

Fourier Analysis

Turja Roy

ID: 2108052

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1 Introduction

1.1 Periodic Functions

Definition 1.1.1: Periodic Functions

A function $f(x)$ is said to be have a *period* P or to be *periodic* with period P if for all x , $f(x + P) = f(x)$ where P is a positive constant. The least value of $P > 0$ is called the *least period* or simply the *period* of $f(x)$.

Example 1.1: Some examples of periodic functions

1. $\sin x$ has periods $2\pi, 4\pi, 6\pi, \dots$ and $-\pi, -3\pi, -5\pi, \dots$ and hence the least period is 2π .
2. $\cos x$ has the least period 2π .
3. $\tan x$ has the least period π .

Some other examples:

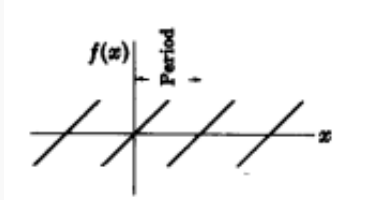


Figure 1.1.1

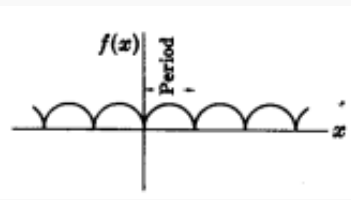


Figure 1.1.2

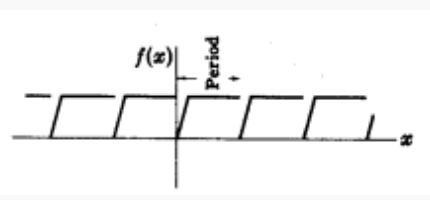


Figure 1.1.3

1.2 Piecewise Continuous Functions

Definition 1.2.1: Piecewise Continuous Functions

A function $f(x)$ is said to be *piecewise continuous* in the interval $[a, b]$ if $f(x)$ is continuous in the interval (a, b) and has a finite number of finite discontinuities in the interval $[a, b]$.

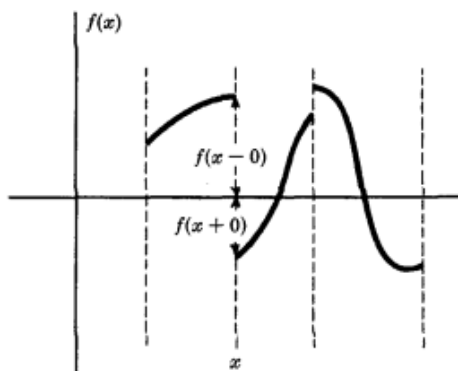


Figure 1.2.1

The right-hand limit of $f(x)$ is often denoted by $\lim_{\epsilon \rightarrow 0} f(x + \epsilon) = f(x + 0)$, where $\epsilon > 0$.

Similarly, the left-hand limit of $f(x)$ is denoted by $\lim_{\epsilon \rightarrow 0} f(x - \epsilon) = f(x - 0)$, where $\epsilon > 0$. The values of $f(x + 0)$ and $f(x - 0)$ at the point x in (1.2.1) are as indicated.

2 Fourier Expansion

2.1 Definition

Definition 2.1.1: Fourier Expansion

Let $f(x)$ be defined in the interval $(-L, L)$ and determined outside of this interval by $f(x+2L) = f(x)$, i.e. assume that $f(x)$ has the period $2L$. The *Fourier series* or *Fourier expansion* corresponding to $f(x)$ is defined to be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \left(\frac{n\pi x}{L} \right) + b_n \sin \left(\frac{n\pi x}{L} \right) \right] \quad (2.1.1)$$

where the *Fourier coefficients* a_n and b_n are given by

$$\begin{cases} a_0 = \frac{1}{L} \int_{-L}^L f(x) dx \\ a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \left(\frac{n\pi x}{L} \right) dx \\ b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \left(\frac{n\pi x}{L} \right) dx \end{cases} \quad n = 0, 1, 2, \dots \quad (2.1.2)$$

2.2 Some pre-derivations

$$\begin{aligned} I &= \int_{-L}^L \sin^2 \frac{n\pi x}{L} dx \\ &= \sin \frac{n\pi x}{L} \cdot \frac{L}{n\pi} (\cos n\pi - \cos n\pi) + \frac{n\pi}{L} \cdot \frac{L}{n\pi} \int_{-L}^L \cos \frac{n\pi x}{L} \cos \frac{n\pi x}{L} dx \\ &= \int_{-L}^L \cos^2 \frac{n\pi x}{L} dx \\ &= \frac{1}{2} \int_{-L}^L \left(\cos \frac{2n\pi x}{L} + 1 \right) dx \\ &= \frac{1}{2} \int_{-L}^L \cos \frac{2n\pi x}{L} dx + \frac{1}{2} \int_{-L}^L dx \\ &= 0 + \frac{1}{2} \cdot 2L \\ &= L \end{aligned}$$

$$\begin{aligned}
I_1 &= \int_{-L}^L \cos \frac{m\pi x}{L} \cos \frac{n\pi x}{L} dx \quad [m \neq 0] \\
&= \cos \frac{m\pi x}{L} \int_{-L}^L \cos \frac{n\pi x}{L} dx + \frac{m}{n} \int_{-L}^L \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L} dx \\
&= \frac{L}{n\pi} \cos \frac{m\pi x}{L} (\sin n\pi + \sin n\pi) + \frac{m}{n} I_2 \\
&= 0 + \frac{m}{n} \left[\int_{-L}^L \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L} dx \right] \\
&= \frac{m}{n} \left[\sin \frac{m\pi x}{L} \int_{-L}^L \sin \frac{n\pi x}{L} dx + \frac{m}{n} \int_{-L}^L \cos \frac{m\pi x}{L} \cos \frac{n\pi x}{L} dx \right] \\
&= \frac{m}{n} \left[\frac{L}{n\pi} \sin \frac{m\pi x}{L} (-\cos n\pi + \cos n\pi) + \frac{m}{n} \right] \\
&= 0 + \frac{m^2}{n^2} I_1 \\
I_1 &= 0 = I_2
\end{aligned}$$

To summarize, we have

$$\int_{-L}^L \sin^2 \frac{n\pi x}{L} dx = \int_{-L}^L \cos^2 \frac{n\pi x}{L} dx = L \quad (2.2.1)$$

$$\int_{-L}^L \cos mx dx = \int_{-L}^L \sin mx dx = 0 \quad (2.2.2)$$

$$\int_{-L}^L \sin \frac{n\pi x}{L} \cos \frac{n\pi x}{L} dx = 0 \quad (2.2.3)$$

$$\int_{-L}^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx = \int_{-L}^L \cos \frac{n\pi x}{L} \cos \frac{m\pi x}{L} dx = 0 \quad [m \neq n] \quad (2.2.4)$$

2.3 Derivation of a_0

Taking integral on both sides of (2.1.1) from $-L$ to L , we get

$$\begin{aligned}
\int_{-L}^L f(x) dx &= \frac{a_0}{2} \int_{-L}^L dx + \sum_{n=1}^{\infty} \int_{-L}^L \left[a_n \cos \frac{m\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right] dx \\
&= \frac{a_0}{2} \cdot 2L \quad [\text{All the other terms are 0 according to equation (2.2.2)}]
\end{aligned}$$

$$\boxed{a_0 = \frac{1}{L} \int_{-L}^L f(x) dx}$$

2.4 Derivation of a_n

Multiplying both sides of (2.1.1) by $\cos \frac{m\pi x}{L}$ and integrating from $-L$ to L , we get

$$\begin{aligned}
\int_{-L}^L f(x) \cos \frac{m\pi x}{L} dx &= \frac{a_0}{2} \int_{-L}^L \cos \frac{m\pi x}{L} dx \\
&\quad + \sum_{n=1}^{\infty} \int_{-L}^L \left[a_n \cos \frac{m\pi x}{L} \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \cos \frac{m\pi x}{L} \right] dx \\
&= a_n \int_{-L}^L \cos^2 \frac{m\pi x}{L} dx \\
&= a_n \cdot L \quad [\text{All the other terms are 0 according to equation (2.1.2)}]
\end{aligned}$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{m\pi x}{L} dx$$

2.5 Derivation of b_n

Multiplying both sides of (1) by $\sin \frac{m\pi x}{L}$ and integrating from $-L$ to L , we get

$$\begin{aligned}
\int_{-L}^L f(x) \sin \frac{m\pi x}{L} dx &= \frac{a_0}{2} \int_{-L}^L \sin \frac{m\pi x}{L} dx \\
&\quad + \sum_{n=1}^{\infty} \int_{-L}^L \left[a_n \sin \frac{m\pi x}{L} \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} \right] dx \\
&= a_n \int_{-L}^L \sin^2 \frac{m\pi x}{L} dx \\
&= a_n \cdot L \quad [\text{All the other terms are 0 according to equation (2.1.2)}]
\end{aligned}$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{m\pi x}{L} dx$$

2.6 Examples

Example 2.1: Obtain the fourier series for $f(x) = x - x^2$ in the interval $(-\pi, \pi)$ and hence evaluate

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (x - x^2) dx = \frac{1}{\pi} \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_{-\pi}^{\pi} = -\frac{2\pi^2}{3}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{\pi} \int_{-\pi}^{\pi} (x - x^2) \cos nx dx \\
&= \frac{1}{\pi} \left[\int_{-\pi}^{\pi} x \cos nx dx - \int_{-\pi}^{\pi} x^2 \cos nx dx \right] \\
&= -\frac{2}{\pi} \int_0^{\pi} x^2 \cos nx dx \\
&= -\frac{2}{\pi} \left[\frac{x^2}{n} \sin nx - \frac{2}{n} \int x \sin nx dx \right]_0^{\pi} \\
&= \frac{4}{n\pi} \left[-\frac{x}{n} \cos nx + \frac{1}{n^2} \sin nx \right]_0^{\pi} \\
&= -\frac{4}{n^2} (-1)^n
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} (x - x^2) \sin nx dx \\
&= \frac{1}{\pi} \left[\int_{-\pi}^{\pi} x \sin nx dx - \int_{-\pi}^{\pi} x^2 \sin nx dx \right] \\
&= \frac{2}{\pi} \int_0^{\pi} x \sin nx dx \\
&= \frac{2}{\pi} \left[-\frac{x}{n} \cos nx + \frac{1}{n^2} \sin nx \right]_0^{\pi} \\
&= -\frac{2}{n} (-1)^n
\end{aligned}$$

$$\therefore f(x) = -\frac{\pi^2}{3} - \sum_{n=1}^{\infty} \left(\frac{4}{n^2} (-1)^n \cos nx + \frac{2}{n} (-1)^n \sin nx \right)$$

For $x = 0$, we get

$$0 = -\frac{\pi^2}{3} - \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n$$

$$\frac{\pi^2}{12} = -\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$$

$$\boxed{\therefore 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}}$$

Example 2.2: Find a fourier series to represent the function $f(x) = e^x$ for $-\pi < x < \pi$ and hence derive a series for $\frac{\pi}{\sinh \pi}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x dx = \frac{1}{\pi} e^x \Big|_{-\pi}^{\pi} = \frac{e^{\pi} - e^{-\pi}}{\pi} = \frac{2 \sinh x}{\pi}$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \cos nx \, dx \\ &= \frac{1}{\pi} \left[e^x \frac{\sin nx}{n} - \frac{1}{n} \int e^x \sin nx \, dx \right]_{-\pi}^{\pi} \\ &= -\frac{1}{n\pi} \left[-e^x \frac{\cos nx}{n} + \frac{1}{n} \int e^x \cos nx \, dx \right]_{-\pi}^{\pi} \end{aligned}$$

$$a_n \left(1 + \frac{1}{n^2} \right) = \frac{(-1)^n}{n^2 \pi} (e^{\pi} - e^{-\pi})$$

$$a_n = \frac{(-1)^n}{n^2 \pi} (e^{\pi} - e^{-\pi}) \left(1 + \frac{1}{n^2} \right)^{-1}$$

$$a_n = 2 \frac{(-1)^n}{n^2 \pi} \sinh x \left(1 + \frac{1}{n^2} \right)^{-1}$$

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \sin nx \, dx \\ &= \frac{1}{\pi} \left[-e^x \frac{\cos nx}{n} + \frac{1}{n} \int e^x \cos nx \, dx \right]_{-\pi}^{\pi} \\ &= \frac{1}{\pi} \left[-e^x \frac{\cos nx}{n} + \frac{1}{n} \left\{ e^x \frac{\sin nx}{n} - \frac{1}{n} \int e^x \sin nx \, dx \right\} \right]_{-\pi}^{\pi} \\ &= \frac{1}{\pi} \left[-e^x \frac{\cos nx}{n} - \frac{1}{n^2} \int e^x \sin nx \, dx \right] \end{aligned}$$

$$b_n = -\frac{(-1)^n}{n\pi} (e^{\pi} - e^{-\pi}) \left(1 + \frac{1}{n^2} \right)^{-1}$$

$$b_n = -2 \frac{(-1)^n}{n\pi} \sinh x \left(1 + \frac{1}{n^2} \right)^{-1}$$

$$\boxed{f(x) = e^x = 2 \frac{\sinh x}{\pi} \left[\frac{1}{2} + \sum_{n=1}^{\infty} \left(1 + \frac{1}{n^2} \right)^{-1} \frac{(-1)^n}{n} \left(\frac{1}{n} \cos nx - \sin nx \right) \right]}$$

For $x = 0$, we get

$$\begin{aligned} \frac{\pi}{\sinh x} &= 1 + 2 \sum_{n=1}^{\infty} \left(1 + \frac{1}{n^2} \right)^{-1} \frac{(-1)^n}{n^2} \\ &= 1 + 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + 1} \end{aligned}$$

$$\boxed{\frac{\pi}{\sinh x} = 1 + 2 \left(-\frac{1}{2} + \frac{1}{5} - \frac{1}{10} + \cdots \right)}$$

3 Fourier Integral

3.1 Definition

Definition 3.1.1: Fourier Integral

The Fourier integral of a function f defined on the interval $(-\infty, \infty)$ is given by

$$f(x) = \frac{1}{\pi} \int_0^{\infty} [A(\alpha) \cos \alpha x + B(\alpha) \sin \alpha x] d\alpha \quad (3.1.1)$$

where the coefficients $A(\alpha)$ and $B(\alpha)$ are given by

$$\begin{cases} A(\alpha) = \int_{-\infty}^{\infty} f(x) \cos \alpha x dx \\ B(\alpha) = \int_{-\infty}^{\infty} f(x) \sin \alpha x dx \end{cases} \quad (3.1.2)$$

The Fourier integral can also be written in the form

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(t) \cos \lambda(t-x) dt d\lambda \quad (3.1.3)$$

where $\lambda = \frac{n\pi}{L}$

Fourier series were used to represent a function f defined on the finite interval $(-L, L)$ or $(0, L)$. It converged to f and to its periodic extension. In this sense, Fourier series is associated with periodic functions.

Fourier integral represents a certain type of non-periodic functions that are defined on $(-\infty, \infty)$ or $(0, \infty)$.

3.2 Derivation

Let a function f be defined on $(-L, L)$. The Fourier series of the function is then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \quad (3.2.1)$$

where the coefficients are given by

$$\begin{aligned} a_0 &= \frac{1}{L} \int_{-L}^L f(t) dt \\ a_n &= \frac{1}{L} \int_{-L}^L f(t) \cos \frac{n\pi t}{L} dt \\ b_n &= \frac{1}{L} \int_{-L}^L f(t) \sin \frac{n\pi t}{L} dt \end{aligned}$$

Now, let $a_n = \frac{n\pi}{L}$,

then $\Delta\alpha = \alpha_{n+1} - \alpha_n = \frac{\pi}{L}$

So, we get

$$\Delta f(x) = \frac{1}{2\pi} \left(\int_{-L}^L f(t) dt \right) \Delta\alpha + \frac{1}{\pi} \sum_{n=1}^{\infty} \left[\left(\int_{-L}^L f(t) \cos \alpha_n t dt \right) \cos \alpha_n x + \left(\int_{-L}^L f(t) \sin \alpha_n t dt \right) \sin \alpha_n x \right] \Delta\alpha \quad (3.2.2)$$

We now expand the interval $(-L, L)$ by taking $L \rightarrow \infty$, which implies that $\Delta\alpha \rightarrow 0$. Consequently, we get

$$\lim_{\Delta\alpha \rightarrow 0} \sum_{n=1}^{\infty} F(\alpha_n) \Delta\alpha \rightarrow \int_0^{\infty} F(\alpha) d\alpha \quad (3.2.3)$$

Thus, the limit of the first term in the Fourier series $\int_{-L}^L f(t) dt$ vanishes, and the limit of the sum becomes

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \left[\left(\int_{-\infty}^{\infty} f(t) \cos \alpha t dt \right) \cos \alpha x + \left(\int_{-\infty}^{\infty} f(t) \sin \alpha t dt \right) \sin \alpha x \right] d\alpha \quad (3.2.4)$$

This is the Fourier integral of f on the interval $(-\infty, \infty)$.

3.3 Alternative Derivation

Substituting the values of a_0 , a_n and b_n in (3.1.), we get

$$\begin{aligned} f(x) &= \frac{1}{2L} \int_{-L}^L f(t) dt + \frac{1}{L} \sum_{n=1}^{\infty} \left[\int_{-L}^L f(t) \cos \frac{n\pi t}{L} \cos \frac{n\pi x}{L} dt + \int_{-L}^L f(t) \sin \frac{n\pi t}{L} \sin \frac{n\pi x}{L} dt \right] \\ &= \frac{1}{2L} \int_{-L}^L f(t) dt + \frac{1}{L} \sum_{n=1}^{\infty} \left[\int_{-L}^L f(t) \left\{ \cos \frac{n\pi t}{L} \cos \frac{n\pi x}{L} + \sin \frac{n\pi t}{L} \sin \frac{n\pi x}{L} \right\} dt \right] \\ f(x) &= \frac{1}{2L} \int_{-L}^L f(t) dt + \frac{1}{L} \sum_{n=1}^{\infty} \int_{-L}^L f(t) \cos \frac{n\pi}{L} (t-x) dt \end{aligned} \quad (3.3.1)$$

Now, if we assume that $\int_{-\infty}^{\infty} |f(x)| dx$ converges, the first term on the right side of (3.2.2) approaches 0 as $L \rightarrow \infty$, since

$$\left| \frac{1}{2L} \int_{-L}^L f(t) dt \right| \leq \frac{1}{2L} \int_{-\infty}^{\infty} |f(t)| dt$$

The second term on the right side of (3.2.2) approaches

$$\begin{aligned} &\lim_{L \rightarrow \infty} \frac{1}{L} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} f(t) \cos \frac{n\pi}{L} (t-x) dt \\ &= \lim_{\delta\lambda \rightarrow 0} \frac{1}{\pi} \sum_{n=1}^{\infty} \delta\lambda \int_{-\infty}^{\infty} f(t) \cos n\delta\lambda (t-x) dt \end{aligned}$$

where $\lambda = \frac{n\pi}{L}$ which implies that $\delta\lambda = \frac{\pi}{L}$.

We know,

$$\lim_{\delta\lambda \rightarrow 0} \sum_{n=1}^{\infty} \delta\lambda F(\lambda_n) = \int_0^{\infty} F(\lambda) d\lambda$$

Thus we get

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(t) \cos \lambda(t-x) dt d\lambda \quad (3.3.2)$$

This is another form of the Fourier integral.

3.4 Fourier Sine and Cosine Integrals

We can rewrite (3.1.1) as

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \cos \lambda x \int_{-\infty}^{\infty} f(t) \cos \lambda t dt d\lambda + \frac{1}{\pi} \int_0^{\infty} \sin \lambda x \int_{-\infty}^{\infty} f(t) \sin \lambda t dt d\lambda \quad (3.4.1)$$

If $f(x)$ is an odd function, $f(t) \cos \lambda t$ is also an odd function while $f(t) \sin \lambda t$ is even. Then the first term on the right side of (3.4.1) vanishes and we get

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \sin \lambda x \int_0^{\infty} f(t) \sin \lambda t dt d\lambda \quad (3.4.2)$$

This is known as the Fourier sine integral.

Similarly, if $f(x)$ is an even function, (3.4.1) becomes

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \cos \lambda x \int_0^{\infty} f(t) \cos \lambda t dt d\lambda \quad (3.4.3)$$

This is known as the Fourier cosine integral.

3.5 Complex Form of Fourier Integral

Equation (3.1.3) can be written as

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \cos \lambda(t-x) dt d\lambda \quad (3.5.1)$$

because $\cos \lambda(t-x)$ is an even function of λ . Also, since $\sin \lambda(t-x)$ is an odd function of λ , we have

$$0 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \sin \lambda(t-x) dt d\lambda \quad (3.5.2)$$

Now, multiplying (3.5.2) by i and adding it to (3.5.1), we get

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) e^{i\lambda(t-x)} dt d\lambda \quad (3.5.3)$$

This is the complex form of the Fourier integral.

4 Fourier Transforms

4.1 Definition

Definition 4.1.1: Fourier Transform

The Fourier transform of a function f defined on the interval $(-\infty, \infty)$ is given by

$$F(\lambda) = \int_{-\infty}^{\infty} f(x) e^{i\lambda x} dx \quad (4.1.1)$$

The inverse Fourier transform is given by

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\lambda) e^{-i\lambda x} d\lambda \quad (4.1.2)$$

4.2 Derivation

We know that

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) e^{i\lambda(t-x)} dt d\lambda \quad (4.2.1)$$

We can rewrite (4.2.1) as follows

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda x} d\lambda \int_{-\infty}^{\infty} f(t) e^{i\lambda t} dt$$

It follows that if

$$F(\lambda) = \int_{-\infty}^{\infty} f(t) e^{i\lambda t} dt \quad (4.2.2)$$

then

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\lambda) e^{-i\lambda x} d\lambda \quad (4.2.3)$$

Here, $F(\lambda)$ is called the **Fourier transform** of $f(x)$ and $f(x)$ is called the **inverse Fourier transform** of $F(\lambda)$.

4.3 Fourier Sine and Cosine Transforms

We know

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \sin \lambda x \int_0^{\infty} f(t) \sin \lambda t dt d\lambda \quad (4.3.1)$$

It follows that if

$$F_s(\lambda) = \int_0^{\infty} f(x) \sin \lambda x dx \quad (4.3.2)$$

then

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F_s(\lambda) \sin \lambda x d\lambda \quad (4.3.3)$$

Here, $F_s(\lambda)$ is called the **Fourier sine transform** of $f(x)$ in $0 < x < \infty$. Also the function $f(x)$ is known as the **inverse Fourier sine transform** of $F_s(\lambda)$.

Again, we know

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \cos \lambda x \int_0^{\infty} f(t) \cos \lambda t dt d\lambda \quad (4.3.4)$$

Similarly, if

$$F_c(\lambda) = \int_0^\infty f(x) \cos \lambda x \, dx \quad (4.3.5)$$

then

$$f(x) = \frac{2}{\pi} \int_0^\infty F_c(\lambda) \cos \lambda x \, d\lambda \quad (4.3.6)$$

Here, $F_c(\lambda)$ is called the **Fourier cosine transform** of $f(x)$ in $0 < x < \infty$. Also the function $f(x)$ is known as the **inverse Fourier cosine transform** of $F_c(\lambda)$.

4.4 Finite Fourier Sine and Cosine Transforms

These transforms are useful for BVPs where at least two of the boundaries are parallel and separated by a finite distance.

The **finite Fourier sine transform** of $f(x)$ in $0 < x < L$ is given by

$$F_s(\lambda) = \int_0^L f(x) \sin \lambda x \, dx \quad (4.4.1)$$

and the **inverse finite Fourier sine transform** of $F_s(\lambda)$ is given by

$$f(x) = \frac{2}{L} \sum_{n=1}^{\infty} F_s(\lambda_n) \sin \lambda_n x \quad (4.4.2)$$

The **finite Fourier cosine transform** of $f(x)$ in $0 < x < L$ is given by

$$F_c(\lambda) = \int_0^L f(x) \cos \lambda x \, dx \quad (4.4.3)$$

and the **inverse finite Fourier cosine transform** of $F_c(\lambda)$ is given by

$$f(x) = \frac{2}{L} \sum_{n=1}^{\infty} F_c(\lambda_n) \cos \lambda_n x \quad (4.4.4)$$

4.5 Examples

Example 4.1: Find the fourier integral of the function $f(x) = \begin{cases} 1, & \text{if } |x| \leq 1 \\ 0, & \text{if } |x| > 1 \end{cases}$

Hence evaluate $\int_0^\infty \frac{\sin \lambda \cos \lambda x}{\lambda} d\lambda$

$$\begin{aligned}
f(x) &= \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty f(t) \cos \lambda(t-x) dt d\lambda \\
&= \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty f(t) [\cos \lambda t \cos \lambda x + \sin \lambda t \sin \lambda x] dt d\lambda \\
&= \frac{1}{\pi} \int_0^\infty \int_{-1}^1 \cos \lambda t \cos \lambda x dt d\lambda \\
&= \frac{2}{\pi} \int_0^\infty \cos \lambda x \int_0^1 \cos \lambda t dt d\lambda \\
\therefore f(x) &= \frac{2}{\pi} \int_0^\infty \frac{\cos \lambda x \sin \lambda}{\lambda} d\lambda
\end{aligned}$$

$$\therefore \int_0^\infty \frac{\sin \lambda \cos \lambda x}{\lambda} d\lambda = \begin{cases} \pi/2, & \text{if } |x| \leq 1 \\ 0, & \text{if } |x| > 1 \end{cases}$$

Example 4.2: Find the fourier integral of the function

$$f(x) = \begin{cases} 1 - x^2, & \text{if } |x| \leq 1 \\ 0, & \text{if } |x| > 1 \end{cases}$$

$$f(x) = \frac{1}{\pi} \int_0^\infty \{A(\lambda) \cos \lambda x + B(\lambda) \sin \lambda x\} d\lambda$$

Here,

$$\begin{aligned}
A(\lambda) &= \int_{-\infty}^\infty f(x) \cos \lambda x dx \\
&= \int_{-1}^1 \cos \lambda x dx - \int_{-1}^1 x^2 \cos \lambda x dx \\
&= 2 \frac{\sin \lambda x}{\lambda} \Big|_0^1 - 2 \left[x^2 \frac{\sin \lambda x}{\lambda} - \frac{2}{\lambda} \int x \sin \lambda x dx \right]_0^1 \\
&= \frac{4}{\lambda} \left[-x \frac{\cos \lambda x}{\lambda} + \frac{1}{\lambda} \int \cos \lambda x dx \right]_0^1 \\
&= \frac{4}{\lambda^3} \sin \lambda - \frac{4}{\lambda^2} \cos \lambda
\end{aligned}$$

$$B(\lambda) = \int_{-\infty}^\infty f(x) \sin \lambda x dx = \int_{-1}^1 \sin \lambda x dx - \int_{-1}^1 x^2 \sin \lambda x dx = 0$$

$$\therefore f(x) = \frac{1}{\pi} \int_0^\infty \left(\frac{4}{\lambda^3} \sin \lambda \cos \lambda x - \frac{4}{\lambda^2} \cos \lambda \cos \lambda x \right) d\lambda$$

Example 4.3: Find the Fourier transform of the function

$$\text{(i) } f(x) = \begin{cases} 1, & \text{if } |x| < a \\ 0, & \text{if } |x| > a \end{cases}$$

Hence evaluate $\int_{-\infty}^\infty \frac{\sin ax}{x} dx$

$$(ii) f(x) = \begin{cases} x^2, & \text{if } |x| < a \\ 0, & \text{if } |x| > a \end{cases}$$

(i) The fourier transform of $f(x)$ is given by

$$\begin{aligned} F\{f(x)\} = F(s) &= \int_{-\infty}^{\infty} f(x)e^{isx} dx \\ &= \int_{-a}^a e^{isx} dx \\ &= \frac{e^{isa} - e^{-isa}}{is} \\ &= \frac{2}{s} \sin as \end{aligned}$$

Now, by inverse Fourier transform, we get

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(s)e^{-isx} ds \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{s} \sin as \cdot e^{-isx} ds \end{aligned}$$

For $x = 0$, we get

$$\begin{aligned} f(0) = 1 &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin as}{s} ds \\ \therefore \int_{-\infty}^{\infty} \frac{\sin ax}{x} dx &= \pi \end{aligned}$$

(ii) The Fourier transform of $f(x)$ is

$$\begin{aligned} F\{f(x)\} = F(s) &= \int_{-\infty}^{\infty} f(x)e^{isx} dx \\ &= \int_{-a}^a x^2 e^{isx} dx \\ &= \left[x^2 \frac{e^{isx}}{is} \right]_{-a}^a - \frac{2}{is} \int_{-a}^a x e^{isx} dx \\ &= \frac{2a^2}{s} \sin as - \frac{2}{is} \left[x \frac{e^{isx}}{is} - \frac{e^{isx}}{(is)^2} \right]_{-a}^a \\ &= \frac{2a^2}{s} \sin as - \frac{4a}{is^2} \sin as - \frac{2}{s^3} \sin as \end{aligned}$$