First Order and First Degree Differential Equations

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2.12 Exact differential equation [Dibrugarh 1996]

Definition. If M and N are functions of x and y, the equation $M\,dx + N\,dy = 0$ ... (1)
is called exact when there exists a function $f(x, y)$ of x and y, such that

$$d[f(x, y)] = M\,dx + N\,dy, \qquad \text{... (2)}$$

i.e.,

$$(\partial f/\partial x)\,dx + (\partial f/\partial y)\,dy = M\,dx + N\,dy. \qquad \text{... (3)}$$

Statement. *The necessary and sufficient condition for the differential equation*

$$M\,dx + N\,dy = 0 \qquad \text{... (1)}$$

to be exact is

$$\partial M/\partial y = \partial N/\partial x. \qquad \text{... (2)}$$

$$\underset{\text{[Treating } y \text{ as constant]}}{\int M\,dx} + \int (\text{terms in } N \text{ not containing } x)\,dy = c,$$

Ex. 1. *Solve* $(x^2 - 4xy - 2y^2)\,dx + (y^2 - 4xy - 2x^2)\,dy = 0.$ **[Delhi Maths (H) 1995, 2005]**

Sol. Comparing the given equation with $M\,dx + N\,dy = 0$, we have

$M = x^2 - 4xy - 2y^2$ and $N = y^2 - 4xy - 2x^2$.

$\therefore$ $\partial M/\partial y = -4x - 4y$ and $\partial N/\partial x = -4y - 4x$ so that $\partial M/\partial y = \partial N/\partial x$.

Hence, the given equation is exact and hence its solution is

$$\underset{\text{[Treating } y \text{ as constant]}}{\int M\,dx} + \int (\text{terms in } N \text{ not containing } x)\,dy = c'$$

or $$\underset{\text{[Treating } y \text{ as constant]}}{\int (x^2 - 4xy - 2y^2)\,dx} + \int y^2\,dy = c'$$

or $$x^3/3 - 4y \times (x^2/2) - 2y^2x + y^3/3 = c/3, \text{ taking } c' = c/3$$

or $$x^3 + y^3 - 6xy\,(x + y) = c, \; c \text{ being an arbitrary constant.}$$

Ex. 2. *Test whether the equation* $(x+y)^2\,dx - (y^2 - 2xy - x^2)\,dy = 0$ *is exact and hence solve it.* **[I.A.S. 1995]**

Sol. The given equation can be re-written as $(x^2 + 2xy + y^2)\,dx + (x^2 + 2xy - y^2)\,dy = 0$ ..(1)

Comparing (1) with $M\,dx + N\,dy = 0$, *here* $M = x^2 + 2xy + y^2$, $N = x^2 + 2xy - y^2$.

$\therefore$ $\partial M/\partial y = 2x + 2y$ and $\partial N/\partial x = 2x + 2y$ so that $\partial M/\partial y = \partial N/\partial x$.

Hence (1) is exact and hence its solution is

$$\underset{\text{[Treating } y \text{ as constant]}}{\int M\,dx} + \int (\text{terms in } N \text{ not containing } x)\,dy = c'$$

or
$$\underset{\text{[Treating } y \text{ as constant]}}{\int (x^2 + 2xy + y^2)\,dx} + \int (-y^2)\,dy = c'$$

or
$$x^3/3 + 2y \times (x^2/2) + y^2 x - y^3/3 = c/3, \text{ taking } c' = c/3$$

or
$$x^3 + y^3 + 3xy\,(x+y) = c, \; c \text{ being an arbitrary constant.}$$

Ex. 4. *Solve* $(1 + e^{x/y})\,dx + e^{x/y}\,\{1 - (x/y)\}\,dy = 0$. **[I.A.S. Prel. 2007;Osmania 2005]**

Sol. Comparing the given equation with $M\,dx + N\,dy = 0$, $M = 1 + e^{x/y}$, $N = e^{x/y}\,\{1 - (x/y)\}$.

$\therefore$ $\partial M/\partial y = e^{x/y}\,(-x/y^2)$, $\partial N/\partial x = e^{x/y}\,(-1/y) + (1 - x/y)\,e^{x/y}\,(1/y) = (-x/y^2)\,e^{x/}$

Thus, $\partial M/\partial y = \partial N/\partial x$ and so the given equation is exact.

Its solution is
$$\underset{\text{[Treating } y \text{ as constant]}}{\int M\,dx} + \int (\text{terms in } N \text{ not containing } x)\,dy = c$$

or
$$\underset{\text{[Treating } y \text{ as constant]}}{\int (1 + e^{x/y})\,dx = c} \quad \text{or} \quad x + ye^{x/y} = c.$$

Ex. 5. *Solve* $(y^2 e^{xy^2} + 4x^3)\, dx + (2xy\, e^{xy^2} - 3y^2)\, dy = 0.$

Ex. 7. *Solve* $\{y\,(1 + 1/x) + \cos y\}\, dx + (x + \log x - x \sin y)\, dy = 0$

Ex. 8(*a*) *Solve* $x\, dx + y\, dy + \dfrac{x\, dy - y\, dx}{x^2 + y^2} = 0.$

Ex. 11. *Solve* $(x^2 + y^2 + x)\, dx - (2x^2 + 2y^2 - y)\, dy = 0.$

6. $(e^y + 1) \cos x\, dx + e^y \sin x\, dy = 0.$ **[Agra 2006]**

7. $x\,(x^2 + 3y^2)\, dx + y\,(y^2 + 3x^2)\, dy = 0.$

8. $(a^2 - 2xy - y^2)\, dx - (x + y)^2\, dy = 0.$

2.16 Integrating factor. [Osmania 2005]

Definition. If an equation of the form $M\,dx + N\,dy = 0$ is not exact, it can always be made exact by multiplying by some function of x and y. Such a multiplier is called an integrating factor. We shall write I.F. for integrating factor.

2.23 Linear differential equation

Definition. A first order differential equation is called linear if it can be written in the form

$$(dx/dy) + Py = Q, \quad \dots (1)$$

where P and Q are constants or functions of x alone (*i.e.*, not of y).

Working rule for solving linear equations. First put the given equation in the standard form (1). Next find an integrating factor (I.F.) by using formula

$$\text{I.F.} = e^{\int P\,dx} \quad \dots (5)$$

Ex. 1. *Solve $x \cos x\,(dy/dx) + y\,(x \sin x + \cos x) = 1$.* **[Agra 1994]**

Sol. Re-writing given equation, we have
$$\frac{dy}{dx} + \left(\tan x + \frac{1}{x}\right) y = \frac{\sec x}{x} \qquad \text{... (1)}$$

$$\text{I.F. of (1)} = e^{\int (\tan x + 1/x)\,dx} = e^{\log \sec x + \log x} = e^{\log x \sec x} = x \sec x.$$

Hence the required solution is
$$yx \sec x = \int \sec^2 x\,dx + c,$$
or
$$yx \sec x = \tan x + c, \ c \text{ being arbitoary constants.}$$

Ex. 2. (a) *Solve $(1 - x^2)(dy/dx) + 2xy = x\sqrt{(1 - x^2)}$.* **[Kerala 2001]**

(b) solve $(1-x^2)(dy/dx)+2xy = x\sqrt{1-x^2}, y(0)=1$ **[Delhi Maths (Prog) 2007]**

Sol. The given equation is
$$\frac{dy}{dx} + \frac{2x}{1-x^2} y = \frac{x}{(1-x^2)^{1/2}}. \qquad \text{... (1)}$$

Comparing (1) with $dy/dx + Py = Q$, here $P = 2x/(1 - x^2)$

Here $\int P\,dx = \int \frac{2x}{1-x^2}\,dx = -\log(1-x^2)$ hence $\text{I.F. of (1)} = e^{\int P\,dx} = \frac{1}{1-x^2}$

So the required solution is
$$\frac{y}{1-x^2} = \int \frac{x}{\sqrt{(1-x^2)}} \times \frac{1}{1-x^2}\,dx = -\frac{1}{2}\int t^{-3/2}\,dt + c, \text{ putting } 1 - x^2 = t \text{ and } -2x\,dx = dt$$

or $\frac{y}{1-x^2} = t^{-1/2} + c = c + \frac{1}{\sqrt{t}}$ or $\frac{y}{1-x^2} = \frac{1}{(1-x^2)^{1/2}} + c$, as $t = 1 - x^2$ ... (2)

(b) First do upto equation (2) as in Ex. 2(a). Putting $x = 0$ and $y = 1$ in (2), we have $1 = 1 + c$ so that $c = 0$. Hence (2) becomes

$y/(1-x^2) = 1/(1-x^2)^{1/2}$ or $y = (1 - x^2)^{1/2}$

Ex. 4. *Integrate $(1+x^2)(dy/dx)+2xy-4x^2=0$. Obtain equation of the curve satisfying this equation and passing through the origin.* **[Agra 1993]**

Sol. Re-writing the given equation, $$\frac{dy}{dx}+\frac{2x}{1+x^2}y=\frac{4x^2}{1+x^2}. \qquad \text{... (1)}$$

Comparing (1) with $dy/dx + Py = Q$, here $P = (2x)/(1+x^2)$

Here $\int P\,dx=\int \frac{2x}{1+x^2}dx=\log(1+x^2)$ so I.F. of (1) $=e^{\int P\,dx}=(1+x^2)$.

Hence the required solution is $$y(1+x^2)=\int \frac{4x^2}{1+x^2}\cdot(1+x^2)\,dx+c$$

or $\quad y(1+x^2)=(4/3)x^3+c$, c being an arbitrary constant. ... (1)

Since the required curve passes through origin, (1) must satisfy the condition $x=0$, $y=0$. Putting these in (1), we get $c=0$. Hence the required curve is $4x^3=3y(1+x^2)$.

Ex. 5. *Solve* $(x+2y^3)(dy/dx)=y$. **[Rohilkhand 1993; Agra 1995; Delhi Maths. (G) 1995, 2002; Lucknow 1995; Rajasthan 2010]**

Sol. Here it is possible to put the equation in form $dx/dy + P_1x = Q_1$.
where P_1 and Q_1 are function of y or constants

Thus, we have $$\frac{dx}{dy}=\frac{x+2y^3}{y}, \quad \text{or} \quad \frac{dx}{dy}-\frac{1}{y}x=2y^2. \qquad \text{... (1)}$$

For (1), $\int P_1\,dy=-\int (1/y)\,dy=-\log y$ so I.F. of (1) $=e^{-\log y}=1/y$.

Hence, the required solution is $$x/y=\int 2y^2\cdot(1/y)\,dx+c.$$

or $\quad x/y=y^2+c$, where c is an arbitrary constant.

Ex. 6. (*a*) *Solve* $(1 + y^2)\, dx = (\tan^{-1} y - x)\, dy$.

Ex. 6. (*b*) Solve $(1 + y^2) + (x - e^{-\tan^{-1} y})(dy / dx) = 0$.

Ex. 8. *Solve* $x\,(1 - x^2)\, dy + (2x^2 y - y - ax^3)\, dx = 0$.

2.25A Bernoulli's equation A particular case of Art. 2.25.

An equation of the form $$(dy/dx) + Py = Qy^n \qquad \text{... (1A)}$$

where P and Q are constants or functions of x alone (and not of y) and n is constant except 0 and 1, is called a *Bernoulli's differential equation*.

We first multiply by y^{-n}, thereby expressing it in the form (1) of Art. 2.25

$$y^{-n}(dy/dx) + Py^{1-n} = Q. \qquad \text{... (2 A)}$$

Let $$y^{1-n} = v \qquad \text{... (3 A)}$$

Differentiating w.r.t. x, (3 A) gives $(1-n)\,y^{-n}\dfrac{dy}{dx} = \dfrac{dv}{dx}$, or $\quad y^{-n}\dfrac{dy}{dx} = \dfrac{1}{1-n}\dfrac{dv}{dx} \qquad \text{... (4 A)}$

Using (3 A) and (4 A), (2 A) reduces to

$$\frac{1}{1-n}\frac{dv}{dx} + Pv = Q \qquad \text{or} \qquad \frac{dv}{dx} + P(1-n)\,v = Q(1-n),$$

which is linear in v and x. Its I.F. $= e^{\int P(1-n)dx} = e^{(1-n)\int P dx}$ and hence the required solution is

$$v \cdot e^{(1-n)\int P dx} = \int Q \cdot e^{(1-n)\int P dx}\, dx + c, \text{ } c \text{ being an arbitrary constant}$$

$$y^{1-n}\, e^{(1-n)\int P dx} = \int Q \cdot e^{(1-n)\int P dx}\, dx + c, \text{ using (3A)}$$

Ex. 1. *Solve* $(dy/dx) + x \sin 2y = x^3 \cos^2 y$.

[I.A.S. (Prel.) 2005; I.A.S. 1994; Calcutta 1995; Kanpur 1997; Lucknow 1996]

Sol. Dividing by $\cos^2 y$, $\qquad \sec^2 y\,(dy/dx) + 2x\,(\tan y) = x^3$. $\qquad$ ... (1)

Put $\tan y = v$ so that $\sec^2 y\,(dy/dx) = dv/dx$ Hence the above eqn. becomes $dv/dx + 2xv = x^3$, which is linear in v and x. Hence its I.F. $= e^{\int 2x\,dx} = e^{x^2}$ and its solution is given by

$$v \cdot e^{x^2} = \int x^3 e^{x^2}\,dx + c,\ c \text{ being an arbitrary constant}$$

$$ve^{x^2} = (1/2) \times \int t e^t\,dt + c, \text{ putting } x^2 = t \text{ and } 2x\,dx = dt$$

$$= (1/2) \times [t \times e^t - \int (1 \times e^t)\,dt] + c = (1/2) \times (t e^t - e^t) + c$$

or $\qquad \tan y \cdot e^{x^2} = (1/2) \times e^{x^2}(x^2 - 1) + c$, as $\quad v = \tan y \quad$ and $\quad t = x^2$

or $\qquad \tan y = (1/2) \times (x^2 - 1) + ce^{-x^2}$, dividing by e^{x^2}

Ex. 2. *Solve* $(dy/dx) = e^{x-y}(e^x - e^y)$. $\qquad$ **[Agra 1995; Delhi Maths (G) 1997; Kanpur 1997; Rohilkhand 1997]**

Sol. Re-writing, $\qquad dy/dx = e^{2x} \cdot e^{-y} - e^x \qquad$ or $\qquad dy/dx + e^x = e^{2x} \cdot e^{-y}$.

Now dividing by e^{-y}, we get $\qquad e^y\,(dy/dx) + e^x \cdot e^y = e^{2x}$.

Putting $e^y = v$ so that $e^y\,(dy/dx) = dv/dx$ we get $\qquad dv/dx + e^x v = e^{2x}$.

Its I.F. $= e^{\int P\,dx} = e^{\int e^x\,dx} = e^{e^x}$ and the solution is

$$v \cdot e^{e^x} = \int e^{2x} \cdot e^{e^x}\,dx + c = \int e^x e^{e^x} \cdot e^x\,dx + c = \int t e^t\,dt + c, \text{ putting } e^x = t \text{ so that } e^x\,dx = dt$$

$$= \int t \cdot e^t - \int 1 \cdot e^t\,dt + c = t \cdot e^t - e^t + c = e^t\,(t - 1) + c$$

i.e., $\qquad e^y e^{e^x} = e^{e^x}(e^x - 1) + c \qquad$ or $\qquad e^{e^x}(e^y - e^x + 1) = c$, as $v = e^y$ and $t = e^x$

Ex. 3. *Solve* $\dfrac{dz}{dx}+\dfrac{z}{x}\log z=\dfrac{z}{x^2}\cdot(\log z)^2$. **[I.A.S. 2001; Calcutta 1994]**

Sol. Here we have z in place of y and so the method of solution will remain similar. Dividing by $z\,(\log z)^2$, we get

$$\frac{1}{z\,(\log z)^2}\frac{dz}{dx}+\frac{1}{x}\frac{1}{(\log z)}=\frac{1}{x^2}. \qquad \text{... (1)}$$

Putting $\dfrac{1}{\log z}=v$ so that $\dfrac{(-1)}{(\log z)^2\,z}\dfrac{dz}{dx}=\dfrac{dv}{dx}$, (1) becomes

$$-\frac{dv}{dx}+\frac{1}{x}v=\frac{1}{x^2} \qquad \text{or} \qquad \frac{dv}{dx}-\frac{1}{x}v=-\frac{1}{x^2}, \qquad \text{... (2)}$$

whose I.F. $= e^{-\int(1/x)dx}=e^{-\log x}=1/x$ and so solution is

$$\frac{v}{x}=\int\left(-\frac{1}{x^3}\right)dx+c=\frac{1}{2x^2}+c \qquad \text{or} \qquad \frac{1}{x\,(\log z)}=\frac{1}{2x^2}+c.$$

Ex. 4. $x\,(dy/dx)+y\log y=xy\,e^x$. **[Agra 1994]**

Sol. Dividing by xy, the given equation reduces to $\dfrac{1}{y}\dfrac{dy}{dx}+\dfrac{1}{x}\log y=e^x$. ... (1)

Let $\log y=v$ so that $(1/y)\times(dy/dx)=dv/dx$ ... (2)

Using (2), (1) gives $(dv/dx)+(1/x)\,v=e^x$. ... (3)

Comparing (3) with $dv/dx+Pv=Q$, we have $P=1/x$ and $Q=e^x$. ... (4)

Since $\int P\,dx=\int(1/x)\,dx=\log x$; I.F. of (3) $=e^{\int P\,dx}=e^{\log x}=x$. Hence solution of (3) is

$$v.(\text{I.F.})=\int Q.(\text{I.F.})\,dx+c \qquad \text{or} \qquad vx=\int x\,e^x\,dx+c \qquad \text{or} \qquad vx=x\,e^x-\int e^x\,dx+c=x\,e^x-e^x+c$$

or $x\log y=e^x\,(x-1)+c$, by (2); c being an arbitrary constant.

Ex. 6. (*a*) *Solve* $2xy\,dy - (x^2 + y^2 + 1)\,dx = 0$.

(*b*) *Solve* $\dfrac{dy}{dx} = \dfrac{x^2 + y^2 + 1}{2xy}$, *given* $y = 1$ *when* $x = 1$.

Ex. 7. *Solve* $dy/dx + (1/x)\sin 2y = x^2 \cos^2 y$.

Ex. 8. *Solve* $(\sec x \tan x \tan y - e^x)\,dx + \sec x \sec^2 y\,dy = 0$

Ex. 9. Solve $(xy^2 + e^{-1/x^3})\,dx - x^2 y\,dy = 0$.

Ex. 1. *Solve* $x\,(dy/dx) + y = y^2 \log x$. **[Delhi Maths (H) 2009; Kanpur 2006]**

Sol. Re-writing the given equation $y^{-2}\,(dy/dx) + (1/x) \times y^{-1} = (1/x) \times \log x$. ... (1)

Putting $y^{-1} = v$ so that $-y^{-2}\,(dy/dx) = dv/dx$. Then (1) gives

$$-\frac{dv}{dx} + \frac{1}{x}v = \frac{1}{x}\log x \qquad \text{or} \qquad \frac{dv}{dx} - \frac{1}{x}v = -\frac{1}{x}\log x \qquad \text{... (2)}$$

I.F. of (2) $= e^{-\int (1/x)dx} = e^{-\log x} = x^{-1} = 1/x$. and hence solution of (2) is

$$vx^{-1} = -\int x^{-2} \log x\, dx + c, \text{ c being an arbitrary constant}$$

or $$y^{-1}x^{-1} = -\left[\log x \times \frac{x^{-1}}{(-1)} - \int \frac{1}{x} \times \frac{x^{-1}}{(-1)}\,dx\right] + c \qquad \text{or} \qquad \frac{1}{y} = \log x + 1 + cx.$$

Ex. 2. *Solve* (dy/dx) $-$ *y tan x* $= -$ *y^2 sec x or cos x dy* $=$ *(sin x $-$ y) y dx.* **[Kanpur 1995]**

Sol. Dividing by y^2, the given equation gives $y^{-2}\,(dy/dx) - \tan x \cdot y^{-1} = -\sec x$ ... (1)

Putting $y^{-1} = v$ so that $-y^{-2}\,(dy/dx) = dv/dx$, (1) becomes

$$-\frac{dv}{dx} - \tan x \cdot v = -\sec x \qquad \text{or} \qquad \frac{dv}{dx} + \tan x \cdot v = \sec x \text{ ... (2)}$$

which is linear whose I.F. $= e^{\int \tan x\, dx} = e^{\log \sec x} = \sec x$.

Hence solution of (2) is $v \cdot \sec x = \int \sec x \cdot \sec x\, dx + c$, c being an arbtritray constant.

or $$v \sec x = \tan x + c \qquad \text{or} \qquad y^{-1} \sec x = \tan x + c, \text{ as } v = y^{-1}$$