First Order and First Degree Differential Equations

Parvin Akter

Assistant Professor

Department of Mathematics, CUET.

2.6 Homogeneous equation Definition. A differential equation of first order and first degree is said to be homogeneous if it can be put in the form $dy/dx = f(y/x)$

2.7 Working rule for solving homogeneous equations

Let the given equation be homogeneous. Then, by definition, the given equation can be put in the form

$$dy/dx = f(y/x). \quad \ldots (1)$$

To solve (1), let $y/x = v$, *i.e.*, $y = vx$. ... (2)

Differentiating with respect to x, (2) gives $dy/dx = v + x\,(dv/dx)$. ... (3)

Using (2) and (3), (1) becomes

$$v + x\frac{dv}{dx} = f(v) \quad \text{or} \quad x\frac{dv}{dx} = f(v) - v$$

Separating the variables x and v, we have

$$\frac{dx}{x} = \frac{dv}{f(v) - v} \quad \text{so that} \quad \log x + c = \int \frac{dv}{f(v) - v}$$

where c is an arbitrary constant. After integration, replace v by y/x.

Ex. 1. *Solve* $(x^3 + 3xy^2)\,dx + (y^3 + 3x^2y)\,dy = 0$. **[I.A.S. (Prel.) 2004]**

Sol. Given $$\frac{dy}{dx} = -\frac{x^3 + 3xy^2}{y^3 + 3x^2y} = -\frac{1 + 3(y/x)^2}{(y/x)^3 + 3(y/x)}. \qquad \text{... (1)}$$

Take $y/x = v$, *i.e.*, $y = vx$. so that $dy/dx = v + x\,(dv/dx)$. ... (2)

From (1) and (2), $$v + x\frac{dv}{dx} = -\frac{1 + 3v^2}{v^3 + 3v}$$

or $$x\frac{dv}{dx} = -\frac{1 + 3v^2}{v^3 + 3v} - v = -\frac{v^4 + 6v^2 + 1}{v^3 + 3v} \quad \text{or} \quad 4\frac{dx}{x} = -\frac{4v^3 + 12v}{v^4 + 6v^2 + 1}\,dv.$$

Integrating, $4 \log x = -\log(v^4 + 6v^2 + 1) + \log c$, c being an arbitrary constant.

or $\log x^4 = \log[c/(v^4 + 6v^2 + 1)]$, *i.e.*, $x^4(v^4 + 6v^2 + 1) = c$

or $y^4 + 6x^2y^2 + x^4 = c$ or $(x^2 + y^2)^2 + 4x^2y^2 = c$, as $y/x = v$.

Ex. 2. Solve: $x\,dy - y\,dx = (x^2 + y^2)^{1/2}\,dx$ **[Meerut 2008; Delhi Maths (G) 1999]**

Sol. Here, $$\frac{dy}{dx} = \frac{y + (x^2 + y^2)^{1/2}}{x} = \frac{y}{x} + \left\{1 + (y/x)^2\right\}^{1/2}. \qquad \text{... (1)}$$

Take $y/x = v$, *i.e.*, $y = vx$. so that $dy/dx = v + x\,(dv/dx)$.... (2)

From (1) and (2), $v + x\dfrac{dv}{dx} = v + \sqrt{(1 + v^2)}$ or $\dfrac{dx}{x} = \dfrac{dv}{\sqrt{(1 + v^2)}}$.

Integrating, $\log x + \log c = \log[v + \sqrt{(v^2 + 1)}]$ or $xc = v + \sqrt{(v^2 + 1)}$

or $$x^2c = y + \sqrt{(y^2 + x^2)},\ as\ v = y/x$$

Ex. 4. *Solve:* $x \cos (y/x)\,(y\,dx + x\,dy) = y \sin (y/x)\,(x\,dy - y\,dx)$ ... (1)

or
$$\left(x \cos \frac{y}{x} + y \sin \frac{y}{x}\right) y - \left(y \sin \frac{y}{x} - x \cos \frac{y}{x}\right) x \frac{dy}{dx} = 0. \qquad \text{... (2)}$$

[Mysore 2004; Kanpur 1996; Lucknow 1997]

Sol. Rewriting (1), we get (2). So (1) and (2) are the same equations.

From (2),
$$\frac{dy}{dx} = \frac{\{x \cos (y/x) + y \sin (y/x)\}\, y}{\{y \sin (y/x) - x \cos (y/x)\}\, x}$$

or
$$\frac{dy}{dx} = \frac{[\cos (y/x) + (y/x) \sin (y/x)]\,(y/x)}{[(y/x) \sin (y/x) - \cos (y/x)]} \qquad \text{... (3)}$$

Take $y/x = v$, *i.e.*, $y = vx$, so that $dy/dx = v + x\,(dv/dx)$.... (4)

Using (4), (3) becomes
$$v + x\frac{dv}{dx} = \frac{v\,(\cos v + v \sin v)}{v \sin v - \cos v}$$

or
$$x\frac{dv}{dx} = \frac{v\,(\cos v + v \sin v)}{v \sin v - \cos v} - v = \frac{2v \cos v}{v \sin v - \cos v} \text{ or } 2\frac{dx}{x} = \frac{v \sin v - \cos v}{v \cos v}\,dv = \left[\frac{\sin v}{\cos v} - \frac{1}{v}\right] dv.$$

Integrating, $2 \log x = -\log \cos v - \log v + \log c$, c being an arbitrary constant.

or $\log x^2 = \log (c/v \cos v)$ or $x^2 v \cos v = c$ or $xy \cos (y/x) = c$. $[\because\ v = y/x]$

Ex. 7. *Solve* $(x^3 + y^3)\,dx = (x^2y + xy^2)\,dy$ **[Delhi Maths (H) 2002]**

Sol. Re-writing the given equation, $dy/dx = (x^3 + y^3)/(x^2y + xy^2)$ ... (1)

Putting $y = xv$ and $dy/dx = v + x\,(dv/dx)$, (1) becomes

$$v + x\,\frac{dv}{dx} = \frac{1+v^3}{v+v^2} \qquad \text{or} \qquad x\,\frac{dv}{dx} = \frac{1+v^3}{v+v^2} - v = \frac{(1-v)(1+v)}{v(1+v)}$$

or $$\frac{dx}{x} + \frac{v\,dv}{v-1} = 0 \qquad \text{or} \qquad \frac{dx}{x} + \left(1 + \frac{1}{v-1}\right)dv = 0$$

Integrating, $\log x + v + \log(v-1) - \log c = 0$, c being an arbitrary constant.

or $\log\{x(v-1)/c\} = -v$ or $x(v-1) = ce^{-v}$ or $x(y/x) - x = ce^{-y/x}$

or $y - x = ce^{-y/x}$, c being an arbitrary constant.

Ex. 8. *Solve* $(x^2 - 4xy - 2y^2)\,dx + (y^2 - 4xy - 2x^2)\,dy = 0.$ **[Delhi Maths (G) 2005, 06]**

Sol. Re-writing the given differential equation, we have

$$dy/dx = (x^2 - 4xy - 2y^2)/(2x^2 + 4xy - y^2) \quad \text{... (1)}$$

Putting $y = xv$ and $dy/dx = v + x\,(dv/dx)$, (1) reduces to

$$v + x\,\frac{dv}{dx} = \frac{1-4v-2v^2}{2+4v-v^2} \qquad \text{or} \qquad x\,\frac{dv}{dx} = \frac{1-4v-2v^2}{2+4v-v^2} - v$$

or $$x\frac{dv}{dx} = \frac{1-6v-6v^2+v^3}{2+4v-v^2}$$ or $$3\frac{dx}{x} + \frac{3(v^2-4v-2)}{v^3-6v^2-6v+1}dv = 0$$

Integrating, $3\log x + \log(v^3 - 6v^2 - 6v + 1) = \log c$, *being* an arbitrary constant

or $$x^3(v^3 - 6v^2 - 6v + 1) = c$$ or $$x^3\{(y/x)^3 - 6(y/x)^2 - 6(y/x) + 1\} = c$$

or $$y^3 - 6xy^2 - 6x^2y + x^3 = c,$$ c being an arbitrary constant.

H.W.

Solve the equation

$$(x^2 - 3y^2)\,dx + 2xy\,dy = 0.$$

20. $(x^2 + y^2)\,dx + 2xy\,dy = 0$ **(Guwahati 2007)**

21. $(x^2y - 2xy^2)\,dx - (x^3 - 2x^2y)dy = 0$ [l

2.9 Equations reducible to homogeneous form

Equations of the form $\dfrac{dy}{dx} = \dfrac{ax + by + c}{a'x + b'y + c'}$, where $\dfrac{a}{a'} \neq \dfrac{b}{b'}$, ... (1)

can be reduced to homogeneous form as explained below.

Take $x = X + h$ and $y = Y + k$,... (2)

where X and Y are new variables and h and k are constants to be so chosen that the resulting equation in terms of X and Y may become homogeneous.

From (2), $dx = dX$ and $dy = dY$, so that $dy/dx = dY/dX$.... (3)

Using (2) and (3), (1) becomes

$$\frac{dY}{dX} = \frac{a(X+h) + b(Y+k) + c}{a'(X+h) + b'(Y+k) + c'} = \frac{aX + bY + (ah + bk + c)}{a'X + b'Y + (a'h + b'k + c')}. \quad \text{... (4)}$$

In order to make (4) homogeneous, choose h and k so as to satisfy the following two equations $ah + bk + c = 0$ and $a'h + b'k + c' = 0$. ... (5)

Solving (5), $h = \dfrac{bc' - b'c}{ab' - a'b}$ and $k = \dfrac{ca' - c'a}{ab' - a'b}$. ... (6)

Given that $a/a' \neq b/b'$. Therefore, $(ab' - a'b) \neq 0$. Hence, h and k given by (6) are meaningful, *i.e.*, h and k will exist. Now, h and k are known. So from (2), we get

$$X = x - h \quad \text{and} \quad Y = y - k. \quad \text{... (7)}$$

In view of (5), (4) reduces to $\dfrac{dY}{dX} = \dfrac{aX + bY}{a'X + b'Y} = \dfrac{a + b(Y/X)}{a' + b'(Y/X)}$,

which is surely homogeneous equation in X and Y and can be solved by putting $Y/X = v$ as usual. After getting solution in terms of X and Y, we remove X and Y by using (7) and obtain solution in terms of the original variables x and y.

Ex. 3. *Solve* $dy/dx = (x + y + 4)/(x - y - 6)$. **[I.A.S. 2002]**

Sol. Given $$dy/dx = (x + y + 4)/(x - y - 6) \quad \text{... (1)}$$

Let $x = X + h$, $Y = y + k$ so that $dy/dx = dY/dX$ ... (2)

Using (2), (1) reduces to $$\frac{dy}{dx} = \frac{(X + Y) + (h + k + 4)}{(X - Y) + (h - k - 6)} \quad \text{... (3)}$$

We choose h and k, such that $h + k + 4 = 0$, and $h - k - 6 = 0$ ... (4)

Solving (4), $h = 1$, $k = -5$ and so by (2), $X = x - 1$, $Y = y + 5$. ... (5)

Using (4), (3) reduces to $$\frac{dY}{dX} = \frac{X + Y}{X - Y} = \frac{1 + (Y/X)}{1 - (Y/X)} \quad \text{... (6)}$$

Putting $Y = xV$ and $dY/dX = v + X\,(dv/dX)$, (6) becomes

$$v + X\frac{dv}{dX} = \frac{1 + v}{1 - v} \qquad \text{or} \qquad \frac{dX}{X} = \frac{1 - v}{1 + v^2}\,dv = \frac{dv}{1 + v^2} - \frac{v\,dv}{1 + v^2}$$

Integrating, $$\log X = \tan^{-1} v - (1/2)\log(1 + v^2) + (1/2)\log c$$

or $$2\log X + \log(1 + Y^2/X^2) - \log c = 2\tan^{-1}(Y/X), \text{ as } v = Y/X$$

or $$\log\{(X^2 + Y^2)/c\} = 2\tan^{-1}(Y/X) \qquad \text{or} \qquad X^2 + Y^2 = ce^{2\tan^{-1}(Y/X)}$$

or $(x - 1)^2 + (y + 5)^2 = ce^{2\tan^{-1}\{(y+5)/(x-1)\}}$, c being an arbitrary constant.

Ex. 4. *Solve* $dy/dx = (x - 2y + 5)/(2x + y - 1)$. **[Delhi Maths (H) 2002]**

Sol. Let $x = X + h$, $y = Y + k$ so that $dy/dx = dY/dX$ ... (1)

Then given equation becomes $$\frac{dY}{dX} = \frac{X - 2Y + h - 2k + 5}{2X + Y + 2h + k - 1} \quad \text{... (2)}$$

Choose h and k so that $h - 2k + 5 = 0$ and $2h + k - 1 = 0$ ... (3)

(3) $\Rightarrow$ $h = -3/5$, $k = 11/5$ so by (1) $X = x + 3/5$ and $Y = y - 11/5$... (4)

Using (3), (2) becomes $$\frac{dY}{dX} = \frac{X - 2Y}{2X + Y} = \frac{1 - 2(Y/X)}{2 + (Y/X)} \quad \text{... (5)}$$

Putting $Y = Xv$ and $dY/dX = v + X(dv/dX)$, (5) gives

$$v + X\frac{dv}{dX} = \frac{1 - 2v}{2 + v} \quad \text{or} \quad \frac{dX}{X} + \frac{1}{2}\,\frac{2v + 4}{v^2 + 4v - 1}\,dv = 0$$

Integrating, $\log X = (1/2)\log(v^2 + 4v - 1) = (1/2)\log C$

or $X^2(v^2 + 4v - 1) = C$ or $X^2(Y^2/X^2 + 4Y/X - 1) = C$, as $v = Y/X$

or $Y^2 + 4XY - X^2 = C$ or $(y - 11/5)^2 + 4(x + 3/5)(y - 11/5) - (x + 3/5)^2 = C$

or $x^2 - y^2 - 4xy + 10x + 2y = C_1$, where C_1 is another arbitrary constant.

Ex. 5. *Solve* $dy/dx = (x + y - 2)/(y - x - 4)$ **[Delhi Maths (G) 2004]**

Sol. Let $x = X + h$ and $y = Y + k$ so that $dy/dx = dY/dX$ ... (1)

Then given equation gives $$\frac{dY}{dX} = \frac{X + Y + (h + k - 2)}{Y - X + (k - h - 4)} \quad \text{... (2)}$$

Choose h, k such that $h + k - 2 = 0$ and $k - h - 4 = 0$. .. (3)

Solving (3), $h = -1$, $k = 3$. Then (1) gives $X = x + 1$ and $Y = y - 3$... (4)

Using (3), (2) becomes $$\frac{dY}{dX} = \frac{X + Y}{Y - X} = \frac{1 + (Y/X)}{(Y/X) - 1} \quad \text{... (5)}$$

Let $Y/X = v$, *i.e.*, $Y = vX$ so that $dY/dX = v + X\,(dv/dX)$... (6)

From (5) and (6), $$v + X\frac{dv}{dX} = \frac{1 + v}{v - 1} \quad \text{or} \quad X\frac{dv}{dX} = \frac{1 + 2v - v^2}{v - 1}$$

or $$\frac{(v - 1)\,dv}{1 + 2v - v^2} = \frac{dX}{X} \quad \text{or} \quad \frac{(2 - 2v)\,dv}{1 + 2v - v^2} = -2\frac{dX}{X}$$

Integrating, $\log(1 + 2v - v^2) + 2\log X = \log C$ or $X^2(1 + 2v - v^2) = C$

or $X^2\{1 + 2(Y/X) - (Y/X)^2\} = C$ or $X^2 + 2XY - Y^2 = C$

or $(x + 1)^2 + 2(x + 1)(y - 3) - (y - 3)^2 = C$, using (3)

Solve the initial-value problem

$$(y + \sqrt{x^2 + y^2})\, dx - x\, dy = 0,$$

$$y(1) = 0.$$