Simple Harmonic Motion

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Contents

1 Oscillation and Vibration

1.1 Oscillation

- Oscillation is the repetitive variation, typically in time, of some measure about a central value (often a point of equilibrium) or between two or more different states.
- The term vibration is precisely used to describe mechanical oscillation.
- Familiar examples of oscillation include a swinging pendulum and alternating current.

1.2 Vibration

- Vibration is a mechanical phenomenon whereby oscillations occur about an equilibrium point.
- The word comes from Latin vibrationem (”shaking, brandishing”).
- The oscillations may be periodic, such as the motion of a pendulum—or random, such as the movement of a tire on a gravel road.

1.3 Differences between Oscillation and Vibration

- Oscillation is the definite displacement of a body in terms of distance or time, whereas vibration is the movement brought about in a body due to oscillation.
- Oscillation takes place in physical, biological systems, and often in our society, but vibrations is associated with mechanical systems only.
- Oscillation is about a single body, whereas vibration is the result of collective oscillation of atoms in the body.
- All vibrations are oscillations, but not all oscillations are vibrations.

2 Simple Harmonic Motion

2.1 Definition

Definition 2.1.1: Simple Harmonic Motion

Simple harmonic motion is a type of periodic motion where the restoring force is directly proportional to the displacement and acts in the direction opposite to that of displacement.

A particle is said to execute SHM when it will

(a) Trace and retrace the same path over and over again.

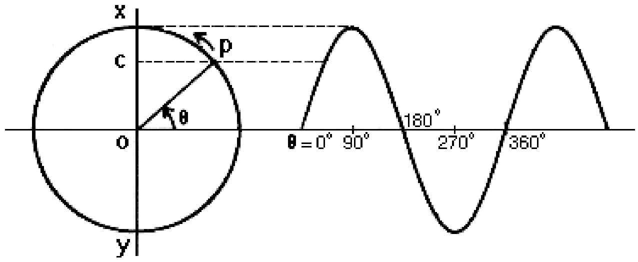
(b) Change direction at a regular interval of time.

(c) Move along a straight line.

(d) Have acceleration proportional to its displacement from the mean position.

A particle which satisfies the condition (a) only is said to execute **periodic motion**. A particle which satisfies condition (a) and (b) is said to execute **vibratory motion**.

Let P be a particle moving on the circumference of a circle of radius r with a uniform velocity v. Let angular velocity be $\omega = v/r$.



Displacement of the particle from the mean position is given by $y = r \sin \omega t$

So, velocity of the particle is given by $v = \dfrac{dy}{dt} = \omega r \cos \omega t$

And acceleration of the particle is given by $a = \dfrac{dv}{dt} = -\omega^2 r \sin \omega t = -\omega^2 y$

Angle	Position of vibrating particle	Displacement $y = r \sin \omega t$	Velocity $\frac{\mathrm{d}y}{\mathrm{d}t} = \omega r \cos \omega t$	Acceleration $-\omega^2 r \sin \omega t = -\omega^2 y$
0	O	0	ωr	0
$\pi/2$	X	r	0	$-\omega^2 a$
π	O	0	$-\omega r$	0
$3\pi/2$	Y	$-r$	0	$\omega^2 r$
2π	O	0	ωr	0

2.2 Differential Equation of SHM

Let y be the displacement of the particle from the mean position at time t, r be the amplitude, and α be the epoch of the vibrating particle

$$y = r \sin(\omega t + \alpha) \tag{1}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = r\omega \cos(\omega t + \alpha) \tag{2}$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = -r\omega^2 \sin(\omega t + \alpha) \tag{3}$$

Hence the differential equation of SHM is

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \omega^2 y = 0 \tag{4}$$

2.3 Solution of the Differential Equation of SHM

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \omega^2 y = 0 \tag{5}$$

Here,

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}y}$$

$$v\,dv + \omega^2 y\,dy = 0$$
$$\int v\,dv + \omega^2 \int y\,dy = 0$$
$$\frac{v^2}{2} + \frac{\omega^2 y^2}{2} = C'$$
$$\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \omega^2 y^2 = C^2$$

At maximum displacement, $y = r$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = 0$
So, $C^2 = \omega^2 r^2$

$$\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = \omega^2(r^2 - y^2)$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = \omega\sqrt{r^2 - y^2}$$
$$\int \frac{1}{\sqrt{r^2 - y^2}}\,dy = \int \omega\,dt$$
$$\sin^{-1}\frac{y}{r} = \omega t + \alpha$$
$$\boxed{y = r\sin(\omega t + \alpha)} \tag{6}$$

By expanding equation (6), we get

$$y = r\sin\omega t\cos\alpha + r\cos\omega t\sin\alpha \tag{7}$$

If $y = 0$ at $t = 0$, then $\alpha = 0$

$$y = r\sin\omega t \tag{8}$$

If $y = r$ at $t = 0$, then $\alpha = \pi/2$

$$y = r\cos\omega t \tag{9}$$

Hence, the general solution of the differential equation of SHM is

$$\boxed{y = A\sin\omega t + B\cos\omega t} \tag{10}$$

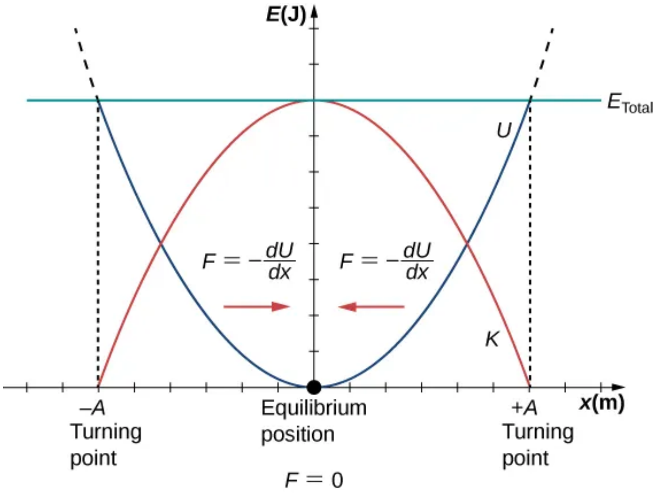
3 Energy in SHM

3.1 Total Energy of a Vibrating Particle

$$\text{Kinetic Energy} = \frac{1}{2}m\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2$$
$$= \frac{1}{2}m\omega^2 r^2\cos^2(\omega t + \alpha)$$
$$\text{Potential Energy} = \frac{1}{2}ky^2$$
$$= \frac{1}{2}m\omega^2 r^2$$
$$= \frac{1}{2}m\omega^2 r^2\sin^2(\omega t + \alpha)$$

Thus, the total energy of the vibrating particle is

$$\boxed{E = \frac{1}{2}kr^2 = \frac{1}{2}m\omega^2 r^2} \tag{11}$$



3.2 Average Kinetic Energy

Kinetic energy of the particle is given by

$$K = \frac{1}{2}m\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = \frac{1}{2}m\omega^2 r^2 \cos^2(\omega t + \alpha) \tag{12}$$

Hence, average kinetic energy is

$$\begin{aligned}
\overline{K} &= \frac{1}{T}\int_0^T K\,dt \\
&= \frac{1}{T}\int_0^T \frac{1}{2}m\omega^2 r^2 \cos^2(\omega t + \alpha)\,dt \\
&= \frac{1}{2}m\omega^2 r^2 \frac{1}{T}\int_0^T \cos^2(\omega t + \alpha)\,dt \\
&= \frac{1}{2}m\omega^2 r^2 \frac{1}{T}\int_0^T \frac{1 + \cos 2(\omega t + \alpha)}{2}\,dt \\
&= \frac{1}{2}m\omega^2 r^2 \frac{1}{T}\left[\frac{t}{2} + \frac{\sin 2(\omega t + \alpha)}{4\omega}\right]_0^T \\
&= \frac{1}{2}m\omega^2 r^2 \frac{1}{T}\left[\frac{T}{2} + \frac{\sin 2(\omega T + \alpha)}{4\omega} - \frac{\sin 2\alpha}{4\omega}\right] \\
&= \frac{1}{2}m\omega^2 r^2 \frac{1}{T}\left[\frac{T}{2} + \frac{\sin 2\alpha}{4\omega} - \frac{\sin 2\alpha}{4\omega}\right] \\
&= \frac{1}{4}m\omega^2 r^2
\end{aligned}$$

$$\boxed{\overline{K} = \frac{1}{4}m\omega^2 r^2 = \frac{1}{4}kr^2 = \frac{1}{2}E} \tag{13}$$

3.3 Average Potential Energy

Potential energy of the particle is given by

$$U = \frac{1}{2}ky^2 = \frac{1}{2}m\omega^2 r^2 \sin^2(\omega t + \alpha) \tag{14}$$

Hence, average potential energy is

$$\overline{U} = \frac{1}{T}\int_0^T U\,dt$$

$$
\begin{aligned}
&= \frac{1}{T}\int_0^T \frac{1}{2}m\omega^2 r^2 \sin^2(\omega t + \alpha)\, dt \\
&= \frac{1}{2}m\omega^2 r^2 \frac{1}{T}\int_0^T \sin^2(\omega t + \alpha)\, dt \\
&= \frac{1}{2}m\omega^2 r^2 \frac{1}{T}\int_0^T \frac{1 - \cos 2(\omega t + \alpha)}{2}\, dt \\
&= \frac{1}{2}m\omega^2 r^2 \frac{1}{T}\left[\frac{t}{2} - \frac{\sin 2(\omega t + \alpha)}{4\omega}\right]_0^T \\
&= \frac{1}{2}m\omega^2 r^2 \frac{1}{T}\left[\frac{T}{2} - \frac{\sin 2(\omega T + \alpha)}{4\omega} + \frac{\sin 2\alpha}{4\omega}\right] \\
&= \frac{1}{2}m\omega^2 r^2 \frac{1}{T}\left[\frac{T}{2} + \frac{\sin 2\alpha}{4\omega} - \frac{\sin 2\alpha}{4\omega}\right] \\
&= \frac{1}{4}m\omega^2 r^2
\end{aligned}
$$

$$
\boxed{\overline{U} = \frac{1}{4}m\omega^2 r^2 = \frac{1}{4}kr^2 = \frac{1}{2}E} \tag{15}
$$