Formation of Partial Differential Equations

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1.2 PARTIAL DIFFERENTIAL EQUATION (P.D.E.) [Delhi Maths (H) 2001]

Definition. An equation containing one or more partial derivatives of an unknown frunction of two or more independent variables is known as a *partial differential equation*.

For examples of partial differential equations we list the following:

$$\partial z/\partial x + \partial z/\partial y = z + xy \quad \ldots (1) \qquad (\partial z/\partial x)^2 + \partial^3 z/\partial y^3 = 2x(\partial z/\partial x) \quad \ldots (2)$$

$$z(\partial z/\partial x) + \partial z/\partial y = x \quad \ldots (3) \qquad \partial u/\partial x + \partial u/\partial y + \partial u/\partial z = xyz \quad \ldots (4)$$

$$\partial^2 z/\partial x^2 = (1 + \partial z/\partial y)^{1/2} \quad \ldots (5) \qquad y\left\{(\partial z/\partial x)^2 + (\partial z/\partial y)^2\right\} = z(\partial z/\partial y) \quad \ldots (6)$$

1.3 ORDER OF A PARTIAL DIFFERENTIAL EQUATION [Delhi Maths (H) 2001]

Definition. The *order* of a partial differential equation is defined as the order of the highest partial derivative occuring in the partial differential equation.

In Art. 1.2, equations (1), (3), (4) and (6) are of the first order, (5) is of the second order and (2) is of the third order.

1.4 DEGREE OF A PARTIAL DIFFERENTIAL EQUATION [Delhi Maths (H) 2001]

The *degree* of a partial differential equation is the degree of the highest order derivative which occurs in it after the equation has been rationalised, *i.e.*, made free from radicals and fractions so far as derivatives are concerned.

In 1.2, equations (1), (2), (3) and (4) are of first degree while equations (5) and (6) are of second degree.

1.5 LINEAR AND NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS

Definitions. A partial differential equation is said to be *linear* if the dependent variable and its partial derivatives occur only in the first degree and are not multiplied. A partial differential equation which is not linear is called a *non-linear* partial differential equation.

In Art. 1.2, equations (1) and (4) are linear while equations (2), (3), (5) and (6) are non-linear.

1.6 NOTATIONS

When we consider the case of two independent variables we usually assume them to be x and y and assume z to be the dependent variable. We adopt the following notations throughout the study of partial differential equations

$$p = \partial z / \partial x, \qquad q = \partial z / \partial y, \qquad r = \partial^2 z / \partial x^2, \qquad s = \partial^2 z / \partial x \partial y \quad \text{and} \quad t = \partial^2 z / \partial y^2$$

Situation I. *When the number of arbitrary constants is less than the number of independent variables, then the elimination of arbitrary constants usually gives rise to more than one partial differential equation of order one.*

For example, consider $$z = ax + y, \qquad \text{... (1)}$$

where a is the only arbitrary constant and x, y are two independent variables.

Differentiating (1) partially w.r.t. 'x', we get $$\partial z / \partial x = a \qquad \text{... (2)}$$

Differentiating (1) partially w.r.t. 'y', we get $$\partial z / \partial y = 1 \qquad \text{... (3)}$$

Eliminating a between (1) and (2) yields $$z = x(\partial z / \partial x) + y \qquad \text{... (4)}$$

Since (3) does not contain arbitrary constant, so (3) is also partial differential under consideration. Thus, we get two partial differential equations (3) and (4).

Situation II. *When the number of arbitrary constants is equal to the number of independent variables, then the elimination of arbitrary constants shall give rise to a unique partial differential equation of order one.*

Example: *Eliminate a and b from* $az + b = a^2x + y$ ... (1)

Differentiating (1) partially w.r.t 'x' and 'y', we have

$a(\partial z/\partial x) = a^2$ ... (2) $\qquad a(\partial z/\partial y) = 1$ ... (3)

Eliminating a from (2) and (3), we have $(\partial z/\partial x)(\partial z/\partial y) = 1$,

which is the unique partial differential equation of order one.

Situation III. *When the number of arbitrary constants is greater than the number of independent variables, then the elimination of arbitrary constants leads to a partial differential equation of order usually greater than one.*

Example: *Eliminate a, b and c from* $z = ax + by + cxy$ ... (1)

Differentiating (1) partially w.r.t., 'x' and 'y', we have

$\partial z/\partial x = a + cy$ ... (2) $\qquad \partial z/\partial y = b + cx$ ... (3)

From (2) and (3), $\partial^2 z/\partial x^2 = 0,$ $\qquad \partial^2 z/\partial y^2 = 0$ ... (4)

and $\partial^2 z/\partial x\partial y = c$ ... (5)

Now, (2) and (3) $\Rightarrow$ $x(\partial z/\partial x) = ax + cxy$ and $y(\partial z/\partial y) = by + cxy$

$\therefore$ $x(\partial z/\partial x) + y(\partial z/\partial y) = ax + by + cxy + cxy$

or $x(\partial z/\partial x) + y(\partial z/\partial y) = z + xy(\partial^2 z/\partial x\partial y)$, using (1) and (5) ... (6)

Thus, we get three partial differential equations given by (4) and (6), which are all of order two.

Ex. 1. *Find a partial differential equation by eliminating a and b from $z = ax + by + a^2 + b^2$.*

Sol. Given $$z = ax + by + a^2 + b^2. \qquad ...(1)$$

Differentiating (1) partially with respect to x and y, we get

$$\partial z/\partial x = a \quad \text{and} \quad \partial z/\partial y = b.$$

Substituting these values of a and b in (1) we see that the arbitrary constants a and b are eliminated and we obtain,

$$z = x(\partial z/\partial x) + y(\partial z/\partial y) + (\partial z/\partial x)^2 + (\partial z/\partial y)^2,$$

which is the required partial differential equation.

Ex. 2. *Eliminate arbitrary constants a and b from $z = (x - a)^2 + (y - b)^2$ to form the partial differential equation.* **[Jiwaji 1999; Banglore 1995]**

Sol. Given $$z = (x - a)^2 + (y - b)^2. \qquad ...(1)$$

Differentiating (1) partially with respect to a and b, we get

$$\partial z/\partial x = 2(x - a) \quad \text{and} \quad \partial z/\partial y = 2(y - b).$$

Squatring and adding these equations, we have

$$(\partial z/\partial x)^2 + (\partial z/\partial y)^2 = 4(x - a)^2 + 4(y - b)^2 = 4\,[(x - a)^2 + (y - b)^2]$$

or $$(\partial z/\partial x)^2 + (dz/\partial y)^2 = 4z, \text{ using (1).}$$

Ex. 3. *Form partial differential equations by eliminating arbitrary constants a and b from the following relations :*

(*a*) $z = a(x + y) + b$. (*b*) $z = ax + by + ab$. **[Bhopal 2010, Rewa 1996]**

(*c*) $z = ax + a^2y^2 + b$. **[Agra 2010]** (*d*) $z = (x + a)(y + b)$. **[Madurai Kamraj 2008]**

Sol. (*a*) Given $$z = a(x + y) + b \qquad ...(1)$$

Differentiating (1) partially with respect to x and y, we get

$$\partial z/\partial x = a \qquad \text{and} \qquad \partial z/\partial y = a.$$

Eliminating a between these, we get $$\partial z/\partial x = \partial z/\partial y,$$
which is the required partial differential equation.

(*b*) Given $$z = ax + by + ab. \qquad ...(1)$$

Differentiating (1) partially with respect to x and y, we get

$$\partial z/\partial x = a \qquad \text{and} \qquad \partial z/\partial y = b \qquad ....(2)$$

Substituting the values of a and b from (2) in (1), we get

$$z = x(\partial z/\partial x) + y(\partial z/\partial y) + (\partial z/dx)(\partial z/\partial y),$$

which is the required partial differential equation.

Ex. 5(*a*). *Form the partial differential equation by eliminating h and k from the equation* $(x - h)^2 + (y - k)^2 + z^2 = \lambda^2$. **[Gulbarga 2005; I.A.S. 1996]**

Sol. Given $$(x - h)^2 + (y - k)^2 + z^2 = \lambda^2. \qquad ...(1)$$

Differentiating (1) partially with respect to x and y, we get

$$2(x - h) + 2z(\partial z/\partial x) = 0 \qquad \text{or} \qquad (x - h) = -z(\partial z/\partial x) \qquad ...(2)$$

and $$2(y - k) + 2z(\partial z/\partial y) = 0 \qquad \text{or} \qquad (y - k) = -z(\partial z/\partial y). \qquad ...(3)$$

Substituting the values of $(x - h)$ and $(y - k)$ from (2) and (3) in (1) gives

$$z^2(\partial z/\partial x)^2 + z^2(\partial z/\partial y)^2 + z^2 = \lambda^2 \qquad \text{or} \qquad z^2[(\partial z/\partial x)^2 + (\partial z/\partial y)^2 + 1] = \lambda^2,$$

which is the required partial differential equation.

Ex. 13. *Find a partial differential equation by eliminating a, b, c, from* $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

[Bhopal 2004; Jabalpur 2000, 03, Jiwaji 2000, Vikram 2002, 04; Ravishanker 2010]

Sol. Given $$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1. \quad \text{... (1)}$$

Differentiating (1) partially with respect to x and y, we get

$$\frac{2x}{a^2} + \frac{2z}{c^2}\frac{dz}{dx} = 0 \quad \text{or} \quad c^2 x + a^2 z \frac{dz}{dx} = 0 \quad \text{... (2)}$$

and $$\frac{2y}{b^2} + \frac{2x}{c^2}\frac{\partial z}{\partial y} = 0 \quad \text{or} \quad c^2 y + b^2 z \frac{\partial z}{\partial y} = 0. \quad \text{... (3)}$$

Differentiating (2) with respect to x and (3) with respect to y, we have

$$c^2 + a^2\left(\frac{\partial z}{\partial x}\right)^2 + a^2 z \frac{\partial^2 z}{\partial x^2} = 0 \quad \text{... (4)} \qquad c^2 + b^2\left(\frac{\partial z}{\partial x}\right)^2 + b^2 z \frac{\partial^2 z}{\partial y^2} = 0. \quad \text{... (5)}$$

From (2), $$c^2 = -(a^2 z/x) \times (\partial z/\partial x) \quad \text{... (6)}$$

Putting this value of c^2 in (4) and dividing by a^2, we obtain

$$-\frac{z}{x}\frac{\partial z}{\partial x} + \left(\frac{\partial z}{\partial x}\right)^2 + z\frac{\partial^2 z}{\partial x^2} = 0 \quad \text{or} \quad zx\frac{\partial^2 z}{\partial x^2} + x\left(\frac{\partial z}{\partial x}\right)^2 - z\frac{\partial z}{\partial x} = 0. \quad \text{... (7)}$$

Similarly, from (3) and (5), $$zy\frac{\partial^2 z}{\partial y^2} + y\left(\frac{\partial z}{\partial y}\right)^2 - z\frac{\partial z}{\partial y} = 0. \quad \text{... (8)}$$

Differentiating (2) partially w.r.t. y, $$0 + a^2\left\{(\partial z/\partial y)(\partial z/\partial x) + z(\partial^2 z/\partial x \partial y)\right\} = 0$$

or $$(\partial z/\partial x)(\partial z/\partial y) + z(\partial^2 z/\partial x\partial y) = 0 \quad \text{... (9)}$$

(7), (8) and (9) are three possible forms of the required partial differential equations.

Ex. 1. *Form a partial differential equation by eliminating the arbitrary function ϕ from $\phi(x + y + z, x^2 + y^2 - z^2) = 0$. What is the order of this partial differential equation ?*

[Bilaspur 2003; Indore 2003; Jiwaji 2003; Vikram 2001]

Sol. Given $$\phi(x + y + z, x^2 + y^2 - z^2) = 0. \qquad ...(1)$$

Let $u = x + y + z$ and $v = x^2 + y^2 - z^2$. ...(2)

Then (1) becomes $$\phi(u, v) = 0. \qquad ...(3)$$

Differentiating (3) w.r.t., 'x' partially, we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right) = 0. \qquad ...(4)$$

From (2), $\dfrac{\partial u}{\partial x} = 1$, $\dfrac{\partial u}{\partial z} = 1$, $\dfrac{\partial v}{\partial x} = 2x$, $\dfrac{\partial v}{\partial z} = -2z$, $\dfrac{\partial u}{\partial y} = 1$, $\dfrac{\partial v}{\partial y} = 2y$. ..(5)

From (4) and (5), $$(\partial\phi/\partial u)(1 + p) + 2(\partial\phi/\partial v)(x - pz) = 0$$

or $$(\partial\phi/\partial u)/(\partial\phi/\partial v) = -2(x - pz)/(1 + p). \qquad ...(6)$$

Again, differentiating (3) w.r.t., 'y' partially, we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right) = 0$$

or $$(\partial\phi/\partial u)(1 + q) + 2(\partial\phi/\partial v)(y - zq) = 0, \text{ by (5)}$$

or $$(\partial\phi/\partial u)/(\partial\phi/\partial v) = -2(y - qz)/(1 + q). \qquad ...(7)$$

Eliminating ϕ from (6) and (7), we obtain

$(x - pz)/(1 + p) = (y - qz)/(1 + q)$ or $(1 + q)(x - pz) = (1 + p)(y - qz)$

or $(y + z)p - (x + z)q = x - y$, which is the desired partial differential equation of first order.

Ex. 2. *Form a partial differential equation by eliminating the arbitrary function f from the equation* $x + y + z = f(x^2 + y^2 + z^2)$. **(Kanpur 2011)**

Sol. Given $$x + y + z = f(x^2 + y^2 + z^2). \quad ...(1)$$

Differentiating partially w.r.t. 'x' and 'y', (1) gives

$$1 + p = f'(x^2 + y^2 + z^2).(2x + 2zp). \quad ...(2)$$

and $$1 + q = f'(x^2 + y^2 + z^2).(2y + 2zq). \quad ...(3)$$

Eliminating $f'(x^2 + y^2 + z^2)$ from (2) and (3), we obtain

$(1 + p)/(2x + 2zp) = (1 + q)/(2y + 2zq)$ or $(1 + p)(y + zq) = (1 + q)(x + zp)$

or $(y - z)p + (z - x)q = x - y$, which is the required partial differential equations.

Ex. 3. *Eliminate the arbitrary functions f and F from* $y = f(x - at) + F(x + at)$.

(Sagar 1997; Vikram 1995; Jabalpur 2002)

Sol. Given $$y = f(x - at) + F(x + at). \quad ...(1)$$

From (1), $$\partial y/\partial x = f'(x - at) + F'(x + at)$$

and hence $$\partial^2 y/\partial x^2 = f''(x - at) + F''(x + at). \quad ...(2)$$

Also, $$\partial y/\partial t = f'(x - at) \,.\, (-a) + F'(x + at) \,.\, (a)$$

and hence $$\partial^2 y/\partial t^2 = f''(x - at) \,.\, (-a)^2 + F''(x + at) \,.\, (a)^2$$

or $$\partial^2 y/\partial t^2 = a^2[f''(x - at) + F''(x + at)]. \quad ...(3)$$

Then, (2) and (3) $\Rightarrow$ $\partial^2 y/\partial t^2 = a^2(\partial^2 y/\partial x^2)$.

Ex. 4. *Eliminate arbitrary function f from*

(*i*) $z = f(x^2 - y^2)$. **[Bilaspur 1996; Sagar 1996; Bangalore 1995]**

(*ii*) $z = f(x^2 + y^2)$. **[Meerut 1995; Pune 2010]**

Sol. (*i*) Given $$z = f(x^2 - y^2). \quad ...(1)$$

Differentiating (1) partially with respect to x and y, we get

$\partial z/\partial x = f'(x^2 - y^2) \times 2x$ so that $f'(x^2 - y^2) = (1/2x) \times (\partial z/\partial x)$ ...(2)

and $\partial z/\partial y = f'(x^2 - y^2) \times (-2y)$ so that $f'(x^2 - y^2) = -(1/2y) \times (\partial z/\partial y)$. ...(3)

Eliminating $f'(x^2 - y^2)$ between (2) and (3), we have

$$\frac{1}{2x}\frac{\partial z}{\partial x} = -\frac{1}{2y}\frac{\partial z}{\partial y} \quad \text{or} \quad y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0.$$

(*ii*) Proceed as in part (1). **Ans.** $y(\partial z/\partial x) - x(\partial z/\partial y) = 0$

Ex. 5. *Form a partial differential equation by eliminating the function f from*

(*i*) $z = f(y/x)$. **[Sagar 2000]** (*ii*) $z = x^n f(y/x)$.

Sol. Given $z = f(y/x)$. ...(1)

Differentiating (1) partially with respect to x and y, we get

$$\partial z/\partial x = f'(y/x) \times (-y/x^2) \quad \text{or} \quad f'(y/x) = -(x^2/y) \times (\partial z/\partial x) \quad \text{...(2)}$$

and $$\partial z/\partial y = f'(y/x) \times (1/x) \quad \text{or} \quad f'(y/x) = x(\partial z/\partial y). \quad \text{...(3)}$$

Eliminating $f'(y/x)$ between (2) and (3), we have

$$-\frac{x^2}{y}\frac{\partial z}{\partial x} = x\frac{\partial z}{\partial y} \quad \text{or} \quad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0.$$

which is the required partial differential equation.

(*ii*) Given $z = x^n f(y/x)$. ...(1)

Differentiating (1) partially with respect to x and y, we get

$$\partial z/\partial x = n\,x^{n-1} f(y/x) + x^n f'(y/x) \times (-y/x^2) \quad \text{...(2)}$$

and $$\partial z/\partial y = x^n f'(y/x) \times (1/x). \quad \text{...(3)}$$

Multiplying both sides of (2) by x, we have $x(\partial z/\partial x) = n\,x^n f(y/x) - yx^{n-1} f'(y/x)$. ...(4)

Multiplying both sides of (3) by y, we have $y(\partial z/\partial y) = y\,x^{n-1} f'(y/x)$. ...(5)

Adding (4) and (5), $x(\partial z/\partial x) + y(\partial z/\partial y) = n\,x^n f(y/x)$

or $$x(\partial z/\partial x) + y(\partial z/\partial y) = nz, \text{ by (1)}$$

Ex. 6. *Form a partial differential equation by eliminating the function* ϕ from $lx + my + nz = \phi(x^2 + y^2 + z^2)$. **[Ravishankar 2003; Vikram 2003]**

Sol. Given $$lx + my + nz = \phi(x^2 + y^2 + z^2). \quad ...(1)$$

Differentiating (1) partially with respect to x and y, we get

$$l + n(\partial z/\partial x) = \phi'(x^2 + y^2 + z^2) \times \{2x + 2z(\partial z/\partial x)\} \quad ...(2)$$

and $$m + n(\partial z/\partial y) = \phi'(x^2 + y^2 + z^2) \times \{2y + 2z(\partial z/\partial y)\} \quad ...(3)$$

Dividing (2) by (3), we get $$\frac{l+n(\partial z/\partial x)}{m+n(\partial z/\partial y)} = \frac{2\{x+z(\partial z/\partial x)\}}{2\{y+z(\partial z/\partial y)\}}$$

or $(ny - mz)(\partial z/\partial x) + (lz - nx)(\partial z/\partial y) = mx - ly$, which is the required partial differential equation.

Ex. 7. *Form partial differential eqn. by eliminating the function f from* $z = e^{ax+by} f(ax - by)$.

Sol. Given $$z = e^{ax+by} f(ax - by). \quad ...(1)$$

Differentiating (1) partially with respect to x and y, we get

$$\partial z/\partial x = e^{ax+by}\, a\, f'(ax - by) + a\, e^{ax+by} f(ax - by) \quad ...(2)$$

and $$\partial z/\partial y = e^{ax+by} \{-b\, f'(ax - by)\} + b\, e^{ax+by} f(ax - by). \quad ...(3)$$

Multiplying (2) by b and (3) by a and adding, we get

$b(\partial z/\partial x) + a(\partial z/\partial y) = 2ab\, e^{ax+by} f(ax - by)$ or $b(\partial z/\partial x) + a(\partial z/\partial y) = 2abz$, by (1)

Ex. 12. *Form a partial differential equation by eliminating the arbitrary function* ϕ *from* $\phi(x^2 + y^2 + z^2, z^2 - 2xy) = 0$. **[Nagpur 1996; 2002]**

Sol. Given $\phi(x^2 + y^2 + z^2, z^2 - 2xy) = 0$. ...(1)

Let $u = x^2 + y^2 + z^2$ and $v = z^2 - 2xy$. ...(2)

Then, (1) becomes $\phi(u, v) = 0$. ...(3)

Differentiating (3) partially w.r.t. 'x', we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0, \quad ...(4)$$

where $p = \partial z/\partial x$ and $q = \partial z/\partial y$. Now, from (2), we have

$\partial u/\partial x = 2x$, $\partial u/\partial y = 2y$, $\partial u/\partial z = 2z$, $\partial v/\partial x = -2y$, $\partial v/\partial y = -2x$, $\partial v/\partial z = 2z$. ...(5)

Using (5), (4) reduces to $(\partial\phi/\partial u)(2x + 2pz) + (\partial\phi/\partial v)(-2y + 2pz) = 0$

or $(x + pz)(\partial\phi/\partial u) = (y - pz)(\partial\phi/\partial v)$. ...(6)

Again, differentiating (3) partially w.r.t. 'y', we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial y}\right) = 0$$

or $(\partial\phi/\partial u)(2y + 2qz) + (\partial\phi/\partial v)(-2x + 2qz) = 0$, by (5)

or $(y + qz)(\partial\phi/\partial u) = (x - qz)(\partial\phi/\partial v)$. ...(7)

Dividing (6) by (7), $(x + pz)/(y + qz) = (y - pz)/(x - qz)$

or $pz(y + x) - qz(y + x) = y^2 - x^2$ or $(p - q)z = y - x$.

H.W. 1. $x^2+y^2+(z-c)^2a^2$ 2. $z=f(x^2-y^2)$ 3. $z=f(x+it)+g(x-it)$

4. $f(x^2+y^2, z-xy)$ 5. $xyz=\varphi(x+y+z)$