# MATH-281
# Complex Variables

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## Contents

# 1 Complex Numbers

## 1.1 Definition

**Definition 1.1.1: Complex Numbers**

A complex number is a number that can be expressed in the form $a + bi$, where $a$ and $b$ are real numbers, and $i$ is a solution of the equation $x^2 = -1$, or simply, $i = \sqrt{-1}$. Because no real number satisfies this equation, $i$ is called an imaginary number. For the complex number $a + bi$, $a$ is called the real part, and $b$ is called the imaginary part.

- The set of all complex numbers is denoted by $\mathbb{C}$.
- The set of all real numbers is denoted by $\mathbb{R}$.

**Definition 1.1.2: Modulus and Amplitude**

Let $z = a + bi$ be a complex number. The modulus of $z$ is the non-negative real number $|z| = \sqrt{a^2 + b^2}$. The amplitude of $z$ is the angle $\theta$ such that $\cos(\theta) = \frac{a}{|z|}$ and $\sin(\theta) = \frac{b}{|z|}$.

If the polar form of the point $(a, b)$ be $(r, \theta)$, then $a = r\cos\theta$ and $b = r\sin\theta$.

$$r = |z| = \sqrt{a^2 + b^2} \quad \text{and} \quad \theta = \arctan\left(\frac{b}{a}\right) \tag{1.1.1}$$

Here, $r$ is the modulus of $z$ and $\theta$ is the amplitude of $z$.
In symbols, we write

$$r = \operatorname{mod}(z) = |a + ib| \quad \text{and} \quad \theta = \arg(z) = \tan^{-1}\left(\frac{b}{a}\right) \tag{1.1.2}$$

## 1.2 De Moivre's Theorem

***Theorem 1.2.1* (De Moivre's Theorem)*:*** *Let $z = r(\cos\theta + i\sin\theta)$ be a complex number. Then, for any positive integer $n$,*

$$z^n = r^n(\cos n\theta + i\sin n\theta) \tag{1.2.1}$$

**Proof:**
**Case 1:** $n \in \mathbb{Z}_+$
We have,

$$\begin{aligned} z_1 z_2 \ldots z_n &= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\ldots(\cos\theta_n + i\sin\theta_n) \\ &= \big\{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\big\}\ldots(\cos\theta_n + i\sin\theta_n) \\ &= \cos(\theta_1 + \theta_2 + \ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \ldots + \theta_n) \end{aligned}$$

Hence, we get

$$z^n = (\cos n\theta + i\sin n\theta) \quad \square$$

**Case 2:** $n \in \mathbb{Z}_-$
Let $n = -m$. We have,

$$\begin{aligned} z^n &= (\cos\theta + i\sin\theta)^{-m} \\ &= \frac{1}{(\cos\theta + i\sin\theta)^m} \\ &= \frac{1}{\cos m\theta + i\sin m\theta} \\ &= \frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta} \\ &= \cos m\theta - i\sin m\theta \\ &= \cos n\theta + i\sin n\theta \end{aligned}$$

Hence, we get

$$z^n = (\cos n\theta + i\sin n\theta) \quad \square$$

**Case 3:** $n \in \mathbb{Q}$, i.e. $n = \frac{p}{q}$, where $p, q \in \mathbb{Z}$ and $q \neq 0$.
Now,

$$\begin{aligned} \left(\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta\right)^q &= \cos\left(q \cdot \frac{p}{q}\theta\right) + i\sin\left(q \cdot \frac{p}{q}\theta\right) \\ &= \cos p\theta + i\sin p\theta \\ &= (\cos\theta + i\sin\theta)^p \end{aligned}$$

Taking the $q^{th}$ root of both sides, we get

$$\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta = (\cos\theta + i\sin\theta)^{\frac{p}{q}} \quad \square$$

**Note:-**

**Some Important Results:**

(i) $1 = e^{i2n\pi} = \cos 2n\pi + i\sin 2n\pi$

(ii) $-1 = \cos\pi + i\sin\pi = e^{i\pi}$

(iii) $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\frac{\pi}{2}}$

(iv) $-i = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2} = e^{-i\frac{\pi}{2}}$

# 2 Analytic Functions

## 2.1 Definitions

**Definition 2.1.1: Complex variable**

A **complex variable** is a variable that can take on complex values. A complex variable is usually denoted by $z$.

If $x$ and $y$ are real variables, then $z = x + iy$ is a complex variable, where $i$ is the imaginary unit.

**Definition 2.1.2: Complex Function**

A **complex function** is a function that takes complex variables as input and returns complex values. A complex function is usually denoted by $f(z)$.

If $z = x + iy$ and $w = u + iv$ are complex variables, then $f(z) = u(x, y) + iv(x, y)$ is a complex function, where $u(x, y)$ and $v(x, y)$ are real functions.

**Definition 2.1.3: Single-valued Function**

A **single-valued function** is a function that returns a unique value for each input.

A complex function $f(z)$ is single-valued if and only if $f(z_1) = f(z_2)$ implies $z_1 = z_2$. In other words, if $z_1 \neq z_2$, then $f(z_1) \neq f(z_2)$.

$$\forall z_1, z_2 \in \mathbb{C} \quad \text{s.t.} \quad z_1 \neq z_2 \quad \text{implies} \quad f(z_1) \neq f(z_2)$$

**Definition 2.1.4: Multiple-valued Function**

A **multiple-valued function** is a function that returns multiple values for each input.

A complex function $f(z)$ is multiple-valued if and only if $f(z_1) = f(z_2)$ for some $z_1 \neq z_2$.

$$\exists z_1, z_2 \in \mathbb{C} \quad \text{s.t.} \quad z_1 \neq z_2 \quad \text{and} \quad f(z_1) = f(z_2)$$

**Definition 2.1.5: Derivative**

The **derivative** of a complex function $f(z)$ is defined as

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

where $\Delta z$ is a complex number.

If the limit exists, then $f(z)$ is said to be **differentiable** at $z$. If $f(z)$ is differentiable at every point in a region $R$, then $f(z)$ is said to be **analytic** in $R$.

**Definition 2.1.6: Analytic Function**

A complex function $f(z)$ is **analytic** in a region $R$ if it is differentiable at every point in $R$.

If $f(z)$ is analytic in a region $R$, then $f(z)$ is also said to be **regular** or **holomorphic** in $R$.

## 2.2 Necessary Conditions for Analyticity

Let $f(z) = u(x, y) + iv(x, y)$ be an analytic function in a region $R$.

That means, $f(z)$ is differentiable at every point in $R$.

$$\implies f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \text{ exists at every point in } R.$$

Now, let $z = x + iy$ and $\Delta z = \Delta x + i\Delta y$.

$$z + \Delta z = (x + \Delta x) + i(y + \Delta y)$$

Then,

$$\begin{aligned}
f'(z) &= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(x + \Delta x + i(y + \Delta y)) - f(x + iy)}{\Delta x + i\Delta y} \\
&= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y) - u(x, y) - iv(x, y)}{\Delta x + i\Delta y} \\
&= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{u(x + \Delta x, y + \Delta y) - u(x, y)}{\Delta x + i\Delta y} + i \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{v(x + \Delta x, y + \Delta y) - v(x, y)}{\Delta x + i\Delta y}
\end{aligned}$$

Along the real axis, $\Delta y = 0$. Hence, the limit is

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x + \Delta x, y) - v(x, y)}{\Delta x}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \tag{2.2.1}$$

Along the imaginary axis, $\Delta x = 0$. Hence, the limit is

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y + \Delta y) - u(x, y)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x, y + \Delta y) - v(x, y)}{i\Delta y}$$

$$f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \tag{2.2.2}$$

## 2.3 Cauchy-Riemann Equations

Since $f'(z)$ exists, (2.2.4) and (2.2.5) must be equal.

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \tag{2.3.1}$$

Comparing real and imaginary parts,

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \quad \text{and} \quad \boxed{\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}} \tag{2.3.2}$$

These are called the **Cauchy-Riemann equations**.

## 2.4 Cauchy-Riemann Equations in Polar Form

Let

$$z = re^{i\theta}$$

$$f(z) = u(r, \theta) + iv(r, \theta)$$

Then

$$f(re^{i\theta}) = u(r, \theta) + iv(r, \theta) \tag{2.4.1}$$

Differentiating (2.4.1) with respect to $r$, we get

$$\frac{\partial f}{\partial r} = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} \tag{2.4.2}$$