} \mathscr{L}\{F(t)\} \\ &= e^{-as} f(s) \quad \square \end{aligned}$$

Example 2.6: Find $\mathscr{L}\{G(t)\}$ where $G(t) = \begin{cases} \cos\left(t - \frac{2\pi}{3}\right) & \text{for } t > \frac{2\pi}{3} \\ 0 & \text{for } t < \frac{2\pi}{3} \end{cases}$

$$\begin{aligned} \mathscr{L}\{G(t)\} &= e^{-\frac{2\pi}{3}s} \cdot \mathscr{L}\{\cos t\} \\ &= e^{-\frac{2\pi}{3}s} \cdot \frac{s}{s^2+1} \\ &= \frac{se^{-\frac{2\pi}{3}s}}{s^2+1} \end{aligned}$$

Example 2.7: Find $\mathscr{L}\{F(t)\}$, if $F(t) = \begin{cases} (t-1)^2 & \text{for } t > 1 \\ 0 & \text{for } t < 0 \end{cases}$

Let

$$G(t) = t^2$$

$$\therefore \mathscr{L}\{G(t)\} = \frac{2!}{s^3}$$

Now,

$$F(t) = \begin{cases} G(t-1) & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases}$$

$$\therefore \mathscr{L}\{F(t)\} = \frac{e^{-s} \cdot 2!}{s^3} = \frac{2e^{-s}}{s^3}$$

2.4 Change of Scale Property

***Theorem 2.4.1* (Change of Scale Property):**
If

$$\mathscr{L}\{F(t)\} = f(s)$$

then

$$\mathscr{L}\{F(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right)$$

Proof:
Let $G(t) = F(at)$, for $t > 0$
Then

$$\begin{aligned}
\mathscr{L}\{G(t)\} &= \int_0^\infty e^{-st} F(at)\, dt \\
&= \int_0^\infty e^{-\frac{s}{a}u} F(u)\, d(u/a) \quad \text{where } u = at \\
&= \frac{1}{a} \int_0^\infty e^{-\frac{s}{a}u} F(u)\, du \\
&= \frac{1}{a} \mathscr{L}\{F(t)\} \\
&= \frac{1}{a} f\left(\frac{s}{a}\right) \quad \square
\end{aligned}$$

Example 2.8: Evaluate $\mathscr{L}\{\sin 3t\}$

$$\begin{aligned}
\mathscr{L}\{\sin 3t\} &= \frac{1}{3} \cdot \frac{1}{\left(\frac{s}{3}\right)^2 + 1} \\
&= \frac{1}{3} \cdot \frac{3^2}{s^2 + 3^2} \\
&= \frac{3}{s^2 + 9}
\end{aligned}$$

Example 2.9: If $\mathscr{L}\{\frac{\sin t}{t}\} = \tan^{-1}\frac{1}{s} = f(s)$, then evaluate $\mathscr{L}\{\frac{\sin at}{t}\}$

$$\begin{aligned}
\mathscr{L}\{\frac{\sin at}{t}\} &= \mathscr{L}\left\{a \cdot \frac{\sin at}{t}\right\} \\
&= a \cdot \mathscr{L}\left\{\frac{\sin at}{at}\right\} \\
&= a \cdot \frac{1}{a} f\left(\frac{s}{a}\right) \\
&= \tan^{-1}\frac{a}{s}
\end{aligned}$$

2.5 Multiplication by t

***Theorem 2.5.1* (Multiplication by t^n):**
If

$$\mathscr{L}\{F(t)\} = f(s)$$

then

$$\mathscr{L}\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s)$$

Proof:
Let $G(t) = t^n F(t)$, for $t > 0$
Then

$$
\begin{aligned}
\mathscr{L}\{G(t)\} &= \int_0^\infty e^{-st} t^n F(t)\, dt \\
&= (-1)^n \int_0^\infty e^{-st} \frac{d^n}{ds^n} F(t)\, dt \\
&= (-1)^n \frac{d^n}{ds^n} \int_0^\infty e^{-st} F(t)\, dt \\
&= (-1)^n \frac{d^n}{ds^n} f(s) \quad \square
\end{aligned}
$$

Alternative Proof:
We have

$$f(s) = \int_0^\infty e^{-st} F(t)\, dt$$

Then by Leibnitz's rule for differentiating under the integral sign, we have

$$
\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}s} = f'(s) &= \frac{\mathrm{d}}{\mathrm{d}s} \int_0^\infty e^{-st} F(t)\, dt \\
&= \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}s} \left(e^{-st} F(t)\right) dt \\
&= \int_0^\infty e^{-st} \left(-te^{-st} F(t) + F'(t)\right) dt \\
&= -\int_0^\infty e^{-st} \{tF(t)\}\, dt \\
&= -\mathscr{L}\{tF(t)\}
\end{aligned}
$$

$$\therefore \mathscr{L}\{tF(t)\} = -\frac{\mathrm{d}}{\mathrm{d}s} f(s) = f'(s)$$

This proves the theorem for $n = 1$.
To establish the theorem in general, we use mathematical induction. Suppose that the theorem is true for $n = k$, i.e

$$\int_0^\infty e^{-st} \{t^k F(t)\}\, dt = (-1)^k f^{(k)}(s) \tag{2.5.1}$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\int_0^\infty e^{-st} \{t^k F(t)\}\, dt\right] = (-1)^k f^{(k+1)}(s)$$

Or by Leibnitz's rule,

$$-\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}s} \left(e^{-st} \{t^{k+1} F(t)\}\right) dt = (-1)^{(k)} f^{(k+1)}(s)$$

i.e

$$\int_0^\infty e^{-st} \{t^{k+1} F(t)\}\, dt = (-1)^{(k+1)} f^{(k+1)}(s) \tag{2.5.2}$$

If follows that if (2.5.1) is true for $n = k$, then (2.5.2) is true for $n = k + 1$. Since (2.5.1) is true for $n = 1$, it follows that (2.5.1) is true for all positive integers n. $\square$

Example 2.10: Find $\mathscr{L}\{t^2 \cos at\}$

$$
\begin{aligned}
\mathscr{L}\{t^2 \cos at\} &= (-1)^2 \cdot \frac{d^2}{dx^2} \left(\frac{s}{s^2 + a^2}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{s^2 + a^2 - 2s^2}{(s^2 + a^2)^2}\right] \\
&= \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{a^2 - s^2}{(s^2 + a^2)^2}\right] \\
&= \frac{(s^2 + a^2)^2(-2s) - (-s^2 + a^2)2(s^2 + a^2) \cdot 2s}{(s^2 + a^2)^4} \\
&= \frac{2s^3 - 6a^2 s}{(s^2 + a^2)^3}
\end{aligned}
$$

2.6 Division by t

***Theorem 2.6.1* (Division by t):**
If

$$\mathscr{L}\{F(t)\} = f(s)$$

then

$$\mathscr{L}\left\{\frac{F(t)}{t}\right\} = \int_s^{\infty} f(x)\, dx$$

Proof:
Let $G(t) = \dfrac{F(t)}{t}$, for $t > 0$. Then $F(t) = tG(t)$. Taking the Laplace Transform of both sides, we get

$$\begin{aligned}\mathscr{L}\{F(t)\} &= \mathscr{L}\{tG(t)\} \\ f(s) &= -\frac{\mathrm{d}}{\mathrm{d}s}\mathscr{L}\{G(t)\} = -\frac{\mathrm{d}}{\mathrm{d}s}g(s)\end{aligned}$$

Then integrating, we have

$$\begin{aligned}g(s) &= -\int_{\infty}^{s} f(u)\, du \\ \mathscr{L}\{G(t)\} &= \int_s^{\infty} f(u)\, du \\ \mathscr{L}\left\{\frac{F(t)}{t}\right\} &= \int_s^{\infty} f(u)\, du \quad \square\end{aligned}$$

Example 2.11: Find $\mathscr{L}\{\frac{\sin t}{t}\}$

$$\begin{aligned}\mathscr{L}\left\{\frac{\sin t}{t}\right\} &= \int_s^{\infty} \frac{1}{s^2+1}\, du \\ &= \tan^{-1} u\Big|_s^{\infty} \\ &= \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s \\ &= \tan^{-1}\frac{1}{s}\end{aligned}$$

2.7 Laplace Transform of Integral

***Theorem 2.7.1* (Laplace transform of Integral):**
If

$$\mathscr{L}\{F(t)\} = f(s)$$

then

$$\mathscr{L}\left\{\int_0^t F(x)\, dx\right\} = \frac{1}{s} f(s)$$

Proof:
Let $G(t) = \displaystyle\int_0^t F(x)dx$, for $t > 0$. Then $G'(t) = F(t)$ and $G(0) = 0$. Taking the Laplace Transform of both sides, we have

$$\begin{aligned}\mathscr{L}\{G'(t)\} &= \mathscr{L}\{F(t)\} \\ s\mathscr{L}\{G(t)\} - G(0) &= f(s) \\ s\mathscr{L}\{G(t)\} &= f(s) \\ \mathscr{L}\{G(t)\} &= \frac{f(s)}{s} \\ \mathscr{L}\left\{\int_0^t F(u)\, du\right\} &= \frac{f(s)}{s} \quad \square\end{aligned}$$

Example 2.12: Evaluate $\mathscr{L}\{\int_0^t \frac{\sin u}{u}\, du\}$

$$\mathscr{L}\left\{\frac{\sin u}{u}\right\} = \tan^{-1}\frac{1}{s}$$

$$\mathscr{L}\left\{\int_0^t \frac{\sin u}{u}\, du\right\} = \frac{f(s)}{s} = \frac{1}{s}\tan^{-1}\frac{1}{s}$$

Example 2.13: Evaluate $\mathscr{L}\{\int_0^t \frac{\sin u}{u}\, du\}$

Let $F(t) = \frac{\sin t}{t}$

$$\begin{aligned}\mathscr{L}\left\{\frac{\sin t}{t}\right\} &= \int_s^\infty \frac{1}{u^2+1}\, du \\ &= \tan^{-1} u\Big|_s^\infty \\ &= \tan^{-1}\frac{1}{s}\end{aligned}$$

$$\therefore \mathscr{L}\left\{\int_0^t \frac{\sin u}{u}\, du\right\} = \frac{1}{s}\cdot\tan^{-1}\frac{1}{s}$$

Example 2.14: Evaluate $\mathscr{L}\{\int_0^t \sin 2u\, du\}$

$$\mathscr{L}\{\sin 2t\} = \frac{2}{s^2+4}$$

$$\therefore \mathscr{L}\left\{\int_0^t \sin 2u\, du\right\} = \frac{2}{s^3+4s}$$

2.8 Laplace Transform of Periodic Functions

***Theorem 2.8.1* (Periodic functions):**

If

$$\mathscr{L}\{F(t)\} = f(s)$$

then

$$\mathscr{L}\{F(t)\} = \frac{1}{1-e^{-Ts}}\int_0^T e^{-st}F(t)\, dt$$

where T is the period of $F(t)$.

Proof:

Let $F(t)$ has period T. Then $F(t) = F(t+T)$ for all t. Then

$$\begin{aligned}\mathscr{L}\{F(t)\} &= \int_0^\infty e^{-st}F(t)\, dt \\ &= \int_0^T e^{-st}F(t)\, dt + \int_T^{2T} e^{-st}F(t)\, dt + \int_{2T}^{3T} e^{-st}F(t)\, dt + \cdots \\ &= \int_0^T e^{-st}F(t)\, dt + \int_0^T e^{-s(t+T)}F(t+T)\, dt + \int_0^T e^{-s(t+2T)}F(t+2T)\, dt + \cdots \\ &= \int_0^T e^{-st}F(t)\, dt + e^{-sT}\int_0^T e^{-st}F(t)\, dt + e^{-2sT}\int_0^T e^{-st}F(t)\, dt + \cdots \\ &= \left[1 + e^{-sT} + e^{-2sT} + \cdots\right]\int_0^T e^{-st}F(t)\, dt \\ &= \frac{1}{1-e^{-sT}}\int_0^T e^{-st}F(t)\, dt \quad \square\end{aligned}$$

Note:-

Sum of an infinite series $1 + r + r^2 + \cdots = \dfrac{1}{1-r}$ for $|r| < 1$.

Example 2.15: Find $\mathscr{L}\{F(t)\}$ for

$$F(t) = \begin{cases} \sin t & \textbf{for } 0 < t < \pi \\ 0 & \textbf{for } \pi < t < 2\pi \end{cases}$$

$$\begin{aligned} \mathscr{L}\{F(t)\} &= \frac{1}{1-e^{-2\pi s}} \int_0^{2\pi} e^{-st} \sin t \, dt \\ &= \frac{1}{1-e^{-2\pi s}} \int_0^{\pi} e^{-st} \sin t \, dt \end{aligned}$$

Example 2.16: Find $\mathscr{L}\{F(t)\}$ for

$$F(t) = \begin{cases} t & \textbf{for } 0 < t < 1 \\ 0 & \textbf{for } 1 < t < 2 \end{cases}$$

$$\begin{aligned} \mathscr{L}\{F(t)\} &= \frac{1}{1-e^{-2s}} \int_0^1 e^{-2t} t \, dt \\ &= \frac{1}{1-e^{-2s}} \left[-t\frac{e^{-2t}}{2} + \frac{1}{2} \int e^{-2t} \, dt \right]_0^1 \\ &= \frac{1}{1-e^{-2s}} \left\{ -\frac{e^{-2}}{2} - \frac{1}{4}(e^{-2} - 1) \right\} \\ &= \frac{1-3e^{-2}}{4(1-e^{-2})} \end{aligned}$$

2.9 Laplace Transform of Derivatives

Theorem 2.9.1* (Laplace transform of derivatives)*:
If

$$\mathscr{L}\{F(t)\} = f(s)$$

then

$$\mathscr{L}\{F'(t)\} = sf(s) - F(0)$$

$$\mathscr{L}\{F''(t)\} = s^2 F(s) - sF(0) - F'(0)$$

$$\mathscr{L}\{F'''(t)\} = s^3 f(s) - s^2 F(0) - sF'(0) - F''(0)$$

$$\boxed{\mathscr{L}\{F^{(n)}(t)\} = s^n f(s) - s^{n-1}F(0) - s^{n-2}F'(0) - \cdots - F^{(n-1)}(0)}$$

$$\boxed{\mathscr{L}\{F^{(n)}(t)\} = s^n f(s) - \sum_{i=n-1}^{0} \sum_{j=0}^{n-1} s^i F^{(j)}(0)}$$

Proof:
Using integration by parts, we have

$$\begin{aligned} \mathscr{L}\{F'(t)\} &= \int_0^\infty e^{-st} F'(t) \, dt \\ &= e^{-st} F(t) \Big|_0^\infty + s \int_0^\infty e^{-st} F(t) \, dt \\ &= -F(0) + s \int_0^\infty e^{-st} F(t) \, dt \\ &= sf(s) - F(0) \quad \square \end{aligned}$$

Similarly,

$$\begin{aligned}
\mathscr{L}\{F''(t)\} &= s\mathscr{L}\{F'(t)\} - F'(0) \\
&= s\left[sf(s) - F(0)\right] - F'(0) \\
&= s^2 f(s) - sF(0) - F'(0)
\end{aligned}$$

Thus using mathematical induction, we get

$$\begin{aligned}
\mathscr{L}\{F^{(n)}(t)\} &= s\mathscr{L}\{F^{(n-1)}(t)\} - F^{(n-1)}(0) \\
&= s\left[s\mathscr{L}\{F^{(n-2)}(t)\} - F^{(n-2)}(0)\right] - F^{(n-1)}(0) \\
&= s^2\mathscr{L}\{F^{(n-2)}(t)\} - sF^{(n-2)}(0) - F^{(n-1)}(0) \\
&= s^2\left[s\mathscr{L}\{F^{(n-3)}(t)\} - F^{(n-3)}(0)\right] - sF^{(n-2)}(0) - F^{(n-1)}(0) \\
&= s^3\mathscr{L}\{F^{(n-3)}(t)\} - s^2F^{(n-3)}(0) - sF^{(n-2)}(0) - F^{(n-1)}(0) \\
&= \cdots \\
&= s^n\mathscr{L}\{F(t)\} - s^{n-1}F(0) - s^{n-2}F'(0) - \cdots - F^{(n-1)}(0) \\
&= s^n f(s) - \sum_{i=n-1}^{0} \sum_{j=0}^{n-1} s^i F^{(j)}(0) \quad \square
\end{aligned}$$

3 Inverse Laplace Transform

3.1 Definition and Existence

Definition 3.1.1: Inverse Laplace Transform

If the Laplace Transform of a function $F(t)$ is $f(s)$, i.e

$$\mathscr{L}\{F(t)\} = f(s)$$

then the Inverse Laplace Transform is defined as

$$\mathscr{L}^{-1}\{f(s)\} = F(t)$$

3.2 Some Functions and their Inverse Laplace Transforms

$f(s)$	$\mathscr{L}^{-1}\{f(s)\} = F(t)$	$f(s)$	$\mathscr{L}^{-1}\{f(s)\} = F(t)$
1	$\delta(t)$	$\frac{n}{s}$	n
$\frac{1}{s^2}$	t	$\frac{1}{s^n}$	$\frac{t^{n-1}}{(n-1)!}$
$\frac{1}{s-a}$	e^{at}	$\frac{1}{s+a}$	e^{-at}
$\frac{1}{s^2+a^2}$	$\frac{\sin at}{a}$	$\frac{1}{s^2-a^2}$	$\frac{\sinh at}{a}$
$\frac{s}{s^2+a^2}$	$\cos at$	$\frac{s}{s^2-a^2}$	$\cosh at$

Table 2: Functions and their Inverse Laplace Transform

Note:-

The Unit Impulse Function or Dirac Delta Function :

Consider the function

$$F_\epsilon(t) = \begin{cases} \frac{1}{\epsilon} & \text{for } 0 \le t \le \epsilon \\ 0 & \text{for } t > \epsilon \end{cases}$$

where $\epsilon > 0$. Then

$$\int_0^\infty F_\epsilon(t)\, dt = 1$$

This idea has led some engineers and physicists to think of a limiting function, denoted by $\delta(t)$, approached by $F_\epsilon(t)$ as $\epsilon \to 0$. This limiting function they have called the unit impulse function or Dirac delta function.
Some properties of $\delta(t)$ are

1. $\displaystyle\int_0^\infty \delta(t)\, dt = 1$

2. $\displaystyle\int_0^\infty \delta(t)G(t)\, dt = G(0)$

3. $\displaystyle\int_0^\infty \delta(t-a)G(t)\, dt = G(a)$

Although mathematically speaking such a function does not exist, manipulations or operations using it can be made rigorous.

4 Basic Properties of Inverse Laplace Transform

4.1 Linearity Property

***Theorem 4.1.1* (Linearity Property):**
If

$$\mathscr{L}^{-1}\{f(s)\} = F(t) \quad \text{and} \quad \mathscr{L}^{-1}\{g(s)\} = G(t)$$

Then,

$$\mathscr{L}^{-1}\{c_1 f(s) + c_2 g(s)\} = c_1\mathscr{L}^{-1}\{f(s)\} + c_2\mathscr{L}^{-1}\{g(s)\}$$

Example 4.1: Evaluate $\mathscr{L}^{-1}\left\{\frac{4}{s-2} - \frac{3s}{s^2+16} + \frac{5}{s^2+4}\right\}$

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{4}{s-2} - \frac{3s}{s^2+16} + \frac{5}{s^2+4}\right\} &= 4\mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\} - 3\mathscr{L}^{-1}\left\{\frac{s}{s^2+4^2}\right\} + 5\mathscr{L}^{-1}\left\{\frac{1}{s^2+2^2}\right\} \\ &= 4e^{2t} - 3\cos 4t + \frac{5}{2}\sin 2t\end{aligned}$$

4.2 First Translation or Shifting Property

***Theorem 4.2.1* (First Translation or Shifting Property):**
If

$$\mathscr{L}^{-1}\{f(s)\} = F(t)$$

then

$$\mathscr{L}^{-1}\{f(s-a)\} = e^{at}F(t)$$

Example 4.2: Evaluate $\mathscr{L}^{-1}\left\{\dfrac{6s-4}{s^2-4s+20}\right\}$

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{6s-4}{s^2-4s+20}\right\} &= \mathscr{L}^{-1}\left\{\frac{6(s-2)+8}{(s-2)^2+16}\right\} \\ &= 6\mathscr{L}^{-1}\left\{\frac{(s-2)}{(s-2)^2+16}\right\} + 2\mathscr{L}^{-1}\left\{\frac{4}{(s-2)^2+16}\right\} \\ &= 6e^{2t}\cos 4t + 2e^{2t}\sin 4t\end{aligned}$$

4.3 Second Translation or Shifting Property

***Theorem 4.3.1* (Second Translation or Shifting Property):**
If

$$\mathscr{L}^{-1}\{f(s)\} = F(t)$$

then

$$\mathscr{L}^{-1}\{e^{-as}f(s)\} = \begin{cases} F(t-a) & \text{for } t > a \\ 0 & \text{for } t < a \end{cases}$$

Example 4.3: Evaluate $\dfrac{e^{-5t}}{(s-2)^4}$

Here

$$\mathscr{L}^{-1}\left\{\frac{1}{(s-2)^4}\right\} = e^{2t}\frac{t^3}{3!} = \frac{e^{2t}}{6}t^3$$

$$\therefore \mathscr{L}^{-1}\left\{\frac{e^{-5s}}{(s-2)^4}\right\} = \frac{e^{2(t-5)}}{6}(t-5)^3, \text{ when } t > 5$$

$$\mathscr{L}^{-1}\left\{\frac{e^{-5s}}{(s-2)^4}\right\} = 0, \text{ when } t < 5$$

4.4 Change of Scale Property

Theorem 4.4.1* (Change of Scale Property)*:
If

$$\mathcal{L}^{-1}\{f(s)\} = F(t)$$

then

$$\mathcal{L}^{-1}\{f(ks)\} = \frac{1}{k}F\left(\frac{t}{k}\right)$$

Example 4.4: Find $\mathcal{L}^{-1}\left\{\frac{2s}{(2s)^2+16}\right\}$

Since

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+4^2}\right\} = \cos 4t$$

we have

$$\mathcal{L}^{-1}\left\{\frac{2s}{(2s)^2+16}\right\} = \frac{1}{2}\cos\frac{4t}{2} = \frac{1}{2}\cos 2t$$

4.5 Inverse Laplace Transform of Derivatives

Theorem 4.5.1* (Inverse Laplace Transform of Derivatives)*:
If

$$\mathcal{L}^{-1}\{f(s)\} = F(t)$$

then

$$\mathcal{L}^{-1}\{f^{(n)}(s)\} = \mathcal{L}^{-1}\left\{\frac{\mathrm{d}^n}{\mathrm{d}s^n}f(s)\right\} = (-1)^n t^n F(t)$$

Example 4.5: Evaluate $\mathcal{L}^{-1}\left\{\ln\left(1+\frac{1}{s^2}\right)\right\}$

Here

$$\begin{aligned}
f(s) &= \ln\left(1+\frac{1}{s^2}\right) = \mathcal{L}\{F(t)\} \\
f'(s) &= \frac{-\frac{2}{s^3}}{1+\frac{1}{s^2}} \\
&= -2\left\{\frac{1}{s(s^2+1)}\right\} \\
&= -2\left(\frac{1}{s} - \frac{s}{s^2+1}\right)
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}^{-1}\{f'(s)\} &= -2(1-\cos t) \\
-tF(t) &= -2(1-\cos t) \\
F(t) &= \frac{2(1-\cos t)}{t} \\
\therefore \mathcal{L}^{-1}\left\{\ln\left(1+\frac{1}{s^2}\right)\right\} &= \frac{2(1-\cos t)}{t}
\end{aligned}$$

4.6 Inverse Laplace Transform of Integrals

***Theorem 4.6.1* (Inverse Laplace Transform of Integrals):**
If

$$\mathscr{L}^{-1}\{f(s)\} = F(t)$$

then

$$\mathscr{L}^{-1}\left\{\int_s^\infty f(u)\,du\right\} = \frac{F(t)}{t}$$

Example 4.6: Find $\mathscr{L}^{-1}\left\{\int_s^\infty \left(\frac{1}{u} - \frac{1}{u+1}\right) du\right\}$

Let

$$f(s) = \frac{1}{s} - \frac{1}{s+1}$$

$$\begin{aligned}
\mathscr{L}^{-1}\{f(s)\} &= \mathscr{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+1}\right\} \\
&= 1 - e^{-t} = F(t) \\
\therefore \mathscr{L}^{-1}\left\{\int_s^\infty f(u)\,du\right\} &= \frac{F(t)}{t} \\
&= \frac{1-e^{-t}}{t}
\end{aligned}$$

4.7 Division by s

***Theorem 4.7.1* (Division by s):**
If

$$\mathscr{L}^{-1}\{f(s)\} = F(t)$$

then

$$\mathscr{L}^{-1}\left\{\frac{f(s)}{s}\right\} = \int_0^t F(u)\,du$$

Example 4.7: Evaluate $\mathscr{L}^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\}$

Let

$$f(s) = \frac{1}{s^2+1}$$

$$\begin{aligned}
\therefore \mathscr{L}^{-1}\{f(s)\} &= \sin t \\
\therefore \mathscr{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} &= \int_0^t \sin u\,du && = 1 - \cos t \\
\therefore \mathscr{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} &= \int_0^t (1-\cos u)\,du && = t - \sin t \\
\therefore \mathscr{L}^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\} &= \int_0^t (u - \sin u)\,du && = \frac{t^2}{2} + \cos t - 1
\end{aligned}$$

Alternative approach:
Let

$$f(s) = \frac{1}{s^4+s^2} = \frac{1}{s^2} - \frac{1}{s^2+1}$$

$$
\begin{aligned}
\mathscr{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} &= \mathscr{L}^{-1}\left\{\frac{1}{s^2}-\frac{1}{s^2+1}\right\} \\
&= t-\sin t \\
&= F(t) \\
\therefore \mathscr{L}^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\} &= \int_0^t F(u)\,du \\
&= \int_0^t (u-\sin u)\,du \\
&= \frac{t^2}{2}+\cos t-1
\end{aligned}
$$

4.8 Multiplication by s^n

***Theorem 4.8.1* (Multiplication by s^n):**
If

$$\mathscr{L}^{-1}\{f(s)\} = F(t)$$

then

$$\mathscr{L}^{-1}\{sf(s)\} = F'(t) + F(0)\delta(t)$$

Example 4.8: Find $\mathscr{L}^{-1}\left\{\frac{s}{s^2+1}\right\}$

Since

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t \quad \text{and} \quad \sin 0 = 0$$

then

$$\mathscr{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \frac{\mathrm{d}}{\mathrm{d}t}\sin t = \cos t$$

4.9 The Convolution Theorem

***Theorem 4.9.1* (The Convolution Theorem):**
If

$$\mathscr{L}^{-1}\{f(s)\} = F(t) \quad \textit{and} \quad \mathscr{L}^{-1}\{g(s)\} = G(t)$$

then

$$\mathscr{L}^{-1}\{f(s)g(s)\} = \int_0^t F(u)G(t-u)\,du = F*G$$

$$G*F = \int_0^t G(u)F(t-u)\,du = F*G$$

Example 4.9: Find $\mathscr{L}^{-1}\left\{\frac{1}{s^2(s^2+1)^2}\right\}$

Here,

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\} = t = F(t)$$

$$\mathscr{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = e^{-t}t = G(t)$$

$$
\begin{aligned}
\therefore \mathscr{L}^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\} &= G * F \\
&= \int_0^t ue^{-u}(t-u)\,du \\
&= \int_0^t (ut-u^2)e^{-u}\,du \\
&= -(ut-u^2)\cdot e^{-u}\Big|_0^t + \int_0^t (t-2u)e^{-u}\,du
\end{aligned}
$$

Example 4.10: Find $\mathscr{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$

Let

$$
f(s) = \frac{s}{s^2+a^2} \qquad \text{and} \qquad g(s) = \frac{1}{s^2+a^2}
$$

$$
\mathscr{L}^{-1}\{f(s)\} = \cos at \qquad \text{and} \qquad \mathscr{L}^{-1}\{g(s)\} = \frac{\sin at}{a}
$$

$$
\begin{aligned}
F * G &= \frac{1}{a}\int_0^t \cos au \cdot \sin au(t-u)\,du \\
&= \frac{1}{a}\int_0^t \cos au(\sin at\cos au - \cos at\sin au)\,du \\
&= \frac{1}{a}\int_0^t \sin at\cos^2 au\,du - \frac{1}{a}\int_0^t \sin au\cos au\,du \\
&= \frac{1}{2a}\sin at\int_0^t (\cos 2au + 1)\,du - \frac{1}{2a}\cos at\int_0^t \sin 2au\,du \\
&= \frac{1}{2a}\left[-\frac{1}{2}\sin 2au + u\right]_0^t - \frac{1}{2a}\left[\frac{1}{2}\cos 2au\right]_0^t \\
&= -\frac{1}{2a}\sin at\cdot\sin 2at - \frac{1}{2a}\cos at\cdot\frac{1}{2}(\cos 2at - 1) + \frac{1}{2a}\sin at\cdot t \\
&= \frac{1}{4a}\big[\cos at - \cos(at-2at)\big] + \frac{t\sin at}{2a} \\
&= \frac{1}{4a}[\cos at - \cos at] + \frac{t\sin at}{2a} \\
&= \frac{t\sin at}{2a}
\end{aligned}
$$

4.10 Methods of Finding Inverse Laplace Transforms

4.10.1 Partial Franctions Method

Theorem 4.10.1* (Partial Fractions Method)*:

Any rational function $\frac{P(s)}{Q(s)}$ where $P(s)$ and $Q(s)$ are polynomials, with degree of $P(s)$ less than that of $Q(s)$, can be written as the sum of rational functions (partial fractions) having the form

$$
\frac{A}{(as+b)^r} \quad \text{or} \quad \frac{As+B}{(as^2+bs+c)^r}
$$

where $r \in \mathbb{N}$ and $A, B, a, b, c \in \mathbb{R}$.
By finding the inverse Laplace transform of each of the partial fractions, we can find $\mathscr{L}^{-1}\left\{\frac{P(s)}{Q(s)}\right\}$

Example 4.11: Find $\mathscr{L}^{-1}\left\{\frac{5^2-15s-11}{(s+1)(s-2)^2}\right\}$

$$\frac{5^2 - 15s - 11}{(s+1)(s-2)^2} = \frac{-\frac{1}{3}}{s+1} - \frac{7}{(s-2)^3} + \frac{4}{(s-2)^2} + \frac{\frac{1}{3}}{s-2}$$

$$\therefore \mathscr{L}^{-1}\left\{\frac{5^2 - 15s - 11}{(s+1)(s-2)^2}\right\} = -\frac{1}{e}e^{-t} - 7\frac{t^2}{2}e^{2t} + 4te^{2t} + \frac{1}{3}e^{2t}$$

4.10.2 Series Method

***Theorem 4.10.2* (Series Method):**
If $f(s)$ has a series expansion in inverse powers of s given by

$$f(s) = \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{a_2}{s^3} + \cdots = \sum_{n=0}^{\infty} a_n s^{-n}$$

then under suitable conditions we can invert them by term to obtain

$$F(t) = a_0 + a_1 t + a_2\frac{t^2}{2!} + a_3\frac{t^3}{3!} + \cdots = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}$$