Wave Motion:

Form of disturbance that travels through a medium and is due to the repeated periodic motion of the particles of the medium.

Progressive wave: A continuous transfer of a particular state from one part of the medium to another part due to similar movements performed successively by the consecutive particles of the medium.

Equation of motion:

$$y = a\sin(\omega t - \varphi)$$

$$y = a\sin\left(\omega t - \frac{2\pi}{\lambda}x\right) \qquad \left[\varphi = \frac{2\pi}{\lambda}x\right]$$

k

$$\boxed{y = a\sin\frac{2\pi}{\lambda}(vt - x)} \quad \text{— (i)}$$

propagation constant $k = \frac{2\pi}{\lambda}$

angular wave number

$$y = a\sin 2\pi n\left(t - \frac{x}{v}\right)$$

$$\omega = \frac{2\pi}{T} = 2\pi n = \frac{2\pi}{\lambda}v$$

$[\because v = n\lambda]$

$\frac{n}{v} = \frac{1}{\lambda}$

$$\boxed{y = a\sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right)} \quad \text{— (ii)}$$

$n = \frac{1}{T}$

Wave traveling in

(+)ve x direction : $y = a\sin\frac{2\pi}{\lambda}(vt - x)$

(−)ve x " : $y = a\sin\frac{2\pi}{\lambda}(vt + x)$

Velocity of wave :

$$\boxed{v = \frac{\lambda}{T} = \frac{\lambda}{\frac{2\pi}{\omega}} = \frac{\omega}{\frac{2\pi}{\lambda}} = \frac{\omega}{k}} \text{ —— (III)}$$

↳ Assuming $t = 0$

For initial phase θ,

$$\boxed{y = a \sin \frac{2\pi}{\lambda}(vt - x + \theta)} \text{ —— (IV)}$$

Wave velocity and particle velocity :

Note : The motion in the eq^n of motion is actually of the particle.

∴ Velocity of wave :

$$\boxed{v = n\lambda} \text{ —— (V)}$$

∴ Velocity of particle

$$V = \dot{y} = \frac{dy}{dt} = \frac{2\pi v}{\lambda} a \cos \frac{2\pi}{\lambda}(vt - x)$$

$$\boxed{V = \frac{dy}{dt} = \frac{2\pi v}{\lambda} a \cos \frac{2\pi}{\lambda}(vt - x)} \text{ —— (VI)}$$

Slope of the wave curve :

$$\boxed{S = \frac{dy}{dx} = -\frac{2\pi}{\lambda} a \cos \frac{2\pi}{\lambda}(vt - x)} \text{ —— (VII)}$$

Now $\frac{(vi)}{(vii)}$:

$$\boxed{\frac{V}{S} = -v \quad \therefore \quad V = -vS} \quad \text{———} \quad (viii)$$

→ Particle velocity = − wave velocity × Curve

Acceleration of particle,

$$\boxed{A = \frac{dV}{dt} = \frac{d^2y}{dt^2} = -\left(\frac{2\pi v}{\lambda}\right)^2 a \sin\frac{2\pi}{\lambda}(vt - x)} \quad \text{———} \quad (ix)$$

Maximum acceleration,

$$A_{max} = \frac{4\pi^2 v^2}{\lambda^2} a = 4\pi^2 n^2 a$$

Wave packet: A group consisting of a number of waves of slightly different frequencies superimposed upon each other.

Phase velocity: The velocity of each individual wave of a wave packet.

$$\boxed{v_p = \frac{\omega}{k} = \frac{\omega\lambda}{2\pi}} \quad \text{———} \quad (x)$$

Relation between phase and group velocity :

$$v_p = \frac{\omega}{k} \quad \text{or} \quad \omega = k v_p$$

Group velocity,

$$v_g = \frac{d\omega}{dk} = \frac{d}{dk}(k v_p) = v_p + k\frac{dv_p}{dk}$$

$$v_g = v_p + k\frac{dv_p}{d\lambda}\cdot\frac{d\lambda}{dk}$$

$$\frac{dk}{d\lambda} = \frac{d}{d\lambda}\left(\frac{2\pi}{\lambda}\right) = -\frac{2\pi}{\lambda^2}$$

$$v_g = v_p - \frac{2\pi}{\lambda}\cdot\frac{\lambda^2}{2\pi}\frac{dv_p}{d\lambda}$$

$$\therefore \boxed{v_g = v_p - \lambda\frac{dv_p}{d\lambda}} \quad \text{———} \quad (xi)$$

Differential Equation of a One dimensional wave motion :

$$y = a\sin\frac{2\pi}{\lambda}(vt - x)$$

$$\frac{dy}{dt} = \frac{2\pi v}{\lambda}\, a\cos\frac{2\pi}{\lambda}(vt - x)$$

$$\frac{d^2y}{dt^2} = -\frac{4\pi^2 v^2}{\lambda^2}\, a\sin\frac{2\pi}{\lambda}(vt - x) \quad \text{———} \quad (xii)$$

$$\frac{dy}{dx} = -\frac{2\pi}{\lambda}\, a\cos\frac{2\pi}{\lambda}(vt - x)$$

$$\frac{d^2y}{dx^2} = -\frac{4\pi^2}{\lambda^2}\, a\sin\frac{2\pi}{\lambda}(vt - x) \quad \text{———} \quad (xiii)$$

(xii) & (xiii) : $$\boxed{\frac{d^2y}{dt^2} = v^2\frac{d^2y}{dx^2}} \quad \text{———} \quad (xiv)$$

Energy density and Energy current (intensity) of a plane progressive wave:

Energy Density: Total energy per unit volume passed through the medium.

Kinetic energy density:

$$\rho_K = \frac{1}{2}(\text{mass density})(\text{velocity})^2$$

$$= \frac{1}{2}\rho V^2$$

$$= \frac{1}{2}\rho\left[\frac{2\pi v}{\lambda} a \cos\frac{2\pi}{\lambda}(vt-x)\right]^2$$

$$\therefore \boxed{\rho_K = \frac{2\pi^2 v^2}{\lambda^2} a^2 \rho \cos^2\frac{2\pi}{\lambda}(vt-x)} \quad \text{——— (xv)}$$

Potential energy density:

$$\rho_U = \int_0^y (\text{Force density})\, y\, dy$$

$$= \int_0^y (\text{mass density})(\text{acceleration})\, y\, dy$$

$$= \int_0^y \rho\left[\left(\frac{2\pi v}{\lambda}\right)^2 a \sin\frac{2\pi}{\lambda}(vt-x)\right] y\, dy$$

$$= \frac{4\pi^2 v^2}{\lambda^2} a \rho \sin\frac{2\pi}{\lambda}(vt-x) \cdot \frac{y^2}{2}$$

$$\therefore \boxed{\rho_U = \frac{2\pi^2 v^2}{\lambda^2} a^2 \rho \sin^2\frac{2\pi}{\lambda}(vt-x)} \quad \text{——— (xvi)}$$

Energy Density:

$$\rho_E = \rho_K + \rho_U$$

$$\therefore \boxed{\rho_E = \frac{2\pi^2 v^2}{\lambda^2} a^2 \rho} \text{ ——— (xvii)}$$

Energy Current: The rate of flow of energy through unit cross-sectional area of the wave front along the direction of the wave propagation.

$$C = \rho_E v \qquad \therefore \boxed{C = 2\pi^2 n^2 a^2 \rho v} \text{ ——— (xviii)}$$

Intensity: Quantity of incident energy per unit area of the wave front per unit time.

$$\therefore \boxed{I = 2\pi^2 n^2 a^2 \rho v} \text{ ——— (xix)}$$

Stationary Wave (Standing Wave)

When two exactly similar progressive wave trains travelling with the same velocity, along the same straight line, but in opposite directions are superimposed upon each other, the resultant wave is confined to the region in which it is produced and is no more progressive. Such wave is known as stationary wave.

Stationary waves in vibrating strings:

$$y_1 = a\sin\frac{2\pi}{\lambda}(vt - x)$$

$$y_2 = -a\sin\frac{2\pi}{\lambda}(vt + x)$$

$$\therefore y = y_1 + y_2$$

$$= a\sin\frac{2\pi}{\lambda}(vt - x) - a\sin\frac{2\pi}{\lambda}(vt + x)$$

$$= 2a\cos\frac{2\pi}{\lambda}vt\,\sin\left(-\frac{2\pi}{\lambda}x\right)$$

$$\therefore \boxed{y = -2a\cos\frac{2\pi}{\lambda}vt\,\sin\frac{2\pi}{\lambda}x} \quad \text{——— (xx)}$$

Now, (xx):

$$y = \left[-2a\cos\frac{2\pi}{\lambda}vt\right]\sin\frac{2\pi}{\lambda}x$$

For zero amplitude:

$$\cos \frac{2\pi}{\lambda} vt = 0$$

$$\frac{2\pi}{\lambda} vt = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

$$\frac{2\pi t}{T} = (2n+1)\frac{\pi}{2}$$

$[n = 0, 1, 2, \ldots]$

$$\boxed{t = (2n+1)\frac{T}{4}}$$

For maximum amplitude:

$$\cos \frac{2\pi}{\lambda} vt = \pm 1$$

$$\frac{2\pi}{\lambda} vt = 0, \pi, 2\pi, \ldots$$

$$2\pi \cdot \frac{t}{T} = n\pi$$

$[n = 0, 1, 2, \ldots]$

$$\boxed{t = \frac{nT}{2}}$$

Again (xx):

$$y = \left[-2a \sin \frac{2\pi}{\lambda} x\right] \cos \frac{2\pi}{\lambda} vt$$

For maximum amplitude:

$$\sin \frac{2\pi}{\lambda} x = \pm 1$$

$$\frac{2\pi}{\lambda} x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

$$\frac{2\pi}{\lambda} x = (2n+1)\frac{\pi}{2} \quad [n = 0, 1, 2, \ldots]$$

$$\therefore \boxed{x = (2n+1)\frac{\lambda}{4}}$$

For minimum amplitude:

$$\sin \frac{2\pi}{\lambda} x = 0$$

$$\frac{2\pi}{\lambda} x = 0, \pi, 2\pi, 3\pi, \ldots$$

$$\frac{2\pi}{\lambda} x = n\pi$$

$[n = 0, 1, 2, \ldots]$

$$\therefore \boxed{x = \frac{n\lambda}{2}}$$