MATH-281
Complex Variables

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1 Complex Numbers

1.1 Definition

Definition 1.1.1: Complex Numbers

A complex number is a number that can be expressed in the form $a + bi$, where a and b are real numbers, and i is a solution of the equation $x^2 = -1$, or simply, $i = \sqrt{-1}$. Because no real number satisfies this equation, i is called an imaginary number. For the complex number $a + bi$, a is called the real part, and b is called the imaginary part.

- The set of all complex numbers is denoted by $\mathbb{C}$.
- The set of all real numbers is denoted by $\mathbb{R}$.

Definition 1.1.2: Modulus and Amplitude

Let $z = a + bi$ be a complex number. The modulus of z is the non-negative real number $|z| = \sqrt{a^2 + b^2}$. The amplitude of z is the angle θ such that $\cos(\theta) = \frac{a}{|z|}$ and $\sin(\theta) = \frac{b}{|z|}$.

If the polar form of the point (a, b) be (r, θ), then $a = r\cos\theta$ and $b = r\sin\theta$.

$$r = |z| = \sqrt{a^2 + b^2} \quad \text{and} \quad \theta = \arctan\left(\frac{b}{a}\right) \tag{1.1.1}$$

Here, r is the modulus of z and θ is the amplitude of z.
In symbols, we write

$$r = \operatorname{mod}(z) = |a + ib| \quad \text{and} \quad \theta = \arg(z) = \tan^{-1}\left(\frac{b}{a}\right) \tag{1.1.2}$$

1.2 De Moivre's Theorem

Theorem 1.2.1* (De Moivre's Theorem)*: *Let $z = r(\cos\theta + i\sin\theta)$ be a complex number. Then, for any positive integer n,*

$$z^n = r^n(\cos n\theta + i\sin n\theta) \tag{1.2.1}$$

Proof:
Case 1: $n \in \mathbb{Z}_+$
We have,

$$\begin{aligned} z_1 z_2 \ldots z_n &= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\ldots(\cos\theta_n + i\sin\theta_n) \\ &= \big\{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\big\}\ldots(\cos\theta_n + i\sin\theta_n) \\ &= \cos(\theta_1+\theta_2+\ldots+\theta_n) + i\sin(\theta_1+\theta_2+\ldots+\theta_n) \end{aligned}$$

Hence, we get

$$z^n = (\cos n\theta + i\sin n\theta) \quad \square$$

Case 2: $n \in \mathbb{Z}_-$
Let $n = -m$. We have,

$$\begin{aligned}
z^n &= (\cos\theta + i\sin\theta)^{-m} \\
&= \frac{1}{(\cos\theta + i\sin\theta)^m} \\
&= \frac{1}{\cos m\theta + i\sin m\theta} \\
&= \frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta} \\
&= \cos m\theta - i\sin m\theta \\
&= \cos n\theta + i\sin n\theta
\end{aligned}$$

Hence, we get

$$z^n = (\cos n\theta + i\sin n\theta) \quad \square$$

Case 3: $n \in \mathbb{Q}$, i.e. $n = \frac{p}{q}$, where $p, q \in \mathbb{Z}$ and $q \neq 0$.
Now,

$$\begin{aligned}
\left(\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta\right)^q &= \cos\left(q\cdot\frac{p}{q}\theta\right) + i\sin\left(q\cdot\frac{p}{q}\theta\right) \\
&= \cos p\theta + i\sin p\theta \\
&= (\cos\theta + i\sin\theta)^p
\end{aligned}$$

Taking the q^{th} root of both sides, we get

$$\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta = (\cos\theta + i\sin\theta)^{\frac{p}{q}} \quad \square$$

Note:-

Some Important Results:

(i) $1 = e^{i2n\pi} = \cos 2n\pi + i\sin 2n\pi$

(ii) $-1 = \cos\pi + i\sin\pi = e^{i\pi}$

(iii) $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\frac{\pi}{2}}$

(iv) $-i = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2} = e^{-i\frac{\pi}{2}}$

2 Analytic Functions

2.1 Definitions

Definition 2.1.1: Complex variable

A **complex variable** is a variable that can take on complex values. A complex variable is usually denoted by z.

If x and y are real variables, then $z = x + iy$ is a complex variable, where i is the imaginary unit.

Definition 2.1.2: Complex Function

A **complex function** is a function that takes complex variables as input and returns complex values. A complex function is usually denoted by $f(z)$.

If $z = x + iy$ and $w = u + iv$ are complex variables, then $f(z) = u(x, y) + iv(x, y)$ is a complex function, where $u(x, y)$ and $v(x, y)$ are real functions.

Definition 2.1.3: Single-valued Function

A **single-valued function** is a function that returns a unique value for each input.

A complex function $f(z)$ is single-valued if and only if $f(z_1) = f(z_2)$ implies $z_1 = z_2$. In other words, if $z_1 \neq z_2$, then $f(z_1) \neq f(z_2)$.

$$\forall z_1, z_2 \in \mathbb{C} \quad \text{s.t.} \quad z_1 \neq z_2 \quad \text{implies} \quad f(z_1) \neq f(z_2)$$

Definition 2.1.4: Multiple-valued Function

A **multiple-valued function** is a function that returns multiple values for each input.

A complex function $f(z)$ is multiple-valued if and only if $f(z_1) = f(z_2)$ for some $z_1 \neq z_2$.

$$\exists z_1, z_2 \in \mathbb{C} \quad \text{s.t.} \quad z_1 \neq z_2 \quad \text{and} \quad f(z_1) = f(z_2)$$

Definition 2.1.5: Derivative

The **derivative** of a complex function $f(z)$ is defined as

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

where Δz is a complex number.

If the limit exists, then $f(z)$ is said to be **differentiable** at z. If $f(z)$ is differentiable at every point in a region R, then $f(z)$ is said to be **analytic** in R.

Definition 2.1.6: Analytic Function

A complex function $f(z)$ is **analytic** in a region R if it is differentiable at every point in R.

If $f(z)$ is analytic in a region R, then $f(z)$ is also said to be **regular** or **holomorphic** in R.

2.2 Necessary Conditions for Analyticity

Let $f(z) = u(x, y) + iv(x, y)$ be an analytic function in a region R.

That means, $f(z)$ is differentiable at every point in R.

$$\implies f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \text{ exists at every point in } R.$$

Now, let $z = x + iy$ and $\Delta z = \Delta x + i\Delta y$.

$$z + \Delta z = (x + \Delta x) + i(y + \Delta y)$$

Then,

$$\begin{aligned}
f'(z) &= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(x + \Delta x + i(y + \Delta y)) - f(x + iy)}{\Delta x + i\Delta y} \\
&= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y) - u(x, y) - iv(x, y)}{\Delta x + i\Delta y} \\
&= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{u(x + \Delta x, y + \Delta y) - u(x, y)}{\Delta x + i\Delta y} + i \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{v(x + \Delta x, y + \Delta y) - v(x, y)}{\Delta x + i\Delta y}
\end{aligned}$$

Along the real axis, $\Delta y = 0$. Hence, the limit is

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x + \Delta x, y) - v(x, y)}{\Delta x}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \tag{2.2.1}$$

Along the imaginary axis, $\Delta x = 0$. Hence, the limit is

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y + \Delta y) - u(x, y)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x, y + \Delta y) - v(x, y)}{i\Delta y}$$

$$f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \tag{2.2.2}$$

2.3 Cauchy-Riemann Equations

Since $f'(z)$ exists, (2.2.4) and (2.2.5) must be equal.

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \tag{2.3.1}$$

Comparing real and imaginary parts,

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \quad \text{and} \quad \boxed{\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}} \tag{2.3.2}$$

These are called the **Cauchy-Riemann equations**.

2.4 Cauchy-Riemann Equations in Polar Form

Let

$$z = re^{i\theta}$$

$$f(z) = u(r, \theta) + iv(r, \theta)$$

Then

$$f(re^{i\theta}) = u(r, \theta) + iv(r, \theta) \tag{2.4.1}$$

Differentiating (2.4.1) with respect to r, we get

$$e^{i\theta} f'(re^{i\theta}) = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} \tag{2.4.2}$$

Differentiating (2.4.1) with respect to θ, we get

$$ire^{i\theta} f'(re^{i\theta}) = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} \tag{2.4.3}$$

Now, from (2.4.2) and (2.4.3),

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = \frac{1}{ir}\left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right)$$
$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = \frac{1}{ir}\frac{\partial u}{\partial \theta} + \frac{1}{r}\frac{\partial v}{\partial \theta}$$
$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = -\frac{i}{r}\frac{\partial u}{\partial \theta} + \frac{1}{r}\frac{\partial v}{\partial \theta}$$

Equating the real and imaginary parts, we get

$$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}} \quad \text{and} \quad \boxed{\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}} \tag{2.4.4}$$

These are the **Cauchy-Riemann equations in polar form**.

3 Harmonic Function

3.1 Laplace's Equation

Definition 3.1.1: Laplace's Equation

An equation of the form

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \text{or} \quad \nabla^2 \phi = 0 \tag{3.1.1}$$

is called **Laplace's equation** (in two dimentions).
Here, ∇^2 is the Laplacian operator.

3.2 Harmonic Function

Definition 3.2.1: Harmonic Function

A function $\phi(x, y)$ is called **harmonic** if it satisfies Laplace's equation

$$\nabla^2 \phi = 0 \tag{3.2.1}$$

where ∇^2 is the Laplacian operator.

Theorem 3.2.2: *If $f(z) = u + iv$ is an analytic function, then u and v are both harmonic functions.*

Proof:
Since $f(z)$ is analytic, it satisfies the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{3.2.2}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{3.2.3}$$

Differentiating (3.2.2) w.r.t. x and (3.2.3) w.r.t. y, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial v}{\partial x \partial y} \tag{3.2.4}$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial v}{\partial y \partial x} \tag{3.2.5}$$

Adding (3.2.4) and (3.2.5), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{3.2.6}$$

Similarly,

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \tag{3.2.7}$$

Hence, both u and v are harmonic functions. $\square$

Definition 3.2.3: Conjugate Harmonic Function

Any two functions ϕ and ψ such that $f(z) = \phi + i\psi$ is analytic, are called **Conjugate Harmonic Functions**.

3.3 Velocity Potential

Consider a two-dimensional flow of an incompressible fluid. The velocity of the fluid at a point (x, y) is given by the vector

$$\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} \tag{3.3.1}$$

The velocity potential $\phi(x, y)$ is defined as the scalar function such that

$$\mathbf{V} = \nabla\phi = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y}\right)\phi = \hat{\mathbf{i}}\frac{\partial\phi}{\partial x} + \hat{\mathbf{j}}\frac{\partial\phi}{\partial y} \tag{3.3.2}$$

Comparing (3.3.1) and (3.3.2), we get

$$v_x = \frac{\partial\phi}{\partial x} \quad \text{and} \quad v_y = \frac{\partial\phi}{\partial y} \tag{3.3.3}$$

The scalar function $\phi(x, y)$ gives the velocity components.
Since the fluid is incompressible,

$$\begin{aligned}
\nabla v &= 0 \\
\left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y}\right)\left(\hat{\mathbf{i}}v_x + \hat{\mathbf{j}}v_y\right) &= 0 \\
\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} &= 0
\end{aligned}$$

Putting the values of v_x and v_y from (3.3.3),

$$\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right) &= 0 \\
\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} &= 0
\end{aligned}$$

This is Laplace's equation. Hence, the velocity potential $\phi(x, y)$ is a harmonic function and is a real part of the analytic function

$$f(z) = \phi + i\psi$$

3.4 Method for Finding Conjugate Harmonic Function

A Method 1: Real or Imaginary Part of an Analytic Function is Given

If $f(z) = u + iv$ and u is known
We know that

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

Using C-R equations,

$$\begin{aligned}
dv &= -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \\
v &= -\int \frac{\partial u}{\partial y}dx + \int \frac{\partial u}{\partial x}dy + C
\end{aligned}$$

Since u is known, v can be found using the above method.

If v is known, then u can be found using the same method.

B Method 2: Milne's Method/ Milne Thomson Method

By this method, $f(z)$ is directly constructed without finding v.
Since

$$z = x + iy \quad \text{and} \quad \bar{z} = x - iy$$
$$x = \frac{z + \bar{z}}{2} \quad \text{and} \quad y = \frac{z - \bar{z}}{2i}$$

Thus,

$$\boxed{f(z) = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)} \tag{3.4.1}$$

Case 1: u is given

Let $f(z) = u + iv$ be an analytic function and u is given.
Then,

$$\frac{\partial u}{\partial x} = u_1(x, y) \quad \text{and} \quad \frac{\partial u}{\partial y} = u_2(x, y)$$

By Milne's method, we get

$$f'(z) = u_1(z, 0) - iu_2(z, 0) \tag{3.4.2}$$

Integrating (3.4.1) w.r.t. z, we get

$$f(z) = \int \big[u_1(z, 0) - iu_2(z, 0)\big]\, dz + C_1 \tag{3.4.3}$$

Case 2: v is given

If v is given, then

$$\frac{\partial v}{\partial y} = v_1(x, y) \quad \text{and} \quad \frac{\partial v}{\partial x} = v_2(x, y)$$

By Milne's method, we get

$$f'(z) = v_1(z, 0) + iv_2(z, 0) \tag{3.4.4}$$

Integrating (3.4.3) w.r.t. z, we get

$$f(z) = \int \big[v_1(z, 0) + iv_2(z, 0)\big]\, dz + C_2 \tag{3.4.5}$$