Application of First Order and First Degree Differential Equations

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A large tank initially contains 50 gal of brine in which there is dissolved 10 lb of salt. Brine containing 2 lb of dissolved salt per gallon flows into the tank at the rate of 5 gal/min. The mixture is kept uniform by stirring, and the stirred mixture simultaneously flows out at the slower rate of 3 gal/min. How much salt is in the tank at any time $t > 0$?

Mathematical Formulation. Let x = the amount of salt at time t. Again we shall use Equation (3.63)

$$\frac{dx}{dt} = \text{IN} - \text{OUT}.$$

Proceeding as in Example 3.10,

$$\text{IN} = (2 \text{ lb/gal})(5 \text{ gal/min}) = 10 \text{ lb/min};$$

also, once again

$$\text{OUT} = (C\ \text{lb/gal})(3\ \text{gal/min}),$$

where C lb/gal denotes the concentration. But here, since the rate of outflow is different from that of inflow, the concentration is not quite so simple. At time $t = 0$, the tank contains 50 gal of brine. Since brine flows in at the rate of 5 gal/min but flows out at the slower rate of 3 gal/min, there is a net gain of $5 - 3 = 2$ gal/min of brine in the tank. Thus at the end of t minutes the amount of brine in the tank is

$$50 + 2t \text{ gal.}$$

Hence the concentration at time t minutes is

$$\frac{x}{50 + 2t} \text{ lb/gal,}$$

and so

$$\text{OUT} = \frac{3x}{50 + 2t} \text{ lb/min.}$$

Thus the differential equation becomes

$$\frac{dx}{dt} = 10 - \frac{3x}{50 + 2t}. \qquad (3.67)$$

Since there was initially 10 lb of salt in the tank, we have the initial condition

$$x(0) = 10. \qquad (3.68)$$

Solution. The differential equation (3.67) is *not* separable but it *is* linear. Putting it in standard form,

$$\frac{dx}{dt} + \frac{3}{2t + 50}x = 10,$$

we find the integrating factor

$$\exp\left(\int \frac{3}{2t + 50}\,dt\right) = (2t + 50)^{3/2}.$$

Multiplying through by this, we have

$$(2t + 50)^{3/2}\frac{dx}{dt} + 3(2t + 50)^{1/2}x = 10(2t + 50)^{3/2}$$

or

$$\frac{d}{dt}[(2t + 50)^{3/2}x] = 10(2t + 50)^{3/2}.$$

Thus

$$(2t + 50)^{3/2}x = 2(2t + 50)^{5/2} + c$$

or

$$x = 4(t + 25) + \frac{c}{(2t + 50)^{3/2}}.$$

Applying condition (3.68), $x = 10$ at $t = 0$, we find

$$10 = 100 + \frac{c}{(50)^{3/2}}$$

or

$$c = -(90)(50)^{3/2} = -22{,}500\sqrt{2}.$$

Thus the amount of salt at any time $t > 0$ is given by

$$x = 4t + 100 - \frac{22{,}500\sqrt{2}}{(2t + 50)^{3/2}}.$$

Example 12.21. *A tank initially contains 50 gallons of fresh water. Brine, containing 2 pounds per gallon of salt, flows into the tank at the rate of 2 gallons per minute and the mixture kept uniform by stirring, runs out at the same rate. How long will it take for the quantity of salt in the tank to increase from 40 to 80 pounds ?* (*Andhra, 1997*)

Solution. Let the salt content at time t be u lb. so that its rate of change is du/dt

$$= 2 \text{ gal.} \times 2 \text{ lb.} = 4 \text{ lb./min.}$$

If C be the concentration of the brine at time t, the rate at which the salt content decreases due to the out-flow

$$= 2 \text{ gal.} \times C \text{ lb.} = 2C \text{ lb./min.}$$

$$\therefore \qquad \frac{du}{dt} = 4 - 2C \qquad \text{...}(i)$$

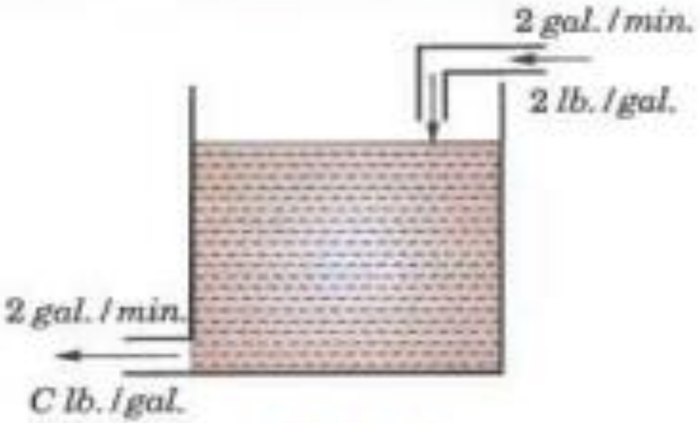


Fig. 12.19

Also since there is no increase in the volume of the liquid, the concentration $C = u/50$.

$\therefore$ (*i*) becomes $\dfrac{du}{dt} = 4 - 2\dfrac{u}{50}$

Separating the variables and integrating, we have

$$\int dt = 25 \int \frac{du}{100 - u} + k \quad \text{or} \quad t = -25 \log_e (100 - u) + k \qquad \text{...}(ii)$$

Initially when $t = 0, u = 0 \quad \therefore \quad 0 = -25 \log_e 100 + k \qquad \text{...}(iii)$

Eliminating k from (*ii*) and (*iii*), we get $t = 25 \log_e \dfrac{100}{100 - u}$.

Taking $t = t_1$ when $u = 40$ and $t = t_2$ when $u = 80$, we have

$$t_1 = 25 \log_e \frac{100}{60} \text{ and } t_2 = 25 \log_e \frac{100}{20}$$

$\therefore$ The required time $(t_2 - t_1) = 25 \log_e 5 - 25 \log_e 5/3$

$$= 25 \log_e 3 = 25 \times 1.0986 = 27 \text{ min. } 28 \text{ sec.}$$

Example 12.18. *A body originally at 80°C cools down to 60°C in 20 minutes, the temperature of the air being 40°C. What will be the temperature of the body after 40 minutes from the original ?*

Solution. If θ be the temperature of the body at any time t, then

$$\frac{d\theta}{dt} = -k(\theta - 40), \qquad \text{where } k \text{ is a constant.}$$

Integrating, $\displaystyle\int \frac{d\theta}{\theta - 40} = -k \int dt + \log c$, where c is a constant.

or $\log(\theta - 40) = -kt + \log c$ *i.e.,* $\theta - 40 = ce^{-kt}$ ...(*i*)

When $t = 0, \theta = 80°$ and when $t = 20, \theta = 60°$. $\therefore$ $40 = c$, and $20 = ce^{-20k}$; $k = \frac{1}{20}\log 2$.

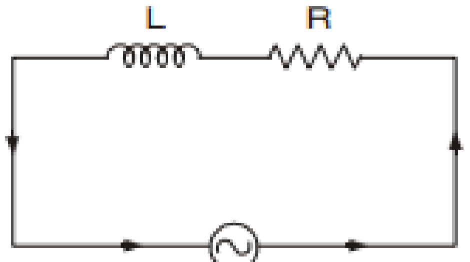
Thus (*i*) becomes $\theta - 40 = 40e^{-\left(\frac{1}{20}\log 2\right)t}$

When $t = 40$ min., $\theta = 40 + 40e^{-2\log 2} = 40 + 40e^{\log(1/4)} = 40 + 40 \times \frac{1}{4} = 50°\text{C}.$

Example 39. *Solve the equation* $L\dfrac{di}{dt} + Ri = E_0 \sin wt$

where L, R and E_0 are constants and discuss the case when t increases indefinitely.

Solution. $L\dfrac{di}{dt} + Ri = E_0 \sin wt$



$$\Rightarrow \quad \frac{di}{dt} + \frac{R}{L} i = \frac{E_0}{L} \sin wt$$

$$\text{I.F.} = e^{\int \frac{R}{L} dt} = e^{\frac{R}{L}t}$$

Solution is $$i.e^{\frac{R}{L}t} = \frac{E_0}{L} \int e^{\frac{R}{L}t} \sin wt \, dt + A$$

$$\left[\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{\sqrt{a^2 + b^2}} \sin\left(bx - \tan^{-1}\frac{b}{a} \right) \right]$$

$$\Rightarrow \quad i.e^{\frac{R}{L}t} = \frac{E_0}{L} \frac{e^{\frac{R}{L}t}}{\sqrt{\frac{R^2}{L^2} + w^2}} \sin\left(wt - \tan^{-1}\frac{Lw}{R} \right) + A$$

$$i = \frac{E_0}{\sqrt{R^2 + L^2 w^2}} \sin\left(wt - \tan^{-1}\frac{Lw}{R} \right) + Ae^{-\frac{R}{L}t}$$

As t increases indefinitely, then $Ae^{-\frac{Rt}{L}}$ tends to zero.

so $$i = \frac{E_0}{\sqrt{R^2 + L^2 w^2}} \sin\left(wt - \tan^{-1}\frac{Lw}{R} \right)$$ **Ans.**

Example 12.17. *Show that the differential equation for the current i in an electrical circuit containing an inductance L and a resistance R in series and acted on by an electromotive force E sin ωt satisfies the equation L di/dt + Ri = E sin ωt.*

Find the value of the current at any time t, if initially there is no current in the circuit.

(*Kurukshetra, 2005*)

Solution. By Kirchhoff's first law, we have sum of voltage drops across R and $L = E \sin \omega t$

i.e.,
$$Ri + L\frac{di}{dt} = E \sin \omega t.$$

This is the required differential equation which can be written as $\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L} \sin \omega t$

This is a *Leibnitz's equation.* Its I.F. $= e^{\int \frac{R}{L} dt} = e^{\frac{Rt}{L}}$

$\therefore$ the solution is $i(\text{I.F.}) = \int \frac{E}{L} \sin \omega t \cdot (\text{I.F.}) dt + c$

or
$$ie^{Rt/L} = \frac{E}{L}\int e^{Rt/L} \sin \omega t \, dt + c = \frac{E}{L} \frac{e^{Rt/L}}{\sqrt{[(R/L)^2 + \omega^2]}} \sin\left(\omega t - \tan^{-1}\frac{L\omega}{R}\right) + c$$

or
$$i = \frac{E}{\sqrt{(R^2 + \omega^2 L^2)}} \sin (\omega t - \phi) + ce^{-Rt/L} \text{ where } \tan \phi = L\omega/R \quad ...(i)$$

Initially when $t = 0$; $i = 0$. $\therefore$ $0 = \frac{E \sin (-\phi)}{\sqrt{(R^2 + \omega^2 L^2)}} + c$, *i.e.*, $c = \frac{E \sin \phi}{\sqrt{(R^2 + \omega^2 L^2)}}$

Thus (*i*) takes the form $i = \frac{E \sin (\omega t - \phi)}{\sqrt{(R^2 + \omega^2 L^2)}} + \frac{E \sin \phi}{\sqrt{(R^2 + \omega^2 L^2)}} \cdot e^{-Rt/L}$

or
$$i = \frac{E}{\sqrt{(R^2 + \omega^2 L^2)}} [\sin (\omega t - \phi) + \sin \phi \, . \, e^{-Rt/L}]$$ which gives the current at any time t.

Example 12.12. *A body of mass m, falling from rest is subject to the force of gravity and an air resistance proportional to the square of the velocity (i.e., kv^2). If it falls through a distance x and possesses a velocity v at that instant, prove that*

$$\frac{2kx}{m} = \log\frac{a^2}{a^2 - v^2}, \text{ where } mg = ka^2.$$

Solution. If the body be moving with the velocity v after having fallen through a distance x, then its equation of motion is

$$mv\frac{dv}{dx} = mg - kv^2 \quad \text{or} \quad mv\frac{dv}{dx} = k(a^2 - v^2). \qquad [\because mg = ka^2] \quad ...(i)$$

$\therefore$ separating the variables and integrating, we get $\int \frac{v\,dv}{a^2 - v^2} = \int \frac{k}{m}dx + c$

or
$$-\frac{1}{2}\log(a^2 - v^2) = \frac{kx}{m} + c \qquad ...(ii)$$

Initially, when $x = 0$, $v = 0$. $\therefore -\frac{1}{2}\log a^2 = c$ ...(iii)

Subtracting (*iii*) from (*ii*), we have $\frac{1}{2}[\log a^2 - \log(a^2 - v^2)] = kx/m$

or
$$\frac{2kx}{m} = \log\left(\frac{a^2}{a^2 - v^2}\right)$$

Example 12.10. Resisted motion. *A moving body is opposed by a force per unit mass of value cx and resistance per unit of mass of value bv^2 where x and v are the displacement and velocity of the particle at that instant. Find the velocity of the particle in terms of x, if it starts from rest.* (Marathwada, 2008)

Solution. By Newton's second law, the equation of motion of the body is $v\dfrac{dv}{dx} = -cx - bv^2$

or
$$v\frac{dv}{dx} + bv^2 = -cx \quad \text{...(i)}$$

This is Bernoulli's equation. $\therefore$ Put $v^2 = z$ and $2v\,dv/dx = dz/dx$, so that (*i*) becomes

$$\frac{dz}{dx} + 2bz = -2cx \quad \text{...(ii)}$$

This is Leibnitz's linear equation and I.F. $= e^{2bx}$.

$\therefore$ the solution of (*ii*) is $\quad ze^{2bx} = -\int 2cxe^{2bx}\,dx + c'$ [Integrate by parts]

$$= -2c\left[x \cdot \frac{e^{2bx}}{2b} - \int 1 \cdot \frac{e^{2bx}}{2b}\,dx\right] + c' = -\frac{cx}{b}e^{2bx} + \frac{c}{2b^2}e^{2bx} + c'$$

or
$$v^2 = \frac{c}{2b^2} + c'e^{-2bx} - \frac{cx}{b} \quad \text{...(iii)}$$

Initially $v = 0$ when $x = 0$ $\therefore$ $0 = c/2b^2 + c'$.

Thus, substituting $c' = -c/2b^2$ in (*iii*), we get $v^2 = \dfrac{c}{2b^2}(1 - e^{-2bx}) - \dfrac{cx}{b}$.

Ex. 27(*a*). *If the population of a country doubles in 50 years, in how many years will it treble under the assumption that the rate of increase is proportional to the number of inhabitants.*

[Delhi B.Sc. I (Hons) 2010; Delhi Maths (H) 1995, 1998, 2000, 08]

Sol. Let the population be x at time t (in years) and x_0 be the population when $t = 0$. Then, given that dx/dt is proportional to x, *i.e.*,

$$dx/dt = kx, \; k \text{ being the constant of proportionality,} \qquad \dots (1)$$

$$(1) \Rightarrow (1/x)\, dx = k\, dt \Rightarrow \int (1/x)\, dx = \int k\, dt \Rightarrow \log x - \log c = kt.$$

$\therefore \log (x/c) = kt$ so that $x = ce^{kt}$ .... (2)

By our assumption, when $t = 0$, $x = x_0$ so that

$(2) \Rightarrow x_0 = c$ and then $(2) \Rightarrow x = x_0\, e^{kt}$ .... (3)

Given $x = 2x_0$ when $t = 50$, so (3) yields

$$2x_0 = x_0\, e^{50k} \Rightarrow 50k = \log 2 \Rightarrow k = (\log 2)/50. \qquad \dots (4)$$

Next, let the population treble in t' years.

$\therefore$ From (3), $3x_0 = x_0\, e^{kt'} \Rightarrow kt' = \log 3 \Rightarrow t' = (\log 3)/k = (50 \log 3)/\log 2$, by (4)

or $t' = (50 \times .47712)/.30103 = 78.25$ years.

Ex. 27(*b*). *The number of bacteria in a yeast culture grows at a rate which is proportional to the number present. If the population of a colony of yeast bacteria triples in 1 hour, find the number of bacteria which will be present at the end of 5 hours.* **[Delhi Maths (H) 1993]**

Sol. Suppose that the number of bacteria is x_0 when $t = 0$, and it is x at time t (in hours). Then given that dx/dt is proportional to x, *i.e.*,

$$(dx/dt) = kx, \ k \text{ being the constant of proportionality,} \qquad \dots (1)$$

$$(1) \Rightarrow (1/x)\, dx = k\, dt \Rightarrow \int (1/x)\, dx = k \int dt \Rightarrow \log x - \log c = kt.$$

$$\therefore \quad \log (x/c) = kt \qquad \text{so that} \qquad x = ce^{kt} \qquad \dots (2)$$

By our assumption, when $t = 0$, $x = x_0$ Therefore,

$$(2) \Rightarrow x_0 = c \quad \text{and then} \quad (2) \Rightarrow \quad x = x_0\, e^{kt}. \qquad \dots (3)$$

Given $x = 3x_0$ when $t = 1$, so (3) yields $\quad 3x_0 = x_0\, e^k \quad \Rightarrow \quad e^k = 3. \qquad \dots (4)$

Next, let $x = x'$ when $t = 5$. Then (3) yields $\quad x' = x_0\, e^{5k} = x_0\, (e^k)^5 = x_0 \cdot 3^5$, by (4)

Hence, the bacteria is expected to grow 3^5 times at the end of 5 hours.

Ex. 29(*a*). *According to Newton's law of cooling, the rate at which a substance cools in moving air is proportional to the difference between the temperature of the substance and that of the air. If the temperature of the air is 290 K and the substance cools from 370 K to 330 K in 10 minutes, find when the temperature will be 295 K.* **[Delhi Maths (H) 1994, 2005]**

Sol. Let T be the temperature of the substance at the time t (in minutes). Then, by hypothesis, we have

$$\frac{dT}{dt} = -\lambda\,(T-290) \qquad \text{or} \qquad \frac{dT}{T-290} = -\lambda\,dt, \qquad \dots (1)$$

where λ is a positive constant of proportionality.

Integrating (1) between the limits $t = 0$, $T = 370$ K and $t = 10$ minutes, $T = 330$ K, we have

$$\int_{370}^{330} \frac{dT}{T-290} = -\lambda \int_0^{10} dt \qquad \text{or} \qquad \Big[\log\,(T-290)\Big]_{370}^{330} = -10\lambda$$

$$\Rightarrow -10\lambda = \log 40 - \log 80 \quad \Rightarrow \quad \lambda = (1/10)\log 2.$$

Again, assuming that $t = t'$ minutes when $T = 295$ K and so integrating (1) between the limits $t = 0$, $T = 370$ K and $t = t'$ (minutes), $T = 295$ K, we have

$$\int_{370}^{295} \frac{dT}{T-290} = -\lambda \int_0^{t'} dt \qquad \text{or} \qquad \Big[\log\,(T-290)\Big]_{370}^{295} = -\lambda\,t'$$

$$\Rightarrow -\lambda t' = \log 5 - \log 80 \quad \Rightarrow \quad \lambda t' = \log 16 \quad \Rightarrow \quad \lambda t' = 4\log 2$$

$$\Rightarrow [(1/10)\log 2]\,t' = 4\log 2, \text{ using (2). So } t' = 40 \text{ minutes.}$$

Ex. 29(c). *A body whose temperature is initially 100°C is allowed to cool in air, whose temperature remains at a constant temperature 20°C. It is given that after 10 minutes, the body has cooled to 40°C. Find the temperature of the body after half an hour.* **[Delhi Maths (H) 2000]**

Sol. Let T be the temperature of the body in degree celsius and t be time in minutes. Then, by Newton's law of cooling, we get

$$\frac{dT}{dt} = -\lambda\,(T - 20) \qquad \text{or} \qquad \frac{dT}{T-20} = -\lambda\, dt \quad \ldots(1)$$

where λ is a positive constant of proportionality.

Integrating (1), $\log (T - 20) - \log C = -\lambda t$ or $T = 20 + Ce^{-\lambda t}$ ... (2)

Initially, when $t = 0$, $T = 100$. So (2) gives $C = 80$

Then (2) reduces to $T = 20 + 80\, e^{-\lambda t}$ ... (3)

Given that $T = 40$ when $t = 10$. (3) gives $40 = 20 + 80\, e^{-10\lambda}$

or $80\, e^{-10\lambda} = 20$ or $e^{-10\lambda} = 4^{-1}$ or $e^{-\lambda} = (4^{-1})^{1/10}$

$\therefore$ (3) reduces to $T = 20 + 80\,(e^{-\lambda})^t = 20 + 80\,(4^{-1})^{t/10}$ ... (4)

Let $T = T'$ when t = half an hour = 30 minutes. Then (4) gives

$T = 20 + 80\,(4^{-1})^3 = 20 + 8\,(1/4^3) = 20 + 1.25 = 21.25$°C.

Example 42. *The rate at which a body cools is proportional to the difference between the temperature of the body and that of the surrounding air. If a body in air at 25°C will cool from 100° to 75° in one minute, find its temperature at the end of three minutes.*

Solution. Let temperature of the body be T°C.

$$\frac{dT}{dt} = k\ (T - 25) \qquad \text{or} \qquad \frac{d\,T}{T-25} = k\,dt$$

$$\log\ (T - 25) = kt + \log A \qquad \text{or} \qquad \log \frac{T-25}{A} = kt$$

$$T - 25 = A\, e^{kt} \qquad ...(1)$$

When $t = 0$, then $T = 100$, from (1) A = 75

When $t = 1$, then $T = 75$ and A =75, From (1) $\frac{2}{3} = e^{k}$

∴ (1) becomes $\quad T = 25 + 75\, e^{kt}$

When $t = 3$, then $T = 25 + 75\, e^{3k} = 25 + 75 \times 8 / 27 = 47.22$ **Ans.**