Lagrange's Method

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2.2. Lagrange's method of solving $Pp + Qq = R$, when P, Q and R are functions of x, y, z (Delhi Maths (H) 2009; Meerut 2003; Poona 2003, 10; Lucknow 2010)

Theorem. *The general solution of Lagrange equation*

$$Pp + Qq = R, \quad \dots (1)$$

is

$$\phi(u, v) = 0 \quad \dots (2)$$

where ϕ *is an arbitrary function and*

$$u(x, y, z) = c_1 \quad \text{and} \quad v(x, y, z) = c_2 \quad \dots (3)$$

are two independent solutions of

$$(dx)/P = (dy)/Q = (dz)/R \quad \dots (4)$$

Here, c_1 and c_2 are arbitrary constants and at least one of u, v must contain z. Also recall that u and v are said to be independent if u/v is not merely a constant.

2.3. Working Rule for solving $Pp + Qq = R$ by Lagrange's method.

[Delhi Maths Hons. 1998]

Step 1. Put the given linear partial differential equation of the first order in the standard form

$$Pp + Qq = R. \quad \dots (1)$$

Step 2. Write down Lagrange's auxiliary equations for (1) namely,

$$(dx)/P = (dy)/Q = (dz)/R \quad \dots (2)$$

Step 3. Solve (2) by using the well known methods (refer Art. 2.5, 2.7, 2.9 and 2.11). Let $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ be two independent solutions of (2).

Step 4. The general solution (or integral) of (1) is then written in one of the following three equivalent forms :

$\phi(u, v) = 0$, $\quad u = \phi(v) \quad$ or $\quad v = \phi(u)$, ϕ being an arbitrary function.

Ex. 1. *Solve* $(y^2z/x)p + xzq = y^2$. **[Indore 2004; Sagar 1994]**

Sol. Given $$(y^2z/x)p + xzq = y^2. \qquad ...(1)$$

The Lagrange's auxiliary equations for (1) are $$\frac{dx}{(y^2z/x)} = \frac{dy}{xz} = \frac{dz}{y^2}. \qquad ..(2)$$

Taking the first two fractions of (2), we have
$$x^2zdx = y^2zdy \quad \text{or} \quad 3x^2dx - 3y^2dy = 0, \qquad ...(3)$$

Integrating (3), $x^3 - y^3 = c_1$, c_1 being an arbitrary constant ...(4)

Next, taking the first and the last fractions of (2), we get
$$xy^2dx = y^2zdz \quad \text{or} \quad 2xdx - 2zdz = 0. \qquad ...(5)$$

Integrating (5), $x^2 - z^2 = c_2$, c_2 being an arbitrary constant ...(6)

From (4) and (6), the required general integral is
$$\phi(x^3 - y^3, x^2 - z^2) = 0, \ \phi \text{ being an arbitrary function.}$$

Ex. 2. *Solve* (*i*) $a(p + q) = z$. **[Bangalore 1997]** (*ii*) $2p + 3q = 1$. **[Bangalore 1995]**

Sol. (*i*) Given $$ap + aq = z. \qquad ...(1)$$

The Lagrange's auxiliary equation for (1) are $(dx)/a = (dy)/a = (dz)/1$. ...(2)

Taking the first two members of (1), $dx - dy = 0$. ...(3)

Integrating (3), $x - y = c_1$, c_1 being an arbitrary constant ...(4)

Taking the last two members of (1), $dy - adz = 0$. ...(5)

Integrating (5), $y - az = c_2$, c_2 being an arbitrary constant. ...(6)

From (4) and (6), the required solution is given by
$$\phi(x - y, y - az) = 0, \ \phi \text{ being an arbitrary function.}$$

Ex. 3. *Solve* $p \tan x + q \tan y = \tan z$. **[Madras 2005 ; Kanpur 2007]**

Sol. Given $(\tan x)p + (\tan y)q = \tan z$. ...(1)

The Lagrange's auxiliary equations for (1) are $\dfrac{dx}{\tan x} = \dfrac{dy}{\tan y} = \dfrac{dz}{\tan z}$. ...(2)

Taking the first two fractions of (2), $\cot x\, dx - \cot y\, dy = 0$.

Integrating, $\log \sin x - \log \sin y = \log c_1$ or $(\sin x)/(\sin y) = c_1$. ...(3)

Taking the last two fractions of (2), $\cot y\, dy - \cot z\, dz = 0$.

Integrating, $\log \sin y - \log \sin z = \log c_2$ or $(\sin y)/(\sin z) = c_2$. ...(4)

From (3) and (4), the required general solution is

$$\sin x/\sin y = \phi(\sin y/\sin z),\ \phi \text{ being an arbitrary function.}$$

Ex. 4. *Solve* $zp = -x$.

Sol. Given $zp + 0.q = -x$. ...(1)

The Lagrange's subsidiary equations for (1) are $(dx)/z = (dy)/0 = (dz)/(-x)$ ...(2)

Taking the first and the last members of (2), we get

$-xdx = zdz$ or $2xdx + 2zdz = 0$. ...(3)

Integrating (3), $x^2 + z^2 = c_1$, c_1 being an arbitrary constant. ...(4)

Next, the second fraction of (2) implies that $dy = 0$ giving $y = c_2$ ...(3)

From (4) and (5), the required solution is $x^2 + z^2 = \phi(y)$, ϕ being an arbitrary function.

Ex. 1. *Solve* $p + 3q = 5z + \tan(y - 3x)$.

[Agra 2006; Meerut 2003; Indore 2002; Ravishankar 2003]

Sol. Given $$p + 3q = 5z + \tan(y - 3x). \qquad \text{...(1)}$$

The Lagrange's subsidary equations for (1) are $$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z + \tan(y - 3x)}. \qquad \text{...(2)}$$

Taking the first two fractions, $$dy - 3dx = 0. \qquad \text{...(3)}$$

Integrating (3), $y - 3x = c_1$, c_1 being an arbitrary constant. ...(4)

Using (4), from (2) we get $$\frac{dx}{1} = \frac{dz}{5z + \tan c_1}. \qquad \text{...(5)}$$

Integrating (5), $x - (1/5) \times \log(5z + \tan c_1) = (1/5) \times c_2$, c_2 being an arbitrary constant.

r $$5x - \log[5z + \tan(y - 3x)] = c_2, \text{ using (4)} \qquad \text{...(6)}$$

From (4) and (6), the required general integral is

$5x - \log[5z + \tan(y - 3x)] = \phi(y - 3x)$, where ϕ is an arbitrary function.

Ex. 2. *Solve* $z(z^2 + xy)(px - qy) = x^4$.

Sol. Given $$xz(z^2 + xy)p - yz(z^2 + xy)q = x^4. \qquad \text{...(1)}$$

The Lagrange's subsidiary equations for (1) are $$\frac{dx}{xz(z^2 + xy)} = \frac{dy}{-yz(z^2 + xy)} = \frac{dz}{x^4}. \qquad \text{...(2)}$$

Cancelling $z(z^2 + xy)$, the first two fractions give

$$(1/x)\,dx = -(1/y)\,dy \quad \text{or} \quad (1/x)\,dx + (1/y)\,dy = 0. \qquad \text{...(3)}$$

Integrating (3), $\log x + \log y = \log c_1$ or $xy = c_1$. ...(4)

Using (4), from (2) we get $$\frac{dx}{xz(z^2 + c_1)} = \frac{dz}{x^4}$$

r $$x^3 dx = z(z^2 + c_1)dz \quad \text{or} \quad x^3 dx - (z^3 + c_1 z)dz = 0. \qquad \text{...(5)}$$

Integrating (5), $x^4/4 - z^4/4 - (c_1 z^2)/2) = c_2/4$ or $x^4 - z^4 - 2c_1 z^2 = c_2$

r $$x^4 - z^4 - 2xy\, z^2 = c_2, \text{ using (4)} \qquad \text{...(6)}$$

From (4) and (6), the required general integral is

$$\phi(xy,\ x^4 - z^4 - 2xy\, z^2) = 0, \quad \phi \text{ being an arbitrary function.}$$

Ex. 3. *Solve* $xyp + y^2q = zxy - 2x^2$. **[Garhwal 2005]**

Sol. Given $$xyp + y^2q = zxy - 2x^2. \qquad ...(1)$$

The Lagrange's subsidiary equations for (1) are $$\frac{dx}{xy} = \frac{dy}{y^2} = \frac{dz}{zxy - 2x^2}. \qquad ...(2)$$

Taking the first two fractions of (2), we have

$(dx)/xy = (dy)/y^2$ or $(1/x)dx - (1/y)dy = 0$ ...(3)

Integrating (3), $\log x - \log y = \log c_1$ or $x/y = c_1$. ...(4)

From (4), $x = c_1y$. Hence from second and third fractions of (2), we get

$$\frac{dy}{y^2} = \frac{dz}{c_1zy^2 - 2c_1^2y^2} \quad \text{or} \quad c_1dy - \frac{dz}{z - 2c_1^2} = 0. \qquad ...(5)$$

Integrating (5), $c_1y - \log(z - 2c_1^2) = c_2$ or $x - \log[z - 2(x^2/y^2)] = c_2$, using (4). ...(6)

From (4) and (6), the required general solution is

$$x - \log[z - 2(x^2/y^2)] = \phi(x/y),\ \phi \text{ being an arbitrary function.}$$

Ex. 4. *Solve* $xzp + yzq = xy$. **[Bhopal 1996; Jabalpur 1999; Jiwaji 2000; Punjab 2005; Agra 2007; Ravishanker 1996; Vikram 2000]**

Sol. Given $$xzp + yzq = xy. \qquad ...(1)$$

The Lagrange's subsidiary equations for (1) are $$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy}. \qquad ...(2)$$

Taking the first two fractions of (2), $(1/x)dx - (1/y)dy = 0$ ...(3)

Integrating (3), $\log x - \log y = \log c_1$ or $x/y = c_1$. ...(4)

From (4), $x = c_1y$. Hence, from second and third fractions of (2), we get

$(1/yz)dy = (1/c_1y^2)dz$ or $2c_1y\,dy - 2z\,dz = 0$. ...(5)

Integrating (5), $c_1y^2 - z^2 = c_2$ or $xy - z^2 = c_2$, using (4). ...(6)

From (4) and (6), the required solution is $\phi(xy - z^2, x/y) = 0$, ϕ being an arbitrary function.

Ex. 5. *Solve* $py + qx = xyz^2 (x^2 - y^2)$.

Sol. Given $$py + qx = xyz^2 (x^2 - y^2). \qquad ...(1)$$

The Lagrange's auxiliary equations for (1) are $$\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{xyz^2(x^2 - y^2)}. \qquad ...(2)$$

Taking the first two fractions of (2), $\quad 2xdx - 2ydy = 0. \qquad ...(3)$

Integrating. $\quad x^2 - y^2 = c_1$, c_1 being an arbitrary constant. $\qquad ...(4)$

Using (4), the last two fractions of (2) give

$(dy)/x = (dz)/(xyz^2c_1) \quad$ or $\quad 2c_1y\, dy - 2z^{-2}dz = 0. \qquad ...(5)$

Integrating (5), $c_1y^2 + (2/z) = c_2$, c_2 being an arbitrary constant.

$$y^2 (x^2 - y^2) + (2/z) = c_2, \text{ using (4).} \qquad ...(6)$$

From (4) and (6), the required general solution is

$$y^2 (x^2 - y^2) + (2/z) = \phi(x^2 - y^2), \text{ where } \phi \text{ is an arbitrary function.}$$

Ex. 6. *Solve* $xp - yq = xy$ **[Madras 2005]**

Sol. The Lagrange's auxiliary equations for the given equation are

$$(dx)/x = (dy)/(-y) = (dz)/(xy) \qquad ...(1)$$

Taking the first two fractions of (1), $\quad (1/x)dx + (1/y)dy = 0$

Intergrating, $\quad \log x + \log y = c_1 \quad$ so that $\quad xy = c_1 \qquad ...(2)$

Using (2), (1) yields $\quad (1/x)dx = (1/c_1)\, dz \quad$ so that $\quad \log x - \log c_2 = z/c_1$

$\log (x/c_2) = z/c_1 \quad$ or $\quad \log (x/c_2) = z/(xy)$, by (2)

Thus, $x/c_2 = e^{z/(xy)}$ or $xe^{-z/(xy)} = c_2$, c_2 being an arbitrary constant. $\qquad ...(3)$

From (2) and (3), the required solution is $x\,e^{-z/(xy)} = \phi(xy)$, ϕ being an arbitrary function