Complex Numbers

1. **Definition:** A number of the form $a+ib$ where a and b are real numbers and $i=\sqrt{-1}$, is called a complex number.

If $z=a+ib$, then a is called the real part of z and b is called the imaginary part of z.

2. **Modulus and Amplitude:**

If the polar coordinates of the point (a,b) be (r,θ), then $a=r\cos\theta$ and $b=r\sin\theta$

$$\therefore\ r=\sqrt{a^2+b^2} \text{ and } \theta=\tan^{-1}\left(\frac{b}{a}\right)$$

The number r is called the modulus or absolute value and θ is called the amplitude or argument of the complex number $z=a+ib$.

In symbols, we write

$$r=\text{mod}\,z=|z|=|a+ib|=\sqrt{a^2+b^2}$$

$$\theta=\text{amp}\,z=\arg z=\tan^{-1}\left(\frac{b}{a}\right)$$

Now, $z=a+ib=r(\cos\theta+i\sin\theta)=re^{i\theta}$

3. State and prove De Moivre's theorem.

Statement: For all rational values of n,

$$(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta.$$

Proof: Case-1: When n is a positive integer.

We have, $(\cos\theta_1+i\sin\theta_1)(\cos\theta_2+i\sin\theta_2)$

$$=\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2+i(\sin\theta_1\cos\theta_2+\cos\theta_1\sin\theta_2)$$

$$=\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)$$

Similarly, $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)(\cos\theta_3 + i\sin\theta_3)$

$$= \{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\}(\cos\theta_3 + i\sin\theta_3)$$

$$= \cos(\theta_1+\theta_2+\theta_3) + i\sin(\theta_1+\theta_2+\theta_3)$$

Proceeding in this way, the product of the n factors

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \cdots\cdots (\cos\theta_n + i\sin\theta_n)$$

$$= \cos(\theta_1+\theta_2+\cdots+\theta_n) + i\sin(\theta_1+\theta_2+\cdots+\theta_n)$$

If we put $\theta_1 = \theta_2 = \cdots = \theta_n = \theta$, then we have

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

<u>Case-2</u>: When n is a negative integer.

Let $n = -m$, where m is a positive integer.

$$(\cos\theta + i\sin\theta)^n = (\cos\theta + i\sin\theta)^{-m}$$

$$= \frac{1}{(\cos\theta + i\sin\theta)^m}$$

$$= \frac{1}{\cos m\theta + i\sin m\theta} \quad [\text{by Case-I}]$$

$$= \frac{1 \times (\cos m\theta - i\sin m\theta)}{(\cos m\theta + i\sin m\theta)(\cos m\theta - i\sin m\theta)}$$

$$= \frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta}$$

$$= \cos m\theta - i\sin m\theta$$

$$= \cos(-n)\theta - i\sin(-n)\theta \quad [\because m = -n]$$

$$= \cos n\theta + i\sin n\theta$$

<u>Case-3</u>: When n is a fraction, +ve or -ve.

Let $n = \frac{p}{q}$, where q is a positive integer and p is any integer, +ve or -ve.

Now, $\left(\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta\right)^q = \cos\left(q\cdot\frac{p}{q}\theta\right) + i\sin\left(q\cdot\frac{p}{q}\theta\right)$

[by Case 1]

$= \cos p\theta + i\sin p\theta$

$= (\cos\theta + i\sin\theta)^p$, since p is any integer.

Taking qth root, we get

$\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta$ = One of the values of $(\cos\theta + i\sin\theta)^{\frac{p}{q}}$

or, $\cos n\theta + i\sin n\theta$ = One of the values of $(\cos\theta + i\sin\theta)^n$

Thus, De Moivre's theorem is completely established for all rational values of n.

4. If $x_r = \cos\frac{\pi}{2^r} + i\sin\frac{\pi}{2^r}$, prove that

$$x_1 x_2 x_3 \ldots\ldots \infty = -1$$

<u>Solution:</u> LHS $= x_1 x_2 x_3 \ldots\ldots \infty$

$= \left(\cos\frac{\pi}{2^1} + i\sin\frac{\pi}{2^1}\right)\left(\cos\frac{\pi}{2^2} + i\sin\frac{\pi}{2^2}\right)\left(\cos\frac{\pi}{2^3} + i\sin\frac{\pi}{2^3}\right)\ldots\infty$

$= \cos\left(\frac{\pi}{2^1} + \frac{\pi}{2^2} + \frac{\pi}{2^3} + \ldots\infty\right) + i\sin\left(\frac{\pi}{2^1} + \frac{\pi}{2^2} + \frac{\pi}{2^3} + \ldots\infty\right)$

$= \cos\left(\frac{\pi/2}{1-1/2}\right) + i\sin\left(\frac{\pi/2}{1-1/2}\right)$ $\left[\because S_\infty = \frac{a}{1-r}\right]$

$= \cos\pi + i\sin\pi$

$= -1 + i\cdot 0$

$= -1$

$=$ R.H.S

5. If $x_r = \cos\frac{\pi}{3^r} + i\sin\frac{\pi}{3^r}$, show that

$$x_1 x_2 x_3 \ldots\ldots \infty = i$$

<u>Solution:</u> LHS $= x_1 x_2 x_3 \ldots\ldots \infty$

$= \left(\cos\frac{\pi}{3^1} + i\sin\frac{\pi}{3^1}\right)\left(\cos\frac{\pi}{3^2} + i\sin\frac{\pi}{3^2}\right)\ldots\ldots\infty$

$= \cos\left(\frac{\pi}{3^1} + \frac{\pi}{3^2} + \frac{\pi}{3^3} + \ldots\infty\right) + i\sin\left(\frac{\pi}{3^1} + \frac{\pi}{3^2} + \frac{\pi}{3^3} + \ldots\infty\right)$

$$\therefore \text{LHS} = \cos\left(\frac{\pi/3}{1-1/3}\right) + i\sin\left(\frac{\pi/3}{1-1/3}\right)$$
$$= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$
$$= 0 + i\cdot 1$$
$$= i$$
$$= \text{R.H.S}$$

6. If n be a positive integer, prove that

(i) $(1+i)^n + (1-i)^n = 2^{\frac{n}{2}+1}\cos\frac{n\pi}{4}$

(ii) $(\sqrt{3}+i)^n + (\sqrt{3}-i)^n = 2^{n+1}\cos\frac{n\pi}{6}$

<u>Solution</u>: (i) Let $1 = r\cos\theta$ --- (1)

$1 = r\sin\theta$ --- (2)

Squaring (1) and (2), then adding

$$2 = r^2(\cos^2\theta + \sin^2\theta)$$
$$= r^2 \cdot 1$$
$$\therefore r = \sqrt{2}$$

Dividing (2) by (1), we get $\tan\theta = 1$

or, $\tan\theta = \tan\frac{\pi}{4}$

$\therefore \theta = \frac{\pi}{4}$

Now, $1+i = 1 + i\cdot 1$
$$= r\cos\theta + ir\sin\theta$$
$$= r(\cos\theta + i\sin\theta)$$
$$= \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$$\therefore (1+i)^n = \left\{\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right\}^n$$
$$= 2^{n/2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^n$$
$$= 2^{n/2}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right) \quad \cdots (3)$$

Similarly, $(1-i)^n = 2^{n/2}\left(\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right) \quad \cdots (4)$

Adding (3) and (4), we get

$$(1+i)^n + (1-i)^n = 2\cdot 2^{n/2}\cos\frac{n\pi}{4}$$
$$= 2^{\frac{n}{2}+1}\cos\frac{n\pi}{4}$$

(ii) Let $\sqrt{3} = r\cos\theta$ ...(1)

$1 = r\sin\theta$ ....(2)

Squaring (1) and (2), then adding

$$3+1 = r^2(\cos^2\theta + \sin^2\theta)$$

or, $4 = r^2 \cdot 1$

$\therefore r = 2$

Dividing (2) by (1), $\tan\theta = \frac{1}{\sqrt{3}}$

or, $\tan\theta = \tan\frac{\pi}{6}$

$\therefore \theta = \frac{\pi}{6}$

Now, $\sqrt{3}+i = \sqrt{3} + i\cdot 1$

$= r\cos\theta + i r\sin\theta$

$= r(\cos\theta + i\sin\theta)$

$= 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$\therefore (\sqrt{3}+i)^n = \left\{2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right\}^n$

$= 2^n\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^n$

$= 2^n\left(\cos\frac{n\pi}{6} + i\sin\frac{n\pi}{6}\right)$ ...(3)

Similarly, $(\sqrt{3}-i)^n = 2^n\left(\cos\frac{n\pi}{6} - i\sin\frac{n\pi}{6}\right)$ ...(4)

Adding (3) and (4), we get

$$(\sqrt{3}+i)^n + (\sqrt{3}-i)^n = 2\cdot 2^n\cos\frac{n\pi}{6}$$

$$= 2^{n+1}\cos\frac{n\pi}{6}$$

7. Determine the locus represented by

(i) $|z-2| = 3$ (ii) $|z-2| = |z+4|$ (iii) $|z-3| + |z+3| = 10$

<u>Solution</u>: (i) $|z-2| = 3$

or, $|x+iy-2| = 3$

or, $|(x-2)+iy| = 3$

or $\sqrt{(x-2)^2+y^2} = 3$

or, $(x-2)^2 + y^2 = 3^2$ which represents a circle of radius 3 and centre (2,0).

(ii) $|z-2| = |z+4|$

or, $|x+iy-2| = |x+iy+4|$

or, $(x-2)^2+y^2 = (x+4)^2+y^2$

or, $x=-1$ which represents a straight line.

(iii) $|z-3|+|z+3| = 10$

or, $\sqrt{(x-3)^2+y^2} + \sqrt{(x+3)^2+y^2} = 10$

or, $\sqrt{(x-3)^2+y^2} = 10 - \sqrt{(x+3)^2+y^2}$

or, $(x-3)^2+y^2 = 100 - 20\sqrt{(x+3)^2+y^2} + (x+3)^2+y^2$

or, $12x+100 = 20\sqrt{(x+3)^2+y^2}$

or, $3x+25 = 5\sqrt{(x+3)^2+y^2}$

or, $9x^2+150x+625 = 25(x^2+6x+9+y^2)$

or, $16x^2+25y^2 = 400$

or, $\frac{x^2}{25}+\frac{y^2}{16} = 1$

or, $\frac{x^2}{5^2}+\frac{y^2}{4^2} = 1$ which represents an ellipse.

8. If $x+\frac{1}{x} = 2\cos\theta$, show that $x^n+\frac{1}{x^n} = 2\cos n\theta$

<u>Solution:</u> We have, $x+\frac{1}{x} = 2\cos\theta$

or, $x^2-2x\cos\theta+1=0$

$\therefore x = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta-4}}{2}$

$= \cos\theta \pm i\sin\theta$

Take +ve sign only, $x = \cos\theta + i\sin\theta$

Now LHS $= x^n + \frac{1}{x^n}$

$= (\cos\theta+i\sin\theta)^n + (\cos\theta+i\sin\theta)^{-n}$

$= \cos n\theta + i\sin n\theta + \cos(-n)\theta + i\sin(-n)\theta$ [By De Moivre's theorem

$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$= 2\cos n\theta$

$=$ RHS

9. If $x = \cos\theta + i\sin\theta$ and $1+\sqrt{1-a^2} = na$, prove that

$$1 + a\cos\theta = \frac{a}{2n}(1+nx)\left(1+\frac{n}{x}\right)$$

<u>Solution:</u> Given, $x = \cos\theta + i\sin\theta$

$$\therefore \frac{1}{x} = \frac{1}{\cos\theta + i\sin\theta}$$

$$= (\cos\theta + i\sin\theta)^{-1}$$

$$= \cos\theta - i\sin\theta$$

$$\therefore x + \frac{1}{x} = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta$$

$$= 2\cos\theta$$

Also, $1+\sqrt{1-a^2} = na$

or, $\sqrt{1-a^2} = na - 1$

or, $1-a^2 = n^2a^2 - 2na + 1$

or, $n^2a^2 + a^2 = 2na$

or, $a^2(1+n^2) = 2na$

or, $\dfrac{a^2(1+n^2)}{2na} = 1$

$$\therefore \frac{a(1+n^2)}{2n} = 1$$

Now, LHS $= 1 + a\cos\theta$

$$= \frac{a(1+n^2)}{2n} + a\cdot\frac{x+\frac{1}{x}}{2}$$

$$= \frac{a}{2n}\left\{(1+n^2) + n\left(x+\frac{1}{x}\right)\right\}$$

$$= \frac{a}{2n}\left(1+n^2 + nx + \frac{n}{x}\right)$$

$$= \frac{a}{2n}\left\{1\cdot(1+nx) + \frac{n}{x}(1+nx)\right\}$$

$$= \frac{a}{2n}(1+nx)\left(1+\frac{n}{x}\right)$$

$$= \text{R.H.S}$$

<u>Important results:</u> (i) $1 = e^{i2n\pi} = \cos 2n\pi + i\sin 2n\pi$

(ii) $-1 = \cos\pi + i\sin\pi = e^{i\pi}$

(iii) $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\pi/2}$

(iv) $-i = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2} = e^{-i\pi/2}$

10. If α, β be the roots of $x^2-2x+4=0$, prove that $\alpha^n+\beta^n = 2^{n+1}\cos\frac{n\pi}{3}$

<u>Solution:</u> Given, $x^2-2x+4=0$

$$\therefore x = \frac{-(-2)\pm\sqrt{(-2)^2-4\cdot1\cdot4}}{2\cdot1}$$
$$= \frac{2\pm\sqrt{-12}}{2}$$
$$= 1\pm\sqrt{-3}$$
$$= 1\pm i\sqrt{3}$$

$\therefore \alpha = 1+i\sqrt{3}$
$$= 2\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$$
$$= 2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$$

and $\beta = 1-i\sqrt{3}$
$$= 2\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)$$
$$= 2\left(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}\right)$$

$$\therefore \alpha^n = 2^n\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^n$$
$$= 2^n\left(\cos\frac{n\pi}{3}+i\sin\frac{n\pi}{3}\right) \quad \cdots (1)$$

and $\beta^n = 2^n\left(\cos\frac{n\pi}{3}-i\sin\frac{n\pi}{3}\right) \quad \cdots (2)$

Adding (1) and (2) we get
$$\alpha^n+\beta^n = 2^n\cdot 2\cos\frac{n\pi}{3}$$
$$= 2^{n+1}\cos\frac{n\pi}{3}$$

11. Show that $\sin(\ln i^i) = -1$

<u>Solution:</u> $i = e^{\frac{i\pi}{2}}\cdot e^{i2n\pi}$ $\quad [\because 1 = e^{i2n\pi}]$
$$= e^{\frac{i\pi}{2}(4n+1)}$$
$$\therefore i^i = e^{-(4n+1)\frac{\pi}{2}}$$
$$\therefore \ln i^i = -(4n+1)\frac{\pi}{2}$$

Now, $\sin(\ln i^i) = \sin\left\{-(4n+1)\frac{\pi}{2}\right\}$
$$= -\sin\left(2n\pi+\frac{\pi}{2}\right)$$
$$= -\sin\frac{\pi}{2}$$
$$= -1$$

12. If $x = \cos\alpha + i\sin\alpha$, $y = \cos\beta + i\sin\beta$, $z = \cos\gamma + i\sin\gamma$ and $x+y+z=0$, then prove that

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=0.$$

<u>Solution</u>: We have, $x+y+z=0$

or, $\cos\alpha + i\sin\alpha + \cos\beta + i\sin\beta + \cos\gamma + i\sin\gamma = 0$

or, $(\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma) = 0$

Equating real and imaginary parts, we get

$\cos\alpha + \cos\beta + \cos\gamma = 0$ ---(1)

and $\sin\alpha + \sin\beta + \sin\gamma = 0$ --(2)

Now $LHS = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$

$= (\cos\alpha + i\sin\alpha)^{-1} + (\cos\beta + i\sin\beta)^{-1} + (\cos\gamma + i\sin\gamma)^{-1}$

$= \cos\alpha - i\sin\alpha + \cos\beta - i\sin\beta + \cos\gamma - i\sin\gamma$

$= (\cos\alpha + \cos\beta + \cos\gamma) - i(\sin\alpha + \sin\beta + \sin\gamma)$

$= 0 - i\cdot 0$

$= 0$

13. Using De Moivre's theorem, solve the equations:

(i) $x^9 = 1$ (ii) $(x+1)^5 + (x-1)^5 = 0$ (iii) $x^4 + x^2 + 1 = 0$

<u>Solution</u>: (i) $x^9 = 1$

or, $x^9 = \cos 2n\pi + i\sin 2n\pi$ $[\because 1 = e^{i2n\pi} = \cos 2n\pi + i\sin 2n\pi]$

or, $x = (\cos 2n\pi + i\sin 2n\pi)^{\frac{1}{9}}$

or, $x = \cos\frac{2n\pi}{9} + i\sin\frac{2n\pi}{9}$

Putting $n = 0, 1, 2, 3, 4, 5, 6, 7, 8$, the required solutions are

$\cos 0 + i\sin 0$, $\cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}$, $\cos\frac{4\pi}{9} + i\sin\frac{4\pi}{9}$,

$\cos\frac{6\pi}{9} + i\sin\frac{6\pi}{9}$, $\cos\frac{8\pi}{9} + i\sin\frac{8\pi}{9}$, $\cos\frac{10\pi}{9} + i\sin\frac{10\pi}{9}$,

$\cos\frac{12\pi}{9} + i\sin\frac{12\pi}{9}$, $\cos\frac{14\pi}{9} + i\sin\frac{14\pi}{9}$, $\cos\frac{16\pi}{9} + i\sin\frac{16\pi}{9}$

(ii) $(x+1)^5 + (x-1)^5 = 0$

or, $(x+1)^5 = -(x-1)^5$

or, $\left(\frac{x+1}{x-1}\right)^5 = -1$

or, $\frac{x+1}{x-1} = (-1)^{\frac{1}{5}}$

or, $\frac{x+1}{x-1} = \left(e^{i\pi} \cdot e^{i2n\pi}\right)^{\frac{1}{5}}$

or, $\frac{x+1}{x-1} = \left\{e^{i(2n\pi+\pi)}\right\}^{\frac{1}{5}}$

or, $\frac{x+1}{x-1} = \left\{\cos(2n\pi+\pi) + i\sin(2n\pi+\pi)\right\}^{\frac{1}{5}}$

or, $\frac{x+1}{x-1} = \cos\left(\frac{2n\pi+\pi}{5}\right) + i\sin\left(\frac{2n\pi+\pi}{5}\right)$

or, $\frac{x+1}{x-1} = \frac{\cos\theta + i\sin\theta}{1}$ where $\theta = \frac{2n\pi+\pi}{5}$

Using componendo and dividendo, we get

$$\frac{x+1+x-1}{x+1-x+1} = \frac{\cos\theta + i\sin\theta + 1}{\cos\theta + i\sin\theta - 1}$$

or, $$\frac{2x}{2} = -\frac{2\cos^2\theta/2 + i\cdot 2\sin\theta/2\cos\theta/2}{2\sin^2\theta/2 - i\cdot 2\sin\theta/2\cos\theta/2}$$

or, $$x = -\frac{2\cos\theta/2\left(\cos\theta/2 + i\sin\theta/2\right)}{2\sin\theta/2\left(\sin\theta/2 - i\cos\theta/2\right)}$$

or, $$x = -i\cot\theta/2\left[\frac{\cos\theta/2 + i\sin\theta/2}{i\sin\theta/2 + \cos\theta/2}\right]$$

or, $x = -i\cot\theta/2$

$\therefore x = -i\cot(2n+1)\frac{\pi}{10}$.

Putting $n = 0, 1, 2, 3, 4$, the required solutions are

$-i\cot\frac{\pi}{10}$, $-i\cot\frac{3\pi}{10}$, $-i\cot\frac{\pi}{2}$, $-i\cot\frac{7\pi}{10}$, $-i\cot\frac{9\pi}{10}$.

(iii) Given, $x^4+x^2+1=0$

or, $(x^2-1)(x^4+x^2+1)=0$ [Multiplying bothsides by (x^2-1)]

or, $(x^2)^3-(1)^3=0$

or, $x^6-1=0$ or, $x^6=1$

or, $x^6=\cos 2n\pi+i\sin 2n\pi$ [$\because 1=\cos 2n\pi+i\sin 2n\pi$]

or, $x=(\cos 2n\pi+i\sin 2n\pi)^{\frac{1}{6}}$

or, $x=\cos\frac{2n\pi}{6}+i\sin\frac{2n\pi}{6}$

$\therefore x=\cos\frac{n\pi}{3}+i\sin\frac{n\pi}{3}$ ... (1)

Putting $n=0,1,2,3,4,5$ in (1), we get

$x=1,\ \cos\frac{\pi}{3}+i\sin\frac{\pi}{3},\ \cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3},\ -1,$

$\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3},\ \cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}$

Among these values $x=\pm 1$ will be omitted as we have multiplied the equation by x^2-1. Hence the four roots of the given equation are

$x=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3},\ \cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3},\ \cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3},$

$\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}$.

14. Find all the values of (i) $(1+i)^{\frac{1}{5}}$ (ii) $(-i)^{\frac{1}{6}}$

<u>Solution:</u> Let us put $1=r\cos\theta$ and $1=r\sin\theta$

$\therefore r=\sqrt{2}$ and $\theta=\frac{\pi}{4}$

Now, $(1+i)^{\frac{1}{5}}=(1+i.1)^{\frac{1}{5}}$

$=(r\cos\theta+ir\sin\theta)^{\frac{1}{5}}$

$=\{r(\cos\theta+i\sin\theta)\}^{\frac{1}{5}}$

$=\{r.e^{i\theta}.e^{i2n\pi}\}^{\frac{1}{5}}$

$\therefore \quad (1+i)^{\frac{1}{5}} = \{ r.\, e^{i(2n\pi+\theta)} \}^{\frac{1}{5}}$

$= \{ \sqrt{2}\, e^{i(2n\pi+\frac{\pi}{4})} \}^{\frac{1}{5}}$

$= 2^{\frac{1}{10}} \{ \cos(2n\pi+\frac{\pi}{4}) + i\sin(2n\pi+\frac{\pi}{4}) \}^{\frac{1}{5}}$

$= 2^{\frac{1}{10}} \{ \cos(8n+1)\frac{\pi}{20} + i\sin(8n+1)\frac{\pi}{20} \}$

Putting $n=0,1,2,3,4$, the required values are

$2^{\frac{1}{10}}(\cos\frac{\pi}{20} + i\sin\frac{\pi}{20})$, $2^{\frac{1}{10}}(\cos\frac{9\pi}{20} + i\sin\frac{9\pi}{20})$,

$2^{\frac{1}{10}}(\cos\frac{17\pi}{20} + i\sin\frac{17\pi}{20})$, $2^{\frac{1}{10}}(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4})$,

$2^{\frac{1}{10}}(\cos\frac{33\pi}{20} + i\sin\frac{33\pi}{20})$

(ii) $(-i)^{\frac{1}{6}} = (\cos\frac{\pi}{2} - i\sin\frac{\pi}{2})^{\frac{1}{6}}$

$= (e^{-i\frac{\pi}{2}})^{\frac{1}{6}}$

$= (e^{-\frac{i\pi}{2}} . e^{i2n\pi})^{\frac{1}{6}} \quad [\because 1 = e^{i2n\pi}]$

$= \{ e^{i(4n-1)\frac{\pi}{2}} \}^{\frac{1}{6}}$

$= \{ \cos(4n-1)\frac{\pi}{2} + i\sin(4n-1)\frac{\pi}{2} \}^{\frac{1}{6}}$

$= \cos(4n-1)\frac{\pi}{12} + i\sin(4n-1)\frac{\pi}{12}$

Putting $n=0,1,2,3,4,5$, the required values are

$\cos\frac{\pi}{12} - i\sin\frac{\pi}{12}$, $\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$, $\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}$,

$\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}$, $\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}$, $\cos\frac{19\pi}{12} + i\sin\frac{19\pi}{12}$

Analytic functions-1

Complex variable: A symbol, such as z, which can stand for any one of a set of complex numbers is called a complex variable.

If x and y are real variables, then $z = x+iy$ is called a complex variable.

Function: If to each value which a complex variable z may assume there corresponds one or more values of a complex variable w, then w is called a function of z, written $w = f(z)$.

Single-valued function: If for each value of z there corresponds only one value of w, then w is called a single-valued function of z or that $f(z)$ is single-valued.

Multiple-valued function: If for each value of z there corresponds more than one value of w, then w is called a multiple-valued or many-valued function of z or that $f(z)$ is multiple-valued.

Derivative: If $f(z)$ is a single-valued function defined in a region R of the z-plane, the derivative of $f(z)$ is defined as

$$f'(z) = \frac{df(z)}{dz} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

provided the limit exists.

1. Define an analytic function. Find the necessary conditions for a function $f(z) = u + iv$ to be analytic in a region R.

Solution:

Analytic function: A single-valued function $f(z)$ which is differentiable at every point of a region R, is called as an analytic function of z in R.

An analytic function can also be called as regular function or holomorphic function.

Necessary conditions for a function $f(z) = u + iv$ to be analytic:

Let $f(z) = u(x,y) + iv(x,y)$ be an analytic function in a region R.

$\therefore f(z)$ is differentiable in R.

$$\Rightarrow f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \text{ exists in R.}$$

Let $z = x + iy$

$\therefore \Delta z = \Delta x + i\Delta y$

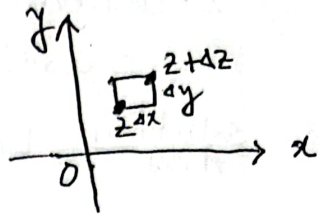
$$\therefore z + \Delta z = (x+iy) + (\Delta x + i\Delta y)$$
$$= (x + \Delta x) + i(y + \Delta y)$$

$$\therefore f(z+\Delta z) = u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)$$

If $\Delta z \to 0$ or, $\Delta x + i\Delta y \to 0$, then $\Delta x \to 0$, $\Delta y \to 0$.

$$\therefore f'(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{[u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)] - [u(x,y) + iv(x,y)]}{\Delta x + i\Delta y}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{[u(x+\Delta x, y+\Delta y) - u(x,y)] + i[v(x+\Delta x, y+\Delta y) - v(x,y)]}{\Delta x + i\Delta y}$$



Along x-axis, $\Delta y = 0$, then the limit is

$$f'(z) = \lim_{\Delta x \to 0} \frac{[u(x+\Delta x, y) - u(x,y)] + i[v(x+\Delta x, y) - v(x,y)]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \quad \text{----} (1)$$

(By definition)

Along y-axis, $\Delta x = 0$, then the limit is

$$f'(z) = \lim_{\Delta y \to 0} \frac{[u(x, y+\Delta y) - u(x,y)] + i[v(x, y+\Delta y) - v(x,y)]}{i \Delta y}$$

$$= \frac{1}{i} \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{\Delta y} + \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{\Delta y}$$

$$= \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad \text{-----} (2)$$

(By definition)

Since $f'(z)$ exists, so (1) and (2) must be equal.

i.e. $\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$

Comparing real and imaginary parts, we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \text{or} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

i.e. $u_x = v_y$ and $u_y = -v_x$

These two equations are called Cauchy-Riemann (C-R) equations.

2. Derive the polar form of Cauchy-Riemann (C-R) equations.

<u>Solution:</u> Let $z = re^{i\theta}$ and $f(z) = u(r,\theta) + iv(r,\theta)$

[By using polar form]

Then $u(r,\theta) + iv(r,\theta) = f(re^{i\theta})$ ... (1)

Differentiating (1) partially wrt r, we get

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta}).e^{i\theta} \quad \ldots (2)$$

Differentiating (1) partially wrt θ, we get

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta}).ire^{i\theta} \quad \ldots (3)$$

Now from (2) and (3), we get

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = \frac{1}{ir}\left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right)$$

or, $$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = \frac{1}{ir}\frac{\partial u}{\partial \theta} + \frac{1}{r}\frac{\partial v}{\partial \theta}$$

or, $$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = -\frac{i}{r}\frac{\partial u}{\partial \theta} + \frac{1}{r}\frac{\partial v}{\partial \theta}$$

Equating the real and imaginary parts, we get

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

The above equations are called Cauchy-Riemann (C-R) equations in Polar form.

<u>Laplace's equation</u>: An equation of the form $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ or $\nabla^2 \phi = 0$, is called a Laplace's equation (in two diemension).

<u>Harmonic function</u>: Any function having continuous second order partial derivatives which satisfies the Laplace's equation is called harmonic function.

<u>Conjugate harmonic functions</u>: Any two harmonic functions u and v such that $f(z) = u + iv$ is analytic, are called conjugate harmonic functions.

3. Examine whether the following functions are analytic or not:

(i) $e^{-x}(\cos y - i \sin y)$ (ii) $\frac{1}{z}$ $(z \neq 0)$ (iii) $\bar{z}$

(iv) $2xy + i(x^2 - y^2)$

<u>Solution</u>: (i) let $f(z) = e^{-x}(\cos y - i \sin y)$

or, $u + iv = e^{-x}\cos y - i e^{-x} \sin y$

Equating real and imaginary parts, we get—

$$u = e^{-x}\cos y, \quad v = -e^{-x}\sin y$$

$$\therefore \frac{\partial u}{\partial x} = -e^{-x}\cos y, \quad \frac{\partial u}{\partial y} = -e^{-x}\sin y$$

$$\frac{\partial v}{\partial x} = e^{-x}\sin y, \quad \frac{\partial v}{\partial y} = -e^{-x}\cos y$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

i.e. C-R equations are satisfied

$\therefore f(z) = e^{-x}(\cos y - i \sin y)$ is analytic.

(ii) Let $f(z) = \frac{1}{z}$ $(z \neq 0)$

$$= \frac{1}{x+iy}$$

$$= \frac{x-iy}{x^2+y^2}$$

or, $u+iv = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$

Equating real and imaginary parts, we get—

$$u = \frac{x}{x^2+y^2}, \quad v = -\frac{y}{x^2+y^2}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{(x^2+y^2)\cdot 1 - x\cdot 2x}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(x^2+y^2)\cdot 0 - x\cdot 2y}{(x^2+y^2)^2}$$

$$= -\frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial x} = -\frac{(x^2+y^2)\cdot 0 - y\cdot 2x}{(x^2+y^2)^2}$$

$$= \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial y} = -\frac{(x^2+y^2)\cdot 1 - y\cdot 2y}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2}$$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

i.e., C-R equations are satisfied

$\therefore f(z) = \frac{1}{z}$ is analytic except at $z=0$.

(iii) Let $f(z) = \bar{z}$

or, $u+iv = x-iy$ $[\because z = x+iy \therefore \bar{z} = x-iy]$

$\therefore u = x, \; v = -y$

$\therefore \frac{\partial u}{\partial x} = 1, \; \frac{\partial u}{\partial y} = 0 \;\Big|\; \frac{\partial v}{\partial x} = 0, \; \frac{\partial v}{\partial y} = -1$

since $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$

$\Rightarrow$ C-R equations are not satisfied.

$\therefore f(z) = \bar{z}$ is not analytic.

(iv) Let $f(z) = 2xy + i(x^2 - y^2)$

or, $u + iv = 2xy + i(x^2 - y^2)$

$\therefore u = 2xy, \; v = x^2 - y^2$

$\therefore \frac{\partial u}{\partial x} = 2y, \; \frac{\partial u}{\partial y} = 2x$

$\frac{\partial v}{\partial x} = 2x, \; \frac{\partial v}{\partial y} = -2y$

Since $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$

$\Rightarrow$ C-R equations are not satisfied.

$\therefore f(z) = 2xy + i(x^2 - y^2)$ is not analytic.

<u>Construction of Conjugate harmonic functions/ an analytic function whose real or imaginary part is given.</u>

<u>Method-1:</u> Suppose u is given.

Then $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ are Known.

By total differentiation,

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$= -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \quad \left[\because \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\right]$$

integrating it, $v = \int \left(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy\right) + c_1$

since v is Known we can construct $f(z) = u + iv$.

Similarly, if v is given we can find $\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$

Then $du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$

$= \frac{\partial v}{\partial y} dx - \frac{\partial v}{\partial x} dy$

$\Rightarrow u = \int \left(\frac{\partial v}{\partial y} dx - \frac{\partial v}{\partial x} dy\right) + c_2$

$\therefore f(z) = u + iv$

Method-2: Milne's method / Milne Thomson method

Let $f(z) = u + iv$ is to be constructed

(i) Suppose the real part 'u' is given.

Then $\frac{\partial u}{\partial x} (= u_1(x,y))$ and $\frac{\partial u}{\partial y} (= u_2(x,y))$ are known.

Then by Milne's method we have,

$$f'(z) = u_1(z,0) - iu_2(z,0)$$

integrating it wrt z, we get

$$f(z) = \int [u_1(z,0) - iu_2(z,0)] dz + c_1$$

(ii) Similarly suppose the imaginary part 'v' is given.

Then $\frac{\partial v}{\partial y} (= v_1(x,y))$ and $\frac{\partial v}{\partial x} (= v_2(x,y))$ are known.

Then by Milne's method we have,

$$f'(z) = v_1(z,0) + iv_2(z,0)$$

integrating it wrt z, we get

$$f(z) = \int [v_1(z,0) + iv_2(z,0)] dz + c_2$$

which is the analytic function.

4. Show that $u = 3x^2y + 2x^2 - y^3 - 2y^2$ is harmonic function. Also find its conjugate harmonic.

Solution: Given, $u = 3x^2y + 2x^2 - y^3 - 2y^2$

$$\therefore \frac{\partial u}{\partial x} = 6xy + 4x, \quad \frac{\partial^2 u}{\partial x^2} = 6y + 4 \quad \cdots (1)$$

$$\frac{\partial u}{\partial y} = 3x^2 - 3y^2 - 4y, \quad \frac{\partial^2 u}{\partial y^2} = -6y - 4 \quad (2)$$

Adding (1) and (2), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

or, $\nabla^2 u = 0$

i.e. u satisfies Laplace's equation.

So, u is a harmonic function.

Let v be the conjugate harmonic function.

Then we have, $dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$

$$= -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$

$$= -(3x^2 - 3y^2 - 4y)dx + (6xy + 4x)dy$$

$$= d(3xy^2 + 4xy - x^3)$$

Integrating it, we get

$$v = 3xy^2 + 4xy - x^3 + c$$

5. In a two dimensional flow, the stream function is $\psi = \tan^{-1}(y/x)$. Find the velocity potential ϕ.

Solution: Given that $\psi = \tan^{-1}(y/x)$

$$\therefore \frac{\partial \psi}{\partial x} = \frac{1}{1 + (y/x)^2} \cdot y \cdot \left(-\frac{1}{x^2}\right)$$

$$= \frac{-y}{x^2 + y^2}$$

$$\frac{\partial \psi}{\partial y} = \frac{1}{1+(y/x)^2} \cdot \frac{1}{x}$$

$$= \frac{x}{x^2+y^2}$$

Then we have,

$$d\varphi = \frac{\partial \varphi}{\partial x} dx + \frac{\partial \varphi}{\partial y} dy \quad \left[\because W = \varphi + i\psi \quad \therefore \frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y}, \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x}\right]$$

$$= \frac{\partial \psi}{\partial y} dx - \frac{\partial \psi}{\partial x} dy$$

$$= \frac{x}{x^2+y^2} dx + \frac{y}{x^2+y^2} dy$$

integrating it, we get

$$\varphi = \frac{1}{2} \ln(x^2+y^2) + \frac{1}{2} \ln(x^2+y^2) + C$$

$$= \ln(x^2+y^2) + C$$ is the required velocity potential.

6. (a) Prove that the function $u = 2x(1-y)$ is harmonic.
(b) Find a function v such that $f(z) = u + iv$ is analytic [i.e. find the conjugate function of u] (c) Express $f(z)$ in terms of z.

Solution: (a) Given that $u = 2x(1-y)$ ---(1)

$$\therefore \frac{\partial u}{\partial x} = 2 - 2y$$

$$= u_1(x,y), \text{ say}$$

$$\frac{\partial^2 u}{\partial x^2} = 0 \quad \text{------} (2)$$

$$\frac{\partial u}{\partial y} = -2x$$

$$= u_2(x,y), \text{ say}$$

$$\frac{\partial^2 u}{\partial y^2} = 0 \quad \text{----} (3)$$

Adding (2) and (3) we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Since u satisfies Laplace's equation, so u is a harmonic function.

(c) By Milne's method we have

$$f'(z) = u_1(z,0) - iu_2(z,0)$$
$$= 2 - i\cdot(-2z)$$
$$= 2 + i2z$$

Integrating it, we get

$$f(z) = 2z + i2\cdot\frac{z^2}{2} + c, \text{ where } c$$

is the complex const.

$\therefore f(z) = 2z + iz^2 + c$ is the required function.

(b) From (c) we have,

$$f(z) = 2z + iz^2 + c$$
$$= 2(x+iy) + i(x+iy)^2 + c_1 + ic_2$$
$$\text{or, } u + iv = 2(x+iy) + i(x^2 + i2xy - y^2) + c_1 + ic_2$$
$$= 2x - 2xy + i(2y + x^2 - y^2) + c_1 + ic_2$$

Equating the imaginary parts, we get

$v = 2y + x^2 - y^2 + c_2$ is the required conjugate harmonic.

7. Show that the function $f(z) = u + iv$, where

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, \quad z \neq 0$$
$$= 0, \quad z = 0$$

satisfies the Cauchy-Riemann conditions at $z=0$. Is the function analytic at $z=0$? Justify your answer.

<u>Solution:</u> Given, $f(z) = \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$

$$\text{or, } u + iv = \frac{x^3 - y^3 + i(x^3+y^3)}{x^2+y^2}$$

$$\therefore u(x,y) = \frac{x^3-y^3}{x^2+y^2}, \quad v(x,y) = \frac{x^3+y^3}{x^2+y^2}$$

Also, $u(0,0)=0$, $v(0,0)=0$ $\quad$ [$\because$ At $z=0$, $f(0)=0$]

At the origin (i.e. at $z=0$)

$$\frac{\partial u}{\partial x} = \lim_{h\to 0} \frac{u(0+h,0)-u(0,0)}{h}$$
$$= \lim_{h\to 0} \frac{h^3/h^2}{h}$$
$$= 1$$

$$\frac{\partial u}{\partial y} = \lim_{K\to 0} \frac{u(0,0+K)-u(0,0)}{K}$$
$$= \lim_{K\to 0} \frac{-K^3/K^2}{K}$$
$$= -1$$

$$\frac{\partial v}{\partial x} = \lim_{h\to 0} \frac{v(0+h,0)-v(0,0)}{h}$$
$$= \lim_{h\to 0} \frac{h^3/h^2}{h}$$
$$= 1$$

$$\frac{\partial v}{\partial y} = \lim_{K\to 0} \frac{v(0,0+K)-v(0,0)}{K}$$
$$= \lim_{K\to 0} \frac{K^3/K^2}{K}$$
$$= 1$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

i.e., Cauchy-Riemann Conditions are satisfied at $z=0$.

Now $$f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z}$$
$$= \lim_{z\to 0} \left[\frac{x^3-y^3+i(x^3+y^3)}{x^2+y^2} \cdot \frac{1}{x+iy}\right]$$

Let $z\to 0$ along $y=x$, then
$$f'(0) = \lim_{x\to 0} \left[\frac{x^3-x^3+i(x^3+x^3)}{x^2+x^2} \cdot \frac{1}{x+ix}\right]$$
$$= \frac{2i}{2(1+i)} = \frac{i(1-i)}{1-i^2} = \frac{1}{2}(1+i)$$

Again let $z\to 0$ along $y=0$, then
$$f'(0) = \lim_{x\to 0} \frac{x^3+ix^3}{x^2} \cdot \frac{1}{x} = 1+i$$

So we see that $f'(0)$ is not unique. Hence the function $f(z)$ is not analytic at $z=0$.

Analytic function-2

The complex potential function

The analytic function $W = \varphi(x,y) + i\psi(x,y)$ is referred to as the complex potential function. Its real part $\varphi(x,y)$ represents the velocity potential function and the imaginary part $\psi(x,y)$ represents the stream function.

Problem-1(a). If $W = \varphi + i\psi$ represents the complex potential for an electric field and $\psi = 3x^2y - y^3$, find the potential function φ.

Solution: Given $\psi = 3x^2y - y^3$

$$\frac{\partial \psi}{\partial y} = 3x^2 - 3y^2$$
$$= \psi_1(x,y), \text{ say}$$

and $$\frac{\partial \psi}{\partial x} = 6xy$$
$$= \psi_2(x,y), \text{ say}$$

By Milne's method we have,

$$W'(z) = \psi_1(z,0) + i\psi_2(z,0)$$
$$= 3z^2 + i \cdot 0$$
$$= 3z^2$$

integrating w.r.t. z, we get

$$W(z) = z^3 + c$$

or $\varphi + i\psi = (x+iy)^3 + c_1 + ic_2$

or, $\varphi + i\psi = x^3 + i3x^2y - 3xy^2 - iy^3 + c_1 + ic_2$

$\therefore \varphi = x^3 - 3xy^2 + c_1$ is the required potential function.

Problem-1(b). An incompressible fluid flowing over the xy-plane has the velocity potential

$$\varphi = x^2 - y^2 + \frac{x}{x^2+y^2}$$

Examine if this is possible and find a stream function ψ.

Solution: Given, $\varphi = x^2 - y^2 + \frac{x}{x^2+y^2}$

$$\therefore \frac{\partial \varphi}{\partial x} = 2x + \frac{(x^2+y^2)\cdot 1 - x\cdot 2x}{(x^2+y^2)^2} = 2x + \frac{y^2-x^2}{(x^2+y^2)^2} = \varphi_1(x,y), \text{ say}$$

$$\frac{\partial^2 \varphi}{\partial x^2} = 2 + \frac{(x^2+y^2)^2\cdot(-2x) - (y^2-x^2)\cdot 2(x^2+y^2)\cdot 2x}{(x^2+y^2)^4}$$

$$= 2 + \frac{2x^3 - 6xy^2}{(x^2+y^2)^3}$$

$$\frac{\partial \varphi}{\partial y} = -2y + \frac{(x^2+y^2)(0) - x(2y)}{(x^2+y^2)^2} = -2y - \frac{2xy}{(x^2+y^2)^2} = \varphi_2(x,y), \text{ say}$$

$$\frac{\partial^2 \varphi}{\partial y^2} = -2 + \frac{(x^2+y^2)^2(-2x) - (-2xy)\cdot 2(x^2+y^2)\cdot 2y}{(x^2+y^2)^4}$$

$$= -2 + \frac{-2x^3 + 6xy^2}{(x^2+y^2)^3}$$

$\therefore \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$ i.e., φ is harmonic.

Hence it can be a possible form of the velocity potential function.

By Milne's method, we have

$$f'(z) = \varphi_1(z,0) - i\varphi_2(z,0)$$

$$= 2z - \frac{1}{z^2} - i\cdot 0$$

Integrating it, we get $f(z) = z^2 + \frac{1}{z} + C$

or, $\varphi + i\psi = (x+iy)^2 + \frac{1}{x+iy} + c_1 + ic_2$

$$= x^2 - y^2 + i2xy + \frac{x-iy}{x^2+y^2} + c_1 + ic_2$$

$\therefore \psi = 2xy - \frac{y}{x^2+y^2} + c_2$ is the required.

Problem-1(c): If $W = \varphi + i\psi$ represents the complex potential for an electric field and $\psi = x^2 - y^2 + \frac{x}{x^2+y^2}$, find φ. Ans. $\varphi = -2xy + \frac{y}{x^2+y^2} + c_1$

Problem-1(d). Prove that the real and imaginary parts of an analytic function $f(z) = u(x,y) + iv(x,y)$ satisfies the Laplace's equation.

<u>Solution</u>: Since $f(z) = u(x,y) + iv(x,y)$ is an analytic function, so we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{---(1) and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{---(2)}$$

Differentiate (1) partially wrt x, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad \text{---(3)}$$

Differentiate (2) partially wrt y, we get

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \quad \text{---(4)}$$

Adding (3) and (4) we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{or,} \quad \nabla^2 u = 0$$

Similarly $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ or $\nabla^2 v = 0$

i.e. u and v satisfy their Laplace's equations.
[Both u and v are harmonic functions]

Problem-2. Show that an analytic function with constant real part is constant.

<u>Solution</u>: Let $f(z) = u + iv$ be an analytic function.

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{---(1) and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{---(2)}$$

Given that $u = \text{constant} = c_1$, say

$$\therefore \frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial y} = 0$$

so from (1) and (2) we get, $\frac{\partial v}{\partial x} = 0$ and $\frac{\partial v}{\partial y} = 0$

i.e. v is independent of x and y

$\Rightarrow v = \text{constant} = c_2$, say

$\therefore f(z) = u + iv = c_1 + ic_2$ is a constant

Problem-3(a). Show than an analytic function with Constant imaginary part is constant.

<u>Solution</u>: Let $f(z) = u + iv$ be an analytic function.

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{--- (1)}$$

Given that $v =$ Constant $= c_1$, say

$$\therefore \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = 0$$

$$\Rightarrow \frac{\partial u}{\partial y} = 0, \quad \frac{\partial u}{\partial x} = 0 \quad [\text{by } (1)]$$

$\Rightarrow u$ is independent of x and y

$\Rightarrow u =$ Constant $= c_2$, say

$$\therefore f(z) = u + iv = c_2 + ic_1 = \text{Constant}$$

Problem-3(b). Determine the analytic function whose real part is $x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$.

<u>Solution</u>: Given that $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$$\therefore \frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x$$

$$= u_1(x, y), \text{ say}$$

$$\frac{\partial u}{\partial y} = -6xy - 6y$$

$$= u_2(x, y), \text{ say}$$

By Milne's method, we have

$$f'(z) = u_1(z, 0) - iu_2(z, 0)$$

$$= 3z^2 + 6z - i \cdot 0$$

Integrating it, we get $f(z) = 3 \cdot \frac{z^3}{3} + 6 \cdot \frac{z^2}{2} + c$

$= z^3 + 3z^2 + c$, where c is the complex constant.

Problem-4: Show that an analytic function with constant <u>absolute value</u>/modulus is constant.

Solution: Let an analytic function be $f(z)=u+iv$

$$\therefore \quad |f(z)| = \sqrt{u^2+v^2}$$

But we are given, $|f(z)| = \text{Constant} = K$, say

$$\therefore \quad u^2+v^2=K^2$$

By differentiation, $uu_x + vv_x = 0$, $uu_y + vv_y = 0$

Now we use $v_x = -u_y$ in the first equation and $v_y = u_x$ in the second, we get

$$uu_x - vu_y = 0 \quad \text{--} \; (1)$$

$$uu_y + vu_x = 0 \quad \text{--} \; (2)$$

Multiplying (1) by u and (2) by v, then adding and also multiplying (1) by $-v$ and (2) by u, then adding we get

$$(u^2+v^2)u_x = 0, \quad (u^2+v^2)u_y = 0$$

If $K^2 = u^2+v^2 = 0$, then $u=0=v$, hence $f=0$.

If $K \neq 0$, then $u_x = u_y = 0$, hence by Cauchy-Riemann equations, also $v_x = v_y = 0$.

Together, $u = \text{Constant}$ and $v = \text{Constant}$, hence $f = \text{Constant}$.

✓ Problem-5: Test whether the function $f(z) = z^3 + z$ is analytic or not.

Solution: We have, $f(z) = z^3 + z$

$$= (x+iy)^3 + (x+iy)$$

$$= x^3 + i3x^2y - 3xy^2 - iy^3 + x + iy$$

or, $u + iv = (x^3 - 3xy^2 + x) + i(3x^2y - y^3 + y)$

Equating the real and imaginary parts, we get

$$u = x^3 - 3xy^2 + x$$

$$v = 3x^2y - y^3 + y$$

$$\therefore \frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 1, \quad \frac{\partial u}{\partial y} = -6xy$$

$$\frac{\partial v}{\partial x} = 6xy, \quad \frac{\partial v}{\partial y} = 3x^2 - 3y^2 + 1$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\Rightarrow$ C-R equations are satisfied.

$\therefore f(z) = z^3 + z$ is analytic.

✓ Problem-6: Find the constants a, b and c if $f(z) = x + ay + i(bx + cy)$ is analytic.

Solution: Given, $f(z) = x + ay + i(bx + cy)$

or, $u + iv = x + ay + i(bx + cy)$

$\therefore u = x + ay, \; v = bx + cy$

$$\therefore \frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = a$$

$$\frac{\partial v}{\partial x} = b, \quad \frac{\partial v}{\partial y} = c$$

Since $f(z)$ is analytic, so Cauchy-Riemann (C-R) equations are satisfied.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\therefore 1 = c, \quad a = -b$$

$$\Rightarrow c = 1, \quad a = -b, \quad b \text{ may be any value.}$$

Problem-7: Determine b such that $u = e^{bx}\cos 5y$ is harmonic.

<u>Solution:</u> Given, $u = e^{bx}\cos 5y$

$$\therefore \frac{\partial u}{\partial x} = b e^{bx}\cos 5y$$

$$\frac{\partial^2 u}{\partial x^2} = b^2 e^{bx}\cos 5y$$

$$\frac{\partial u}{\partial y} = e^{bx}(-5\sin 5y)$$

$$\frac{\partial^2 u}{\partial y^2} = -5e^{bx} \cdot 5\cos 5y$$

$$= -25 e^{bx}\cos 5y$$

$\because u$ is harmonic function, so

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

or, $b^2 e^{bx}\cos 5y - 25 e^{bx}\cos 5y = 0$

or, $e^{bx}\cos 5y\,(b^2 - 25) = 0$

or, $b^2 - 25 = 0 \quad [\because e^{bx}\cos 5y \neq 0]$

$$\therefore b = \pm 5$$

Problem-8: (a) Prove that $u = e^{-x}(x\sin y - y\cos y)$ is harmonic. (b) Find v such that $f(z) = u + iv$ is analytic.

Solution: Given, $u = e^{-x}(x\sin y - y\cos y)$

$$\therefore \frac{\partial u}{\partial x} = e^{-x} \cdot \sin y + (-e^{-x})(x\sin y - y\cos y)$$

$$= e^{-x}(\sin y - x\sin y + y\cos y)$$

$$= u_1(x,y), \text{ say}$$

$$\frac{\partial^2 u}{\partial x^2} = -e^{-x}(\sin y - x\sin y + y\cos y) + e^{-x}(-\sin y)$$

$$= -e^{-x}(2\sin y - x\sin y + y\cos y) \quad \text{--- (1)}$$

$$\frac{\partial u}{\partial y} = e^{-x}(x\cos y - 1\cdot\cos y + y\sin y)$$

$$= u_2(x,y), \text{ say}$$

$$\frac{\partial^2 u}{\partial y^2} = e^{-x}(-x\sin y + \sin y + 1\cdot\sin y + y\cos y)$$

$$= e^{-x}(-x\sin y + 2\sin y + y\cos y) \quad \text{--- (2)}$$

Adding (1) and (2), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Since u satisfies Laplace's equation, so u is harmonic function.

<u>2nd part</u>: (b) By Milne's method we have,

$$f'(z) = u_1(z,0) - iu_2(z,0)$$

$$= 0 - i(ze^{-z} - e^{-z})$$

Integrating, $f(z) = -iz\cdot(-e^{-z}) + i\int 1\cdot(-e^{-z})\,dz + i\int e^{-z}\,dz + c$

$$= ize^{-z} + c$$

or, $u+iv = i(x+iy)\,e^{-(x+iy)} + C$

$= i(x+iy)\,e^{-x}.\,e^{-iy} + C$

$= i(x+iy).e^{-x}(\cos y - i\sin y) + C$

$= ixe^{-x}\cos y + xe^{-x}\sin y - ye^{-x}\cos y + iye^{-x}\sin y + C_1 + iC_2$

$= (xe^{-x}\sin y - ye^{-x}\cos y + C_1) + i(xe^{-x}\cos y + ye^{-x}\sin y + C_2)$

Equating imaginary parts, we get

$v = e^{-x}(x\cos y + y\sin y) + C_2$

Problem-9. In a two dimensional flow of a fluid, the velocity potential $\varphi = x^2 - y^2$. Find the stream function ψ.

<u>Solution:</u> Given that $\varphi = x^2 - y^2$

$\therefore \frac{\partial \varphi}{\partial x} = 2x$

$= \varphi_1(x,y)$, say

and $\frac{\partial \varphi}{\partial y} = -2y$

$= \varphi_2(x,y)$, say

By Milne's method we have,

$W'(z) = \varphi_1(z,0) - i\varphi_2(z,0)$

$= 2z - i.0$

$= 2z$

Integrating it, we get $W(z) = \cancel{2}\frac{z^2}{\cancel{2}} + C$

or, $W(z) = z^2 + C$

or, $\phi + i\psi = (x+iy)^2 + c_1 + ic_2$

$= x^2 - y^2 + i2xy + c_1 + ic_2$

Equating imaginary parts, we get

$\psi = 2xy + c_2$ is the required stream function.

Problem-10. Show that xy^2 cannot be real part of an analytic function.

<u>Solution</u>: Given $u = xy^2$

$\therefore \frac{\partial u}{\partial x} = y^2$, $\frac{\partial^2 u}{\partial x^2} = 0$ ---(1)

$\frac{\partial u}{\partial y} = 2xy$, $\frac{\partial^2 u}{\partial y^2} = 2x$ --(2)

Adding (1) and (2) we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2x$$

$$\neq 0$$

$\therefore$ u is not harmonic function.

i.e. u cannot be a real part of an analytic function.

Complex Integration

1. State and prove Cauchy's theorem / Cauchy's integral theorem.

Statement: If $f(z)$ is analytic inside and on a simple closed curve C, then $\oint_C f(z)\,dz = 0$.

Proof: Let $f(z) = u(x,y) + iv(x,y)$ be analytic.

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \cdots (1)$$

Since $z = x+iy$, so $dz = dx + i\,dy$

Now $\oint_C f(z)\,dz = \oint_C (u+iv)(dx+i\,dy)$

$$= \oint_C (u\,dx + iu\,dy + iv\,dx - v\,dy)$$

$$= \oint_C (u\,dx - v\,dy) + i\oint_C (v\,dx + u\,dy) \quad \cdots (1)$$

By Green's theorem we have,

$$\oint_C (u\,dx - v\,dy) = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

and $\oint_C (v\,dx + u\,dy) = \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$

where R is the region bounded by C.

Hence (1) becomes,

$$\oint_C f(z)\,dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

$$= \iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\right) dx\,dy$$

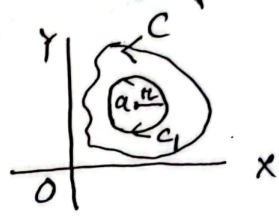
$$= 0 + i \cdot 0$$

$$= 0$$

2. State and prove Cauchy's integral formula.

<u>Statement</u>: If $f(z)$ is analytic inside and on a simple closed curve C, and 'a' is any point within C, then $\oint_C \frac{f(z)dz}{z-a} = 2\pi i f(a)$

<u>Proof</u>: Since $f(z)$ is analytic inside and on C, $\frac{f(z)}{z-a}$ is also analytic inside and on C, except at the point $z=a$. Hence, we draw a small circle with centre at $z=a$ and radius r lying entirely inside C.



Now, $\frac{f(z)}{z-a}$ is analytic in the region enclosed between C and C_1.

Hence, by Cauchy's extended theorem,

$$\oint_C \frac{f(z)dz}{z-a} = \oint_{C_1} \frac{f(z)dz}{z-a} \quad \text{---} (1)$$

On C_1, any point z is given by $z = a + re^{i\theta}$

$\therefore dz = ire^{i\theta} d\theta$

where θ varies from 0 to 2π.

$$\therefore \oint_{C_1} \frac{f(z)dz}{z-a} = \int_{\theta=0}^{2\pi} \frac{f(a+re^{i\theta}) \cdot ire^{i\theta} d\theta}{re^{i\theta}}$$

$$= i\int_{\theta=0}^{2\pi} f(a+re^{i\theta})\, d\theta$$

As $r \to 0$, the circle tends to a point.

Taking limit $r \to 0$, we get

$$\oint_{C_1} \frac{f(z)dz}{z-a} = i\int_{\theta=0}^{2\pi} f(a)\, d\theta$$

$$= if(a)\,[\theta]_{\theta=0}^{2\pi}$$

$$= 2\pi i f(a)$$

So from (1), we get $\oint_C \frac{f(z)dz}{z-a} = 2\pi i f(a)$

* In general, $\oint_C \frac{f(z)\,dz}{(z-a)^{n+1}} = \frac{2\pi i}{\lfloor n} f^{(n)}(a)$

where $n = 0, 1, 2, 3, \ldots$

and $f^{(0)}(a) = f(a)$

* <u>Cauchy's extended theorem</u>: If $f(z)$ is analytic within and on the boundary of a region bounded by two closed curves C_1 and C_2, then

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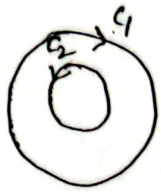


$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$$

3. Evaluate $\frac{1}{2\pi i}\oint_C \frac{e^z\,dz}{z-2}$ if C is (a) the circle $|z| = 3$, (b) the circle $|z| = 1$.

<u>Solution</u>: (a) Here $f(z) = e^z$ is analytic inside and on the circle $|z| = 3$ and $z = a = 2$ is a point inside the given circle.

Then by using the Cauchy's integral formula, we have

$$\oint_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

$$\therefore \oint_C \frac{e^z}{z-2}\,dz = 2\pi i \cdot e^2$$

$$\text{or, } \frac{1}{2\pi i}\oint_C \frac{e^z}{z-2}\,dz = e^2$$

(b) Here $f(z) = \frac{e^z}{z-2}$ is analytic inside and on the circle $|z| = 1$ and $z = 2$ is a point outside the given circle.

Then by using the Cauchy's integral theorem, $\oint_C f(z)\,dz = 0$ we get

$$\oint_C \frac{e^z}{z-2}\,dz = 0 \quad \text{or, } \frac{1}{2\pi i}\oint_C \frac{e^z}{z-2}\,dz = 0$$

4. Evaluate $\oint_C \frac{\sin 3z}{z+\pi/2} dz$ if C is the circle $|z|=5$.

<u>Solution</u>: Here $f(z) = \sin 3z$ is analytic inside and on the circle $|z|=5$ and $z = -\frac{\pi}{2}$ lies inside the given circle.

Then by using Cauchy's integral formula,

$\oint_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$ we get

$$\oint_C \frac{\sin 3z}{z-(-\pi/2)} dz = 2\pi i f(-\pi/2)$$

$$\text{or, } \oint_C \frac{\sin 3z}{z+\pi/2} = 2\pi i$$

$\therefore f(-\frac{\pi}{2}) = \sin 3(-\frac{\pi}{2})$
$= -(-1)$
$= 1$

5. Evaluate $\oint_C \frac{e^{3z}}{z-\pi i} dz$ if C is (a) the circle $|z-1| = 4$, (b) the ellipse $|z-2|+|z+2| = 6$.

<u>Solution</u>: (a) Here $f(z) = e^{3z}$ is analytic inside and on the given circle $|z-1|=4$, and $z = \pi i$ is a point inside the given circle.

Then by using Cauchy's integral formula,

$\oint_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$ we get

$$\oint_C \frac{e^{3z}}{z-\pi i} dz = 2\pi i f(\pi i)$$
$$= 2\pi i \cdot e^{3\pi i}$$
$$= 2\pi i (\cos 3\pi + i \sin 3\pi)$$
$$= 2\pi i (-1 + i \cdot 0)$$
$$= -2\pi i$$

(b) Here $f(z) = \frac{e^{3z}}{z-\pi i}$ is analytic inside and on the ellipse C, and $z = \pi i$ lies outside the given ellipse C.

Then by using Cauchy's integral theorem,

$\oint_C f(z)\, dz = 0$ we get

$$\oint_C \frac{e^{3z}}{z-\pi i} dz = 0$$

** Locus of $|z-2|+|z+2| = 6$ is $\frac{x^2}{3^2} + \frac{y^2}{(\sqrt{5})^2} = 1$.

Its foci $(\pm ae, 0) = (\pm 3 \cdot \frac{2}{3}, 0) = (\pm 2, 0)$, $e = \sqrt{1-\frac{5}{9}} = \frac{2}{3}$

and length of major axis $= 2 \cdot 3 = 6$

6. Evaluate $\frac{1}{2\pi i}\oint_C \frac{\cos\pi z}{z^2-1}dz$ around a rectangle with vertices at: (a) $2\pm i, -2\pm i$ (b) $-i, 2-i, 2+i, i$.

<u>Solution:</u> We have $\frac{1}{2\pi i}\oint_C \frac{\cos\pi z}{z^2-1}dz = \frac{1}{4\pi i}\oint_C\left[\frac{\cos\pi z}{z-1} - \frac{\cos\pi z}{z+1}\right]dz$

$$= \frac{1}{4\pi i}\left[\oint_C \frac{\cos\pi z}{z-1}dz - \oint_C \frac{\cos\pi z}{z+1}dz\right] \quad \cdots (1)$$

(a) Here $f(z) = \cos\pi z$ is analytic inside and on C, and also both points $z = \pm 1$ lie inside the rectangle $2\pm i$, $-2\pm i$.

Then by using Cauchy's integral formula, $\oint_C \frac{f(z)}{z-a}dz = 2\pi i f(a)$, we get from (1)

$$\frac{1}{2\pi i}\oint_C \frac{\cos\pi z}{z^2-1}dz = \frac{1}{4\pi i}\left[2\pi i\cos\pi - 2\pi i\cos(-\pi)\right]$$

$$= \frac{1}{4\pi i}\left[-2\pi i + 2\pi i\right]$$

$$= 0$$

(b) Here only the point $z=1$ lies inside the rectangle $\pm i$, $2\pm i$.

Then by using the Cauchy's integral formula and also the Cauchy's integral theorem, we get from (1),

$$\frac{1}{2\pi i}\oint_C \frac{\cos\pi z}{z^2-1}dz = \frac{1}{4\pi i}\left[2\pi i\cos\pi - 0\right]$$

$$= -\frac{1}{2}$$

7. Show that $\frac{1}{2\pi i}\oint_C \frac{e^{zt}dz}{z^2+1} = \sin t$ if $t>0$ and C is the circle $|z|=3$

<u>Solution:</u> We have $\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2+1}dz = \frac{1}{2\pi i}\oint_C \frac{e^{zt}dz}{(z+i)(z-i)}$

$$= \frac{1}{2\pi i}\cdot\frac{1}{2i}\left[\oint_C \frac{e^{zt}}{z-i}dz - \oint_C \frac{e^{zt}dz}{z+i}\right] \quad \cdots (1)$$

Here $f(z) = e^{zt}$ is analytic inside and on the given circle $|z|=3$ and $z=\pm i$ are inside C.

Then by using Cauchy's integral formula, we get from (1)

$$\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2+1}\,dz = \frac{1}{2\pi i}\cdot\frac{1}{2i}\left[2\pi i f(i) - 2\pi i f(-i)\right]$$

$$= \frac{1}{2i}\left[e^{it} - e^{-it}\right]$$

$$= \frac{1}{2i}\cdot 2i\sin t \quad \left[\because e^{it} - e^{-it} = 2i\sin t\right]$$

$$= \sin t$$

8. Evaluate $\oint_C \frac{e^{iz}}{z^3}\,dz$ where C is the circle $|z|=2$.

<u>Solution</u>: Here $f(z) = e^{iz}$ is analytic inside and on the circle $|z|=2$ and $z=a=0$ is a point inside the given circle.

Then by using Cauchy's integral formula,

$$\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{\lfloor n}\, f^{(n)}(a), \text{ we get}$$

$$\oint_C \frac{f(z)}{(z-0)^3}\,dz = \frac{2\pi i}{\lfloor 2}\, f^{(2)}(0)$$

$$\therefore \oint_C \frac{e^{iz}}{z^3}\,dz = \frac{2\pi i}{2}\cdot(-1)$$

$$= -\pi i$$

$f(z) = e^{iz}$

$\therefore f'(z) = i e^{iz}$

$f''(z) = -e^{iz}$

$f''(0) = -1 = f^{(2)}(0)$

9. Find the value of (a) $\oint_C \frac{\sin^6 z}{z-\frac{\pi}{6}}\,dz$, (b) $\oint_C \frac{\sin^6 z\,dz}{(z-\frac{\pi}{6})^3}$ if C is the circle $|z|=1$.

<u>Solution</u>: Here $f(z) = \sin^6 z$ is analytic inside and on the circle $|z|=1$ and $z=a=\frac{\pi}{6}$ is a point inside the given circle.

(a) Then by using Cauchy's integral formula,

$$\oint_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a) \text{ we get}$$

$$\oint_C \frac{\sin^6 z}{z-\frac{\pi}{6}}\,dz = 2\pi i\left(\sin\frac{\pi}{6}\right)^6$$

$$= 2\pi i\cdot\left(\frac{1}{2}\right)^6$$

$$= 2\pi i\cdot\frac{1}{64}$$

$$= \frac{\pi i}{32}$$

(b) Then by using Cauchy's integral formula,

$$\oint_C \frac{f(z)}{(z-a)^{n+1}}dz = \frac{2\pi i}{\lfloor n} f^{(n)}(a) \text{ we get}$$

$$\oint_C \frac{\sin^6 z}{(z-\frac{\pi}{6})^3}dz = \frac{2\pi i}{\lfloor 2} f^{(2)}(\tfrac{\pi}{6}) \quad \cdots (1)$$

We have $f(z) = \sin^6 z$

$\therefore f'(z) = 6\sin^5 z \cdot \cos z$

$f''(z) = 30\sin^4 z \cos^2 z + 6\sin^5 z(-\sin z)$

$$\therefore f''(\tfrac{\pi}{6}) = f^{(2)}(\tfrac{\pi}{6}) = 30\cdot(\tfrac{1}{2})^4\cdot(\tfrac{\sqrt{3}}{2})^2 - 6\cdot(\tfrac{1}{2})^6$$

$$= 30\cdot\frac{1}{16}\cdot\frac{3}{4} - 6\cdot\frac{1}{64}$$

$$= \frac{90-6}{64}$$

$$= \frac{84}{64}$$

$$= \frac{21}{16}$$

So from (1) we get

$$\oint_C \frac{\sin^6 z\, dz}{(z-\frac{\pi}{6})^3} = \frac{2\pi i}{2}\cdot\frac{21}{16}$$

$$= \frac{21\pi i}{16}$$

10. Evaluate $\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{(z^2+1)^2}dz$ if $t>0$ and C is the circle $|z|=3$.

<u>Solution:</u> We have, $\frac{1}{(z^2+1)^2} = \frac{1}{(z+i)^2(z-i)^2}$

$$= \frac{1}{4iz}\left[\frac{1}{(z-i)^2} - \frac{1}{(z+i)^2}\right]$$

$$\therefore \oint_C \frac{e^{zt}dz}{(z^2+1)^2} = \frac{1}{4i}\left[\oint_C \frac{\frac{e^{zt}}{z}dz}{(z-i)^2} - \oint_C \frac{\frac{e^{zt}}{z}dz}{(z+i)^2}\right]$$

Here $f(z) = \frac{e^{zt}}{z}$ is analytic inside on the given circle $|z|=3$ and $z=\pm i$ are inside C.

Then by using Cauchy's integral theorem,

$$\oint_C \frac{f(z)}{(z-a)^{n+1}}dz = \frac{2\pi i}{\lfloor n} f^{(n)}(a) \quad \text{we get}$$

$$\oint_C \frac{e^{zt}dz}{(z^2+1)^2} = \frac{1}{4i}\left[\frac{2\pi i}{\lfloor 1} f'(i) - \frac{2\pi i}{\lfloor 1} f'(-i)\right] \cdots (1)$$

we have, $f(z) = \dfrac{e^{zt}}{z}$

$$\therefore f'(z) = \frac{z \cdot e^{zt} \cdot t - 1 \cdot e^{zt}}{z^2}$$

$$\therefore f'(i) = \frac{ite^{it} - e^{it}}{i^2}$$

$$= e^{it} - ite^{it} \qquad [\because i^2 = -1]$$

Also, $f'(-i) = \dfrac{-ite^{-it} - e^{-it}}{(-i)^2}$

$$= ite^{-it} + e^{-it}$$

So from (1) we get

$$\frac{1}{2\pi i}\oint_C \frac{e^{zt}dz}{(z^2+1)^2} = \frac{1}{4i}\left[e^{it} - ite^{it} - ite^{-it} - e^{-it}\right]$$

$$= \frac{1}{4i}\left[(e^{it} - e^{-it}) - it(e^{it} + e^{-it})\right]$$

$$= \frac{1}{4i}\left[2i\sin t - it \cdot 2\cos t\right]$$

$$= \frac{1}{4i} \cdot 2i\left[\sin t - t\cos t\right]$$

$$= \frac{1}{2}(\sin t - t\cos t) \qquad \left[\begin{array}{l} \because e^{i\theta} = \cos\theta + i\sin\theta \\ e^{-i\theta} = \cos\theta - i\sin\theta \\ \Rightarrow 2\cos\theta = e^{i\theta} + e^{-i\theta} \\ 2i\sin\theta = e^{i\theta} - e^{-i\theta} \end{array}\right]$$

11. Evaluate $\oint_C \dfrac{e^z dz}{z(1-z)^3}$ if (i) 0 lies inside C and 1 lies outside C, (ii) 1 lies inside C and 0 lies outside C, (iii) 0 and 1 lie inside C.

<u>Solution</u>: (i) Since 0 lies inside C and 1 lies outside C.

$$\therefore \oint_C \frac{e^z dz}{z(1-z)^3} = \oint_C \frac{f(z)}{z-0}dz \quad \text{where } f(z) = \frac{e^z}{(1-z)^3}$$

$$= 2\pi i f(0)$$

$$= 2\pi i \cdot 1$$

$$= 2\pi i$$

(ii) Since 1 lies inside C and 0 lies outside C.

$$\therefore \oint_C \frac{e^z dz}{z(1-z)^3} = \oint_C \frac{f(z)\,dz}{(z-1)^3}$$

$$= \frac{2\pi i}{\lfloor 2} f''(1)$$

Where, $f(z) = -\frac{e^z}{z}$

$$\therefore f'(z) = \frac{e^z}{z^2} - \frac{e^z}{z}$$

$$f''(z) = -\frac{2e^z}{z^3} + \frac{e^z}{z^2} + \frac{e^z}{z^2} - \frac{e^z}{z}$$

$$\therefore f''(1) = -2e^1 + 2e^1 - e^1$$

$$= -e$$

$$\therefore \oint_C \frac{e^z dz}{z(1-z)^3} = \frac{2\pi i}{2} \cdot (-e)$$

$$= -\pi i e$$

(iii) Since 0 and 1 lie inside C, so we express $\frac{1}{z(1-z)^3}$ in partial fractions.

Let $\frac{1}{z(1-z)^3} = \frac{1}{z} + \frac{1}{1-z} + \frac{A}{(1-z)^2} + \frac{B}{(1-z)^3}$ ---(1)

$\Rightarrow 1 = (1-z)^3 + z(1-z)^2 + Az(1-z) + Bz$ ---(2)

Putting $z=1$ in (2), we get $1 = B$

Equating the coefficients of z^2 from the bothsides of (2), we get

$$0 = 3 - 2 - A \quad \text{or, } A = 1$$

So from (1) we get

$$\frac{1}{z(1-z)^3} = \frac{1}{z} + \frac{1}{1-z} + \frac{1}{(1-z)^2} + \frac{1}{(1-z)^3}$$

$$\therefore \oint_C \frac{e^z dz}{z(1-z)^3} = \oint_C \frac{e^z dz}{z} + \oint_C \frac{e^z dz}{1-z} + \oint_C \frac{e^z dz}{(1-z)^2} + \oint_C \frac{e^z dz}{(1-z)^3}$$

$$= 2\pi i \cdot (e^0) - 2\pi i \cdot (e^1) + \frac{2\pi i}{\lfloor 1} f'(1) - \frac{2\pi i}{\lfloor 2} f''(1)$$

$$= 2\pi i - 2\pi i e + 2\pi i e - \frac{2\pi i}{2} \cdot e$$

$$= \pi i (2 - e)$$

12. What is the value of $\oint_C \frac{z^2+1}{z^2-1}\,dz$ if C is a circle of unit radius with centre at (i) $z=1$ and (ii) $z=-1$.

<u>Solution</u>: (i) If C is a circle of unit radius with centre at $z=1$, then

$$\oint_C \frac{(z^2+1)\,dz}{z^2-1} = \oint_C \frac{\frac{z^2+1}{z+1}}{z-1}\,dz$$

$$= 2\pi i f(1) \quad \text{where } f(z) = \frac{z^2+1}{z+1}$$

$$= 2\pi i \cdot 1$$

$$= 2\pi i$$

(ii) If C is a circle of unit radius with centre $z=-1$, then

$$\oint_C \frac{z^2+1}{z^2-1}\,dz = \oint_C \frac{\frac{z^2+1}{z-1}}{z+1}\,dz$$

$$= 2\pi i f(-1) \quad \text{where } f(z) = \frac{z^2+1}{z^2-1}$$

$$= 2\pi i(-1)$$

$$= -2\pi i$$

13. Using Cauchy's integral formula, evaluate $\oint_C \frac{z\,dz}{(z-1)(z-2)}$ where C is the circle $|z-2| = \frac{1}{2}$

<u>Solution</u>: Since $z=2$ is the only point lies inside the circle $|z-2| = \frac{1}{2}$,

$$\therefore \oint_C \frac{z\,dz}{(z-1)(z-2)} = \oint_C \frac{\left(\frac{z}{z-1}\right)dz}{z-2}$$

$$= 2\pi i f(2) \quad \text{where } f(z) = \frac{z}{z-1}$$

$$= 2\pi i \cdot 2$$

$$= 4\pi i$$

14. Evaluate $\oint_C \frac{dz}{(z^2+4)^2}$, where C is the circle $|z-i| = 2$

<u>Solution</u>: Let $F(z) = \frac{1}{(z^2+4)^2}$

$\therefore$ Singular points of $f(z)$ are $z = \pm 2i$. Among this only $z = 2i$ lies inside the circle $|z-i| = 2$.

$$\therefore \oint_C \frac{dz}{(z^2+4)^2} = \oint_C \frac{\frac{1}{(z+2i)^2}\,dz}{(z-2i)^2}$$

$$= \frac{2\pi i}{1!} f'(2i)$$

$$= 2\pi i \cdot \frac{1}{32i}$$

$$= \frac{\pi}{16}$$

where $f(z) = \frac{1}{(z+2i)^2}$

$$\therefore f'(z) = -\frac{2}{(z+2i)^3}$$

$$\therefore f(2i) = -\frac{2}{(4i)^3}$$

$$= \frac{1}{32i}$$

Singular points, poles, Residues

Singular points: All the points of the z-plane at which an analytic function does not have a unique derivative are said to be singular points.

If $f(z) = \frac{1}{(z-3)^2}$, then $z=3$ is a singularity of $f(z)$.

Poles: If $f(z) = \frac{\varphi(z)}{(z-a)^n}$, $\varphi(a) \neq 0$, where $\varphi(z)$ is analytic everywhere in a region including $z=a$, and if n is a positive integer, then $f(z)$ has a singularity at $z=a$ which is called a pole of order n. If $n=1$, the pole is often called a simple pole; if $n=2$ it is called a double pole, etc.

If $f(z) = \frac{z}{(z-3)^2(z+1)}$ has two singularities; a pole of order 2 or double pole at $z=3$ and a pole of order 1 or simple pole at $z=-1$.

Residues: If $f(z)$ has a pole of order n at $z=a$ but is analytic at every other point inside and on a circle C with centre at a, then the Laurent's series about $z=a$ is given by

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$$

$$= \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} a_n (z-a)^{-n}$$

$$\therefore f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots\cdots$$
$$+ \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots\cdots$$

The part $a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots\cdots$ is called the analytic part, while the remainder consisting of inverse powers of $z-a$ is called the principal part.

The coefficient a_{-1}, called the residue of $f(z)$ at the pole $z=a$.

<u>Method of finding residues:</u>

(i) Residue of $f(z)$ at simple pole $z=a$ is

$$\lim_{x \to a} (z-a) f(z)$$

(ii) Residue of $f(z)$ at $z=a$ with pole of order n is

$$\lim_{x \to a} \frac{1}{\lfloor n-1} \frac{d^{n-1}}{dz^{n-1}} \left\{ (z-a)^n f(z) \right\}$$

<u>Problem-1:</u> Determine the residues of each function at its poles:

(i) $\dfrac{z^2}{(z-2)(z^2+1)}$

Let $f(z) = \dfrac{z^2}{(z-2)(z^2+1)}$

poles of $f(z)$ are given by

$$(z-2)(z^2+1) = 0$$

$\therefore \quad z = 2, \pm i$

Residue at simple pole $z = 2$ is

$$\lim_{z \to 2} (z-2) f(z)$$

$$= \lim_{z \to 2} (z-2) \cdot \frac{z^2}{(z-2)(z^2+1)}$$

$$= \frac{4}{5}$$

Residue at simple pole $z = i$ is

$$\lim_{z \to i} (z-i) \cdot \frac{z^2}{(z-2)(z+i)(z-i)}$$

$$= \frac{i^2}{(i-2) \cdot 2i}$$

$$= \frac{-1}{2i^2 - 4i}$$

$$= \frac{-1}{-2(1+2i)}$$

$$= \frac{1-2i}{10}$$

Residue at simple pole $z = -i$ is

$$\lim_{z \to -i} (z+i) \frac{z^2}{(z-2)(z+i)(z-i)}$$

$$= \frac{1+2i}{10}$$

(ii) $f(z) = \dfrac{1}{z(z+2)^3}$

poles of $f(z)$ are given by $z(z+2)^3 = 0$

$\therefore z = 0, -2$

$\therefore$ $z = 0$ is a simple pole and $z = -2$ is a pole of order 3.

Residue at simple pole $z=0$ is

$$\lim_{z\to 0} z \cdot \frac{1}{z(z+2)^3}$$

$$= \frac{1}{8}$$

Residue at $z=-2$ (pole of order 3) is

$$\lim_{z\to -2} \frac{1}{\lfloor 2} \frac{d^2}{dz^2}\left\{(z+2)^3 \cdot \frac{1}{z(z+2)^3}\right\}$$

$$= \lim_{z\to -2} \frac{1}{2} \frac{d^2}{dz^2}\left(\frac{1}{z}\right)$$

$$= \lim_{z\to -2} \frac{1}{2}\left(\frac{2}{z^3}\right)$$

$$= -\frac{1}{8}$$

Problem-2: Determine the residues of each function at its poles!

(i) $\frac{2z+3}{z^2-4}$ (ii) $\frac{z-3}{z^3+5z^2}$ (iii) $\frac{e^{zt}}{(z-2)^3}$

(iv) $\frac{z}{(z^2+1)^2}$

<u>Cauchy's residue theorem</u>: If $f(z)$ is analytic within and on a simple closed curve C except at a number of poles $a, b, c, \ldots$ interior to C at which the residues $a_{-1}, b_{-1}, c_{-1}, \ldots$ respectively, then

$$\oint_C f(z)\,dz = 2\pi i\,(a_{-1}+b_{-1}+c_{-1}+\cdots)$$

$$= 2\pi i\,(\text{sum of residues})$$

Problem-3: Evaluate $\oint_C \frac{e^z dz}{(z-1)(z+3)^2}$ where C is given by (i) $|z| = \frac{3}{2}$, (ii) $|z| = 10$.

Solution: Here $f(z) = \frac{e^z}{(z-1)(z+3)^2}$

poles of $f(z)$ are given by $(z-1)(z+3)^2 = 0$

$\therefore z = 1, -3$

Residue at simple pole $z=1$ is

$$\lim_{z \to 1} \cancel{(z-1)} \cdot \frac{e^z}{\cancel{(z-1)}(z+3)^2}$$

$$= \frac{e}{16}$$

Residue at double pole $z=-3$ is

$$\lim_{z \to -3} \frac{1}{1!} \frac{d^1}{dz^1} \left\{ (z+3)^2 \cdot \frac{e^z}{(z-1)(z+3)^2} \right\}$$

$$= \lim_{z \to -3} \frac{d}{dz} \left(\frac{e^z}{z-1} \right)$$

$$= \lim_{z \to -3} \frac{(z-1)e^z - e^z}{(z-1)^2}$$

$$= \frac{-5e^{-3}}{16}$$

(i) Since $|z| = \frac{3}{2}$ encloses only the pole $z=1$,

the required integral $= 2\pi i \left(\frac{e}{16} \right)$

$$= \frac{\pi i e}{8}$$

(ii) Since $|z| = 10$ encloses both poles $z=1$ and $z=-3$, the required integral

$$= 2\pi i \left(\frac{e}{16} - \frac{5e^{-3}}{16} \right)$$

$$= \frac{\pi i (e - 5e^{-3})}{8}$$

Problem-4: Evaluate $\oint_C \frac{z^2\,dz}{(z+1)(z+3)}$, where C is a simple closed curve enclosing all the poles.

<u>Solution</u>: Here $f(z) = \frac{z^2}{(z+1)(z+3)}$

poles of $f(z)$ are given by $(z+1)(z+3)=0$

$\therefore z=-1,-3$

Both poles are simple.

Residue at simple pole $z=-1$ is

$$\lim_{z\to -1} \cancel{(z+1)}\cdot\frac{z^2}{\cancel{(z+1)}(z+3)}$$

$$= \frac{1}{2}$$

Residue at simple pole $z=-3$ is

$$\lim_{z\to -3} \cancel{(z+3)}\cdot\frac{z^2}{(z+1)\cancel{(z+3)}}$$

$$= -\frac{9}{2}$$

Hence by Cauchy's residue theorem, we get

$$\oint_C f(z)\,dz = 2\pi i\left(\frac{1}{2}-\frac{9}{2}\right)$$

$$\therefore \oint_C \frac{z^2\,dz}{(z+1)(z+2)} = -8\pi i$$

<u>OR (Use by Cauchy's integral formula)</u>

$$\oint_C \frac{z^2\,dz}{(z+1)(z+3)} = \oint_C \left\{1+\frac{-4z-3}{(z+1)(z+3)}\right\}dz$$

$$= \oint_C \left\{1+\frac{\frac{1}{2}}{z+1}+\frac{-9/2}{z+3}\right)dz$$

$$= \oint_C dz + \frac{1}{2}\oint_C \frac{1}{z+1}\,dz - \frac{9}{2}\oint_C \frac{1}{z+3}\,dz$$

$$= 0 + \frac{1}{2}\cdot 2\pi i\cdot 1 - \frac{9}{2}\cdot 2\pi i\cdot 1$$

$$= -8\pi i$$

Laurent's series

SOS-91 / P-166 Expand $f(z) = \frac{1}{z-3}$ in a Laurent series valid for (a) $|z| < 3$, (b) $|z| > 3$.

Solution: We have, $f(z) = \frac{1}{z-3}$ --- (1)

(a) Since $|z| < 3$, so from (1) we get

$$f(z) = -\frac{1}{3}\left(1 - \frac{z}{3}\right)^{-1}$$
$$= -\frac{1}{3}\left(1 + \frac{z}{3} + \frac{z^2}{9} + \frac{z^3}{27} + \cdots\cdots\right)$$
$$= -\frac{1}{3} - \frac{z}{9} - \frac{z^2}{27} - \frac{z^3}{81} - \cdots\cdots$$

(b) Since $|z| > 3$, so from (1) we get

$$f(z) = \frac{1}{z}\left(1 - \frac{3}{z}\right)^{-1}$$
$$= \frac{1}{z}\left(1 + \frac{3}{z} + \frac{9}{z^2} + \frac{27}{z^3} + \cdots\cdots\right)$$
$$= \frac{1}{z} + \frac{3}{z^2} + \frac{9}{z^3} + \frac{27}{z^4} + \cdots\cdots$$

SOS-92 / P-166 Expand $f(z) = \frac{z}{(z-1)(2-z)}$ in a Laurent series valid for:

(a) $|z| < 1$, (b) $1 < |z| < 2$, (c) $|z| > 2$, (d) $|z-1| > 1$

(e) $0 < |z-2| < 1$

Solution: We have, $f(z) = \frac{z}{(z-1)(2-z)}$

$$= \frac{1}{z-1} + \frac{2}{2-z} \quad \text{---(1)}$$

(a) Since $|z| < 1$, so from (1) we get

$$f(z) = -\frac{1}{1-z} + \frac{1}{1-z/2}$$
$$= -(1-z)^{-1} + \left(1 - \frac{z}{2}\right)^{-1}$$
$$= -(1 + z + z^2 + z^3 + \cdots\cdots) + \left(1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \cdots\right)$$
$$= -\frac{z}{2} - \frac{3z^2}{4} - \frac{7z^3}{8} - \cdots\cdots$$

(b) Since $1 < |z| < 2$, so from (1) we get

$$f(z) = \frac{1}{z\left(1 - \frac{1}{z}\right)} + \frac{1}{1 - z/2}$$
$$= \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-1} + \left(1 - \frac{z}{2}\right)^{-1}$$
$$= \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right) + \left(1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \cdots\right)$$

$$\therefore f(z) = \cdots\cdots + \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \cdots\cdots$$

(c) Since $|z| > 2$, so from (1) we get

$$f(z) = \frac{1}{z(1-\frac{1}{z})} - \frac{2}{z(1-2/z)}$$

$$= \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} - \frac{2}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$$= \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right) - \frac{2}{z}\left(1+\frac{2}{z}+\frac{4}{z^2}+\cdots\right)$$

$$= \frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\cdots - \frac{2}{z} - \frac{4}{z^2} - \frac{8}{z^3} - \cdots$$

$$= -\frac{1}{z} - \frac{3}{z^2} - \frac{7}{z^3} - \cdots\cdots$$

(d) Since $|z-1| > 1$, so from (1) we get

$$f(z) = \frac{1}{z-1} - \frac{2}{z-2}$$

$$= \frac{1}{z-1} - \frac{2}{(z-1)-1}$$

$$= \frac{1}{z-1} - \frac{2}{(z-1)\left(1-\frac{1}{z-1}\right)}$$

$$= \frac{1}{z-1} - \frac{2}{z-1}\left(1-\frac{1}{z-1}\right)^{-1}$$

$$= \frac{1}{z-1} - \frac{2}{z-1}\left\{1+\frac{1}{z-1}+\frac{1}{(z-1)^2}+\cdots\cdots\right\}$$

$$= \frac{1}{z-1} - \frac{2}{z-1} - \frac{2}{(z-1)^2} - \frac{2}{(z-1)^3} - \cdots\cdots$$

$$= -\frac{1}{z-1} - \frac{2}{(z-1)^2} - \frac{2}{(z-1)^3} - \cdots\cdots$$

(e) Since $0 < |z-2| < 1$, so from (1) we get

$$f(z) = \frac{1}{(z-2)+1} - \frac{2}{z-2}$$

$$= \{1+(z-2)\}^{-1} - \frac{2}{z-2}$$

$$= 1 - (z-2) + (z-2)^2 - (z-2)^3 + \cdots\cdots - \frac{2}{z-2}$$

$$= 1 - \frac{2}{z-2} - (z-2) + (z-2)^2 - (z-2)^3 + \cdots\cdots$$

SOS-93
P-166

Expand $f(z) = \frac{1}{z(z-2)}$ in a Laurent series valid for (a) $0<|z|<2$, (b) $|z|>2$.

<u>Solution</u>: We have, $f(z) = \frac{1}{z(z-2)}$

$$= \frac{-1/2}{z} + \frac{1/2}{z-2}$$

$$= \frac{1}{2}\left[\frac{1}{z-2} - \frac{1}{z}\right] \cdots (1)$$

(a) Since $0<|z|<2$, so from (1) we get

$$f(z) = \frac{1}{2}\left[\frac{1}{-2(1-z/2)} - \frac{1}{z}\right]$$

$$= \frac{1}{2}\left[-\frac{1}{2}(1-z/2)^{-1} - \frac{1}{z}\right]$$

$$= \frac{1}{2}\left[-\frac{1}{2}\left(1+z/2+\frac{z^2}{4}+\frac{z^3}{8}+\cdots\right) - \frac{1}{z}\right]$$

$$= -\frac{1}{2z} - \frac{1}{4} - \frac{z}{8} - \frac{z^2}{16} - \frac{z^3}{32} - \cdots\cdots$$

(b) Since $|z|>2$, so from (1) we get

$$f(z) = \frac{1}{2}\left[\frac{1}{z}\left(1-\frac{2}{z}\right)^{-1} - \frac{1}{z}\right]$$

$$= \frac{1}{2}\left[\frac{1}{z}\left(1+\frac{2}{z}+\frac{4}{z^2}+\frac{8}{z^3}+\cdots\right) - \frac{1}{z}\right]$$

$$= \frac{1}{z^2} + \frac{2}{z^3} + \frac{4}{z^4} + \cdots\cdots$$

SOS-94
P-166

Find an expression of $f(z) = \frac{z}{z^2+1}$ valid for $|z-3|>2$.

<u>Solution</u>: Since $|z-3|>2$, so from $f(z) = \frac{z}{z^2+1}$, we get

$$f(z) = \frac{z}{z^2\left(1+\frac{1}{z^2}\right)}$$

$$= \frac{1}{z}\left(1+\frac{1}{z^2}\right)^{-1}$$

$$= \frac{1}{z}\left(1-\frac{1}{z^2}+\frac{1}{z^4}-\frac{1}{z^6}+\frac{1}{z^8}-\cdots\right)$$

$$= \frac{1}{z} - \frac{1}{z^3} + \frac{1}{z^5} - \frac{1}{z^7} + \frac{1}{z^9} - \cdots\cdots$$

SOS-95 P-166 Expand $f(z) = \frac{1}{(z-2)^2}$ in a Laurent series valid for (a) $|z| < 2$, (b) $|z| > 2$.

Solution: We have, $f(z) = \frac{1}{(z-2)^2}$ --- (1)

(a) Since $|z| < 2$, so from (1) we get

$$f(z) = \frac{1}{\{-2(1-z/2)\}^2}$$

$$= \frac{1}{4(1-z/2)^2}$$

$$= \frac{1}{4}(1-z/2)^{-2}$$

$$= \frac{1}{4}\{1 + 2\cdot\left(\frac{z}{2}\right) + 3\cdot(z/2)^2 + 4(z/2)^3 + \cdots \infty\}$$

$$= \frac{1}{4} + \frac{z}{4} + \frac{3z^2}{16} + \frac{z^3}{8} + \cdots\cdots$$

(b) Since $|z| > 2$, so from (1) we get

$$f(z) = \frac{1}{\{z(1-\frac{2}{z})\}^2}$$

$$= \frac{1}{z}\left(1-\frac{2}{z}\right)^{-2}$$

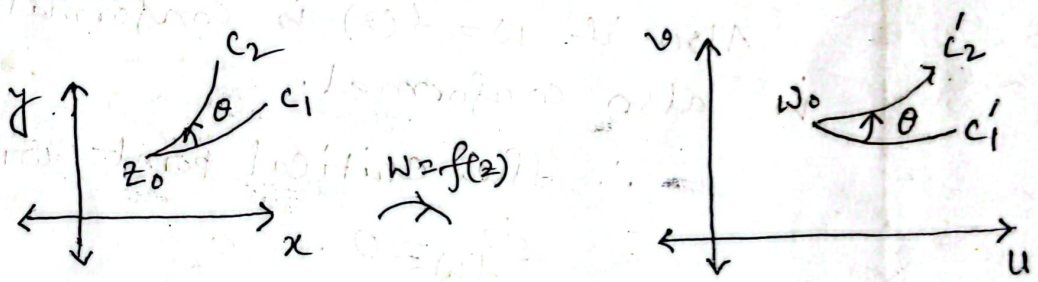
$$= \frac{1}{z}\left\{1 + 2\left(\frac{2}{z}\right) + 3\left(\frac{2}{z}\right)^2 + 4\left(\frac{2}{z}\right)^3 + \cdots \infty\right\}$$

$$= \frac{1}{z} + \frac{4}{z^2} + \frac{12}{z^3} + \frac{32}{z^4} + \cdots$$

Conformal Mapping

Mapping or Transformation: The set of equations $u = u(x,y)$, $v = v(x,y)$ defines a mapping or transformation which establishes correspondence between the points in the uv and xy planes.

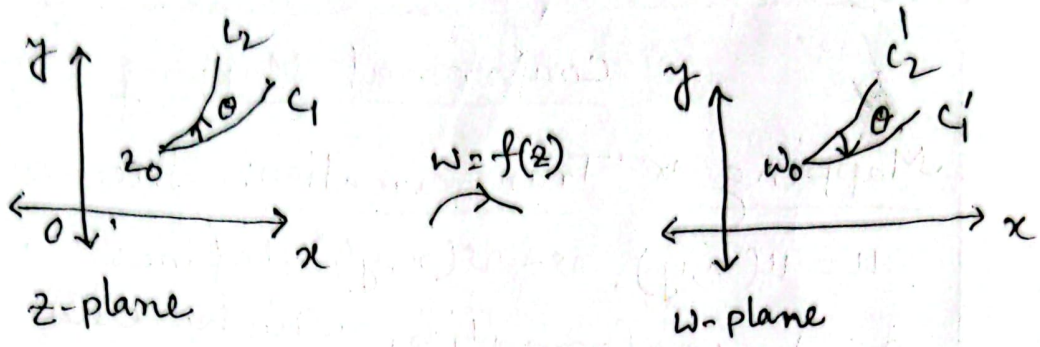
Conformal Mapping: A mapping $w = f(z)$ is said to be conformal at $z = z_0$ if it preserves the angle between any two curves through z_0 in z plane both in magnitude and direction.



z - plane

conformal mapping

Isogonal Mapping: A mapping $w = f(z)$ is said to be isogonal if it preserves the angle between any two curves through z_0 in z plane only in magnitude but not necessarily in direction.



Isogonal mapping

<u>Critical point:</u> * A point z_0 is said to be a critical point of a transformation $w = f(z)$ if is not conformal at z_0 where $f'(z_0) = 0$.

(i) The critical point will occur at $\frac{dw}{dz} = 0$.

Also if $w = f(z)$ is conformal then $z = f^{-1}(w)$ is also conformal.

$\therefore$ The critical point will occur at $\frac{dz}{dw} = 0$.

Hence the critical points of $w = f(z)$ will occur at $\frac{dw}{dz} = 0$, $\frac{dz}{dw} = 0$.

(ii) If $f(z)$ is not analytic, then $f(z)$ is not conformal.

(iii) An analytic function $f(z)$ is conformal everywhere except at its critical points where $f'(z) \neq 0$.

* A point at which $f'(z) = 0$ is called a critical point of the transformation.

Problem-1. Find the critical points of the transformation $w = z + \frac{1}{z}$.

Solution: Given $w = z + \frac{1}{z}$

$$\therefore \frac{dw}{dz} = 1 - \frac{1}{z^2} = \frac{z^2-1}{z^2}$$

Now the critical points occur at $\frac{dw}{dz} = 0, \frac{dz}{dw} = 0$.

$$\therefore \frac{z^2-1}{z^2} = 0, \quad \frac{z^2}{z^2-1} = 0$$

or, $z^2 - 1 = 0$, and $z^2 = 0$

$$\therefore z = \pm 1, 0$$

$\therefore$ The critical points are $z = \pm 1, 0$

Problem-2. Find the points such that $w = f(z) = \sin z$ is not conformal.

Solution: The points at which $f(z)$ is not conformal are called critical points.

Hence we require the critical points of $w = \sin z$.

$$\therefore \frac{dw}{dz} = \cos z, \quad \frac{dz}{dw} = \frac{1}{\cos z}$$

The critical points occur at $\frac{dw}{dz} = 0, \frac{dz}{dw} = 0$

$$\Rightarrow \cos z = 0, \quad \frac{1}{\cos z} = 0$$

$$\Rightarrow z = (2n+1)\frac{\pi}{2}, \; n = 0, \pm 1, \pm 2, \ldots$$

and $1 = 0$ which is impossible.

$\therefore$ The critical points are

$$z = (2n+1)\frac{\pi}{2}, \quad n = 0, \pm 1, \pm 2, \ldots$$

Problem-3. Find the critical points of the transformation $w^2 = (z-\alpha)(z-\beta)$

Solution: Given $w^2 = (z-\alpha)(z-\beta) \quad \ldots (1)$

$$\therefore \; 2w\frac{dw}{dz} = (z-\alpha)\cdot 1 + (z-\beta)\cdot 1$$

The critical points of $w = f(z)$ is given by $\frac{dw}{dz} = 0$ or, $\frac{(z-\alpha)+(z-\beta)}{2w} = 0$.

or, $(z-\alpha)+(z-\beta) = 0$

or, $z = \frac{\alpha+\beta}{2}$

Also, $\frac{dz}{dw} = 0$ or, $\frac{2w}{(z-\alpha)+(z-\beta)} = 0$

or, $w = 0$

or, $w^2 = 0$

or, $(z-\alpha)(z-\beta) = 0$ [from (1)]

$\therefore \; z = \alpha, \beta$

$\therefore$ The critical points are $z = \alpha, \beta, \frac{\alpha+\beta}{2}$.

15 <u>Linear transformation</u>: A transformation of the form $W = az + b$ where a and b are complex constants, is called a linear transformation.

15 <u>Bilinear transformation</u>: A transformation of the form $W = \frac{az+b}{cz+d}$ where a, b, c, d are complex constants such that $ad - bc \neq 0$, is called a bilinear transformation.

** (i) The bilinear transformation is also called Mobius transformation or linear fractional transformation.

(ii) The transformation $W = \frac{az+b}{cz+d}$ can be expressed as

$$cWz + dW - az - b = 0.$$

It is linear both in W and z. That is why, it is called bilinear.

(iii) The inverse of the transformation $W = \frac{az+b}{cz+d}$ is $z = \frac{-dW+b}{cW-a}$ which is also a bilinear transformation except $W = \frac{a}{c}$.

(iv) The expression $ad - bc$ is called the determinant of the transformation $W = \frac{az+b}{cz+d}$ or $z = \frac{-dW+b}{cW-a}$.

(v) The transformation is conformal only when $\frac{dW}{dz} \neq 0$ or $\frac{dz}{dW} \neq 0$ i.e. $ad - bc \neq 0$.

(vi) If $ad-bc=0$, every point in the z or w-plane is a critical point.

Invariant or fixed points:

If z maps into itself (i.e., $w=z$), then $w = \frac{az+b}{cz+d}$ gives

$$z = \frac{az+b}{cz+d}$$

$$\text{or, } cz^2 + (d-a)z - b = 0$$

The roots of this equation are defined as the invariant or fixed points of the bilinear transformation.

Cross ratio: The cross ratio of four points z_1, z_2, z_3, z_4 is defined by

$$\frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$$

Problem-1. Prove that cross ratio of four points is invariant (unchanged) in a bilinear transformation.

Proof: Let the points z_1, z_2, z_3 and z_4 of the z-plane map onto the points w_1, w_2, w_3 and w_4 of w-plane respectively under the bilinear transformation $w = \frac{az+b}{cz+d}$, then to prove that

$$\frac{(w_1-w_2)(w_3-w_4)}{(w_2-w_3)(w_4-w_1)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$$

Since W_i is image of z_i, $i=1,2,3,4$

$$\therefore W_1 - W_2 = \frac{az_1+b}{cz_1+d} - \frac{az_2+b}{cz_2+d}$$

$$= \frac{(ad-bc)(z_1-z_2)}{(cz_1+d)(cz_2+d)}$$

Similarly $W_3 - W_4 = \dfrac{(ad-bc)(z_3-z_4)}{(cz_3+d)(cz_4+d)}$

$$\therefore (W_1-W_2)(W_3-W_4) = \frac{(ad-bc)^2(z_1-z_2)(z_3-z_4)}{(cz_1+d)(cz_2+d)(cz_3+d)(cz_4+d)} \quad \cdots (1)$$

Similarly $(W_2-W_3)(W_4-W_1) = \dfrac{(ad-bc)^2(z_2-z_3)(z_4-z_1)}{(cz_1+d)(cz_2+d)(cz_3+d)(cz_4+d)}$ $\cdots (2)$

Now divide (1) by (2) we get

$$\frac{(W_1-W_2)(W_3-W_4)}{(W_2-W_3)(W_4-W_1)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$$

Thus the cross ratio of four points is invariant under bilinear transformation.

Problem-2: Find a bilinear transformation which maps points z_1, z_2, z_3 of the z-plane into points w_1, w_2, w_3 of the w-plane respectively.

Solution: We know that the bilinear transformation is given by $w = \dfrac{az+b}{cz+d}$

Since W_1, W_2, W_3 are the images of z_1, z_2, z_3 respectively.

Now $W_1 - W_2 = \dfrac{(ad-bc)(z_1-z_2)}{(cz_1+d)(cz_2+d)}$

$$W_2 - W_3 = \frac{(ad-bc)(z_2-z_3)}{(cz_2+d)(cz_3+d)}$$

$$w - w_1 = \frac{(ad-bc)(z-z_1)}{(cz+d)(cz_1+d)}$$

and $$w_3 - w = \frac{(ad-bc)(z_3-z)}{(cz_3+d)(cz+d)}$$

$$\therefore \frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{\dfrac{(z-z_1)(z_2-z_3)(ad-bc)^2}{(cz+d)(cz_1+d)(cz_2+d)(cz_3+d)}}{\dfrac{(z_1-z_2)(z_3-z)(ad-bc)^2}{(cz+d)(cz_1+d)(cz_2+d)(cz_3+d)}}$$

or, $$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

Solving for w in terms of z we get the required transformation.

15 Problem-3. Find a bilinear transformation which maps points $z=0, -i, -1$ into $w=i, 1, 0$ respectively.

<u>Solution</u>: Let $z_1 = 0,\ z_2 = -i,\ z_3 = -1$ and
$w_1 = i,\ w_2 = 1,\ w_3 = 0$

The required transformation is given by

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

or, $$\frac{(w-i)(1-0)}{(i-1)(0-w)} = \frac{(z-0)(-i+1)}{(0+i)(-1-z)}$$

or, $$\frac{(w-i)}{(1-i)w} = \frac{z(1-i)}{-i(1+z)}$$

or, $$-i(w+wz-i-iz) = wz(1-2i-1)$$

or, $$-iw - iwz - 1 - z = -2iwz$$

or, $iw(z-1) = z+1$

or, $w = \dfrac{z+1}{i(z-1)}$

$\therefore\ w = -\dfrac{i(z+1)}{z-1}$ which is the required transformation.

15 Problem-4. Find a bilinear transformation which maps the points $i, -i, 1$ of the z-plane into $0, 1, \infty$ of the w-plane respectively.

Solution: Since $w = \dfrac{az+b}{cz+d}$ ---(1)

So we have, $\dfrac{a\cdot i+b}{c\cdot i+d} = 0$ --(2)

$\dfrac{a(-i)+b}{c(-i)+d} = 1$ --(3)

$\dfrac{a\cdot 1+b}{c\cdot 1+d} = \infty$ --(4)

From (4) we get $c+d=0$ or, $c=-d$

(2) gives, $ai+b=0$ or $b=-ai$

Thus from (3), $\dfrac{-ai-ai}{di+d} = 1$

or, $\dfrac{-2ai}{d(1+i)} = 1$

or, $d = \dfrac{-2ai}{1+i} = \dfrac{-2ai(1-i)}{2}$

$\therefore d = -a(i+1)$

$\therefore c = a(i+1)$

So from (1) we get $w = \dfrac{az-ai}{a(i+1)z-a(i+1)}$

$= \dfrac{\cancel{a}(z-i)}{\cancel{a}(i+1)(z-1)}$

$= \dfrac{(z-i)(1-i)}{2(z-1)}$ is the required transformation.

15 Problem-5. Find the bilinear transformation which maps the points $z = \infty, i, 0$ into $w = 0, i, \infty$ respectively. What are the invariant points of this transformation?

Solution: Let $z_1 = \infty$, $z_2 = i$, $z_3 = 0$ and

$w_1 = 0$, $w_2 = i$, $w_3 = \infty$

The required bilinear transformation is given by

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

or, $$\frac{(w-w_1)w_3\left(\frac{w_2}{w_3}-1\right)}{(w_1-w_2)w_3\left(1-\frac{w_1}{w_3}\right)} = \frac{z_1\left(\frac{z}{z_1}-1\right)(z_2-z_3)}{z_1\left(1-\frac{z_2}{z_1}\right)(z_3-z)}$$

or, $$\frac{(w-0)\left(\frac{i}{\infty}-1\right)}{(0-i)\left(1-\frac{w}{\infty}\right)} = \frac{\left(\frac{z}{\infty}-1\right)(i-0)}{\left(1-\frac{i}{\infty}\right)(0-z)}$$

or, $$\frac{-w}{-i} = \frac{-i}{-z}$$

or, $$w = \frac{i^2}{z}$$

$\therefore w = \frac{-1}{z}$ which is the required transformation.

The invariant points of $w = \frac{-1}{z}$ are given by

$$z = \frac{-1}{z}$$

or, $z^2 = -1$

or, $z^2 = i^2$

$\therefore z = \pm i$ are the required invariant points.

Problem-6. Find the fixed points of the transformation $W = \frac{3z-5}{1+z}$.

Solution: The fixed points of $W = \frac{3z-5}{1+z}$ are given by

$$z = \frac{3z-5}{1+z}$$

$$\text{or, } z + z^2 = 3z - 5$$

$$\text{or, } z^2 - 2z + 5 = 0$$

$$\therefore z = \frac{2 \pm \sqrt{4-20}}{2}$$

$$= \frac{2 \pm 4i}{2}$$

$$= 1 \pm 2i$$

Therefore, the fixed points are $1+2i$ and $1-2i$.

15 Problem-7. Find the bilinear transformation which maps the points $z = 0, 1, \infty$ onto $w = -5, -1, 3$ respectively. What are the invariant points of this transformation?

Solution: Given $z_1 = 0$, $z_2 = 1$, $z_3 = \infty$ and $w_1 = -5$, $w_2 = -1$, $w_3 = 3$

The required bilinear transformation is given by

$$\frac{(W - W_1)(W_2 - W_3)}{(W_1 - W_2)(W_3 - W)} = \frac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)}$$

$$\text{or, } \frac{(W - W_1)(W_2 - W_3)}{(W_1 - W_2)(W_3 - W)} = \frac{(z - z_1) z_3 \left(\frac{z_2}{z_3} - 1\right)}{(z_1 - z_2) z_3 \left(1 - \frac{z}{z_3}\right)}$$

$$\text{or, } \frac{(W + 5)(-1 - 3)}{(-5 + 1)(3 - W)} = \frac{(z - 0)(0 - 1)}{(0 - 1)(1 - 0)}$$

$$\text{or, } \frac{(W + 5)(-4)}{(-4)(3 - W)} = z$$

or, $\frac{w+5}{3-w} = \frac{z}{1}$

or, $w+5 = 3z - wz$

or, $w(1+z) = 3z-5$

or, $w = \frac{3z-5}{1+z}$

To find the invariant points, put $w = z$.

$\therefore z = \frac{3z-5}{1+z}$

or, $z^2 + z = 3z - 5$

or, $z^2 - 2z + 1 = -4$

or, $(z-1)^2 = i^2 4$

or, $z - 1 = \pm i2$

$\therefore z = 1 \pm i2$ are the invariant points.

Problem-8. Define fixed points. Prove that a bilinear transformation has atmost two fixed points.

Solution: A point $z = a$ is said to be a fixed point of a mapping $w = f(z)$ if its image under $f(z)$ is itself. Thus it is given by the equation $z = f(z)$.

The fixed points of the general bilinear transformation $w = \frac{az+b}{cz+d}$ are given by $z = \frac{az+b}{cz+d}$

or, $cz^2 + dz = az + b$

or, $cz^2 + (d-a)z - b = 0$

This is a quadratic equation in z. Hence a bilinear transformation has atmost two fixed points.

Contour Integration-1

Evaluate by Contour integration:

(i) $\int_0^{2\pi} \frac{\cos 3\theta \, d\theta}{5-4\cos\theta}$ (ii) $\int_0^{\pi} \frac{\cos 2\theta \, d\theta}{1-2a\cos\theta+a^2}$, $a^2<1$

(iii) $\int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta+a^2}$, $|a|<1$

(iv) $\int_0^{\pi} \frac{\sin^2\theta \, d\theta}{5-4\cos\theta}$ (v) $\int_0^{2\pi} \frac{d\theta}{5+3\sin\theta}$

Solution: (i) let $I = \int_0^{2\pi} \frac{\cos 3\theta \, d\theta}{5-4\cos\theta}$

$$= \text{Real part of } \int_0^{2\pi} \frac{e^{i3\theta} d\theta}{5-4\cos\theta}$$

$$= \text{R P of } \int_0^{2\pi} \frac{e^{i3\theta} d\theta}{5-4\cdot\frac{1}{2}(e^{i\theta}+e^{-i\theta})}$$

Put $e^{i\theta} = z$

$\therefore e^{i\theta} \cdot i d\theta = dz$

or, $d\theta = \frac{dz}{iz}$

$\therefore I = \text{R.P. of } \oint_C \frac{z^3 \cdot \frac{dz}{iz}}{5-2(z+\frac{1}{z})}$, where C is the unit circle, $|z|=1$.

$$= \text{RP of } \frac{1}{i}\oint_C \frac{z^3 dz}{5z-2z^2-2}$$

$$= \text{RP of } \frac{-1}{i}\oint_C \frac{z^3 dz}{2z^2-5z+2}$$

$$= \text{RP of } \frac{-1}{i}\oint_C f(z)\,dz \quad \cdots (1)$$

where $f(z) = \frac{z^3}{2z^2-5z+2}$

poles of $f(z)$ are given by

$2z^2 - 5z + 2 = 0$

or, $(z-2)(2z-1) = 0$

$\therefore z = 2, \frac{1}{2}$

Only simple pole $z = \frac{1}{2}$ within C.

$= 0$ $\quad 1-a^2$

Residue at $z=\frac{1}{2}$ is

$$\lim_{z\to\frac{1}{2}} (z-\tfrac{1}{2})\cdot\frac{z^3}{2z^2-5z+2}$$

$$=\lim_{z\to\frac{1}{2}} (z-\tfrac{1}{2})\cdot\frac{z^3}{(z-2)(2z-1)}$$

$$=\lim_{z\to\frac{1}{2}} \cancel{(z-\tfrac{1}{2})}\cdot\frac{z^3}{2(z-2)\cancel{(z-\tfrac{1}{2})}}$$

$$=\frac{\frac{1}{8}}{2(\frac{1}{2}-2)}$$

$$=-\frac{1}{24}$$

By Cauchy's residue theorem we have,

$$\oint_C f(z)\,dz = 2\pi i\left(-\frac{1}{24}\right)$$

$$=-\frac{\pi i}{12}$$

$$\therefore I = \text{R.P. of } \frac{-1}{i}\oint_C f(z)\,dz$$

$$=\text{R.P. of } \frac{-1}{i}\cdot\left(-\frac{\pi i}{12}\right)$$

$$=\frac{\pi}{12}$$

(ii) Let $I=\int_0^{\pi}\frac{\cos 2\theta\, d\theta}{1-2a\cos\theta+a^2}$

$$=\frac{1}{2}\int_0^{2\pi}\frac{\cos 2\theta\, d\theta}{1-2a\cos\theta+a^2}\quad\left[\because \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx \text{ if } f(2a-x)=f(x)\right]$$

$$=\text{R.P of } \frac{1}{2}\int_0^{2\pi}\frac{e^{i2\theta}\,d\theta}{1-2a\cos\theta+a^2}$$

$$=\text{RP of } \frac{1}{2}\int_0^{2\pi}\frac{e^{i2\theta}\,d\theta}{1-2a\cdot\frac{1}{2}(e^{i\theta}+e^{-i\theta})+a^2}$$

Put $e^{i\theta}=z \therefore d\theta=\frac{dz}{iz}$

$$\therefore I = \text{RP of } \frac{1}{2}\oint_C \frac{z^2 \cdot \frac{dz}{iz}}{1-a(z+\frac{1}{z})+a^2}$$

$$= \text{RP of } \frac{1}{2i}\oint_C \frac{z^2\,dz}{z-az^2-a+a^2z}$$

$$= \text{R.P of } \frac{-1}{2i}\oint_C \frac{z^2\,dz}{az^2-a^2z-z+a}$$

$$= \text{R.P. of } \frac{-1}{2i}\oint_C f(z)\,dz$$

where $f(z) = \dfrac{z^2}{az^2-a^2z-z+a}$

poles of $f(z)$ are given by

$$az^2-a^2z-z+a=0$$

$$\text{or, } (z-a)(az-1)=0$$

$$\therefore z=a, \frac{1}{a}$$

Only simple pole $z=a$ lies inside C.

Residue at $z=a$ is,

$$\lim_{z\to a} \cancel{(z-a)} \frac{z^2}{\cancel{(z-a)}(az-1)}$$

$$= \frac{a^2}{a^2-1}$$

Hence by Cauchy's residue theorem we get

$$\oint_C f(z)\,dz = 2\pi i\left(\frac{a^2}{a^2-1}\right)$$

$$\therefore I = \text{R.P. of } \frac{-1}{2i}\oint_C f(z)\,dz$$

$$= \text{R.P. of } \frac{-1}{2i}\cdot 2\pi i\left(\frac{a^2}{a^2-1}\right)$$

$$= \frac{\pi a^2}{1-a^2}$$

(iii) let $I = \int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta+a^2}$

$$= \int_0^{2\pi} \frac{d\theta}{1-a(e^{i\theta}+e^{-i\theta})+a^2}$$

$$= \oint_C \frac{dz/iz}{1-a(z+1/z)+a^2}$$

Put $e^{i\theta} = z$
$\therefore d\theta = \frac{dz}{iz}$

Where C is the unit circle, $|z| = 1$.

$$= \frac{1}{i}\oint_C \frac{dz}{z-az^2-a+a^2z}$$

$$= \frac{-1}{i}\oint_C \frac{dz}{az^2-a^2z-z+a}$$

$$= \frac{-1}{i}\oint_C f(z)\,dz$$

where $f(z) = \frac{1}{az^2-a^2z-z+a}$

Poles of $f(z)$ are given by

$$az^2 - a^2z - z + a = 0$$

or, $(z-a)(az-1) = 0$

$\therefore z = a, 1/a$

Only simple pole $z=a$ lies inside C,

Residue at $z=a$ is $\lim_{z\to a} (z-a)\cdot\frac{1}{(z-a)(az-1)}$

$$= \frac{1}{a^2-1}$$

Hence by Cauchy's residue theorem we get

$$\oint_C f(z)\,dz = 2\pi i\left(\frac{1}{a^2-1}\right)$$

$$\therefore I = -\frac{1}{i}\oint_C f(z)\,dz$$

$$= -\frac{1}{i}\cdot 2\pi i\left(\frac{1}{a^2-1}\right)$$

$$= \frac{2\pi}{1-a^2}$$

(iv) Let $I = \int_0^{\pi} \frac{\sin^2\theta \, d\theta}{5-4\cos\theta}$

$$= \frac{1}{2}\int_0^{2\pi} \frac{\sin^2\theta \, d\theta}{5-4\cos\theta} \quad \left[\because \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx \text{ if } f(2a-x)=f(x)\right]$$

$$= \frac{1}{2}\cdot\frac{1}{2}\int_0^{2\pi} \frac{(1-\cos 2\theta)\,d\theta}{5-4\cos\theta}$$

$$= \text{R.P of } \frac{1}{4}\int_0^{2\pi} \frac{(1-e^{i2\theta})\,d\theta}{5-4\cos\theta}$$

$$= \text{R.P of } \frac{1}{4}\int_0^{2\pi} \frac{(1-e^{i2\theta})\,d\theta}{5-2(e^{i\theta}+e^{-i\theta})}$$

$$= \text{RP of } \frac{1}{4}\oint_C \frac{(1-z^2)\cdot\frac{dz}{iz}}{5-2(z+\frac{1}{z})}, \quad \text{Put } e^{i\theta}=z \quad \therefore d\theta = \frac{dz}{iz}$$

$$= \text{R.P of } \frac{1}{4i}\oint_C \frac{(1-z^2)\,dz}{5z-2z^2-2}$$ where C is the unit circle, $|z|=1$.

$$= \text{R.P. of } \frac{-1}{4i}\oint_C \frac{(1-z^2)\,dz}{2z^2-5z+2}$$

$$= \text{R.P. of } \frac{-1}{4i}\oint_C f(z)\,dz$$

where $f(z) = \frac{1-z^2}{2z^2-5z+2}$

poles of $f(z)$ are given by $2z^2-5z+2=0$

or, $(z-2)(2z-1)=0$

$\therefore z = 2, \frac{1}{2}$

Only simple pole $z=\frac{1}{2}$ lies inside C.

Residue at simple pole $z=\frac{1}{2}$ is

$$\lim_{z\to\frac{1}{2}} (z-\tfrac{1}{2})\frac{1-z^2}{(z-2)(2z-1)}$$

$$= \lim_{z\to\frac{1}{2}} \frac{1-z^2}{2(z-2)}$$

$$= \lim_{z\to\frac{1}{2}} \frac{1-\frac{1}{4}}{1-4}$$

$$= -\frac{1}{4}$$

By Cauchy's residue theorem we get

$$\oint_C f(z)\,dz = 2\pi i\left(-\frac{1}{4}\right)$$

$\therefore I = \text{R.P. of } \frac{-1}{4i} \cdot 2\pi i \left(-\frac{1}{4}\right)$

$= \frac{\pi}{8}$

(v) Let $I = \int_0^{2\pi} \frac{d\theta}{5+3\sin\theta}$

$= \int_0^{2\pi} \frac{d\theta}{5+3\cdot\frac{1}{2i}(e^{i\theta}-e^{-i\theta})}$

$= \oint_C \frac{dz/iz}{5+\frac{3}{2i}\left(z-\frac{1}{z}\right)}$ Put $e^{i\theta} = z$ $\therefore d\theta = \frac{dz}{iz}$

$= 2\oint_C \frac{dz}{10iz+3z^2-3}$

$= 2\oint_C f(z)\,dz$

where $f(z) = \frac{1}{3z^2+10iz-3}$

Poles of $f(z)$ are given by $3z^2+10iz-3=0$

or, $3z^2+10iz+i^2 3=0$

or, $3z^2+9iz+iz+i^2 3=0$

or, $3z(z+3i)+i(z+3i)=0$

or, $(z+3i)(3z+i)=0$

$\therefore z = -3i, -\frac{i}{3}$

Since $\left|-\frac{i}{3}\right| = \frac{1}{3}$, so only simple pole $z=-\frac{i}{3}$ lies inside C.

Residue at simple pole $z=-\frac{i}{3}$ is

$\lim_{z\to -i/3} \left(z+\frac{i}{3}\right)\cdot\frac{1}{(z+3i)(3z+i)}$

$= \lim_{z\to -i/3} \frac{1}{3(z+3i)}$

$= \frac{1}{3(-i/3+3i)}$

$= \frac{1}{8i}$

Hence by Cauchy's residue theorem, we get

$\oint_C f(z)dz = 2\pi i\left(\frac{1}{8i}\right)$

$\therefore I = 2\cdot 2\pi i\left(\frac{1}{8i}\right)$

$= \frac{\pi}{2}$

Evaluate by contour integration: $\int_0^{2\pi} \frac{\sin 2\theta \, d\theta}{1-2a\cos\theta+a^2}$, $a^2<1$

<u>Solution</u>: Let $I = \int_0^{2\pi} \frac{\sin 2\theta \, d\theta}{1-2a\cos\theta+a^2}$

$= \text{imaginary part of } \int_0^{2\pi} \frac{e^{i2\theta} d\theta}{1-2a\cos\theta+a^2}$

$= \text{I.P. of } \int_0^{2\pi} \frac{e^{i2\theta} d\theta}{1-2a\cdot\frac{1}{2}(e^{i\theta}+e^{-i\theta})+a^2}$

Let us put $e^{i\theta} = z$ $\therefore$ $e^{i\theta} i d\theta = dz$

or, $d\theta = \frac{dz}{iz}$

$\therefore I = \text{I.P. of } \oint_C \frac{z^2 \cdot \frac{dz}{iz}}{1-a(z+\frac{1}{z})+a^2}$, where C is the unit circle $|z|=1$.

$= \text{I.P. of } \frac{1}{i}\oint_C \frac{z^2 dz}{z-az^2-a+a^2z}$

$= \text{I.P. of } \frac{-1}{i}\oint_C \frac{z^2 dz}{az^2-z-a^2z+a}$

$= \text{I.P. of } \frac{-1}{i}\oint_C f(z)\, dz \quad \cdots (1)$

where $f(z) = \frac{z^2}{az^2-z-a^2z+a}$

poles of $f(z)$ are given by

$az^2-z-a^2z+a=0$

or, $z(az-1)-a(az-1)=0$

or, $(az-1)(z-a)=0$

$\therefore z = \frac{1}{a}, a$

Since $a^2<1$, so $z=a$ lies inside C.

Residue at simple pole $z=a$ is

$\lim_{z\to a} \cancel{(z-a)} \cdot \frac{z^2}{(az-1)\cancel{(z-a)}}$

$= \frac{a^2}{a^2-1}$

Hence by Cauchy's residue theorem, we get

$\oint_C f(z)\, dz = 2\pi i \left(\frac{a^2}{a^2-1}\right)$

So from (1) we get

$I = \text{I.P. of } \frac{-1}{i} \cdot 2\pi i \left(\frac{a^2}{a^2-1}\right)$

$= \text{I.P. of } \left(\frac{2\pi a^2}{1-a^2} + i\cdot 0\right)$

$= 0$

Contour Integration-2

Lemma related to Cauchy's residue theorem:

(i) Cauchy's Lemma I: If γ is the arc of a small semicircle $|z-a|=r$ between the angles θ_1 and θ_2, $\theta_1 \le \theta \le \theta_2$ and if

$$\lim_{z \to a} (z-a) f(z) = A, \text{ then}$$

$$\lim_{r \to 0} \int_{\gamma} f(z)\,dz = -iA(\theta_2 - \theta_1)$$

where $f(z)$ is a continuous function.

(ii) Cauchy's Lemma II: If Γ be an arc $(\theta_1 \le \theta \le \theta_2)$ of the big semicircle $|z|=R$ and if $\lim_{z \to \infty} z f(z) = A$, then

$$\lim_{R \to \infty} \int_{\Gamma} f(z)\,dz = iA(\theta_2 - \theta_1)$$

(iii) Jordan's Lemma: If $f(z)$ is analytic except at finite numbere of singularities and if $\lim_{z \to \infty} f(z) = 0$, then

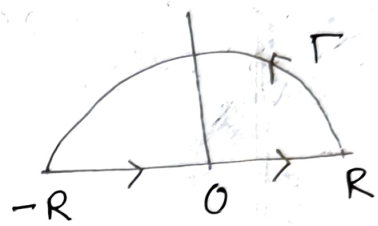
$$\lim_{R \to \infty} \int_{\Gamma} e^{imz} f(z)\,dz = 0, \quad m > 0$$

Evaluate by contour integration: (i) $\int_0^\infty \frac{dx}{x^4+a^4}$.

Solution: Let us consider the integral

$$\oint_C f(z)\,dz,$$

where $f(z) = \frac{1}{z^4+a^4}$ and C is the closed contour consisting of semicircle Γ and the real axis from $-R$ to R.



Then by Cauchy's residue theorem we have,

$$\oint_C f(z)\,dz = 2\pi i \text{ (sum of residues)}$$

or, $$\int_\Gamma f(z)\,dz + \int_{-R}^{R} f(x)\,dx = 2\pi i \text{ (sum of residues)} \quad \text{--- (1)}$$

since $$\lim_{z\to\infty} z f(z) = \lim_{z\to\infty} \frac{z}{z^4+a^4}$$

$$= 0$$

$$= A$$

$$\therefore \lim_{R\to\infty} \int_\Gamma f(z)\,dz = iA(\theta_2-\theta_1)$$

$$= i\cdot 0(\pi - 0)$$

$$= 0$$

Poles of $f(z)$ are given by $z^4+a^4=0$

or, $z^4 = -a^4$ or, $z^4 = e^{(2n+1)\pi i} a^4$

$\therefore$ $z = a e^{(2n+1)\pi i/4}$, $n = 0, 1, 2, 3,$

Only $z = ae^{\pi i/4}$ and $z = ae^{3\pi i/4}$ lie inside C.

Residue at $z = ae^{\pi i/4}$ is

$$\lim_{z \to ae^{\pi i/4}} \left\{ \left(z - ae^{\pi i/4}\right) \cdot \frac{1}{z^4 + a^4} \right\}$$

$$= \lim_{z \to ae^{\pi i/4}} \frac{1}{4z^3}$$

$$= \frac{1}{4a^3} e^{-3\pi i/4}$$

$$= -\frac{1}{4a^3} e^{\pi i/4}$$

Residue at $z = ae^{3i\pi/4}$ is

$$\lim_{z \to ae^{3\pi i/4}} \left\{ \left(z - ae^{3\pi i/4}\right) \cdot \frac{1}{z^4 + a^4} \right\}$$

$$= \lim_{z \to ae^{3\pi i/4}} \frac{1}{4z^3}$$

$$= \frac{1}{4a^3} e^{-9\pi i/4}$$

$$= \frac{1}{4a^3} e^{-\pi i/4} \quad [\because e^{-2\pi i} = 1]$$

Hence by making $R \to \infty$, relation (1) becomes

$$0 + \int_{-\infty}^{\infty} \frac{dx}{x^4+a^4} = 2\pi i\left(-\frac{1}{4a^3}e^{\pi i/4} + \frac{1}{4a^3}e^{-\pi i/4}\right)$$

or, $$2\int_0^{\infty} \frac{dx}{x^4+a^4} = \frac{-2\pi i}{4a^3}\left(e^{\pi i/4} - e^{-\pi i/4}\right)$$

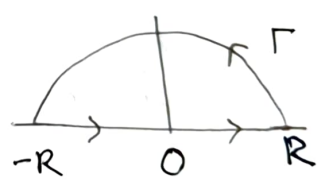
$$= \frac{-2\pi i}{4a^3} \cdot 2i \sin\frac{\pi}{4}$$

$$= \frac{\pi}{a^3} \cdot \frac{1}{\sqrt{2}}$$

$$= \frac{\pi\sqrt{2}}{2a^3}$$

$$\therefore \int_0^{\infty} \frac{dx}{x^4+a^4} = \frac{\pi\sqrt{2}}{4a^3}$$

(ii) $$\int_0^{\infty} \frac{dx}{(x^2+1)(x^2+4)^2}$$



Let us consider the integral $\oint_C f(z)\,dz$,

where $f(z) = \dfrac{1}{(z^2+1)(z^2+4)^2}$ and C is the closed contour consisting of semicircle Γ and the real axis from $-R$ to R.

Thus $\oint_C f(z)\,dz = 2\pi i$ (sum of residues)

or, $$\int_\Gamma f(z)\,dz + \int_{-R}^{R} f(x)\,dx = 2\pi i \text{ (sum of residues)} \quad \text{--- (1)}$$

$\because \lim_{z\to\infty} z f(z) = \lim_{z\to\infty} \frac{z}{(z^2+1)(z^2+4)^2}$

$= 0$

$= A$

$\therefore \lim_{R\to\infty} \int_\Gamma f(z)dz = iA(\theta_2-\theta_1)$

$= i\cdot 0(\pi-0)$

$= 0$

Hence by making $R\to\infty$, relation (1) becomes,

$$0 + \int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)^2} = 2\pi i \text{ (sum of residues)} \quad \text{--- (2)}$$

Poles of $f(z)$ are given by

$(z^2+1)(z^2+4)^2 = 0$

$\therefore z = \pm i, \pm 2i$

But only two poles $z=i$, $z=2i$ (pole of 2nd order) lie inside C.

Residue at $z=i$ is $\lim_{z\to i} (z-i)\cdot \frac{1}{(z^2+1)(z^2+4)^2}$

$= \lim_{z\to i} \frac{1}{(z+i)(z^2+4)^2}$

$= \frac{1}{2i(-1+4)^2}$

$= \frac{1}{18i}$

Residue at double pole $z = 2i$ is

$$\lim_{z \to 2i} \frac{1}{\lfloor 2-1} \frac{d}{dz}\left\{(z-2i)^2 \cdot \frac{1}{(z^2+1)(z^2+4)^2}\right\}$$

$$= \lim_{z \to 2i} \frac{d}{dz}\left\{\frac{1}{(z^2+1)(z+2i)^2}\right\}$$

$$= \lim_{z \to 2i} \frac{0 - [2z(z+2i)^2 + (z^2+1)\cdot 2(z+2i)\cdot 1]}{(z^2+1)^2 (z+2i)^4}$$

$$= \frac{-[(4i)(4i)^2 + (-4+1)\cdot 2\cdot 4i]}{(-4+1)^2 (4i)^4}$$

$$= \frac{64i + 24i}{9 \times 16 \times 16}$$

$$= \frac{88i}{9 \times 16 \times 16}$$

$$= \frac{11i}{9 \times 2 \times 16}$$

$$= \frac{11i}{18 \times 16}$$

So from (2) we get

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)^2} = 2\pi i\left(\frac{1}{18i} + \frac{11i}{18 \times 16}\right)$$

$$\text{or, } 2\int_0^{\infty} \frac{dx}{(x^2+1)(x^2+4)^2} = 2\pi i\left(\frac{16-11}{18i \times 16}\right)$$

$$\therefore \int_0^{\infty} \frac{dx}{(x^2+1)(x^2+4)^2} = \frac{5\pi}{288}$$

(iii) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x-1)(x^2+1)}$

Let us consider the integral $\oint_C f(z)\,dz$, where $f(z) = \dfrac{1}{(z-1)(z^2+1)}$ and C is the closed contour consisting of a big semicircle Γ and the real axis $-R$ to R indented at $z=1$. The small semicircle is denoted by γ, its centre is at $z=1$ and radius r.

Hence by Cauchy's residue theorem we have,

$$\oint_C f(z)\,dz = 2\pi i \text{ (sum of residues)}$$

$$\text{or, } \int_\Gamma f(z)\,dz + \int_{-R}^{1-r} f(x)\,dx + \int_\gamma f(z)\,dz + \int_{1+r}^{R} f(x)\,dx$$

$$= 2\pi i \text{ (sum of residues)} \quad (1)$$

$$\because \lim_{z\to\infty} z f(z) = \lim_{z\to\infty} \frac{z}{(z-1)(z^2+1)}$$

$$= 0$$

$$= A$$

$$\therefore \lim_{R\to\infty} \int_\Gamma f(z)\,dz = iA(\theta_2 - \theta_1)$$

$$= i\cdot 0\,(\pi - 0)$$

$$= 0$$

Also $\lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1}\cancel{(z-1)}\cdot\frac{1}{\cancel{(z-1)}(z^2+1)}$

$$= \frac{1}{2}$$

$$= A$$

$\therefore \lim_{r\to 0}\int_{\gamma_1} f(z)\,dz = -iA(\theta_2-\theta_1)$

$$= -i\cdot\frac{1}{2}(\pi-0)$$

$$= \frac{-\pi i}{2}$$

Poles of $f(z)$ are given by

$$(z-1)(z^2+1)=0$$

$$\therefore z = 1, \pm i$$

Both $z=1, -i$ lie outside the contour but $z=i$ lies in C.

Residue at $z=i$ is $\lim_{z\to i}(z-i)\cdot\frac{1}{(z-1)(z^2+1)}$

$$= \lim_{z\to i}\frac{1}{(z-1)(z+i)}$$

$$= \frac{1}{(i-1)\cdot 2i}$$

$$= -\frac{1-i}{4}$$

Hence by making $R\to\infty$, $r\to 0$, relation(1) becomes,

$$0 + \int_{-\infty}^{1} f(x)\,dx - \frac{\pi i}{2} + \int_{1}^{\infty} f(x)\,dx = 2\pi i\left(-\frac{1-i}{4}\right)$$

or, $\int_{-\infty}^{\infty} f(x)\,dx = \pi i\left(-\frac{1-i}{2}\right) + \frac{\pi i}{2}$

$$\therefore \int_{-\infty}^{\infty}\frac{dx}{(x-1)(x^2+1)} = -\frac{\pi}{2}$$

(iv) $\int_0^\infty \frac{x \sin x \, dx}{x^2+a^2}$

Let us consider the integral $\oint_C f(z)\,dz$, where $f(z) = \frac{z e^{iz}}{z^2+a^2}$ and C is the closed contour consisting of semicircle Γ and the real axis $-R$ to R.

Hence by Cauchy's residue theorem we have, $\oint_C f(z)\,dz = 2\pi i$ (sum of residues)

or, $\int_\Gamma f(z)\,dz + \int_{-R}^{R} f(x)\,dx = 2\pi i$ (sum of residues)

$\because \lim_{z\to\infty} \varphi(z) = \lim_{z\to\infty} \frac{z}{z^2+a^2}$ $\left[\because f(z) = \frac{z e^{iz}}{z^2+a^2} = e^{iz}\varphi(z),\right.$ where $\varphi(z) = \frac{z}{z^2+a^2}$

$= \lim_{z\to\infty} \frac{1}{2z}$ [using L. Hospital's rule

$= 0$

Hence by Jordan's Lemma we get

$$\lim_{R\to\infty} \int_\Gamma f(z)\,dz = \lim_{R\to\infty} \int_\Gamma e^{iz}\varphi(z)\,dz$$

$$= 0$$

poles of $f(z)$ are given by $z^2+a^2=0$

$\therefore z = \pm ia$

Only simple pole $z = ia$ lies inside C.

Residue at $z = ai$ is $\lim_{z \to ai} (z - ai) \cdot \frac{z e^{iz}}{z^2 + a^2}$

$$= \lim_{z \to ai} \frac{z e^{iz}}{z + ai}$$

$$= \frac{i a e^{-a}}{2ai}$$

$$= \frac{e^{-a}}{2}$$

Hence by making $R \to \infty$, relation (1) becomes

$$0 + \int_{-\infty}^{\infty} f(x)\, dx = 2\pi i \left(\frac{e^{-a}}{2} \right)$$

or, $$\int_{-\infty}^{\infty} \frac{x e^{ix}\, dx}{x^2 + a^2} = \pi i e^{-a}.$$

or, $$\int_{-\infty}^{\infty} \frac{x (\cos x + i \sin x)\, dx}{x^2 + a^2} = \pi i e^{-a}$$

Equating imaginary parts, we get

$$\int_{-\infty}^{\infty} \frac{x \sin x\, dx}{x^2 + a^2} = \pi e^{-a}$$

or, $$2 \int_{0}^{\infty} \frac{x \sin x\, dx}{x^2 + a^2} = \pi e^{-a}$$

$$\therefore \int_{0}^{\infty} \frac{x \sin x\, dx}{x^2 + a^2} = \frac{\pi}{2} e^{-a}.$$