Math-183
Differential Equations

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Contents

1 Differential Equations and Their Solutions

1.1 Classification of Differential Equations

Definition 1.1.1: Differential Equation

Differential equation is an equation involving derivatives of one or more dependent variables with respect to one or more independent variables.

Definition 1.1.2: Ordinary Differential Equation

A differential equation involving ordinary derivatives of one or more dependent variables with respect to a single independent variable is called an ordinary differential equation.

Example 1.1.1: Ordinary Differential Equations:

$$\frac{dy}{dx} + xy\left(\frac{d}{dx}\right)^2 = 0 \tag{1.1.1}$$

$$\frac{d^4x}{dt^4} + 5\frac{d^2x}{dt^2} + 3x = \sin t \tag{1.1.2}$$

Definition 1.1.3: Partial Differential Equation

A differential equation involving partial derivatives of one or more dependent variables with respect to more than one independent variables is called an partial differential equation.

Example 1.1.2: Partial Differential Equations:

$$\frac{\partial v}{\partial s} + \frac{\partial v}{\partial t} = v \tag{1.1.3}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \tag{1.1.4}$$

Definition 1.1.4: Order and Degree of Differential Equations

Order of DE: The order of the highest ordered derivative involved in a differential equation is called the order of the differential equation.

Degree of DE: The power of the highest order derivative involved in a differential equation is called the degree of the differential equation.

Definition 1.1.5: Linearity of Differential Equations

If the dependent variable and its various derivatives occur to the first degree only, the DE is a linear DE. Otherwise it's a non-linear DE.

$$a_0(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_1(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + a_{n-1}(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_n(x)y = b(x)$$

Linear DE can also be classified as linear with *constant* and *variable* coefficients.

Example 1.1.3: Ordinary Differential Equations: Orders, Degree, Linearity

$$\begin{aligned}
\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} - 3\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 3\frac{\mathrm{d}y}{\mathrm{d}x} - 6y = \sin x & \quad \text{3rd ord 1st deg Lin} \\
\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + y = 0 & \quad \text{2nd ord 1st deg Non-Lin} \\
y = x\frac{\mathrm{d}y}{\mathrm{d}x} + \sqrt{1 + \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}} & \quad \text{2nd ord 1st deg Non-Lin} \\
\frac{\mathrm{d}^4 x}{\mathrm{d}t^4} + t^2\frac{\mathrm{d}^3 x}{\mathrm{d}t^3} + \frac{\mathrm{d}y}{\mathrm{d}x} = \sin t & \quad \text{4th ord 1st deg Lin} \\
\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 5\frac{\mathrm{d}y}{\mathrm{d}x} + 6y^2 = 0 & \quad \text{2nd ord 1st deg Non-Lin} \\
\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 5\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3 + 6y = 0 & \quad \text{2nd ord 1st deg Non-Lin} \\
\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 5y\frac{\mathrm{d}y}{\mathrm{d}x} + 6y = 0 & \quad \text{2nd ord 1st deg Lin}
\end{aligned}$$

1.2 Solutions

A Nature of Solutions

An nth-order Differential Equation:

$$F\left[x, y, \frac{\mathrm{d}y}{\mathrm{d}x}, \cdots, \frac{\mathrm{d}^n y}{\mathrm{d}x^n}\right] = 0 \tag{1.2.1}$$

Definition 1.2.1: Explicit solution

f is an explicit solution of (1.2.1) if

$$\forall x \in I, F\left[x, f(x), f'(x), \cdots, f^{(n)}(x)\right] = 0$$

where I is a real interval.

Definition 1.2.2: Implicit solution

$g(x, y) = 0$ is an implicit solution if this relation defines at least one real function $f(x)$ on an interval I such that f is an explicit solution of (1.2.1)

Example 1.2.1: Explicit and Implicit Solutions

$$x^2 + y^2 - 25 = 0 \quad : \quad \text{Implicit solution}$$

$$2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$x + y\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \quad : \quad \text{Differential Equation}$$

$$y = \pm\sqrt{25 - x^2}\ ;\ -5 \leq x \leq 5 \quad : \quad \text{Explicit solution}$$

B Methods of Solution

The study of a Differential Equation consists of 3 phases:

1. Formulation of DE from the given physical situation.
2. Solutions of DE, evaluating the arbitrary constants from the given condition.
3. Physical interpretation of the solution.

Example 1.2.2: Show that the function $f(x) = e^x + 2x^2 + 6x + 7$ **is a solution to the DE** $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 4x^2$

$$f(x) = e^x + 2x^2 + 6x + 7$$

$$f'(x) = e^x + 4x + 6$$

$$f"(x) = e^x + 4$$

$$\begin{aligned}\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 3\frac{\mathrm{d}y}{\mathrm{d}x} + 2y &= (e^x + 4) - 3(e^x + 4x + 6) + 2(e^x + 2x^2 + 6x + 7) \\ &= 0 \cdot e^x + 0 \cdot x + (4 - 18 + 14) + 4x^2 \\ &= 4x^2\end{aligned}$$

□

Example 1.2.3: Show that the function $f(x) = \dfrac{1}{1 + x^2}$ **is a solution to the DE** $(1 + x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0$

$$f(x) = \frac{1}{1 + x^2}$$

$$(1 + x^2)f(x) = 1$$

$$(1 + x^2)f'(x) + 2xf(x) = 0$$

$$(1 + x^2)f''(x) + 2xf'(x) + 2xf'(x) + 2f(x) = 0$$

$$(1 + x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0$$

□

Example 1.2.4: Show that the function $y = (2x^2 + 2e^{3x} + 3)e^{-2x}$ satisfies the DE

$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 6e^x + 4xe^{-2x}$$

$$y = (2x^2 + 2e^{3x} + 3)e^{-2x}$$
$$y_1 = (4x + 6e^{3x})e^{-2x} - (2x^2 + 2e^{3x} + 3)2e^{-2x}$$
$$y_1 = 4xe^{-2x} + 6e^x - 2y$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 6e^x + 4e^{-2x}$$

□

1.3 Initial-Value and Boundary-Value Problems, and Existence of Solutions

A Initial-value Problems and Boundary-value Problems

One of the most frequently encountered type of problems in Differential Equations involves both a DE and one or more supplementary conditions which the solution of the given DE must satisfy.

Definition 1.3.1: IVP and BVP

Consider the first-order DE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$$

where f is a continuous function of x and y in some domain D of the xy plane; and let (x_0, y_0) be a point of D. The **initial-value problem** associated with the DE is to find a solution ϕ of the DE, defined on some real interval containing x_0, and satisfying the initial condition

$$\phi(x_0) = y_0$$

In the customary abbreviated notation, this initial-value problem may be written

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$$

$$y(x_0) = y_0$$

If the conditions relate to two different x values (the extreme or boundary values), the proble is called a **Two-Point Boundary-Value Problem** or simply a **Boundary-Value Problem (BVP)**.

Example 1.3.1: Find the solution of the DE $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$ such that $\forall x \in I, f'(x) = 2x$ and $f(1) = 4$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$$
$$\int \frac{\mathrm{d}y}{\mathrm{d}x}\, dx = \int 2x\, dx$$
$$y = x^2 + c$$

Substituting $y = 4$ and $x = 1$,

$$4 = 1 + c \text{ or } c = 3$$
$$\therefore \text{ Solution: } y^2 = x + 3$$

□

Example 1.3.2: $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$, $y(3) = 4$

$$x + y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$
$$\int x\, dx + \int y\frac{\mathrm{d}y}{\mathrm{d}x}\, dx = 0$$
$$\frac{x^2}{2} + \frac{y^2}{2} = c'$$
$$x^2 + y^2 = c$$

Substituting $x = 3$ and $y = 4$,

$$16^2 + 3^2 = c \text{ or } c = 25$$
$$\therefore \text{ Solution: } x^2 + y^2 - 25 = 0$$

B Existence of Solutions

Not all initial-value and boundary-value problems have solutions. For example,

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y = 0$$
$$y(0) = 1 \quad , \quad y(\pi) = 5$$

has no solutions! Thus arises the question of *existence* of solutions. We can say, every initial-value problem that satisfies definition (1.3.1) has *at least one* solution. However, there arises another question. Can a problem have more than one solution?
Let's consider the initial-value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^{1/3} \;\; ; \;\; y(0) = 0$$

One may verify that the functions f_1 and f_2 defined, respectively, by

$$\forall x \in \mathbb{R}, \quad f_1(x) = 0$$

and

$$f_2(x) = (\frac{2}{3}x)^{3/2}, \quad x \geq 0; \quad f_2(x) = 0, \quad x \leq 0$$

are both solutions of this initial-value problem. In fact, this problem has infinitely many solutions. Hence, we can state that the initial-value problem need not have a *unique* solution. In order to ensure uniqeness, some additional requiremnt must certainly be imposed.

***Theorem 1.3.1* (Basic Existence and Uniqueness Theorem):**
Hypothesis: *Consider the differential equation*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) \tag{1.3.1}$$

where

- *The function f is a continuous function of x and y in some domain D of the xy plane, and*
- *The partial derivative $\frac{\partial f}{\partial y}$ is also a continuous function of x and y in D; and let (x_0, y_0) be a point in D.*

Conclusion: *There exists a unique solution ϕ of the differential equation (1.3.1), defined on some interval $|x - x_0| \leq h$, where h is sufficiently small, that satisfies the condition*

$$\phi(x_0) = y_0$$

Example 1.3.3: Show that

$$y = 4e^{2x} + 2e^{-3x}$$

is a solution of the initial-value problem

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} - 6y = 0$$

$$y(0) = 6$$

$$y'(0) = 2$$

Is $y = 2e^{2x} + 4e^{-3x}$ also a solution of this problem? Explain why or why not.

$$y = 4e^{2x} + 2e^{-3x}$$
$$y_1 = 8e^{2x} - 6e^{-3x}$$
$$y_2 = 16e^{2x} + 18e^{-3x}$$

$$\begin{aligned} y_2 + y_1 - 6y &= (16e^{2x} + 18e^{-3x}) + (8e^{2x} - 6e^{-3x}) - 6(4e^{2x} + 2e^{-3x}) \\ &= 0 \cdot e^{2x} + 0 \cdot e^{-3x} \\ &= 0 \end{aligned}$$

The solution also satisfies $y(0) = 6$ and $y'(0) = 2$

Now, for $y = 2e^{2x} + 4e^{-3x}$,

$$y_1 = 4e^{2x} - 12e^{-3x} \;\; ; \;\; y_2 = 8e^{2x} + 36e^{-3x}$$

$$
\begin{aligned}
y_2 + y_1 - 6y &= (8e^{2x} + 36e^{-3x}) + (4e^{2x} - 12e^{-3x}) - 6(2e^{2x} + 4e^{-3x}) \\
&= 0 \cdot e^{2x} + 0 \cdot e^{-3x} \\
&= 0
\end{aligned}
$$

However, in this case,

$$y(0) = 6 \ ; \ y'(0) = -8$$

As we can see, this solution doesn't satisfy the initial-value problem. Hence $y = 2e^{2x} + 4e^{-3x}$ is not a solution of this problem.

Example 1.3.4: Given that every solution of

$$x^3 \frac{\mathrm{d}^3 y}{\mathrm{d}x^3} - 3x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 6x \frac{\mathrm{d}y}{\mathrm{d}x} - 6y = 0$$

may be written in the form $y = c_1 x + c_2 x^2 + c_3 x^3$ for some choice of the arbitrary constants c_1, c_2, and c_3, solve the initial-value problem consisting of the above DE plus the three conditions

$$y(2) = 0 \ , \ y'(2) = 2 \ , \ y''(2) = 6$$

$$y = c_1 x + c_2 x^2 + c_3 x^3$$

$$y(2) = 0 \text{ or, } 8c_3 + 4c_2 + 2c_1 = 0 \tag{1.3.2}$$

$$y' = c_1 + 2c_2 x + 3c_3 x^2$$

$$y'(2) = 2 \text{ or, } 12c_3 + 4c_2 + c_1 = 2 \tag{1.3.3}$$

$$y'' = 0 + 2c_2 + 6c_3 x$$

$$y''(2) = 6 \text{ or, } 12c_3 + 2c_2 + 0c_1 = 6 \tag{1.3.4}$$

Solving (1.3.1), (1.3.2), and (1.3.3) we get,

$$c_1 = 2 \ , \ c_2 = -3 \ , \ c_3 = 1$$

$$\therefore \text{ Solution: } y = 2x - 3x^2 + x^3$$

2 First Order Equations for Which Exact Solutions Are Obtainable

2.1 Exact Differential Equations and Integrating Factors

A Standard Forms of First-Order Differential Equations

The first-order differential equations may be expressed in either the **Derivative Form**

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y) \tag{2.1.1}$$

or the **Differential Form**

$$M(x, y)\,dx + N(x, y)\,dy = 0 \tag{2.1.2}$$

Example 2.1.1: Standard Forms

The equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2 + y^2}{x - y}$$

is the form (2.1.1). It may be written as

$$(x^2 + y^2)\,dx + (y - x)\,dy = 0$$

which is of the form (2.1.2).

Again, the equation

$$(\sin x + y)\,dx + (x + 3y)\,dy = 0$$

is of the form (2.1.2), which can also be written as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\sin x + y}{x + 3y}$$

B Exact Differential Equations

Definition 2.1.1: Exact Differential

Let F be a function of two real variables such that F has continuous first partial derivatives in a domain D. The total differential dF of the function F is defined by the formula

$$dF(x, y) = \frac{\partial F(x, y)}{\partial x}\,dx + \frac{\partial F(x, y)}{\partial y}\,dy$$

for all $(x, y) \in D$.

Comparing $dF(x, y)$ with the form (2.1.2), we get

$$\frac{\partial F(x, y)}{\partial x} = M(x, y) \quad \textit{and} \quad \frac{\partial F(x, y)}{\partial y} = N(x, y)$$

Example 2.1.2

Let F be a function

$$F(x, y) = xy^2 + 2x^3y$$

for all real (x, y). Then

$$\frac{\partial F(x,y)}{\partial x} = y^2 + 6x^2 y, \quad \frac{\partial F(x,y)}{\partial y} = 2xy + 2x^3$$

and the total differential dF is defined by

$$dF(x,y) = (y^2 + 6x^2 y)\,dx + (2xy + 2x^3)\,dy$$

for all real (x, y)

Definition 2.1.2: Exact Differential Equation

The expression

$$M(x,y)\,dx + N(x,y)\,dy \tag{2.1.3}$$

is called an exact differential in a domain D if there exists a function F of two real variables such that this expression equals the total differential $dF(x,y)$ for all $(x,y) \in D$. That is, expression (2.1.3) is an exact differential in D if there exists a function F such that

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) \quad \textit{and} \quad \frac{\partial F(x,y)}{\partial y} = N(x,y)$$

for all $(x,y) \in D$.
If $M(x,y)\,dx + N(x,y)\,dy$ is an exact differential, then the differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is called an **Exact Differential Equation**.

Theorem 2.1.1* (Exact Differential Equation)*:

1. *If the DE $M(x,y)\,dx + N(x,y)\,dy = 0$ is exact in D, then*

$$\forall (x,y) \in D, \quad \frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

2. *Conversely, if*

$$\forall (x,y) \in D, \quad \frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

then the DE is exact in D.

Proof (1):

$$\frac{\partial F(x,y)}{\partial x} = M(x,y) \quad , \quad \frac{\partial F(x,y)}{\partial y} = N(x,y)$$

$$\frac{\partial^2 F(x,y)}{\partial x \partial y} = \frac{\partial M(x,y)}{\partial y} \quad , \quad \frac{\partial^2 F(x,y)}{\partial x \partial y} = \frac{\partial N(x,y)}{\partial x}$$

$$\because \frac{\partial^2 F(x,y)}{\partial y \partial x} = \frac{\partial^2 F(x,y)}{\partial x \partial y}$$

$$\therefore \frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x} \quad \square$$

C The Solution of Exact Differential Equations

***Theorem 2.1.2* (Solution of Exact DE):**
If $M(x,y)\,dx + N(x,y)\,dy = 0$ is exact in domain D, then

$$\forall (x,y) \in D, \exists F(x,y) : \frac{\partial F(x,y)}{\partial x} = M(x,y) \quad \textit{and} \quad \frac{\partial F(x,y)}{\partial y} = N(x,y)$$

Then the equation may be written

$$\frac{\partial F(x,y)}{\partial x}\,dx + \frac{\partial F(x,y)}{\partial y}\,dy = 0$$

or simply,

$$dF(x,y) = 0$$

Here, $F(x,y) = c$ is a one-parameter family of solutions of this DE, where c is an arbitrary constant.

Example 2.1.3: Solve the equation

$$(3x^2 + 4xy)\,dx + (2x^2 + 2y)\,dy = 0$$

Standard Method:

$$\begin{aligned}
\frac{\partial F(x,y)}{\partial x} &= M(x,y) = 3x^2 + 4xy \\
F(x,y) &= \int (3x^2 + 4xy)\,\partial x + \phi(y) \\
&= x^3 + 2x^2y + \phi(y)
\end{aligned}$$

Again,

$$\begin{gathered}
\frac{\partial F(x,y)}{\partial y} = 2x^2 + \frac{\partial \phi(y)}{\partial y} = 2x^2 + 2y \\
\frac{\mathrm{d}\phi(y)}{\mathrm{d}y} = 2y \\
\int \frac{\mathrm{d}\phi(y)}{\mathrm{d}y}\,dy = \int 2y\,dy \\
\phi(y) = y^2 + c_0
\end{gathered}$$

Thus, we get

$$F(x,y) = x^3 + 2x^2y + y^2 + c_0$$

Hence, a one-parameter family of the solution is $F(x,y) = c_1$ or,

$$x^3 + 2x^2y + y^2 + c_0 = c_1$$

$$\boxed{x^3 + 2x^2y + y^2 = c}$$

Method of Grouping:

$$\begin{aligned}(3x^2 + 4xy)\,dx + (2x^2 + 2y)\,dy &= 0\\ 3x^2\,dx + (4xy\,dx + 2x^2\,dy) + 2y\,dy &= 0\\ d(x^3) + d(2x^2y) + d(y^2) &= d(c)\\ \boxed{x^3 + 2x^2y + y^2 = c}\end{aligned}$$

Example 2.1.4: Solve the initial-value problem

$$(2x\cos y + 3x^2y)\,dx + (x^3 - x^2\sin y - y)\,dy = 0 \quad ; \quad y(0) = 2$$

$$\begin{aligned}(2x\cos y\,dx - x^2\sin y\,dy) + (3x^2y\,dx + x^3\,dy) - y\,dy &= 0\\ d(x^2\cos y) + d(x^3y) + d(\frac{y^2}{2}) &= d(c_1)\\ 2x^2\cos y + x^3y + y^2 &= c\end{aligned}$$

Substituting $x = 0$ and $y = 2$,

$$2^2 = c$$

Hence, the solution is:

$$2x^2\cos y + x^3y + y^2 = 4$$

D Integrating Factors

Definition 2.1.3: Integrating Factor (IF)

If the DE

$$M(x, y)\,dx + N(x, y)\,dy = 0 \tag{2.1.4}$$

is not exact in a domain D but the DE

$$\mu(x, y)M(x, y)\,dx + \mu(x, y)N(x, y)\,dy = 0 \tag{2.1.5}$$

is exact in D, then $\mu(x, y)$ is called an **Integrating Factor** of the DE.

Example 2.1.5: Integrating factor

Consider the DE

$$(3y + 4xy^2)\,dx + (2x + 3x^2y)\,dy = 0 \tag{2.1.6}$$

This equation is of the form (2.1.4), where

$$M(x, y) = 3y + 4xy^2, \qquad N(x, y) = 2x + 3x^2y$$

$$\frac{\partial M(x, y)}{\partial y} = 3 + 8xy, \qquad \frac{\partial N(x, y)}{\partial x} = 2 + 6xy$$

Since

$$\frac{\partial M(x, y)}{\partial y} \neq \frac{\partial N(x, y)}{\partial x}$$

except for (x, y) such that $2xy + 1 = 0$, Equation (2.1.4) is not exact in any rectangular domain D.

Let $\mu(x,y) = x^2y$. Then the corresponding DE of the form (2.1.5) is

$$(3x^2y^2 + 4x^3y^3)\,dx + (2x^3y + 3x^4y^2)\,dy = 0$$

This equation is exact in every rectangular domain D, since

$$\frac{\partial[\mu(x,y)M(x,y)]}{\partial y} = 6x^2y + 12x^3y^2 = \frac{\partial[\mu(x,y)N(x,y)]}{\partial x}$$

For all real (x,y). Hence, $\mu(x,y) = x^2y$ is an integrating factor of Equation (2.1.6).

Example 2.1.6: Determine whether or not the following equation is exact

$$\left(\frac{x}{y^2} + x\right)\,dx + \left(\frac{x^2}{y^3} + y\right)\,dy = 0$$

$$\frac{\partial M(x,y)}{\partial y} = -\frac{x}{2y^3}$$
$$\frac{\partial N(x,y)}{\partial x} = \frac{2x}{y^3}$$

Here, $\frac{\partial M(x,y)}{\partial y} \neq \frac{\partial N(x,y)}{\partial x}$. Hence, the equation is not exact.

Example 2.1.7: Determine the constant A in the following equations such that the equation is exact

1. $(Ax^2y + 2y^2)\,dx + x^3 + 4xy\,dy = 0$
2. $\left(\frac{Ay}{x^3} + \frac{y}{x^2}\right)\,dx + \left(\frac{1}{x^2} - \frac{1}{x}\right)\,dy = 0$

1.

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial(Ax^2y + 2y^2)}{\partial y} = Ax^2 + 4y$$
$$\frac{\partial N(x,y)}{\partial x} = \frac{\partial(x^3 + 4xy)}{\partial x} = 3x^2 + 4y$$

Equating the coefficients of x^2, we get

$$\boxed{A = 3}$$

2.

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial\left(\frac{Ay}{x^3} + \frac{y}{x^2}\right)}{\partial y} = \frac{A}{x^3} + \frac{1}{x^2}$$
$$\frac{\partial N(x,y)}{\partial x} = \frac{\partial\left(\frac{1}{x^2} - \frac{1}{x}\right)}{\partial x} = -\frac{1}{2x^3} + \frac{1}{x^2}$$

Equating the coefficients of $\frac{1}{x^3}$, we get

$$\boxed{A = -\frac{1}{2}}$$

2.2 Separable Equations and Equations Reducible to this Form

A Separable Equations

Definition 2.2.1: Separable Equations

An equations of the form

$$F(x)G(y)\,dx + f(x)g(y)\,dy = 0 \tag{2.2.1}$$

is called an equation with separable variables or simply a separable equation.

***Theorem 2.2.1* (Solution of Separable Differential Equations):**
In general, the separable equations are not exact, but they possess an obvious integrating factor $\dfrac{1}{f(x)G(y)}$
Thus the equation (2.2.1) becomes

$$\frac{F(x)}{f(x)}\,dx + \frac{g(y)}{G(y)}\,dy = 0 \tag{2.2.2}$$

which is exact, because

$$\frac{\partial}{\partial y}\left(\frac{F(x)}{f(x)}\right) = 0 = \frac{\partial}{\partial x}\left(\frac{g(y)}{G(y)}\right)$$

We can write the equation (2.2.2) as

$$M(x)\,dx + N(y)\,dy = 0$$

where $M(x) = \dfrac{F(x)}{f(x)}$ *and* $N(y) = \dfrac{g(y)}{G(y)}$
A one-parameter family solution to the DE is

$$\int M(x)\,dx + \int N(y)\,dy = c \tag{2.2.3}$$

Example 2.2.1: Solve the equation

$$(x-4)y^4\,dx - x^3(y^2-3)\,dy = 0$$

The equation is separable; dividing by x^3y^4 we obtain

$$\frac{x-4}{x^3}\,dx - \frac{y^2-3}{y^4}\,dy = 0$$

$$\int (x^{-2} - 4x^{-3})\,dx - \int (y^{-2} - 3y^{-4})\,dy = 0$$

$$\boxed{-\frac{1}{x} + \frac{2}{x^2} + \frac{1}{y} - \frac{1}{y^3} = c}$$

The DE in derivative form:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x-4)y^4}{x^3(y^2-3)}$$

Here, $y = 0$ is a solution which was lost in the separation process.

Example 2.2.2: Solve the initial-value problem that consists of the DE

$$x \sin y\, dx + (x^2 + 1) \cos y\, dy = 0$$

and the initial condition $y(1) = \frac{\pi}{2}$

$$\frac{x}{x^2+1}\, dx + \frac{\cos y}{\sin y}\, dy = 0$$

$$\frac{1}{2} \int \frac{d(x^2+1)}{x^2+1} + \int \cot y\, dy = 0$$

$$\frac{1}{2} \ln |x^2 + 1| + \ln |\sin y| = \ln |c_1|$$

$$\ln |(x^2+1) \sin^2 y| = \ln |c|$$

$$\therefore (x^2+1) \sin^2 y = c$$

Applying the initial condition, we get

$$2 \sin^2 \frac{\pi}{2} = c \text{ or, } c = 2$$

Thus, the solution is

$$\boxed{(x^2+1) \sin^2 y = 2}$$

B Homogeneous Equations

Definition 2.2.2: Homogeneous Equations

The first-orger differential equation

$$M(x, y)\, dx + N(x, y)\, dy = 0$$

is said to be homogeneous if, when written in the derivative form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$$

there exists a function g such that $f(x, y)$ can be expressed in the form $g(y/x)$

Example 2.2.3

The DE

$$x^2 - 3y^2\, dx + 2xy\, dy = 0$$

is homogeneous. To see this, we first write the derivative form of the equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3y^2 - x^2}{2xy} = \frac{3y}{2x} - \frac{x}{2y}$$

We see that the DE can be written as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{2}\left(\frac{y}{x}\right) - \frac{1}{2}\left(\frac{1}{y/x}\right)$$

in which the right side of the equation is of the form $g(y/x)$ for a certain function g.

Example 2.2.4: The equation

$$(y + \sqrt{x^2 + y^2})\, dx - x\, dy = 0$$

is homogeneous.

Derivative form:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{y + \sqrt{x^2 + y^2}}{x} \\
&= \frac{y}{x} + \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2}} \\
&= \frac{y}{x} + \sqrt{1 + \left(\frac{y}{x}\right)^2} = g\left(\frac{y}{x}\right)
\end{aligned}$$

Definition 2.2.3: Homogeneous Equation of degree n

A function F is called homogeneous of degree n if

$$F(tx, ty) = t^n F(x, y)$$

This means that if the tx and ty are substituted for x and y respectively in $F(x, y)$, and if t^n is then factored out, the other factor that remains is the original expression $F(x, y)$ itself.

For example, the function given by $F(x, y) = x^2 + y^2$ is homogeneous of degree 2, since

$$F(tx, ty) = (tx)^2 + (ty)^2 = t^2(x^2 + y^2) = t^2 F(x, y)$$

Now, suppose both $M(x, y)$ and $N(x, y)$ in the DE

$$M(x, y)\, dx + N(x, y)\, dy = 0$$

are homogeneous of the same degree n. Since $M(tx, ty) = t^n M(x, y)$, for $t = \frac{1}{x}$, we have

$$M\left(1, \frac{y}{x}\right) = \left(\frac{1}{x}\right)^n M(x, y)$$

$$M(x, y) = \left(\frac{1}{x}\right)^{-n} M\left(1, \frac{y}{x}\right)$$

Similarly,

$$N(x, y) = \left(\frac{1}{x}\right)^{-n} N\left(1, \frac{y}{x}\right)$$

Now, writing the DE in derivative form, we get

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{M(x, y)}{N(x, y)} \\
&= -\frac{\left(\frac{1}{x}\right)^{-n} M\left(1, \frac{y}{x}\right)}{\left(\frac{1}{x}\right)^{-n} N\left(1, \frac{y}{x}\right)} \\
&= -\frac{M\left(1, \frac{y}{x}\right)}{N\left(1, \frac{y}{x}\right)} \\
&= g\left(\frac{y}{x}\right)
\end{aligned}$$

Note:-

If $M(x, y)$ and $N(x, y)$ in

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

are both homogeneous functions of the same degree n, then the differential equation is a homogeneous differential equation.

Theorem 2.2.2:

If

$$M(x, y)\,dx + N(x, y)\,dy = 0 \tag{2.2.4}$$

is a homogeneous equation, then the change of variables $y = vx$ transforms the equation into a separable equation in the variables v and x.

Proof:

Since $M(x, y)\,dx + N(x, y)\,dy = 0$ is homogeneous, it may be written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g\left(\frac{y}{x}\right)$$

Let $y = vx$. Then

$$\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$$

and the initial equation becomes

$$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = g(v)$$

or,

$$[v - g(v)]\,dx + x\,dv = 0$$

This equation is separable. Separating the variables we obtain

$$\frac{dv}{v - g(v)} + \frac{dx}{x} = 0 \quad \square \tag{2.2.5}$$

Theorem 2.2.3 **(Solution of a Homogeneous Differential Equation)***: To solve a DE of the form (2.2.4), we let $y = vx$ and transform the homogeneous equation into a separable equation of the form (2.2.5). From this, we have*

$$\int \frac{dv}{v - g(v)} + \int \frac{dx}{x} = c$$

where c is an arbitrary constant. Letting $F(v)$ denote

$$\int \frac{dv}{v - g(v)}$$

and returning to the original dependent variable y, the solution takes the form

$$F\left(\frac{y}{x}\right) + \ln|x| = c$$

Example 2.2.5: Solve the equation

$$(x^2 - 3y^2)\, dx + 2xy\, dy = 0$$

Derivative form:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{2y} + \frac{3y}{2x}$$

Letting $y = vx$, we get

$$\begin{aligned} v + x\frac{\mathrm{d}v}{\mathrm{d}x} &= -\frac{1}{2v} + \frac{3v}{2} \\ x\frac{\mathrm{d}v}{\mathrm{d}x} &= \frac{v^2 - 1}{2v} \\ \frac{2v\, dv}{v^2 - 1} &= \frac{dx}{x} \end{aligned}$$

Integrating, we find

$$\begin{aligned} \ln|v^2 - 1| &= \ln|x| + \ln|c| \\ |v^2 - 1| &= |cx| \\ |\frac{y^2}{x^2} - 1| &= |cx| \\ |y^2 - x^2| &= x^2|cx| \end{aligned}$$

For $y \geq x \geq 0$, it can be written as

$$\boxed{y^2 - x^2 = cx^3}$$

Example 2.2.6: Solve the initial-value problem

$$(y + \sqrt{x^2 + y^2})\, dx - x\, dy = 0, y(1) = 0$$

Derivative form:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} + \sqrt{1 + \left(\frac{y}{x}\right)^2}$$

Letting $y = vx$, we get

$$\begin{aligned} v + x\frac{\mathrm{d}v}{\mathrm{d}x} &= v + \sqrt{1 + v^2} \\ \frac{dv}{\sqrt{1 + v^2}} &= \frac{dx}{x} \\ \ln|v + \sqrt{1 + v^2}| &= ln|x| + ln|c| \\ v + \sqrt{v^2 + 1} &= cx \\ \frac{y}{x} + \frac{1}{x}\sqrt{y^2 + x^2} &= cx \\ y + \sqrt{x^2 + y^2} &= cx^2 \end{aligned}$$

Applying the initial condition, we get

$$0 + \sqrt{1} = c \cdot 1 \text{ or, } c = 1$$

Hence, the solution:

$$\boxed{y + \sqrt{x^2 + y^2} = x^2} \text{ or, } \boxed{y = \frac{1}{2}(x^2 - 1)}$$

Exercise 2.1: Solve the following differential equations

1. $(xy + 2x + y + 2)\, dx + (x^2 + 2x)\, dy = 0$
2. $(2x \cos y + 3x^2 y)\, dx + (x^3 - x^2 - y)\, dy = 0, y(0) = 2$
3. $(e^v + 1) \cos u\, du + e^v (\sin u + 1)\, dv = 0$
4. $(x + 4)(y^2 + 1)\, dx + y(x^2 + 3x + 2)\, dy = 0$
5. $(2xy + 3y^2)\, dx - (2xy + x^2)\, dy = 0$
6. $(x + y)\, dx - x\, dy = 0$
7. $v^3\, du + (u^3 - uv^2)\, dv = 0$
8. $\left(x \tan \frac{y}{x} + y\right)\, dx - x\, dy = 0$
9. $(2s^2 + 2st + t^2)\, ds + (s^2 + 2st - t^2)\, dt = 0$
10. $(x^3 + y^2 \sqrt{x^2 + y^2})\, dx - xy\sqrt{x^2 + y^2}\, dy = 0$
11. $(\sqrt{x + y} + \sqrt{x - y})\, dx + (\sqrt{x - y} - \sqrt{x + y})\, dy = 0$
12. $(3x + 8)(y^2 + 4)\, dx - 4y(x^2 + 5x + 6)\, dy = 0, y(1) = 2$
13. $(2x^2 + 2xy + y^2)\, dx + (x^2 + 2xy)\, dy = 0$

1.

$$\begin{aligned} (xy + 2x + y + 2)\, dx + (x^2 + 2x)\, dy &= 0 \\ (x + 1)(y + 2)\, dx + x(x + 2)\, dy &= 0 \\ \int \frac{x + 1}{x(x + 2)}\, dx + \int \frac{dy}{y + 2} &= 0 \\ \frac{1}{2} \ln |x^2 + 2x| + \ln |y + 2| &= \ln |c_1| \\ \ln |(x^2 + 2x)(y + 2)^2| &= \ln |c| \\ \boxed{(x^2 + 2x)(y + 2)^2 = c} & \end{aligned}$$

2.

$$(2x \cos y + 3x^2 y)\, dx + (x^3 - x^2 \sin^2 y - y)\, dy = 0$$

$$\begin{aligned} F(x, y) &= \int (2x \cos y + 3x^2 y)\, \partial x + \phi(y) \\ &= x^2 \cos y + x^3 y + \phi(y) \end{aligned}$$

Now,

$$\frac{\partial F(x,y)}{\partial y} = x^3 - x^2 \sin y - y = x^3 - x^2 \sin y + \frac{\mathrm{d}}{\mathrm{d}x}\phi(y)$$

$$\therefore \phi(y) = -\int y\, dy = -\frac{y^2}{2} + c_0$$

$$F(x,y) = x^2 \cos y + x^3 y - \frac{y^2}{2} + c_0 = c_1$$

$$2x^2 \cos y + 2x^3 y - y^2 = c$$

Applying initial value,

$$c = -4$$

$$\boxed{2x^2 \cos y + 2x^3 y - y^2 + 4 = 0}$$

3.

$$(e^v + 1) \cos u\, du + e^v (\sin u + 1)\, dv = 0$$

$$(e^v \cos u\, du + e^v \sin u\, dv) + \cos u\, du + e^v\, dv = 0$$

$$d(e^v \sin u) + d(\sin u) + de^v = d(c)$$

$$\boxed{\sin u + e^v(\sin u + 1) = c}$$

4.

$$(x+4)(y^2+1)\, dx + y(x^2 + 3x + 2)\, dy = 0$$

$$\int \frac{x+4}{x^2+3x+2}\, dx + \int \frac{y}{y^2+1}\, dy = 0$$

$$\int \frac{x+4}{(x+2)(x+1)}\, dx + \frac{1}{2}\int \frac{2y}{y^2+1}\, dy = 0$$

$$\int \frac{3}{x+1}\, dx - \int \frac{2}{x+2}\, dx + \frac{1}{2}\ln|y^2+1|$$

$$3\ln|x+1| - 2\ln|x+2| + \frac{1}{2}\ln|y^2+1| = \ln|c_1|$$

$$\ln\left|\frac{(x+1)^6}{(x+2)^4} \cdot (y^2+1)\right| = \ln|c|$$

$$\boxed{(x+6)^6(y^2+1) = c(x+2)^4}$$

5.

$$(2xy + 3y^2)\, dx - (2xy + x^2)\, dy = 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2xy + 3y^2}{2xy + x^2} = \frac{2\left(\frac{y}{x}\right) + 3\left(\frac{y}{x}\right)^2}{2\left(\frac{y}{x}\right) + 1}$$

Letting $y = vx$, we get

$$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{2 + 3v^2}{2v + 1}$$

$$x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{v^2 + v}{2v + 1}$$

$$\int \frac{2v + 1}{v^2 + v}\, dv = \int \frac{dx}{x}$$

$$\int \frac{d(v^2 + v)}{v^2 + v} = \int \frac{dx}{x}$$

$$\ln|v^2 + v| = \ln|cx|$$

$$\frac{y^2}{x^2} + \frac{y}{x} = cx$$

$$\boxed{y^2 + xy = cx^3}$$

6.

$$(x + y)\, dx - x\, dy = 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x + y}{x} = 1 + \frac{y}{x}$$

Letting $y = vx$, we get

$$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = 1 + v$$

$$\int dv = \int \frac{dx}{x}$$

$$\frac{y}{x} = \ln|cx|$$

$$\boxed{cx = e^{y/x}}$$

7.

$$v^3\, du + (u^3 - uv^2)\, dv = 0$$

$$\frac{\mathrm{d}u}{\mathrm{d}v} = \frac{uv^2 - u^3}{v^3} = \frac{u}{v} - \left(\frac{u}{v}\right)^3$$

Letting $u = wv$, we get

$$w + v\frac{\mathrm{d}w}{\mathrm{d}v} = w - w^3$$

$$-\int \frac{dw}{w^3} = \int \frac{dv}{v}$$

$$\frac{1}{2w^2} = \ln|v| + c_1$$

$$\boxed{v^2 = u^2(\ln v^2 + c)}$$

8.

$$\left(x \tan\frac{y}{x} + y\right)\, dx - x\, dy = 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \tan\frac{y}{x} + \frac{y}{x}$$

Letting $y = vx$, we get

$$v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \tan v + v$$

$$\int \frac{dv}{\tan v} = \int \frac{dx}{x}$$

$$\ln|\sin v| = \ln|cx|$$

$$\boxed{\sin\frac{y}{x} = cx}$$

9.

$$(2s^2 + 2st + t^2)\, ds + (s^2 + 2st - t^2)\, dt = 0$$

$$\begin{aligned}\frac{\mathrm{d}s}{\mathrm{d}t} &= \frac{t^2 - 2st - s^2}{2s^2 + 2st + t^2} \\ &= \frac{1 - 2\left(\dfrac{s}{t}\right) - \left(\dfrac{s}{t}\right)^2}{2\left(\dfrac{s}{t}\right)^2 + 2\left(\dfrac{s}{t}\right) + 1}\end{aligned}$$

Letting $s = vt$, we get

$$
\begin{aligned}
v + t\frac{\mathrm{d}v}{\mathrm{d}t} &= \frac{1-2v-v^2}{2v^2+2v+1} \\
t\frac{\mathrm{d}v}{\mathrm{d}t} &= -\frac{2v^3+3v^2+3v-1}{2v^2+2v+1} \\
-\int \frac{2v^2+2v+1}{2v^3+3v^2+3v-1}\,dv &= \int \frac{dt}{t} \\
-\frac{1}{3}\int \frac{d(2v^3+3v^2+3v-1)}{2v^3+3v^2+3v-1} &= \ln|t| + \ln|c_1|
\end{aligned}
$$

$$\boxed{2v^3+3v^2+3v-1 = \frac{c}{t^3}}$$

10.

$$(x^3 + y^2\sqrt{x^2+y^2})\,dx - xy\sqrt{x^2+y^2}\,dy = 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^3 + y^2\sqrt{x^2+y^2}}{xy\sqrt{x^2+y^2}} = \frac{1 + \left(\frac{y}{x}\right)^2\sqrt{1+\left(\frac{y}{x}\right)^2}}{\frac{y}{x}\sqrt{1+\left(\frac{y}{x}\right)^2}}$$

Letting $y = vx$,

$$
\begin{aligned}
v + x\frac{\mathrm{d}v}{\mathrm{d}x} &= \frac{1+v^2\sqrt{1+v^2}}{v\sqrt{1+v^2}} \\
x\frac{\mathrm{d}v}{\mathrm{d}x} &= \frac{1}{v\sqrt{1+v^2}} \\
\int v\sqrt{1+v^2}\,dv &= \int \frac{dx}{x} \\
\frac{1}{2}\int \sqrt{1+v^2}\,d(1+v^2) &= \ln|c_1x| \\
(1+v^2)^{3/2} &= 3\ln|c_1x| \\
\left(1+\frac{y^2}{x^2}\right)\sqrt{1+\frac{y^2}{x^2}} &= \ln|cx^3|
\end{aligned}
$$

$$\boxed{(x^2+y^2)\sqrt{x^2+y^2} = x^3\ln|cx^3|}$$

11.

$$(\sqrt{x+y} + \sqrt{x-y})\,dx + (\sqrt{x-y} - \sqrt{x+y})\,dy = 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sqrt{\frac{x}{y}+1} - \sqrt{x-y} - 1}{\sqrt{\frac{x}{y}+1} + \sqrt{\frac{x}{y}-1}}$$

Letting $x = vy$,

$$
\begin{aligned}
v + y\frac{\mathrm{d}v}{\mathrm{d}y} &= \frac{\sqrt{v+1}-\sqrt{v-1}}{\sqrt{v+1}+\sqrt{v-1}} \\
&= \frac{v+1+v-1-2\sqrt{v^2-1}}{v+1-v+1} \\
v + y\frac{\mathrm{d}v}{\mathrm{d}y} &= v - \sqrt{v^2-1} \\
\int \frac{dv}{\sqrt{v^2-1}} &= -\int \frac{dy}{y} \\
\ln|v+\sqrt{v^2-1}| &= \ln\left|\frac{c}{y}\right| \\
\frac{x}{y} + \sqrt{\frac{x^2}{y^2}-1} &= \frac{c}{y}
\end{aligned}
$$

$$\boxed{x + \sqrt{x^2 - y^2} = c}$$

12.

$$(3x+8)(y^2+4)\,dx + 4y(x^2+5x+6)\,dy = 0$$

$$\frac{3x+8}{x^2+5x+6}\,dx - \frac{4y}{y^2+4}\,dy = 0$$

Here,

$$\frac{3x+8}{(x+3)(x+2)} = \frac{1}{x+3} + \frac{2}{x+2}$$

$$\therefore \int \frac{dx}{x+3} + 2\int \frac{dx}{x+2} - 2\ln|y^2+4| = c_2$$

$$\ln|x+3| + 2\ln|x+2| = 2\ln|c_1(y^2+4)|$$

$$(x+3)(x+2)^2 = c(y^2+4)$$

Applying initial value,

$$c = \frac{9}{16}$$

$$\boxed{16(x+3)(x+2)^2 = 9(y^2+4)}$$

13.

$$(2x^2+2xy+y^2)\,dx + (x^2+2xy)\,dy = 0$$

$$(2xy\,dx + x^2\,dy) + (y^2\,dx + 2xy\,dy) + 2x^2\,dx = 0$$

$$d(x^2y) + d(xy^2) + d\left(\frac{2}{3}x^3\right) = d(c_1)$$

$$x^2y + xy^2 + \frac{2}{3}x^3 = c_1$$

$$\boxed{2x^3 + 3x^2y + 3xy^2 = c}$$

Exercise 2.2: Show that the homogeneous equation

$$(Ax^2 + Bxy + Cy^2)\, dx + (Dx^2 + Exy + Fy^2)\, dy = 0$$

is exact if and only if $B = 2D$ and $E = 2C$.

The equation is exact if and only if

$$\frac{\partial(Ax^2 + Bxy + Cy^2)}{\partial y} = \frac{\partial(Dx^2 + Exy + Fy^2)}{\partial x}$$

$$Bx + 2Cy = 2Dx + Ey$$

Equating the coefficients of x, $B = 2D$
Equating the coefficients of y, $E = 2C$

2.3 Linear Equations and Bernoulli Equations

A Linear Equation

Definition 2.3.1: Linear Equation

A first-order ordinary differential equation is linear in the dependent variable y and independent variable x if it is, or can be, written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x) \tag{2.3.1}$$

Equation (2.3.1) can also be written as

$$[P(x)y - Q(x)]\, dx + \, dy = 0 \tag{2.3.2}$$

Here,

$$\frac{\partial}{\partial y} M(x,y) = P(x,y) \quad \text{and} \quad \frac{\partial}{\partial x} N(x,y) = 0$$

The equation is not exact. So we multiply both sides of (2.3.2) by an integrating factor:

$$[\mu(x)P(x)y - \mu(x)Q(x)]\, dx + \mu(x)\, dy = 0$$

Now,

$$\begin{aligned}
\frac{\partial}{\partial y}\,[\mu(x)M(x,y)] &= \frac{\partial}{\partial x}\,[\mu(x)N(x,y)] \\
\frac{\partial}{\partial y}\,[\mu(x)P(x)y - \mu(x)Q(x)] &= \frac{\partial}{\partial x}\,[\mu(x)] \\
\mu P(x) &= \frac{d}{dx}\,\mu \\
\int P(x)\, dx &= \int \frac{d\mu}{\mu} \\
\ln|\mu| &= \int P(x)\, dx
\end{aligned}$$

$$\boxed{\mu = e^{\int P(x)\, dx}}$$

Theorem 2.3.1 **(Solution of Linear Differential Equation)*:***
The linear differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$$

has an integrating factor of the form

$$\mu = e^{\int P(x)\, dx} \tag{2.3.3}$$

A one-parameter family of solution of this equation is

$$\mu\, y = \int \mu\, Q(x)\, dx + c$$

or

$$y\, e^{\int P(x)\, dx} = \int e^{\int P(x)\, dx}\, Q(x)\, dx + c \tag{2.3.4}$$

That is,

$$y = e^{-\int P(x)\, dx} \left[\int e^{\int P(x)\, dx}\, Q(x)\, dx + c \right] \tag{2.3.5}$$

Example 2.3.1: Solve the Linear Differential Equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \left(\frac{2x+1}{x}\right) y = e^{-2x}$$

$$P(x) = 2 + \frac{1}{x}$$

$$\therefore \mathrm{IF} = e^{\int P(x)\, dx} = e^{2x + \ln x} = xe^{2x}$$

Now,

$$\begin{aligned} xe^{2x}\frac{\mathrm{d}y}{\mathrm{d}x} + e^{2x}(2x+1)y &= x \\ \frac{d}{dx}\left(xe^{2x}y\right) &= x \\ xe^{2x}y &= \frac{x^2}{2} + c_1 \end{aligned}$$

$$\boxed{y = \frac{1}{2}xe^{-2x} + \frac{c}{x}e^{-2x}}$$

Example 2.3.2: Solve the initial value problem

$$(x^2+1)\frac{\mathrm{d}y}{\mathrm{d}x} + 4xy = x \quad , \quad y(2) = 1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \left(\frac{4x}{x^2+1}\right) y = \frac{x}{x^2+1}$$

$$\therefore\ \mathrm{IF}\ = \exp\left(2\int \frac{2x}{x^2+1}\, dx\right) = \exp\left(2\ln|x^2+1|\right) = (x^2+1)^2$$

Therefore, the solution is

$$\begin{aligned}(x^2+1)^2 y &= \int (x^2+1)^2 \cdot \frac{x}{x^2+1}\,dx + c_1 \\ &= \int (x^3+x)\,dx + c_1 \\ &= \frac{x^4}{4} + \frac{x^2}{2} + c\end{aligned}$$

Applying initial value,

$$c = 19$$

$$\boxed{(x^2+1)^2 y = \frac{x^4}{4} + \frac{x^2}{2} + 19}$$

Example 2.3.3: Solve the linear DE

$$y^2\,dx + (3xy-1)\,dy = 0$$

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}y} &= -\frac{3x-1}{y^2} = -\frac{3}{y}x + \frac{1}{y^2} \\ \therefore \frac{\mathrm{d}x}{\mathrm{d}y} + \frac{3}{y}x &= \frac{1}{y^2} \\ \therefore\ \text{IF } &= e^{3\int \frac{dy}{y}} = y^3\end{aligned}$$

Now,

$$\begin{aligned}y^3\frac{\mathrm{d}x}{\mathrm{d}y} + 3xy^2 &= y \\ y^3\,dx + (3xy^2 - y)\,dy &= 0 \\ (y^3\,dx + 3xy^2\,dy) - y\,dy &= 0 \\ d(xy^3) - d\left(\frac{y^2}{2}\right) &= d(c_1) \\ 2xy^3 - y^2 &= c\end{aligned}$$

$$\boxed{x = \frac{1}{2y} + \frac{c}{y^3}}$$

B Bernoulli Equations

Definition 2.3.2: Bernoulli Equations

An equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)y^n$$

is called a Bernoulli Equation.

If $n = 0$ or $n = 1$, the equation is simply a linear DE. However, in general case in which $n \neq 0$ or $n \neq 1$, we must proceed in a different manner.

***Theorem 2.3.2* (Transformation of Bernoulli Equation to Linear Equation):** *Suppose $n \neq 0$ or $n \neq 1$. Then the transformation*

$$v = y^{1-n}$$

reduces the Bernoulli Equation to a linear equation in v.

Proof:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)y^n$$

$$y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y^{1-n} = Q(x)$$

Substituting $v = y^{1-n}$,

$$\frac{\mathrm{d}v}{\mathrm{d}x} = (1-n)y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\frac{1}{1-n}\frac{\mathrm{d}v}{\mathrm{d}x} + P(x)v = Q(x)$$

$$\frac{\mathrm{d}v}{\mathrm{d}x} + (1-n)P(x)v = (1-n)Q(x)$$

Letting $P_1(x) = (1-n)P(x)$ and $Q_1(x) = (1-n)Q(x)$ we get,

$$\frac{\mathrm{d}v}{\mathrm{d}x} + P_1(x)v = Q_1(x) \quad \square$$

Example 2.3.4:

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y = xy^3$$

$$y^{-3}\frac{\mathrm{d}y}{\mathrm{d}x} + y^{-2} = x$$

Letting $v = y^{-2}$,

$$\begin{aligned} \frac{\mathrm{d}v}{\mathrm{d}x} &= -2y^{-3}\frac{\mathrm{d}y}{\mathrm{d}x} \\ -\frac{1}{2}\frac{\mathrm{d}v}{\mathrm{d}x} + v &= x \\ \frac{\mathrm{d}v}{\mathrm{d}x} - 2v &= x \end{aligned}$$

$$\therefore\ \text{IF} = e^{-\int 2\,dx} = e^{-2x}$$

$$\begin{aligned} e^{-2x}v &= \int e^{-2x}(-2x)\,dx + c \\ e^{-2x}\frac{1}{y^2} &= -2\int xe^{-2x}\,dx + c \\ &= xe^{-2x} + \frac{1}{2}e^{-2x} + c \end{aligned}$$

$$\boxed{\frac{1}{y^2} = x + \frac{1}{2} + ce^{2x}}$$

Exercise 2.3: Solve the Differential Equations

1. $x^4\frac{\mathrm{d}y}{\mathrm{d}x} + 2x^3y = 1$
2. $\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = 3x^2e^{-3x}$
3. $(x^2 + x - 2)\frac{\mathrm{d}y}{\mathrm{d}x} + 3(x+1)y = x - 1$
4. $y\,dx + (xy^2 + x - y)\,dy = 0$
5. $\cos\theta\,dr + (r\sin\theta - \cos^4\theta)\,d\theta = 0$
6. $(y\sin 2x - \cos x)\,dx + (1 + \sin^2 x)\,dy = 0$
7. $x\frac{\mathrm{d}y}{\mathrm{d}x} + y = -2x^6y^4$
8. $\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{y}{x} = -\frac{y^2}{x}$
9. $e^x\left[y - 3(e^x+1)^2\right]\,dx + (e^x + 1)\,dy = 0$
10. $\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{y}{2x} = \frac{x}{y^3}$

Handwritten solutions on notebook. No time for updating in LaTeXnow. If anyone is interested, contact; I can provide handwritten solutions.

Exercise 2.4: Consider the equation

$$a\frac{\mathrm{d}y}{\mathrm{d}x} + by = ke^{-\lambda x}$$

where a, b, and k are positive constants and λ is a non-negative constant.

(a) Solve this equation.

(b) Show that if $\lambda = 0$, every solution approaches $\frac{k}{b}$ as $x \to \infty$, but if $\lambda > 0$ every solution approaches 0 as $x \to \infty$.

Handwritten solutions on notebook. No time for updating in LaTeXnow. If anyone is interested, contact; I can provide handwritten solutions.

Exercise 2.5:

(a) Prove that if f and g are two different solutions of

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x) \tag{A}$$

then $f - g$ is a solution of the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = 0$$

(b) Thus show that if f and g are two different solutions of Equation (A) and c is an arbitrary constant, then

$$c(f - g) + f$$

is a one-parameter family of solutions of (A).

Handwritten solutions on notebook. No time for updating in LaTeXnow. If anyone is interested, contact; I can provide handwritten solutions.

2.4 Special Integrating Factors and Transformations

The five basic types of differential equations we've encountered so far:

- Exact $\to$ Direct solution
- Separable $\to$ Integrating Factor $\to$ Exact DE
- Homogeneous $\to$ Integrating Factor $\to$ Exact DE
- Linear $\to$ Appropriate Transformation $\to$ Separable DE
- Bernoulli $\to$ Appropriate Transformation $\to$ Linear DE

How to solve a DE that is not of one of the five types?

1. Either multiply by proper IF $\rightarrow$ Exact DE
2. Or, appropriate transformation $\rightarrow$ One of the five basic forms.

A Finding Integrating Factors

Separable equations always possess integrating factors that can be determined by immediate inspection. However, some non-separable equations also possess such integrating factors that can be determined.
Suppose a non-exact DE

$$M(x,y)\,dx + N(x,y)\,dy = 0 \tag{2.4.1}$$

has an IF $\mu(x,y)$. Then the equation is

$$\mu(x,y)M(x,y)\,dx + \mu(x,y)N(x,y)\,dy = 0 \tag{2.4.2}$$

is exact. Now, we can say the equation (2.4.2) is exact if and only if

$$\frac{\partial}{\partial y}\,\left[\mu(x,y)M(x,y)\right] = \frac{\partial}{\partial x}\,\left[\mu(x,y)N(x,y)\right]$$

$$M(x,y)\frac{\partial \mu(x,y)}{\partial y} + \mu(x,y)\frac{\partial M(x,y)}{\partial y} = N(x,y)\frac{\partial \mu(x,y)}{\partial x} + \mu(x,y)\frac{\partial N(x,y)}{\partial x}$$

$$N(x,y)\frac{\partial \mu}{\partial x} - M(x,y)\frac{\partial \mu}{\partial x} = \left[\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right]\mu \tag{2.4.3}$$

Equation (2.4.3) is a PDE fo rthe general IF μ, and we're in no position to attempt to solve such an equation. Let's attepts to determine IF of certain special types instead.
If M and N are functions of x and y, but the IF μ depends only upon x, then equation (2.4.3) reduces to

$$N(x,y)\,\frac{\mathrm{d}\mu(x)}{\mathrm{d}x} = \mu(x)\,\frac{\partial M(x,y)}{\partial y} - \mu(x)\,\frac{\partial N(x,y)}{\partial x}$$

or,

$$\frac{d\mu(x)}{\mu(x)} = \frac{1}{N(x,y)}\left[\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right]\,dx \tag{2.4.4}$$

Here, if

$$\frac{1}{N(x,y)}\left[\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right]$$

depends upon x only, equation (2.4.4) is a separable ordinary equation in the single independent, equation (2.4.4) is a separable ordinary equation in the single independent variable x and the single dependent variable μ. In this case, we may integrate to obtain the IF

$$\mu(x) = \exp\left\{\int \frac{1}{N(x,y)}\left[\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right]\,dx\right\}$$

Likewise, if

$$\frac{1}{M(x,y)}\left[\frac{\partial N(x,y)}{\partial x} - \frac{\partial M(x,y)}{\partial y}\right]$$

depends upon y only, then we may obtain an IF that depends only on y.

***Theorem 2.4.1* (Integrating Factors):**
Consider the differential equation

$$M(x,y)\,dx + N(x,y)\,dy = 0 \tag{2.4.5}$$

If

$$\frac{1}{N(x,y)}\left[\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right] \tag{2.4.6}$$

depends upon x only, then IF

$$\mu(x) = \exp\left\{\int \frac{1}{N(x,y)}\left[\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x}\right] dx\right\} \tag{2.4.7}$$

And if

$$\frac{1}{M(x,y)}\left[\frac{\partial N(x,y)}{\partial x} - \frac{\partial M(x,y)}{\partial y}\right] \tag{2.4.8}$$

depends upon y only, then IF

$$\mu(y) = \exp\left\{\int \frac{1}{M(x,y)}\left[\frac{\partial N(x,y)}{\partial x} - \frac{\partial M(x,y)}{\partial y}\right] dy\right\} \tag{2.4.9}$$

Example 2.4.1:

$$(2x^2 + y)\,dx + (x^2 y - x)\,dy = 0$$

This equation is not any of the five basic types of differential equations. We can apply Theorem 2.4.1 in this case. Here, $M(x,y) = 2x^2 + y$ and $N(x,y) = x^2y - x$, and the equation (2.4.6) becomes

$$\frac{1}{x^2y - x}[1 - (2xy - 1)] = \frac{2(1 - xy)}{x(xy - y)} = -\frac{2}{x}$$

This depends upon x only, so

$$\text{IF } = \exp\left(-\int \frac{2}{x}\,dx\right) = \exp\bigl(-2\ln|x|\bigr) = \frac{1}{x^2}$$

Thus we obtain the equation

$$\left(2 + \frac{y}{x^2}\right)\,dx + \left(y - \frac{1}{x}\right)\,dy = 0$$

This equation is exact, and the solution is

$$2x + \frac{y^2}{2} - \frac{y}{x} = c$$

B A Special Transformation

***Theorem 2.4.2* (A Special Transformation):**
Consider the equation

$$(a_1x + b_1y + c_1)\,dx + (a_2x + b_2y + c_2)\,dy = 0 \tag{2.4.10}$$

where $a_1, b_1, c_1, a_2, b_2, c_2$ are constants.

Case 1: *If $\frac{a_2}{a_1} \neq \frac{b_2}{b_1}$, then the transformation*

$$\begin{aligned} x &= X + h \\ y &= Y + k \end{aligned}$$

where (h, k) is the solution of the system

$$\begin{aligned} a_1h + b_1k + c_1 &= 0 \\ a_2h + b_2k + c_2 &= 0 \end{aligned}$$

reduces the equation (2.4.10) to the Homogeneous Equation

$$(a_1X + b_1Y)\,dX + (a_2X + b_2Y)\,dY = 0 \tag{2.4.11}$$

Case 2: *If $\frac{a_2}{a_1} = \frac{b_2}{b_1}$, then the transformation*

$$z = a_1x + b_1y$$

reduces the equation (2.4.10) to a separable equation in the variables x and z.

Example 2.4.2:

$$(x - 2y + 1)\,dx + (4x - 3y - 6)\,dy = 0$$

Here,

$$\frac{a_2}{a_1} = 4 \neq \frac{3}{2} = \frac{b_2}{b_1}$$

Therefore, we make the transformation

$$\begin{aligned} x &= X + h \\ y &= Y + k \end{aligned}$$

where (h, k) is the solution of the system

$$\begin{aligned} h - 2k + 1 &= 0 \\ 4h - 3k - 6 &= 0 \end{aligned}$$

The solution of the system is $(3, 2)$, and so the transformation is

$$\begin{aligned} x &= X + 3 \\ y &= Y + 2 \end{aligned}$$

This reduces the given equation to the homogeneous equation

$$(X - 2Y)\,dX + (4X - 3Y)\,dY = 0$$

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{1-2(Y/X)}{3(Y/X)-4}$$

Letting $Y = vX$,

$$\begin{aligned}
v + X\frac{\mathrm{d}v}{\mathrm{d}X} &= \frac{1-2v}{3v-4} \\
\int \frac{3v-4}{3v^2-2v-1}\,dv &= -\int \frac{dX}{X} \\
\frac{1}{2}\int \frac{d(3v^2-2v-1)}{3v^2-2v-1} - 3\int \frac{dv}{3v^2-2v-1} &= -\int \frac{dX}{X} \\
\frac{1}{2}\int \frac{d(3v^2-2v-1)}{3v^2-2v-1} - 3\int \left[\frac{1}{4}\int \frac{dv}{v-1} - \frac{3}{4}\int \frac{dv}{3v+1}\right] &= -\int \frac{dX}{X} \\
\frac{1}{2}\ln|3v^2-2v-1| - \frac{3}{4}\ln|v-1| + \frac{9}{4}\ln|3v+1| + \ln|X| &= \ln|c_1| \\
\ln\left|X^4 \cdot \frac{(v-1)^2(3v+1)^{11}}{(v-1)^3}\right| &= \ln|c| \\
|3Y+X|^{11} &= X^6 c|Y-X|
\end{aligned}$$

$$\boxed{|x+3y-9|^{11} = c(x-3)^6|y-x+1|}$$

Example 2.4.3:

$$(x+2y+3)\,dx + (2x+4y-1)\,dy = 0$$

Here,

$$\frac{a_2}{a_1} = 2 = \frac{b_2}{b_1}$$

Therefore, we apply the transformation

$$z = x + 2y$$

$$\begin{aligned}
\therefore (z+3)\,dx + (2z-1)\left(\frac{dz-dx}{2}\right) &= 0 \\
7\,dx + (2z-1)\,dz &= 0 \\
7x + z^2 - z &= c \\
7x + x^2 + 4y^2 + 4xy - x - 2y &= c
\end{aligned}$$

$$\boxed{x^2 + 4xy + 4y^2 + 6x - 2y = c}$$

3 Explicit Methods of Solving Higher-Order Linear Differential Equations

3.1 Basic Theory of Linear Differential Equations

A Definition and Basic Existence Theorem

Definition 3.1.1: Linear ODE and Homogeneous DE of Order n

A **linear ordinary differential equation of order** n in the dependent variable y and the independent variable x is an equation that is in, or can be expressed in the form

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)}n + \cdots + a_{n-1}(x)y' + a_n(x)y = F(x) \tag{3.1.1}$$

where a_0 is not identically zero. In the equation, $a_0, a_1, \cdots, a_n$ and F are continuous real functions on a real interval $a \leq x \leq b$ and that $a_0(x) \neq 0$ for any x on $a \leq x \leq b$. The $F(x)$ is called the nonhomogeneous term. If F is identically zero, Equation (3.1.1) reduces to

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)}n + \cdots + a_{n-1}(x)y' + a_n(x)y = 0 \tag{3.1.2}$$

Equation (3.1.2) is a **homogeneous differential equation of order** n.

Example 3.1.1

The equation

$$y'' + 3xy' + x^3y = e^x$$

is a linear ordinary differential equation.
The equation

$$y''' + xy'' + 3x^2y' - 5y = \sin x$$

is a linear ODE of third order.

Theorem 3.1.1* (Basic Existence Theorem)*:
Hypothesis:

1. *Consider the nth-order linear differential equation*

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)}n + \cdots + a_{n-1}(x)y' + a_n(x)y = F(x) \tag{3.1.3}$$

where $a_0, a_1, \cdots, a_n$ and F are real functions on a real interval $a \leq x \leq b$ and $a_0(x) \neq 0$ for any x on $a \leq x \leq b$.

2. *Let x_0 be any point of the interval $a \leq x \leq b$, and let $c_0, c_1, \cdots, c_{n-1}$ be n arbitrary real constants.*

Conclusion: *There exists a unique solution f of (3.1.3) such that*

$$f(x_0) = c_0, f'(x_0) = c_1, \cdots, f^{(n-1)}(x_0) = c_{n-1}$$

and this solution is defined over the entire interval $a \leq x \leq b$.

Example 3.1.2: Consider the initial-value problem

$$2y''' + xy'' + 3x^2y' - 5y = \sin x$$

$$y(4) = 3$$
$$y'(4) = 5$$
$$y''(4) = -\frac{7}{2}$$

Here we have a third-order problem. The coefficients $2, x, 3x^2$, and -5, as well as the nonhomogeneous term $\sin x$, are all continuous for all $x \in (-\infty, \infty)$. The point $x_0 = 4$ certainly belongs to this intervall; the real numbers c_0, c_1, and c_2 in this problem are 3, 5, and $-\frac{7}{2}$ respectively. Theorem 3.1.1 assures is that this problem also has a unique solution which is defined for all $x \in (-\infty, \infty)$

Corollary 3.1.2:
Hypothesis: *Let f be a solution of the nth-order homogeneous linear DE*

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = 0 \tag{3.1.4}$$

such that

$$f(x_0) = 0, f'(x_0) = 0, \cdots, f^{(n-1)}(x_0) = 0,$$

where x_0 is a point of the interval $a \leq x \leq b$ in which the coefficients $a_0, a_1, \cdots, a_n$ are all continuous and $a_0(x) \neq 0$.

Conclusion: *Then $f(x) = 0$ for all x on $a \leq x \leq b$.*

Example 3.1.3

The unique solution of f of the third-order homogeneous equation

$$y''' + 2y'' + 4xy' + x^2y = 0$$

which is such that

$$f(2) = f'(2) = f''(2) = 0$$

is the trivial solution f such that $f(x) = 0$ for all x.

B The Homogeneous Equation

We now consider the fundamental results concerning the homogeneous equation

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = 0 \tag{3.1.5}$$

Theorem 3.1.3* (Basic Theorem on Linear Homogeneous Differential Equations)*:
Hypothesis: *Let $f_1, f_2, \cdots, f_m$ be any m solutions of the homogeneous linear differential equation (3.1.5).*

Conclusion: *Then*

$$c_1 f_1 + c_2 f_2 + \cdots + c_m f_m$$

is also a solution of (3.1.5), where $c_1, c_2, \cdots, c_m$ are m arbitrary constants.

In other words: Any linear combination of solutions of the homogeneous linear differential equation (3.1.5) is also a solution of (3.1.5).

Definition 3.1.2: Linear Combination

If $f_1, f_2, \cdots, f_m$ are m given functions, and $c_1, c_2, \cdots, c_m$ are m constants, then the expression

$$c_1 f_1 + c_2 f_2 + \cdots + c_m f_m$$

is called a linear combination of $f_1, f_2, \cdots, f_m$.

Example 3.1.4

e^x, e^{-x}, e^{2x} are solutions of

$$y''' - 2y'' - y' + 2y = 0$$

Theorem 3.1.3 states that the linear combination $c_1e^x + c_2e^{-x} + c_3x^{2x}$ is also a solution for any constants c_1, c_2, c_3. For example, the particular linear combination

$$2e^x - 3e^{-x} + \frac{2}{3}e^{2x}$$

is a solution.

Definition 3.1.3: Linear Dependence

The n functions $f_1, f_2, \cdots, f_n$ are called *linearly dependent* on $a \leq x \leq b$ if there exist constants $c_1, c_2, \cdots, c_n$, not all zero, such that

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_m f_m(x) = 0$$

for all x such that $a \leq x \leq b$.

Definition 3.1.4: Linear Independence

The n functions $f_1, f_2, \cdots, f_n$ are called linearly independent on the interval $a \le x \le b$ if the relation

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

for all x such that $a \le x \le b$ implies that

$$c_1 = c_2 = \cdots = c_n = 0$$

In other words, the only linear combination of $f_1, f_2, \cdots, f_n$ that is identically zero on $a \le x \le b$ is the trivial linear combination

$$0 \cdot f_1 + 0 \cdot f_2 + \cdots + 0 \cdot f_n$$

Theorem 3.1.4* (Linearly Independent Solutions of n-th Order Linear Differential Equation)*: *The n-th order homogeneous linear differential equation*

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = 0 \tag{3.1.6}$$

always possesses n solutions that are linearly independent. Further, if $f_1, f_2, \cdots f_n$ are n linearly independent solutions of (3.1.6), then every solution f of (3.1.6) can be expressed as a linear combination

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

of these n linearly independent soulutions by proper choice of the constants $c_1, c_2, \cdots, c_n$.

Example 3.1.5

We have observed that $\sin x$ and $\cos x$ are solutions of

$$y'' + y = 0 \tag{3.1.7}$$

for all $x \in (-\infty, \infty)$. Further, we can show that these two solutions are linearly independent. Suppose f is any solution of (4.7). Then by Theorem 3.1.4 f can be expressed as a certain linear combination $c_1 \sin x + c_2 \cos x$ of the two linearly independent solutions $\sin x$ and $\cos x$ by proper choice of c_1 and c_2. That is, there exist two particular constants c_1 and c_2 such that

$$f(x) = c_1 \sin x + c_2 \cos x \tag{3.1.8}$$

for all $x \in (-\infty, \infty)$. For example, it can be easily verified that $f(x) = \sin\big(x + \pi/6\big)$ is a solution of the equation (3.1.7). Since

$$\sin\left(x + \frac{\pi}{6}\right) = \sin x \cos\frac{\pi}{6} + \cos x \sin\frac{\pi}{6} = \frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x,$$

we see that the solution $\sin\big(x + \pi/6\big)$ can be expressed as the linear combination

$$\frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x$$

of the two linearly independent solutions $\sin x$ and $\cos x$. Here, $c_1 = \sqrt{3}/2$ and $c_2 = 1/2$

Definition 3.1.5: Fundamental Set of Solutions

If $f_1, f_2, \cdots, f_n$ are n linearly independent solutions of the n-th order homogeneous linear differential equation

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = 0 \tag{3.1.9}$$

on $a \le x \le b$, then the set $f_1, f_2, \cdots, f_n$ is called a fundamental set of solutions of (3.1.9) and the function f defined by

$$f(x) = c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x), \quad a \le x \le b,$$

where $c_1, c_2, \cdots, c_n$ are arbitrary constants, is called a general solution of (3.1.9 on $a \le x \le b$.

Therefore, if we can find n linearly independent solutions of (3.1.9), we can at once write the general solution of (3.1.9) as a general linear combination of these n solutions.

Example 3.1.6

The solutions e^x, e^{-x}, and e^{2x} of

$$y''' - 2y'' + y' + 2y = 0$$

may be shown to be linearly independent for all $x \in (-\infty, \infty)$. Thus, e^x, e^{-x}, and e^{2x} constitute a fundamental set of the given DE, and its general solution may be expressed as the linear combination

$$c_1 e^x + e^{-x} + c_3 e^{2x}$$

where c_1, c_2, and c_3 are arbitrary constants. We can write this as

$$y = c_1 e^x + e^{-x} + c_3 e^{2x}$$

Definition 3.1.6: Wronskian

Let $f_1, f_2, \cdots, f_3$ be n real functions each of which has an $(n-1)$th derivative on a real interval $a \le x \le b$. The determinant

$$W(f_1, f_2, \cdots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

is called the Wronskian of these n functions. We observe that $W(f_1, f_2, \cdots, f_n)$ is itself a real function defined on $a \le x \le b$. Its value at x is denoted by $W(f_1, f_2, \cdots, f_n)(x)$ or by $W[f_1(x), f_2(x), \cdots, f_n(x)]$.