Laplace Transform

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1 Definition and Existence

1.1 Definition and Existence

> **Definition 1.1.1: Laplace Transform**
>
> Let F be a real-valued function of the real variable t, defined for $t > 0$. Let s be a variable that we shall assume to be real, and consider the function f defined by
>
> $$f(s) = \int_0^\infty e^{-st} F(t)\, dt \tag{1}$$
>
> for all values of s for which this integral exists. The function f defined by the integral (1) is called the Laplace Transform of the function F. We shall denote the Laplace transform of F by $\mathcal{L}\{F(t)\}$.
>
> Thus the Laplace transform of a function f is given by
>
> $$\mathcal{L}\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\, dt = \lim_{R \to \infty} \int_0^R e^{-st} F(t)\, dt \tag{2}$$

Some ways to write Laplace transforms:

$$\mathcal{L}F(t) = f(s) = \int_0^\infty e^{-st} F(t)\, dt$$

$$\mathcal{L}G(t) = g(s)$$

$$\mathcal{L}u(t) = \tilde{u}(s)$$

> ***Theorem 1.1.2: Hypothesis:*** *Let F be a real function that has the following properties:*
>
> *1. F is a piecewise continuous in every finite closed interval $0 \le t \le a$ $(b > 0)$.*
>
> *2. F is of exponential order, i.e, there exists α, $M > 0$, and $t_0 > 0$ such that*
>
> $$e^{-\alpha t}|F(t)| < M \ \text{for } t > t_0$$
>
> ***Conclusion:*** *The Laplace transform of F exists for $s > \alpha$.*
>
> $$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt$$

1.2 Some Functions and Their Laplace Transforms

$F(t)$	$\mathcal{L}\{F(t)\} = f(s)$	$F(t)$	$\mathcal{L}\{F(t)\} = f(s)$
1	$\dfrac{1}{s}$	n	$\dfrac{n}{s}$
t	$\dfrac{1}{s^2}$	t^n	$\dfrac{n!}{s^{n+1}}$
e^{at}	$\dfrac{1}{s-a}$	e^{-at}	$\dfrac{1}{s+a}$
$\sin at$	$\dfrac{a}{s^2 + a^2}$	$\sinh at$	$\dfrac{a}{s^2 - a^2}$
$\cos at$	$\dfrac{s}{s^2 + a^2}$	$\cosh at$	$\dfrac{s}{s^2 - a^2}$

Table 1: Functions and their Laplace Transform

Proofs :

$$\mathcal{L}\{n\} = \frac{n}{s}$$

Let $F(t) = n$, for $t > 0$
Then

$$\mathcal{L}\{n\} = \int_0^\infty e^{-st} \cdot n \, dt$$

$$= n \frac{-e^{st}}{s} \Big|_0^\infty$$

$$= \frac{n}{s} \quad \square$$

$$\mathcal{L}\{t\} = \frac{1}{s^2}$$

Let $F(t) = t$, for $t > 0$
Then

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st} \cdot t \, dt$$

$$= -t \frac{e^{-st}}{s} + \int_0^\infty \frac{e^{-st}}{s} \, dt$$

$$= -e^{-st} \frac{t}{s} \Big|_0^\infty - e^{-st} \frac{1}{s^2} \Big|_0^\infty$$

$$= \frac{1}{s^2} \quad \square$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

Let $F(t) = t^n$, for $t > 0$
Then

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-st} t^n \, dt$$

$$= -t^n \frac{e^{st}}{s} + \int_0^\infty n t^{n-1} \frac{e^{-st}}{s} \, dt$$

$$= -n t^{n-1} \frac{e^{-st}}{s^2} \Big|_0^\infty + \int_0^\infty n(n-1) t^{n-2} \frac{e^{-st}}{s^2} \, dt$$

$$= -n(n-1) t^{n-2} \left(\frac{e^{-st}}{s^3} \right) \Big|_0^\infty + \int_0^\infty n(n-1)(n-2) t^{n-3} \frac{e^{-st}}{s^3} \, dt$$

$$= \cdots$$

$$= n! t^{n-n} \frac{e^{-st}}{s^{n+1}} \Big|_0^\infty + \int_0^\infty n(n-1) \cdots (n-n) \frac{e^{-st}}{s^{n+1}} \, dt$$

$$= \frac{n!}{s^{n+1}} \quad \square$$

Let $F(t) = e^{at}$, for $t > 0$
Then

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at} \, dt$$

$$= \int_0^\infty e^{(a-s)t} \, dt$$

$$= \frac{e^{(a-s)t}}{a-s} \Big|_0^\infty$$

$$= \frac{1}{s-a} \quad \square$$

Let $F(t) = e^{-at}$, for $t > 0$
Then

$$\mathcal{L}\{e^{-at}\} = \int_0^\infty e^{-st} e^{-at} \, dt$$

$$= \int_0^\infty e^{-(a+s)t} \, dx$$

$$= \frac{e^{-(a+s)t}}{s+a} \Big|_0^\infty$$

$$= \frac{1}{s+a} \quad \square$$

Let $F(t) = \sin at$, for $t > 0$
Then

$$
\begin{aligned}
\mathscr{L}\{\sin at\} &= \int_0^\infty e^{-st} \sin at \, dt \\
&= -\frac{e^{-st}}{s^2 + a^2} \left(s \sin at + a \cos at\right) \Big|_0^\infty \\
&= \frac{a}{s^2 + a^2} \quad \square
\end{aligned}
$$

Let $F(t) = \cos at$, for $t > 0$
Then

$$
\begin{aligned}
\mathscr{L}\{\cos at\} &= \int_0^\infty e^{-st} \cos at \, dt \\
&= \frac{e^{-st}}{s^2 + a^2} \left(-s \cos at + a \sin at\right) \Big|_0^\infty \\
&= \frac{s}{s^2 + a^2} \quad \square
\end{aligned}
$$

2 Basic Properties of the Laplace Transform

2.1 Linearity Property

> ***Theorem* (The Linearity Property):**
> Let F_1 and F_2 be functions whose Laplace transform exist, and let c_1 and c_2 be constants.
> Then
> $$\mathscr{L}\{c_1 F_1(t) + c_2 F_2(t)\} = c_1 \mathscr{L}\{F_1(t)\} + c_2 \mathscr{L}\{F_2(t)\}$$

Proof :
Let $F(t) = c_1 F_1(t) + c_2 F_2(t)$, for $t > 0$
Then

$$
\begin{aligned}
\mathscr{L}\{c_1 F_1(t) + c_2 F_2(t)\} &= \int_0^\infty e^{-st} \left[c_1 F_1(t) + c_2 F_2(t)\right] dt \\
&= c_1 \int_0^\infty e^{-st} F_1(t) \, dt + c_2 \int_0^\infty e^{-st} F_2(t) \, dt \\
&= c_1 \mathscr{L}\{F_1(t)\} + c_2 \mathscr{L}\{F_2(t)\} \quad \square
\end{aligned}
$$

Example 2.1:

$$\mathscr{L}\{4t^2 - 3\cos 2t + 5e^{-t}\}$$

$$
\begin{aligned}
\mathscr{L}\{4t^2 - 3\cos 2t + 5e^{-t}\} &= 4\mathscr{L}\{t^2\} - 3\mathscr{L}\{\cos 2t\} + 5\mathscr{L}\{e^{-t}\} \\
&= 4 \cdot \frac{2!}{s^3} - 3 \cdot \frac{s}{s^2 + 4} + 5 \cdot \frac{1}{s+1} \\
&= \frac{8}{s^3} - \frac{3s}{s^2 + 4} + \frac{5}{s+1}
\end{aligned}
$$

Example 2.2: Find $\mathscr{L}\{F(w)\}$, when $F(t) = \begin{cases} 5 & \text{for } 0 < t < 3 \\ 0 & \text{for } t > 3 \end{cases}$

$$
\begin{aligned}
\mathscr{L}\{F(t)\} &= \int_0^\infty e^{-st} F(t) \, dt = \int_0^3 e^{-st} \cdot 5 \, dt + \int_3^\infty 0 \, dt \\
&= \int_0^3 e^{-st} \cdot 5 \, dt \\
&= \frac{5e^{-st}}{s} \Big|_0^3 \\
&= \frac{5}{s}\left(1 - e^{-3s}\right)
\end{aligned}
$$

Example 2.3: Find $\mathscr{L}\{F(t)\}$, when $F(t = \begin{cases} (t-1)^2 & \text{for } t > 1 \\ 0 & \text{for } 0 < t < 1 \end{cases})$

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$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt = \int_0^1 0\, dt + \int_1^\infty e^{-st}(t-1)^2\, dt$$

$$= \int_1^\infty e^{-st}(t^2 - 2t + 1)\, dt$$

$$= -t^2 \frac{e^{-st}}{s}\Big|_1^\infty - 2\int_1^\infty t \cdot \frac{e^{-st}}{s}\, dt - 2\int_1^\infty t \cdot e^{-st}\, dt - \frac{e^{-st}}{s}$$

$$= \frac{e^{-s}}{s} + 2\left[-\frac{t}{s}\left(\frac{e^{-st}}{s}\right)\Big|_1^\infty + \int_1^\infty \frac{e^{-st}}{s^2}\, dt + 2t\frac{e^{-st}}{s}\Big|_1^\infty - \int_1^\infty \frac{e^{-st}}{s}\, dt\right] + \frac{e^{-st}}{s}$$

$$= \frac{e^{-s}}{s} - 2\frac{e^{-s}}{s^2} + 2\frac{e^{-s}}{s^2} - 2\frac{e^{-st}}{s} + 2\frac{e^{-s}}{s^2} + \frac{e^{-st}}{s}$$

$$= 2\frac{e^{-s}}{s^3}$$

2.2 First Translation Property

> **Theorem 2.2.1 (First Translation of Shifting Property):**
> *If*
> $$\mathcal{L}\{F(t)\} = f(s)$$
> *then*
> $$\mathcal{L}\{e^{at} F(t)\} = f(s-a)$$

Proof:
Let $G(t) = e^{at} F(t)$, for $t > 0$
Then

$$\mathcal{L}\{e^{at} F(t)\} = \int_0^\infty e^{-st} e^{at} F(t)\, dt$$

$$= \int_0^\infty e^{-(s-a)t} F(t)\, dt$$

$$= \mathcal{L}\{F(t)\}\Big|_{s-a}$$

$$= f(s-a) \quad \square$$

Example 2.4:
$$\mathcal{L}\{e^{-t} \cos 2t\}$$

Since $\mathcal{L}\{\cos 2t\} = \dfrac{s}{s^2 + 4}$, we have

$$\mathcal{L}\{e^{-t} \cos 2t\} = \frac{s+1}{(s+1)^2 + 4} = \frac{s+1}{s^2 + 2s + 5}\}$$

Example 2.5: Evaluate $\mathcal{L}\{e^{-2t}(3\cos 6t - 5\sin 6t)\}$
Now,

$$f(s) = \mathcal{L}\{3\cos 6t - 5\sin 6t\}$$

$$= \frac{3s}{s^2 + 36} - \frac{30}{s^2 + 36}$$

$$= \frac{3s - 30}{s^2 + 36}$$

$$\therefore \mathcal{L}\{e^{-2t}(3\cos 6t - 5\sin 6t)\} = f(s+2)$$

$$= \frac{3(s+2) - 30}{(s+2)^2 + 36}$$

$$= \frac{3s - 24}{s^2 + 4s + 40}$$

2.3   Second Translation Property

> **Theorem 2.3.1 (Second Translation or Shifting Property):**
> *If*
> $$\mathcal{L}\{F(t)\} = f(s)$$
> *and*
> $$G(t) = \begin{cases} F(t-a) & for\ t > a \\ 0 & for\ t < a \end{cases}$$
> *then*
> $$\mathcal{L}\{G(t)\} = e^{-as} f(s)$$

Proof:

Let $G(t) = \begin{cases} F(t-a) & \text{for } t > a \\ 0 & \text{for } t < a \end{cases}$

Then

$$\mathcal{L}\{G(t)\} = \int_0^\infty e^{-st} F(t-a)\, dt$$
$$= \int_a^\infty e^{-st} F(t-a)\, dt$$
$$= \int_0^\infty e^{-s(u+a)} F(u)\, du \quad \text{where } u = t - a$$
$$= e^{-as} \int_0^\infty e^{-su} F(u)\, du$$
$$= e^{-as} \mathcal{L}\{F(t)\}$$
$$= e^{-as} f(s) \quad \square$$

Example 2.6:   Find $\mathcal{L}\{G(t)\}$ **where** $G(t) = \begin{cases} \cos\left(t - \frac{2\pi}{3}\right) & \textbf{for } t > \frac{2\pi}{3} \\ 0 & \textbf{for } t < \frac{2\pi}{3} \end{cases}$

$$\mathcal{L}\{G(t)\} = e^{-\frac{2\pi}{3}s} \cdot \mathcal{L}\{\cos t\}$$
$$= e^{-\frac{2\pi}{3}s} \cdot \frac{s}{s^2 + 1}$$
$$= \frac{s e^{-\frac{2\pi}{3}s}}{s^2 + 1}$$

Example 2.7: Find $\mathcal{L}\{F(t)\}$**, if** $F(t) = \begin{cases} (t-1)^2 & \textbf{for } t > 1 \\ 0 & \textbf{for } t < 0 \end{cases}$

Let

$$G(t) = t^2$$
$$\therefore \mathcal{L}\{G(t)\} = \frac{2!}{s^3}$$

Now,

$$F(t) = \begin{cases} G(t-1) & \text{for } t > 0 \\ 0 & \text{for } t < 0 \end{cases}$$

$$\therefore \mathcal{L}\{F(t)\} = \frac{e^{-s} \cdot 2!}{s^3} = \frac{2e^{-s}}{s^3}$$

2.4   Change of Scale Property

> ***Theorem 2.4.1* (Change of Scale Property):**
> If
> $$\mathcal{L}\{F(t)\} = f(s)$$
> then
> $$\mathcal{L}\{F(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right)$$

Proof:
Let $G(t) = F(at)$, for $t > 0$
Then

$$\mathcal{L}\{G(t)\} = \int_0^\infty e^{-st} F(at)\, dt$$
$$= \int_0^\infty e^{-\frac{s}{a}u} F(u)\, d(u/a) \quad \text{where } u = at$$
$$= \frac{1}{a} \int_0^\infty e^{-\frac{s}{a}u} F(u)\, du$$
$$= \frac{1}{a} \mathcal{L}\{F(t)\}$$
$$= \frac{1}{a} f\left(\frac{s}{a}\right) \quad \square$$

Example 2.8: Evaluate $\mathcal{L}\{\sin 3t\}$

$$\mathcal{L}\{\sin 3t\} = \frac{1}{3} \cdot \frac{1}{\left(\frac{s}{3}\right) + 1}$$
$$= \frac{1}{3} \cdot \frac{3^2}{s^2 + 3^2}$$
$$= \frac{3}{s^2 + 9}$$

Example 2.9: If $\mathcal{L}\{\frac{\sin t}{t} = \tan^{-1}\frac{1}{s} = f(s)\}$**, then evaluate** $\mathcal{L}\{\frac{\sin at}{t}\}$

$$\mathcal{L}\{\frac{\sin at}{t}\} = \mathcal{L}\left\{a \cdot \frac{\sin at}{t}\right\}$$
$$= a \cdot \mathcal{L}\left\{\frac{\sin at}{at}\right\}$$
$$= a \cdot \frac{1}{a} f\left(\frac{s}{a}\right)$$
$$= \tan^{-1}\frac{a}{s}$$

2.5   Multiplication by t

> ***Theorem 2.5.1* (Multiplication by t^n):**
> If
> $$\mathcal{L}\{F(t)\} = f(s)$$
> then
> $$\mathcal{L}\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s)$$

Proof:
Let $G(t) = t^n F(t)$, for $t > 0$
Then

$$\mathcal{L}\{G(t)\} = \int_0^\infty e^{-st} t^n F(t)\, dt$$

$$= (-1)^n \int_0^\infty e^{-st} \frac{d^n}{ds^n} F(t)\, dt$$

$$= (-1)^n \frac{d^n}{ds^n} \int_0^\infty e^{-st} F(t)\, dt$$

$$= (-1)^n \frac{d^n}{ds^n} f(s) \quad \square$$

Alternative Proof:

We have

$$f(s) = \int_0^\infty e^{-st} F(t)\, dt$$

Then by Leibnitz's rule for differentiating under the integral sign, we have

$$\frac{df}{ds} = f'(s) = \frac{d}{ds} \int_0^\infty e^{-st} F(t)\, dt$$

$$= \int_0^\infty \frac{d}{ds} \left(e^{-st} F(t) \right) dt$$

$$= \int_0^\infty e^{-st} \left(-t e^{-st} F(t) + F'(t) \right) dt$$

$$= -\int_0^\infty e^{-st} \{ t F(t) \}\, dt$$

$$= -\mathcal{L}\{ t F(t) \}$$

$$\therefore \mathcal{L}\{ t F(t) \} = -\frac{d}{ds} f(s) = f'(s)$$

This proves the theorem for $n = 1$.

To establish the theorem in general, we use mathematical induction. Suppose that the theorem is true for $n = k$, i.e

$$\int_0^\infty e^{-st} \{ t^k F(t) \}\, dt = (-1)^k f^{(k)}(s) \tag{2.5.1}$$

Then

$$\frac{d}{ds} \left[\int_0^\infty e^{-st} \{ t^k F(t) \}\, dt \right] = (-1)^k f^{(k+1)}(s)$$

Or by Leibnitz's rule,

$$-\int_0^\infty \frac{d}{ds} \left(e^{-st} \{ t^{k+1} F(t) \} \right) dt = (-1)^{(k)} f^{(k+1)}(s)$$

i.e

$$\int_0^\infty e^{-st} \{ t^{k+1} F(t) \}\, dt = (-1)^{(k+1)} f^{(k+1)}(s) \tag{2.5.2}$$

If follows that if (2.5.1) is true for $n = k$, then (2.5.2) is true for $n = k + 1$. Since (2.5.1) is true for $n = 1$, it follows that (2.5.1) is true for all positive integers n. $\qquad \square$

Example 2.10: Find $\mathcal{L}\{t^2 \cos at\}$

$$\mathcal{L}\{t^2 \cos at\} = (-1)^2 \cdot \frac{d^2}{dx^2} \left(\frac{s}{s^2 + a^2} \right)$$

$$= \frac{d}{ds} \left[\frac{s^2 + a^2 - 2s^2}{(s^2 + a^2)^2} \right]$$

$$= \frac{d}{ds} \left[\frac{a^2 - s^2}{(s^2 + a^2)^2} \right]$$

$$= \frac{(s^2 + a^2)^2(-2s) - (-s^2 + a^2)2(s^2 + a^2) \cdot 2s}{(s^2 + a^2)^4}$$

$$= \frac{2s^3 - 6a^2 s}{(s^2 + a^2)^3}$$

2.6 Division by t

> **Theorem 2.6.1 (Division by t):**
> If
> $$\mathcal{L}\{F(t)\} = f(s)$$
> then
> $$\mathcal{L}\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(x)\,dx$$

Proof:

Let $G(t) = \dfrac{F(t)}{t}$, for $t > 0$. Then $F(t) = tG(t)$. Taking the Laplace Transform of both sides, we get

$$\mathcal{L}\{F(t)\} = \mathcal{L}\{tG(t)\}$$
$$f(s) = -\frac{d}{ds}\mathcal{L}\{G(t)\} = -\frac{d}{ds}g(s)$$

Then integrating, we have

$$g(s) = -\int_\infty^s f(u)\,du$$
$$\mathcal{L}\{G(t)\} = \int_s^\infty f(u)\,du$$
$$\mathcal{L}\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(u)\,du \quad \square$$

> **Example 2.11: Find $\mathcal{L}\left\{\frac{\sin t}{t}\right\}$**
>
> $$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{s^2 + 1}\,du$$
> $$= \tan^{-1} u \Big|_s^\infty$$
> $$= \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s$$
> $$= \tan^{-1}\frac{1}{s}$$

2.7 Laplace Transform of Integral

> **Theorem 2.7.1 (Laplace transform of Integral):**
> If
> $$\mathcal{L}\{F(t)\} = f(s)$$
> then
> $$\mathcal{L}\left\{\int_0^t F(x)\,dx\right\} = \frac{1}{s}f(s)$$

Proof:

Let $G(t) = \displaystyle\int_0^t F(x)\,dx$, for $t > 0$. Then $G'(t) = F(t)$ and $G(0) = 0$. Taking the Laplace Transform of both sides, we have

$$\mathcal{L}\{G'(t)\} = \mathcal{L}\{F(t)\}$$
$$s\mathcal{L}\{G(t)\} - G(0) = f(s)$$
$$s\mathcal{L}\{G(t)\} = f(s)$$
$$\mathcal{L}\{G(t)\} = \frac{f(s)}{s}$$
$$\mathcal{L}\left\{\int_0^t F(u)\,du\right\} = \frac{f(s)}{s} \quad \square$$

Example 2.12: Evaluate $\mathcal{L}\{\int_0^t \frac{\sin u}{u}\, du\}$

$$\mathcal{L}\left\{\frac{\sin u}{u}\right\} = \tan^{-1}\frac{1}{s}$$

$$\mathcal{L}\left\{\int_0^t \frac{\sin u}{u}\, du\right\} = \frac{f(s)}{s} = \frac{1}{s}\tan^{-1}\frac{1}{s}$$

Example 2.13: Evaluate $\mathcal{L}\{\int_0^t \frac{\sin u}{u}\, du\}$

Let $F(t) = \frac{\sin t}{t}$

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{u^2+1}\, du$$

$$= \tan^{-1} u \Big|_s^\infty$$

$$= \tan^{-1}\frac{1}{s}$$

$$\therefore \mathcal{L}\left\{\int_0^t \frac{\sin u}{u}\, du\right\} = \frac{1}{s}\cdot\tan^{-1}\frac{1}{s}$$

Example 2.14: Evaluate $\mathcal{L}\{\int_0^t \sin 2u\, du\}$

$$\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4}$$

$$\therefore \mathcal{L}\left\{\int_0^t \sin 2u\, du\right\} = \frac{2}{s^3+4s}$$

2.8 Laplace Transform of Periodic Functions

> **Theorem 2.8.1 (Periodic functions):**
> If
> $$\mathcal{L}\{F(t)\} = f(s)$$
> then
> $$\mathcal{L}\{F(t)\} = \frac{1}{1-e^{-Ts}}\int_0^T e^{-st}F(t)\, dt$$
> where T is the period of $F(t)$.

Proof:

Let $F(t)$ has period T. Then $F(t) = F(t+T)$ for all t. Then

$$\mathcal{L}\{F(t)\} = \int_0^\infty e^{-st}F(t)\, dt$$

$$= \int_0^T e^{-st}F(t)\, dt + \int_T^{2T} e^{-st}F(t)\, dt + \int_{2T}^{3T} e^{-st}F(t)\, dt + \cdots$$

$$= \int_0^T e^{-st}F(t)\, dt + \int_0^T e^{-s(t+T)}F(t+T)\, dt + \int_0^T e^{-s(t+2T)}F(t+2T)\, dt + \cdots$$

$$= \int_0^T e^{-st}F(t)\, dt + e^{-sT}\int_0^T e^{-st}F(t)\, dt + e^{-2sT}\int_0^T e^{-st}F(t)\, dt + \cdots$$

$$= \left[1 + e^{-sT} + e^{-2sT} + \cdots\right]\int_0^T e^{-st}F(t)\, dt$$

$$= \frac{1}{1-e^{-sT}}\int_0^T e^{-st}F(t)\, dt \quad \square$$

Example 2.15: **Find** $\mathcal{L}\{F(t)\}$ **for**

$$F(t) = \begin{cases} \sin t & \text{for } 0 < t < \pi \\ 0 & \text{for } \pi < t < 2\pi \end{cases}$$

$$\mathcal{L}\{F(t)\} = \frac{1}{1 - e^{-2\pi s}} \int_0^{2\pi} e^{st} \sin t \, dt$$

$$= \frac{1}{1 - e^{-2\pi s}} \int_0^{\pi} e^{-st} \sin t \, dt$$

Example 2.16: Find $\mathcal{L}\{F(t)\}$ **for**

$$F(t) = \begin{cases} t & \text{for } 0 < t < 1 \\ 0 & \text{for } 1 < t < 2 \end{cases}$$

$$\mathcal{L}\{F(t)\} = \frac{1}{1 - e^{-2s}} \int_0^1 t \, dt$$

$$= \frac{t^2 \Big|_0^1}{2 - 2e^{-2s}}$$

$$= \frac{1}{2 - 2e^{-2s}}$$

2.9   Laplace Transform of Derivatives

> *Theorem 2.9.1* **(Laplace transform of derivatives)***:*
> *If*
> $$\mathcal{L}\{F(t)\} = f(s)$$
> *then*
> $$\mathcal{L}\{F'(t)\} = sf(s) - F(0)$$
> $$\mathcal{L}\{F''(t)\} = s^2 F(s) - sF(0) - F'(0)$$
> $$\mathcal{L}\{F'''(t) = s^3 f(s) - s^2 F(0) - sF'(0) - F''(0)\}$$
> $$\boxed{\mathcal{L}\{F^{(n)}(t)\} = s^n f(s) - s^{n-1}F(0) - s^{n-2}F'(0) - \cdots - F^{(n-1)}(0)}$$
> $$\boxed{\mathcal{L}\{F^{(n)}(t)\} = s^n f(s) - \sum_{i=n-1}^{0} \sum_{j=0}^{n-1} s^i F^{(j)}(0)}$$

Proof:
Using integration by parts, we have

$$\mathcal{L}\{F'(t)\} = \int_0^{\infty} e^{-st} F'(t) \, dt$$

$$= e^{-st} F(t) \Big|_0^{\infty} + s \int_0^{\infty} e^{-st} F(t) \, dt$$

$$= -F(0) + s \int_0^{\infty} e^{-st} F(t) \, dt$$

$$= sf(s) - F(0) \quad \square$$

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Similarly,

$$\mathscr{L}\{F''(t)\} = s\mathscr{L}\{F'(t)\} - F'(0)$$
$$= s\left[sf(s) - F(0)\right] - F'(0)$$
$$= s^2 f(s) - sF(0) - F'(0)$$

Thus using mathematical induction, we get

$$\mathscr{L}\{F^{(n)}(t)\} = s\mathscr{L}\{F^{(n-1)}(t)\} - F^{(n-1)}(0)$$
$$= s\left[s\mathscr{L}\{F^{(n-2)}(t)\} - F^{(n-2)}(0)\right] - F^{(n-1)}(0)$$
$$= s^2\mathscr{L}\{F^{(n-2)}(t)\} - sF^{(n-2)}(0) - F^{(n-1)}(0)$$
$$= s^2\left[s\mathscr{L}\{F^{(n-3)}(t)\} - F^{(n-3)}(0)\right] - sF^{(n-2)}(0) - F^{(n-1)}(0)$$
$$= s^3\mathscr{L}\{F^{(n-3)}(t)\} - s^2 F^{(n-3)}(0) - sF^{(n-2)}(0) - F^{(n-1)}(0)$$
$$= \cdots$$
$$= s^n\mathscr{L}\{F(t)\} - s^{n-1}F(0) - s^{n-2}F'(0) - \cdots - F^{(n-1)}(0)$$
$$= s^n f(s) - \sum_{i=n-1}^{0}\sum_{j=0}^{n-1} s^i F^{(j)}(0) \quad \square$$

3 Inverse Laplace Transform

Theorem 3.0.1 (Inverse Laplace Transform):

If

$$\mathscr{L}\{F(t)\} = f(s)$$

then

$$\mathscr{L}^{-1}\{f(s)\} = F(t)$$