

Laplace Transform

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1 Definition, Existence, and Basic Properties of the Laplace Transform

1.1 Definition and Existence

Definition 1.1.1: Laplace Transform

Let F be a real-valued function of the real variable t , defined for $t > 0$. Let s be a variable that we shall assume to be real, and consider the function f defined by

$$f(s) = \int_0^{\infty} e^{-st} F(t) dt \quad (1)$$

for all values of s for which this integral exists. The function f defined by the integral (1) is called the Laplace Transform of the function F . We shall denote the Laplace transform of F by $\mathcal{L}\{F(t)\}$.

Thus the Laplace transform of a function f is given by

$$\mathcal{L}\{F(t)\} = f(s) = \int_0^{\infty} e^{-st} F(t) dt = \lim_{R \rightarrow \infty} \int_0^R e^{-st} F(t) dt \quad (2)$$

Some ways to write Laplace transforms:

$$\mathcal{L}F(t) = f(s) = \int_0^{\infty} e^{-st} F(t) dt$$

$$\mathcal{L}G(t) = g(s)$$

$$\mathcal{L}u(t) = \tilde{u}(s)$$

$F(t)$	$\mathcal{L}\{F(t)\} = f(s)$	$F(t)$	$\mathcal{L}\{F(t)\} = f(s)$
1	$\frac{1}{s}$	n	$\frac{n}{s}$
t	$\frac{1}{s^2}$	t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$	e^{-at}	$\frac{1}{s+a}$
$\sin at$	$\frac{a}{s^2 + a^2}$	$\sinh at$	$\frac{a}{s^2 - a^2}$
$\cos at$	$\frac{s}{s^2 + a^2}$	$\cosh at$	$\frac{s}{s^2 - a^2}$

Table 1: Functions and their Laplace Transform

Proofs :

Let $F(t) = n$, for $t > 0$

Then

$$\begin{aligned}\mathcal{L}\{n\} &= \int_0^\infty e^{-st} \cdot n \, dt \\ &= n \frac{-e^{-st}}{s} \Big|_0^\infty \\ &= \frac{n}{s} \quad \square\end{aligned}$$

Let $F(t) = t$, for $t > 0$

Then

$$\begin{aligned}\mathcal{L}\{t\} &= \int_0^\infty e^{-st} \cdot t \, dt \\ &= -t \frac{e^{-st}}{s} + \int_0^\infty \frac{e^{-st}}{s} \, dt \\ &= -e^{-st} \frac{t}{s} \Big|_0^\infty - e^{-st} \frac{1}{s^2} \Big|_0^\infty \\ &= \frac{1}{s^2} \quad \square\end{aligned}$$

Let $F(t) = t^n$, for $t > 0$

Then

$$\begin{aligned}\mathcal{L}\{t^n\} &= \int_0^\infty e^{-st} t^n \, dt \\ &= -t^n \frac{e^{-st}}{s} + \int_0^\infty n t^{n-1} \frac{e^{-st}}{s} \, dt \\ &= -n t^{n-1} \frac{e^{-st}}{s^2} \Big|_0^\infty + \int_0^\infty n(n-1) t^{n-2} \frac{e^{-st}}{s^2} \, dt \\ &= -n(n-1) t^{n-2} \left(\frac{e^{-st}}{s^3} \right) \Big|_0^\infty + \int_0^\infty n(n-1)(n-2) t^{n-3} \frac{e^{-st}}{s^3} \, dt \\ &= \dots \\ &= n! t^{n-n} \frac{e^{-st}}{s^{n+1}} \Big|_0^\infty + \int_0^\infty n(n-1) \dots (n-n) \frac{e^{-st}}{s^{n+1}} \, dt \\ &= \frac{n!}{s^{n+1}} \quad \square\end{aligned}$$

Let $F(t) = e^{at}$, for $t > 0$

Then

$$\begin{aligned}\mathcal{L}\{e^{at}\} &= \int_0^\infty e^{-st} e^{at} \, dt \\ &= \int_0^\infty e^{(a-s)t} \, dt \\ &= \frac{e^{(a-s)t}}{a-s} \Big|_0^\infty \\ &= \frac{1}{s-a} \quad \square\end{aligned}$$

Let $F(t) = e^{-at}$, for $t > 0$

Then

$$\begin{aligned}\mathcal{L}\{e^{-at}\} &= \int_0^\infty e^{-st} e^{-at} \, dt \\ &= \int_0^\infty e^{-(a+s)t} \, dt \\ &= \frac{e^{-(a+s)t}}{s+a} \Big|_0^\infty \\ &= \frac{1}{s+a} \quad \square\end{aligned}$$

Let $F(t) = \sin at$, for $t > 0$
Then

$$\begin{aligned}\mathcal{L}\{\sin at\} &= \int_0^{\infty} e^{-st} \sin at \, dt \\ &= -\frac{e^{-st}}{s^2 + a^2} (s \sin at + a \cos at) \Big|_0^{\infty} \\ &= \frac{a}{s^2 + a^2} \quad \square\end{aligned}$$

Let $F(t) = \cos at$, for $t > 0$
Then

$$\begin{aligned}\mathcal{L}\{\cos at\} &= \int_0^{\infty} e^{-st} \cos at \, dt \\ &= \frac{e^{-st}}{s^2 + a^2} (-s \cos at + a \sin at) \Big|_0^{\infty} \\ &= \frac{s}{s^2 + a^2} \quad \square\end{aligned}$$

Theorem 1.1.2: Hypothesis: Let F be a real function that has the following properties:

1. F is a piecewise continuous in every finite closed interval $0 \leq t \leq a$ ($a > 0$).
2. F is of exponential order, i.e, there exists α , $M > 0$, and $t_0 > 0$ such that

$$e^{-\alpha t} |F(t)| < M \text{ for } t > t_0$$

Conclusion: The Laplace transform of F exists for $s > \alpha$.

$$\mathcal{L}\{F(t)\} = \int_0^{\infty} e^{-st} F(t) \, dt$$

1.2 Basic Properties of the Laplace Transform

Theorem 1.2.1 (The Linear Property):

Let F_1 and F_2 be functions whose Laplace transform exist, and let c_1 and c_2 be constants. Then

$$\mathcal{L}\{c_1 F_1(t) + c_2 F_2(t)\} = c_1 \mathcal{L}\{F_1(t)\} + c_2 \mathcal{L}\{F_2(t)\}$$

Theorem 1.2.2: Hypothesis:

1. Let F be a real function that is continuous for $t \geq 0$ and of exponential order $e^{\alpha t}$.
2. Let F' be piecewise continuous in every finite closed interval $0 \leq t \leq a$.

Conclusion: Then $\mathcal{L}\{F'\}$ exists for $s > \alpha$; and

$$\mathcal{L}\{F'(t)\} = s\mathcal{L}\{F(t)\} - F(0)$$