Math-183
Differential Equations

Note taken by: Turja Roy
ID: 2108052

Contents

1 Differential Equations and Their Solutions

1.1 Classification of Differential Equations

Definition 1.1.1: Differential Equation

Differential equation is an equation involving derivatives of one or more dependent variables with respect to one or more independent variables.

Definition 1.1.2: Ordinary Differential Equation

A differential equation involving ordinary derivatives of one or more dependent variables with respect to a single independent variable is called an ordinary differential equation.

Example 1.1.1: Ordinary Differential Equations:

$$\frac{dy}{dx} + xy\left(\frac{d}{dx}\right)^2 = 0 \tag{1.1.1}$$

$$\frac{d^4x}{dt^4} + 5\frac{d^2x}{dt^2} + 3x = \sin t \tag{1.1.2}$$

Definition 1.1.3: Partial Differential Equation

A differential equation involving partial derivatives of one or more dependent variables with respect to more than one independent variables is called an partial differential equation.

Example 1.1.2: Partial Differential Equations:

$$\frac{\partial v}{\partial s} + \frac{\partial v}{\partial t} = v \tag{1.1.3}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \tag{1.1.4}$$

Definition 1.1.4: Order and Degree of Differential Equations

Order of DE: The order of the highest ordered derivative involved in a differential equation is called the order of the differential equation.

Degree of DE: The power of the highest order derivative involved in a differential equation is called the degree of the differential equation.

Definition 1.1.5: Linearity of Differential Equations

If the dependent variable and its various derivatives occur to the first degree only, the DE is a linear DE. Otherwise it's a non-linear DE.

$$a_0(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_1(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + a_{n-1}(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_n(x)y = b(x)$$

Linear DE can also be classified as linear with *constant* and *variable* coefficients.

Example 1.1.3: Ordinary Differential Equations: Orders, Degree, Linearity

$$\begin{aligned}
\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} - 3\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 3\frac{\mathrm{d}y}{\mathrm{d}x} - 6y = \sin x &\quad \text{3rd ord 1st deg Lin} \\
\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + y = 0 &\quad \text{2nd ord 1st deg Non-Lin} \\
y = x\frac{\mathrm{d}y}{\mathrm{d}x} + \sqrt{1 + \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}} &\quad \text{2nd ord 1st deg Non-Lin} \\
\frac{\mathrm{d}^4 x}{\mathrm{d}t^4} + t^2\frac{\mathrm{d}^3 x}{\mathrm{d}t^3} + \frac{\mathrm{d}y}{\mathrm{d}x} = \sin t &\quad \text{4th ord 1st deg Lin} \\
\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 5\frac{\mathrm{d}y}{\mathrm{d}x} + 6y^2 = 0 &\quad \text{2nd ord 1st deg Non-Lin} \\
\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 5\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3 + 6y = 0 &\quad \text{2nd ord 1st deg Non-Lin} \\
\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 5y\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = 0 &\quad \text{2nd ord 1st deg Lin}
\end{aligned}$$

1.2 Solutions

A Nature of Solutions

An nth-order Differential Equation:

$$F\left[x, y, \frac{\mathrm{d}y}{\mathrm{d}x}, \cdots, \frac{\mathrm{d}^n y}{\mathrm{d}x^n}\right] = 0 \tag{1.2.1}$$

Definition 1.2.1: Explicit solution

f is an explicit solution of (1.2.1) if

$$\forall x \in I, F\left[x, f(x), f'(x), \cdots, f^{(n)}(x)\right] = 0$$

where I is a real interval.

Definition 1.2.2: Implicit solution

$g(x, y) = 0$ is an implicit solution if this relation defines at least one real function $f(x)$ on an interval I such that f is an explicit solution of (1.2.1)

Example 1.2.1: Explicit and Implicit Solutions

$$x^2 + y^2 - 25 = 0 \quad : \quad \text{Implicit solution}$$
$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$x + y\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \quad : \quad \text{Differential Equation}$$
$$y = \pm\sqrt{25 - x^2}\ ;\ -5 \le x \le 5 \quad : \quad \text{Explicit solution}$$

B Methods of Solution

The study of a Differential Equation consists of 3 phases:

1. Formulation of DE from the given physical situation.
2. Solutions of DE, evaluating the arbitrary constants from the given condition.
3. Physical interpretation of the solution.

Example 1.2.2: Show that the function $f(x) = e^x + 2x^2 + 6x + 7$ is a solution to the DE $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 4x^2$

$$f(x) = e^x + 2x^2 + 6x + 7$$
$$f'(x) = e^x + 4x + 6$$
$$f''(x) = e^x + 4$$

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y &= (e^x + 4) - 3(e^x + 4x + 6) + 2(e^x + 2x^2 + 6x + 7)\\ &= 0 \cdot e^x + 0 \cdot x + (4 - 18 + 14) + 4x^2\\ &= 4x^2\end{aligned}$$

□

Example 1.2.3: Show that the function $f(x) = \dfrac{1}{1+x^2}$ is a solution to the DE $(1 + x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0$

$$f(x) = \frac{1}{1+x^2}$$
$$(1+x^2)f(x) = 1$$
$$(1+x^2)f'(x) + 2xf(x) = 0$$
$$(1+x^2)f''(x) + 2xf'(x) + 2xf'(x) + 2f(x) = 0$$
$$(1+x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0$$

□

Example 1.2.4: Show that the function $y = (2x^2 + 2e^{3x} + 3)e^{-2x}$ satisfies the DE

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 6e^x + 4xe^{-2x}$$

$$y = (2x^2 + 2e^{3x} + 3)e^{-2x}$$
$$y_1 = (4x + 6e^{3x})e^{-2x} - (2x^2 + 2e^{3x} + 3)2e^{-2x}$$
$$y_1 = 4xe^{-2x} + 6e^x - 2y$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 6e^x + 4e^{-2x}$$

□

1.3 Initial-Value and Boundary-Value Problems, and Existence of Solutions

A Initial-value Problems and Boundary-value Problems

One of the most frequently encountered type of problems in Differential Equations involves both a DE and one or more supplementary conditions which the solution of the given DE must satisfy.

Definition 1.3.1: IVP and BVP

Consider the first-order DE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$$

where f is a continuous function of x and y in some domain D of the xy plane; and let (x_0, y_0) be a point of D. The **initial-value problem** associated with the DE is to find a solution ϕ of the DE, defined on some real interval containing x_0, and satisfying the initial condition

$$\phi(x_0) = y_0$$

In the customary abbreviated notation, this initial-value problem may be written

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$$

$$y(x_0) = y_0$$

If the conditions relate to two different x values (the extreme or boundary values), the proble is called a **Two-Point Boundary-Value Problem** or simply a **Boundary-Value Problem (BVP)**.

Example 1.3.1: Find the solution of the DE $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$ such that $\forall x \in I, f'(x) = 2x$ and $f(1) = 4$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$$

$$\int \frac{\mathrm{d}y}{\mathrm{d}x}\, dx = \int 2x\, dx$$

$$y = x^2 + c$$

Substituting $y = 4$ and $x = 1$,

$$4 = 1 + c \text{ or } c = 3$$

$$\therefore \text{ Solution: } y^2 = x + 3$$

□

Example 1.3.2: $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$, $y(3) = 4$

$$x + y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\int x\, dx + \int y\frac{\mathrm{d}y}{\mathrm{d}x}\, dx = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} = c'$$

$$x^2 + y^2 = c$$

Substituting $x = 3$ and $y = 4$,

$$16^2 + 3^2 = c \text{ or } c = 25$$

$$\therefore \text{ Solution: } x^2 + y^2 - 25 = 0$$

B Existence of Solutions

Not all initial-value and boundary-value problems have solutions. For example,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y = 0$$

$$y(0) = 1 \quad , \quad y(\pi) = 5$$

has no solutions! Thus arises the question of *existence* of solutions. We can say, every initial-value problem that satisfies definition (1.3.1) has *at least one* solution. However, there arises another question. Can a problem have more than one solution?
Let's consider the initial-value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^{1/3} \;\; ; \;\; y(0) = 0$$

One may verify that the functions f_1 and f_2 defined, respectively, by

$$\forall x \in \mathbb{R}, \;\; f_1(x) = 0$$

and

$$f_2(x) = (\frac{2}{3}x)^{3/2}, \;\; x \geq 0; \;\; f_2(x) = 0, \;\; x \leq 0$$

are both solutions of this initial-value problem. In fact, this problem has infinitely many solutions. Hence, we can state that the initial-value problem need not have a *unique* solution. In order to ensure uniqeness, some additional requiremnt must certainly be imposed.

***Theorem 1.3.1* (Basic Existence and Uniqueness Theorem):**
Hypothesis: *Consider the differential equation*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) \tag{1.3.1}$$

where

- *The function f is a continuous function of x and y in some domain D of the xy plane, and*
- *The partial derivative $\frac{\partial f}{\partial y}$ is also a continuous function of x and y in D; and let (x_0, y_0) be a point in D.*

Conclusion: *There exists a unique solution ϕ of the differential equation (1.3.1), defined on some interval $|x - x_0| \leq h$, where h is sufficiently small, that satisfies the condition*

$$\phi(x_0) = y_0$$

Example 1.3.3: Show that

$$y = 4e^{2x} + 2e^{-3x}$$

is a solution of the initial-value problem

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} - 6y = 0$$

$$y(0) = 6$$

$$y'(0) = 2$$

Is $y = 2e^{2x} + 4e^{-3x}$ also a solution of this problem? Explain why or why not.

$$y = 4e^{2x} + 2e^{-3x}$$
$$y_1 = 8e^{2x} - 6e^{-3x}$$
$$y_2 = 16e^{2x} + 18e^{-3x}$$

$$\begin{aligned} y_2 + y_1 - 6y &= (16e^{2x} + 18e^{-3x}) + (8e^{2x} - 6e^{-3x}) - 6(4e^{2x} + 2e^{-3x}) \\ &= 0 \cdot e^{2x} + 0 \cdot e^{-3x} \\ &= 0 \end{aligned}$$

The solution also satisfies $y(0) = 6$ and $y'(0) = 2$

Now, for $y = 2e^{2x} + 4e^{-3x}$,

$$y_1 = 4e^{2x} - 12e^{-3x} \ \ ; \ \ y_2 = 8e^{2x} + 36e^{-3x}$$

$$\begin{aligned} y_2 + y_1 - 6y &= (8e^{2x} + 36e^{-3x}) + (4e^{2x} - 12e^{-3x}) - 6(2e^{2x} + 4e^{-3x}) \\ &= 0 \cdot e^{2x} + 0 \cdot e^{-3x} \\ &= 0 \end{aligned}$$

However, in this case,

$$y(0) = 6 \ ; \ y'(0) = -8$$

As we can see, this solution doesn't satisfy the initial-value problem. Hence $y = 2e^{2x} + 4e^{-3x}$ is not a solution of this problem.

Example 1.3.4: Given that every solution of

$$x^3 \frac{\mathrm{d}^3 y}{\mathrm{d}x^3} - 3x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 6x \frac{\mathrm{d}y}{\mathrm{d}x} - 6y = 0$$

may be written in the form $y = c_1 x + c_2 x^2 + c_3 x^3$ for some choice of the arbitrary constants c_1, c_2, and c_3, solve the initial-value problem consisting of the above DE plus the three conditions

$$y(2) = 0 \ , \ y'(2) = 2 \ , \ y''(2) = 6$$

$$y = c_1 x + c_2 x^2 + c_3 x^3$$

$$y(2) = 0 \text{ or, } 8c_3 + 4c_2 + 2c_1 = 0 \tag{1.3.2}$$

$$y' = c_1 + 2c_2 x + 3c_3 x^2$$

$$y'(2) = 2 \text{ or, } 12c_3 + 4c_2 + c_1 = 2 \tag{1.3.3}$$

$$y'' = 0 + 2c_2 + 6c_3 x$$

$$y''(2) = 6 \text{ or, } 12c_3 + 2c_2 + 0c_1 = 6 \tag{1.3.4}$$

Solving (1.3.1), (1.3.2), and (1.3.3) we get,

$$c_1 = 2 \ , \ c_2 = -3 \ , \ c_3 = 1$$

$$\therefore \text{ Solution: } y = 2x - 3x^2 + x^3$$

2 First Order Equations for Which Exact Solutions Are Obtainable

2.1 Exact Differential Equations and Integrating Factors

A Standard Forms of First-Order Differential Equations

The first-order differential equations may be expressed in either the **Derivative Form**

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) \tag{2.1.1}$$

or the **Differential Form**

$$M(x, y)\, dx + N(x, y)\, dy = 0 \tag{2.1.2}$$

Example 2.1.1: Standard Forms

The equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2 + y^2}{x - y}$$

is the form (2.1.1). It may be written as

$$(x^2 + y^2)\, dx + (y - x)\, dy = 0$$

which is of the form (2.1.2).
Again, the equation

$$(\sin x + y)\, dx + (x + 3y)\, dy = 0$$

is of the form (2.1.2), which can also be written as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\sin x + y}{x + 3y}$$

B Exact Differential Equations

Definition 2.1.1: Exact Differential

Let F be a function of two real variables such that F has continuous first partial derivatives in a domain D. The total differential dF of the function F is defined by the formula

$$dF(x, y) = \frac{\partial F(x, y)}{\partial x}\, dx + \frac{\partial F(x, y)}{\partial y}\, dy$$

for all $(x, y) \in D$.

Comparing $dF(x, y)$ with the form (2.1.2), we get

$$\frac{\partial F(x, y)}{\partial x} = M(x, y) \quad \textit{and} \quad \frac{\partial F(x, y)}{\partial y} = N(x, y)$$

Example 2.1.2

Let F be a function

$$F(x, y) = xy^2 + 2x^3 y$$

for all real (x, y). Then

$$\frac{\partial F(x,y)}{\partial x} = y^2 + 6x^2 y, \quad \frac{\partial F(x,y)}{\partial y} = 2xy + 2x^3$$

and the total differential dF is defined by

$$dF(x,y) = (y^2 + 6x^2 y)\, dx + (2xy + 2x^3)\, dy$$

for all real (x, y)

Definition 2.1.2: Exact Differential Equation

The expression

$$M(x,y)\, dx + N(x,y)\, dy \tag{2.1.3}$$

is called an exact differential in a domain D if there exists a function F of two real variables such that this expression equals the total differential $dF(x, y)$ for all $(x, y) \in D$. That is, expression (2.1.3) is an exact differential in D if there exists a function F such that

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) \quad \textit{and} \quad \frac{\partial F(x,y)}{\partial y} = N(x,y)$$

for all $(x, y) \in D$.
If $M(x, y)\, dx + N(x, y)\, dy$ is an exact differential, then the differential equation

$$M(x,y)\, dx + N(x,y)\, dy = 0$$

is called an **Exact Differential Equation**.

Theorem 2.1.1* (Exact Differential Equation)*:

1. *If the DE $M(x,y)\, dx + N(x,y)\, dy = 0$ is exact in D, then*

$$\forall (x,y) \in D, \quad \frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

2. *Conversely, if*

$$\forall (x,y) \in D, \quad \frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

then the DE is exact in D.

Proof (1):

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) \quad , \quad \frac{\partial F(x,y)}{\partial y} = N(x,y)$$

$$\frac{\partial^2 F(x,y)}{\partial x \partial y} = \frac{\partial M(x,y)}{\partial y} \quad , \quad \frac{\partial^2 F(x,y)}{\partial x \partial y} = \frac{\partial N(x,y)}{\partial x}$$

$$\because \frac{\partial^2 F(x,y)}{\partial y \partial x} = \frac{\partial^2 F(x,y)}{\partial x \partial y}$$

$$\therefore \frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x} \quad \square$$

C The Solution of Exact Differential Equations

***Theorem 2.1.2* (Solution of Exact DE):**
If $M(x,y)\,dx + N(x,y)\,dy = 0$ is exact in domain D, then

$$\forall (x,y) \in D, \exists F(x,y) : \frac{\partial F(x,y)}{\partial x} = M(x,y) \quad \textit{and} \quad \frac{\partial F(x,y)}{\partial y} = N(x,y)$$

Then the equation may be written

$$\frac{\partial F(x,y)}{\partial x}\,dx + \frac{\partial F(x,y)}{\partial y}\,dy = 0$$

or simply,

$$dF(x,y) = 0$$

Here, $F(x,y) = c$ is a one-parameter family of solutions of this DE, where c is an arbitrary constant.

Example 2.1.3: Solve the equation

$$(3x^2 + 4xy)\,dx + (2x^2 + 2y)\,dy = 0$$

Standard Method:

$$\begin{aligned}
\frac{\partial F(x,y)}{\partial x} &= M(x,y) = 3x^2 + 4xy \\
F(x,y) &= \int (3x^2 + 4xy)\,\partial x + \phi(y) \\
&= x^3 + 2x^2y + \phi(y)
\end{aligned}$$

Again,

$$\frac{\partial F(x,y)}{\partial y} = 2x^2 + \frac{\partial \phi(y)}{\partial y} = 2x^2 + 2y$$

$$\frac{\mathrm{d}\phi(y)}{\mathrm{d}y} = 2y$$

$$\int \frac{\mathrm{d}\phi(y)}{\mathrm{d}y}\,dy = \int 2y\,dy$$

$$\phi(y) = y^2 + c_0$$

Thus, we get

$$F(x,y) = x^3 + 2x^2y + y^2 + c_0$$

Hence, a one-parameter family of the solution is $F(x,y) = c_1$ or,

$$x^3 + 2x^2y + y^2 + c_0 = c_1$$

$$\boxed{x^3 + 2x^2y + y^2 = c}$$

Method of Grouping:

$$
\begin{aligned}
(3x^2 + 4xy)\,dx + (2x^2 + 2y)\,dy &= 0 \\
3x^2\,dx + (4xy\,dx + 2x^2\,dy) + 2y\,dy &= 0 \\
d(x^3) + d(2x^2y) + d(y^2) &= d(c) \\
\boxed{x^3 + 2x^2y + y^2 = c}
\end{aligned}
$$

Example 2.1.4: Solve the initial-value problem

$$(2x\cos y + 3x^2y)\,dx + (x^3 - x^2\sin y - y)\,dy = 0 \quad ; \quad y(0) = 2$$

$$
\begin{aligned}
(2x\cos y\,dx - x^2\sin y\,dy) + (3x^2y\,dx + x^3\,dy) - y\,dy &= 0 \\
d(x^2\cos y) + d(x^3y) + d(\frac{y^2}{2}) &= d(c_1) \\
2x^2\cos y + x^3y + y^2 &= c
\end{aligned}
$$

Substituting $x = 0$ and $y = 2$,

$$2^2 = c$$

Hence, the solution is:

$$2x^2\cos y + x^3y + y^2 = 4$$

D Integrating Factors

Definition 2.1.3: Integrating Factor (IF)

If the DE

$$M(x,y)\,dx + N(x,y)\,dy = 0 \tag{2.1.4}$$

is not exact in a domain D but the DE

$$\mu(x,y)M(x,y)\,dx + \mu(x,y)N(x,y)\,dy = 0 \tag{2.1.5}$$

is exact in D, then $\mu(x,y)$ is called an **Integrating Factor** of the DE.

Example 2.1.5: Integrating factor

Consider the DE

$$(3y + 4xy^2)\,dx + (2x + 3x^2y)\,dy = 0 \tag{2.1.6}$$

This equation is of the form (2.1.4), where

$$
\begin{aligned}
M(x,y) &= 3y + 4xy^2, & N(x,y) &= 2x + 3x^2y \\
\frac{\partial M(x,y)}{\partial y} &= 3 + 8xy, & \frac{\partial N(x,y)}{\partial x} &= 2 + 6xy
\end{aligned}
$$

Since

$$\frac{\partial M(x,y)}{\partial y} \neq \frac{\partial N(x,y)}{\partial x}$$

except for (x,y) such that $2xy + 1 = 0$, Equation (2.1.4) is not exact in any rectangular domain D.

Let $\mu(x, y) = x^2y$. Then the corresponding DE of the form (2.1.5) is

$$(3x^2y^2 + 4x^3y^3)\,dx + (2x^3y + 3x^4y^2)\,dy = 0$$

This equation is exact in every rectangular domain D, since

$$\frac{\partial[\mu(x,y)M(x,y)]}{\partial y} = 6x^2y + 12x^3y^2 = \frac{\partial[\mu(x,y)N(x,y)]}{\partial x}$$

For all real (x, y). Hence, $\mu(x, y) = x^2y$ is an integrating factor of Equation (2.1.6).

Example 2.1.6: Determine whether or not the following equation is exact

$$\left(\frac{x}{y^2} + x\right)\,dx + \left(\frac{x^2}{y^3} + y\right)\,dy = 0$$

$$\frac{\partial M(x,y)}{\partial y} = -\frac{x}{2y^3}$$
$$\frac{\partial N(x,y)}{\partial x} = \frac{2x}{y^3}$$

Here, $\frac{\partial M(x,y)}{\partial y} \neq \frac{\partial N(x,y)}{\partial x}$. Hence, the equation is not exact.

Example 2.1.7: Determine the constant A in the following equations such that the equation is exact

1. $(Ax^2y + 2y^2)\,dx + x^3 + 4xy\,dy = 0$
2. $\left(\frac{Ay}{x^3} + \frac{y}{x^2}\right)\,dx + \left(\frac{1}{x^2} - \frac{1}{x}\right)\,dy = 0$

1.

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial(Ax^2y + 2y^2)}{\partial y} = Ax^2 + 4y$$
$$\frac{\partial N(x,y)}{\partial x} = \frac{\partial(x^3 + 4xy)}{\partial x} = 3x^2 + 4y$$

Equating the coefficients of x^2, we get

$$\boxed{A = 3}$$

2.

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial\left(\frac{Ay}{x^3} + \frac{y}{x^2}\right)}{\partial y} = \frac{A}{x^3} + \frac{1}{x^2}$$
$$\frac{\partial N(x,y)}{\partial x} = \frac{\partial\left(\frac{1}{x^2} - \frac{1}{x}\right)}{\partial x} = -\frac{1}{2x^3} + \frac{1}{x^2}$$

Equating the coefficients of $\frac{1}{x^3}$, we get

$$\boxed{A = -\frac{1}{2}}$$

2.2 Separable Equations and Equations Reducible to this Form

A Separable Equations

Definition 2.2.1: Separable Equations

An equations of the form

$$F(x)G(y)\,dx + f(x)g(y)\,dy = 0 \tag{2.2.1}$$

is called an equation with separable variables or simply a separable equation.

***Theorem 2.2.1* (Solution of Separable Differential Equations):**
In general, the separable equations are not exact, but they possess an obvious integrating factor $\dfrac{1}{f(x)G(y)}$
Thus the equation (2.2.1) becomes

$$\frac{F(x)}{f(x)}\,dx + \frac{g(y)}{G(y)}\,dy = 0 \tag{2.2.2}$$

which is exact, because

$$\frac{\partial}{\partial y}\left(\frac{F(x)}{f(x)}\right) = 0 = \frac{\partial}{\partial x}\left(\frac{g(y)}{G(y)}\right)$$

We can write the equation (2.2.2) as

$$M(x)\,dx + N(y)\,dy = 0$$

where $M(x) = \dfrac{F(x)}{f(x)}$ *and* $N(y) = \dfrac{g(y)}{G(y)}$
A one-parameter family solution to the DE is

$$\int M(x)\,dx + \int N(y)\,dy = c \tag{2.2.3}$$

Example 2.2.1: Solve the equation

$$(x-4)y^4\,dx - x^3(y^2-3)\,dy = 0$$

The equation is separable; dividing by x^3y^4 we obtain

$$\frac{x-4}{x^3}\,dx - \frac{y^2-3}{y^4}\,dy = 0$$

$$\int (x^{-2} - 4x^{-3})\,dx - \int (y^{-2} - 3y^{-4})\,dy = 0$$

$$\boxed{-\frac{1}{x} + \frac{2}{x^2} + \frac{1}{y} - \frac{1}{y^3} = c}$$

The DE in derivative form:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x-4)y^4}{x^3(y^2-3)}$$

Here, $y = 0$ is a solution which was lost in the separation process.

Example 2.2.2: Solve the initial-value problem that consists of the DE

$$x \sin y \, dx + (x^2 + 1) \cos y \, dy = 0$$

and the initial condition $y(1) = \frac{\pi}{2}$

$$\frac{x}{x^2+1}\,dx + \frac{\cos y}{\sin y}\,dy = 0$$

$$\frac{1}{2}\int \frac{d(x^2+1)}{x^2+1} + \int \cot y \, dy = 0$$

$$\frac{1}{2}\ln|x^2+1| + \ln|\sin y| = \ln|c_1|$$

$$\ln|(x^2+1)\sin^2 y| = \ln|c|$$

$$\therefore (x^2+1)\sin^2 y = c$$

Applying the initial condition, we get

$$2\sin^2\frac{\pi}{2} = c \text{ or, } c = 2$$

Thus, the solution is

$$\boxed{(x^2+1)\sin^2 y = 2}$$

B Homogeneous Equations

Definition 2.2.2: Homogeneous Equations

The first-orger differential equation

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is said to be homogeneous if, when written in the derivative form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$$

there exists a function g such that $f(x, y)$ can be expressed in the form $g(y/x)$

Example 2.2.3

The DE

$$x^2 - 3y^2\,dx + 2xy\,dy = 0$$

is homogeneous. To see this, we first write the derivative form of the equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3y^2 - x^2}{2xy} = \frac{3y}{2x} - \frac{x}{2y}$$

We see that the DE can be written as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{2}\left(\frac{y}{x}\right) - \frac{1}{2}\left(\frac{1}{y/x}\right)$$

in which the right side of the equation is of the form $g(y/x)$ for a certain function g.

Example 2.2.4: The equation

$$(y + \sqrt{x^2 + y^2})\, dx - x\, dy = 0$$

is homogeneous.

Derivative form:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{y + \sqrt{x^2 + y^2}}{x} \\
&= \frac{y}{x} + \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2}} \\
&= \frac{y}{x} + \sqrt{1 + \left(\frac{y}{x}\right)^2} = g\left(\frac{y}{x}\right)
\end{aligned}$$

Definition 2.2.3: Homogeneous Equation of degree n

A function F is called homogeneous of degree n if

$$F(tx, ty) = t^n F(x, y)$$

This means that if the tx and ty are substituted for x and y respectively in $F(x, y)$, and if t^n is then factored out, the other factor that remains is the original expression $F(x, y)$ itself.

For example, the function given by $F(x, y) = x^2 + y^2$ is homogeneous of degree 2, since

$$F(tx, ty) = (tx)^2 + (ty)^2 = t^2(x^2 + y^2) = t^2 F(x, y)$$

Now, suppose both $M(x, y)$ and $N(x, y)$ in the DE

$$M(x, y)\, dx + N(x, y)\, dy = 0$$

are homogeneous of the same degree n. Since $M(tx, ty) = t^n M(x, y)$, for $t = \frac{1}{x}$, we have

$$M\left(1, \frac{y}{x}\right) = \left(\frac{1}{x}\right)^n M(x, y)$$

$$M(x, y) = \left(\frac{1}{x}\right)^{-n} M\left(1, \frac{y}{x}\right)$$

Similarly,

$$N(x, y) = \left(\frac{1}{x}\right)^{-n} N\left(1, \frac{y}{x}\right)$$

Now, writing the DE in derivative form, we get

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{M(x, y)}{N(x, y)} \\
&= -\frac{\left(\frac{1}{x}\right)^{-n} M\left(1, \frac{y}{x}\right)}{\left(\frac{1}{x}\right)^{-n} N\left(1, \frac{y}{x}\right)} \\
&= -\frac{M\left(1, \frac{y}{x}\right)}{N\left(1, \frac{y}{x}\right)} \\
&= g\left(\frac{y}{x}\right)
\end{aligned}$$

Note:-

If $M(x,y)$ and $N(x,y)$ in

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

are both homogeneous functions of the same degree n, then the differential equation is a homogeneous differential equation.

Theorem 2.2.2:
If

$$M(x,y)\,dx + N(x,y)\,dy = 0 \tag{2.2.4}$$

is a homogeneous equation, then the change of variables $y = vx$ transforms the equation into a separable equation in the variables v and x.

Proof:
Since $M(x,y)\,dx + N(x,y)\,dy = 0$ is homogeneous, it may be written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g\left(\frac{y}{x}\right)$$

Let $y = vx$. Then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$$

and the initial equation becomes

$$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = g(v)$$

or,

$$[v - g(v)]\,dx + x\,dv = 0$$

This equation is separable. Separating the variables we obtain

$$\frac{dv}{v - g(v)} + \frac{dx}{x} = 0 \quad \square \tag{2.2.5}$$

Theorem 2.2.3* (Solution of a Homogeneous Differential Equation)*: To solve a DE of the form (2.2.4), we let $y = vx$ and transform the homogeneous equation into a separable equation of the form (2.2.5). From this, we have

$$\int \frac{dv}{v - g(v)} + \int \frac{dx}{x} = c$$

where c is an arbitrary constant. Letting $F(v)$ denote

$$\int \frac{dv}{v - g(v)}$$

and returning to the original dependent variable y, the solution takes the form

$$F\left(\frac{y}{x}\right) + \ln|x| = c$$

Example 2.2.5: Solve the equation

$$(x^2 - 3y^2)\,dx + 2xy\,dy = 0$$

Derivative form:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{2y} + \frac{3y}{2x}$$

Letting $y = vx$, we get

$$\begin{aligned} v + x\frac{\mathrm{d}v}{\mathrm{d}x} &= -\frac{1}{2v} + \frac{3v}{2} \\ x\frac{\mathrm{d}v}{\mathrm{d}x} &= \frac{v^2-1}{2v} \\ \frac{2v\,dv}{v^2-1} &= \frac{dx}{x} \end{aligned}$$

Integrating, we find

$$\begin{aligned} \ln|v^2-1| &= \ln|x| + \ln|c| \\ |v^2-1| &= |cx| \\ |\frac{y^2}{x^2} - 1| &= |cx| \\ |y^2 - x^2| &= x^2|cx| \end{aligned}$$

For $y \geq x \geq 0$, it can be written as

$$\boxed{y^2 - x^2 = cx^3}$$

Example 2.2.6: Solve the initial-value problem

$$(y + \sqrt{x^2+y^2})\,dx - x\,dy = 0, y(1) = 0$$

Derivative form:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} + \sqrt{1 + \left(\frac{y}{x}\right)^2}$$

Letting $y = vx$, we get

$$\begin{aligned} v + x\frac{\mathrm{d}v}{\mathrm{d}x} &= v + \sqrt{1+v^2} \\ \frac{dv}{\sqrt{1+v^2}} &= \frac{dx}{x} \\ \ln|v + \sqrt{1+v^2}| &= ln|x| + ln|c| \\ v + \sqrt{v^2+1} &= cx \\ \frac{y}{x} + \frac{1}{x}\sqrt{y^2+x^2} &= cx \\ y + \sqrt{x^2+y^2} &= cx^2 \end{aligned}$$

Applying the initial condition, we get

$$0 + \sqrt{1} = c \cdot 1 \text{ or, } c = 1$$

Hence, the solution:

$$\boxed{y + \sqrt{x^2 + y^2} = x^2} \text{ or, } \boxed{y = \frac{1}{2}(x^2 - 1)}$$

Exercise 2.1: Solve the following differential equations

1. $(xy + 2x + y + 2)\,dx + (x^2 + 2x)\,dy = 0$
2. $(2x\cos y + 3x^2 y)\,dx + (x^3 - x^2 - y)\,dy = 0, y(0) = 2$
3. $(e^v + 1)\cos u\,du + e^v(\sin u + 1)\,dv = 0$
4. $(x + 4)(y^2 + 1)\,dx + y(x^2 + 3x + 2)\,dy = 0$
5. $(2xy + 3y^2)\,dx - (2xy + x^2)\,dy = 0$
6. $(x + y)\,dx - x\,dy = 0$
7. $v^3\,du + (u^3 - uv^2)\,dv = 0$
8. $\left(x\tan\frac{y}{x} + y\right)\,dx - x\,dy = 0$
9. $(2s^2 + 2st + t^2)\,ds + (s^2 + 2st - t^2)\,dt = 0$
10. $(x^3 + y^2\sqrt{x^2 + y^2})\,dx - xy\sqrt{x^2 + y^2}\,dy = 0$
11. $(\sqrt{x + y} + \sqrt{x - y})\,dx + (\sqrt{x - y} - \sqrt{x + y})\,dy = 0$
12. $(3x + 8)(y^2 + 4)\,dx - 4y(x^2 + 5x + 6)\,dy = 0, y(1) = 2$
13. $(2x^2 + 2xy + y^2)\,dx + (x^2 + 2xy)\,dy = 0$

1.

$$\begin{aligned}
(xy + 2x + y + 2)\,dx + (x^2 + 2x)\,dy &= 0 \\
(x + 1)(y + 2)\,dx + x(x + 2)\,dy &= 0 \\
\int \frac{x + 1}{x(x + 2)}\,dx + \int \frac{dy}{y + 2} &= 0 \\
\frac{1}{2}\ln|x^2 + 2x| + \ln|y + 2| &= \ln|c_1| \\
\ln|(x^2 + 2x)(y + 2)^2| &= \ln|c| \\
\boxed{(x^2 + 2x)(y + 2)^2 = c}
\end{aligned}$$

2.

$$(2x\cos y + 3x^2 y)\,dx + (x^3 - x^2\sin^2 y - y)\,dy = 0$$

$$\begin{aligned}
F(x, y) &= \int (2x\cos y + 3x^2 y)\,\partial x + \phi(y) \\
&= x^2\cos y + x^3 y + \phi(y)
\end{aligned}$$

Now,

$$\frac{\partial F(x,y)}{\partial y} = x^3 - x^2 \sin y - y = x^3 - x^2 \sin y + \frac{\mathrm{d}}{\mathrm{d}x}\phi(y)$$

$$\therefore \phi(y) = -\int y\, dy = -\frac{y^2}{2} + c_0$$

$$F(x,y) = x^2 \cos y + x^3 y - \frac{y^2}{2} + c_0 = c_1$$
$$2x^2 \cos y + 2x^3 y - y^2 = c$$

Applying initial value,

$$c = -4$$

$$\boxed{2x^2 \cos y + 2x^3 y - y^2 + 4 = 0}$$

3.

$$(e^v + 1)\cos u\, du + e^v(\sin u + 1)\, dv = 0$$
$$(e^v \cos u\, du + e^v \sin u\, dv) + \cos u\, du + e^v\, dv = 0$$
$$d(e^v \sin u) + d(\sin u) + de^v = d(c)$$

$$\boxed{\sin u + e^v(\sin u + 1) = c}$$

4.

$$(x+4)(y^2+1)\, dx + y(x^2+3x+2)\, dy = 0$$
$$\int \frac{x+4}{x^2+3x+2}\, dx + \int \frac{y}{y^2+1}\, dy = 0$$
$$\int \frac{x+4}{(x+2)(x+1)}\, dx + \frac{1}{2}\int \frac{2y}{y^2+1}\, dy = 0$$
$$\int \frac{3}{x+1}\, dx - \int \frac{2}{x+2}\, dx + \frac{1}{2}\ln|y^2+1|$$
$$3\ln|x+1| - 2\ln|x+2| + \frac{1}{2}\ln|y^2+1| = \ln|c_1|$$
$$\ln\left|\frac{(x+1)^6}{(x+2)^4}\cdot(y^2+1)\right| = \ln|c|$$

$$\boxed{(x+6)^6(y^2+1) = c(x+2)^4}$$

5.

$$(2xy+3y^2)\,dx-(2xy+x^2)\,dy=0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2xy+3y^2}{2xy+x^2}=\frac{2\left(\frac{y}{x}\right)+3\left(\frac{y}{x}\right)^2}{2\left(\frac{y}{x}\right)+1}$$

Letting $y=vx$, we get

$$v+x\frac{\mathrm{d}v}{\mathrm{d}x}=\frac{2+3v^2}{2v+1}$$

$$x\frac{\mathrm{d}v}{\mathrm{d}x}=\frac{v^2+v}{2v+1}$$

$$\int\frac{2v+1}{v^2+v}\,dv=\int\frac{dx}{x}$$

$$\int\frac{d(v^2+v)}{v^2+v}=\int\frac{dx}{x}$$

$$\ln|v^2+v|=\ln|cx|$$

$$\frac{y^2}{x^2}+\frac{y}{x}=cx$$

$$\boxed{y^2+xy=cx^3}$$

6.

$$(x+y)\,dx-x\,dy=0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{x+y}{x}=1+\frac{y}{x}$$

Letting $y=vx$, we get

$$v+x\frac{\mathrm{d}v}{\mathrm{d}x}=1+v$$

$$\int dv=\int\frac{dx}{x}$$

$$\frac{y}{x}=\ln|cx|$$

$$\boxed{cx=e^{y/x}}$$

7.

$$v^3\,du+(u^3-uv^2)\,dv=0$$

$$\frac{\mathrm{d}u}{\mathrm{d}v}=\frac{uv^2-u^3}{v^3}=\frac{u}{v}-\left(\frac{u}{v}\right)^3$$

Letting $u=wv$, we get

$$w+v\frac{\mathrm{d}w}{\mathrm{d}v}=w-w^3$$

$$-\int\frac{dw}{w^3}=\int\frac{dv}{v}$$

$$\frac{1}{2w^2}=\ln|v|+c_1$$

$$\boxed{v^2=u^2(\ln v^2+c)}$$

8.

$$\left(x\tan\frac{y}{x}+y\right)\,dx-x\,dy=0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\tan\frac{y}{x}+\frac{y}{x}$$

Letting $y=vx$, we get

$$v+x\frac{\mathrm{d}v}{\mathrm{d}x}=\tan v+v$$

$$\int\frac{dv}{\tan v}=\int\frac{dx}{x}$$

$$\ln|\sin v|=\ln|cx|$$

$$\boxed{\sin\frac{y}{x}=cx}$$

9.

$$(2s^2+2st+t^2)\,ds+(s^2+2st-t^2)\,dt=0$$

$$\begin{aligned}\frac{\mathrm{d}s}{\mathrm{d}t}&=\frac{t^2-2st-s^2}{2s^2+2st+t^2}\\&=\frac{1-2\left(\dfrac{s}{t}\right)-\left(\dfrac{s}{t}\right)^2}{2\left(\dfrac{s}{t}\right)^2+2\left(\dfrac{s}{t}\right)+1}\end{aligned}$$

Letting $s = vt$, we get

$$
\begin{aligned}
v + t\frac{\mathrm{d}v}{\mathrm{d}t} &= \frac{1 - 2v - v^2}{2v^2 + 2v + 1} \\
t\frac{\mathrm{d}v}{\mathrm{d}t} &= -\frac{2v^3 + 3v^2 + 3v - 1}{2v^2 + 2v + 1} \\
-\int \frac{2v^2 + 2v + 1}{2v^3 + 3v^2 + 3v - 1}\, dv &= \int \frac{dt}{t} \\
-\frac{1}{3}\int \frac{d(2v^3 + 3v^2 + 3v - 1)}{2v^3 + 3v^2 + 3v - 1} &= \ln|t| + \ln|c_1|
\end{aligned}
$$

$$\boxed{2v^3 + 3v^2 + 3v - 1 = \frac{c}{t^3}}$$

10.

$$(x^3 + y^2\sqrt{x^2 + y^2})\, dx - xy\sqrt{x^2 + y^2}\, dy = 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^3 + y^2\sqrt{x^2 + y^2}}{xy\sqrt{x^2 + y^2}} = \frac{1 + \left(\frac{y}{x}\right)^2\sqrt{1 + \left(\frac{y}{x}\right)^2}}{\frac{y}{x}\sqrt{1 + \left(\frac{y}{x}\right)^2}}$$

Letting $y = vx$,

$$
\begin{aligned}
v + x\frac{\mathrm{d}v}{\mathrm{d}x} &= \frac{1 + v^2\sqrt{1 + v^2}}{v\sqrt{1 + v^2}} \\
x\frac{\mathrm{d}v}{\mathrm{d}x} &= \frac{1}{v\sqrt{1 + v^2}} \\
\int v\sqrt{1 + v^2}\, dv &= \int \frac{dx}{x} \\
\frac{1}{2}\int \sqrt{1 + v^2}\, d(1 + v^2) &= \ln|c_1 x| \\
(1 + v^2)^{3/2} &= 3\ln|c_1 x| \\
\left(1 + \frac{y^2}{x^2}\right)\sqrt{1 + \frac{y^2}{x^2}} &= \ln|cx^3|
\end{aligned}
$$

$$\boxed{(x^2 + y^2)\sqrt{x^2 + y^2} = x^3\ln|cx^3|}$$

11.

$$(\sqrt{x + y} + \sqrt{x - y})\, dx + (\sqrt{x - y} - \sqrt{x + y})\, dy = 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sqrt{\frac{x}{y} + 1} - \sqrt{x - y} - 1}{\sqrt{\frac{x}{y} + 1} + \sqrt{\frac{x}{y} - 1}}$$

Letting $x = vy$,

$$\begin{aligned}
v + y\frac{\mathrm{d}v}{\mathrm{d}y} &= \frac{\sqrt{v+1} - \sqrt{v-1}}{\sqrt{v+1} + \sqrt{v-1}} \\
&= \frac{v+1+v-1-2\sqrt{v^2-1}}{v+1-v+1} \\
v + y\frac{\mathrm{d}v}{\mathrm{d}y} &= v - \sqrt{v^2-1} \\
\int \frac{dv}{\sqrt{v^2-1}} &= -\int \frac{dy}{y} \\
\ln|v + \sqrt{v^2-1}| &= \ln\left|\frac{c}{y}\right| \\
\frac{x}{y} + \sqrt{\frac{x^2}{y^2} - 1} &= \frac{c}{y}
\end{aligned}$$

$$\boxed{x + \sqrt{x^2 - y^2} = c}$$

12.

$$(3x+8)(y^2+4)\,dx + 4y(x^2+5x+6)\,dy = 0$$
$$\frac{3x+8}{x^2+5x+6}\,dx - \frac{4y}{y^2+4}\,dy = 0$$

Here,

$$\frac{3x+8}{(x+3)(x+2)} = \frac{1}{x+3} + \frac{2}{x+2}$$
$$\therefore \int \frac{dx}{x+3} + 2\int \frac{dx}{x+2} - 2\ln|y^2+4| = c_2$$
$$\ln|x+3| + 2\ln|x+2| = 2\ln|c_1(y^2+4)|$$
$$(x+3)(x+2)^2 = c(y^2+4)$$

Applying initial value,

$$c = \frac{9}{16}$$
$$\boxed{16(x+3)(x+2)^2 = 9(y^2+4)}$$

13.

$$(2x^2 + 2xy + y^2)\,dx + (x^2 + 2xy)\,dy = 0$$
$$(2xy\,dx + x^2\,dy) + (y^2\,dx + 2xy\,dy) + 2x^2\,dx = 0$$
$$d(x^2y) + d(xy^2) + d\left(\frac{2}{3}x^3\right) = d(c_1)$$
$$x^2y + xy^2 + \frac{2}{3}x^3 = c_1$$
$$\boxed{2x^3 + 3x^2y + 3xy^2 = c}$$

Exercise 2.2: Show that the homogeneous equation

$$(Ax^2 + Bxy + Cy^2)\, dx + (Dx^2 + Exy + Fy^2)\, dy = 0$$

is exact if and only if $B = 2D$ and $E = 2C$.

The equation is exact if and only if

$$\frac{\partial(Ax^2 + Bxy + Cy^2)}{\partial y} = \frac{\partial(Dx^2 + Exy + Fy^2)}{\partial x}$$

$$Bx + 2Cy = 2Dx + Ey$$

Equating the coefficients of x, $B = 2D$
Equating the coefficients of y, $E = 2C$

2.3 Linear Equations and Bernoulli Equations

A Linear Equation

Definition 2.3.1: Linear Equation

A first-order ordinary differential equation is linear in the dependent variable y and independent variable x if it is, or can be, written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x) \tag{2.3.1}$$

Equation (2.3.1) can also be written as

$$[P(x)y - Q(x)]\, dx + \, dy = 0 \tag{2.3.2}$$

Here,

$$\frac{\partial}{\partial y}M(x, y) = P(x, y) \quad \text{and} \quad \frac{\partial}{\partial x}N(x, y) = 0$$

The equation is not exact. So we multiply both sides of (2.3.2) by an integrating factor:

$$[\mu(x)P(x)y - \mu(x)Q(x)]\, dx + \mu(x)\, dy = 0$$

Now,

$$\begin{aligned}
\frac{\partial}{\partial y}\,[\mu(x)M(x, y)] &= \frac{\partial}{\partial x}\,[\mu(x)N(x, y)] \\
\frac{\partial}{\partial y}\,[\mu(x)P(x)y - \mu(x)Q(x)] &= \frac{\partial}{\partial x}\,[\mu(x)] \\
\mu P(x) &= \frac{d}{dx}\,\mu \\
\int P(x)\, dx &= \int \frac{d\mu}{\mu} \\
\ln|\mu| &= \int P(x)\, dx
\end{aligned}$$

$$\boxed{\mu = e^{\int P(x)\, dx}}$$

***Theorem 2.3.1* (Solution of Linear Differential Equation):**
The linear differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$$

has an integrating factor of the form

$$\mu = e^{\int P(x)\,dx} \tag{2.3.3}$$

A one-parameter family of solution of this equation is

$$\mu\, y = \int \mu\, Q(x)\, dx + c$$

or

$$y\, e^{\int P(x)\,dx} = \int e^{\int P(x)\,dx}\, Q(x)\, dx + c \tag{2.3.4}$$

That is,

$$y = e^{-\int P(x)\,dx}\left[\int e^{\int P(x)\,dx}\, Q(x)\, dx + c\right] \tag{2.3.5}$$

Example 2.3.1: Solve the Linear Differential Equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \left(\frac{2x+1}{x}\right) y = e^{-2x}$$

$$P(x) = 2 + \frac{1}{x}$$

$$\therefore \mathrm{IF} = e^{\int P(x)\,dx} = e^{2x+\ln x} = xe^{2x}$$

Now,

$$xe^{2x}\frac{\mathrm{d}y}{\mathrm{d}x} + e^{2x}(2x+1)y = x$$

$$\frac{d}{dx}\left(xe^{2x}y\right) = x$$

$$xe^{2x}y = \frac{x^2}{2} + c_1$$

$$\boxed{y = \frac{1}{2}xe^{-2x} + \frac{c}{x}e^{-2x}}$$

Example 2.3.2: Solve the initial value problem

$$(x^2+1)\frac{\mathrm{d}y}{\mathrm{d}x} + 4xy = x \quad , \quad y(2) = 1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \left(\frac{4x}{x^2+1}\right) y = \frac{x}{x^2+1}$$

$$\therefore \ \mathrm{IF} \ = \exp\left(2\int \frac{2x}{x^2+1}\, dx\right) = \exp\left(2\ln|x^2+1|\right) = (x^2+1)^2$$

Therefore, the solution is

$$\begin{aligned}(x^2+1)^2 y &= \int (x^2+1)^2 \cdot \frac{x}{x^2+1}\,dx + c_1\\ &= \int (x^3+x)\,dx + c_1\\ &= \frac{x^4}{4} + \frac{x^2}{2} + c\end{aligned}$$

Applying initial value,

$$c = 19$$

$$\boxed{(x^2+1)^2 y = \frac{x^4}{4} + \frac{x^2}{2} + 19}$$

Example 2.3.3: Solve the linear DE

$$y^2\,dx + (3xy-1)\,dy = 0$$

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}y} &= -\frac{3x-1}{y^2} = -\frac{3}{y}x + \frac{1}{y^2}\\ \therefore \frac{\mathrm{d}x}{\mathrm{d}y} + \frac{3}{y}x &= \frac{1}{y^2}\\ \therefore \text{ IF } &= e^{3\int \frac{dy}{y}} = y^3\end{aligned}$$

Now,

$$\begin{aligned}y^3\frac{\mathrm{d}x}{\mathrm{d}y} + 3xy^2 &= y\\ y^3\,dx + (3xy^2 - y)\,dy &= 0\\ (y^3\,dx + 3xy^2\,dy) - y\,dy &= 0\\ d(xy^3) - d\left(\frac{y^2}{2}\right) &= d(c_1)\\ 2xy^3 - y^2 &= c\end{aligned}$$

$$\boxed{x = \frac{1}{2y} + \frac{c}{y^3}}$$

B Bernoulli Equations

Definition 2.3.2: Bernoulli Equations

An equation of the form
$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)y^n$$
is called a Bernoulli Equation.

If $n = 0$ or $n = 1$, the equation is simply a linear DE. However, in general case in which $n \neq 0$ or $n \neq 1$, we must proceed in a different manner.

Theorem 2.3.2* (Transformation of Bernoulli Equation to Linear Equation)*: *Suppose $n \neq 0$ or $n \neq 1$. Then the transformation*
$$v = y^{1-n}$$
reduces the Bernoulli Equation to a linear equation in v.

Proof:
$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)y^n$$
$$y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y^{1-n} = Q(x)$$
Substituting $v = y^{1-n}$,
$$\frac{\mathrm{d}v}{\mathrm{d}x} = (1-n)y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x}$$
$$\frac{1}{1-n}\frac{\mathrm{d}v}{\mathrm{d}x} + P(x)v = Q(x)$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} + (1-n)P(x)v = (1-n)Q(x)$$
Letting $P_1(x) = (1-n)P(x)$ and $Q_1(x) = (1-n)Q(x)$ we get,
$$\frac{\mathrm{d}v}{\mathrm{d}x} + P_1(x)v = Q_1(x) \quad \square$$

Example 2.3.4:
$$\frac{\mathrm{d}y}{\mathrm{d}x} + y = xy^3$$
$$y^{-3}\frac{\mathrm{d}y}{\mathrm{d}x} + y^{-2} = x$$

Letting $v = y^{-2}$,

$$
\begin{aligned}
\frac{\mathrm{d}v}{\mathrm{d}x} &= -2y^{-3}\frac{\mathrm{d}y}{\mathrm{d}x} \\
-\frac{1}{2}\frac{\mathrm{d}v}{\mathrm{d}x} + v &= x \\
\frac{\mathrm{d}v}{\mathrm{d}x} - 2v &= x
\end{aligned}
$$

$$\therefore \ \text{IF} \ = e^{-\int 2\,dx} = e^{-2x}$$

$$
\begin{aligned}
e^{-2x}v &= \int e^{-2x}(-2x)\,dx + c \\
e^{-2x}\frac{1}{y^2} &= -2\int xe^{-2x}\,dx + c \\
&= xe^{-2x} + \frac{1}{2}e^{-2x} + c
\end{aligned}
$$

$$\boxed{\frac{1}{y^2} = x + \frac{1}{2} + ce^{2x}}$$

Exercise 2.3: Solve the Differential Equations

1. $x^4\frac{\mathrm{d}y}{\mathrm{d}x} + 2x^3y = 1$
2. $\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = 3x^2e^{-3x}$
3. $(x^2 + x - 2)\frac{\mathrm{d}y}{\mathrm{d}x} + 3(x+1)y = x - 1$
4. $y\,dx + (xy^2 + x - y)\,dy = 0$
5. $\cos\theta\,dr + (r\sin\theta - \cos^4\theta)\,d\theta = 0$
6. $(y\sin 2x - \cos x)\,dx + (1 + \sin^2 x)\,dy = 0$
7. $x\frac{\mathrm{d}y}{\mathrm{d}x} + y = -2x^6y^4$
8. $\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{x} = -\frac{y^2}{x}$
9. $e^x\left[y - 3(e^x+1)^2\right]\,dx + (e^x + 1)\,dy = 0$
10. $\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{2x} = \frac{x}{y^3}$

Handwritten solutions on notebook. No time for updating in LaTeXnow. If anyone is interested, contact; I can provide handwritten solutions.

Exercise 2.4: Consider the equation

$$a\frac{\mathrm{d}y}{\mathrm{d}x} + by = ke^{-\lambda x}$$

where a, b, and k are positive constants and λ is a non-negative constant.

(a) Solve this equation.

(b) Show that if $\lambda = 0$, every solution approaches $\frac{k}{b}$ as $x \to \infty$, but if $\lambda > 0$ every solution approaches 0 as $x \to \infty$.

Handwritten solutions on notebook. No time for updating in LaTeXnow. If anyone is interested, contact; I can provide handwritten solutions.

Exercise 2.5:

(a) Prove that if f and g are two different solutions of

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x) \tag{A}$$

then $f - g$ is a solution of the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = 0$$

(b) Thus show that if f and g are two different solutions of Equation (A) and c is an arbitrary constant, then

$$c(f - g) + f$$

is a one-parameter family of solutions of (A).

Handwritten solutions on notebook. No time for updating in LaTeXnow. If anyone is interested, contact; I can provide handwritten solutions.

2.4 Special Integrating Factors and Transformations

The five basic types of differential equations we've encountered so far:

- Exact $\to$ Direct solution
- Separable $\to$ Integrating Factor $\to$ Exact DE
- Homogeneous $\to$ Integrating Factor $\to$ Exact DE
- Linear $\to$ Appropriate Transformation $\to$ Separable DE
- Bernoulli $\to$ Appropriate Transformation $\to$ Linear DE

How to solve a DE that is not of one of the five types?

1. Either multiply by proper IF $\rightarrow$ Exact DE
2. Or, appropriate transformation $\rightarrow$ One of the five basic forms.

A Finding Integrating Factors

Separable equations always possess integrating factors that can be determined by immediate inspection. However, some non-separable equations also possess such integrating factors that can be determined.
Suppose a non-exact DE

$$M(x,y)\,dx + N(x,y)\,dy = 0 \tag{2.4.1}$$

has an IF $\mu(x,y)$. Then the equation is

$$\mu(x,y)M(x,y)\,dx + \mu(x,y)N(x,y)\,dy = 0 \tag{2.4.2}$$

is exact. Now, we can say the equation (2.4.2) is exact if and only if

$$\frac{\partial}{\partial y}\,\left[\mu(x,y)M(x,y)\right] = \frac{\partial}{\partial x}\,\left[\mu(x,y)N(x,y)\right]$$

$$M(x,y)\frac{\partial\mu(x,y)}{\partial y} + \mu(x,y)\frac{\partial M(x,y)}{\partial y} = N(x,y)\frac{\partial\mu(x,y)}{\partial x} + \mu(x,y)\frac{\partial N(x,y)}{\partial x}$$

$$N(x,y)\frac{\partial\mu}{\partial x} - M(x,y)\frac{\partial\mu}{\partial x} = \left[\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right]\mu \tag{2.4.3}$$

Equation (2.4.3) is a PDE fo rthe general IF μ, and we're in no position to attempt to solve such an equation. Let's attepts to determine IF of certain special types instead.
If M and N are functions of x and y, but the IF μ depends only upon x, then equation (2.4.3) reduces to

$$N(x,y)\,\frac{\mathrm{d}\mu(x)}{\mathrm{d}x} = \mu(x)\,\frac{\partial M(x,y)}{\partial y} - \mu(x)\,\frac{\partial N(x,y)}{\partial x}$$

or,

$$\frac{d\mu(x)}{\mu(x)} = \frac{1}{N(x,y)}\left[\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right]\,dx \tag{2.4.4}$$

Here, if

$$\frac{1}{N(x,y)}\left[\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right]$$

depends upon x only, equation (2.4.4) is a separable ordinary equation in the single independent, equation (2.4.4) is a separable ordinary equation in the single independent variable x and the single dependent variable μ. In this case, we may integrate to obtain the IF

$$\mu(x) = \exp\left\{\int \frac{1}{N(x,y)}\left[\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right]\,dx\right\}$$

Likewise, if

$$\frac{1}{M(x,y)}\left[\frac{\partial N(x,y)}{\partial x} - \frac{\partial M(x,y)}{\partial y}\right]$$

depends upon y only, then we may obtain an IF that depends only on y.

Theorem 2.4.1* (Integrating Factors)*:
Consider the differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0 \tag{2.4.5}$$

If

$$\frac{1}{N(x,y)}\left[\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right] \tag{2.4.6}$$

depends upon x only, then IF

$$\mu(x) = \exp\left\{\int \frac{1}{N(x,y)}\left[\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right] dx\right\} \tag{2.4.7}$$

And if

$$\frac{1}{M(x,y)}\left[\frac{\partial N(x,y)}{\partial x} - \frac{\partial M(x,y)}{\partial y}\right] \tag{2.4.8}$$

depends upon y only, then IF

$$\mu(y) = \exp\left\{\int \frac{1}{M(x,y)}\left[\frac{\partial N(x,y)}{\partial x} - \frac{\partial M(x,y)}{\partial y}\right] dy\right\} \tag{2.4.9}$$

Example 2.4.1:

$$(2x^2 + y)\,dx + (x^2y - x)\,dy = 0$$

This equation is not any of the five basic types of differential equations. We can apply Theorem 2.4.1 in this case. Here, $M(x,y) = 2x^2 + y$ and $N(x,y) = x^2y - x$, and the equation (2.4.6) becomes

$$\frac{1}{x^2y - x}[1 - (2xy - 1)] = \frac{2(1 - xy)}{x(xy - y)} = -\frac{2}{x}$$

This depends upon x only, so

$$\text{IF } = \exp\left(-\int \frac{2}{x}\,dx\right) = \exp\bigl(-2\ln|x|\bigr) = \frac{1}{x^2}$$

Thus we obtain the equation

$$\left(2 + \frac{y}{x^2}\right)\,dx + \left(y - \frac{1}{x}\right)\,dy = 0$$

This equation is exact, and the solution is

$$2x + \frac{y^2}{2} - \frac{y}{x} = c$$

B A Special Transformation

***Theorem 2.4.2* (A Special Transformation):**
Consider the equation

$$(a_1x + b_1y + c_1)\,dx + (a_2x + b_2y + c_2)\,dy = 0 \tag{2.4.10}$$

where $a_1, b_1, c_1, a_2, b_2, c_2$ are constants.

Case 1: *If $\frac{a_2}{a_1} \neq \frac{b_2}{b_1}$, then the transformation*

$$\begin{aligned} x &= X + h \\ y &= Y + k \end{aligned}$$

where (h, k) is the solution of the system

$$\begin{aligned} a_1h + b_1k + c_1 &= 0 \\ a_2h + b_2k + c_2 &= 0 \end{aligned}$$

reduces the equation (2.4.10) to the Homogeneous Equation

$$(a_1X + b_1Y)\,dX + (a_2X + b_2Y)\,dY = 0 \tag{2.4.11}$$

Case 2: *If $\frac{a_2}{a_1} = \frac{b_2}{b_1}$, then the transformation*

$$z = a_1x + b_1y$$

reduces the equation (2.4.10) to a separable equation in the variables x and z.

Example 2.4.2:

$$(x - 2y + 1)\,dx + (4x - 3y - 6)\,dy = 0$$

Here,

$$\frac{a_2}{a_1} = 4 \neq \frac{3}{2} = \frac{b_2}{b_1}$$

Therefore, we make the transformation

$$\begin{aligned} x &= X + h \\ y &= Y + k \end{aligned}$$

where (h, k) is the solution of the system

$$\begin{aligned} h - 2k + 1 &= 0 \\ 4h - 3k - 6 &= 0 \end{aligned}$$

The solution of the system is $(3, 2)$, and so the transformation is

$$\begin{aligned} x &= X + 3 \\ y &= Y + 2 \end{aligned}$$

This reduces the given equation to the homogeneous equation

$$(X - 2Y)\,dX + (4X - 3Y)\,dY = 0$$

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{1-2(Y/X)}{3(Y/X)-4}$$

Letting $Y = vX$,

$$\begin{aligned}
v + X\frac{\mathrm{d}v}{\mathrm{d}X} &= \frac{1-2v}{3v-4} \\
\int \frac{3v-4}{3v^2-2v-1}\,dv &= -\int \frac{dX}{X} \\
\frac{1}{2}\int \frac{d(3v^2-2v-1)}{3v^2-2v-1} - 3\int \frac{dv}{3v^2-2v-1} &= -\int \frac{dX}{X} \\
\frac{1}{2}\int \frac{d(3v^2-2v-1)}{3v^2-2v-1} - 3\int \left[\frac{1}{4}\int \frac{dv}{v-1} - \frac{3}{4}\int \frac{dv}{3v+1}\right] &= -\int \frac{dX}{X} \\
\frac{1}{2}\ln|3v^2-2v-1| - \frac{3}{4}\ln|v-1| + \frac{9}{4}\ln|3v+1| + \ln|X| &= \ln|c_1| \\
\ln\left|X^4 \cdot \frac{(v-1)^2(3v+1)^{11}}{(v-1)^3}\right| &= \ln|c| \\
|3Y+X|^{11} &= X^6 c|Y-X|
\end{aligned}$$

$$\boxed{|x+3y-9|^{11} = c(x-3)^6|y-x+1|}$$

Example 2.4.3:

$$(x+2y+3)\,dx + (2x+4y-1)\,dy = 0$$

Here,

$$\frac{a_2}{a_1} = 2 = \frac{b_2}{b_1}$$

Therefore, we apply the transformation

$$z = x + 2y$$

$$\begin{aligned}
\therefore (z+3)\,dx + (2z-1)\left(\frac{dz-dx}{2}\right) &= 0 \\
7\,dx + (2z-1)\,dz &= 0 \\
7x + z^2 - z &= c \\
7x + x^2 + 4y^2 + 4xy - x - 2y &= c
\end{aligned}$$

$$\boxed{x^2 + 4xy + 4y^2 + 6x - 2y = c}$$

3 Explicit Methods of Solving Higher-Order Linear Differential Equations

3.1 Basic Theory of Linear Differential Equations

A Definition and Basic Existence Theorem

Definition 3.1.1: Linear Ordinary Differential Equation of Order n

A **linear ordinary differential equation of order** n in the dependent variable y and the independent variable x is an equation that is in, or can be expressed in the form

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)}n + \cdots + a_{n-1}(x)y' + a_n(x)y = F(x) \tag{3.1.1}$$

where a_0 is not identically zero. In the equation, $a_0, a_1, \cdots, a_n$ and F are continuous real functions on a real interval $a \leq x \leq b$ and that $a_0(x) \neq 0$ for any x on $a \leq x \leq b$. The $F(x)$ is called the nonhomogeneous term. If F is identically zero, Equation (3.1.1) reduces to

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)}n + \cdots + a_{n-1}(x)y' + a_n(x)y = 0 \tag{3.1.2}$$

Equation (3.1.2) is a **homogeneous differential equation of order** n.

Example 3.1.1

The equation

$$y'' + 3xy' + x^3y = e^x$$

is a linear ordinary differential equation.

The equation

$$y''' + xy'' + 3x^2y' - 5y = \sin x$$

is a linear ODE of third order.

Theorem 3.1.1* (Basic Existence Theorem)*:

Hypothesis:

1. *Consider the nth-order linear differential equation*

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)}n + \cdots + a_{n-1}(x)y' + a_n(x)y = F(x) \tag{3.1.1}$$

   where $a_0, a_1, \cdots, a_n$ and F are real functions on a real interval $a \leq x \leq b$ and $a_0(x) \neq 0$ for any x on $a \leq x \leq b$.

2. *Let x_0 be any point of the interval $a \leq x \leq b$, and let $c_0, c_1, \cdots, c_{n-1}$ be n arbitrary real constants.*

Conclusion: *There exists a unique solution f of (3.1.1) such that*

$$f(x_0) = c_0, f'(x_0) = c_1, \cdots, f^{(n-1)}(x_0) = c_{n-1}$$

and this solution is defined over the entire interval $a \leq x \leq b$.

Example 3.1.2: Consider the initial-value problem

$$2y''' + xy'' + 3x^2y' - 5y = \sin x$$

$$y(4) = 3$$
$$y'(4) = 5$$
$$y''(4) = -\frac{7}{2}$$

Here we have a third-order problem. The coefficients $2, x, 3x^2$, and -5, as well as the nonhomogeneous term $\sin x$, are all continuous for all $x \in (-\infty, \infty)$. The point $x_0 = 4$ certainly belongs to this intervall; the real numbers c_0, c_1, and c_2 in this problem are 3, 5, and $-\frac{7}{2}$ respectively. Theorem 3.1.1 assures is that this problem also has a unique solution which is defined for all $x \in (-\infty, \infty)$

Corollary 3.1.1.1:
Hypothesis: *Let f be a solution of the nth-order homogeneous linear DE*

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = 0 \tag{3.1.2}$$

such that

$$f(x_0) = 0, f'(x_0) = 0, \cdots, f^{(n-1)}(x_0) = 0,$$

where x_0 is a point of the interval $a \leq x \leq b$ in which the coefficients $a_0, a_1, \cdots, a_n$ are all continuous and $a_0(x) \neq 0$.

Conclusion: *Then $f(x) = 0$ for all x on $a \leq x \leq b$.*

Example 3.1.3

The unique solution of f of the third-order homogeneous equation

$$y''' + 2y'' + 4xy' + x^2y = 0$$

which is such that

$$f(2) = f'(2) = f''(2) = 0$$

is the trivial solution f such that $f(x) = 0$ for all x.

B The Homogeneous Equation

We now consider the fundamental results concerning the homogeneous equation

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = 0 \tag{3.1.2}$$

Theorem 3.1.2 **(Basic Theorem on Linear Homogeneous Differential Equations)*:***
Hypothesis: *Let $f_1, f_2, \cdots, f_m$ be any m solutions of the homogeneous linear differential equation (3.1.2).*

Conclusion: *Then*

$$c_1 f_1 + c_2 f_2 + \cdots + c_m f_m$$

is also a solution of (3.1.2), where $c_1, c_2, \cdots, c_m$ are m arbitrary constants.

In other words: Any linear combination of solutions of the homogeneous linear differential equation (3.1.2) is also a solution of (3.1.2).

Definition 3.1.2: Linear Combination

If $f_1, f_2, \cdots, f_m$ are m given functions, and $c_1, c_2, \cdots, c_m$ are m constants, then the expression

$$c_1 f_1 + c_2 f_2 + \cdots + c_m f_m$$

is called a linear combination of $f_1, f_2, \cdots, f_m$.

Example 3.1.4

e^x, e^{-x}, e^{2x} are solutions of

$$y''' - 2y'' - y' + 2y = 0$$

Theorem 3.1.3 states that the linear combination $c_1 e^x + c_2 e^{-x} + c_3 x^{2x}$ is also a solution for any constants c_1, c_2, c_3. For example, the particular linear combination

$$2e^x - 3e^{-x} + \frac{2}{3}e^{2x}$$

is a solution.

Definition 3.1.3: Linear Dependence

The n functions $f_1, f_2, \cdots, f_n$ are called *linearly dependent* on $a \leq x \leq b$ if there exist constants $c_1, c_2, \cdots, c_n$, not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_m f_m(x) = 0$$

for all x such that $a \leq x \leq b$.

Definition 3.1.4: Linear Independence

The n functions $f_1, f_2, \cdots, f_n$ are called linearly independent on the interval $a \leq x \leq b$ if the relation

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

for all x such that $a \leq x \leq b$ implies that

$$c_1 = c_2 = \cdots = c_n = 0$$

In other words, the only linear combination of $f_1, f_2, \cdots, f_n$ that is identically zero on $a \leq x \leq b$ is the trivial linear combination

$$0 \cdot f_1 + 0 \cdot f_2 + \cdots + 0 \cdot f_n$$

Theorem 3.1.3* (Linearly Independent Solutions of nth-Order Linear Differential Equation)*: *The nth-order homogeneous linear differential equation*

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = 0 \tag{3.1.2}$$

always possesses n solutions that are linearly independent. Further, if $f_1, f_2, \cdots f_n$ are n linearly independent solutions of (3.1.2), then every solution f of (3.1.2) can be expressed as a linear combination

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

of these n linearly independent soulutions by proper choice of the constants $c_1, c_2, \cdots, c_n$.

Example 3.1.5

We have observed that $\sin x$ and $\cos x$ are solutions of

$$y'' + y = 0 \tag{3.1.3}$$

for all $x \in (-\infty, \infty)$. Further, we can show that these two solutions are linearly independent. Suppose f is any solution of (3.1.3). Then by Theorem 3.1.4 f can be expressed as a certain linear combination $c_1 \sin x + c_2 \cos x$ of the two linearly independent solutions $\sin x$ and $\cos x$ by proper choice of c_1 and c_2. That is, there exist two particular constants c_1 and c_2 such that

$$f(x) = c_1 \sin x + c_2 \cos x \tag{3.1.4}$$

for all $x \in (-\infty, \infty)$. For example, it can be easily verified that $f(x) = \sin\bigl(x + \pi/6\bigr)$ is a solution of the equation (3.1.3). Since

$$\sin\left(x + \frac{\pi}{6}\right) = \sin x \cos \frac{\pi}{6} + \cos x \sin \frac{\pi}{6} = \frac{\sqrt{3}}{2} \sin x + \frac{1}{2} \cos x,$$

we see that the solution $\sin\bigl(x + \pi/6\bigr)$ can be expressed as the linear combination

$$\frac{\sqrt{3}}{2} \sin x + \frac{1}{2} \cos x$$

of the two linearly independent solutions $\sin x$ and $\cos x$. Here, $c_1 = \sqrt{3}/2$ and $c_2 = 1/2$

Definition 3.1.5: Fundamental Set of Solutions

If $f_1, f_2, \cdots, f_n$ are n linearly independent solutions of the nth-order homogeneous linear differential equation

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = 0 \tag{3.1.2}$$

on $a \leq x \leq b$, then the set $f_1, f_2, \cdots, f_n$ is called a fundamental set of solutions of (3.1.2) and the function f defined by

$$f(x) = c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x), \quad a \leq x \leq b,$$

where $c_1, c_2, \cdots, c_n$ are arbitrary constants, is called a general solution of (3.1.2) on $a \leq x \leq b$.

Therefore, if we can find n linearly independent solutions of (3.1.2), we can at once write the general solution of (3.1.2) as a general linear combination of these n solutions.

Example 3.1.6

The solutions e^x, e^{-x}, and e^{2x} of

$$y''' - 2y'' + y' + 2y = 0$$

may be shown to be linearly independent for all $x \in (-\infty, \infty)$. Thus, e^x, e^{-x}, and e^{2x} constitute a fundamental set of the given DE, and its general solution may be expressed as the linear combination

$$c_1 e^x + e^{-x} + c_3 e^{2x}$$

where c_1, c_2, and c_3 are arbitrary constants. We can write this as

$$y = c_1 e^x + e^{-x} + c_3 e^{2x}$$

Definition 3.1.6: Wronskian

Let $f_1, f_2, \cdots, f_3$ be n real functions each of which has an $(n-1)$th derivative on a real interval $a \leq x \leq b$. The determinant

$$W(f_1, f_2, \cdots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

is called the Wronskian of these n functions. We observe that $W(f_1, f_2, \cdots, f_n)$ is itself a real function defined on $a \leq x \leq b$. Its value at x is denoted by $W(f_1, f_2, \cdots, f_n)(x)$ or by $W[f_1(x), f_2(x), \cdots, f_n(x)]$.

Theorem 3.1.4: *The n solutions $f_1, f_2, \cdots, f_n$ of the nth-order homogeneous linear DE (3.1.2) are linearly independent on $a \leq x \leq b$ if and only if the Wronskian of $f_1, f_2, \cdots, f_n$ is different from zero for some x on the interval $a \leq x \leq b$.*

Theorem 3.1.5: *The Wronskian of n solutions $f_1, f_2, \cdots, f_n$ of (3.1.2) is either identically zero on $a \le x \le b$ or else is never zero on $a \le x \le b$.*

Example 3.1.7
We can apply Theorem 3.1.2 to show that the solutions $\sin x$ and $\cos x$ of

$$y'' + y = 0$$

are linearly independent. We find that

$$W(\sin x, \cos x) = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x = -1 \neq 0$$

for all real x. Thus, since $W(\sin x, \cos x) \neq 0$ for all real x, we conclude that $\sin x$ and $\cos x$ are indeed linearly independent solutions of the given DE on every real interval.

Example 3.1.8
The solutions e^x, e^{-1}, and e^{2x} of

$$y''' - 2y'' - y' + 2y = 0$$

are linearly independent on every real interval, for

$$W(e^x, e^{-1}, e^{2x}) = \begin{vmatrix} e^x & e^{-1} & e^{2x} \\ e^x & -e^{-1} & 2e^{2x} \\ e^x & e^{-1} & 4e^{2x} \end{vmatrix} = e^{2x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 4 \end{vmatrix} = -6e^{2x} \neq 0$$

for all real x.

C Reduction of Order

Theorem 3.1.6:
Hypothesis: *Let f be a nontrivial solution of the nth-order homogeneous linear differential equation*

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = 0 \tag{3.1.2}$$

Conclusion: *The transformation $y = f(x)v$ reduces Equation (3.1.2) to an $(n-1)$th-order homogeneous linear different equation in the dependent variable $w = dv/dx$.*

According to this theorem, id one nonzero solution of the nth-order homogeneous linear DE (3.1.2) is known, then by making the appropriate transformation we may reduce the given equation to another homogeneous linear DE that is one order lower than the original. We'll now investigate the second-order ($n = 2$) DE in detal.
Suppose f is a known nontrivial solution of the second-order homogeneous linear equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0 \tag{3.1.5}$$

Let's make the transformation

$$y = f(x)v \tag{3.1.6}$$

, where f is the known solution of (3.1.5) and v is a function of x that will be determined. Then, differentiating, we obtain

$$y' = f(x)v' + f'(x)v, \tag{3.1.7}$$

$$y'' = f(x)v'' + 2f'(x)v' + f''(x)v. \tag{3.1.8}$$

Substituting (3.1.6), (3.1.7), and (3.1.8) into (3.1.5), we obtain

$$a_0(x)[y'' = f(x)v'' + 2f'(x)v' + f''(x)v] + a_1(x)[y' = f(x)v' + f'(x)v] + a_2(x)f(x)v = 0$$

or

$$a_0(x)f(x)v'' + [2a_0(x)f'(x) + a_1 f(x)]v' + [a_0(x)f''(x) + a_1(x)f'(x) + a_2(x)f(x)]v = 0$$

Since f is a solution of (3.1.5), the coefficient of v is zero, and so the last equation reduces to

$$a_0(x)f(x)v'' + [2a_0(x)f'(x) + a_1(x)f(x)]v' = 0$$

Letting $w = v'$, this becomes

$$a_0(x)f(x)\frac{dw}{dx} + [2a_0(x)f'(x) + a_1(x)f(x)]w = 0 \tag{3.1.9}$$

This is a first-order homogeneous linear DE in the dependent variable w. The equation is separable; thus, assuming $f(x) \neq 0$ and $a_0(x) \neq 0$, we may write

$$\frac{dw}{w} = -2\left[2\frac{f'(x)}{f(x)} + \frac{a_1(x)}{a_0(x)}\right] dx$$

Thus integrating, we obtain

$$\ln|w| = -\ln[f(x)]^2 - \int \frac{a_1(x)}{a_0(x)}\, dx + \ln|c|$$

or

$$w = \frac{c \cdot \exp\left[-\int \frac{a_1(x)}{a_0(x)}\, dx\right]}{[f(x)]^2}$$

This is the general solution of Equation (3.1.9); choosing the particular solution for which $c = 1$, recalling that $dv/dw = w$, and integrating again, we now obtain

$$v = \int \frac{\exp\left[-\int \frac{a_1(x)}{a_0(x)}\, dx\right]}{[f(x)]^2}\, dx$$

Finally, from (3.1.6), we obtain

$$y = f(x)\int \frac{\exp\left[-\int \frac{a_1(x)}{a_0(x)}\, dx\right]}{[f(x)]^2}\, dx$$

If we denote the right side of the function by g, which is a solution of (3.1.5), we can write the general solution of (3.1.5) as the following linear combination

$$c_1 f + c_2 g$$

where c_1 and c_2 are arbitrary constants.
We also observe that g and f are linearly independent, since

$$W(f,g)(x) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = \begin{vmatrix} f(x) & f(x)v \\ f'(x) & f(x)v' + f'(x)v \end{vmatrix} = [f(x)]^2 v' = \exp\left[-\int \frac{a_1(x)}{a_0(x)}\,dx\right] \neq 0$$

Theorem 3.1.7:
Hypothesis: *Let f be a nontrivial solution of the second-order homogeneous linear differential equation*

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0 \tag{3.1.10}$$

Conclusion 1: *The transformation $y = f(x)v$ reduces (3.1.10) to the first-order linear homogeneous DE*

$$a_0(x)f(x)\frac{dw}{dx} + [2a_0(x)f'(x) + a_1(x)f(x)]w = 0$$

in the dependent variable w, where $w = v'$
Conclusion 2: *The particular solution*

$$w = \frac{\exp\left[-\int \frac{a_1(x)}{a_0(x)}\,dx\right]}{[f(x)]^2}$$

of equation (3.1.10) gives rise to the function v, where

$$v(x) = \int \frac{\exp\left[-\int \frac{a_1(x)}{a_0(x)}\,dx\right]}{[f(x)]^2}\,dx$$

The function $g(x)$ defined by $g(x) = f(x)v(x)$ is then a solution of the second-order equation (3.1.10).
Conclusion 3: *The original known solution f and the new solution g are linearly independent solutions of (3.1.10), and hence the general solution of (3.1.10) is*

$$c_1 f + c_2 g$$

where c_1 and c_2 are arbitrary constants.

Example 3.1.9: Given that $y = x$ is a solution of

$$(x^2+1)y'' - 2xy' + 2y = 0 \tag{A}$$

find a linearly independent solution by reducing the order.
First applying the transformation

$$y = xv$$

Then

$$y' = xv' + v, \text{ and } y'' = xv'' + 2v'$$

Substituting these into (A) we obtain

$$(x^2+1)(xv''+2v') - 2x(xv'+v) + 2xv = 0$$

$$x(x^2+1)v'' + 2v' = 0$$

Letting $w = v'$,

$$x(x^2+1)\frac{dw}{dx} + 2w = 0$$

$$\int \frac{dw}{w} = -\int \frac{2dx}{x(x^2+1)} = \int \left(-\frac{2}{x} + \frac{2x}{x^2+1}\right) dx$$

$$\ln|w| = \ln|x^2+1| - \ln|x|^2 + \ln|c|$$

$$w = \frac{c(x^2+1)}{x^2}$$

Letting $c = 1$,

$$v(x) = \int \left(1 + \frac{1}{x^2}\right) dx = x - \frac{1}{x}$$

$$\therefore g = f(x)v(x) = x \cdot \left(x - \frac{1}{x}\right) = x^2 - 1$$

Hence, the general solution of (A) is

$$\boxed{y = c_1 x + c_2(x^2 - 1)}$$

D The Nonhomogeneous Equation

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = F(x) \tag{3.1.1}$$

Theorem 3.1.8:
Hypothesis:

1. *Let v be any solution of the given (nonhomogeneous) nth-order linear differential equation (3.1.1)*

2. *Let u be any solution of the corresponding homogeneous equation*

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = 0 \tag{3.1.2}$$

Conclusion: *Then $u + v$ is also a solution of the given (nonhomogeneous) equation (3.1.1)*

Theorem 3.1.9:
Hypothesis:

1. *Let y_p be a given solution of the nth-order nonhomogeneous linear equation (3.1.1) involving no arbitrary constants.*

2. *Let*
$$y_c = c_1y_1 + c_2y_2 + \cdots + c_ny_n$$
be the general solution of the corresponding homogeneous equation (3.1.2)

Conclusion: *Then every solution ϕ of the nth-order nonhomogeneous equation (3.1.1) can be expressed in the form*
$$y_c + y_p$$
that is,
$$c_1y_1 + c_2y_2 + \cdots + c_ny_n + y_p$$
for suitable choice of the n arbitrary constants $c_1, c_2, \cdots, c_n$.

Definition 3.1.7

Consider the nth-order nonhomogeneous linear DE

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = F(x) \tag{3.1.1}$$

and the corresponding homogeneous equation

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = 0 \tag{3.1.2}$$

Complimentary Function: y_c = general solution of (3.1.2)
Particular Solution: y_p = any solution of (3.1.1) involving no arbitrary constant
General Solution of (3.1.1): $y_c + y_p$

Example 3.1.10
Consider the DE
$$y'' + y = x$$
The complimentary function
$$y_c = c_1 \sin x + c_2 \cos x$$
of the corresponding homogeneous DE
$$y'' + y = 0$$
A particular integral is
$$y_p = x$$
Thus the general solution is
$$y = y_c + y_p = c_1 \sin x + c_2 \cos x + x$$

Theorem 3.1.10:
Hypothesis:

1. *Let f_1 be a particular integral of*

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = F_1(x) \tag{3.1.11}$$

2. *Let f_2 be a particular integral of*

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = F_2(x) \tag{3.1.12}$$

Conclusion: *Then $k_1f_1 + k_2f_2$ is a particular integral of*

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = k_1F_1(x) + k_2F_2(x) \tag{3.1.13}$$

3.2 The Homogeneous Linear Equation with Constant Coefficients

A Introduction

In this section we'll consider the special case of the nth-order homogeneous linear differential equation in which all the coefficients are real constants. That is, we'll deal with the following equation

$$a_0y^{(n)} + a_1y^{(n-1)} + \cdots + a_{n-1}y' + a_ny = 0 \tag{3.2.1}$$

where $a_0, a_1, \cdots, a_{n-1}, a_n$ are real constants.
The DE (3.2.1) requires a function f having the property such that if it and its various derivatives are each multiplied by certain constants and the resultant products are then added, the result will be equal to zero for the appropriate domain of this equation.

Therefore, we need a function such that its derivatives are constant multiples of itself. That is, having the following property

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}f(x) = cf(x)$$

We all know such a function is e^{mx}, and

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}e^{mx} = m^ke^{mx}$$

Thus we'll seek solutions of (3.2.1) of the form $y = e^{mx}$.

Suppose $y = e^{mx}$ is a solution for certain m. Then we have:

$$\begin{aligned} y' &= me^{mx} \\ y'' &= m^2e^{mx} \\ &\vdots \\ y^{(n)} &= m^ne^{mx} \end{aligned}$$

Substituting in (3.2.1) we obtain

$$a_0m^ne^{mx} + a_1m^{n-1}e^{mx} + \cdots + a_{n-1}me^{mx} + a_ne^{mx} = 0$$

or

$$e^{mx}(a_0m^n + a_1m^{n-1} + \cdots + a_{n-1}m + a_n) = 0$$

Since $e^{mx} \neq 0$, we obtain the polynomial equation in the unknown m :

$$a_0m^n + a_1m^{n-1} + \cdots + a_{n-1}m + a_n = 0 \tag{3.2.2}$$

This equation is called the **Auxiliary Equation** or the **Characteristic Equation** of the given differential equation (3.2.1).

Note that (3.2.2) is formally obtained from (3.2.1). If $y = e^{mx}$ is a solution of (3.2.1), then the constant m must satisfy (3.2.2). Hence, to solve (3.2.1), we write the auxiliary equation (3.2.2) and solve it for m.

B Case 1: Distinct Real Roots

If the AE of (3.2.1) has the n distinct real roots $m_1, m_2, \cdots, m_n$, then $e^{m_1x}, e^{m_2x}, \cdots, e^{m_nx}$ are n distinct solutions of (3.2.1).

Theorem 3.2.1: *Consider the nth-order homogeneous linear differential equation (3.2.1) with constant coefficients. If the auxiliary equation (3.2.2) has the n distinct real roots*

$$m_1, m_2, \cdots, m_n$$

, then the general solution of (3.2.1) is

$$y = c_1e^{m_1x} + c_2e^{m_2x} + \cdots + c_ne^{m_nx}$$

, where $c_1, c_2, \cdots, c_n$ are arbitrary constants.

Example 3.2.1: Solve the differential equation

$$y'' - 3y' + 2y = 0$$

The auxiliary equation is

$$m^2 - 3m + 2y = 0$$

$$(m-1)(m-2) = 0$$

$$m = 1, 2$$

The roots are real and distinct. Thus e^x and e^{2x} are the solutions, and the general solution is

$$y = c_1e^x + c_2e^{2x}$$

We can also verify that the solutions are linearly independent. Their Wronskian :

$$W(e^x, e^{2x}) = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^{3x} \neq 0$$

Example 3.2.2: Solve the differential equation

$$y''' - 4y'' + y' + 6y = 0$$

The auxiliary equation is

$$m^3 - 4m^2 + m + 6 = 0$$
$$(m+1)(m^2 - 5m + 6) = 0$$
$$(m+1)(m-2)(m-3)$$
$$m = -1, 2, 3$$

Thus the general solution is

$$y = c_1 e^{-1} + c_2 e^{2x} + c_3 e^{3x}$$

C Case 2: Repeated Real Roots

Example 3.2.3: Consider the equation

$$y'' - 6y' + 9y = 0 \tag{3.2.3}$$

The auxiliary equation is

$$m^2 - 6m + 9 = 0$$
$$(m-3)^2 = 0$$

The roots of this equation are

$$m_1 = 3, m_2 = 3$$

(real but not distinct)

From Example 3.2.3 we find the linear combination

$$c_1 e^{3x} + c_2 e^{3x}$$

$$\text{or, } (c_1 + c_2)e^{3x} = c_0 e^{3x}$$

However, the solutions are clearly not the general solution of the given DE, because the solutions are not linearly independent since the Wronskian is zero:

$$W(e^{3x}, e^{3x}) = \begin{vmatrix} e^{3x} & e^{3x} \\ 3e^{3x} & 3e^{3x} \end{vmatrix} = 0$$

So, we must find a linearly independent solution. Since we already know one solution e^{3x}, we may apply Theorem 3.1.7 and reduce the order. Let

$$y = e^{3x} v$$

Then

$$y' = e^{3x} v' + 3e^{3x} v$$
$$y'' = e^{3x} v'' + 6e^{3x} v' + 9e^{3x} v$$

Substituting into (3.2.3) we obtain

$$(e^{3x} v'' + 6e^{3x} v' + 9e^{3x} v) - 6(e^{3x} v' + 3e^{3x} v) + 9e^{3x} v = 0$$

$$\text{or, } e^{3x} v'' = 0$$

Letting $w = v'$, we have the first-order equation

$$e^{3x}\frac{dw}{dx} = 0$$

$$\text{or, } \frac{dw}{dx} = 0$$
$$\text{or, } w = c$$

Choosing the particular solution $w = 1$, we get

$$v(x) = \int w\, dx = \int dx = x + c_0$$

By Theorem 3.1.7 we know that for any choice of c_0, a solution of the given second-order equation (3.2.3) is

$$y = v(x)e^{3x} = (x + c_0)e^{3x}$$

We also observe that this solution and the previously known solution e^{3x} are linearly independent. Choosing $c_0 = 0$, we get $y = xe^{3x}$. And thus corresponding to the double root 3, we find the linearly independent solutions e^{3x} and xe^{3x}. Thus the general solution of (3.2.3) may be written as

$$y = c_1e^{3x} + c_2xe^{3x}$$
$$\text{or, } y = (c_1 + c_2x)e^{3x}$$

Theorem 3.2.2:

1. *Consider the nth-order homogeneous linear differential equation (3.2.1) with constant coefficients. If the auxiliary equation (3.2.2) has the real root m occurring k times, then the part of the general solution of (3.2.1) corresponding to this k-fold repeated root is*
$$(c_1 + c_2x + c_3x^2 + \cdots + c_kx^{k-1})e^{mx}$$
2. *If, further, the remaining roots of the auxiliary equation (3.2.2) are the distinct real numbers $m_{k+1}, \cdots, m_n$, then the general solution of (3.2.1) is*
$$y = (c_1 + c_2x + c_3x^2 + \cdots + c_kx^{k-1})e^{mx} + c_{k+1}e^{m_{k+1}x} + \cdots + c_ne^{m_nx}$$
3. *If, however, any of the remaining roots are also repeated, then the parts of the general solution of (3.2.1) corresponding to each of these other repeated roots are expressions similar to that corresponding to m in part 1.*

Example 3.2.4: Find the solution of

$$y^{iv} - 5y''' + 6y'' + 4y' - 8y = 0$$

The AE is

$$m^4 - 5m^3 + 6m^2 + 4m - 8 = 0$$

with roots $2, 2, 2, -1$. Thus, the general equation is

$$y = (c_1 + c_2x + c_3x^2)e^{2x} + c_4e^{-x}$$

D Case 3: Conjugate Complex Roots

Now we suppose that the auxiliary equation has the complex number $a + bi$ as a nonrepeated root. Since the coefficients are real, the conjugate complex number $a - bi$ is also a nonrepeated root. The corresponding part of the solution is

$$k_1 e^{(a+bi)x} + k_2 e^{(a-bi)x}$$

We can replace the complex functions $e^{(a+bi)x}$ and $e^{(a-bi)x}$ by two real linearly independent solutions using Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Then we have

$$\begin{aligned}
k_1 e^{(a+bi)x} + k_2 e^{(a-bi)x} &= k_1 e^{ax} e^{bix} + k_2 e^{ax} e^{-bix} \\
&= e^{ax}[k_1 e^{bix} + k_2 e^{-bix}] \\
&= e^{ax}[k_1(\cos bx + i\sin bx) + k_2(\cos bx - i\sin bx)] \\
&= e^{ax}[(k_1 + k_2)\cos bx + i(k_1 - k_2)\sin bx] \\
&= e^{ax}[c_1 \sin bx + c_2 \cos bx]
\end{aligned}$$

Thus the part of the general solution corresponding to the nonrepeated conjugate complex roots $a \pm bi$ is

$$e^{ax}[c_1 \sin bx + c_2 \cos bx]$$

Theorem 3.2.3:

1. *Consider the nth-order homogeneous linear DE (3.2.1) with constant coefficients. If the auxiliary equation (3.2.2) has the nonrepeated conjugate complex roots $a \pm bi$, then the corresponding part of the general solution of (3.2.2) may be written as*

$$y = e^{ax}(c_1 \sin bx + c_2 \cos bx)$$

2. *If, however, $a + bi$ and $a - bi$ are each k-fold roots of the auxiliary equation (3.2.2), then the corresponding part of the general solution of (3.2.1) may be written as*

$$y = e^{ax}[(c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1})\sin bx + (c_{k+1} + c_{k+2} x + c_{k+3} x^2 + \cdots + c_{2k} x^{k-1})\cos bx]$$

Example 3.2.5: Find the general solution of

$$y'' - 6y' + 25y = 0$$

The AE is

$$m^2 - 6m + 25 = 0$$

Solving it, we find

$$m = \frac{6 \pm \sqrt{36 - 100}}{2} = \frac{6 \pm 8i}{2} = 3 \pm 4i$$

Here, the roots are conjugate complex numbers. Hence, the general solution is

$$y = e^{3x}(c_1 \sin 4x + c_2 \cos 4x)$$

Example 3.2.6: Find the general solution of

$$y^{iv} - 4y''' + 14y'' - 20y' + 25y = 0$$

The AE is

$$m^4 - 4m^3 + 14m^2 - 20m + 25 = 0$$

The roots are

$$m = 1 + 2i, 1 - 2i, 1 + 2i, 1 - 2i$$

Hence, the general solution is

$$y = e^x[(c_1 + c_2 x)\sin 2x + (c_3 + c_4 x)\cos 2x]$$

or

$$y = e^x[c_1 \sin 2x + c_2 x \sin 2x + c_3 \cos 2x + c_4 x \cos 2x]$$

E An Initial-Value Problem

Example 3.2.7: Solve the initial-value problem

$$y'' - 6y' + 25y = 0$$

$$y(0) = -3, y'(0) = -1$$

We already found the general solution of the DE in Example 3.2.5. It is

$$y = e^{3x}(c_1 \sin 4x + c_2 \cos 4x)$$

From this, we find

$$y' = e^{3x}[(3c_1 - 4c_2)\sin 4x + (4c_1 + 3c_2)\cos]$$

Now, applying the initial condition $y(0) = -3$, we obtain

$$-3 = e^0(c_1 \sin 0 + c_2 \cos 0)$$

$$\text{or, } c_2 = -3$$

Applying initial condition $y'(0) = -1$, we obtain

$$-1 = e^0[(3c_1 - 4c_2)\sin 0 + (4c_1 + 3c_2)\cos 0]$$

$$\text{or, } 4c_1 + 3c_2 = -1$$

$$\text{or, } c_1 = 2$$

Thus we can write the solution

$$y = e^{3x}(2\sin 4x - 3\cos 4x)$$

We can further modify it by multiplying $\sqrt{(2)^2 + (-3)^2} = \sqrt{13}$ with nominator and denominator

$$y = \sqrt{13}e^{3x}\left[\frac{2}{\sqrt{13}}\sin 4x - \frac{3}{\sqrt{13}}\cos 4x\right]$$

From this, we may express the solution in the alternative form

$$y = \sqrt{13}e^{3x}\sin(4x + \phi)$$

, where

$$\sin\phi = -\frac{3}{13,} \text{ and } \cos\phi = \frac{2}{\sqrt{13}}$$