Math-183
Differential Equations

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Contents

1 Differential Equations and Their Solutions

1.1 Classification of Differential Equations

Definition 1.1.1: Differential Equation

Differential equation is an equation involving derivatives of one or more dependent variables with respect to one or more independent variables.

Definition 1.1.2: Ordinary Differential Equation

A differential equation involving ordinary derivatives of one or more dependent variables with respect to a single independent variable is called an ordinary differential equation.

Example 1.1.1: Ordinary Differential Equations:

$$\frac{dy}{dx} + xy\left(\frac{d}{dx}\right)^2 = 0 \tag{1.1.1}$$

$$\frac{d^4x}{dt^4} + 5\frac{d^2x}{dt^2} + 3x = \sin t \tag{1.1.2}$$

Definition 1.1.3: Partial Differential Equation

A differential equation involving partial derivatives of one or more dependent variables with respect to more than one independent variables is called an partial differential equation.

Example 1.1.2: Partial Differential Equations:

$$\frac{\partial v}{\partial s} + \frac{\partial v}{\partial t} = v \tag{1.1.3}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \tag{1.1.4}$$

Definition 1.1.4: Order and Degree of Differential Equations

Order of DE: The order of the highest ordered derivative involved in a differential equation is called the order of the differential equation.

Degree of DE: The power of the highest order derivative involved in a differential equation is called the degree of the differential equation.

Definition 1.1.5: Linearity of Differential Equations

If the dependent variable and its various derivatives occur to the first degree only, the DE is a linear DE. Otherwise it's a non-linear DE.

$$a_0(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_1(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + a_{n-1}(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_n(x)y = b(x)$$

Linear DE can also be classified as linear with *constant* and *variable* coefficients.

Example 1.1.3: Ordinary Differential Equations: Orders, Degree, Linearity

$$\begin{aligned}
\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} - 3\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 3\frac{\mathrm{d}y}{\mathrm{d}x} - 6y = \sin x & \quad \text{3rd ord 1st deg Lin} \\
\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + y = 0 & \quad \text{2nd ord 1st deg Non-Lin} \\
y = x\frac{\mathrm{d}y}{\mathrm{d}x} + \sqrt{1 + \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}} & \quad \text{2nd ord 1st deg Non-Lin} \\
\frac{\mathrm{d}^4 x}{\mathrm{d}t^4} + t^2\frac{\mathrm{d}^3 x}{\mathrm{d}t^3} + \frac{\mathrm{d}y}{\mathrm{d}x} = \sin t & \quad \text{4th ord 1st deg Lin} \\
\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 5\frac{\mathrm{d}y}{\mathrm{d}x} + 6y^2 = 0 & \quad \text{2nd ord 1st deg Non-Lin} \\
\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 5\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3 + 6y = 0 & \quad \text{2nd ord 1st deg Non-Lin} \\
\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 5y\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = 0 & \quad \text{2nd ord 1st deg Lin}
\end{aligned}$$

1.2 Solutions

A Nature of Solutions

An nth-order Differential Equation:

$$F\left[x, y, \frac{\mathrm{d}y}{\mathrm{d}x}, \cdots, \frac{\mathrm{d}^n y}{\mathrm{d}x^n}\right] = 0 \tag{1.2.1}$$

Definition 1.2.1: Explicit solution

f is an explicit solution of (1.2.1) if

$$\forall x \in I, F\left[x, f(x), f'(x), \cdots, f^{(n)}(x)\right] = 0$$

where I is a real interval.

Definition 1.2.2: Implicit solution

$g(x, y) = 0$ is an implicit solution if this relation defines at least one real function $f(x)$ on an interval I such that f is an explicit solution of (1.2.1)

Example 1.2.1: Explicit and Implicit Solutions

$$x^2 + y^2 - 25 = 0 \quad : \quad \text{Implicit solution}$$
$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$x + y\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \quad : \quad \text{Differential Equation}$$
$$y = \pm\sqrt{25 - x^2}\ ;\ -5 \leq x \leq 5 \quad : \quad \text{Explicit solution}$$

B Methods of Solution

The study of a Differential Equation consists of 3 phases:

1. Formulation of DE from the given physical situation.
2. Solutions of DE, evaluating the arbitrary constants from the given condition.
3. Physical interpretation of the solution.

Example 1.2.2: Show that the function $f(x) = e^x + 2x^2 + 6x + 7$ **is a solution to the DE** $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 4x^2$

$$f(x) = e^x + 2x^2 + 6x + 7$$
$$f'(x) = e^x + 4x + 6$$
$$f"(x) = e^x + 4$$

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y &= (e^x + 4) - 3(e^x + 4x + 6) + 2(e^x + 2x^2 + 6x + 7) \\ &= 0 \cdot e^x + 0 \cdot x + (4 - 18 + 14) + 4x^2 \\ &= 4x^2\end{aligned}$$

□

Example 1.2.3: Show that the function $f(x) = \dfrac{1}{1+x^2}$ **is a solution to the DE** $(1 + x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0$

$$f(x) = \frac{1}{1+x^2}$$
$$(1 + x^2)f(x) = 1$$
$$(1 + x^2)f'(x) + 2xf(x) = 0$$
$$(1 + x^2)f''(x) + 2xf'(x) + 2xf'(x) + 2f(x) = 0$$
$$(1 + x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0$$

□

Example 1.2.4: Show that the function $y = (2x^2 + 2e^{3x} + 3)e^{-2x}$ satisfies the DE

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 6e^x + 4xe^{-2x}$$

$$\begin{aligned} y &= (2x^2 + 2e^{3x} + 3)e^{-2x} \\ y_1 &= (4x + 6e^{3x})e^{-2x} - (2x^2 + 2e^{3x} + 3)2e^{-2x} \\ y_1 &= 4xe^{-2x} + 6e^x - 2y \\ \frac{\mathrm{d}y}{\mathrm{d}x} &+ 2y = 6e^x + 4e^{-2x} \end{aligned}$$

□

1.3 Initial-Value and Boundary-Value Problems, and Existence of Solutions

A Initial-value Problems and Boundary-value Problems

One of the most frequently encountered type of problems in Differential Equations involves both a DE and one or more supplementary conditions which the solution of the given DE must satisfy.

Definition 1.3.1: IVP and BVP

Consider the first-order DE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$$

where f is a continuous function of x and y in some domain D of the xy plane; and let (x_0, y_0) be a point of D. The **initial-value problem** associated with the DE is to find a solution ϕ of the DE, defined on some real interval containing x_0, and satisfying the initial condition

$$\phi(x_0) = y_0$$

In the customary abbreviated notation, this initial-value problem may be written

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$$

$$y(x_0) = y_0$$

If the conditions relate to two different x values (the extreme or boundary values), the proble is called a **Two-Point Boundary-Value Problem** or simply a **Boundary-Value Problem (BVP)**.

Example 1.3.1: Find the solution of the DE $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$ such that $\forall x \in I, f'(x) = 2x$ and $f(1) = 4$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$$

$$\int \frac{\mathrm{d}y}{\mathrm{d}x}\, dx = \int 2x\, dx$$

$$y = x^2 + c$$

Substituting $y = 4$ and $x = 1$,

$$4 = 1 + c \text{ or } c = 3$$

$$\therefore \text{ Solution: } y^2 = x + 3$$

□

Example 1.3.2: $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$**,** $y(3) = 4$

$$x + y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$\int x\, dx + \int y\frac{\mathrm{d}y}{\mathrm{d}x}\, dx = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} = c'$$

$$x^2 + y^2 = c$$

Substituting $x = 3$ and $y = 4$,

$$16^2 + 3^2 = c \text{ or } c = 25$$

$$\therefore \text{ Solution: } x^2 + y^2 - 25 = 0$$

B Existence of Solutions

Not all initial-value and boundary-value problems have solutions. For example,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y = 0$$

$$y(0) = 1 \quad , \quad y(\pi) = 5$$

has no solutions! Thus arises the question of *existence* of solutions. We can say, every initial-value problem that satisfies definition (1.3.1) has *at least one* solution. However, there arises another question. Can a problem have more than one solution?
Let's consider the initial-value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^{1/3} \quad ; \quad y(0) = 0$$

One may verify that the functions f_1 and f_2 defined, respectively, by

$$\forall x \in \mathbb{R}, \quad f_1(x) = 0$$

and

$$f_2(x) = (\frac{2}{3}x)^{3/2}, \quad x \geq 0; \quad f_2(x) = 0, \quad x \leq 0$$

are both solutions of this initial-value problem. In fact, this problem has infinitely many solutions. Hence, we can state that the initial-value problem need not have a *unique* solution. In order to ensure uniqeness, some additional requiremnt must certainly be imposed.

***Theorem 1.3.1* (Basic Existence and Uniqueness Theorem):**
Hypothesis: *Consider the differential equation*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) \tag{1.3.1}$$

where

- *The function f is a continuous function of x and y in some domain D of the xy plane, and*
- *The partial derivative $\frac{\partial f}{\partial y}$ is also a continuous function of x and y in D; and let (x_0, y_0) be a point in D.*

Conclusion: *There exists a unique solution ϕ of the differential equation (1.3.1), defined on some interval $|x - x_0| \leq h$, where h is sufficiently small, that satisfies the condition*

$$\phi(x_0) = y_0$$

Example 1.3.3: Show that

$$y = 4e^{2x} + 2e^{-3x}$$

is a solution of the initial-value problem

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} - 6y = 0$$

$$y(0) = 6$$

$$y'(0) = 2$$

Is $y = 2e^{2x} + 4e^{-3x}$ also a solution of this problem? Explain why or why not.

$$y = 4e^{2x} + 2e^{-3x}$$
$$y_1 = 8e^{2x} - 6e^{-3x}$$
$$y_2 = 16e^{2x} + 18e^{-3x}$$

$$\begin{aligned} y_2 + y_1 - 6y &= (16e^{2x} + 18e^{-3x}) + (8e^{2x} - 6e^{-3x}) - 6(4e^{2x} + 2e^{-3x}) \\ &= 0 \cdot e^{2x} + 0 \cdot e^{-3x} \\ &= 0 \end{aligned}$$

The solution also satisfies $y(0) = 6$ and $y'(0) = 2$

Now, for $y = 2e^{2x} + 4e^{-3x}$,

$$y_1 = 4e^{2x} - 12e^{-3x} \;\; ; \;\; y_2 = 8e^{2x} + 36e^{-3x}$$

$$\begin{aligned} y_2 + y_1 - 6y &= (8e^{2x} + 36e^{-3x}) + (4e^{2x} - 12e^{-3x}) - 6(2e^{2x} + 4e^{-3x}) \\ &= 0 \cdot e^{2x} + 0 \cdot e^{-3x} \\ &= 0 \end{aligned}$$

However, in this case,

$$y(0) = 6 \;\; ; \;\; y'(0) = -8$$

As we can see, this solution doesn't satisfy the initial-value problem. Hence $y = 2e^{2x} + 4e^{-3x}$ is not a solution of this problem.

Example 1.3.4: Given that every solution of

$$x^3 \frac{\mathrm{d}^3 y}{\mathrm{d}x^3} - 3x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 6x \frac{\mathrm{d}y}{\mathrm{d}x} - 6y = 0$$

may be written in the form $y = c_1 x + c_2 x^2 + c_3 x^3$ **for some choice of the arbitrary constants** c_1**,** c_2**, and** c_3**, solve the initial-value problem consisting of the above DE plus the three conditions**

$$y(2) = 0 \;\; , \;\; y'(2) = 2 \;\; , \;\; y''(2) = 6$$

$$y = c_1 x + c_2 x^2 + c_3 x^3$$

$$y(2) = 0 \text{ or, } 8c_3 + 4c_2 + 2c_1 = 0 \tag{1.3.2}$$

$$y' = c_1 + 2c_2 x + 3c_3 x^2$$

$$y'(2) = 2 \text{ or, } 12c_3 + 4c_2 + c_1 = 2 \tag{1.3.3}$$

$$y'' = 0 + 2c_2 + 6c_3 x$$

$$y''(2) = 6 \text{ or, } 12c_3 + 2c_2 + 0c_1 = 6 \tag{1.3.4}$$

Solving (1.3.1), (1.3.2), and (1.3.3) we get,

$$c_1 = 2 \;\; , \;\; c_2 = -3 \;\; , \;\; c_3 = 1$$

$$\therefore \text{ Solution: } y = 2x - 3x^2 + x^3$$

2 First Order Equations for Which Exact Solutions Are Obtainable

2.1 Exact Differential Equations and Integrating Factors

A Standard Forms of First-Order Differential Equations

The first-order differential equations may be expressed in either the **Derivative Form**

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) \tag{2.1.1}$$

or the **Differential Form**

$$M(x, y)\,dx + N(x, y)\,dy = 0 \tag{2.1.2}$$

Example 2.1.1: Standard Forms

The equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2 + y^2}{x - y}$$

is the form (2.1.1). It may be written as

$$(x^2 + y^2)\,dx + (y - x)\,dy = 0$$

which is of the form (2.1.2).
Again, the equation

$$(\sin x + y)\,dx + (x + 3y)\,dy = 0$$

is of the form (2.1.2), which can also be written as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\sin x + y}{x + 3y}$$

B Exact Differential Equations

Definition 2.1.1: Exact Differential

Let F be a function of two real variables such that F has continuous first partial derivatives in a domain D. The total differential dF of the function F is defined by the formula

$$dF(x, y) = \frac{\partial F(x, y)}{\partial x}\,dx + \frac{\partial F(x, y)}{\partial y}\,dy$$

for all $(x, y) \in D$.

Comparing $dF(x, y)$ with the form (2.1.2), we get

$$\frac{\partial F(x, y)}{\partial x} = M(x, y) \quad and \quad \frac{\partial F(x, y)}{\partial y} = N(x, y)$$

Example 2.1.2

Let F be a function

$$F(x, y) = xy^2 + 2x^3y$$

for all real (x, y). Then

$$\frac{\partial F(x,y)}{\partial x} = y^2 + 6x^2 y, \quad \frac{\partial F(x,y)}{\partial y} = 2xy + 2x^3$$

and the total differential dF is defined by

$$dF(x,y) = (y^2 + 6x^2 y)\, dx + (2xy + 2x^3)\, dy$$

for all real (x, y)

Definition 2.1.2: Exact Differential Equation

The expression

$$M(x,y)\, dx + N(x,y)\, dy \tag{2.1.3}$$

is called an exact differential in a domain D if there exists a function F of two real variables such that this expression equals the total differential $dF(x, y)$ for all $(x, y) \in D$. That is, expression (2.1.3) is an exact differential in D if there exists a function F such that

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) \quad \textit{and} \quad \frac{\partial F(x,y)}{\partial y} = N(x,y)$$

for all $(x, y) \in D$.
If $M(x, y)\, dx + N(x, y)\, dy$ is an exact differential, then the differential equation

$$M(x,y)\, dx + N(x,y)\, dy = 0$$

is called an **Exact Differential Equation**.

Theorem 2.1.1* (Exact Differential Equation)*:

1. *If the DE $M(x, y)\, dx + N(x, y)\, dy = 0$ is exact in D, then*

$$\forall (x,y) \in D, \quad \frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

2. *Conversely, if*

$$\forall (x,y) \in D, \quad \frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

then the DE is exact in D.

Proof (1):

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) \quad , \quad \frac{\partial F(x,y)}{\partial y} = N(x,y)$$

$$\frac{\partial^2 F(x,y)}{\partial x \partial y} = \frac{\partial M(x,y)}{\partial y} \quad , \quad \frac{\partial^2 F(x,y)}{\partial x \partial y} = \frac{\partial N(x,y)}{\partial x}$$

$$\because \frac{\partial^2 F(x,y)}{\partial y \partial x} = \frac{\partial^2 F(x,y)}{\partial x \partial y}$$

$$\therefore \frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x} \quad \square$$

C The Solution of Exact Differential Equations

***Theorem 2.1.2* (Solution of Exact DE):**
If $M(x,y)\,dx + N(x,y)\,dy = 0$ is exact in domain D, then

$$\forall (x,y) \in D, \exists F(x,y) : \frac{\partial F(x,y)}{\partial x} = M(x,y) \quad \textit{and} \quad \frac{\partial F(x,y)}{\partial y} = N(x,y)$$

Then the equation may be written

$$\frac{\partial F(x,y)}{\partial x}\,dx + \frac{\partial F(x,y)}{\partial y}\,dy = 0$$

or simply,

$$dF(x,y) = 0$$

Here, $F(x,y) = c$ is a one-parameter family of solutions of this DE, where c is an arbitrary constant.

Example 2.1.3: Solve the equation

$$(3x^2 + 4xy)\,dx + (2x^2 + 2y)\,dy = 0$$

Standard Method:

$$\begin{aligned}\frac{\partial F(x,y)}{\partial x} &= M(x,y) = 3x^2 + 4xy \\ F(x,y) &= \int (3x^2 + 4xy)\,\partial x + \phi(y) \\ &= x^3 + 2x^2y + \phi(y)\end{aligned}$$

Again,

$$\begin{gathered}\frac{\partial F(x,y)}{\partial y} = 2x^2 + \frac{\partial \phi(y)}{\partial y} = 2x^2 + 2y \\ \frac{\mathrm{d}\phi(y)}{\mathrm{d}y} = 2y \\ \int \frac{\mathrm{d}\phi(y)}{\mathrm{d}y}\,dy = \int 2y\,dy \\ \phi(y) = y^2 + c_0\end{gathered}$$

Thus, we get

$$F(x,y) = x^3 + 2x^2y + y^2 + c_0$$

Hence, a one-parameter family of the solution is $F(x,y) = c_1$ or,

$$x^3 + 2x^2y + y^2 + c_0 = c_1$$

$$\boxed{x^3 + 2x^2y + y^2 = c}$$

Method of Grouping:

$$\begin{aligned}(3x^2+4xy)\,dx+(2x^2+2y)\,dy&=0\\ 3x^2\,dx+(4xy\,dx+2x^2\,dy)+2y\,dy&=0\\ d(x^3)+d(2x^2y)+d(y^2)&=d(c)\\ \boxed{x^3+2x^2y+y^2=c}\end{aligned}$$

Example 2.1.4: Solve the initial-value problem

$$(2x\cos y+3x^2y)\,dx+(x^3-x^2\sin y-y)\,dy=0\ ;\quad y(0)=2$$

$$\begin{aligned}(2x\cos y\,dx-x^2\sin y\,dy)+(3x^2y\,dx+x^3\,dy)-y\,dy&=0\\ d(x^2\cos y)+d(x^3y)+d(\frac{y^2}{2})&=d(c_1)\\ 2x^2\cos y+x^3y+y^2&=c\end{aligned}$$

Substituting $x=0$ and $y=2$,

$$2^2=c$$

Hence, the solution is:

$$2x^2\cos y+x^3y+y^2=4$$

D Integrating Factors

Definition 2.1.3: Integrating Factor (IF)

If the DE

$$M(x,y)\,dx+N(x,y)\,dy=0 \tag{2.1.4}$$

is not exact in a domain D but the DE

$$\mu(x,y)M(x,y)\,dx+\mu(x,y)N(x,y)\,dy=0 \tag{2.1.5}$$

is exact in D, then $\mu(x,y)$ is called an **Integrating Factor** of the DE.

Example 2.1.5: Integrating factor

Consider the DE

$$(3y+4xy^2)\,dx+(2x+3x^2y)\,dy=0 \tag{2.1.6}$$

This equation is of the form (2.1.4), where

$$M(x,y)=3y+4xy^2,\qquad N(x,y)=2x+3x^2y$$

$$\frac{\partial M(x,y)}{\partial y}=3+8xy,\qquad \frac{\partial N(x,y)}{\partial x}=2+6xy$$

Since

$$\frac{\partial M(x,y)}{\partial y}\neq\frac{\partial N(x,y)}{\partial x}$$

except for (x,y) such that $2xy+1=0$, Equation (2.1.4) is not exact in any rectangular domain D.

Let $\mu(x,y) = x^2 y$. Then the corresponding DE of the form (2.1.5) is

$$(3x^2y^2 + 4x^3y^3)\,dx + (2x^3y + 3x^4y^2)\,dy = 0$$

This equation is exact in every rectangular domain D, since

$$\frac{\partial[\mu(x,y)M(x,y)]}{\partial y} = 6x^2y + 12x^3y^2 = \frac{\partial[\mu(x,y)N(x,y)]}{\partial x}$$

For all real (x,y). Hence, $\mu(x,y) = x^2y$ is an integrating factor of Equation (2.1.6).

Example 2.1.6: Determine whether or not the following equation is exact

$$\left(\frac{x}{y^2} + x\right)\,dx + \left(\frac{x^2}{y^3} + y\right)\,dy = 0$$

$$\frac{\partial M(x,y)}{\partial y} = -\frac{x}{2y^3}$$
$$\frac{\partial N(x,y)}{\partial x} = \frac{2x}{y^3}$$

Here, $\frac{\partial M(x,y)}{\partial y} \neq \frac{\partial N(x,y)}{\partial x}$. Hence, the equation is not exact.

Example 2.1.7: Determine the constant A in the following equations such that the equation is exact

1. $(Ax^2y + 2y^2)\,dx + x^3 + 4xy\,dy = 0$
2. $\left(\frac{Ay}{x^3} + \frac{y}{x^2}\right)\,dx + \left(\frac{1}{x^2} - \frac{1}{x}\right)\,dy = 0$

1.

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial(Ax^2y + 2y^2)}{\partial y} = Ax^2 + 4y$$
$$\frac{\partial N(x,y)}{\partial x} = \frac{\partial(x^3 + 4xy)}{\partial x} = 3x^2 + 4y$$

Equating the coefficients of x^2, we get

$$\boxed{A = 3}$$

2.

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial\left(\frac{Ay}{x^3} + \frac{y}{x^2}\right)}{\partial y} = \frac{A}{x^3} + \frac{1}{x^2}$$
$$\frac{\partial N(x,y)}{\partial x} = \frac{\partial\left(\frac{1}{x^2} - \frac{1}{x}\right)}{\partial x} = -\frac{1}{2x^3} + \frac{1}{x^2}$$

Equating the coefficients of $\frac{1}{x^3}$, we get

$$\boxed{A = -\frac{1}{2}}$$

2.2 Separable Equations and Equations Reducible to this Form

A Separable Equations

Definition 2.2.1: Separable Equations

An equations of the form

$$F(x)G(y)\,dx + f(x)g(y)\,dy = 0 \tag{2.2.1}$$

is called an equation with separable variables or simply a separable equation.

Theorem 2.2.1* (Solution of Separable Differential Equations)*:
In general, the separable equations are not exact, but they possess an obvious integrating factor $\frac{1}{f(x)G(y)}$
Thus the equation (2.2.1) becomes

$$\frac{F(x)}{f(x)}\,dx + \frac{g(y)}{G(y)}\,dy = 0 \tag{2.2.2}$$

which is exact, because

$$\frac{\partial}{\partial y}\left(\frac{F(x)}{f(x)}\right) = 0 = \frac{\partial}{\partial x}\left(\frac{g(y)}{G(y)}\right)$$

We can write the equation (2.2.2) as

$$M(x)\,dx + N(y)\,dy = 0$$

where $M(x) = \frac{F(x)}{f(x)}$ *and* $N(y) = \frac{g(y)}{G(y)}$
A one-parameter family solution to the DE is

$$\int M(x)\,dx + \int N(y)\,dy = c \tag{2.2.3}$$

Example 2.2.1: Solve the equation

$$(x-4)y^4\,dx - x^3(y^2-3)\,dy = 0$$

The equation is separable; dividing by x^3y^4 we obtain

$$\frac{x-4}{x^3}\,dx - \frac{y^2-3}{y^4}\,dy = 0$$

$$\int (x^{-2} - 4x^{-3})\,dx - \int (y^{-2} - 3y^{-4})\,dy = 0$$

$$\boxed{-\frac{1}{x} + \frac{2}{x^2} + \frac{1}{y} - \frac{1}{y^3} = c}$$

The DE in derivative form:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x-4)y^4}{x^3(y^2-3)}$$

Here, $y = 0$ is a solution which was lost in the separation process.

Example 2.2.2: Solve the initial-value problem that consists of the DE

$$x \sin y \, dx + (x^2+1)\cos y \, dy = 0$$

and the initial condition $y(1) = \frac{\pi}{2}$

$$\frac{x}{x^2+1}\, dx + \frac{\cos y}{\sin y}\, dy = 0$$

$$\frac{1}{2}\int \frac{d(x^2+1)}{x^2+1} + \int \cot y \, dy = 0$$

$$\frac{1}{2}\ln|x^2+1| + \ln|\sin y| = \ln|c_1|$$

$$\ln|(x^2+1)\sin^2 y| = \ln|c|$$

$$\therefore (x^2+1)\sin^2 y = c$$

Applying the initial condition, we get

$$2\sin^2\frac{\pi}{2} = c \text{ or, } c = 2$$

Thus, the solution is

$$\boxed{(x^2+1)\sin^2 y = 2}$$

B Homogeneous Equations

Definition 2.2.2: Homogeneous Equations

The first-orger differential equation

$$M(x,y)\, dx + N(x,y)\, dy = 0$$

is said to be homogeneous if, when written in the derivative form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x,y)$$

there exists a function g such that $f(x,y)$ can be expressed in the form $g(y/x)$

Example 2.2.3

The DE

$$x^2 - 3y^2\, dx + 2xy\, dy = 0$$

is homogeneous. To see this, we first write the derivative form of the equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3y^2 - x^2}{2xy} = \frac{3y}{2x} - \frac{x}{2y}$$

We see that the DE can be written as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{2}\left(\frac{y}{x}\right) - \frac{1}{2}\left(\frac{1}{y/x}\right)$$

in which the right side of the equation is of the form $g(y/x)$ for a certain function g.

Example 2.2.4: The equation

$$(y + \sqrt{x^2 + y^2})\, dx - x\, dy = 0$$

is homogeneous.

Derivative form:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{y + \sqrt{x^2 + y^2}}{x} \\ &= \frac{y}{x} + \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2}} \\ &= \frac{y}{x} + \sqrt{1 + \left(\frac{y}{x}\right)^2} = g\left(\frac{y}{x}\right)\end{aligned}$$

Definition 2.2.3: Homogeneous Equation of degree n

A function F is called homogeneous of degree n if

$$F(tx, ty) = t^n F(x, y)$$

This means that if the tx and ty are substituted for x and y respectively in $F(x, y)$, and if t^n is then factored out, the other factor that remains is the original expression $F(x, y)$ itself.

For example, the function given by $F(x, y) = x^2 + y^2$ is homogeneous of degree 2, since

$$F(tx, ty) = (tx)^2 + (ty)^2 = t^2(x^2 + y^2) = t^2 F(x, y)$$

Now, suppose both $M(x, y)$ and $N(x, y)$ in the DE

$$M(x, y)\, dx + N(x, y)\, dy = 0$$

are homogeneous of the same degree n. Since $M(tx, ty) = t^n M(x, y)$, for $t = \frac{1}{x}$, we have

$$M\left(1, \frac{y}{x}\right) = \left(\frac{1}{x}\right)^n M(x, y)$$

$$M(x, y) = \left(\frac{1}{x}\right)^{-n} M\left(1, \frac{y}{x}\right)$$

Similarly,

$$N(x, y) = \left(\frac{1}{x}\right)^{-n} N\left(1, \frac{y}{x}\right)$$

Now, writing the DE in derivative form, we get

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{M(x, y)}{N(x, y)} \\ &= -\frac{\left(\frac{1}{x}\right)^{-n} M\left(1, \frac{y}{x}\right)}{\left(\frac{1}{x}\right)^{-n} N\left(1, \frac{y}{x}\right)} \\ &= -\frac{M\left(1, \frac{y}{x}\right)}{N\left(1, \frac{y}{x}\right)} \\ &= g\left(\frac{y}{x}\right)\end{aligned}$$

Note:-

If $M(x, y)$ and $N(x, y)$ in

$$M(x, y)\, dx + N(x, y)\, dy = 0$$

are both homogeneous functions of the same degree n, then the differential equation is a homogeneous differential equation.

Theorem 2.2.2:
If

$$M(x, y)\, dx + N(x, y)\, dy = 0 \tag{2.2.4}$$

is a homogeneous equation, then the change of variables $y = vx$ transforms the equation into a separable equation in the variables v and x.

Proof:
Since $M(x, y)\, dx + N(x, y)\, dy = 0$ is homogeneous, it may be written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g\left(\frac{y}{x}\right)$$

Let $y = vx$. Then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$$

and the initial equation becomes

$$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = g(v)$$

or,

$$[v - g(v)]\, dx + x\, dv = 0$$

This equation is separable. Separating the variables we obtain

$$\frac{dv}{v - g(v)} + \frac{dx}{x} = 0 \tag{2.2.5}$$

Theorem 2.2.3* (Solution of a Homogeneous Differential Equation)*: To solve a DE of the form (2.2.4), we let $y = vx$ and transform the homogeneous equation into a separable equation of the form (2.2.5). From this, we have

$$\int \frac{dv}{v - g(v)} + \int \frac{dx}{x} = c$$

where c is an arbitrary constant. Letting $F(v)$ denote

$$\int \frac{dv}{v - g(v)}$$

and returning to the original dependent variable y, the solution takes the form

$$F\left(\frac{y}{x}\right) + \ln|x| = c$$

Example 2.2.5: Solve the equation

$$(x^2 - 3y^2)\,dx + 2xy\,dy = 0$$

Derivative form:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{2y} + \frac{3y}{2x}$$

Letting $y = vx$, we get

$$\begin{aligned} v + x\frac{\mathrm{d}v}{\mathrm{d}x} &= -\frac{1}{2v} + \frac{3v}{2} \\ x\frac{\mathrm{d}v}{\mathrm{d}x} &= \frac{v^2-1}{2v} \\ \frac{2v\,dv}{v^2-1} &= \frac{dx}{x} \end{aligned}$$

Integrating, we find

$$\begin{aligned} \ln|v^2-1| &= \ln|x| + \ln|c| \\ |v^2-1| &= |cx| \\ |\frac{y^2}{x^2} - 1| &= |cx| \\ |y^2 - x^2| &= x^2|cx| \end{aligned}$$

For $y \geq x \geq 0$, it can be written as

$$\boxed{y^2 - x^2 = cx^3}$$

Example 2.2.6: Solve the initial-value problem

$$(y + \sqrt{x^2+y^2})\,dx - x\,dy = 0, y(1) = 0$$

Derivative form:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} + \sqrt{1 + \left(\frac{y}{x}\right)^2}$$

Letting $y = vx$, we get

$$\begin{aligned} v + x\frac{\mathrm{d}v}{\mathrm{d}x} &= v + \sqrt{1+v^2} \\ \frac{dv}{\sqrt{1+v^2}} &= \frac{dx}{x} \\ \ln|v + \sqrt{1+v^2}| &= ln|x| + ln|c| \\ v + \sqrt{v^2+1} &= cx \\ \frac{y}{x} + \frac{1}{x}\sqrt{y^2+x^2} &= cx \\ y + \sqrt{x^2+y^2} &= cx^2 \end{aligned}$$

Applying the initial condition, we get

$$0 + \sqrt{1} = c \cdot 1 \text{ or, } c = 1$$

Hence, the solution:

$$\boxed{y + \sqrt{x^2 + y^2} = x^2} \text{ or, } \boxed{y = \frac{1}{2}(x^2 - 1)}$$

Exercise 2.1: Solve the following differential equations

1. $(xy + 2x + y + 2)\,dx + (x^2 + 2x)\,dy = 0$
2. $(2x\cos y + 3x^2y)\,dx + (x^3 - x^2 - y)\,dy = 0, y(0) = 2$
3. $(e^v + 1)\cos u\,du + e^v(\sin u + 1)\,dv = 0$
4. $(x + 4)(y^2 + 1)\,dx + y(x^2 + 3x + 2)\,dy = 0$
5. $(2xy + 3y^2)\,dx - (2xy + x^2)\,dy = 0$
6. $(x + y)\,dx - x\,dy = 0$
7. $v^3\,du + (u^3 - uv^2)\,dv = 0$
8. $\left(x\tan\frac{y}{x} + y\right)\,dx - x\,dy = 0$
9. $(2s^2 + 2st + t^2)\,ds + (s^2 + 2st - t^2)\,dt = 0$
10. $(x^3 + y^2\sqrt{x^2 + y^2})\,dx - xy\sqrt{x^2 + y^2}\,dy = 0$
11. $(\sqrt{x + y} + \sqrt{x - y})\,dx + (\sqrt{x - y} - \sqrt{x + y})\,dy = 0$
12. $(3x + 8)(y^2 + 4)\,dx - 4y(x^2 + 5x + 6)\,dy = 0, y(1) = 2$
13. $(2x^2 + 2xy + y^2)\,dx + (x^2 + 2xy)\,dy = 0$

1.

$$\begin{aligned}
(xy + 2x + y + 2)\,dx + (x^2 + 2x)\,dy &= 0\\
(x + 1)(y + 2)\,dx + x(x + 2)\,dy &= 0\\
\int \frac{x + 1}{x(x + 2)}\,dx + \int \frac{dy}{y + 2} &= 0\\
\frac{1}{2}\ln|x^2 + 2x| + \ln|y + 2| &= \ln|c_1|\\
\ln|(x^2 + 2x)(y + 2)^2| &= \ln|c|\\
\boxed{(x^2 + 2x)(y + 2)^2 = c}
\end{aligned}$$

2.

$$(2x\cos y + 3x^2y)\,dx + (x^3 - x^2\sin^2 y - y)\,dy = 0$$

$$\begin{aligned}
F(x, y) &= \int (2x\cos y + 3x^2y)\,\partial x + \phi(y)\\
&= x^2\cos y + x^3y + \phi(y)
\end{aligned}$$

Now,

$$\frac{\partial F(x,y)}{\partial y} = x^3 - x^2 \sin y - y = x^3 - x^2 \sin y + \frac{\mathrm{d}}{\mathrm{d}x}\phi(y)$$

$$\therefore \phi(y) = -\int y\,dy = -\frac{y^2}{2} + c_0$$

$$\begin{aligned} F(x,y) = x^2 \cos y + x^3 y - \frac{y^2}{2} + c_0 &= c_1 \\ 2x^2 \cos y + 2x^3 y - y^2 &= c \end{aligned}$$

Applying initial value,

$$c = -4$$

$$\boxed{2x^2 \cos y + 2x^3 y - y^2 + 4 = 0}$$

3.

$$\begin{aligned} (e^v + 1)\cos u\,du + e^v(\sin u + 1)\,dv &= 0 \\ (e^v \cos u\,du + e^v \sin u\,dv) + \cos u\,du + e^v\,dv &= 0 \\ d(e^v \sin u) + d(\sin u) + de^v &= d(c) \end{aligned}$$

$$\boxed{\sin u + e^v(\sin u + 1) = c}$$

4.

$$(x+4)(y^2+1)\,dx + y(x^2+3x+2)\,dy = 0$$

$$\int \frac{x+4}{x^2+3x+2}\,dx + \int \frac{y}{y^2+1}\,dy = 0$$

$$\int \frac{x+4}{(x+2)(x+1)}\,dx + \frac{1}{2}\int \frac{2y}{y^2+1}\,dy = 0$$

$$\int \frac{3}{x+1}\,dx - \int \frac{2}{x+2}\,dx + \frac{1}{2}\ln|y^2+1|$$

$$3\ln|x+1| - 2\ln|x+2| + \frac{1}{2}\ln|y^2+1| = \ln|c_1|$$

$$\ln\left|\frac{(x+1)^6}{(x+2)^4}\cdot(y^2+1)\right| = \ln|c|$$

$$\boxed{(x+6)^6(y^2+1) = c(x+2)^4}$$

5.

$$(2xy+3y^2)\,dx-(2xy+x^2)\,dy=0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2xy+3y^2}{2xy+x^2}=\frac{2\left(\frac{y}{x}\right)+3\left(\frac{y}{x}\right)^2}{2\left(\frac{y}{x}\right)+1}$$

Letting $y=vx$, we get

$$v+x\frac{\mathrm{d}v}{\mathrm{d}x}=\frac{2+3v^2}{2v+1}$$

$$x\frac{\mathrm{d}v}{\mathrm{d}x}=\frac{v^2+v}{2v+1}$$

$$\int\frac{2v+1}{v^2+v}\,dv=\int\frac{dx}{x}$$

$$\int\frac{d(v^2+v)}{v^2+v}=\int\frac{dx}{x}$$

$$\ln|v^2+v|=\ln|cx|$$

$$\frac{y^2}{x^2}+\frac{y}{x}=cx$$

$$\boxed{y^2+xy=cx^3}$$

6.

$$(x+y)\,dx-x\,dy=0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{x+y}{x}=1+\frac{y}{x}$$

Letting $y=vx$, we get

$$v+x\frac{\mathrm{d}v}{\mathrm{d}x}=1+v$$

$$\int dv=\int\frac{dx}{x}$$

$$\frac{y}{x}=\ln|cx|$$

$$\boxed{cx=e^{y/x}}$$

7.

$$v^3\,du+(u^3-uv^2)\,dv=0$$

$$\frac{\mathrm{d}u}{\mathrm{d}v}=\frac{uv^2-u^3}{v^3}=\frac{u}{v}-\left(\frac{u}{v}\right)^3$$

Letting $u=wv$, we get

$$w+v\frac{\mathrm{d}w}{\mathrm{d}v}=w-w^3$$

$$-\int\frac{dw}{w^3}=\int\frac{dv}{v}$$

$$\frac{1}{2w^2}=\ln|v|+c_1$$

$$\boxed{v^2=u^2(\ln v^2+c)}$$

8.

$$\left(x\tan\frac{y}{x}+y\right)\,dx-x\,dy=0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\tan\frac{y}{x}+\frac{y}{x}$$

Letting $y=vx$, we get

$$v+x\frac{\mathrm{d}v}{\mathrm{d}x}=\tan v+v$$

$$\int\frac{dv}{\tan v}=\int\frac{dx}{x}$$

$$\ln|\sin v|=\ln|cx|$$

$$\boxed{\sin\frac{y}{x}=cx}$$

9.

$$(2s^2+2st+t^2)\,ds+(s^2+2st-t^2)\,dt=0$$

$$\frac{\mathrm{d}s}{\mathrm{d}t}=\frac{t^2-2st-s^2}{2s^2+2st+t^2}$$

$$=\frac{1-2\left(\frac{s}{t}\right)-\left(\frac{s}{t}\right)^2}{2\left(\frac{s}{t}\right)^2+2\left(\frac{s}{t}\right)+1}$$

Letting $s = vt$, we get

$$
\begin{aligned}
v + t\frac{\mathrm{d}v}{\mathrm{d}t} &= \frac{1 - 2v - v^2}{2v^2 + 2v + 1} \\
t\frac{\mathrm{d}v}{\mathrm{d}t} &= -\frac{2v^3 + 3v^2 + 3v - 1}{2v^2 + 2v + 1} \\
-\int \frac{2v^2 + 2v + 1}{2v^3 + 3v^2 + 3v - 1}\, dv &= \int \frac{dt}{t} \\
-\frac{1}{3}\int \frac{d(2v^3 + 3v^2 + 3v - 1)}{2v^3 + 3v^2 + 3v - 1} &= \ln|t| + \ln|c_1|
\end{aligned}
$$

$$\boxed{2v^3 + 3v^2 + 3v - 1 = \frac{c}{t^3}}$$

10.

$$(x^3 + y^2\sqrt{x^2 + y^2})\, dx - xy\sqrt{x^2 + y^2}\, dy = 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^3 + y^2\sqrt{x^2 + y^2}}{xy\sqrt{x^2 + y^2}} = \frac{1 + \left(\frac{y}{x}\right)^2\sqrt{1 + \left(\frac{y}{x}\right)^2}}{\frac{y}{x}\sqrt{1 + \left(\frac{y}{x}\right)^2}}$$

Letting $y = vx$,

$$
\begin{aligned}
v + x\frac{\mathrm{d}v}{\mathrm{d}x} &= \frac{1 + v^2\sqrt{1 + v^2}}{v\sqrt{1 + v^2}} \\
x\frac{\mathrm{d}v}{\mathrm{d}x} &= \frac{1}{v\sqrt{1 + v^2}} \\
\int v\sqrt{1 + v^2}\, dv &= \int \frac{dx}{x} \\
\frac{1}{2}\int \sqrt{1 + v^2}\, d(1 + v^2) &= \ln|c_1 x| \\
(1 + v^2)^{3/2} &= 3\ln|c_1 x| \\
\left(1 + \frac{y^2}{x^2}\right)\sqrt{1 + \frac{y^2}{x^2}} &= \ln|cx^3|
\end{aligned}
$$

$$\boxed{(x^2 + y^2)\sqrt{x^2 + y^2} = x^3\ln|cx^3|}$$