

Math-183

Differential Equations

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1 Differential Equations and Their Solutions

1.1 Classification of Differential Equations

Definition 1.1.1: Differential Equation

Differential equation is an equation involving derivatives of one or more dependent variables with respect to one or more independent variables.

Definition 1.1.2: Ordinary Differential Equation

A differential equation involving ordinary derivatives of one or more dependent variables with respect to a single independent variable is called an ordinary differential equation.

Example 1.1.1: Ordinary Differential Equations:

$$\frac{dy}{dx} + xy \left(\frac{d}{dx} \right)^2 = 0 \quad (1.1.1)$$

$$\frac{d^4x}{dt^4} + 5\frac{d^2x}{dt^2} + 3x = \sin t \quad (1.1.2)$$

Definition 1.1.3: Partial Differential Equation

A differential equation involving partial derivatives of one or more dependent variables with respect to more than one independent variables is called a partial differential equation.

Example 1.1.2: Partial Differential Equations:

$$\frac{\partial v}{\partial s} + \frac{\partial v}{\partial t} = v \quad (1.1.3)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \quad (1.1.4)$$

Definition 1.1.4: Order and Degree of Differential Equations

Order of DE: The order of the highest ordered derivative involved in a differential equation is called the order of the differential equation.

Degree of DE: The power of the highest order derivative involved in a differential equation is called the degree of the differential equation.

Definition 1.1.5: Linearity of Differential Equations

If the dependent variable and its various derivatives occur to the first degree only, the DE is a linear DE. Otherwise it's a non-linear DE.

$$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = b(x)$$

Linear DE can also be classified as linear with *constant* and *variable* coefficients.

Example 1.1.3: Ordinary Differential Equations: Orders, Degree, Linearity

$$\begin{aligned}\frac{d^3 y}{dx^3} - 3\frac{d^2 y}{dx^2} + 3\frac{dy}{dx} - 6y &= \sin x && \text{3rd ord 1st deg Lin} \\ \frac{d^2 y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + y &= 0 && \text{2nd ord 1st deg Non-Lin} \\ y = x\frac{dy}{dx} + \sqrt{1 + \frac{d^2 y}{dx^2}} &&& \text{2nd ord 1st deg Non-Lin} \\ \frac{d^4 x}{dt^4} + t^2\frac{d^3 x}{dt^3} + \frac{dy}{dx} &= \sin t && \text{4th ord 1st deg Lin} \\ \frac{d^2 y}{dx^2} + 5\frac{dy}{dx} + 6y^2 &= 0 && \text{2nd ord 1st deg Non-Lin} \\ \frac{d^2 y}{dx^2} + 5\left(\frac{dy}{dx}\right)^3 + 6y &= 0 && \text{2nd ord 1st deg Non-Lin} \\ \frac{d^2 y}{dx^2} + 5y\frac{dy}{dx} + 6y &= 0 && \text{2nd ord 1st deg Lin}\end{aligned}$$

1.2 Solutions

A Nature of Solutions

An nth-order Differential Equation:

$$F\left[x, y, \frac{dy}{dx}, \dots, \frac{d^n y}{dx^n}\right] = 0 \quad (1.2.1)$$

Definition 1.2.1: Explicit solution

f is an explicit solution of (1.2.1) if

$$\forall x \in I, F\left[x, f(x), f'(x), \dots, f^{(n)}(x)\right] = 0$$

where I is a real interval.

Definition 1.2.2: Implicit solution

$g(x, y) = 0$ is an implicit solution if this relation defines at least one real function $f(x)$ on an interval I such that f is an explicit solution of (1.2.1)

Example 1.2.1: Explicit and Implicit Solutions

$$x^2 + y^2 - 25 = 0 \quad : \quad \text{Implicit solution}$$

$$2x + 2y \frac{dy}{dx} = 0$$

$$x + y \frac{dy}{dx} = 0 \quad : \quad \text{Differential Equation}$$

$$y = \pm \sqrt{25 - x^2} ; -5 \leq x \leq 5 \quad : \quad \text{Explicit solution}$$

B Methods of Solution

The study of a Differential Equation consists of 3 phases:

1. Formulation of DE from the given physical situation.
2. Solutions of DE, evaluating the arbitrary constants from the given condition.
3. Physical interpretation of the solution.

Example 1.2.2: Show that the function $f(x) = e^x + 2x^2 + 6x + 7$ is a solution to the DE $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 4x^2$

$$f(x) = e^x + 2x^2 + 6x + 7$$

$$f'(x) = e^x + 4x + 6$$

$$f''(x) = e^x + 4$$

$$\begin{aligned} \frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y &= (e^x + 4) - 3(e^x + 4x + 6) + 2(e^x + 2x^2 + 6x + 7) \\ &= 0 \cdot e^x + 0 \cdot x + (4 - 18 + 14) + 4x^2 \\ &= 4x^2 \end{aligned}$$

□

Example 1.2.3: Show that the function $f(x) = \frac{1}{1+x^2}$ is a solution to the DE $(1+x^2)\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = 0$

$$f(x) = \frac{1}{1+x^2}$$

$$(1+x^2)f(x) = 1$$

$$(1+x^2)f'(x) + 2xf(x) = 0$$

$$(1+x^2)f''(x) + 4xf'(x) + 2f(x) = 0$$

$$(1+x^2)\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = 0$$

□

Example 1.2.4: Show that the function $y = (2x^2 + 2e^{3x} + 3)e^{-2x}$ satisfies the DE

$$\frac{dy}{dx} + 2y = 6e^x + 4xe^{-2x}$$

$$\begin{aligned} y &= (2x^2 + 2e^{3x} + 3)e^{-2x} \\ y_1 &= (4x + 6e^{3x})e^{-2x} - (2x^2 + 2e^{3x} + 3)2e^{-2x} \\ y_1 &= 4xe^{-2x} + 6e^x - 2y \\ \frac{dy}{dx} + 2y &= 6e^x + 4e^{-2x} \end{aligned}$$

□

1.3 Initial-Value and Boundary-Value Problems, and Existence of Solutions

A Initial-value Problems and Boundary-value Problems

One of the most frequently encountered type of problems in Differential Equations involves both a DE and one or more supplementary conditions which the solution of the given DE must satisfy.

Definition 1.3.1: IVP and BVP

Consider the first-order DE

$$\frac{dy}{dx} = f(x, y)$$

where f is a continuous function of x and y in some domain D of the xy plane; and let (x_0, y_0) be a point of D . The **initial-value problem** associated with the DE is to find a solution ϕ of the DE, defined on some real interval containing x_0 , and satisfying the initial condition

$$\phi(x_0) = y_0$$

In the customary abbreviated notation, this initial-value problem may be written

$$\frac{dy}{dx} = f(x, y)$$

$$y(x_0) = y_0$$

If the conditions relate to two different x values (the extreme or boundary values), the problem is called a **Two-Point Boundary-Value Problem** or simply a **Boundary-Value Problem (BVP)**.

Example 1.3.1: Find the solution of the DE $\frac{dy}{dx} = 2x$ such that $\forall x \in I, f'(x) = 2x$ and $f(1) = 4$

$$\begin{aligned}\frac{dy}{dx} &= 2x \\ \int \frac{dy}{dx} dx &= \int 2x dx \\ y &= x^2 + c\end{aligned}$$

Substituting $y = 4$ and $x = 1$,

$$4 = 1 + c \text{ or } c = 3$$

$$\therefore \text{Solution: } y^2 = x + 3$$

□

Example 1.3.2: $\frac{dy}{dx} = -\frac{x}{y}$, $y(3) = 4$

$$\begin{aligned}x + y \frac{dy}{dx} &= 0 \\ \int x dx + \int y \frac{dy}{dx} dx &= 0 \\ \frac{x^2}{2} + \frac{y^2}{2} &= c' \\ x^2 + y^2 &= c\end{aligned}$$

Substituting $x = 3$ and $y = 4$,

$$16 + 9 = c \text{ or } c = 25$$

$$\therefore \text{Solution: } x^2 + y^2 - 25 = 0$$

B Existence of Solutions

Not all initial-value and boundary-value problems have solutions. For example,

$$\begin{aligned}\frac{d^2y}{dx^2} + y &= 0 \\ y(0) &= 1, \quad y(\pi) = 5\end{aligned}$$

has no solutions! Thus arises the question of *existence* of solutions. We can say, every initial-value problem that satisfies definition (1.3.1) has *at least one* solution. However, there arises another question. Can a problem have more than one solution?

Let's consider the initial-value problem

$$\frac{dy}{dx} = y^{1/3}; \quad y(0) = 0$$

One may verify that the functions f_1 and f_2 defined, respectively, by

$$\forall x \in \mathbb{R}, \quad f_1(x) = 0$$

and

$$f_2(x) = \left(\frac{2}{3}x\right)^{3/2}, \quad x \geq 0; \quad f_2(x) = 0, \quad x \leq 0$$

are both solutions of this initial-value problem. In fact, this problem has infinitely many solutions. Hence, we can state that the initial-value problem need not have a *unique* solution. In order to ensure uniqueness, some additional requirement must certainly be imposed.

Theorem 1.3.1 (Basic Existence and Uniqueness Theorem):

Hypothesis: Consider the differential equation

$$\frac{dy}{dx} = f(x, y) \quad (1.3.1)$$

where

- The function f is a continuous function of x and y in some domain D of the xy plane, and
- The partial derivative $\frac{\partial f}{\partial y}$ is also a continuous function of x and y in D ; and let (x_0, y_0) be a point in D .

Conclusion: There exists a unique solution ϕ of the differential equation (1.3.1), defined on some interval $|x - x_0| \leq h$, where h is sufficiently small, that satisfies the condition

$$\phi(x_0) = y_0$$

Example 1.3.3: Show that

$$y = 4e^{2x} + 2e^{-3x}$$

is a solution of the initial-value problem

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$$

$$y(0) = 6$$

$$y'(0) = 2$$

Is $y = 2e^{2x} + 4e^{-3x}$ also a solution of this problem? Explain why or why not.

$$y = 4e^{2x} + 2e^{-3x}$$

$$y_1 = 8e^{2x} - 6e^{-3x}$$

$$y_2 = 16e^{2x} + 18e^{-3x}$$

$$\begin{aligned} y_2 + y_1 - 6y &= (16e^{2x} + 18e^{-3x}) + (8e^{2x} - 6e^{-3x}) - 6(4e^{2x} + 2e^{-3x}) \\ &= 0 \cdot e^{2x} + 0 \cdot e^{-3x} \\ &= 0 \end{aligned}$$

The solution also satisfies $y(0) = 6$ and $y'(0) = 2$

Now, for $y = 2e^{2x} + 4e^{-3x}$,

$$y_1 = 4e^{2x} - 12e^{-3x} ; \quad y_2 = 8e^{2x} + 36e^{-3x}$$

$$\begin{aligned}
y_2 + y_1 - 6y &= (8e^{2x} + 36e^{-3x}) + (4e^{2x} - 12e^{-3x}) - 6(2e^{2x} + 4e^{-3x}) \\
&= 0 \cdot e^{2x} + 0 \cdot e^{-3x} \\
&= 0
\end{aligned}$$

However, in this case,

$$y(0) = 6 \ ; \ y'(0) = -8$$

As we can see, this solution doesn't satisfy the initial-value problem. Hence $y = 2e^{2x} + 4e^{-3x}$ is not a solution of this problem.

Example 1.3.4: Given that every solution of

$$x^3 \frac{d^3y}{dx^3} - 3x^2 \frac{d^2y}{dx^2} + 6x \frac{dy}{dx} - 6y = 0$$

may be written in the form $y = c_1x + c_2x^2 + c_3x^3$ for some choice of the arbitrary constants c_1, c_2 , and c_3 , solve the initial-value problem consisting of the above DE plus the three conditions

$$y(2) = 0 \ , \ y'(2) = 2 \ , \ y''(2) = 6$$

$$y = c_1x + c_2x^2 + c_3x^3$$

$$y(2) = 0 \text{ or, } 8c_3 + 4c_2 + 2c_1 = 0 \tag{1.3.2}$$

$$y' = c_1 + 2c_2x + 3c_3x^2$$

$$y'(2) = 2 \text{ or, } 12c_3 + 4c_2 + c_1 = 2 \tag{1.3.3}$$

$$y'' = 0 + 2c_2 + 6c_3x$$

$$y''(2) = 6 \text{ or, } 12c_3 + 2c_2 + 0c_1 = 6 \tag{1.3.4}$$

Solving (1.3.1), (1.3.2), and (1.3.3) we get,

$$c_1 = 2 \ , \ c_2 = -3 \ , \ c_3 = 1$$

$$\therefore \text{ Solution: } y = 2x - 3x^2 + x^3$$

2 First Order Equations for Which Exact Solutions Are Obtainable

2.1 Exact Differential Equations and Integrating Factors

A Standard Forms of First-Order Differential Equations

The first-order differential equations may be expressed in either the **Derivative Form**

$$\frac{dy}{dx} = f(x, y) \quad (2.1.1)$$

or the **Differential Form**

$$M(x, y) dx + N(x, y) dy = 0 \quad (2.1.2)$$

Example 2.1.1: Standard Forms

The equation

$$\frac{dy}{dx} = \frac{x^2 + y^2}{x - y}$$

is the form (2.1.1). It may be written as

$$(x^2 + y^2) dx + (y - x) dy = 0$$

which is of the form (2.1.2).

Again, the equation

$$(\sin x + y) dx + (x + 3y) dy = 0$$

is of the form (2.1.2), which can also be written as

$$\frac{dy}{dx} = -\frac{\sin x + y}{x + 3y}$$

B Exact Differential Equations

Definition 2.1.1: Exact Differential

Let F be a function of two real variables such that F has continuous first partial derivatives in a domain D . The total differential dF of the function F is defined by the formula

$$dF(x, y) = \frac{\partial F(x, y)}{\partial x} dx + \frac{\partial F(x, y)}{\partial y} dy$$

for all $(x, y) \in D$.

Comparing $dF(x, y)$ with the form (2.1.2), we get

$$\frac{\partial F(x, y)}{\partial x} = M(x, y) \quad \text{and} \quad \frac{\partial F(x, y)}{\partial y} = N(x, y)$$

Example 2.1.2

Let F be a function

$$F(x, y) = xy^2 + 2x^3y$$

for all real (x, y) . Then

$$\frac{\partial F(x, y)}{\partial x} = y^2 + 6x^2y, \quad \frac{\partial F(x, y)}{\partial y} = 2xy + 2x^3$$

and the total differential dF is defined by

$$dF(x, y) = (y^2 + 6x^2y) dx + (2xy + 2x^3) dy$$

for all real (x, y)

Definition 2.1.2: Exact Differential Equation

The expression

$$M(x, y) dx + N(x, y) dy \quad (2.1.3)$$

is called an exact differential in a domain D if there exists a function F of two real variables such that this expression equals the total differential $dF(x, y)$ for all $(x, y) \in D$.

That is, expression (2.1.3) is an exact differential in D if there exists a function F such that

$$\frac{\partial F(x, y)}{\partial x} = M(x, y) \quad \text{and} \quad \frac{\partial F(x, y)}{\partial y} = N(x, y)$$

for all $(x, y) \in D$.

If $M(x, y) dx + N(x, y) dy$ is an exact differential, then the differential equation

$$M(x, y) dx + N(x, y) dy = 0$$

is called an **Exact Differential Equation**.

Theorem 2.1.1 (Exact Differential Equation):

1. If the DE $M(x, y) dx + N(x, y) dy = 0$ is exact in D , then

$$\forall (x, y) \in D, \quad \frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x}$$

2. Conversely, if

$$\forall (x, y) \in D, \quad \frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x}$$

then the DE is exact in D .

Proof (1):

$$\begin{aligned} \frac{\partial F(x, y)}{\partial x} &= M(x, y) \quad , \quad \frac{\partial F(x, y)}{\partial y} = N(x, y) \\ \frac{\partial^2 F(x, y)}{\partial x \partial y} &= \frac{\partial M(x, y)}{\partial y} \quad , \quad \frac{\partial^2 F(x, y)}{\partial x \partial y} = \frac{\partial N(x, y)}{\partial x} \\ \therefore \frac{\partial^2 F(x, y)}{\partial y \partial x} &= \frac{\partial^2 F(x, y)}{\partial x \partial y} \\ \therefore \frac{\partial M(x, y)}{\partial y} &= \frac{\partial N(x, y)}{\partial x} \quad \square \end{aligned}$$

C The Solution of Exact Differential Equations

Theorem 2.1.2 (Solution of Exact DE):

If $M(x, y) dx + N(x, y) dy = 0$ is exact in domain D , then

$$\forall (x, y) \in D, \exists F(x, y) : \frac{\partial F(x, y)}{\partial x} = M(x, y) \quad \text{and} \quad \frac{\partial F(x, y)}{\partial y} = N(x, y)$$

Then the equation may be written

$$\frac{\partial F(x, y)}{\partial x} dx + \frac{\partial F(x, y)}{\partial y} dy = 0$$

or simply,

$$dF(x, y) = 0$$

Here, $F(x, y) = c$ is a one-parameter family of solutions of this DE, where c is an arbitrary constant.

Example 2.1.3: Solve the equation

$$(3x^2 + 4xy) dx + (2x^2 + 2y) dy = 0$$

Standard Method:

$$\begin{aligned} \frac{\partial F(x, y)}{\partial x} &= M(x, y) = 3x^2 + 4xy \\ F(x, y) &= \int (3x^2 + 4xy) dx + \phi(y) \\ &= x^3 + 2x^2y + \phi(y) \end{aligned}$$

Again,

$$\begin{aligned} \frac{\partial F(x, y)}{\partial y} &= 2x^2 + \frac{\partial \phi(y)}{\partial y} = 2x^2 + 2y \\ \frac{d\phi(y)}{dy} &= 2y \\ \int \frac{d\phi(y)}{dy} dy &= \int 2y dy \\ \phi(y) &= y^2 + c_0 \end{aligned}$$

Thus, we get

$$F(x, y) = x^3 + 2x^2y + y^2 + c_0$$

Hence, a one-parameter family of the solution is $F(x, y) = c_1$ or,

$$x^3 + 2x^2y + y^2 + c_0 = c_1$$

$$\boxed{x^3 + 2x^2y + y^2 = c}$$

Method of Grouping:

$$\begin{aligned}(3x^2 + 4xy) dx + (2x^2 + 2y) dy &= 0 \\ 3x^2 dx + (4xy dx + 2x^2 dy) + 2y dy &= 0 \\ d(x^3) + d(2x^2y) + d(y^2) &= d(c) \\ \boxed{x^3 + 2x^2y + y^2} &= c\end{aligned}$$

Example 2.1.4: Solve the initial-value problem

$$(2x \cos y + 3x^2y) dx + (x^3 - x^2 \sin y - y) dy = 0 ; \quad y(0) = 2$$

$$\begin{aligned}(2x \cos y dx - x^2 \sin y dy) + (3x^2y dx + x^3 dy) - y dy &= 0 \\ d(x^2 \cos y) + d(x^3y) + d\left(\frac{y^2}{2}\right) &= d(c_1) \\ 2x^2 \cos y + x^3y + y^2 &= c\end{aligned}$$

Substituting $x = 0$ and $y = 2$,

$$2^2 = c$$

Hence, the solution is:

$$2x^2 \cos y + x^3y + y^2 = 4$$

D Integrating Factors

Definition 2.1.3: Integrating Factor (IF)

If the DE

$$M(x, y) dx + N(x, y) dy = 0 \quad (2.1.4)$$

is not exact in a domain D but the DE

$$\mu(x, y)M(x, y) dx + \mu(x, y)N(x, y) dy = 0 \quad (2.1.5)$$

is exact in D , then $\mu(x, y)$ is called an **Integrating Factor** of the DE.

Example 2.1.5: Integrating factor

Consider the DE

$$(3y + 4xy^2) dx + (2x + 3x^2y) dy = 0 \quad (2.1.6)$$

This equation is of the form (2.1.4), where

$$\begin{aligned}M(x, y) &= 3y + 4xy^2, & N(x, y) &= 2x + 3x^2y \\ \frac{\partial M(x, y)}{\partial y} &= 3 + 8xy, & \frac{\partial N(x, y)}{\partial x} &= 2 + 6xy\end{aligned}$$

Since

$$\frac{\partial M(x, y)}{\partial y} \neq \frac{\partial N(x, y)}{\partial x}$$

except for (x, y) such that $2xy + 1 = 0$, Equation (2.1.4) is not exact in any rectangular domain D .

Let $\mu(x, y) = x^2y$. Then the corresponding DE of the form (2.1.5) is

$$(3x^2y^2 + 4x^3y^3) dx + (2x^3y + 3x^4y^2) dy = 0$$

This equation is exact in every rectangular domain D , since

$$\frac{\partial[\mu(x, y)M(x, y)]}{\partial y} = 6x^2y + 12x^3y^2 = \frac{\partial[\mu(x, y)N(x, y)]}{\partial x}$$

For all real (x, y) . Hence, $\mu(x, y) = x^2y$ is an integrating factor of Equation (2.1.6).

Example 2.1.6: Determine whether or not the following equation is exact

$$\left(\frac{x}{y^2} + x\right) dx + \left(\frac{x^2}{y^3} + y\right) dy = 0$$

$$\begin{aligned}\frac{\partial M(x, y)}{\partial y} &= -\frac{x}{2y^3} \\ \frac{\partial N(x, y)}{\partial x} &= \frac{2x}{y^3}\end{aligned}$$

Here, $\frac{\partial M(x, y)}{\partial y} \neq \frac{\partial N(x, y)}{\partial x}$. Hence, the equation is not exact.

Example 2.1.7: Determine the constant A in the following equations such that the equation is exact

1. $(Ax^2y + 2y^2) dx + x^3 + 4xy dy = 0$
2. $\left(\frac{Ay}{x^3} + \frac{y}{x^2}\right) dx + \left(\frac{1}{x^2} - \frac{1}{x}\right) dy = 0$

1.

$$\begin{aligned}\frac{\partial M(x, y)}{\partial y} &= \frac{\partial(Ax^2y + 2y^2)}{\partial y} = Ax^2 + 4y \\ \frac{\partial N(x, y)}{\partial x} &= \frac{\partial(x^3 + 4xy)}{\partial x} = 3x^2 + 4y\end{aligned}$$

Equating the coefficients of x^2 , we get

$$\boxed{A = 3}$$

2.

$$\begin{aligned}\frac{\partial M(x, y)}{\partial y} &= \frac{\partial\left(\frac{Ay}{x^3} + \frac{y}{x^2}\right)}{\partial y} = \frac{A}{x^3} + \frac{1}{x^2} \\ \frac{\partial N(x, y)}{\partial x} &= \frac{\partial\left(\frac{1}{x^2} - \frac{1}{x}\right)}{\partial x} = -\frac{1}{2x^3} + \frac{1}{x^2}\end{aligned}$$

Equating the coefficients of $\frac{1}{x^3}$, we get

$$\boxed{A = -\frac{1}{2}}$$

2.2 Separable Equations and Equations Reducible to this Form

A Separable Equations

Definition 2.2.1: Separable Equations

An equations of the form

$$F(x)G(y) dx + f(x)g(y) dy = 0 \quad (2.2.1)$$

is called an equation with separable variables or simply a separable equation.

Theorem 2.2.1 (Solution of Separable Differential Equations):

In general, the separable equations are not exact, but they possess an obvious integrating factor $\frac{1}{f(x)G(y)}$

Thus the equation (2.2.1) becomes

$$\frac{F(x)}{f(x)} dx + \frac{g(y)}{G(y)} dy = 0 \quad (2.2.2)$$

which is exact, because

$$\frac{\partial}{\partial y} \left(\frac{F(x)}{f(x)} \right) = 0 = \frac{\partial}{\partial x} \left(\frac{g(y)}{G(y)} \right)$$

We can write the equation (2.2.2) as

$$M(x) dx + N(y) dy = 0$$

where $M(x) = \frac{F(x)}{f(x)}$ and $N(y) = \frac{g(y)}{G(y)}$

A one-parameter family solution to the DE is

$$\int M(x) dx + \int N(y) dy = c \quad (2.2.3)$$

Example 2.2.1: Solve the equation

$$(x - 4)y^4 dx - x^3(y^2 - 3) dy = 0$$

The equation is separable; dividing by x^3y^4 we obtain

$$\frac{x - 4}{x^3} dx - \frac{y^2 - 3}{y^4} dy = 0$$
$$\int (x^{-2} - 4x^{-3}) dx - \int (y^{-2} - 3y^{-4}) dy = 0$$

$$\boxed{-\frac{1}{x} + \frac{2}{x^2} + \frac{1}{y} - \frac{1}{y^3} = c}$$

The DE in derivative form:

$$\frac{dy}{dx} = \frac{(x - 4)y^4}{x^3(y^2 - 3)}$$

Here, $y = 0$ is a solution which was lost in the separation process.

Example 2.2.2: Solve the initial-value problem that consists of the DE

$$x \sin y \, dx + (x^2 + 1) \cos y \, dy = 0$$

and the initial condition $y(1) = \frac{\pi}{2}$

$$\begin{aligned}\frac{x}{x^2 + 1} \, dx + \frac{\cos y}{\sin y} \, dy &= 0 \\ \frac{1}{2} \int \frac{d(x^2 + 1)}{x^2 + 1} + \int \cot y \, dy &= 0 \\ \frac{1}{2} \ln |x^2 + 1| + \ln |\sin y| &= \ln |c_1| \\ \ln |(x^2 + 1) \sin^2 y| &= \ln |c| \\ \therefore (x^2 + 1) \sin^2 y &= c\end{aligned}$$

Applying the initial condition, we get

$$2 \sin^2 \frac{\pi}{2} = c \text{ or, } c = 2$$

Thus, the solution is

$$(x^2 + 1) \sin^2 y = 2$$

B Homogeneous Equations

Definition 2.2.2: Homogeneous Equations

The first-order differential equation

$$M(x, y) \, dx + N(x, y) \, dy = 0$$

is said to be homogeneous if, when written in the derivative form

$$\frac{dy}{dx} = f(x, y)$$

there exists a function g such that $f(x, y)$ can be expressed in the form $g(y/x)$

Example 2.2.3

The DE

$$x^2 - 3y^2 \, dx + 2xy \, dy = 0$$

is homogeneous. To see this, we first write the derivative form of the equation:

$$\frac{dy}{dx} = \frac{3y^2 - x^2}{2xy} = \frac{3y}{2x} - \frac{x}{2y}$$

We see that the DE can be written as

$$\frac{dy}{dx} = \frac{3}{2} \left(\frac{y}{x} \right) - \frac{1}{2} \left(\frac{1}{y/x} \right)$$

in which the right side of the equation is of the form $g(y/x)$ for a certain function g .

Example 2.2.4: The equation

$$(y + \sqrt{x^2 + y^2}) dx - x dy = 0$$

is homogeneous.

Derivative form:

$$\begin{aligned} \frac{dy}{dx} &= \frac{y + \sqrt{x^2 + y^2}}{x} \\ &= \frac{y}{x} + \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2}} \\ &= \frac{y}{x} + \sqrt{1 + \left(\frac{y}{x}\right)^2} = g\left(\frac{y}{x}\right) \end{aligned}$$

Definition 2.2.3: Homogeneous Equation of degree n

A function F is called homogeneous of degree n if

$$F(tx, ty) = t^n F(x, y)$$

This means that if the tx and ty are substituted for x and y respectively in $F(x, y)$, and if t^n is then factored out, the other factor that remains is the original expression $F(x, y)$ itself.

For example, the function given by $F(x, y) = x^2 + y^2$ is homogeneous of degree 2, since

$$F(tx, ty) = (tx)^2 + (ty)^2 = t^2(x^2 + y^2) = t^2 F(x, y)$$

Now, suppose both $M(x, y)$ and $N(x, y)$ in the DE

$$M(x, y) dx + N(x, y) dy = 0$$

are homogeneous of the same degree n . Since $M(tx, ty) = t^n M(x, y)$, for $t = \frac{1}{x}$, we have

$$\begin{aligned} M\left(1, \frac{y}{x}\right) &= \left(\frac{1}{x}\right)^n M(x, y) \\ M(x, y) &= \left(\frac{1}{x}\right)^{-n} M\left(1, \frac{y}{x}\right) \end{aligned}$$

Similarly,

$$N(x, y) = \left(\frac{1}{x}\right)^{-n} N\left(1, \frac{y}{x}\right)$$

Now, writing the DE in derivative form, we get

$$\begin{aligned} \frac{dy}{dx} &= -\frac{M(x, y)}{N(x, y)} \\ &= -\frac{\left(\frac{1}{x}\right)^{-n} M\left(1, \frac{y}{x}\right)}{\left(\frac{1}{x}\right)^{-n} N\left(1, \frac{y}{x}\right)} \\ &= -\frac{M\left(1, \frac{y}{x}\right)}{N\left(1, \frac{y}{x}\right)} \\ &= g\left(\frac{y}{x}\right) \end{aligned}$$

Note:-

If $M(x, y)$ and $N(x, y)$ in

$$M(x, y) dx + N(x, y) dy = 0$$

are both homogeneous functions of the same degree n , then the differential equation is a homogeneous differential equation.

Theorem 2.2.2:

If

$$M(x, y) dx + N(x, y) dy = 0 \quad (2.2.4)$$

is a homogeneous equation, then the change of variables $y = vx$ transforms the equation into a separable equation in the variables v and x .

Proof:

Since $M(x, y) dx + N(x, y) dy = 0$ is homogeneous, it may be written in the form

$$\frac{dy}{dx} = g\left(\frac{y}{x}\right)$$

Let $y = vx$. Then

$$\frac{dy}{dx} = v + x \frac{dv}{dx}$$

and the initial equation becomes

$$v + x \frac{dv}{dx} = g(v)$$

or,

$$[v - g(v)] dx + x dv = 0$$

This equation is separable. Separating the variables we obtain

$$\frac{dv}{v - g(v)} + \frac{dx}{x} = 0 \quad (2.2.5)$$

Theorem 2.2.3 (Solution of a Homogeneous Differential Equation): To solve a DE of the form (2.2.4), we let $y = vx$ and transform the homogeneous equation into a separable equation of the form (2.2.5). From this, we have

$$\int \frac{dv}{v - g(v)} + \int \frac{dx}{x} = c$$

where c is an arbitrary constant. Letting $F(v)$ denote

$$\int \frac{dv}{v - g(v)}$$

and returning to the original dependent variable y , the solution takes the form

$$F\left(\frac{y}{x}\right) + \ln|x| = c$$

Example 2.2.5: Solve the equation

$$(x^2 - 3y^2) dx + 2xy dy = 0$$

Derivative form:

$$\frac{dy}{dx} = -\frac{x}{2y} + \frac{3y}{2x}$$

Letting $y = vx$, we get

$$\begin{aligned}v + x \frac{dv}{dx} &= -\frac{1}{2v} + \frac{3v}{2} \\x \frac{dv}{dx} &= \frac{v^2 - 1}{2v} \\ \frac{2v dv}{v^2 - 1} &= \frac{dx}{x}\end{aligned}$$

Integrating, we find

$$\begin{aligned}\ln |v^2 - 1| &= \ln |x| + \ln |c| \\|v^2 - 1| &= |cx| \\\left|\frac{y^2}{x^2} - 1\right| &= |cx| \\|y^2 - x^2| &= x^2 |cx|\end{aligned}$$

For $y \geq x \geq 0$, it can be written as

$$\boxed{y^2 - x^2 = cx^3}$$