Solution of differential Equations with Undetermined Coefficient

Parvin Akter
Assistant Professor
Department of Mathematics, CUET.

Ex. 1. *(a) By the method of undetermined coefficients, solve* $(D^2 + 4)\, y = x^2$.

Sol. Here given that $\qquad (D^2 + 4)\, y = x^2. \qquad \ldots (1)$

Its auxiliary equation is $\qquad D^2 + 4 = 0 \qquad$ so that $\qquad D = \pm\, i.$

$\therefore$ C.F. $= c_1 \cos 2x + c_2 \sin 2x$, c_1, c_2 being arbitrary constants. $\qquad \ldots (2)$

Let the trial solution be $y^* = A_0 + A_1 x + A_2 x^2$. [Refer result 1 in table of Art. 5.26] $\ldots (3)$

Since y^* must satisfy (1), we have $\qquad (D^2 + 4)\, y^* = x^2 \quad$ or $\quad D^2 y^* + 4y^* = x^2. \qquad \ldots (4)$

Now, $\qquad (3) \Rightarrow D\, y^* = A_1 + 2A_2 x \qquad$ and $\qquad D^2 y^* = 2A_2. \qquad \ldots (5)$

Using (3) and (5), (4) reduces to $\qquad 2A_2 + 4\,(A_0 + A_1 x + A_2 x^2) = x^2$

$$2A_2 + 4A_0 + 4A_1 x + 4A_2\, x^2 = x^2. \qquad \ldots (6)$$

(6) is an identify. Comparing the coefficients of like terms, we get

$2A_2 + 4A_0 = 0, \qquad 4A_1 = 0, \qquad 4A_2 = 1. \qquad \ldots (7)$

Solving (7), $A_1 = 0$, $A_2 = 1/4$. $A_0 = -\,1/8$. Then, from (3), we have

$$y^* = -\,(1/8) + x^2/4 = (1/8)\,(2x^2 - 1).$$

Hence the required general solution is $y =$ C.F. + P.I. = C.F. + y^*

or $$y = c_1 \cos 2x + c_2 \sin 2x + (1/8)\,(2x^2 - 1).$$

Ex. 1(*b*). *Using the method of undetermined coefficients, solve* $\qquad y_2 - 2y_1 + y = x^2$.

[Delhi Maths (G) 1995]

Sol. Let $D \equiv d/dx$. Then, we have $\qquad (D^2 - 2D + 1)\, y = x^2. \qquad \ldots (1)$

Its auxiliary equation is $\qquad D^2 - 2D + 1 = 0 \qquad$ so that $\qquad D = 1, 1.$

$\therefore$ C.F. $= (c_1 + c_2 x)\, e^x$, c_1, c_2 being arbitrary constants. $\qquad \ldots (2)$

Let the trial solution be $\quad y^* = A_0 + A_1 x + A_2 x^2$. [Refer result 1 in table of Art. 5.26] $\ldots (3)$

Since y^* must satisfy (1), we have $\qquad D^2 y^* - 2D\, y^* + y^* = x^2. \qquad \ldots (4)$

Now, $\qquad (3) \quad \Rightarrow D\, y^* = A_1 + 2A_2 x \qquad$ and $\qquad D^2 y^* = 2A_2. \qquad \ldots (5)$

Using (3) and (5), (4) reduces to $\qquad 2A_2 - 2(A_1 + 2A_2 x) + A_0 + A_1 x + A_2 x^2 = x^2$

or $$(A_0 - 2A_1 + 2A_2) + x\,(A_1 - 4A_2) + A_2 x^2 = x^2.$$

Comparing the coefficients of like terms in above identity, we have

$A_0 - 2A_1 + 2A_2 = 0, \quad A_1 - 4A_2 = 0 \quad$ and $\quad A_2 = 1 \quad$ so that $\quad A_2 = 1, \quad A_1 = 4, \quad A_0 = 6.$

From (3), $\qquad y^* = 6 + 4x + x^2$ and so solution is $\qquad y =$ C.F. $+ y^* = (c_1 + c_2 x)\, e^x + 6 + 4x + x^2.$

Ex. 2. *Using the method of undetermined coefficients, solve* $y_2 + 2y_1 + y = x - e^x$.

[Delhi Maths (G) 1996]

Sol. Re-writing the given equation, $(D^2 + 2D + 1)\, y = x - e^x$ ... (1)

Its auxiliary equation $D^2 + 2D + 1 = 0$ so that $D = -1, -1$.

$\therefore$ C.F. $= (c_1 + c_2 x)\, e^{-x}$, c_1, c_2 being arbitrary constants. ... (2)

Let the trial solution be $y^* = Ax + B + C\, e^x$. [Refer results 1 and 2 in table of Art. 5.26] ... (3)

Since y^* must satisfy (1), $(D^2 + 2D + 1)\, y^* = x - e^x$ or $D^2 y^* + 2D\, y^* + y^* = x - e^x$. ... (4)

From (3), $D\, y^* = A + C\, e^x$ and $D^2\, y^* = Ce^x$. ... (5)

Using (3) and (5), (4) gives $C\, e^x + 2(A + C\, e^x) + Ax + B + C\, e^x = x - e^x$

or $(2A + B) + Ax + 4C\, e^x = x - e^x$.

Equating the coefficients of like terms in the above identity, we get

$2A + B = 0$, $A = 1$, $4C = -1$ so that $A = 1$, $B = -2$, $C = -1/4$.

$\therefore$ from (3), $y^* = x - 2 - (1/4)\, e^x$ and so the general solution is

$y = \text{C.F.} + y^*$ or $y = (c_1 + c_2 x)\, e^{-x} + x - 2 - (1/4)\, e^x$.

Ex. 4. *Using the method of undetermined coefficients to solve* $(d^2y/dx^2) - 2(dy/dx) - 3y = 2e^x - 10 \sin x$. **[Delhi Maths Hons. 1997]**

Sol. Given $(D^2 - 2D - 3)\, y = 2\, e^x - 10 \sin x$, where $D \equiv d/dx$. ... (1)

Its auxiliary equation is $D^2 - 2D - 3 = 0$ so that $D = -1, 3$.

$\therefore$ C.F. $= c_1 e^{-x} + c_2 e^{3x}$, c_1, c_2 being arbitrary constants. ... (2)

Let the trial solution be $y^* = A\, e^x + B \sin x + C \cos x$. ... (3)

Since y^* must satisfy (1), $D^2 y^* - 2D\, y^* - 3\, y^* = 2\, e^x - 10 \sin x$. ... (4)

From (3), $D\, y^* = Ae^x + B \cos x - C \sin x$, $D^2 y^* = Ae^x - B \sin x - C \cos x$ ... (5)

Using (3) and (5), (4) reduces to

$$(Ae^x - B \sin x - C \cos x) - 2\, (Ae^x + B \cos x - C \sin x) - 3\, (Ae^x + B \sin x + C \cos x) = 2e^x - 10 \sin x$$

or $$-4\, Ae^x - (4B - 2C) \sin x - (4C + 2B) \cos x = 2e^x - 10 \sin x.$$

Equating the coefficients of like terms in above identity, we have

$-4A = 2$, $-(4B - 2C) = -10$ and $-(4C + 2B) = 0$ $\Rightarrow$ $A = -1/2$, $B = 2$, $C = -1$

$\therefore$ From (3), $y^* = (-1/2)\, e^x + 2 \sin x - \cos x$.

and general solution is $y =$ C.F. $+ y^*$ *i.e.*, $y = c_1\, e^{-x} + c_2 e^{3x} - (1/2)\, e^x + 2 \sin x - \cos x$.

H.W. 3,5,6,8