Laplace Transform

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1 Definition, Existence, and Basic Properties of the Laplace Transform

1.1 Definition and Existence

Definition 1.1.1: Laplace Transform

Let F be a real-valued function of the real variable t, defined for $t > 0$. Let s be a variable that we shall assume to be real, and consider the function f defined by

$$f(s) = \int_0^\infty e^{-st} F(t)\, dt \tag{1}$$

for all values of s for which this integral exists. The function f defined by the integral (1) is called the Laplace Transform of the function F. We shall denote the Laplace transform of F by $\mathscr{L}\{F(t)\}$.
Thus the Laplace transform of a function f is given by

$$\mathscr{L}\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\, dt = \lim_{R\to\infty} \int_0^R e^{-st} F(t)\, dt \tag{2}$$

Some ways to write Laplace transforms:

$$\mathscr{L}F(t) = f(s) = \int_0^\infty e^{-st} F(t)\, dt$$
$$\mathscr{L}G(t) = g(s)$$
$$\mathscr{L}u(t) = \tilde{u}(s)$$

$F(t)$	$\mathscr{L}\{F(t)\} = f(s)$	$F(t)$	$\mathscr{L}\{F(t)\} = f(s)$
1	$\frac{1}{s}$	n	$\frac{n}{s}$
t	$\frac{1}{s^2}$	t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$	e^{-at}	$\frac{1}{s+a}$
$\sin at$	$\frac{a}{s^2+a^2}$	$\sinh at$	$\frac{a}{s^2-a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$	$\cosh at$	$\frac{s}{s^2-a^2}$

Table 1: Functions and their Laplace Transform

Proofs :

Let $F(t) = n$, for $t > 0$
Then

$$\begin{aligned}\mathscr{L}\{n\} &= \int_0^\infty e^{-st} \cdot n \, dt \\ &= n \frac{-e^{st}}{s}\bigg|_0^\infty \\ &= \frac{n}{s} \quad \square\end{aligned}$$

Let $F(t) = t$, for $t > 0$
Then

$$\begin{aligned}\mathscr{L}\{t\} &= \int_0^\infty e^{-st} \cdot t \, dt \\ &= -t\frac{e^{-st}}{s} + \int_0^\infty \frac{e^{-st}}{s} \, dt \\ &= -e^{-st}\frac{t}{s}\bigg|_0^\infty - e^{-st}\frac{1}{s^2}\bigg|_0^\infty \\ &= \frac{1}{s^2} \quad \square\end{aligned}$$

Let $F(t) = t^n$, for $t > 0$
Then

$$\begin{aligned}\mathscr{L}\{t^n\} &= \int_0^\infty e^{-st} t^n \, dt \\ &= -t^n \frac{e^{st}}{s} + \int_0^\infty n t^{n-1} \frac{e^{-st}}{s} \, dt \\ &= -n t^{n-1} \frac{e^{-st}}{s^2}\bigg|_0^\infty + \int_0^\infty n(n-1) t^{n-2} \frac{e^{-st}}{s^2} \, dt \\ &= -n(n-1)t^{n-2}\left(\frac{e^{-st}}{s^3}\right)\bigg|_0^\infty + \int_0^\infty n(n-1)(n-2)t^{n-3}\frac{e^{-st}}{s^3} \, dt \\ &= \cdots \\ &= n! t^{n-n} \frac{e^{-st}}{s^{n+1}}\bigg|_0^\infty + \int_0^\infty n(n-1)\cdots(n-n)\frac{e^{-st}}{s^{n+1}} \, dt \\ &= \frac{n!}{s^{n+1}} \quad \square\end{aligned}$$

Let $F(t) = e^{at}$, for $t > 0$
Then

$$\begin{aligned}\mathscr{L}\{e^{at}\} &= \int_0^\infty e^{-st} e^{at} \, dt \\ &= \int_0^\infty e^{(a-s)t} \, dt \\ &= \frac{e^{(a-s)t}}{a-s}\bigg|_0^\infty \\ &= \frac{1}{s-a} \quad \square\end{aligned}$$

Let $F(t) = e^{-at}$, for $t > 0$
Then

$$\begin{aligned}\mathscr{L}\{e^{-at}\} &= \int_0^\infty e^{-st} e^{-at} \, dt \\ &= \int_0^\infty e^{-(a+s)t} \, dx \\ &= \frac{e^{-(a+s)t}}{s+a}\bigg|_0^\infty \\ &= \frac{1}{s+a} \quad \square\end{aligned}$$

Let $F(t) = \sin at$, for $t > 0$
Then

$$\mathscr{L}\{\sin at\} = \int_0^\infty e^{-st \sin at}\, dt$$
$$= -\frac{e^{-st}}{s^2 + a^2}(s \sin at + a \cos at)\Big|_0^\infty$$
$$= \frac{a}{s^2 + a^2} \quad \square$$

Let $F(t) = \cos at$, for $t > 0$
Then

$$\mathscr{L}\{\cos at\} = \int_0^\infty e^{-st} \cos at\, dt$$
$$= \frac{e^{-st}}{s^2 + a^2}(-s \cos at + a \sin at)\Big|_0^\infty$$
$$= \frac{s}{s^2 + a^2} \quad \square$$

Definition 1.1.2: Piecewise continuous or Sectionally continuous Function

A function f is said to be piecewise continuous on a finite interval $a \leq t \leq b$ if this interval can be divided into a finite number of subintervals such that

1. f is continuous in the interior of each of these subintervals, and
2. $f(t)$ approaches finite limits as t approaches either endpoint of each of the subintervals from its interior

Example 1.1: Consider the function f defined by

$$f(t) = \begin{cases} -1, & \textbf{if } 0 < t < 2 \\ 1, & \textbf{if } t > 2 \end{cases}$$

f is piecewise continuous on every finite interval $0 \leq t \leq b$, for every positive number b. At $t = 2$, we have

$$f(2-) = \lim_{t \to 2-} f(t) = -1$$
$$f(2+) = \lim_{t \to 2+} f(t) = +1$$