# Fourier Series

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## Contents

# 1 Introduction

## 1.1 Periodic Functions

**Definition 1.1.1: Periodic Functions**

A function $f(x)$ is said to be have a *period* $P$ or to be *periodic* with period $P$ if for all $x$, $f(x+P) = f(x)$ where $P$ is a positive constant. The least value of $P > 0$ is called the *least period* or simply the *period* of $f(x)$.

**Example 1.1: Some examples of periodic functions**

1. $\sin x$ has periods $2\pi, 4\pi, 6\pi, \cdots$ and $-\pi, -3\pi, -5\pi, \cdots$ and hence the least period is $2\pi$.

2. $\cos x$ has the least period $2\pi$.

3. $\tan x$ has the least period $\pi$.

Some other examples:

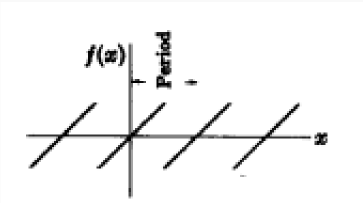


Figure 1.1.1

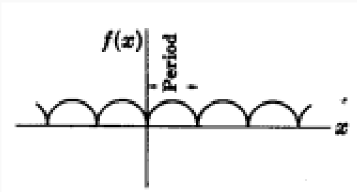


Figure 1.1.2

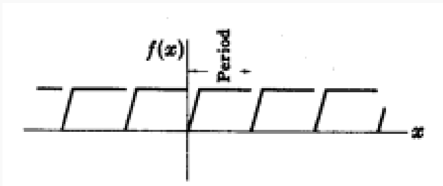


Figure 1.1.3

## 1.2 Piecewise Continuous Functions

**Definition 1.2.1: Piecewise Continuous Functions**

A function $f(x)$ is said to be *piecewise continuous* in the interval $[a,b]$ if $f(x)$ is continuous in the interval $(a,b)$ and has a finite number of finite discontinuities in the interval $[a,b]$.

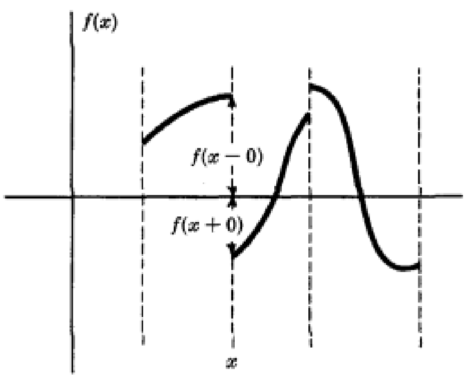


Figure 1.2.1

The right-hand limit of $f(x)$ is often denoted by $\lim_{\epsilon\to 0} f(x+\epsilon) = f(x+0)$, where $\epsilon > 0$.

Similarly, the left-hand limit of $f(x)$ is denoted by $\lim_{\epsilon\to 0} f(x-\epsilon) = f(x-0)$, where $\epsilon > 0$. The values of $f(x+0)$ and $f(x-0)$ at the point $x$ in (1.2.1) are as indicated.

# 2 Fourier Expansion

## 2.1 Definition and Derivation

### 2.1.1 Definition

**Definition 2.1.1: Fourier Expansion**

Let $f(x)$ be defined in the interval $(-L, L)$ and determined outside of this interval by $f(x+2L) = f(x)$, i.e. assume that $f(x)$ has the period $2L$. The *Fourier series* or *Fourier expansion* corresponding to $f(x)$ is defined to be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right] \tag{1}$$

where the *Fourier coefficients* $a_n$ and $b_n$ are given by

$$\begin{cases} a_0 = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)dx \\ a_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \qquad n = 0, 1, 2, \cdots \\ b_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \end{cases} \tag{2}$$

### 2.1.2 Some pre-derivations

$$\begin{aligned} I &= \int_{-L}^{L} \sin^2\frac{n\pi x}{L}\,dx \\ &= \sin\frac{n\pi x}{L}\cdot\frac{L}{n\pi}(\cos n\pi - \cos n\pi) + \frac{n\pi}{L}\cdot\frac{L}{n\pi}\int_{-L}^{L}\cos\frac{n\pi x}{L}\cos\frac{n\pi x}{L}\,dx \\ &= \int_{-L}^{L}\cos^2\frac{n\pi x}{L}\,dx \\ &= \frac{1}{2}\int_{-L}^{L}\left(\cos\frac{2n\pi x}{L} + 1\right)dx \\ &= \frac{1}{2}\int_{-L}^{L}\cos\frac{2n\pi x}{L}\,dx + \frac{1}{2}\int_{-L}^{L}dx \\ &= 0 + \frac{1}{2}\cdot 2L \\ &= L \end{aligned}$$

$$\begin{aligned}
I_1 &= \int_{-L}^{L} \cos\frac{m\pi x}{L} \cos\frac{n\pi x}{L}\, dx \qquad [m \neq 0] \\
&= \cos\frac{m\pi x}{L} \int_{-L}^{L} \cos\frac{n\pi x}{L}\, dx + \frac{m}{n} \int_{-L}^{L} \sin\frac{m\pi x}{L} \sin\frac{n\pi x}{L}\, dx \\
&= \frac{L}{n\pi} \cos\frac{m\pi x}{L} (\sin n\pi + \sin n\pi) + \frac{m}{n} I_2 \\
&= 0 + \frac{m}{n} \left[ \int_{-L}^{L} \sin\frac{m\pi x}{L} \sin\frac{n\pi x}{L}\, dx \right] \\
&= \frac{m}{n} \left[ \sin\frac{m\pi x}{L} \int_{-L}^{L} \sin\frac{n\pi x}{L}\, dx + \frac{m}{n} \int_{-L}^{L} \cos\frac{m\pi x}{L} \cos\frac{n\pi x}{L}\, dx \right] \\
&= \frac{m}{n} \left[ \frac{L}{n\pi} \sin\frac{m\pi x}{L} (-\cos n\pi + \cos n\pi) + \frac{m}{n} \right] \\
&= 0 + \frac{m^2}{n^2} I_1 \\
I_1 &= 0 = I_2
\end{aligned}$$

To summarize, we have

$$\int_{-L}^{L} \sin^2\frac{n\pi x}{L}\, dx = \int_{-L}^{L} \cos^2\frac{n\pi x}{L}\, dx = L \tag{3}$$

$$\int_{-L}^{L} \cos mx\, dx = \int_{-L}^{L} \sin mx\, dx = 0 \tag{4}$$

$$\int_{-L}^{L} \sin\frac{n\pi x}{L} \cos\frac{n\pi x}{L}\, dx = 0 \tag{5}$$

$$\int_{-L}^{L} \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L}\, dx = \int_{-L}^{L} \cos\frac{n\pi x}{L} \cos\frac{m\pi x}{L}\, dx = 0 \qquad [m \neq n] \tag{6}$$

### 2.1.3 Derivation of $a_0$

Taking integral on both sides of (1) from $-L$ to $L$, we get

$$\begin{aligned}
\int_{-L}^{L} f(x)\, dx &= \frac{a_0}{2} \int_{-L}^{L} dx + \sum_{n=1}^{\infty} \int_{-L}^{L} \left[ a_n \cos\frac{m\pi x}{L} + b_n \sin\frac{n\pi x}{L} \right] dx \\
&= \frac{a_0}{2} \cdot 2L \qquad \text{[All the other terms are 0 according to equation (4)]}
\end{aligned}$$

$$\boxed{a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\, dx}$$

### 2.1.4 Derivation of $a_n$

Multiplying both sides of (1) by $\cos\frac{m\pi x}{L}$ and integrating from $-L$ to $L$, we get

$$
\begin{aligned}
\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx &= \frac{a_0}{2}\int_{-L}^{L}\cos\frac{m\pi x}{L}\,dx \\
&\quad + \sum_{n=1}^{\infty}\int_{-L}^{L}\left[a_n\cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\right]dx \\
&= a_n\int_{-L}^{L}\cos^2\frac{m\pi x}{L}\,dx \\
&= a_n\cdot L \qquad \text{[All the other terms are 0 according to equation (2)]}
\end{aligned}
$$

$$
\boxed{a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx}
$$

#### 2.1.5 Derivation of $b_n$

Multiplying both sides of (1) by $\sin\dfrac{m\pi x}{L}$ and integrating from $-L$ to $L$, we get

$$
\begin{aligned}
\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\,dx &= \frac{a_0}{2}\int_{-L}^{L}\sin\frac{m\pi x}{L}\,dx \\
&\quad + \sum_{n=1}^{\infty}\int_{-L}^{L}\left[a_n\sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\right]dx \\
&= a_n\int_{-L}^{L}\sin^2\frac{m\pi x}{L}\,dx \\
&= a_n\cdot L \qquad \text{[All the other terms are 0 according to equation (2)]}
\end{aligned}
$$

$$
\boxed{b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\,dx}
$$

**Example 2.1: Obtain the F.S for $f(x) = x - x^2$ in the interval $(-\pi, \pi)$ and hence evaluate**

$$
\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots
$$

$$
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)
$$

where

$$
a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)dx = \frac{1}{\pi}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-\pi}^{\pi} = -\frac{2\pi^2}{3}
$$

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\cos nx dx \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\cos nx\,dx - \int_{-\pi}^{\pi} x^2\cos nx\,dx\right] \\
&= -\frac{2}{\pi}\int_{0}^{\pi} x^2\cos nx\,dx \\
&= -\frac{2}{\pi}\left[\frac{x^2}{n}\sin nx - \frac{2}{n}\int x\sin nx\,dx\right]_0^{\pi} \\
&= \frac{4}{n\pi}\left[-\frac{x}{n}\cos nx + \frac{1}{n^2}\sin nx\right]_0^{\pi} \\
&= -\frac{4}{n^2}(-1)^n
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} (x-x^2)\cos nx\,dx \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\sin nx\,dx - \int_{-\pi}^{\pi} x^2\sin nx\,dx\right] \\
&= \frac{2}{\pi}\int_{0}^{\pi} x\sin nx\,dx \\
&= \frac{2}{\pi}\left[-\frac{x}{n}\cos nx + \frac{1}{n^2}\sin nx\right]_0^{\pi} \\
&= -\frac{2}{n}(-1)^n
\end{aligned}
$$

$$
\therefore f(x) = -\frac{\pi^2}{3} - \sum_{n=1}^{\infty}\left(\frac{4}{n^2}(-1)^n\cos nx + \frac{2}{n}(-1)^n\sin nx\right)
$$

For $x = 0$, we get

$$
\begin{aligned}
0 &= -\frac{\pi^2}{3} - \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n \\
\frac{\pi^2}{12} &= -\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^2}
\end{aligned}
$$

$$
\boxed{\therefore 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}}
$$

**Example 2.2: Find a fourier series to represent the function $f(x) = e^x$ for $-\pi < x < \pi$ and hence derive a series for $\dfrac{\pi}{\sinh \pi}$**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi} \int_{-\pi}^{\pi} e^x dx = \frac{1}{\pi} \left. e^x \right|_{-\pi}^{\pi} = \frac{e^{\pi} - e^{-\pi}}{\pi} = \frac{2\sinh x}{\pi}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \cos nx \, dx \\
&= \frac{1}{\pi} \left[ e^x \frac{\sin nx}{n} - \frac{1}{n} \int e^x \sin nx \, dx \right]_{-\pi}^{\pi} \\
&= -\frac{1}{n\pi} \left[ -e^x \frac{\cos nx}{n} + \frac{1}{n} \int e^x \cos nx \, dx \right]_{-\pi}^{\pi} \\
a_n \left(1 + \frac{1}{n^2}\right) &= \frac{(-1)^n}{n^2 \pi} \left(e^{\pi} - e^{-\pi}\right) \\
a_n &= \frac{(-1)^n}{n^2 \pi} \left(e^{\pi} - e^{-\pi}\right) \left(1 + \frac{1}{n^2}\right)^{-1} \\
a_n &= 2 \frac{(-1)^n}{n^2 \pi} \sinh x \left(1 + \frac{1}{n^2}\right)^{-1}
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} e^x \sin nx \, dx \\
&= \frac{1}{\pi} \left[ -e^x \frac{\cos nx}{n} + \frac{1}{n} \int e^x \cos nx \, dx \right]_{-\pi}^{\pi} \\
&= \frac{1}{\pi} \left[ -e^x \frac{\cos nx}{n} + \frac{1}{n} \left\{ e^x \frac{\sin nx}{n} - \frac{1}{n} \int e^x \sin nx \, dx \right\} \right]_{-\pi}^{\pi} \\
&= \frac{1}{\pi} \left[ -e^x \frac{\cos nx}{n} - \frac{1}{n^2} \int e^x \sin nx \, dx \right] \\
b_n &= -\frac{(-1)^n}{n\pi} \left(e^{\pi} - e^{-\pi}\right) \left(1 + \frac{1}{n^2}\right)^{-1} \\
b_n &= -2 \frac{(-1)^n}{n\pi} \sinh x \left(1 + \frac{1}{n^2}\right)^{-1}
\end{aligned}$$

$$\boxed{f(x) = e^x = 2\,\frac{\sinh x}{\pi} \left[ \frac{1}{2} + \sum_{n=1}^{\infty} \left(1 + \frac{1}{n^2}\right)^{-1} \frac{(-1)^n}{n} \left( \frac{1}{n} \cos nx - \sin nx \right) \right]}$$

For $x = 0$, we get

$$\begin{aligned}
\frac{\pi}{\sinh x} &= 1 + 2 \sum_{n=1}^{\infty} \left(1 + \frac{1}{n^2}\right)^{-1} \frac{(-1)^n}{n^2} \\
&= 1 + 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + 1}
\end{aligned}$$

$$\boxed{\frac{\pi}{\sinh x} = 1 + 2 \left( -\frac{1}{2} + \frac{1}{5} - \frac{1}{10} + \cdots \right)}$$

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