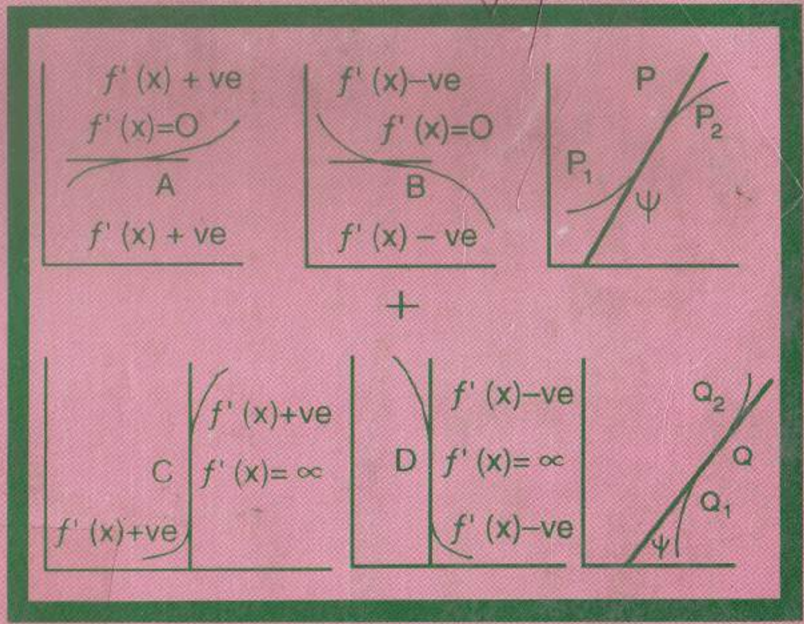
A TEXT BOOK ON DIFFERENTIAL CALCULUS

(PASS & HONOURS)



MOHAMMAD, BHATTACHARJEE & LATIF

Paper I (প্রথম পত্র)

(Marks 50)

ANALYTIC AND VECTOR GEOMETRY

See Rahman and Bhattacharjee's. A Text Book on Coordinate Geometry and vector Analysis.

A-I **Two-dimensional Geometry :** (দ্বিমাত্রিক স্থানাঙ্ক জ্যামিতি)

Transformation of Coordinates etc : Chapter IV. Pairs of Straightlines (যুগল সরলরেখা অধ্যায়-৫ (chapter-V). (Homogeneous 2nd degree equation Art 36.General 2nd degree equation Art 40. Angle between pairs of St. lines Art 38. Art 41. Bisectors of angles between pairs of St. lines Art 39. Art 41 (d).

General Eq. of 2nd degree Chapter VI (অধ্যায়-৬)

(Reduction to standard forms Art 47. Art 48. At 49. Art 51 Art 51 (a).

A-2 **Three dimensional Geometry chapter I,** অধ্যায়-১ (ত্রিমাত্রিক জ্যামিতি) Coordinates in three dimensions (Art 1. Art 2) Distance (Art 3) Direction Cosines (দিক কোসাইন) (Art 8: Art 9. Art 10. Art 11). Planes (সমতল) ঃ Equation of plane Art 18. Art 19. Angle between two planes (Art 22). Distance of a point from a plane (Art 25 (a)

Straight lines : chapter III (অধ্যায়-৩)

Equation of a line (Art 28. 29. 30(a) 32(a)

Relationship between planes and lines

Art 33. Art 34. Art 35

Shortest distance Art 38

Sphere (গোলক) ঃ Chapter IV অধ্যায়-৪ Art 41. 42. 43

Tangent to the sphere Art 46, Art 47

A-3 **Vector Analysis :**

অধ্যায়-১ অধ্যায়-২, Art 18. অধ্যায়-৩

B CALCULUS- I (Marks 50)

A Text Book on Differential calculus By Mohammed, Bhattacharjee and Latif.

অধ্যায়-১, অধ্যায়-২ অধ্যায়-৩ (বাদ Uniform continuity) বিবিধ প্রশ্নমালা, অধ্যায়-iv অধ্যায়-IV(a) IV(b) অধ্যায়-(৫) Art 5.5, Art 5.6, Art 5.7, Art 5.8 etc.

অধ্যায়-৭ Art 7.2 Art 7.3 Art 7.4 Art 7.5 এবং এদের নিয়ে অংক অধ্যায়-X (A) Art 102, Art 103 অধ্যায়-X (B) Art 108. অধ্যায়-XI Art 11.1 Art 11.2 Art 11.3 Art 113(a), Art 11.5

Integral Calculus: A Text Book on Intgral Calculus By Mohammed, Bhattacharjee and Latif.

অধ্যায়-১ থেকে অধ্যায়-৬ Definite Integrals অধ্যায়-৭ Art 22 অধ্যায়-৭(B) অধ্যায়-VII(c)

Paper II (দ্বিতীয় পত্র)

A **BASIC ALGEBRA :** (Marks 50)

See Shahidullah and Bhattacharjee's A Text book on Higher Algebra, Modern Algebra, Theory of Numbers and Trigonometry.

Elements of Logic (মৌলিক যুক্তিবিদ্যা) (Mathematical statements etc ....Deductic reasoning) see Appendix (উল্লিখিত বই এর প্রথমে) Basic Algebra এর জন্য।

Elements of set Theory : উল্লিখিত বই এর মডার্ণ এলজেবরা (Sets and Subsets, Relation-Orders Equivalence; Functions etc. অধ্যায় পাঁচ পর্যন্ত)

Real Number System : Field and order properties Natural Numbers, Absolute value (See First Chapter of **A Test Book on Differential Calculus by Mohammed, Bhattacharjee and Latif)**

Inequalities : Basic Inequalities :

Weier strass's; Tcheby chef's Cauchy's; A. M. (Arithmetic Mean) and Geometric Means উল্লিখিত এলজেবরা বইএর অসমতা (Inequalities অধ্যায় এক, Art 2(i), 2(ii), 2(iii) Art 3 Art 4. Art 5, Art 8, Art 9. Holder's Inequality Art 14. এর অঙ্কগুলি)

Complex number: System, field of complex numbers De moiver's theorem and its application- See Algebra বই এর **Trigonometry**. (ত্রিকোণমিতি) অধ্যায় এক) অধ্যায় দুই, অধ্যায় তিন।

Elementary Number Theory : Divisibility, Fundamental Theorem Arithmetic, Congruence. See Theory of Number of Higher Algebra-এর সংখ্যা তত্ত্ব (Theory of Numbers) অধ্যায়।

Summation of finite Series : Arithmetico-Geometric Series, Method of Difference, Successive differences. Use of mathematical induction : **See Higher Algebra** (অধ্যায়-৩ III (A), III(B), III(C):

Theory of Equations : Synthetic division, Number of roots of Polynomial, equations. Relation between roots and coefficients. Multiplicitiy of roots, Symmetric functions of roots. Transformation of equations. **See Higher Algebra** অধ্যায়-৪ (ch. IV) Art 50 থেকে প্রশ্নমালা IV পর্যন্ত।

B **Linear Algebra :**

Matrix, Determinant. থেকে Applications of matrices and determinants in solving system of linear equtions—**See Higher Algebra by Shahidulla & Bhattacharjee :**

অধ্যায় নির্ণয়াক (Determinants) অধ্যায়-৬ ঃ মেট্রিক্স (Matrix)

General vector space থেকে Cayley-Hameltion's Theorem. Application-এর জন্য যে কোন একটি Linear algebra বই যথা Schaums Serice এর বই আলোচনা করা যেতে পারে। দেশী বইও দেখা যেতে পারে, তবে কেলী-হেমিষ্টন উপপাদ্য ও ব্যবহার **See Shahidullah and Bhattacharjee's Higher Algebra** ঃ তে Matrix অধ্যায়ের Art 115.11 থেকে Art 115.14 পর্যন্ত পাওয়া যাবে।

প্রকাশকের কথা

জাতীয় বিশ্ববিদ্যালয়ের পাস ও সম্মান শ্রেণীর উচ্চতর গণিতের ১ম পত্র, ২য় পত্র ও ৩য় পত্রের বিষয়াবলী আমাদের প্রকাশিত বইগুলোতে যেভাবে ব্যবহৃত হয়েছে তার একটি নির্দেশিকা দেওয়া হল। বর্তমানে জাতীয় বিশ্ববিদ্যালয়ের পাঠ্যসূচি অনুযায়ী ১ম পত্রের জন্য যেমন, জ্যামিতি ও ভেক্টর এবং ক্যালকুলাসের কিছু অংশ আছে। অবশিষ্ট অংশ ২য় ও ৩য় পত্রে আছে। ছাত্র//ছাত্রীদের বর্তমান পাঠ্যসূচি বুঝার সহজ উপায় নির্দ্ধারণের জন্য এই নির্দেশিকা দেয়া হল।

আশা করি এতে ছাত্র/ছাত্রীদের বিশেষ সুবিধা হলে আমাদের শ্রমি সার্থক হবে।

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PREFACE TO 10TH EDITION

In this edition Set Algebra has been used in defining Functions, Relations and related problems. A list of formulae of Differential Calculus and set Algebra is placed at the beginning of the subject matters. Almost all the Chapters are thoroughly revised, altered and adjusted. New sums both worked out and in exercise have been added. Due to readjustment, addition and alteration there may be some errors and maladjustments which we desire to correct in the next edition of the book.

Suggestions for the improvement of the book and intimation of errors will be cordially recieeved.

May, 2001 **AUTHORS**

PUBLISHER'S NOTE

The Present edition of "A Text book on Differential Calculus" is published after thorough revesion of the whole book with a veiw to enricing the contents with modern ideas at home and abroad.

Md. Abdul Latif Dept. of Mathematics, Rajshahi University, Rajshahi. has extended his whole hearted active cooperation to us in this respect.

With a mark of appereciation to Mr. Abdul Latif. he is included as the third author of the book.

1992 **Publisher**

PREFACE TO THE 1ST EDITION

The book is intended to serve as a companion book to our "Integral Calculus" and is designed to meet the requirements of the Pass and Honours Students of our Universities.

The book consists of sixteen chapters covering entire syllabus of Universities. Attempts have been made to elaborate each article with a good numbers of examples. The pass students may drop harder sums and the sums marked with asterisks. Honours students should read the book thoroughly. A good number of examples are given in the book and for their collection foreign books. University question papers of various examinations have been consulted.

We are much thankful to Prof. A. B. Mitra and Prof. H. N. Datta and many others for their reviewing of some chapters of the book.

Suggestions for the improvement of the book, intimation of errors and misprints will be thankfully recieved.

1968 **AUTHORS**

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APPENDIX

Uniform continuity

University Questions

1. **Convergency and Divergency see Higher Algebra**
BY Shahidullah & Bhattacharjee

Convergency and Divergency of Imporper Integrals. see Integral Calculus
BY Mohammad and Bhattacharjee

2. **For Vector Analysis see Co-ordinate Geometry and Vector Analysis.**
By Rahman & Bhattacharjee

APPENDIX

UNIFORM CONTINUITY

Definition : A function $f(x)$ is said to be uniformly continuous in $[a, b]$ if and only if (iff) for a given arbitrary small positive number ε, there exists a number δ, depending only on ε, such that $x_1, x_2 \varepsilon [a,b]$ and $|x_1-x_2|<\delta \Rightarrow |f(x_1)-f(x_2)| < \varepsilon$

Difference between continuity and uniform continuity.

Ans: *Uniform continuity is a property associated with an interval not with a single point.*

Note: The uniform continuity is defined with reference to a finite closed interval $[a,b]$. But no such restriction is necessary for uniform continuity. The definition is also true for intervals (a,b), (a,∞) Open intervals.

Art. A function which is continuous in a closed and bounded interval [a,b] is uniformly continuous in [a,b].

Proof: Let the interval $[a,b]$ be divided into sub intervals $[a,x_1]$, $[x_1,x_2]$ .... $[x_{n-1},b]$ ... (1)

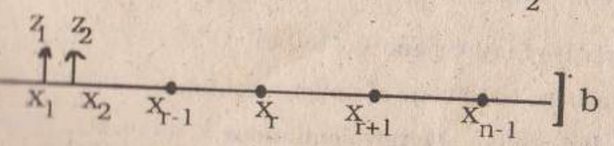
such that if $z, z_2 \varepsilon [x_1,x_2]$ or belong to any subinterval, we have

$$|f(z_1)-f(z_2)| < \frac{1}{2}\varepsilon \; ..... (2)$$

Let us consider a small positive number δ such that it is not greater than the least of the numbers

$$x_1-a,\ x_2-x_1,\ x_3-x_2,\ ....\ b-x_{n-1}\ ....\ (3)$$

Let us consider z_1, z_2 two points in the same subinterval say x_2-x_1, then by (2) $|z_2-z_1| < \delta\ (> 0)$ and $|f(z_1)-f(z_2)| < \frac{1}{2}\varepsilon$



If z_1, z_2 do not belong to the same interval $(x_2-x_1) = [x_1,x_2]$, let z_1 lies in $x_2-x_1= [x_1,x_2]$ and z_2 in $(x_3-x_2) = [x_2,x_3]$

i, e; one in each of the two econsecutive intervals

$[x_1, x_2]$ and $[x_2, x_3]$. then we have

$x_1< z_1<x_2<z_2< x_3$ i, e, in general if z_1,z_2 be the points in

$[x_{r-1}, x_r]$ and $[x_r, x_{r+1}]$ respectively, i, e; $x_{r-1}\ z_1<x_r<z_2<x_{r+1}$;

then $|f(z_1)-f(z_2)| = | f(z_1)-f(x_r) + f(x_r) - f(z_2)|$

$\leq|f(z_1) - f(x_r)| + | f(x_r) - f(z_2)| \quad \leq\frac{1}{2}\varepsilon+\frac{1}{2}\varepsilon=\varepsilon$

But we have noticed that for given $\varepsilon>0$,there exists a small positive number $\delta>0$ such that $|f(z_1) -f(z_2)|<\varepsilon$ for any two points in $[a,b]$ such that $|z_1-z_2| < \delta$

Hence $f(x)$ is uniformly continuous in $[a,b]$

Ex.1. Prove that $f(x) = x^2$ for all $x \varepsilon$ R is uniformly continuous on (0,1].

Ans: Let x_1, x_2 be any two points in (0,1], then

$|f(x_1) -f(x_2)| =|x^2_1-x^2_2|=|x_1-x_2|\ |x_1+x_2|$ ....(1)

$x_1,x_2 \varepsilon (0, 1] \Rightarrow |x_1+x_2|<2$, whatever be the values of x_1 and x_2 between 0 and 1. Now from (1)

$\therefore\quad |f(x_1) - f(x_2)|<2|x_1-x_2|$ .... (2)

Again if $|x_1-x_2|<\delta$, then from (2)

$|f(x_1)-f(x_2)| <2| x_1-x_2| <2\delta$ ...... from (3)

If $\varepsilon>0$ is given, let $\delta=\frac{1}{2}\varepsilon$, then from (3)

$|f(x_1) - f(x_2)|<2.\frac{1}{2}\varepsilon=\varepsilon$ and $|x_1-x_2|<\frac{1}{2}\varepsilon = \delta$

or, $| f(x_1) - f(x_2) < \varepsilon$ and $|x_1-x_2|<\delta$

which are the conditions of uniform continuity.

Hence $f(x) = x^2$ is uniformly continuous.

Ex. 2. Show that $f(x) = \frac{1}{x}$: $(x>0)$ is not uniformly ontinuous' in (0,1]

Ans: Let us consider two points x_1 and x_2 such that

$x_1, x_2 \varepsilon (0,1]$, $|x_1 -x_2|<\delta\Rightarrow|f(x_1) - f(x_2|< \varepsilon$

If $\varepsilon=\frac{1}{2}$ and $\delta>0$, any positive number for $n\geq 1/\delta$.

If we take $x_1=1/n$, $x_2=1/(n+1)$; $x_1,x_2 \varepsilon (0,1]$, then

$|f(x_1) -f(x_2)| = |\frac{1}{x_1}-\frac{1}{x_2}| =|n-(n+1)| = 1> \varepsilon$

when $|x_1-x_2| < \delta$

Hence $f(x) = 1/x$ is not uniformly continuous on (0,1]

Note. 1. The function is uniformly continuous on $[a, \infty)$, where $a>0$

Ex. Show that $f(x) = x^3+3x^2-2x+7$ in [-2,3] is uniformly continuous or not.

Ex. Let x_1x_2 be any two points on [-2,3], then

$|f(x_1) - f(x_2)|= |x_1^3+3x_1^2-2x_1+7-x^3_2-3x_2^2+2x_2-7|$

$= | (x_1^3-x_2^3)-3(x_1^2-x_2^2)-2(x_1-x_2)|$

$= | (x_1-x_2)\ (x_1^2+x_1x_2+x_2^2)-3(x_1+x_2)\ (x_1-x_2)\ -2(x_1-x_2)|$

$= |(x_1-x_2)\ \{(x_1+x_2)^2-x_1x_2\}-3(x_1+x_2)\ (x_1-x_2)\ -2(x_1-x_2)|$...(1)

Now $x_1,x_2 \varepsilon [-2,3] \rightarrow |x_1+x_2|<3$, $x_1x_2<3$.

If $\quad |x_1-x_2|<|\delta$, then from (1)

$|f(x_1) -f(x_2)| <| \delta(3-3) -3.3\ \delta - 2\delta| = |\delta.0 - 9\delta-2\delta|=11\delta$

or, $|f(x_1) -f(x)|<11\delta$

$| f(x_1-f(x_2)| < 11.\ \varepsilon/11= \varepsilon$ where $\varepsilon= 11\delta$

and $| x_1-x_2|<\delta$

If $\varepsilon=\frac{1}{2}$, then $\frac{1}{2}=11\delta$ or, $\delta=1/22$ which lies between-2 and 3.

Hence $f(x)$ is uniformly continuous in [-2,3]

Exercise. Test the uniform continuity of the following functions

(i) $f(x) = x^2+3x$ in $[-2,2]$

(ii) $f(x) = \dfrac{x+2}{2x+3}$ in $[-1,3]$

(iii) Show that $f(x) = x^2$ for all $x \varepsilon$ R, $f(x)$ is uniformly continuous on R

(iv) $f(x) = \sqrt{x}$ in $[0,2]$ Ans. yes

(v) $f(x) = \dfrac{x}{1+x^2}$ on R Ans. yes

(vi) $f(x) = \sin\pi x$ for $x \varepsilon (1,2)$ Ans. yes

$= x^2-1$ for $x \varepsilon (1,2)$

(vii) $f(x) = \sin x$ on $[0,\infty)$ Ans. yes

(viii) $f(x) = \tan^{-1}x$ on R Ans. yes

2. Prove that

$$f(x) = \sin\frac{1}{x}, \; x\neq 0$$
$$= 0, \; x=0$$

is not uniformly continuous on $[0,\infty)$

Sol: Let $\varepsilon=1/3$ and $\delta>0$ be such that $1/z<\delta$ for all $n\geq m$.

Let $x=\dfrac{4}{4m\pi}$, $y=\dfrac{4}{(4m+1)\pi}$ be any two points of $[0,\infty)$, then.

$$|x-y|=\left|\frac{4}{4m\pi}-\frac{4}{(4m+1)\pi}\right|=\left|\frac{4m+1-4m}{m(4m+1)\pi}\right|=\frac{1}{m(4m+1)\pi}=\frac{1}{z}<\delta$$

$$\text{Now } |f(x)-f(y)|=\left|\sin m\pi-\sin\frac{(4m+1)\pi}{4}\right|$$

$$=\left|\sin m\pi-\sin\left(m\pi+\frac{\pi}{4}\right)\right|<|\sin\pi/4| = 1/\sqrt{2}=\frac{\sqrt{2}}{2}=0.7/2$$

but $\varepsilon=\dfrac{1}{3}=0.33 \not\subset 0.7/2$ i. e.,

$|f(x)-f(y)| > \varepsilon$ which does not satisfy the condition of uniform continuity

Ex. Show that the function

$f(x) = 1-x+[x]-[1-x]$ is not continuous at $x=0$. $[x]$ denotes the greatest integer positive or negative but not numerically greater than x.

Sol: We know

$[x] < x$, $[1-x] < 1-x$ or, $-[1-x] > -1+x$

$\therefore$ $[0] = 0$, $0\leq x<1$	$[1-0] = 1-0=1$ or, $-[1-0]=-1, 0\leq x\leq 1$
$[1] = 1$, $1\leq x<2$	$[1-2] = 1-2=-1$, or, $-[1-2]= 1, 1\leq x\leq 2$
$[2] = 2$, $2\leq x<3$	$[1-3] = 1-3=-2$ or, $-[1-3]= 2, 2\leq x<3$
$[3] = 3$, $3\leq x<4$	

$f(x)=1-x+[x]-[1-x] = 1-x+x-(1-x) = 1-1+x=x$

$f(h) = 1-h+h-1+h=h$ $\quad\therefore \underset{h\to 0}{\text{Lt}}\, h = 0$

$f(-h) = 1+h+0-h-(1-0+h) = 1-1-h=-h$ $\quad\therefore \underset{h\to 0}{\text{Lt}}\, -h = 0$

$f(0) = 1-0+[0]-[1-0] = 1-0+0-1= 0$

$\therefore$ $f(0) = f(0+h) = f(0-h) = 0$

At $x=0$, $f(x)$ is continuous.

✓✓

Ex—iv(B)

142. If $y = \sin^{-1}\{x\sqrt{1-x} - \sqrt{x}(1-x^2)\}$ then show that

$$\frac{dy}{dx} = \frac{1}{\sqrt{(1-x^2)}} - \frac{1}{2\sqrt{(x-x^2)}}$$

143. If $\tan y = \dfrac{\sqrt{(1+x^2)} + \sqrt{(1-x^2)}}{\sqrt{(1+x^2)} - \sqrt{(1-x^2)}}$

then $\dfrac{dy}{dx} = -\dfrac{x}{\sqrt{(1-x^4)}}$

$\therefore$ if $y = \tan^{-1}\dfrac{\sqrt{(1-x^2)} + \sqrt{(1-x^2)}}{\sqrt{(1=x^2)} - \sqrt{(1=-x^2)}}$

then $\dfrac{dy}{dx} = -\dfrac{x}{\sqrt{(1-x^4)}}$

144. If $\cot y = \dfrac{\sqrt{(1+\sin x)} + \sqrt{(1-\sin x)}}{\sqrt{(1+\sin x)} + \sqrt{(1-\sin x)}}$, then

$$\frac{dy}{dx} = \frac{1}{2}$$

145. If $y = \tan^{-1}\dfrac{\sqrt{(1-x)} - \sqrt{(1-x)}}{\sqrt{(1+x)} + \sqrt{(1-x)}}$, then

$$\frac{dy}{dx} = \frac{1}{2\sqrt{(1-x^4)}}$$

146. If $\sqrt{(1-x^4)} + \sqrt{(1-y^4)} = a(x^2-y^2)$, then

$$\frac{dy}{dx} = \frac{x\sqrt{(1-y^4)}}{y\sqrt{(1-x^4)}}$$

147. If $y = \log\dfrac{\sqrt{(1+x)} + \sqrt{(1-x)}}{\sqrt{(1+x)} - \sqrt{(1-x)}}$, then

$$\frac{dy}{dx} = \frac{-1}{x\sqrt{(1-x^2)}}$$

148. If $y = \tan\text{-}1\left(\sqrt{\dfrac{a-b}{a+b}}\tan\dfrac{2}{2}\right)$, then

$$\frac{dy}{dx} = \frac{\sqrt{(a^2-b^2)}}{2(a+b\cos x)}$$

149. If $\sqrt{(1-x^2)} + \sqrt{(1-y^2)} = a(x-y)$, then

$$\frac{dy}{dx} = \frac{\sqrt{(1-y^2)}}{\sqrt{(1-x^2)}}$$

150. If $u\sqrt{(1+v)} + v\sqrt{(1+u)} = 0$,

then show that $\dfrac{du}{dv} = -\dfrac{1}{(1+u)^2}$, $u \neq v$

151. If $y = \sqrt{[\sin x + \sqrt{\{\sin x + \ldots\ldots \text{ to infinity}\}}]}$

Show that $\dfrac{dy}{dx} = \dfrac{\cos x}{2y-1}$

152. If $y = \cos^{-1}\dfrac{3+5\cos x}{5+3\cos x}$, then

Show that $\dfrac{dy}{dx} = \dfrac{4}{5+3\cos x}$

153. If $y = x^{x\ldots\ldots ad.\ infinity}$, prove that

$$\frac{xdy}{dx} = \frac{y^2}{1-y\log x}$$

154. If $y = (\sin x)^{x}$ ....ad. infinity.).

Show that $\dfrac{dy}{dx} = \dfrac{y^2\cot x}{1-y\log\sin x}$

155. If $y = e^{x+e^{x+e^{x}}}$ .... *ad.* infinity.

Show that $\dfrac{dy}{dx} = \dfrac{y}{1-y}$

Ex—v এর Example

Ex 5. Separate the intervals in which the polynomials $2x^3 - 15x^2 + 36x + 1$ is increasing or decreasing.

Ans Let $f(x) = 2x^3 - 15x^2 + 36x + 1$

$\therefore\quad f'(x) = 6x^2 - 30x + 36 = 6(x^2 - 5x + 6)$
$\qquad = 6(x-3)(x-2)$

Now $f'(x) > 0$ for $x > 2$

$f'(x) < 0$ for $2 < x < 3$

$f'(x) > 0$ for $x > 3$

$f'(x) = 0$ for $x = 2$ and 3.

Hence we see that $f'(x)$ is positive in the interval $]-\infty, 2[$ and $]3, \infty[$ and negative in the interval $]2, 3[$ Thus $f(x)$ is monotonically increasing in the intervals $]-\infty, 2[$. $]3, \infty[$ and monotonically decreasing in the intervaL $]2, 3[$

Ex : 6. 'et $f(x) = (x^4 + 6x^3 + 17x^2 + 32x + 32)\,e^{-x}$

$f'(x) = (x^4+6x^3+17x^2+32x+32)(-e^{x} + (4x^3+18x^2+34x+32)\,e^{-x}$
$\quad = -e^{-x}(x^4+2x^3-x^2-2x) = -xe^{-x}(x^3+2x^2-x-2)$
$\quad = -x(x+2)(x-1)(x+1)\,e^{-x}$
$\quad = x(1-x)(1+x)(2+x)\,e^{-x}$

The function $f'(x)$ is positive in the ntervals $[-2, -1]$ and $[0,1]$ and negative in $]-\infty, -2[$, $[-1, 0]$ and $[1, \infty[$ Hence $f(x)$ is monotoniclly increasing in the intervals $[-2, -1]$ and $[0, 1]$ and monotonically decreasing in intervals $]-\infty, -2[$, $[-1', 0]$ and $[1, \infty]$

Ex. 7. Find the intervals in which the function $f(x) = 8x^3 - 60x^2 + 144x + 15$ is increasing or decreasing.

Sol : $f(x) = 8x^3 - 60x^2 + 144x + 5$

$$f'(x) = 24x^2 - 120x + 144$$
$$= 24(x^2 - 5x + 16)$$
$$= 24(x-2)(x-3)$$

Now $f'(x) > 0$ if $x < 2$ .... (1)

$f'(x) < 0$ if $2 < x < 3$ .... (2)

$f'(x) > 0$ if $x > 3$ .... (3)

and $f'(x) = 0$ for $x = 2, 3$ .... (4)

Hence $f'(x)$ is positive in the intervals from (1) and (3), $(-\infty, 2)$ and $(3, \infty)$ i. e: negative excluding $x = 2$ and $x = 3$ Thus $f(x)$ is increasing monotonically in $(-\infty, 2)$, $(3, \infty)$ open intervals and monotonically decreasing in $(2, 3)$ open interval.

Art. 17.23. A Function is twice differentiable and satisfies the inequalities.

$|f(x)| < A$, $|f''(x)| < B$. in the range $x > a$;

Where A and B are constants. Prove that $|f(x)| < 2\sqrt{(AB)}$.

Ans. For positive number h, and $x > a$:

$$f(x+h) = f(x) + hf''(x) + \frac{h^2}{\lfloor 2} f''(x+\theta h),\ 0<\theta<1$$

$$\therefore \quad |hf'(x)| = |f(x+h) - f(x) - \frac{h^2}{\lfloor 2} f''(x+\theta h)|$$

$$\leq |f(x+h)| + |f(x)| + |\frac{h^2}{\lfloor 2} f''(x+\theta h)$$

$$< A + A + Bh^2/2$$

or: $|f'(x)| < \frac{2A}{h} + Bh/2$: h is $+ve$

$|f'(x)$ is free from h and also less than $(2A/h + Bh/2)$ for all positive value of h. Thus $|f'(x)|$ must be less than the least value of $(2A/h + Bh/2)$

Thus $(2A/h + Bh/2) = \{\sqrt{(2A/h)} - \sqrt{(Bh/2)}\}^2 + 2\sqrt{(AB)}$

such that $2\sqrt{(AB)} = (2A/h) + Bh/2$ least value

Hence $|f'(x)| < 2A/h + Bh/2 \geq 2\sqrt{(AB)}$

$\therefore \quad |f'(x)| < 2\sqrt{(AB)}$

Ex—vii এর Art. 22, 23, 24, 25.

22. A function f(x) is defined in [0;2] as follows

$f(x) = 2$ for $0 < x < 1$

$= 3$ for $1 \leq x \leq 2$

Show that $f(x)$ satisfies none of the conditions of Rolle's Theorem yet $f'(x) = 1$ for many points in [1,2]

Sol

Here we note that $f(a-h) = f(1-0) = 2$, $f(1+0) = 3$

i.e. $f(1-0) \neq f(1+0)$

Though $f(1+0) = 3 = f(1) \neq f(1-0)$ ....(1)

Hence $f(x)$ is discontinuous at $x = 1$

But we know that continuity is a necessary condition for a finite derivative, so the function $f'(x)$ does not exist for every point in $1 \leq x \leq 2$ ....(2)

Also $f(1) = 2$ and $f(2) = 3$ given

So $f(1) \neq f(2)$ .... (3)

Hence all the three conditions of Rolle's Theorem are not satisfied by $f(x)$ in $[1,2]$

Here $f(x)$ is a function free from x i.e. a constant in $[1,2]$. There is possibitlity of value of $f'(x)$ to be one at many points in $[1,2]$

23. Discuss the applicability of Rolle's Theorem to the function

$f(x) = x^2 + 1$ when $0 < x \leq 1$

$= 3 - x$ when $1 \leq x \leq 2$

Sol : $f(0) = 0^2 + 1 = 1$, $f(2) = 3 - x = 3 - 2 = 1$

$f(0) = 2 = f(2)$ ....(1)

Let us test the continuity of $f(x)$ at $x = 1$.

$$f(1+0) = \underset{x\to 0}{\text{Lt}} (3-x) = \underset{h\to 0}{\text{Lt}} (3-1-h) = 2$$

$$f(1-0) = \underset{x\to 0}{\text{Lt}} x^2 + 1 = \underset{h\to 0}{\text{Lt}} (1-h)^2 + 1 = 2$$

$\therefore \quad f(1-0) = f(1+0) = f(0) = 2$

It is continuous at $x = 1$, so we infer that $f(x)$ is continuous in the interval $[0,2]$

Again $f'(x) = 2x,\ 0 \le x \le 1$
$= -1,\ 1 < x \le 2$

Let us suppose that f(x) is differentiable in the interval (0,2) exceft at x = 1

one

$$Rf'(1) = \underset{h\to 0}{Lt} \frac{f(1+h) - f(1)}{h} = \underset{h\to 0}{Lt} \frac{\{3 - (1+h)\} - 2}{h}$$

$$= \underset{h\to 0}{Lt} \frac{2-h-2}{h} = \underset{h\to 0}{Lt} (-1) = -1$$

$$Lf'(1) = \underset{h\to 0}{Lt} \frac{f(1+h) - f(1)}{-h} = \underset{h\to 0}{Lt} \frac{(1-h)^2 + 1 - 2}{-h}$$

$$= \underset{h\to 0}{Lt} \frac{2h-h^2}{h} = 2$$

$\therefore Rf'(1) \ne lf'(1)$; so $f'(1)$ does not exist. $f(x)$ is not differentiable in the entire region (0,2) and therefore Rolle's Theorem is not applicable to the given function $f(x)$ in (0,2)-

24. A function $f(x)$ is continuous in closed interval [2,3] and differentiable in the open interval (2,3). Prove that

$f'(\theta) = f(3) - f(2)$ where $2 < \theta < 3$.

The conditions of Mean value Theorem are $f(x)$ is continuous in the closed intersol [2,3] and differentiable in the open interval (2,3). if c a value of x such that $2 < c < 3$.

$f(b)\text{-}f(a) = (b-a)\, f'((c)$

or; $f(3) - f(2) = (3-2)\, f'(c)$

or; $f'(c) = f(3) - f(2)$

Ex. 25. Meaning of the sign of derivative

Let c be a interior point a < c < b of the function $f(x)$. if $f'(c) > 0$

$$\underset{x\to c}{Lt} \frac{f(x) - f(c)}{x - c} = f'(c) > 0$$

If $\varepsilon > 0$ be any number $< f'(c)$. there exists a positive number $\partial > 0$ such that

$$|x - c| < \partial \Rightarrow \left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < \epsilon$$

Where $c - \partial < x < c + \partial$

$\Rightarrow \dfrac{f(x) - f(c)}{x - c} \subset c\,(f'(c) - \varepsilon,\ f'(c) + \varepsilon)$ in the open interval. Since $\varepsilon <$ f'(c). then we conclude that

$$\frac{f(x) - f(c)}{x - c} > 0 \text{ when } x\,\varepsilon\,[c - \partial, c+\delta],\ x \ne c$$

We infer that

$f(x) - f(c) > 0$ when $c < x \le x + \partial$

$f(x) - f(c) < 0$ When $c - \partial \le x < c$

If $f'(c) > 0$, there exists a neighbourhood $[c - \partial,\ c+\partial]$ of c such that $f(x) > f(c)$ for all $x\,\varepsilon\,(c,\ c+\partial)$

$f(x) < f(c)$ for all $x\,\varepsilon\,(c, -\partial, c)$

If $f'(c) < 0$, then there exist a neighbourhood $[c-\delta, c+\delta]$ of c such that $f(x) > f(c)$ for all $x\varepsilon[c-\delta, c]$

$f(x) < f(c)$ for all $x\,\varepsilon[c, c+\delta]$

For the end points a and b, it can be shown that there exist intervals $(a, a+\partial]$, $(b, b-\partial]$ such that

$f'(a) > 0 \Rightarrow f(x) > f(a)$ for all $x\,\varepsilon\,(a,\ a+\partial]$

$f'(a) < 0 \Rightarrow f(x) < f(a)$ for all $x\,\varepsilon\,(a,\ a+\partial]$

$f'(b) > 0 \Rightarrow f(x) < f(b)$ for all $x\,\varepsilon\,[b-\partial,\ b)$

$f'(b) < 0 \Rightarrow f(x) > f(b)$ for all $x\,\varepsilon\,[b-\partial,\ b)$

Ex—vii

52. Prove that $\phi'(x) = F'(\{f(x)\}\, f'(x).\ \phi(x) = f\{f(x)\}$ assume that the derivatives which are continuous and apply the mean value theorem.

53. Examine whether all the conditions of Rolle's Theorem are satisfied by the function $f(x) = 1 - |x|$ in $[-1, 1]$. What is your conclusion?

54. A function $f(x)$ is continuous in the closed interval $0 < x < 1$ and differentiable in the open interval $0 < x < 1$ Prove that $f'(x_1) = f(1) - f(0)$ where $0 < x_1 < 1$

55. Show that Rolle's theorem is not valid for the function $f(x) = x$ in $[-1, 1]$ as $f'(x)$ does not exist for a value of x in $[-1, 1]$

56. If $f''(x)$ is continuous and not zero at $x = a$. Show that $\underset{h\to 0}{Lt}\ \theta = \frac{1}{2}$ Where $f(x+h) = f(x) + hf'((x+\theta h);\ 0 < \theta < 1$

57. Prove that if $f''(x)$ is continuous

$$\underset{h\to 0}{Lt} \frac{f(x+2h) - 2f(x+h) + f(x)}{h^2} = f''(x)$$

Ex—viii

75. Examine whether the function $f(x)$ is continuous at $x=0$

Where $f(x)=x^{2x}, x\neq 0$. $f(0)=1$

Sol. প্রদত্ত রাশি, $f(x)=x^{2x}$

$$\therefore \log f(x) = 2x\log x = 2\frac{\log x}{1/x}$$

$$\therefore \mathop{Lt}_{x\to 0} \log f(x) = 2\mathop{Lt}_{x\to 0} \frac{\log x}{1/x} \therefore \text{form } \frac{\infty}{\infty}$$

$$= -2\mathop{Lt}_{x\to 0} \frac{1/x}{1/x^2} = 0$$

এখন $\mathop{Lt}_{x\to 0} \log f(x) = \log \mathop{Lt}_{x\to 0} f(x)$

অতএব $\log \mathop{Lt}_{x\to 0} \log f(x) = 0$, ফলে

$$\mathop{Lt}_{x\to 0} f(x) = e^{\circ} = 1$$

দেওয়া আছে $f(0)=1$ ফলে $\mathop{Lt}_{x\to 0} f(x) = f(0)$

অতএব $x=0$ বিন্দুতে $f(x)$ অবিচ্ছিন্ন।

76. মান নির্ণয় কর $\frac{a^x \sin bx - b^x \sin ax}{\tan bx - \tan ax}$ R.U. 1988

Ex—ix

97. If $z = x + f(u)$ and $u = xy$, the show that $x\frac{\delta z}{\delta x} - \frac{\delta z}{\delta y} = x$.

[যদি $z = x + f(u)$ এবং $u = xy$ হয়, তাহা হইলে দেখাও যে $x\frac{\delta z}{\delta x} - y\frac{\delta z}{\delta y} = x$]

98. If $\phi(x,y) = 0$, $\psi(y,z) = 0$, show that $\frac{\delta\phi}{dy}\frac{\delta\psi}{\delta z}\frac{dz}{dx} = \frac{\delta\alpha}{\delta x}\frac{\delta\psi}{\delta y}$

Interpret the result geometrically (জ্যামিতির সাহায্যে ব্যাখ্যা কর)

99. If $F(x,y) = x^4y^2\sin\frac{y}{x} + \tan^{-1}\frac{y}{x}$, then $x\frac{\delta F}{\delta x} + y\frac{\delta F}{\delta y} = 6F$

100. If $z = \frac{x}{y} + x\tan y$, then $\frac{\delta^2 z}{\delta x\delta y} = \frac{\delta^2 z}{\delta y\delta x}$

INTRODUCTION

"The word **Calculus** is the latin name for a stone which was employed by the Romans for reckoning i. e, for "**Calculation**". When used as in the title of the book, it is an abbreviation for "**Infinitesimal Calculus**" which implies a reckoning or Calculation with numbers which are infinitesimally small.

One of the most powerful methods in modern mathematics is that of the Calculus, the ideas of which were conceived by Archimedes in the third century B. C. By dividing a segment of a parabola into thin strips and adding together thier areas, he found an approximation to the area of the segment. He then obtained closer and closer approximations by taking more and thinner strips. By this method of exhaustion he found the area of the segment exactly.

About 1586 Stevinus of Bruges used a method similar to that of Archimedes to find the thrust of a liquid on a surface, and a little later a Jesuit priest, Cavalieri, extended the method to find the volumes of solids. The methods used by these mathematicians to find the whole area or, volume, by dividing it up into small parts is now called "integration' i. e. finding the whole.

In the beginning of the 17th century the French mathematician Fermat considered the ratio of infinitesimally small increments so laid the foundation on which Newton (1642—1727) and Leibnitz (1646—1716) later built the theory of "differentiation" or finding rates of change from the ratios of small differences It was due to the genius of Newton and Leibnitz that a great advance was made. Newton conceived the idea of continuous change and rate of change at an instant or flux, and he described his new subject as "fluxions". He found that his knowledge of rates of change could be applied to calculate areas and volumes that is to perform integration much more easily than by the method of exhaustion described above, Leibnitz discoverd the method of differentiation about the same time and we are specially indebted to him for his notation which is essentially that now in general use. In the last two centuries calculus has been developed to such an extent that it is now used to deal with problems in branch of technical Science.'

Before taking up the subject matters of Calculus let us first discuss something about numbers and related quantities.

উপরে বর্ণিত ক্যালকুলাসের উৎপত্তি ও ব্যাখ্যা হইতে আমরা নিম্নলিখিত মূল অংশে মনোনিবেশ করিতে পারি। ক্যালকুলাসের সাহায্যে আমরা মূলত দুইটি গুরুত্বপূর্ণ বিষয়ের অনুসন্ধান করিতে পারি।

(ক) একটি রেখার (curve) ঢালুতার (solpe) অর্থের অনুসন্ধান করিয়া উহার মান নির্ণয় করিতে পারি।

(খ) কোন রেখার দ্বারা পরিবেষ্টিত স্থানের ক্ষেত্রফলের অর্থ বাহির করিয়া উহার মান নির্ণয় করিতে পারি।

ডিফারেন্সিয়াল ক্যালকুলাস (ক) প্রথমোক্ত বিষয়ে এবং (খ) দ্বিতীয়োক্ত বিষয়ে আমরা ইন্টিগ্র্যাল ক্যালকুলাসে আলোচনা করিয়া থাকি।

পরিভাষা

ইংরেজী-বাংলা

Abscissa —ভুজ
Absolute value —পরম মান
Acceleration —ত্বরণ
Algebraic —বীজগাণিতিক
Alternative —বিকল্প
Analytical —বিশ্লেষক
Angle of incidence —আপতণ কোণ
Angle of reflection —প্রতিফলন কোণ
Angular displacement — কৌণিক সরণ
Applied —ফলিত
Arbitrary —অবাধ
Arch —খিলান
Arc length —চাপ– দৈর্ঘ্য
Assumption —প্রতিজ্ঞা
Astroid —আসট্রোয়েড
asymptote —অসীমতট
axiom —স্বতঃসিদ্ধ
Beam —কড়িকাঠ
Binomial —দ্বিপদী
Bisector —দিখণ্ডক
Chain rule —শৃঙ্খল নিয়ম
Counter clockwise —বামাবর্ত
Critical value —সন্ধিমান
Cross multiplication —বজ্রগুণন
Cross secton —আড়চ্ছেদ
Change of variables —চলক পরিবর্তন
Chord of curvature —বক্রতা জ্যা
Circular function —বৃত্তীয় ফাংশন
Clockwise —দক্ষিণাবর্ত
Co-efficiant —সহগ

Co-efficent of viscosity —আঠালুতা সহগ
Coincidence —সমাপতন
Co-ordinate —স্থানাঙ্ক
Corollary —অনুসিদ্ধান্ত
Concave downward —নিম্নদিকে চন্দ্রাকৃতি/ নিম্ন অবতল
Concave upward —উর্ধ্বদিকে চন্দ্রাকৃতি/উর্ধ্ব অবতল
Cone — কোণক
Conic section — কোণচ্ছেদ
Constant —ধ্রুবক
Continuity — ছেদহীন
Continuity — ছেদহীনতা
Convergence —সীমামুখিতা
Convergent —সীমামুখী
Cubical parabola —ত্রিঘাত পরাবৃত্ত
Curvature —বক্রতা
Curve — রেখা, বক্র
Cycle —চক্র
Cube —ঘনক
Cylinder —সিলিন্ডার/স্তম্ভক
Decreasing —হ্রাসমানী
Deflection —ব্যত্যয়
Determinant —নির্ণায়ক
Differentiability —ডিফারেন্সিয়েশন যোগ্যতা / অন্তরীকরণীয়ে
Differential —ডিফারেন্সিয়েল /অন্তরক
Differentiate —ডিফারেন্সিয়েশন করা/ অন্তরীকরণ করা
Differentiation —ডিফারেন্সিয়েশন/অন্তরকরণ
Dimension —মাত্রা
Direction cosines —গতিকোষাইন
Discontinuity — ছেদযুক্ততা
discontinuous — ছেদযুক্ত
Discrete — ছেদযুক্ত
Divergence —সীমাবিমুখিত
divergent —সীমাবিমুখ

Domain —এলাকা
Eccentricity —বিকেন্দ্রিকতা
Dlectromotive force —তড়িৎ চালক শক্তি
Ellipsoid —উপবৃত্তক
Element —উপাদান
Ellipse —উপবৃত্ত
Eliminate —অপসারণ করা
Envelope —আচ্ছাদন
Equiangular spiral —সদৃশকোণী স্পাইরেল
Even — জোড়া/যুগ্ম
Expansion —বিস্তারন
Exlicit —ব্যক্ত
Exponential —সূচক
Cycloid —সাইক্লয়েড
Focal chord —ফোকাস জ্যা
Focus — ফোকাস/নাভিবিন্দু
folium —ফলিয়াম
Formula —সূত্র
function —ফাংশন / নির্ণায়ক
General formula —সাধারণ সুত্র
Genralised —ব্যাপকীকৃত
Generalization —ব্যাপকীকরণ
Gradient —ঢালুতা/গ্র্যাডিয়েন্ট
Graph —চিত্র/লেখচিত্র
Homogenous —সমমাত্রিক/সুষম
Horizontal —আনুভূমিক
Hyperbola —অধিবৃত্তি
Hyperboloid —অধিবৃত্তক
Illumination —দীপন
Imaginary number —কাল্পনিক সংখ্যা
Implicit —অব্যক্ত
Improper fraction —অপ্রকৃত ভাগ্নাংশ
Indeterminate form —অনিশ্চিত আকার

Inequality —অসমতা
Infinite —অপরিমেয়/অসীম
Infinitesimal —অণু
Inscribed —অন্তলখিত
Intensity —তীব্রতা
Integer —পূর্ণসংখ্যা
Interval —ব্যবধি
Finite number —পরিমেয় সংখ্যা
Finite series —সসীম ধারা
Intrinsi —স্বকীয়
Inverse cicular function —বিপরীত বৃত্তীয় ফাংশন
Invrsely proportional —বিপরীতক্রমে সমানুপাতিক
Involute —ইনভোলিউট
Irrational —অবাস্তব
Kinetic energy —গতিশক্তি
Left hand limit —বামসীমা
Limit —সীমা
Limiting point —সীমায়িত বিন্দু
Limitsign —সীমা চিহ্ন
Limiting value —সীমায়িত মান
Limiting value of a limit —সীমার সীমায়িত মান
Loop —লুপ/ফাঁস
Major axis —বৃহৎ অক্ষ
Maximum —গুরুমান
Mean value theorem —গড়মান উপপাদ্য
Method of induction —আরোহ পদ্ধতি
Minor axis —ক্ষুদ্র অক্ষ
Moment —ঘূরণ বল
Motion —গতি
Multiple valued —বহুমানী
Negative —বিয়োগ বোধক
Node, নোড, বা গিট
Numerical —সঙ্কেত

Interval of convergence —সীমা মুখিতায় ব্যবধি
Operator —অপারেটর / ঘটক
Order —ক্রম
Ordinate — কোটি
Parabola —পরাবৃত্ত
Paraboloid —পরাবৃত্তক
Parametric —প্যারামিটারযুক্ত
Differentiation —ডিফারেন্সিশন, অন্তরীকরণ
Partial fraction —আংশিক ভগ্নাংশ
Particular solution —নির্দিষ্ট সমাধান
Pendulum — দোলক
Perimeter —পরিসীমা
Periodic —আবর্তনকাল
Plane —সমতল
Point of inflection —ইনফ্লেকশন বিন্দু
Point of intersection — ছেদবিন্দু
Polygon —বহুভুজ
Polynomial —বহুপদী
Positive — যোগবোধক
Powere series —শক্তি ধারা
Probabltity curve —সম্ভাব্যতা রেখা
Process of summation — যোগপ্রক্রিয়া
Proper fraction —প্রকৃত ভগ্নাংশ
Properties —ধর্মাবলী
Proportional— সমানুপাতিক
Projection —প্রক্ষেপ
Range —ব্যাপ্তি
Rational —আনুপাতিক
Rationalization —আনুপাতিকরণ
Real number —বাস্তব সংখ্যা
Numerical —সাংখ্যিক
Odd — বেজোড়/অযুগ্ম
One dimensional —একমাত্রিক

One to one correspondence —এক এক সম্পর্ক
Operation —প্রক্রিয়া/অপরেশন
Resistance —প্রতিবন্ধকতা
Repeated —পুনরাবৃত্তিক
Retardation —মন্দন
Regular pyramid —সুষম পিরামিড
Right hand limit —ডানসীমা
Root —মূল
Sag —ঝুলন
Secant — ছেদক
Semi-axis —অর্ধক্ষ
Series —ধারা
Set — সেট
Sequence —অনুক্রম
Sign —চিহ্ন
Similar —সদৃশ
Single valued —একমানী
singular —বিশিষ্ট, ব্যাত্রিম
Solution —সমাধান
slope —ঢালুতা/ঢাল
Space —স্থান
Speed —দ্রুতি
Sphere — গোলক
Spherical shell — গোলকীয় খোলস
Spiral —স্পাইরেল/ কুণ্ডলী
Standard —প্রামান্য
Strenght —শক্তিমাত্রা
Strophoid — স্ট্রোফয়েড
Rectangular hyperbola —আয়াতাকর অধিবৃত্ত
" Parallelopiped " বাক্স
Reference line —নিয়ন্ত্রণ রেখা
Relative —আপেক্ষিক
Remainder —অবশিষ্ট

Sub-interval —উপব্যবধি
Substitution —প্রতিস্থাপন
Subset —উপসেট
Successive differentiation—ক্রমিক ডিফারেন্সিয়েশন ব অন্তরীকরণ
Surface —তল
Suffix —নিম্নসূচক
Symbolic —সাংকেতিক
Symmetrical —প্রতিসম
Symmetry —প্রতিসাম্য
Tangent —স্পর্শক
Temperature —উষ্ণতা
Tetrahedron — টেট্রাহেড্রন
Theorem —উপপাদ্য
Total differential —সার্বিক অন্তরকীয়
Transcendental —তুরীয়/অবীজগণিতীয়
Uniformly —সমভাবে
Variable —চলক
Velocity —বেগ
Vice-versa —বিপরীতক্রমে
Viscosity —আঠালুতা
Viscous —আঠালু
Vertex —শীর্ষবিন্দু
Vertical —ভূলম্ব রেখা

Important Formulae

1. $|x + y| \le |x| + |y|$
2. $|x - y| \ge |x| - |y|$
3. Continuity at $x = a$ of $f(x)$

$$\lim_{x\to a+} f(x) = \lim_{x\to a-} f(x) = f(a)$$

3. (a) $\lim_{h\to 0} f(a+h) = \lim_{h\to 0} f(a-h) = f(a)$

3. (b) or, $|f(x)-f(a)| < \varepsilon, |x-a| \le \delta, \varepsilon > 0$

4. Limit

If $\lim_{x\to a} f(x) = l$, $\lim_{x\to a} \phi(x) = m$, then

(a) $\lim_{x\to a} \{f(x) \pm \phi(x)\} = l \pm m$

(b) $\lim_{x\to a} f(x)\,\phi(x) = lm$

(c) $\lim_{x\to a} \frac{f(x)}{\phi(x)} = \frac{l}{m}$ if $\lim_{x\to a} \phi(x) \ne 0$

Limit of a function of a function

$$\lim_{x\to a} \phi(f(x)) = \phi\left\{\lim_{x\to a} f(x)\right\}$$

5. (a) $\lim_{x\to 0} \frac{\sin x}{x} = 1$ (b) $\lim_{x\to 0} \left(1 + \frac{1}{x}\right)^x = e$

(c) $\lim_{x\to 0} \frac{a^x - 1}{x} = \log_e a$ (d) $\lim_{x\to 0} \frac{1}{x} \log(1 + x) = 1$

(e) $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$ (f) $\lim_{x\to 0} \frac{x^n - a^n}{x - a} = na^{n-1}$

(g) $\lim_{x\to 0} \frac{(1+x)^n - 1}{x} = n$

6. Dfferential Co-efficient of elementary functions

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \frac{d}{dx}\{f(x)\} = f'(x)$$

7. $\frac{d}{dx}(x^n) = nx^{n-1}$ $\frac{d}{dx}(x^{-n}) = -\frac{n}{x^{n+1}}$

$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$ $\frac{d}{dx}(e^x) = e^x$

$\frac{d}{dx}(a^x) = a^x \log a$ $\frac{d}{dx}(\log x) = 1/x$

$\frac{d}{dx}(\sin x) = \cos x$ $\frac{d}{dx}(\cos x) = -\sin x$

$\frac{d}{dx}(\tan x) = \sec^2 x$ $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

$\frac{d}{dx}(\sec x) = \sec x \tan x$ $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{(1-x^2)}}$ $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{(1-x^2)}}$

$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$ $\frac{d}{dx}(\cot^{-1}) = \frac{-1}{1+x^2}$

$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{(x^2-1)}}$ $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{x\sqrt{(x^2-1)}}$

$\frac{d}{dx}(\sin hx) = \cos hx$ $\frac{d}{dx}(\cos hx) = \sin hx$

$\frac{d}{dx}(\tan hx) = \operatorname{sech}^2 x$ $\frac{d}{dx}(\cot hx) = -\operatorname{cosec} h^2 x.$

$\frac{d}{dx}(\sec hx) = -\sec hx \tan hx$ $\frac{d}{dx}(\operatorname{cosec} hx) = -\operatorname{cosec} hx \cot hx.$

$\frac{d}{dx}(\sin h^{-1} x) = \frac{1}{\sqrt{(1+x^2)}}$ $\frac{d}{dx}(\cosh^{-1} x) = \frac{-1}{\sqrt{(x^2-1)}}, x > 1$

$\frac{d}{dx}(\tan h^{-1} x) = \frac{1}{1-x^2}; x < 1$ $\frac{d}{dx}(\cot h^{-1} x) = \frac{-1}{\sqrt{(x^2-1)}}, x > 1$

$\frac{d}{dx}(\sec h^{-1} x) = \frac{-1}{x\sqrt{(1-x^2)}}, x < 1$

$\frac{d}{dx}(\operatorname{cosec} h^{-1} x) = \frac{-1}{x\sqrt{(x^2+1)}}$

8. $\frac{d}{dx}(c)=0,\ \frac{d}{dx}cf(x)=c\frac{d}{dx}f(x),\quad \frac{d}{dx}[f(x)\pm\phi(x)]=f'(x)\pm\phi'(x)$

9. $\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$

10. $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$; $v\neq 0$

11. If $y=f(t),\ t=\psi(x)$

$\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}$

Successive Differentiation

12. $D^n(ax+b)^m=\frac{m!}{(m-n)!}a^n(ax+b)^{m-n};\ (m>n)$

$D^n(ax+b)^m=0,\ (n>m)$

$D^n(ax+b)^n=a^n n!$ if $m=n$.

13. $D^n(e^{ax})=a^n e^{ax}$

$D^n(a^x)=(\log a)^n a^x$

14. $D^n\frac{1}{(ax+b)}=\frac{(-1)^n n!\,a^n}{(ax+b)^{n+1}}$

15. $D^n\left[\frac{1}{(ax+b)^m}\right]=\frac{(-1)^n(m+n-1)!\,a^n}{(m-1)!\,(ax+b)^{m+n}}$

16. $D^n\log(ax+b)=\frac{(-1)^{n-1}(n-1)!\,a^n}{(ax+b)^n}$

17. $D^n\sin(ax+b)=a^n\sin\left(ax+b+\frac{1}{2}n\Pi\right)$

18. $D^n\cos(ax+b)=a^n\cos\left(ax+b+\frac{1}{2}n\Pi\right)$

19. $D^n(e^{ax}\cos bx)=\sqrt{(a^2+b^2)^n}\,e^{ax}\cos(bx+n\tan^{-1}b/a)$

20. $D^n(e^{ax}\sin bx)=\sqrt{(a^2+b^2)^n}\,e^{ax}\sin(bx+n\tan^{-1}b/a)$

21. $D^n\left(\frac{1}{x^2+a^2}\right)=\frac{(-1)^n n!}{a^{n+2}}\sin^{n+1}\theta(\sin(n+1)\theta$

Where $\theta=\tan^{-1}a/x$ or, $\cot^{-1}x/a$.

22. $D^n\left(\tan^{-1}\frac{x}{a}\right)=\frac{(-1)^{n-1}(n-1)!}{a^n}\sin^n\theta\sin n\theta$

Where $\theta=\tan^{-1}a/x$ or, $\cot^{-1}x/a$

23. Leibnitz's Theorem

$D^n(uv)=u_nv+{}^nc_1u_{n-1}v_1+{}^nc_2u_{n-2}v_2+\ldots+{}^nc_ru_{n-2}v_r+\ldots+uv_n$

Expansions

24. Machlaurin's Series

$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\ldots+\frac{x^nf^n(0)}{n!}+\ldots$

25. Taylor's Series

$f(x+h)=f(x)+hf'(x)+\frac{h^2}{|\underline{2}}f''(x)+\ldots+\frac{h^nf^n(x)}{n!}+\ldots$

26. $f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f(a)+\ldots$

$+\frac{(x-a)^n}{n!}f^n(a)+\ldots$

27. $e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^r}{r!}+\ldots$

$e^{-x}=1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+\ldots+(-1)^r\frac{x^r}{r!}+\ldots$

$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots$

$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots$

$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}-\ldots$

$\log(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}+\ldots$

$\tan^{-1}x=x-\frac{x^3}{x}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots$

$$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots\ldots\ldots\ldots$$

$$\sin^{-1}x = x + \frac{x^3}{2.3} + \frac{1.3x^5}{2.4.5} + \ldots\ldots\ldots\ldots$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \ldots\ldots\ldots\ldots$$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \ldots\ldots\ldots\ldots$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots\ldots\ldots$$

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots\ldots\ldots\ldots$$

$$(1-x)^{-1/2} = 1 + \frac{1}{2}x + \frac{1.3}{2.4}x^2 + \frac{1.3.5}{2.4.6}x^3 + \ldots\ldots\ldots$$

$$(1+x)^{-1/2} = 1 - \frac{1}{2} + \frac{1.3}{2.4}x^2 - \frac{1.3.5}{2.4.6}x^3 + \ldots.$$

Partial Differentiation

28. If $u = f(x,y)$,

$$\frac{\delta^2 u}{\delta x \delta y} = \frac{\delta^2 u}{\delta y \delta x}$$

29. if $u = f(x,y)$, $x = f(t)$, $y = \psi(t)$, then

$$\frac{du}{dt} = \frac{\delta u}{\delta x}\frac{dx}{dt} + \frac{\delta u}{\delta y}\frac{dy}{dx}$$ (Totall differntial co-efficient for a variable t.)

cor : If $u = f(x,y)$, $y = f(x)$, then

30. $$\frac{\delta u}{\delta x} = \frac{\delta u}{\delta x} + \frac{\delta u}{\delta y} = \frac{\delta v}{\delta x}$$

30. (a) Total Differential of $u = f(x,y)$

$$du = \frac{\delta u}{\delta x}dx + \frac{\delta u}{\delta y}dy.$$

If $x = \phi(r, s, t)$, $y = \psi(r, s, t)$, then

$$dx = \frac{\delta x}{\delta r}dr + \frac{\delta x}{\delta s}ds + \frac{\delta x}{\delta t}dt$$

$$dy = \frac{\delta y}{\delta r}dr + \frac{\delta y}{\delta s}ds + \frac{\delta y}{\delta t}dt$$

31. If $u = f(x, y)$, $x = \phi(r, s, t)$, $y = \psi(r, s, t)$, the Partial differential co-efficient of u are

$$\frac{\delta u}{\delta r} = \frac{\delta u}{\delta x}\frac{\delta x}{\delta r} + \frac{\delta u}{\delta y}\frac{\delta v}{\delta r}$$

$$\frac{\delta u}{\delta s} = \frac{\delta u}{\delta x}\frac{\delta x}{\delta s} + \frac{\delta u}{\delta y}\frac{\delta y}{\delta s}$$

$$\frac{\delta u}{\delta t} = \frac{\delta u}{\delta x}\frac{\delta x}{\delta t} + \frac{\delta u}{\delta y}\frac{\delta y}{\delta t}$$

32. If $f(x,y) = 0$ or c, then $\frac{dy}{dx} = -\frac{f_x}{f_y}$. $f_y \neq 0$.

33. Euler Theorem

If $f(x,y)$ be a homogeneous function of degree n, then

$$x\frac{\delta f}{\delta x} + y\frac{\delta f}{\delta y} = nu$$

34. If $u = f(x,y)$, then $du = \left(\frac{\delta f}{\delta x}dx + \frac{\delta f}{\delta x}\ \frac{\delta}{\delta y}dy\right)u.$

and $d^n u = \left(\frac{\delta f}{\delta x}dx + \frac{\delta f}{\delta y}dy\right)^n u.$

35. Taylor's Theorem for n variables

$$\phi(x+h, y+k, z+l) = e^{\left(x\frac{\delta f}{\delta x} + y\frac{\delta f}{\delta y} + z\frac{\delta f}{\delta z} + \ldots\right)}\ \phi(x, y, z, \ldots\ldots)$$

36. Jacobian

$$J_n(u_1, u_2, \ldots u_n) = \frac{\delta(u_1, x_2, \ldots\ldots u_n)}{\delta(x_1, x_2 \ldots\ldots x_n)} = \begin{vmatrix} \frac{\delta u_1}{\delta x_1} & \frac{\delta u_1}{\delta x_2} & \cdots\cdots\cdots & \frac{\delta u_1}{\delta x_n} \\ \frac{\delta u_2}{\delta x_1} & \frac{\delta u_2}{\delta x_2} & \cdots\cdots\cdots & \frac{\delta u_2}{\delta x_n} \\ \cdots & \cdots & & \cdots \\ \frac{\delta u_n}{\delta x_1} & \frac{\delta u_n}{\delta x_2} & \cdots\cdots\cdots & \frac{\delta u_1}{\delta x_n} \end{vmatrix}$$

37. Equation of tangent at (x,y)

$$Y - y = \frac{dy}{dx}(X - x) \text{ for } y = f(x)$$

$$(Y - y) f_y + (X - x) f_x = 0 \text{ for } f(x,y) = 0$$

(a) Slope for the tangent, $\frac{dy}{dx}$ or, $f'(x)$

(b) tangent is parallel to x—axis : $f'(x) = 0$ or, $f_x = 0$

(c) tangent is perpendicular to x—axis is

$f'(x) = \infty$ or, $f_y = 0$

38. Angle of intersection of two curves

$y = f(x)$, $y = \varphi(x)$

$$\tan\alpha = \frac{f_x \varphi_y - \varphi_x f_y}{f_x \varphi_x + f_y \varphi_y}$$, α is the angle of intersection

(a) Two curves touch

$\frac{f_x}{f_y} = \frac{\varphi_x}{\varphi_y}$ or, $f'(x) = \varphi'(x)$

(b) Two curves intersect orthogonally

$f_x \varphi_x + f_y \varphi_y = 0$ or, $f'(x)\,\varphi'(x) = -1$.

39. Length of the tangent$= y \operatorname{cosec}\psi = \frac{y}{y_1}\sqrt{(1+y_1^2)}$; $y_1 = dy/dx$

Length of the normal $= y \sec\psi$

40. Subtangent $= y \cot\psi = y/y_1$

Subnormal$= y\tan\psi = yy_1$

41. ϕ, the angle between tne radius vector and tangent,

$\tan\phi = r\frac{d\theta}{dr} = \frac{f(\theta)}{f_1(\theta)}$, $f(\theta) = r$

42. Pedal Equation

$p = r\sin\phi$, p is the perpendicular from pole to the tangent.

43. $\frac{1}{p^2} = \frac{1}{r^2} + \left(\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 = u^2 + \left(\frac{du}{d\theta}\right)^2$, $u = \frac{1}{r}$

44. Polar subtangent $= r\tan\phi = r^2\, d\theta/dr$

Polar Subbnormal $= r\cot\phi = dr/d\theta$

45. $\alpha = \phi_1 \sim \phi_2$, α is the angle of intersection between two curves $r = f(\theta)$, $r = \phi(\theta)$

(a) Two curves tauch if $\phi_1 = \phi_2$

(b) Two curves cut orthogonally $\phi_r \sim \phi_2 = \frac{1}{2}\pi$

46. Arc-length

$\frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}$, $\frac{ds}{dy} = \sqrt{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}}$

$(ds)^2 = (dx)^2 + (dy)^2$, $(ds)^2 = (rd\theta)^2 + dr^2$

47. $\tan\psi = \frac{dy}{dx}$, $\sin\psi = \frac{dy}{dx}$, $\cos\psi = \frac{dx}{ds}$

$\tan\varphi = r\frac{d\theta}{dr}$, $\sin\varphi = r\frac{d\theta}{ds}$, $\cos\varphi = \frac{dr}{ds}$

48. Negative Pedals

Put $p = r$ and $r = p^2/r$ in $f(p,r) = 0$ i. e,

$f(r, p^2/r) = 0$ is the First Negative pedal.

Repeat this proves for 2nd negative pedal, 3rd negative pedal etc.

48. (a) Inverse curve

$f\left(\frac{k^2 x}{x^2 + y^2}, \frac{k^2}{x^2 + y^2}\right) = 0$ for $f(x,y) = 0$

(b) $f\left(\frac{k^2}{r}, \theta\right) = 0$ for $f(r, \theta) = 0$

(c) $p = \frac{r^2}{k^2} f\left(\frac{k^2}{r}\right)$ for $p = f(r)$.

48. (d) Pedal Equations of well known curve

Circle : $x^2 + y^2 = a^2$ (centre), $r = p$

Circle : $x^2 + y^2 = a^2$ (point on the circumference, $r^2 = 2ap$

Parabola : $y^2 = 4ax$ (focus), $p^2 = ar$

Ellipse : $\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$ (focus), $\frac{b^2}{p^2} = \frac{2a}{r} - 1$

Ellipse : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (centre), $\frac{a^2 b^2}{p^2} + r^2 = a^2 + b^2$

Hyperbola, $\frac{y^2}{a^2}-\frac{y^2}{b^2}=1$ (focus), $\frac{b^2}{p^2}=\frac{2a}{r}+1$

Hyperbola, $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ (centre), $\frac{a^2b^2}{p^2}-r^2=a^2-b^2$

Rect. Hyperbola, $x^2-y^2=a^2$(centre), $pr=a^2$

Parabola, $r=\frac{2a}{1\pm\cos}$ (focous), $p^2=ar$.

Cardioide $r = a(1 \pm \cos\theta)$ (pole), $r^3=2ap^2$

Lemniscate $\left.\begin{array}{l} r^2=a^2\cos 2\theta \\ r^2=a^2\sin 2\theta \end{array}\right\} r^3=a^2p$

$$\left.\begin{array}{l} r^n=a^n\cos n\theta \\ r^n=a^n\sin n\theta \end{array}\right\} \text{(pole)},\ r^{n+1}=a^np.$$

49. Indeterminate Forms

(a) If $f(a)=\phi(a)=0$, then

$$\underset{x\to a}{Lt}\frac{f(x)}{\phi(x)}=\underset{x\to a}{Lt}\frac{f'(x)}{\phi'(x)} \text{ form } \frac{0}{0}$$

(b) If $f(a)=\o(a)=\infty$, then

$$\underset{x\to a}{Lt}\frac{f(x)}{\o(x)}=\underset{x\to a}{Lt}\frac{f'(x)}{\o'(x)} \text{ for } \frac{\propto}{\propto}$$

Maxima and Minima

50. If $f'(a)=0$, $f''(a)\neq 0$ for $y=f(x)$, then

(a) $f(a)$ is maximum if $f''(a)$ is negative

(b) $f(a)$ is minimum if $f''(a)$ is positive.

(c) If $f'(a)=f''(a)=$ .... $=f^{n-1}(a)=0$ and $f^n(a)\neq 0$,

(i) $f(x)$ is maximum if $f^n(a)$ is negative n is even

(ii) $f(x)$ is minimum if $f^n(a)$ is negaive n is even

(iii) $f(x)$ is neither maximum or minimum if n is odd.

51. $\phi(x, y)$ be a functions of two variables x and y and $r=\frac{\delta^2\o}{\delta x^2}$

$s=\frac{\delta^2\o}{\delta x\delta y}$, $t=\frac{\delta^2\o}{\delta y^2}$, then for (a, b)

If $rt-s^2$ is positive, $\o(a, b)$ is maximum or minimum according as r and t are both negative or both positive.

If $rt-s^2$ is negative, $\o(a, b)$ is neither maximum or minimum.

52. For $\o(x, y, z)$ for a point $\o(a, b, c)$,

$$A=\frac{\delta^2\o}{\delta x^2},\ B=\frac{\delta^2\o}{\delta y^2},\ C=\frac{\delta^2\varphi}{\delta z^2},\ F=\frac{\delta^2\varphi}{\delta y\delta z},\ G=\frac{\delta^2\varphi}{\delta z\delta x},\ H=\frac{\delta^2\varphi}{\delta x\delta y}.$$

$\varphi(a, b, c)$ is minimum

if A, $\begin{vmatrix} A & H \\ H & B \end{vmatrix}$, $\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$ are all positive

and $\varphi(a, b, c)$ is maximum if

A, $\begin{vmatrix} A & H \\ H & B \end{vmatrix}$ $\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$ are alternately negative and positive.

Asymptotes

53. $y=mx+c$ be the asymptote of the curve

$y=f(x)$ if

$m=\underset{x\to\propto}{Lt}\ y/x$ and $c=\underset{x\to\propto}{Lt}\ (y-mx)$

54 $\varphi(x, y)=p_n+P_{n-1}+P_{n-2}+$ .... $+P_o$, n indicated the homogeneous function in x and y of degree n.

Asymptotes.

$$y-m_1x+\underset{x\to\propto}{Lt}\frac{F_{n-1}}{\varphi_{n-1}}=0$$

(a) If $y-m_1x$ is repeated, then asymptotes are

$$(y-m_1x)^2+(y-m_1x)\underset{x\to\propto}{Lt}\frac{R_{n-1}}{\varphi_{n-1}}+\underset{x\to\propto}{Lt}\frac{F_{n-2}}{\varphi_{n-2}}=0$$

(b) In $\varphi_n(y/x)$, Put $y=m$, $x=1$, find $\varphi_n(m)$

Differentiate, $\varphi'(m)$.

Then put $\varphi_n(m)=0$, then m_1, m_2 etc are obtained.

For the roots of m, $c\,\varphi'_n(m) + \varphi_{n-1}(m) = 0$, then c, s are obtained.

Thus $y = mx + c$ be the asymptote.

(c) For repeated roots of m, say two equal roots, the

$$\frac{c^2}{2\,!}\varphi''_n(m) + c\,\varphi'_{n-1}(m) + \varphi_{n-2}(m) = 0$$

then put the valus of $c's$ in $y = mx + c$. two asymptotes will be obtained.

In the same way for three equal roots etc.

55. Asymptotes in Polar Coordinates

If α be a root of $f(\theta) = 0$, then

$r \sin(\theta - \alpha) = 1/f'(\alpha)$ is an asymptote of the curve $1/r = f(\theta)$

Curvature

$$\rho = \frac{ds}{d\psi} \quad \therefore s = f(\psi)$$

$$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2} \qquad \therefore y = f(x),\ y_2 \neq 0$$

$$\rho = \frac{(1 + x^2)^{3/2}}{x_2} \qquad \therefore x = f(y),\ x_2 \neq 0$$

$$\rho = \frac{(x_1^2 + y_1^2)^{3/2}}{y_2}\ .\ x = \varphi(t),\ y = \psi(t)$$

$$\rho = \frac{(f_x^2 + f_y^2)^{3/2}}{f_{xx}f_y^2 - 2f_{xy}f_xf_y + f_{yy}f_x^2} \ \therefore f(x, y) = 0$$

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2},\ r = f(\theta)$$

$$\rho = \frac{(u^2 + u_1^2)^{3/2}}{u^3(u + u_2)},\ u = f(\theta)$$

$$\rho = \frac{rdr}{dp},\ r = f(r)\ ;\ \rho = p + \frac{d^2p}{d\psi^2} \quad \rho = f(\psi)$$

$\rho = \text{Lt}\,\frac{y^2}{2x}$ (at the origin, y–axis ($x = 0$) being tangent)

$\rho = \text{Lt}\,\frac{x^2}{2y}$ (at the origin, x axis ($y = 0$) being tangent.

$\rho = \sqrt{(a^2 + b^2)}\ \text{Lt}\,\frac{x^2 + y^2}{ax + by}$ (at the origin $ax + by = 0$ being tangent)

Chord of curvature through the pole = $2\rho \sin \psi$

Chord of curvature Parallel to x–axis is $2\rho \sin \psi$

Parallel to y–axis is $2\rho \cos \psi$

Centre of curvature

$$\overline{x} = x - \frac{y_1(1 + y_1^2)}{y_2}\ ;\ \overline{y} = y + \frac{1 + y_1^2}{y_2} \quad \therefore y = f(x)$$

$$\overline{x} = x + \frac{1 + x_1^2}{x_2},\ \overline{y} = y - \frac{x_1(1 + x_1^2)}{x_2} \quad \therefore x = f(y)$$

$\overline{x} = x - \rho \sin y$, $y = \overline{y} + \rho \cos \psi$

Sets

1.1. Finite sets : $A = \{1, 2, 3, 4\}$
2. Infinite sets : $B = \{1, 2, 3, \ldots\ldots\ldots\ldots\}$
3. Empty set : $A = A\ (\phi)$ or, ϕ or $\{0\}$
4. $a \in A$ means a contains in A or belongs to A.
5. $a \notin A$ means a does not belong to A.
6. Unit set : $A = \{a\}$
7. Subset : $A\ \ B$ Proper subset : $A \subset B$ if $A \neq B$
8. Union of sets : $A \cup B = \{x \in A \text{ or } x \in B\}$ $\therefore$ Read A cup B
9. Intersection of sets : $A \cap B = \{x \in A \text{ and} x \in B\}$ $\therefore$ Read A cup B
10. Disjoint sets : $A \cap B = \phi = \{x \notin A \text{ and } x \notin B\}$
11. Difference of two sets : A and B. $A - B = \{x \in A \text{ and } x \notin B\}$
12. Universal set = U (Union of all sets)
13. complement of a set : $A^c + A' = \{\, x \in X : x \notin A\}$ i. e, $x \in A^c = A'$ i. e; $x \notin A$.
14. Power set of $S = 2^s$

15. Countable set : A set is countable if it finite or denumerable.

16. $A \cup B = B \cup A, A \cap B = B \cap A,$

17. $(A \cup B) \cup C = A \cup (B \cup C);$

18. $(A \cap B) \cap C = A \cap (B \cap C)$

19. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

20. $A \cup (B \cap C) = A \cup B) \cap (A \cup C)$

21. $(A')' = A, (A \cup B)' = A' \cap B'; (A \cap B)' = A' \cup B'$

22. $A-B = A-(A \cap B)$

23. $(A-B) \cap A=A, (A-B) \cap B = \emptyset$

24. $A -B = A \cap B' = B'-A'$

25. $B-A' = B \cap A, A-B= A \cup B, A \cup (B-A) = B$

Set Theory

11. Meaning of Sets : An object which belongs to a given set is called a number or an element of the set. We designate sets by the Capital letters *A, B, C* etc and elements of a set by small letters *a, b, c* etc. Generally we say a is an element or member of *A i. e;* $a \in A$. Sets may be finite and infinite. Set which does not contain any element is called an empty set and is denoted by φ.

Description of Sets : *A* set generally dscribed by two methods :

(i) Roster method, (ii) Rule method :

In **Roster method,** We include a set by listing the elements and enclosing them with braces { }. Thus the set consisting of Rahim, Jack, Ram be written as {Rahim, Jack, Ram]

In the **Rule method,** we describe the set by a phrase "the set of all books of Rajshahi University Library." It is written as {*x* is a book in the Rajshahi University Library} or, as {x/x is a book in the Rajshahi University Library}

The oblique line standing for **"such that"**

Roster method is used for finite sets while Rule method is used for infinite set or sets containg large number of elements.

Subsets : A set A is a subset of B if every element of A is also as element of *B i. e;* $A \subseteq B$ or $B \subseteq A$

Example : Let A = {1, 2, 3, 4} and B={1, 2, 3, 4, 5}

Here every element of *A* is in *B i. e.* $A \subsetneq B$

Every set is a subset of itself thus $A \subseteq A$, because *A* certainly contains *A i. e.,* $x \subset A$ implies *x* in *A'*

The null set $\emptyset$ is a subset of every set. Thus $\emptyset \subseteq A$ because *A* certainly 'contains' $\emptyset$.

2.2 Equality of sets :

Two sets A and B are equal (svmbolically $A = B$) if and only if $A \subseteq B$ and $B \subseteq A$ *i. e.,* *A* is a subset of *B* and B is also a subset of A.

Example : Let A = {2, 3, 5} and {3, 5, 2}

Here every element of A is in B and every evlement of B is in A. Here $A \subseteq B$ and $B \subseteq A \therefore A = B$

2.3 Proper subsets :

A set A is a Proper subset of a set B (symbloically, $A \subsetneq B$) if $A \subseteq B$ and $A \neq B$

Thus $A \subset B$ means that A is a subset of B but B is not a subset of *A, i. e.* every element of *A* is in B but B has at least one element which is not in *A*.

Example : Let A = {1, 2, 3, 4} and *B* ={ 1,2, 3, 4, 5}

Here every element of *A* is in *B*, but *B*, has the element 5 which is not in A.

Here $A \subseteq B$ but $A \neq B. \therefore A \subsetneq B,$

Example : Let *A*={ 2, 3, 5}, *B*={3, 5,2} and *C*= {3, 2}.

Here every element of A is in *B* and every element of *B* is in *A*.

$A \subseteq B$ and $B \subseteq A$, and hence $A = B$

But though every element of C is in A or B the element 5 of / or B is not in C.

$\therefore C \subseteq A$ and $C \subseteq B$ but $C \neq A$ or $C \neq B \therefore C \subset A, C \subset B$.

2.4 disjoint sets :

Two sets A and B are disjoint if A and B have no commo elements.

Example : The sets {0, 2, 3} and {4, 5, 6} are disjoint.

2.5 Number of Subsets or a set; Power set P (S)

The set of all subsets s of a set S is called the power set of S and is denoted by $P(S) = \{s / s \subset S\}$ if S contains n elements, the power (S) = P set 2^n or the number of subsets s of a set S.

Example : Let $A = \{a, b\}$ and $B = \{a, b, c\}$

(i) the subsets of A are ø, {a}, {b}, {a, b}

there being ${}^2C_0 = 1$ subset with no element. ${}^2C_1 = 2$ subsets with one element and ${}^2C_2 = 1$ subset with two elements.

Thus the set A containing 2 elements

has $4 (= 2^2 = {}^2C_0 + {}^2C_1 + {}^2C_2)$ subsets.

(ii) the subsets of B are

ø, {a}, {b}, {c} : {a, b} : {a, c}; {b, c}; {a, b, c}

there being ${}^3C_0 = 1$ subset with no elements, and ${}^3C_1 = 1$ subset with three elements.

Thus the set B containing 2 elements

has $8 (= 2^3 = {}^3C_0 + {}^3C_1 + {}^3C_2 + {}^3C_3)$ elements)

from the above consideration, it is clear that a set containing n elements has

${}^nC_0 = 1$ subset with no element.

${}^nC_1 = n$ subsets with one element.

${}^nC_2 = \frac{n(n-1)}{\lfloor 2}$ subsets with two elements and so on, the total number of subsets thus being

${}^nC_0 + {}^nC_1 + {}^nC_2 \ldots\ldots\ldots\ldots + {}^nC_n = (1 + 1)^n \; 2^n.$

2.6. about the Symbols ε and ⊂

The two symbols ε and ⊂ should not be confused with each other. The symbols ε denotes the relationship between a set and its elements. where as the symbol ⊂ denotes the relationship between two sets.

Thus if $D = \{5, 2, -3\}$, then it is true that $\{5\} \subset D$ It is not, however, correct to write 5 c D or, | 5 | CD.

2.7 Finite Set : A set is finite if it consists of a specific number of different elements.

Infinite set : A set is infinite if it is equivalent to a proper subswet of itself otherwise, a set is finite.

Example : A = { S.M.T.W.Th. F sat} A is finite

B = { x | x is an integer}, B contains infinite numbers of intergers, B is infinite.

Power Set : The family of all the subsets of any set S is called the power set of S. The power set of S is denoted by 2^s

Let S = {2, 3}, then subsets are { 2, 3}, {2}, {3}, φ, there are four subsets = 2^s, Hence Power set $2^s = 2^2 = 4$.

Countable Set : a set is called countable if it is finite or denumerable *e.g.* {(1,1), (4,8), (9,27)} ....$(n^2 n^3)$}.

3.1 Union of two Sets :

The union of two sets A and B is the set of elements which are in at least one of the sets A and B *i.e.*, which belongs to either A or B.

Symbolically we write the union of A and B as $A \cup B$. read **'A union B, or ' A cup B'**

Thus $A \cup B = B \{x / x \in A$ or, $x \in B)\}$ It follows from the definition that, $A \cup B = B \cup A$ i ε., the equation of union is commutative.

Example : Let $A = \{a,b,c\}$, $B = \{d,e\}$, $C = (b,e)$, $D = (a,c)$ then $A \cup B = \{a, b, c, d, e\}$, $B \cup C = \{b, d, e\}$ $A \cup D = \{a, b, c\}$,

Note—In the above example.

$D \cup A$ and $A \cup D$ = A. This is true for two such sets. In particular A ∪ A = A and A ∪ ø = A for any set A.

3.2 Intersection of two sets;

The intersection of two sets A and B is the set of elements which belong to both A and B.

Symbolically we write the intersection of two sets A and B as ***A ∩ B; A* intersection B or A cup B.**

Thus $A \cap B = \{x/x \in A$ and $x \in B\}$. If follows form the definition that $A \cap B = B \cap A$, i.e., the operation of its intersection is commutative.

Example : Let A, B, C, D be sets as in the previous example in Art. 3.1

Then $A \cap B = \emptyset$. $A \cap C = \{b\}$, $A \cap D = \{a, c\}$

Note- in the sbove example

A and B are disjoint sets and $A \cap B = \emptyset$. This is true for any two such sets. Also $D \subseteq A$ and $A \cap D = D$. This is also true for aby two such sets. In particular $A \cap A = A$ and $A \cap \emptyset = \emptyset$ for any set A.

3. 3. Complement of a set.

If A is a subset of a universal set U then the complement of A is the set of elements which belongs to U are not contained in A.

The complement of a set A is denoted by A' read 'A prime' and is defined relative to a particular universal set U.

Thus $A' = \{x/x \in U$ and x, does not belong to A$\}$

Example — Let $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$A = \{1, 7, 6\}$, and $B = \{2, 5, 8, 9, 0, 3, 1\}$

Thus $A' = \{0, 2, 3, 4, 5, 8, 9\}$ and $B' = \{4, 6, 7\}$

Note- In the above example.

we change U, A' and B' will be changed. This is true in all cases. the following results directly obtained from definitions.

(1) $U' = \emptyset$ i. e., the complement of the universal set is the null set

(2) $\emptyset' = U$, i. e., the complement of the null set is the universal set

(3) $(A')' = A$, i, e. the complement of the complement of any set is the set itself.

(4) $A \cup A' = U$, i. e., the union of any set and complement of the universal set.

(5) $A \cap A' = \emptyset$, i.e., the intersection of any set and its complement is of the null set.

4. 1. Venn Diagrams.

It is often convenient to draw diagrams to represent relation between sets or operations on sets. Such diagrams are called Venn diagrams.

In the following illustions, we have represented the univer- sal set U as the set of points in a rectangle and other sets as sets of points a circle within the rectangle.

Laws of Algebra of Sets

Art.4.2. A, B, C are sets A', B', C' are the complements of A,B and C respectively U-universal set, $\emptyset$ = null set

Identity Laws

1. (i) $A \cup \emptyset = A$ (ii) $A \cap \emptyset = \emptyset$

(iii) $A \cup U = U$ (iv) $A \cap U = A$

Idempotent Laws

2. (i) $A \cup A = A$ (ii) $A \cap A = A$

Complement Laws

3. (i) $A \cup A' = U$ (ii) $A \cap A' = \emptyset$

(iii) $(A')' = A$ (iv) $\emptyset' = U$ (v) $U' = \emptyset$

Commutative Laws

4. (i) $A \cup B = B \cup A$ (ii) $A \cap B = B \cap A$

Associative Laws

5. (i) $(A \cup B) \cup C = A \cup (B \cup C)$

(ii) $A \cap (B \cap C) = (A \cap B) \cap C$

Distributive Laws

6. (i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cup C)$

The distributive laws take on the general forms

$A \cap (B_1 \cup B_2 \cup \ldots\ldots\ldots \cup B_n) = (A \cup B_1) \cup (A \cup B_2) \cup \ldots \cup (A \cap B_n)$.

$A \cup (B_1 \cap B_2 \cap \ldots\ldots\ldots \cap B_n) = (A \cup B_1) \cap (A \cup B_2) \cap \ldots\ldots \cap (A \cup B_n)$

De Morgan's Laws

7. (i) $(B \cup B)' = B' \cap B'$

(ii) $(B \cap B)' = B' \cup B'$

Art. 4.3 (1) Prove De Morgan's Theorem :

$(A \cup B)' = A' \cap B' = \{x / x \in (A \cup B)'\}$

Proof : $(A \cup B)' = \{x \mid x \notin A \cup B\} = \{x \mid -(x \in A \text{ or } x \in B\}$.

$= \{x \mid x \notin A' \text{and } x \notin B\} = x \mid \{x \notin A \text{ and } x \notin B\} = A \cap B$

Alternative Method:-Let $x \notin (A \cup B)$, then $x \notin (A \cup B)$. Therefore $x \notin A$ or $x \notin B$ thus $x \notin A'$ and $x \in B'$. So $a \in A' \cap B'$ But our assumption is that $x \in (A \cup B')$ Hence $(A \cup B)' \subseteq A' \cup B$. (1)

Again if $y \in A' \cap B'$ then, $y \in A'$ and $y \in B'$ then $y \notin A$ and $y \notin B$. hence $y \notin A \cup B)$ so $y \in (A \cup B)'$ we have shown that $y \in A' \cap B'$ implies that $y \in (A \cup B)'$ Hence $A' \cap B' \subseteq (A \cup B)'$ .... (2)

From (1) and (2). $(A \cup B)' = A' \cap B'$ proved.

44.(2) Prove that $(A \cap B)' A' \cup B'$

Proof : $(A \cap B)' = \{x \mid (x \in (A \cap B)')$

$= \{x \mid x \notin (A \cup B)\} - \{x \mid (x \in = A \text{ and } x \in B)\}$

$= x \mid x \notin A \text{ or } x \notin B\} - (x \mid x \in A' \text{ or } x \notin ŒB') = A' \cup B'$

De Morgan's Laws can be extended and this has been śtated without proof here :-

$(A_1 \cup A_2 \cup A_3 \cup ......A_n)' = (A_1)' \cap (A_2)' \cap (A_3)' \cap ....\cap (A_n)'$

$(A_1 \cap A_2 \cap A_3 \cap ......\cap A_n)' = (A_1)' \cup (A_2)' \cup (A_3)' \cup ...\cup (A_n)'$

4.5 (1) difference : The difference of two sets A and B (relative to A) is the set of elements which belong to A but do not belong to B. The difference is writen as $A-B$.

which reads' "A difference B" or, simple, 'A minus B"

i. e. $A-B = \{x . \mid x \in A \text{ and } x \notin B\}$

Similarly, the difference of two sets A and B (relative to B) is defined as $B-A = \{x \mid x \in B \text{ and } x \notin A\}$

For example, if $A = \{a, b, c, d\}$ and $B = \{a, e, f, d\}$, then

$A-B = \{b, c\}$

But $B-A = \{e, f\}$

Art. 2. In Venn Diagram, the difference $A-B$ and $B-A$ are shown. The dotted area is the difference $A-B$ and cross cut area is the difference $B-A$. U represents the universe.

Art. 2. (i) Prove that $A-B = A-(A \cap B)$

Let $x \in (A-B)$ (if and only if)$\leftrightarrow x \in A$ and $A \notin B \leftrightarrow x \in A$ and $x \notin (A \cap B) \leftrightarrow x \in A-(A \cap B)$

$\therefore A-B = A-(A \cap B)$

Art. 3. Prove that $(A-B) \subset A$

i. e, set A contains A-B as a subset

Proof : Let x be any element of $A-B$. Then we have $x \in A$ and $x \notin B$ i.e. x belong to A but we have shown that $x \in A-B$ implies that $x \in A$. Hence $(A-B) \subset A$

Art. 4. Prove that $(A-B) \cap B = \emptyset$

Let x belong to $(A-B) \cap B$ i. e., $x \in (A-B) \cap B$ by the intersection of two sets. we have $x \in A-B$ and $x \in B$ but by the definition of difference $(A-B)$, we have $x \in A$ and $x \in B$. Hence there is no element satisfies both $x \in B$ and $x \notin B$ then

$(A-B) \cap B = \emptyset$

Art. 5. rove that A-B = A $\cap$ B$'$ = B$'$-A$'$ where A$'$ is the complement of A, B$'$ is the complement of B.

Proof : $A-B = \{x : x \in A \text{ and } x \notin B\}$

$= \{x : x \in A \text{ and } x \in B'\} = A \cap B'$

$= \{x : x \in A' \text{ and } x \in B'\}$

$= \{x : x \in B' \text{ and } x \notin A'\} = B'-A'$

Alse $B-A = A'-B'$

Note : The complement of a set A is the set of elements which are not present in A, i. e, the difference of the universal set U and A. we represent the complement of A by A', concisely we define A' By

$A' = \{ x \text{ is such that } x \in U, x \notin A\}$

$= \{x \mid x \in U, x \notin A\}$

or, simply, $A' = \{x \mid x \notin A\}$.

Art. 6 Prove B–A is a subset of A′

Proof : Suppose $x \in B-A$. . e. $x \in B$ and $x \notin A$. Hence $x \in B$ and $x \in A'$ i. e, x belongs to B' but $x \in B-A$ impies that $x \in A'$. $B-A$ is a subset of A', i. e.. $B-A \subseteq A'$.

Art. 7. Prove that B–A′ = B ∩ A.

Proof : Let $x \in B-A'$ which means that $x \in B$ and $x \notin A'$., i. e, $x \in B$ and $x \in A$ which means that $x \in B \cap A$. Hence $B-A' = \{x \in B$ and $x \notin A\} = \{x \mid x \in B$ and $A\} = B \cap A$

$\therefore$ $B-A' = B \cap a$.

Art. 8 Prove that A–B is subset of AUB.

Proof : Let $x \in A-B$ means $x \in A$ and $x \notin B$, therefore $x \in AUB$ i. e, $A-B \subseteq AUB$.

Art. 9. Prove that A ⊆ B inplies that subset of AU(B–A) = B

Proof : $A \; U \; (B-A) = \{ x \mid x \in a \text{ or}, x \in (B-A)\}$

$= \{ x \mid x \in A \text{ or}, (x \in B \text{ and } x \notin A)\}$

$= \{ x \mid x \in A \cap x \in B\}$

$= \{ x \mid x \in B\}$ *since* $A \subset B$

Hence A is the Proper subset of B *i. e*, $A \subset B$.

Example 10. To show thar for any three sets A, B and C $(A \cap B) \cap C = A \cap (B \cap C)$.

we show that (1) $(A \cap B) \cap C \subseteq A \cap (B \cap C)$

and (ii) $A \cap (B \cap C) \subseteq (A \cap B) \cap C$

(i) Let x be any element of the set $(A \cap B) \cap C$, Then $x \in (A \cap B)$ and $x \in C$. But $x \in (A \cap B)$ implies that $x \in A$ and $x \in B$. Thus $x \in A$ and $x \in B$ and $x \in C$ and as such $x \in (B \cap C)$ as well. Hence $x \in A \cap (B \cap C)$. Therefore

$(A \cap B) \cap C \subseteq A \cap (B \cap C)$.

(ii) Let y be any element of the set $A \cap (B \cap C)$. Then, $y \in (B \cap C)$. But $y \in (B \cap C)$ implies that $y \in B$ and $y \in C$. Thus $y \in A$ and $y \in B$ and $y \in C$, and as such $y \in (A \cap B)$ as well.

Hence $(A \cap B) \cap C$

Therefore $A \cap (B \cap C) \subseteq (A \cap B) \cap C$

Hence the conclusion

Ex. 11. prove that $(A-B) \cup (B-A) = (A \cup B) = (A \cup B) - (A \cap B)$ when $A = \{1, 2, 3, 4, 5\}$ and $B = \{2, 3, 5, 6, 7\}$ *C.U.* 1982

Ans. $A - B = \{ x : x \in A$ and $x \notin B = \{1, 4\}$

$B-A = \{x : x \in B$ and $x \notin A\} = \{6, 7\}$

Now $(A-B) \cup (B-A) = \{1, 4\} \cup \{6, 7\} = \{1, 4, 6, 7\}$ ....(1)

Again $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$

$A \cap B = \{2, 3, 5\}$

$(A \cup B) - (A \cap B) = \{x : x \in (A \cup B)$ and $x \in (A \cap B)$

$= \{ 1, 4, 6, 7\}$...........(2)

From (1) and (2)

$(A - B) \cup (B-A) = (A \cup B) - (A \cap B)$.

Ex. 11. (*a*) If $U = \{1, 2, 3, ...9\}$, $A = \{1, 2, 3, 4\}$, $B = \{2, 3, 4, 5\}$

Find $A \cap B$ and the complement of $A-B$. *C. U.* 1982.

2. A few examples :

Example. 1. Find the union and intersection of the sets

$A = \{n/n$ is and integer and $n \geq 9\}$.

and $B = \{n/n$ is an interger and $n \geq 2\}$.

Answer. Since A is the set of integers ≥ 9 and B is the set of integers ≥ 2 . $\therefore$ $A \subseteq B$.

Every element of A being also in B.

Hence $A \cup B = B$ and $A \cap B = A$.

Example 2. For the set $A = \{a, b, c\}$, $B = \{b, d, e\}$ and $C = \{d, f, g\}$, veryfy that

$A \cap (A \cup C) = (A \cap B) \cup (A \cap C)$,

Answer : Here $B \cup C = \{a, b, e\} \; U \; \{d, f, g\} = \{b, d, e, f, g\}$

$\therefore$ $A \cap (B \cup C) = \{a, b, c\} \; U \{b, d, f, g\} = \{b\}$

Also $A \cap B = \{a, b, c\} \cap \{b, d, e, f, g\} = \{b\}$

And $A \cap C = \{a, b, c\} \cap \{ d, f, g\} = \emptyset$

$\therefore$ $(A \cap B) \cup (A \cap C) = (b \cup \emptyset) = \{b\}$.

Hernce the conclusion.

Example 3. Given any set A in the universe $\cup$. show that there exists a unique set X satisfying the conditions

(a) $A \cup X = U$ and (b) $A \cap X = \emptyset$.

Ans : There exists a set $X = A$ satisfying the conditions because $A \cup A' = U$ and $A \cap A' = \emptyset$

[**Art.** 3,3; Properties (4) and (5)]

If Y be any other set satisfying the conditions, then

$A \cup Y = \cup$ and $A \cap Y = \emptyset$

But $Y = Y \cap U$ [since $Y \subset U$]

$= Y \cap (A \cup X) = (Y \cap A) \cup (Y \cap X)$ [Property (5) **Art. 4.2**]

$= \emptyset \cup (Y \cap X)$ (since $Y \cap A = A \cap Y = \emptyset$)

$= Y \cap X$ (since $\emptyset \cup k = k$ for any set K)

$\therefore Y \subseteq X$ Art. 4.1

Also, by similar argument, $X \subseteq Y$

$Y = X$

Hence there the set X satisfying the given condition is unique.

Example 4. Show that for any two sets A and $B : A \subseteq B$ if and only if $B' \subseteq A'$

Let $A \subseteq B$. Then $x \in A$ implies that $x \in B$ and as such x is not in B'. Hence A and B are disjoint sets. Having no common elements so that $B' \cap A = \emptyset$

Now $B' = B \cap U$	(U is the universal set)
$= B' \cap (A \cup A)'$	(since $A \cup A' = U$)
$= (B' \cap A) \cup (B' \cap A')$	(property (5) Art. (4.2)
$= \emptyset \cup (B' \cap A') = (B' \cap A')$	($\emptyset \cup K$ = k for any set (K)
$\therefore B' \subseteq A'$	

Example 5. show by means of Venn Diagram that is n(x) denotes the number of elements of the finite set x, then

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

A and B being two finite sets. How is the result modified if A and B are disjoint sets?

Ans : Visualising the sets A and B as the sets of points within the two circles, let p, q, r denote the number of points in the regions.

P, Q, remarked in the diagram. Then

n(A) = number of points in the regions P and $Q = p + q$

n (B) = number of points in the regions Q and $R = q + r$

$n (A \cap B)$ = number of points in the regions $Q = q$

$n (A \cap B)$ = number of points in the regions P. Q and $R = p + q + r$.

Hence $n(A \cup B) = p + q + r = (p + q) + (q + r) - q = n(A) + n(B) - n(A \cap B)$

If A and B are disjoint sets then $A \cap B = \emptyset$ so that $n(A \cap B) = o$, In that case, $n(A \cup B) = n(A) + n(B)$ simply.

6.2 Ordered Pairs :

A pair element is said to form an ordered pair if it is specified which of the two comes first and which comes second.

An ordered pair in which a is the first component and b is the second component is denoted by the notation (a, b)

It should be noted that $(a, b) \neq (b, a)$

unless $a = b$ although $\{a, b\} = \{b, a\}$. Morover. we speak of orderd pairs of the form (a, b) even though tlere is no set (a, a)

Two ordered pairs (a, b) and (c, d) are said to be equal *i, e.* $(a, b) = (c, d)$ if and only if a =c and $b = d$.

Example : In plane analytic geometry, the position of a point in the plane is determined by an ordered pair (x,y) of real numbers, the first component x being the abscissa of the point and the second component y being the ordinate of the point. The ordered pair (2, 3) is certainly different from the ordered pair (3, 2) as they represent differents points in the plane.

Example. Given a set A ={a, b} the ordered pairs which formed with elements of A are (a, a). (a, b). (b, a). (b, b)

Example. Given two sets $a = \{a, b\}$ and $X = \{x, y\}$ the ordered pairs which can be formed with an element of A as the first component and an element of X as the second component are (a, x). (a, y). (b, x) . (b, y)

7.1 MAPPINGS

Let X and Y be two sets not necessarily distinct. *A* mapping of X into Y is a correspondence that associates with each element of X with unique element of Y.

A mapping is usually denoted by a single letter such as f, g, a, etc. The fact that f is a mapping of X into Y is often indicated by the symbol $f : X \to Y$.

If $f : X \to Y$, then for each $x \in X$, the corresponding element of Y is called the image of x under the mapping f and is denoted by (x). The set X is called the domain of the mapping f and the set Y is called its image space. The subset of Y consisting of those elements of Y which are image of some $x \in X$ *i e.* $\{ y/y \in Y$ and $y = f(x)$ for some $x \in X\}$ is called the range of the mapping f.

If may be noted that under a mapping f of X into Y, every $x \in X$ has one and exactly one image in Y. Where as the same $y \in Y$ may be the image of more than one $x \in X$ and there may be some $y \in Y$ which is not the image of any $x \in X$.

A mapping of X into Y is defined if we know the image of of each $x \in X$. The notation $f : x \to f(x)$ is used to indicate that under the mapping of X into Y, x is mapped into f(x), *i. e.* $f(x)$ is the image of x.

A mapping $f : X \to Y$ can be pictorially represented by listing the elements of X and Y inside two closed curves and drawing arrows from each $x \in X$ the corresponking image $y \in Y$.

Example : Let a mapping $f : Y \to Y$ be defined as follows. $1 \to 3$, $2 \to 2$, $3 \to 1$, $4 \to 1$, $3 \to 3$, $7 \to 2$, $9 \to 5$.

Here the domain of f is $x = \{1, 2, 4, 7, 9\}$ and the range of f is $\{1, 2, 3, 5\}$.

Example : Let $x = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $f : x \to X$ be a mapping defined as follows :

$f(1) = 1$, $f(2) = 5$, $f(3)=4$, $f(4)=8$.
$f(5)=6$, $f(6)=3$, $f(7)=7$, $f(8)=2$.

Hence the domain as range of f is the x.

Example : Let $X= \{1, 2, 3\}$ and $Y = \{a, b, c\}$.

The correspondence defined as.

$1 \to a$, $1 \to b$, $2 \to b$, $3 \to c$, is not *a* mapping because under the correspondence, two distinct elements Y correspond to the elements 1 of X.

CHAPTER–1

NUMBERS

1.1 The set of Real Numbers

(a) **Integers** (পূর্ণসংখ্যা) **:** The numbers 1, 2, 3, ... are known as the ***natural*** or ***counting*** numbers. The natural numbers, their negatives and zero form the set of integers Z. Thus

$$Z = \{ \ldots\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\ldots \}$$

(b) **Rational numbers** (আনুপাতিক সংখ্যা) **:** Any number which can be expressed in the form $\frac{p}{q}$, where p and q are integers with $q \neq 0$, is called a rational number. Clearly any integer is a rational number (correspo nding to $q = 1$).

Examples of rational numbers are 2, $\frac{5}{7}$, $\frac{-3}{8}$, 1.2, etc.

In *decimal representation* of a rational number, the steps will either *terminate* or *a certain part of the steps will repeat.* For example,

$\frac{1}{8} = 0{\cdot}125$; here the steps terminate.

$\frac{1}{3} = 0{\cdot}3333 \ldots\ldots\ldots = 0.\dot{3}$; here the steps do not terminate, but 3 is repeated which is indicated by putting a dot over 3·

$\frac{1}{7} = 0.142857\ 142857\ 142857 \ldots\ldots = 0{\cdot}\dot{1}4285\dot{7}$; the dots over 1 and 7 show that the part '142857' is repeated.

(c) **Irrational numbers** (অমেয় সংখ্যা) **:** A number which represents a certain length on a straight line but cannot be represented in the form $\frac{p}{q}$ (p, q being integers $q \neq 0$) is called an irrational number.

In decimal representation of an irrational number the steps involved one non-terminating and non-recurring.

$\sqrt{2}$, $\sqrt{3}$, π, e, etc. are irrational numbers.

All the rational and irrational number together are said to form the **continuum of Real numbers** or **the Set of real numbers,** denoted by $\mathbb{R}$.

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All the rational and irrational number together are said to form the **continuum of Real numbers** or **the Set of real numbers,** denoted by $\mathbb{R}$.

All the real numbers, positive or negative, rational or irrational can be represented on a straight line, say, the axis of X which is called the *real axis*.

Let XOX′ be the axis of x (or the real axis), the point O being the origin or the point of reference.

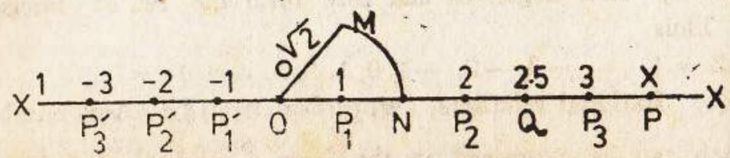


Fig. 1

Let P be any point on the real line. Let, on certain scale, the algebraic distance OP$=x$; that is, $x > o$ if P lies to the right of O (on OX side), while $x < o$, if is to the left of O (or OX′ side). We say that the point P represents the real number x. Evidently O represents the number 0 (zero). For different positions of P, x takes different values and we call x to be a real variable. If the actual distances of P_1 or P'_1, P_2 or P'_2, P_3 or P'_3, ... measured from O be 1, 2, 3, ... respectively, then $P_1, P_2, P_3, \ldots$ ...represent the numbers 1, 2, 3, ... while $P'_1, P'_2, P'_3, \ldots \ldots$ represent the numbers $-1, -2, -3, \ldots$ as shown in Fig. 1. The point Q which is the mid point of the lines segment $P_2 P_3$ denotes the number 2·5. In the triangle OP_1M right-angled at P_1, $P_1M = OP_1 = 1$, $OM = \sqrt{2}$. A circle drawn with OM as radius cuts the x-axis at N which is to the right of O. Hence N represents the number $\sqrt{2}$.

1.2 Aboslute value of a real number.

The *absolute value* or *modulus* of a real number x is the magnitude or the numerical value of the number with positive sign. It is denoted by $|x|$. Thus

$$|x| = x, \text{ if } x \geq o,$$

and $$|x| = -x, \text{ if } x < o.$$

Since $(-x)^2 = x^2 = |x|^2 \geq 0$, we see that *the square of a real number is never negative.*

It is to be noted that

(i) $\sqrt{x^2} = |x|$

and (ii) $|x| \geq x$.

Ex. $|5| = 5$, $|0| = 0$, $|-5| = -(-5) = 5$;

$|5| = 5$, but $|-5| = 5 > -5$.

$\sqrt{9} = \sqrt{(\pm 3)^2} = |\pm 3| = 3$.

If the negative square root of 9 or any other positive real number is wanted then the radical sign should be preceeded by a '−' (minus) sign ; e.g. $-\sqrt{9} = -3$.

1.3 Imaginary and complex numbers.

Since the square of a real number is never negative, we introduce the number i, such that

$$i^2 = -1 \quad \text{or} \quad i = \sqrt{-1}$$

with $|i| = 1$.

"i" is called the *unit imaginary number*. Any number z expressed as $z = x + iy$ (where x and y are real) is called a *complex number*. If $y = 0$, the number z is purely real ; if $x = 0$, then z is purely imaginary. $-1 + i$, $2 - \sqrt{3}\,i$, $\sqrt{-7}$ or $\sqrt{7}\,i$ are all complex numbers.

Art 1.3 : Properties of Absolute values :

(a) If x and y are any two real numbers, prove that

$$|x+y| \leq |x| + |y|$$

Proof : Let $x + y \geq 0$. Then

$$|x+y| = x+y \leq |x| + |y| \quad [\because x \leq |x|,\ y \leq |y|]$$

If $x+y < 0$, then

$$|x+y| = -(x+y) = (-x) + (-y) \leq |x| + |y|$$

$$[\because (-x) \leq |-x| = |x|,\ (-y) \leq |-y| = |y|]$$

Hence, in any case,

$$|x+y| \leq |x| + |y|$$

By repeated application of the above result, we can prove that

$$|x \pm y \pm z + \ldots\ldots| \leq |x| + |y| + |z| + \ldots\ldots$$

Ex (i) $|-5+7| = |2| = 2$; $|-5| = 5$, $|7| = 7$; $5+7=12$.

Now $2 < 12 \Rightarrow |-5+7| < |-5| + |7|$.

(ii) $|-5-7| = |-12| = 12$;

$|-5| + |-7| = 5+7 = 12$

$\therefore |-5-7| = 12 = |-5| + |-7|$.

(b) **Prove that**

$$|\mathbf{x}-\mathbf{y}| \geq |\mathbf{x}| - |\mathbf{y}|$$

Proof : We have

$|x| = |(x-y)+y| \leq |x-y| + |y|$ [*by (a)*]

or $|x| - |y| \leq |x-y|$

$\Rightarrow |x-y| \geq |x| - |y|$

Ex. (i) $|5-7| = 2$; $|5| - |7| = 5-7 = -2$

$\therefore |5-7| > |5| - |7|$ $(\therefore 2 > -2)$

(ii) $|7-5| = 2$; $|7| - |5| = 7-5 = 2$

$\therefore |7-5| = |7| - |5| = 2$.

(iii) Prove that $|\mathbf{xy}| = |\mathbf{x}|\,|\mathbf{y}|$.

Proof : Let x, y be both positive. Then $xy > 0$.

So $|xy| = xy = |x|\,|y|$ $(\because |x| = x$, when $x > 0$,

$|y| = y$, when $y > 0$.)

If x, y are both negative, then $xy > 0$ and

So $|xy| = xy = (-x)(-y) = |x|\,|y|$.

If. $x > 0$, $y < 0$, then $xy < 0$ and

$|xy| = -xy = x(-y) = |x|\,|y|$

Simillarly, when $x < 0$ and $y > 0$

$|xy| = -xy = (-x)(y) = |x|\,|y|$.

Hence in any case, $|xy| = |x|\,|y|$.

Art. 1·4. Meaning of $|x-\alpha| < \delta$.

(a) Let $x-\alpha > 0$.

Then $|x-\alpha| = x-\alpha$.

$\therefore |x-\alpha| < \delta \Rightarrow x-\alpha < \delta$, or $x < \alpha + \delta \cdots (i)$

(ii) If $x-\alpha < 0$, *then*

$|x-\alpha| < \delta \Rightarrow -(x-\alpha) < \delta$

or $-x+\alpha < \delta$

or $-x < -\alpha+\delta$ *or* $x > \alpha-\delta \cdots (ii)$

Hence form (i) and (ii), We have,

$$\left.\begin{array}{c}\alpha-\delta < x < \alpha+\delta \\ \text{if } |x-\alpha| < \delta\end{array}\right\}$$

Cor. $\left.\begin{array}{l}|x-\alpha| \leq \delta \\ \Rightarrow \alpha-\delta \leq x \leq \alpha+\delta\end{array}\right\}$

While $|x| \leq \delta \Rightarrow -\delta \leq x \leq \delta$.

Ex. Give the equivalent of $|x+1| \leqq 2$.

By removing the absolute notation.

$|x+1| \leqq 2$ is written as $-2 \leqq x+1 \leqq 2$

or : $-2-1 \leqq x+1-1 \leqq 2-1$ or $-3 \leqq x \leqq 1$

Ex. Give the equivalents of the statement $-3 \leqq x \leqq 7$ in the terms of the the absolute notation.

Let the relation be

$|x-\alpha| \leqq \delta$ or $\alpha-\delta \leqq x \leqq \alpha+\delta$

If we compare it with $-3 \leqq x \leqq 7$, then we have

$\alpha-\delta = -3$ and $\alpha+\delta = 7$.

Solving these $\alpha = 2$ $\delta = 5$

Hence the expression $-3 \leqq x \leqq 7$ is equivalent to $|x-2| \leqq 5$.

1. 5. Draw the graph of $y = |x|$

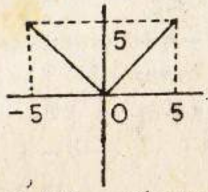
$y = |x|$ means

$y = x$ if $x \geq 0$ and $y = -x$ if $x < 0$

So, we are to draw two graphs for the two equations. Let us restrict our attention to the graph corress ponding to the interval $-5 \leqq x \leqq 5$.

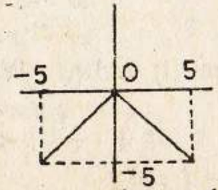
x	-5	-3	0	3	5
y	5	3	0	3	5

The graph of $y = |x|$ is given by the Fig. (1) The graph of $y = -|x|$ is given in Fig. 2.



$y=|x|, -5 \leq x \leq 5$

Fig. (1)



$y=-|x|, -5 \leq x \leq 5$

Fig. (2)

Ex. 1 Draw the graph of $y=\frac{1}{2}(x+|x|)$

Let us restrict our attention to the graph corressponding to the interval $-5 \leqq x \leqq 5$.

For $x \geqq 0$, $|x| = x$

and so $y=\frac{1}{2}(x+x)=x$ (1)

But for $x<0$, $|x|=-x$, and

$y=\frac{1}{2}(x-x)=0$ (2)

This can be written as

$$y=\begin{cases} x, & x \geqslant 0 \\ 0, & x<0 \end{cases}$$

The graph of $y=\frac{1}{2}(x+|x|)$ for $-5 \leqq x \leqq 5$ is shown in fig (3).

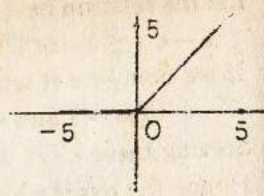


Fig. (3)

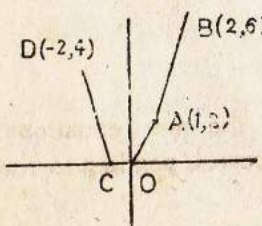


Fig. (4)

Ex 2 (a) Draw the graph of $y=2|x+1|+|x|+|x-1|-3$ in the interval $-2 \leqq x \leqq 2$

For $x \geqslant 1$,

$2|x+1| = 2x+2$, $|x| = x$, $|x-1| = x-1$,

$y = 2x+2+x+x-1-3 = 4x-2$

or, $y=4x-2$ represents segment AB in Fig. (4)

For $0<x<1$,

$y=2|x+1|+|x|+|x-1|-3$

$=2x+2+x-(x-1)-3=2x$ or $y=2x$ represents OA Fig (2)

For $-1 \leqq x \leqq 0$,

$y=2|x+1|+|x|+|x-1|-3=2x+2-x-(x-1)-3 = 0$

or, $y=0$ reprsents OC (3)

For $x<-1$,

$y=2|x+1|+|x|+|x-1|-3$

$=-2(x+1)-x-(x-1)-3=-4x-4$

or, $y=-4x-4$ represents CD (4)

We draw four graphs for four equations (1)–(4). The four graphs are all straight lines. The combined graph is continuous and is shown in fig. (4).

Ex2 (c) show that Ex2 (b) may be expressed as

$$y\begin{cases} 4x-2, & x \geqslant 1 \\ 2x, & 0<x<1 \\ 0, & 1 \leq x \leq 0 \\ -4x-4, & x<-1 \end{cases}$$

Art. 1.6. Graphs of Inequalities (অসমতার লেখচিত্র)

Let us consider the following cases involving absolute values,

For the equation $|x| = 1$

x has only two solutions viz, $x=+1$ and $x=-1$

What will happen to $|x| \leq 1$

In this inequality x has solutions in the entire interval $-1 \leq x \leq 1$.

For the equation $|x-3| = 5$,

x has only two solutions x such as $x-3=5$ and $x-3=-5$ or $x=8$ and $x=-2$

Ex. Solve the inequality $|x|+|y| \gtreqless 1$ ... ...(1)

Let us first consider the values of x and y in the first quadrant so that $x \geqq 0$ and $y \geqq 0$.

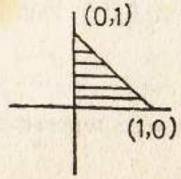


Fig (7)

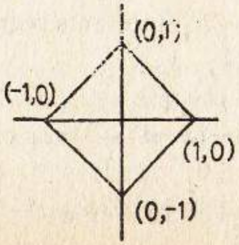


Fig (8)

The inequality (1) becomes $x+y \leqq 1$

For $x+y=1$, we get the line segment in the first quadrant, joining the points (1, 0) and (0, 1).

Hence $x+y \leqq 1$, $x \geqslant 0$, $y \geqslant 0$, $|x|+|y| \leqq 1$ is a graph consisting of all points in and on the triangle with vertices (0,0), (1, 0) and (0, 1) [Fig. (7)]

Considering the valus of (x, y) in the other quadrants, we see that the solution of the inequality

$$|x|+|y| \leq 1$$

is the set of all points lying in and on the quadrilateral with vertices at (1, 0), (0, 1), (−1, 0), (0, −1), [Fig. (8))

Exercise 1

1. If $|a-b|<l$, $|b-c|<m$, show that $|a-c|<l+m$.

2. Give the equivalents of the following terms in the absolute notation.

(i) $-7 \leqq x \leqq 13$ Ans. $|x-3| \leqq 10$

(ii) $-3<x<7$ Ans. $|x-2|<5$

(iii) $l-\varepsilon \leqq x \leqq l+\varepsilon$ Ans. $|x-\varepsilon| \leqq l$.

3. Find the equivalents of the following by removing the absolute notations.

(i) $|x-5|<7$ (ii) $|x+2| \leqq 5$ (iii) $0<|x-2|<3$

Ans. $-2<x<12$ Ans. $-7 \leqq x \leqq 3$ Ans. $-1<x<5$, $x \neq 2$

4. Draw the graphs of the following equations and express them free from modulus system.

(নিম্নলিখিত সমীকরণগুলির লেখচিত্র অঙ্কন কর এবং পরমমান বর্জন করে y এর ফাংশানের আকারে প্রকাশ কর)

(i) $y=\frac{1}{2}(x-|x|)$ in $-5 \leqq x \leqq 5$

(ii) $y=-\frac{1}{2}(x+|x|)$ in $-3 \leqq x \leqq 3$

(iii) $y=\frac{1}{2}(|x|-x)$ in $-5 \leqq x \leqslant 5$

(iv) $y=2|(x-1)|-|x|+2|x+1|-5$ in $-3 \leqq x \leqq 3$.

5. Draw the graphs of the following inequalities. (অসমতার)

(i) $|x|-|y| \geqq 1$ for $-2 \leqq x \leqq 2$

(ii) $|x|+2|y| \leqslant 1$

(iii) $2x+y \leqq 5$

$x-y \leqq 1$ for $-3 \leqq x \leqq 3$

$x+2y \leqq 7$.

5 (iv) Show that areas shown in the figure are represented by the following equation.

(a) $x \leqq 0$, $y \leqq 0$, $x+y \leqq 1$

(b) $x \leqq 0$, $y \geqslant 0$, $x+y \geqq 1$

(c) $x \geqq 0$, $y \geqslant 0$, $x+y \geqq 1$

(d) $x \geqslant 0$, $y \leqq 0$, $x+y \geqq 1$ Ans. (i)

(e) $x \geqq 0$, $y \leqq 0$, $x+y \leqq 1$

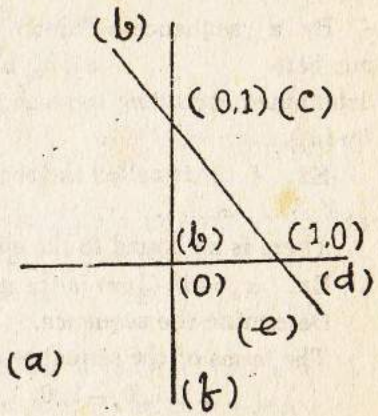
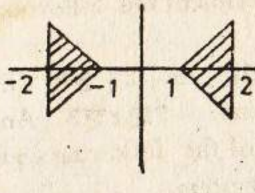
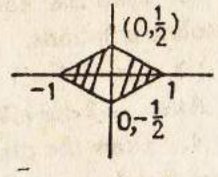


Fig. (9)

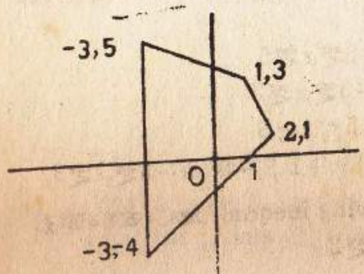
Ans. 5 (i)



5 (i)



5 (ii)



5 (iii)

6. If $y = |x| + |x-1|$, then show that

$$y = \begin{cases} 1-2x, & \text{for } x \leqslant 0 \\ 1, & \text{for } 0 < x < 1 \\ 2x-1 & \text{for } x \leqslant 1 \end{cases}$$

Sequence (ক্রম)

1.7. Definition : *A sequence is a set of numbers arranged in order so that there is a first one, a second one, third one and so on.*

By a sequence we mean an ordered infinite succession of numbers $a_1, a_2, a_3 \dots \dots a_m, \dots \dots$ determined according to some rule. Symbolically it is represented by $\{a_n\}$

Ex. $\{n\}$ is called the sequence of all positive integers *i. e* ; $1, 2, 3, \dots \dots n \dots \dots$

There is no bound to the size of the numbers ; $\{n\}$ is endless.

Ex. $x_n = \sin(\frac{1}{2} n\pi)$ is the general term of a sequence.

Determine the sequence.

The terms of the sequence are as follows.

$$\dots \dots 1, 0, -1, 0, 1, 0, -1, \dots \dots \dots$$

i. e. : the numbers are repeated in pairs with signs changed.

1. 8. **Constant (ধ্রুবক) :**—A constant is a quantity which remains unchanged during any mathematical operations or any investigation.

There are two types of constants.

(*i*) *Arbitrary constants*

(*ii*) *Absolute constants*

Thus quantities which have the same value under all circumstences are called **absolute constants.**

e. g, $1, 5, 9, 10, \pi, e$

Again quantities which have the same value under one investigation but are different for different investigation are called **arbitrary constant**, *e. g.* in the equation of the straight line.

$$x \cos \alpha + y \sin \alpha = p$$

α and p are same for the same straight line. but they will be different for different straight lines. So α and p are arbitrary constants.

Art. 1.9 Variable (চলক) : A changing entity is called a variable. Generally we use the letters $x, y, z, u, t, \alpha, \beta$, etc for variables. There are two types of variables.

(*i*) *Independent variables* (স্বাধীন চলক)

(*ii*) *Dependent variables* (অধীন চলক)

1.10 Independent variable : A variable which may take any arbitrary value assigned to it is called an independent variable.

1.11. Dependent variable : A variable whose value depends on the values of second variable or on the values of a system of variables is called a dependent variable.

Ex. (i) $y = \sin x$.

For each value of x, there exists a value of $\sin x$ or y. Here y is the dependent variable and x is the independent variable.

(ii) $u=xyz$.

For every set of values x, y, z, there is a definite value of u. Hence in this case u is the dependent variable and x, y, z are independent variables.

1.12 Domain or Interval of a variable.

The set of all permissible values of a variable x is called the domain of the variable x.

Suppose the variable x assumes all values between two given numbers a and b including the values a and b (with $b>a$). Then the domain of x is called an **closed domain** or **closed interval** denoted by

$$a \leq x \leq b \quad \text{or} \quad x \in [\,a, b\,].$$

If one of the end points, say a, is not included in the interval $[\,a, b\,]$, then we say that the domain of x is open at the left and closed at the right ; we denote it by $a < x \leq b$ or $x \in (\,a, b\,]$. Similarly, when $a \leq x < b$, we write $x \in [\,a, b\,.)$

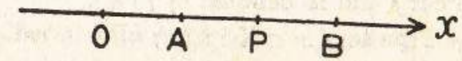
If both a and b are excluded from the interval $[\,a, b\,]$, then the domain or interval is called an **open domain** or **open interval.** It is decoted by

$$a < x < b, \quad \text{or} \quad x \in (\,a, b\,)'$$

We can respresent an interval geometrically on the real line.

Let A, B represent the given numbers a, b respectively on the real line OX and let the point P representing the variable x can take any position on the line segment AB.

The interval $[\,a, b\,]$ is represented by the whole segment AB and P can take any position from the end A to the end B ; in the intervals $(\,a, b\,]$, $[\,a, b\,)$ and $(\,a, b\,)$ the point A, the point B both of the points A and B are respectively excluded from the segment AB. The length of each of thise intervals is



Neighbourhood of a point : Let δ be an infinitesimally small orbitrary positive number. Then for a given number a the interval $(\,a-\delta, a+\delta\,)$ is called a *neighbourhood* of a for the variable x. If we say that 'x tends to a' denoted by '$x \to a$', we mean that *x is in the neighbourhood of a with $x \neq a$.*

Art. 2. : Function : Suppose that we have two non-empty sets X, Y and a rule f establishing a correspondence between the members of X and Y. If the rule f is such that it assigns to each element $x \in X$ a unique element $y \in Y$, then f is called a **function.** This is denoted by $f : X \to Y$ and read as 'f is a function of X to Y' or f is a mapping from X to Y'. We also express this as

$$f : x \to y \text{ or } y = f(x).$$

2.1. Function defined as sets of ordered pairs.

Let X and Y be two nonempty ordered sets. A subset f of $X \times Y$ is called a function from X to Y if and only if to each $x \in X$, there exists a unique y in Y such that $(x,y) \in f$.

i. e : (i) $\mathbf{x \in X, (x, y) \in f.}$ **for some** $\mathbf{y \in Y}$

(ii) $\mathbf{(x, y_1) \in f}$ **and** $\mathbf{(x, y_2) \in f \to}$**(implies that),** $y_1 = y_2$

The first condition exerts that a rule f which gives a image for each element of X and the second condition states the existence of unique image.

The graph of f is the subset of $X \times Y$ defined by $\{x, f(x) : x \in X\}$. The range of f is the set of all images under f and is denoted by $f[X]$

$\therefore f[X] = \{y \in Y : y = f(x) \text{ for some } x \in X\}$
$= \{f(x) ; x \in X\}$

Similarly if $A \subset X$, then the set $\{f(x) ; x \in A\}$ is called the image of A under f and is denoted by $f[A]$.

If $B \subset Y$, then the set $\{x \in X : f(x) \in B\}$ is called the inverse image of B under f and is denoted by $f^{-1}[B]$.

Art. 3. Equivalence relation : A relation R in a set is an equivalence relation if

(i) R *is* reflexive i. e ; for every $a \in A$, $(a, a) \in R$

(ii) R *is* symmetric i. e ; for every $(a, b) \in R$ implies $(b, a) \in R$

(iii) R *is* transitive i. e ; for every $(a, b) \in R$. and $(b, c) \in R$ implies that $(a, c) \in R$

Art. 4 Equivalence set : Set A is equivalent to set B represented by $A \sim B$ if there exists a fnnction $f : A \to B$ whieh is one—one and onto.

Art. 5. Types of functions :—

Let A and B be two sets and $f : A \to B$.

(i) f is one-to-one if $(x_1, y) \in f$ and $(x_2, y) \in f$, then $x_1 = x_2$. We express it by $f : A \xrightarrow[1-1]{} B$, (fig -1)

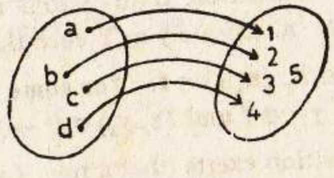


Fig. 1

One-one functinn from A **into** B

(2) If f is not one-one, f is called many-to-one.

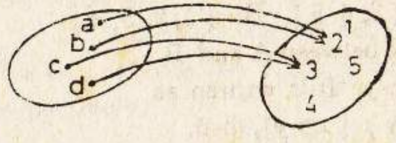
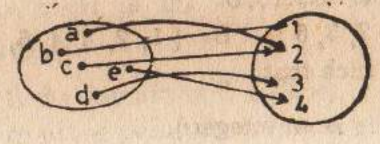


Fig. 2

A many-to-one function from **A into B** (Fig-2)

(3) f is said to be a function from **A onto B** if $R(f) = B$ i.e. if the range of f contains all the elements of B (fig-3)



$f : A \xrightarrow{\text{onto}} B$

Fig. 3

A function from **A onto B**

(4) f is one-one and onto B, then f is said to be a one-one correspondence between A and B (fig-4)

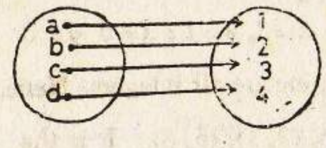


Fig. 4

Relation : Any nonempty subset R of a cartesian product $A \times B$ is called **a relation** from A to B. If $(x, y) \in R$, it is often written as ${}_xR_y$ read. "x is related to y."

In general a relation R from A to B, between two sets A and B is a subset of $A \times B$ i e., $R \subset A \times B$.

A relation from A to A i. e : a given subset of $A \times A$, is called **a relation on A.**

Ex. 1 Let $A = \{3, 4, 5\}$, $B = \{1, 2, 3, 4\}$

Then a relation R between A and B.

exists in such that $x > y$. It is written as

Define $R = \{(x, y) \mid x > y\}$, then.

$R = \{(3, 1,) (3, 2), (4, 3), (4, 1)\}$......(1)

Here each Ist element for x is greater than the 2nd element.

It is the relation R. But $A \times B = \{3, 4, 5\} \times \{1, 2, 3, 4\}$

$= \{(3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), (4, 4), (5, 1), (5, 2), (5, 3), (5, 4)\}$ ... ... (2)

From (1) and (2), we see that R is the subset of $A \times B$ *i e.* $R \subset A \times B$.

Ex. 2. Let $A = \{1, 2, 3, 5\}$, $B = \{1, 2, 3, 4, 5, 6\}$

Then a relation R is such that

$$R = \{x, y) \mid \frac{3x+1}{y+1} \text{ is an integer }\}.$$

Select the values of x from A

and y from B such that $(3x+1) / (y+1)$

is an integer. If $x = 1$, $y = 1$, then

$(3x+1) / (y+1) = (3.1+1) / (1+1) = 4/2 = 2$ is an integer.

so the pair is $(1, 1)$

For $x = 3, y = 4$; $x = 3, y = 1$; $x = 5, y = 3$

$\frac{3x+1}{y+1} = 2, 5, 4$, etc are all integers. Hence

$R = \{(1, 1), (3, 4), (3, 1), (5, 3\}$. It is the subset of $A \times B$ i. e, $R \subset A \times B$

Ex. 3. Let $A = \{2, 3, 4\}$

such that $R = \{(x, y) \in A \times A \mid x + 2y = 10\}$

$A \times A = \{2, 3, 4\} \times \{2, 3, 4\}$

$x = 2, y = 4,$; $x = 4, y = 3$

$2 + 2.4 = 10$; $4 + 2.3 = 10$.

$\therefore$ $R = \{(2, 4), (4, 3)\}. \subset A \times A$.

Domain and Range of f**:** For $f : X \to Y$, the permissible values which x can take from the set x is called the **domain** of f, while the set of corresponding values of $y \in Y$ is called the **range** of f. We will denote the domain and the range of f by D_f and R_f respectively.

Domian and Range of a Relation.

If A and B be sets and R is a relation from A to B, the **domain of** R is the set of all first elements (or first co-ordinates) of the pairs (x, y). which belongs to R.

The Range of R is the set of all second co-ordinates of of the pair (x, y).

From Ex. 1, the domain of R is $\{3, 4\}$, if an element is present more than once in the pair, take only one element for its domain The Range is $\{1, 2, 3\}$

From Ex. 2, the domain of R is $\{1, 3, 5\}$ and the range is $\{1, 4, 3\}$

From Ex. 3. ; the domain of R is $\{2, 4\}$

and the range of is $\{4, 3\}$

Art. 6. Inverse function.

Let $f : A \to B$ be any function. Then $f^{-1}(B) = A$, since every element in A has its image in B. If $f(A)$ denotes the range of f, then $\quad f^{-1}(f(A)) = A$

If $b \in B$, then $f^{-1}(b) = f^{-1}(\{b\})$

Here f^{-1} has two meaning as the inverse of an element of B and as the inverse of a subset of B-

2—

Def.: Let f be a function A into B and let $b \in B$.
If $f: A \to B$, then
$f^{-1}(b) = \{x \mid x \in A, (x) = b\}$

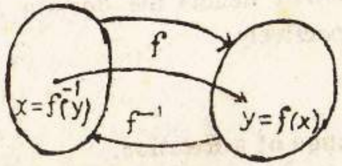


Fig. 5

Inverse of a Relation :

If R be a relation from a set A to another set B, then the inverse relation of R denoted R^{-1} is defined as the inverse relation i. e, R^{-1} from B to A if and only if

$$R^{-1} = \{(x), x\} : (x, y) \in R\}$$

It is clear that domain of R^{-1} is equal to the range of R and range of R^{-1} is equal to the domain of R

i. e ; $D(R^{-1}) = R(R)$
$R(R^{-1}) = D(R)$

Ex. If $A = \{a, b, c,\}$, $B = \{1, 2\}$

$R = \{(a, 1), (a, 1), (b, 2)\ (b, 2\}$ is a relation from A to B. so that $R^{-1} = \{(1, a), (1, a), (2, b), (2, b)\}$

will be a relation from B to A i. e, R^{-1} is the inverse relation on. R.

Difference between Function and Relation :

Any non-empty subset R of a cartesian product $A \times B$ is called a relation from A to B.

If $(x, y) \in R$, then Ex 1, Ex 2, Ex 3 are all the examples of relations. There is no restriction on the elements of Domain of Relation. Same element of A may occur any number of times as first element in (x, y). But in Function it cannot be. A relation is then a function if all elements of the set A are the first elements of (x, y) without Repeatation. *i. e.* **a function is a relation but a relation may not be a function.**

A relation from A to B is a function if

(*i*) Domain $= A$. (*ii*) $(x, y) \in R$, and $(x, z) \in R$, then $y = z$.

or, it may be stated that 'A function from A to B is a relation which associates each element of A with one and only one element of B''

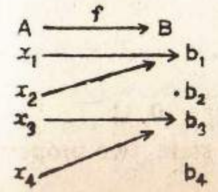


fig (6)
It is a function

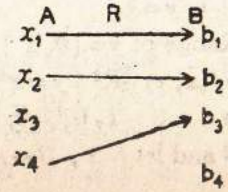


fig (7)
It is a relation as x_3 has no relation in B.

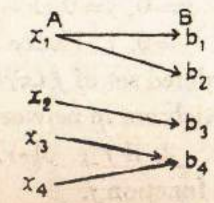


fig (8)
It is not a function as x_1 has images in B.

Different types of function will be discussed in the next Article.

Types of Relations

1. **Reflexive :** A relation R on a set A is known as reflexive if and only if each member of A is R related to itself.
$(x, x) \in R$ for each $x \in A$.
or, ${}_xR_x$ for each $x \in A$.
2. **Symmetric :** A relation R on a set A is known as Symmetric if $(x, y) \in R \to (y, x) \in R$
or, ${}_xR_y \to {}_yR_x$; $x, y \in A$.
If is also known as if $R^{-1} = R$, then a relation R is symmetric.
3. **Transitive :** A relation R on a set A is known as transitive if and only if $(x, y) \in R$ and $(y, z) \in R \to (x, z) \in R$
or, ${}_xR_y$ and ${}_xR_z \to {}_xR_z$, $x, y, z \in A$.

Ex. 1. If $f(x)$ is a function whose domain is the interval the set of all x, $-1 \leq x \leq 1$ and a rule of f is given by the equation $f(x)=x^2$, what set of ordered pairs is f?

Let $y=f(x)$ $\therefore$ $-1 \leq x \leq 1$

Domain of x is $[-1, 1]$

Range of $y=f(x)$ is obtained from the rule $f(x)=x^2$

or, $y=x^2$, for all value of x in $[-1, 1] \rightarrow [-1, 0] \cup [0, 1]$

If $x=0, y=0, x=\pm 1, y=1$

$\therefore$ $y=0, 1$. Hence Range of $y=[0, 1]$

Ordered set of $f(x)=\{(x, y) \in R^2 \mid y=x^2\}$

which are in between $=\{[-1, 0] \cup [0, 1]\} \times \{0, 1\}$

Ex. 2. If $f: A \rightarrow B$ and let $f^{-1}; B \rightarrow A$. state two properties of the function f.

Ans. The function f must be both one—one and onto.

Ex. 3. Let the function $f=\{(x, y) \in R \mid y=x^2\}$

Find $f^{-1}(36), f^{-1}(-16), f^{-1}([-1, 1]), f[(-\infty, 0)], f^{-1}(9, 36)]$

$f^{-1}([-\infty, 0])=\{x \mid -\infty \leq f(x) \leq 0\}=\{x \mid -\infty < x^2 \leq 0\}$

$=\{0\}$ since no other number squared belongs to $[-\infty, 0]$

$f^{-1}([9, 36])=\{x \mid 9 \leq f(x) \leq 36\}=\{x \mid 9 \mid \leq x^2 \leq 36\}$

$=\{x \mid 3 \leq x \leq 6, -6 \leq x \leq -3\}=[3, 6] \cup [-6, -3]$

$f^{-1}(-16)=\{x \mid f(x)=-16\}=\{x \mid x^2=-16\}=\emptyset$ as there is no number whose squared is negative.

$f^{-1}(36)=\{x \mid f(x)=36\}=\{x \mid x^2=36\}$

$=\{x \mid x=\pm 6\}$

$=\{6, -6\}$ only two values of x.

$f(x)=x^2$

$f^{-1}([-1, 1]$

$=\{x \mid -1 \leq f(x) \leq 1\}=\{x \mid -1 \leq x^2 \leq 1\}$

$=\{x \mid x^2 \leq 1\}=\{x \mid x \mid \leq 1\}$

$=[-1, 1]$

Ex. 4. If $f=\{(x, y) \in R \times R \mid y=2x-3\}$. Is it a function? Has its inverse? Ans. yes.

Ex. 5. Let $A=\{x \mid \in R$ and $\mid x-1 \mid \leq 1\}$ and $B=\{y \mid \in R$ and $\mid y \mid \leq 1\}$.

Describe $A \times B$ without absolute value sign and sketch $A \times B$ in the Cartesian plane. [D. U. 1984]

Ans $A=\{x \mid x \in R$ and $-1 \leq x-1 \leq 1$ or, $0 \leq x \leq 2\}$

$B=\{y \mid y \in R$ and $\mid y \mid \leq 1$ or, $-1 \leq y \leq 1\}$

Draw lines $x=0$ and $x=2$

$y=-1$ and $y=1$

The four lines form a rectangle. Thus $A \times B$ represents the area of $PNML$ together with bordersides as Product set and sketch $A \times B$ is the shaded area.

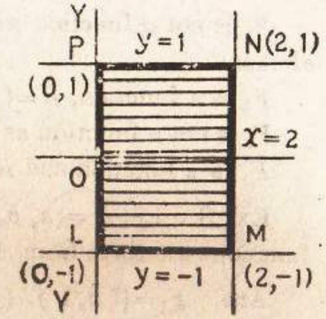


Fig. 9.

Ex. 6. Let $A=\{x \mid x \in R, -3 < x < 3\}$, Find the product set $A \times A$. such that $R=\{(x, y) \in A \times A \mid y=x^2\}$ Is it a function? The table of ordered pairs in $A \times A$ are obtained from $y=x^2$ is

x	0	1	-1	2	-2
y	0	1	1	4	4

Since x lies between 3 and -3, so these values are neglected.

Now the product set $A \times A$

$=[(0, 0), (1, 1), (-1, 1), (2, 4), (-2, 4)\}$

It is a function as no two pairs contain the same first element.

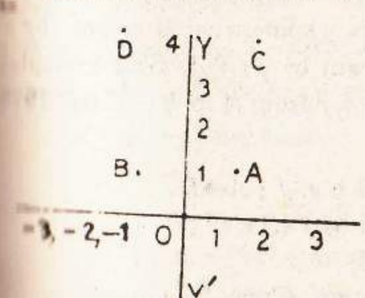


Fig. 10.

Ex. 7. Which are functions ?

Let $X=\{1, 2, 3\}$ and $Y=\{a, b, c, d\}$

If $F_1=\{(1, a), (2, a), (3, a)\}$, $F_2=\{(1, a), (1, b), (2, c), (3, d)\}$

$F_3=\{(1, a), (2, b), (3, d)\}$. $F_4=\{(1, b), (2, a)\}$

$F_5=\{(1, a), (2, b)\ (3, c)\}$ [R. U. 1978]

Solve F_1 is a function, Range, $R=\{a\}$

F_2 is not *a* function as $(1, a), (1, b)$ contain the same first element.

F_3 is a function, $R=\{a, b, d\}$

F_4 is not a function as element 3 of X is not present.

F_5 is a function and $R=\{a, b, c\}$

Ex. 8. Let $A=\{a, b, c\}$ and $B=\{1, 0\}$. How many different functions are there from A to B and what are they ? R. U. 1979

Ans. $F_1=\{(a, 1), (b, 1). (c, 1)\}$, $F_4(a, 0), (b, 0), (c, 0)\}$,
$F_2=\{(a, 1), (b, 1)\}, (c, 0)\}$, $F_5=\{(a, 1), (b, 0), (c, 0\}$
$F_4=\{a, 0), (b, 1), (c, 1)\}$, $F_6=\{(a, 1), (b, 0), (c, 0\}$
$F_3=\{(a, 1), (b, 0), (c, 1)\}$, $F_7=\{(a, 0), (b, 0), (c, 1)\}$

Ranges $R_1=\{1\}$, $R_2=\{0\}$, $R_3=\{1, 0\}$, $R_4=\{0, 1\}$, $R_5=\{0\ 1\}$,
$R_6=\{0, 1\}$, $R_7=\{0, 1\}$

From the ranges, F_1, F_2 are constant functions and the remaining are onto functions.

Ex. 9. Let A and B be two sets. Define what is meant by a function f from A to B. What is meant by $f(A)$ }. Give examples of two sets A and B and of a function f from A to B. R. U. 1975

Solve. Fuuctions :—

Let f be a functions of A into B *i, e.* $f: A\to B$.

The renge of : $A\to B$ is denoted by $f(A)$

If $A=\{a, b, c, d\}$, $B=\{1, 2, 3\}$

The range of $f(A)$ of the function f may be a subset of B *i e*, $f(A)\subset B$ or, $f(A)=B$. Ranges of A

are $f(A)=\{1\}$, $f(A)=\{1,2\}$, $f(A)=\{1, 2, 3\}$

Domain of f in each range $f(A)$ is $D=\{a, b, c, d\}$

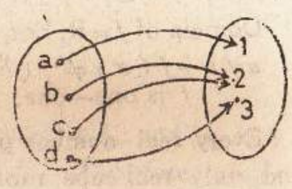
So all of them are functions f from A to B.

Ex. 10. Give an example in each of the following function f.

i) f is many—one into
(ii) f is many—one onto
(iii) f is one—one into
(iv) f is one—one onto
(v) f is neither one—one nor onto

Ans. R is the set of real numbers.

$f=\{(x, y)\in R\times R, x\in R, y\in R \mid y=x^2\}$
Range of $f=R^+$, Domain of $f=R$



f—image is the subset of its domain

i. e., $\{f(x)\}\subset R$, $x\in R$

f is a mapping of R into R

Again $a, b\in R$,

$a\neq b\to f(a)\neq f(b)$ i, e, $a^2\neq b^2$

Fig. 11

f is many—one into.

(ii) Let $f: A\to B$ where $A=\{a, b, c, d\}$. $B=\{1, 2, 3\}$

$\therefore$ $f(a)=1$, $f(b)=1$, $2=f(c)$, $f(d)=3$.

Since element a, b have the same image 1 of B, hence the mapping is many—one. Again there is no element in B which is not an image of A. Thus f is onto

$\therefore$ f is many—one onto

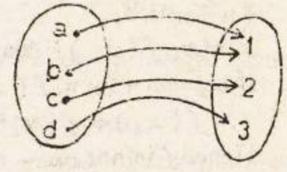


Fig. 12

(iii) $f\{(x, y)\in I\times I, x\in I, y\in I \mid f(x)=x^2\}$

I is set of $+ve$ integers.

Domain of $f=\{1, 2, 3, \ldots\ldots\}$,
Range of $f=\{1, 4, 9, \ldots\ldots\}$,

f image is the subset of its domain i. e, $\{ f(x) \subset I, x \in I \}$

Thus f is into mapping since all elements of the range are not the image of Domain such as 4 of range is not the f image of any element of the domain.

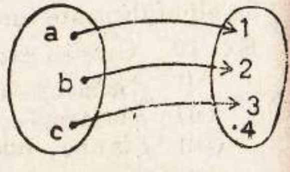


Fig. 13

For, $a, b \in I$, $a \neq b \rightarrow f(a) \neq f(b)$ $i, e, a^2 \neq b^2$

It is one—one.

Hence f is one—one and into

(iv) f $(x, y) \in R \times R, x \in R, y \in R \mid f(x) = x^3\}$

Domain of $f = R$, For, $a, b, \in R$

$a \neq b \rightarrow f(a) \neq f(b) \rightarrow a^3 \neq b^3$

f is one—one.

Every real number possesses one and only real cube root, all the elements in the range set R are the f-image of any element in the domain set R. Thus f is onto $f: R \rightarrow R$.

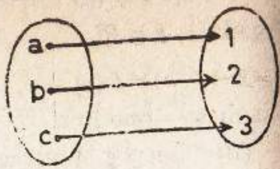


Fig. 14

$\therefore$ f is one—one and onto

(v) $f: \{(x, y) \in R \mid f(x) = \cos x\}$

$x_1, x_2, \in R$,

$x_1 \neq x_2, f(x_1) = \cos x_1, f(x_2) = \cos x_2$

If $x_2 = x_1 + 2n\pi, f(x_1 + n\pi) = \cos(x_1 + 2n\pi) = \cos x_1$

$\therefore f(x_1) = f(x_2)$

Hence f is not one—one

Again any element $x \in R$, say $f(x) = 2$

$f(x) = 2 \neq \cos x$, since $|\cos x| \leq 1$

Thus all the element of R in the range are not the f image of elements of the Domain so f is not onto

Hence f is neither one—one nor onto

Ex. 11. In each of the following cases, decide whether the given relation f is a function. If f is a function, determine its domain and range.

Which are onto and which are one—one

(i) $f = \{(x, y) \in R \times R \mid x^2 + y^2 = 1\}$ Ans. It is not a function.

(ii) $f = \{(x, y) \in R^2 : x - y = 2\}$ Ans 1—1 and onto

(iii) $f = (x, y) \in R^2 \mid y = 1\}$ Ans a function but not 1—1 and onto

(iv) $f = \{(x, y) \in A \times B \mid y = \sqrt{(1 - x^2)}$

$A = \{x \mid x$ real number $-1 \leq x \leq 1$ and B denotes all real numbers.

Ex. 12. Prove that the identity relation I_A is a function, hence called the identity function.

Proof: I_A is a relation by definition.

For each $x \in A$, $(x\ x) \in I_A$

So, domain of $I_A = A$. Finally if $(x, y) \in I_A$, and $(x, z) \in I_A$, then $x = y$ and $x = z$ by definition of I_A. Therefore, $y = z$ and I_A is a function.

Ex. 13. Show that f: $\{(x, y) \in R \times R \mid y = 2x\}$ is an 1—1 function but not onto function, R is the set integers.

Sol.: $x \in R$, $y \in R$, $(x, y) \in f$. Hence $D(f) = R$.

i. e. domain of f is R.

If $(x, y) \in f$, $(x, z) \in f$, then $y = 2x$, $z = 2x$, i. e, $y = z$.

Hence f is a function.

f is not onto R since $y = 3 \in R$ but $3 \neq 2x$ as no integer x such that $3 = 2x$.

Function of many (more than one) variables

Let $(x_1, x_2 \ldots\ldots, x_n)$ be an ordered n—tuple of real mumbers belonging to the set E_n and $y \in R$. If under a rule f, there exists a unique value of y corresponding to each n— tuple $(x_1, x_2 \cdot x_n)$ belonging to the set X_n, where $X_n \subset E_n$, then f is a function of n independend variables $x_1, x_2, \ldots\ldots, x_n$; we express this as,

$f: (x_1, x_2, \ldots \ldots, x_n) \rightarrow y$ or $y = f(x_1, x_2, \ldots, x_n)$.

The set X_n is the domain of f and the set y consisting of the corresponding values of y obtained by the rule f is the range of f.

Ex 13. Let $f:(x, y)\to\sqrt{x^2+y^2-1}$

or, $z=f(x, y)=\sqrt{x^2+y^2-1}$; $x \in \mathbb{R}$, $y \in \mathbb{R}$, $z \in \mathbb{R}$.

Ans. Since z is real, $z^2 \geqslant 0$ or $x^2+y^2-1 \geqslant 0$ or $x^2+y^2 \geqslant 1$.

Thus z is real for all points (x, y) lying on or outside the circle $x^2+y^2=1$ in the xy-plane. So

$D_f : \{(x, y) : x^2+y^2 \geqslant 1\}$

For $=(x, y) \in D_f$, $z \geqslant 0$.

Hence $R_f=$ range of $f=\{z : z \in R^+\}=[0, \infty)$.

Inverse function : If a function f given by $y=f(x)$ is such that for any two elements x_1 and x_2 belonging to the domain of f,

$f(x_1) \neq f(x_2)$ when $x_1 \neq x_2$,

Then there exists a unique function f^{-1} called the inverse of f with the properties

$f^{-1}(y)=x$ or $f^{-1}f(x)=x$

and $ff^{-1}(y)=f(x)=y \Rightarrow ff^{-1}(x)=x$ (interchanging the roles of x and y)

Note : For the function $y=f(x)$ to have its inverse f^{-1}, there exists a one-to-one correspodence between x and y under the rule f. We sometimes express this by saying that f is a **Single-valued function** of x.

A function f does not have an inverse, if there exist elements x_1, x_2 ... ... belonging to Df such that

$f(x_1)=f(x_2)=$ ... ...

even when $x_1 \neq x_2 \neq$ ... .

In such a case $f(x)$ is termed as a **many-valued function**.

It is clear from the definition that

$Df^{-1}=Rf$ and $Rf^{-1}=D_f$

Ex 14 Find the inverse of the function $y=f(x)=2x+3$,

Ans. $Df=$ Domain of $f=\mathbb{R}=(-\infty, \infty)$

$Rf=$ range of $f=\mathbb{R}=(-\infty, \infty)$

If $x_1, x_2 \in Df$ and $x_1 \neq x_2$, then

$2x_1+3 \neq 2x_2+3$ in $f(x_1) \neq f(x_2)$.

Hence f^{-1}, the inverse of f exists.

Solving for x, we get

$2x=y-3$ or $x=\dfrac{y-3}{2}$

Hence $f^{-1}(x)=\dfrac{x-3}{2}$.

It can be seen that

$$ff^{-1}(x)=f\left(\frac{x-3}{2}\right)=2\left(\frac{x-3}{2}\right)+3=x,$$

also $f^{-1}f(x)=f^{-1}(2x+3)=\dfrac{(2x+3)-3}{2}=x$.

$D_f^{-1}=Rf=(-\infty, \infty)$, $R_f^{-1}=D_f=(-\infty, \infty)$.

Ex. 15. Does f^{-1} exist for the function $y=f(x)=x^2, x \in \mathbb{R}$.

We have, $D_f=(-\infty, \infty)$.

Let us take two numbers x_1 and x_2, such that $x_2=-x_1$.

Then $x_1 \neq x_2$, but

$f(x_2)=x_2^2=(-x_1)^2=x_1^2=f(x_1)$.

Hence the function $y=x^2$, does not have its inverse.

Inverse of trigonometric functions : These one periodic circular functions. So their inverses can be defined only on their principal parts.

(i) The principal part of $y=f(x)=\sin x$ has domain $D_f=[-\frac{1}{2}\pi, \frac{1}{2}\pi]$ in $-\frac{1}{2}\pi \leqslant x \leqslant \frac{1}{2}\pi$

and range $R_f=[-1, 1]$ in $-1 \leqslant y \leqslant 1$

$\therefore f^{-1}(x)=\sin^{-1}x$

has domain $D_f^{-1}=R_f=[-1, 1]$ and $R_f^{-1}=[-\frac{1}{2}\pi, \frac{1}{2}\pi]$

(ii) The principal part of $y=f(x)=\cos x$

has domain $D_f=[0, \pi]$ and range $R_f=[-1, 1]$;

so $f^{-1}(x)=\cos^{-1}x$

has domain $Df^{-1}=Rf=[-1, 1]$ and range $Rf^{-1}=[0, \pi]$

(iii) The Principal part of $y=f(x)=\tan x$

has domain $D_f=(-\frac{1}{2}\pi, \frac{1}{2}\pi)$ i. e ; $-\frac{1}{2}\pi, < x < \frac{1}{2}\pi$

and range $Rf=(\infty, \infty)$.

Hence $f^{-1}(x)=\tan^{-1}x$

has domain $D_f^{-1}=R_f=(-\infty, \infty)$

and range $R_f^{-1}=D_f=(-\frac{1}{2}\pi, \frac{1}{2}\pi)$

Inverse of the exponential function or e^x ;

Since $e^{x_1} \neq e^{x_2}$ when $x_1 \neq x_2$

So e^x has a unique inverse.

Let $y = f(x) = e^x$

Then $x = \log_e y = \ln y$ ($\because$ log is written as l_n)

Hence $f^{-1}(x) =$ inverse of $(e^x) = l_{nx}$.

Similarly, we can show that

the inverse of $(\log_e x) = e^x$, $x > 0$.

$\therefore$ $\log(e^x) = x$; $e^{\log x} = x$

Note : (I) : If a function $y = f(x)$ is of the form

$$f(x) = \frac{f_1(x)}{f_2(x)}$$

Then $f(x)$ is not defined for values of x for which $f_2(x) = 0$.

Since $\frac{1}{0}$ is undefined.

Again, if for some values of x, $f_1(x) = 0$ as well as $f_2(x) = 0$, then $f(x) = \frac{0}{0}$ for those values of x.

Now $\frac{0}{0}$ has no definite value ; in such cases, we say that **$f(x) = \frac{0}{0}$ is indeterminate and so undefined.**

(II) : **Reflecetion of a point about the line y = x**

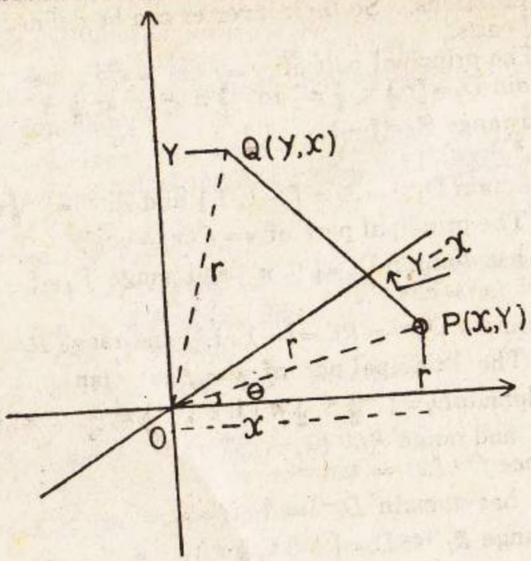


Fig 15

Let P(x, y) be any point on the xy plane. Then its reflection on the line $y = x$ is the point Q(y, x) which is evident from the geometry of the Fig. 15.

If $y = f(x)$, then $x = f^{-1}(y)$, whenever f^{-1} exists. Hence when (x, y) is a point on the graph of $y = f(x)$, (y, x) will be a point on the graph of $y = f^{-1}(x)$. Thus the graph of $y = f^{-1}(x)$ is obtained by reflecting the graph of $y = f(x)$ about the line $y = x$.

Ex. (i) $y = \frac{x^3 - a^3}{x - a}$. **If $x = a$, then**

$y = \frac{0}{0}$ which is indeterminate and undefined.

(ii) $y = (x - a)^2$ is defined for all values of x, while $y = \frac{1}{(x - a)^2}$ is undefined for $x = a$, since

then $y = \frac{1}{0}$

(iii) $y = \frac{\sin x}{x}$. If $x \neq 0$, then y has a definite value but when $x = 0$, then $y = 0/0$ which is indeterminate and so y is not defined at $x = 0$.

Similarly there are many more examples of these types.

1. 13. Classification of Functions (ফাংশানের বিভিন্ন রূপ)

Any given function is either **Algebraic or Transcendental**

(a) **Algebraic Function (বীজগণিতীয় ফাংশান)**

A function is said to be an algebraic functions which consists of a definite number of terms involving only the operations of addition, subtraction, multiplication, division, root extraction and raising to powers of one or more variables

Examples are

$y = x^3 + 3x^2 + x + 5$, $y = \frac{2x + 1}{3x - 1}$

$y = \sqrt{(x+3)}$ and $y = x + 5$

There are two types of Algebraical functions. Such as

(i) Polynomial functions

(ii) Rational functions

(a) **Polynomials (বহুপদী) :** A function of the type

$y=a_0+a_1x+a_2x^2+\ldots+a_nx^n$, where a_0, a_1, $a_2\cdots a_n$ are all constants and n is a positive integer, is called a polynomials in x of degree n.

(ii) **Rational (আনুপাতিক) functions**

A function which appears as a quotiont of two polynomials.

Such as $\dfrac{a_0+a_1x+\cdots+a_nx^n}{b_0+b_1x+\ldots+b_nx^m}$ is called a rational function

If P (x) and Q (x) be two polynomials, the ratio

$y=\dfrac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ have no common factor, is said to be a rational function.

Irrational (অমেয়) function is an algebrical function of x when it involves root extraction of terms involving x.

e. g. $\sqrt{(x+2)}$, $\sqrt{(x^2+2)}+3$ are the examples of irrational functions.

(b **Transcendental (তুরীয়) Fnnctions.**

Functions which are not algebraic are said to be transcendental. The following are transcendental functions.

(i) *Trigonometric functions*

(ii) *Inverse Trigonometric functions*

(iii) *Exponential functions*

(iv) *Logarithmic functions*

(i) Trigonometric Functions :—

$y=\sin x$, $y=\cos x$, $y=\tan x$, $y=\cot x$, $y=\sec x$ etc, are all Trigonometric functions of x.

(ii) **Inveres (বিপরীত) Trigonometric Functions.**

$y=\sin^{-1}x$, $y=\cos^{-1}x$, $y=\tan^{-1}x$, $y=\cot^{-1}x$, $y=\sec^{-1}x$ etc are all inverse trigonometrical functions of x, or, inverse circular functions of x.

(iii) **Exponential functions, (সূচকীয় ফাংশন)**

$y=2^x$, 10^x, a^x, x^x, e^x, etc are all exponential functions of x.

(iv) **Logarithmic Functions**

$y=\log_e x$, $\log_{10}x$, $\log_a x$, $\log\sin x$ etc. are all logarithmic functions of x.

(c) **Explicit (ব্যক্ত) Functions :** A function which is directly expressed in terms of the independent variable is called an explicit function e. g. $y=x\ \sin x$, $y=a\cos\theta$, $y=e^{ax}\cos bx$, etc are explicit functions.

(d) **Implicit (অব্যক্ত) Function :** A function which is not expressed directly in terms of the independent variable is called an implicit function e. g.

$$x^2+y^2=a^2,\ ax^2+2hxy+by^2=0.$$

$$3x^2y+2xy^2+4xy+5x+7y+3=0$$

In these examples, y is marked as explicit functions of x.

(e) **Periodic (পিরিয়ডযুক্ত) Functions**

If $f(x)=f(x+d)$ for all values of x for which the function is defined, $f(x)$ is called a periodic function with the period d, where d is minimum positive change in x for which $f(x)=f(x+d)$ holds.

Trigonometrical functions such as $\sin x$ and $\cos x$ are periodic with 2π as period, while $\tan x$ is periodic with period π

(f) **Monotone (মানের ক্রম অনুসারী) function :** A function $f(x)$ is said to be monotonic in a given interval $(a, b.)$ such that for any two values x_1 and x_2 $(x_1 < x_2)$ of the variable x in the interval either (i) $f(x_2) \leq f(x_1)$ or, (ii) $f(x_2) \geq f(x_1)$

If $f(x_2) \leq f(x_1)$, then the functions $f(x)$ is continually deceasing and so $f(x)$ is called a **monotonically decreasing function.**

Again if $f(x_2) \geq f(x_1)$, the function $f(x)$ is continually increasing In this case, $f(x)$, is called a monotonically increasing function.

(g) Odd or even Function : (অযুগ্ম ও যুগ্ম)

A function $f(x)$ is said to be odd if it changes sign with the change of sign of the variable x. That is $f(x)$ is odd, if

$$f(-x) = -f(x).$$

e. g. $f(x) = \sin x = -\sin(-x) = -f(-x)$ $\therefore f(-x) = -f(x)$

Again let $f(x) = \sin^5 x \cos^2 x$, then $f(-x) = \sin^5(-x) \cos^2(-x)$

$= -\sin^5 x \cos^2 x$ $\therefore$ $f(-x) = -f(x)$

So $\sin x$ and $\sin^3 x \cos^2 x$ are both odd function.

A function $f(x)$ is called an even function if it does not change sign with the change of sign of x. If $f(x)$ is even, then

$$f(-x) = f(x).$$

e. g., $f(x) = \cos x = \cos(-x) = f(-x)$ i. e., $f(x) = f(-x)$.

$\cos x$ is an even function.

(i) $f(x) = \sin^6 x \tan x$

$f(-x) = \sin^6(-x) \tan(-x) = \sin^6 x \tan x$

$\therefore$ $f(-x) = f(x)$

(ii) $f(x) = = ax^4 + bx^2 + c$

$f(-x) = a(-x)^4 + b(-x)^2 + c = ax^4 + bx^2 + c = f(x)$

$\therefore$ $f(-x) = f(x)$ These are the examples of even functions.

(i) Continuous and discontinuous (অবিচ্ছিন্ন ও বিচ্ছিন্ন) functions.

A function of x is said to be continuous in an interval, if it has a definite value for every value of x in the given interval e. g., $y = \sin x$; $y = x^2$

If the function is undefined for any value of the variable in the interval then the function is said to be discontinuous for that value of the variable in the given domain or interval.

e. g. $y = \dfrac{x^2 - a^2}{x - a}$ ....(1), $y = \dfrac{\sin x}{x}$ ....(2)

In the example (1), the function y is not defined for $x = a$ and from the example (2) we see that the function x is not defined for $x = 0$. Hence the functions are discontinuous at $x = a$ and $x = 0$ respectively.

For detail discussion see chapter on the continuity of a function.

1 14. Graphical Representation of Functions (ফাংশানের লেখচিত্র)

If y is a function of x i. e; $y = f(x)$ then we can represent the function graphically using cartesian co-ordinates. Generally we use the independent variable as a abscissa and dependent variable as ordinate. Thus each number pair (x, y), for $x \in D_f$, is a point on the xy plane and the collection of all points so obtained represents the graph of the function. From the graph of the function we can understand the nature of the function, that is how it changes with the change of the variable x.

1. 1'. Graphs of the function.

(a) Graph of $y=a^x$

Case 1 When $a>1$ and x is any positive or negative integer, the function $y=a^x$ is always positive.

For positive values of x, y increases indefinitely with the increasing values of x

When $x=0$, $y=a^0=1$, thus (0, 1 is on the curve.

For negative values of x, y will gradually decrease with the increasing values of $|x|$ and ultimately y tends to zero as $|x|$ tends to infinity. i. e. negative x axis is the assmptote of the curve $y=a^x$.

The curve is continuous.

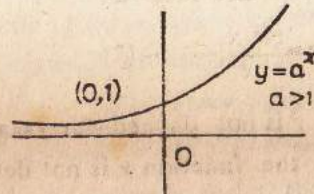


Fig 16

Now the graph of $y=a^x$, $a>1$ is shown in fig. 13

Case 11 When $0<a<1$, x is positive or, negative the function, $y=a^x$ is always positive.

We observe that for x, positive and $x\to\infty$ from the right hand sides of the origin. $y\to 0$, i. e., positive x axis is the asymptote of the curve.

When $x=0$, $y=a^0=1$, Thus (0, 1) lies on the curve.

For negative value of x as $a<1$, y will increase indefinitely with the increasing values of $|x|$ i. e. y will increase without any limi . Let $a=\frac{1}{2}<1$, $x=-2,-3\ldots-n$ then $=a^x=(\frac{1}{2})^{-2}=2^2$.

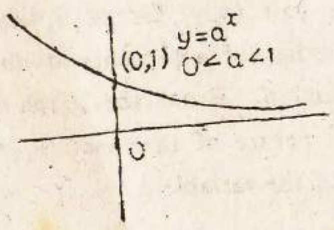


Fig 17

$y=2^3,\ldots\ldots y^n=2^n$. If $n\to\infty$ then $y\to\infty$. Hence the case The curve is **monotonically** decreasing.

Now we can draw the graph. see fig. 17.

(b) Draw the graph of $y=e^x$

In this function e is positive and greater than 2. So y is positive for all values of x positive or negative. The graph is similar to that at $y=e^x$ with $a=e>1$ (Fig 16)

(c) Graph of the function $y=\log_a x$, $x>0$, $a>0$.

we can write $y=\log_a x$ as $x=a^y$

Case I When $a>1$, x is positive for all values of y.

x is monotonically increasing with the increase of y, conversely y increases monotonically with $y\to\infty$ as $x\to\infty$.

When $y=0$, then $x=a^0=1$, the point (1, 0) is on the curve. For negative values of y, x is decreasing with the increasing values of $|y|$ i. e. $y\to\infty$ as $x\to 0$. i. e. negative y axis is the asymptote of the curve.

The curve is monotonically increasing from $-\infty$ to $+\infty$ when $a>1$, the graph of the function is shown below in fig. 18

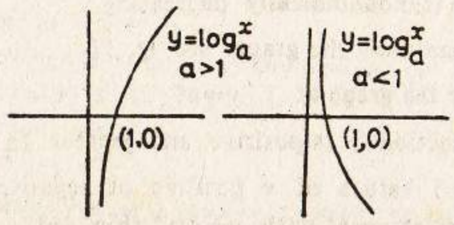
Case II. When $0<a<1$. the function $x=a^y$ is monotonically decreasing, and $x\to 0$ as $y\to\infty$

The curve passes through (1, 0)

For $a<1$, x is monotonically increasing when y is negative i. e. $x\to\infty$, is $y\to-\infty$

Foot Note : Asymptote—If a straight line cuts a eurve in two coincident points at an infinite distance from the origin and yet is not itself wholly at infinity is called an asymptote to the curve. See Chapter on Asymptotes of this book.

The curve is shown in fig. 19



Fig—18 Fig—19

(d) Graph of the Function

$y = \log_{10} x$

Form a table, plot the points in the graph papers and draw the graph through these points.

x	—.5	1	2	3	4	5	10	12
y	—.3	0	0.3	0.5	.6	.7	1	1·07

when $x \to 0$, then y tends to $-\infty$ i e. negative y axis will be the asymptote of the cu ve. The graph of the function can be easily drawn. The graph is similar to Fig. 18 with $a=10$.

(e) Draw the graph of $y = \log_e x$

Form a table of logarithm to the base e. The table of x and y is

x	.5	1	2	.3	4	5	etc.
y	6	0	.7	1.1	1.4	1.9	ete.

From the tables we see that $y \to -\infty$ when $x \to 0$ i e.. negative y axis will be the asymptote of the curve.

$x \to \infty$ with the increasing value of y, plotting the above points on a graph paper. The graph of the function $y = \log_{10} x$ is similar to Fig. 18 with $a=e>1$

(f) Graphs of $y = x^n$, n being any positive or negative integer.

Case 1. When n is positive even integer. Let $n=2m$

Then $y = x^n = x^{2m}$

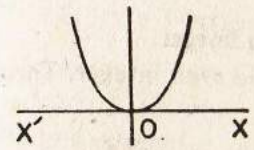


Fig 20

$y = x^{2m}$ is an even function of x. So its graph is symmetrical about the y=axis

Put $x=0$ then $y=0$.
$x=1$. then $y=1$.
$x=-1$, then $y=1$.

thus we see that the curve passes through (0, 0), (1, 1) and (—1, 1); y is always positive whether x is positive or negative.

So, there will be no branch of the curve in the 3rd or 4th quadrants.

Again $y = x^{2m} = (-x)^{2m}$

i. e. the curve is symmetrical about y axis.

y tends to infinity with the increasing values of $|x|$ with $-\infty < x < \infty$.

The graph of the function $y = x^n = x^{2m}$ is shown above (some) particular value of(n).

Case 11 When n is a positive odd integer.

Let $n = 2m+1$, then $y = x^n = x^{2m+1} = x. (x)^{2m}$

Put $x=0$. then $y=0$
$x=1$. then $y=1$
$x=-1$. then $y=-1$

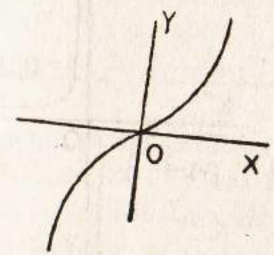


Fig 21

The graph passes through the points (0,0), (1,1), (—1, —1) Moreover y is positive or negative according as x is positive or negative. So there will be no branch of the curve in the 2nd and 4th quadrants.

The function $y = x^{2m+1}$ is odd and so it is symmetrical about the origin. That is, if (x,y) is a point on the graph $(-x,-y)$ is also a point on it. The graph of the function is shown above.

Case. III. When n is a negative even integer.

Let $n=-2m$, where m is any positive even integer. Then $y=x^n=x^{-2m}=1/x^{2m}$

The function is defined for all valuse of x except at $x=0$,

The function $y=1/x^{2m}$ even and so its graph is symmetrieal about y-axis.

when $x=1$, then $y=1$,

$x=-1$, then $y=1$,

The curve passes through (1,1) and (−1,1)

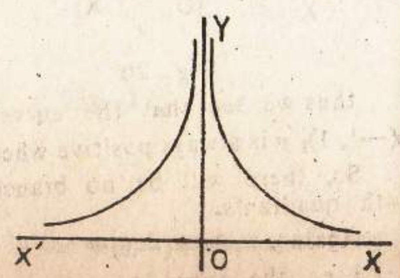


Fig. 22

y is always positive whether x is positive or negative as m is an even integer.

So, there is no part of the curve in the 3rd or 4th quadrants.

Again $y\to 0$ as $|x|\to\infty$ and $y\to\infty$ as $|x|\to 0$ Hence the axes of coordinates are asymptotes of the curve.

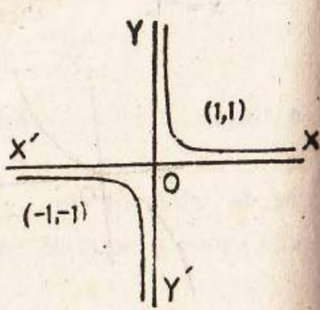
The graph of the function is shown in Fig. 22

Case. IV. When n is a negative odd integer.

Let $n=-(2m+1)$.

Then $y=x^n=\dfrac{1}{x^{2m+1}}$

The graph is not defined at $x=0$, Since $y=\dfrac{1}{x^{2m+1}}$ an odd function, the graph is symmetrieal about the origin, The graph passes through (1,1) and (−1,−1).



Fig—23

y is positive or negative according as x is positive or negative

So, there is no branch of the graph in the 2nd and 4th quadrants.

In $y=\dfrac{1}{x^{2m+1}}$ if x increases from 1 to $+\infty$, then y decreases from 1 to zero. Again if x decreases from 1 to zero, then y increases from 1 to ∞.

Similarly for negative value of x, y decreases from $-\infty$ to zero (numerically) for x lying between 0 and $-\infty$ also x decreases from $-\infty$ to zero (numerically for y lying between 0 and ∞,)

The graph of the function is shown in fig. 23,

(g) Draw the graphs of $y=x^n$ when n is fractional

A few cases are given below.

Let $n=2/3,\ 1/3$, etc.

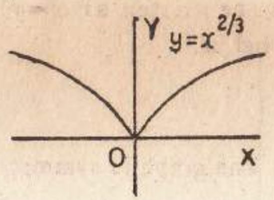


Fig. 24

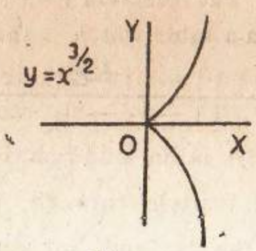


Fig. 25

[i] For the function $y=x^{2/3}=(x^{1/3})^2$ an even function.

we can form the table

x	0	1	−1	$2\sqrt{2}$	$-2\sqrt{2}$	ect.
y	0	1	1	2	2	etc.

The graph passes through (0,0), (1,1),(−1,1) etc. No branch of the graph lies in 3rd and 4th quadrants. With the increase of $|x|$, y will also tends to $+\infty$,

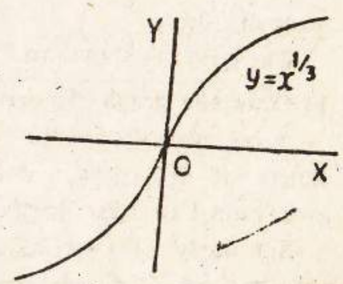


Fig. 26

The graph is shown in Fig 24

(ii) **For the function** $y=x^{3/2}$

we can form the following table.

x	0	1	2	3	4	etc.
y	0	1	$2\sqrt{2}$	$3\sqrt{3}$	8	etc.

$y=x^{3/2}=\sqrt{x^3}$, The function is defined for $x \geq 0$ with $y \geq 0$. So the graph lies entirely in the first quadrants and it passes through the points (0,0), (,1) $(2, 2\sqrt{2})$, $(3, 3\sqrt{3})$, (4,8).

The graph is the part above x axis in fig. 25

The graph of $y^2=x^3$ or $y=\pm x^{3/2}$ consists of both the parts above and below the x axis in fig. 25. One part is the reflection of the other about the x—axis.

(iii) **The function** $y=x^{1/3}$ can be written as $x=y^3$. Now we can form a table which is shown below.

y	0	1	-1	-2	etc.
x	0	1	-1	-8	etc.

$y=x^{1/3}$ is an odd function. The graph is symmetrical about the orgin it passes through, (0,0),(1,1),(—1,—1) etc. The graph lies in the 1st and 3rd quadrants, no part of it lies in the 2nd and 4th quadrants. The graph ex ends from—∞ to $+\infty$ through (0,0),

The curve is shown in Fig 26.

[h] Draw the graph of $y=\sin x$

y increases from 0 to 1 for values of $0 \leq x \leq \frac{1}{2}\pi$, y decreases from 1 to 0 for $\frac{1}{2}\pi \leq x \leq \pi$

Similarly for $-\pi/2 \leq x \leq 0$ and $-\pi \leq x \leq -\frac{1}{2}\pi$, y decreases.

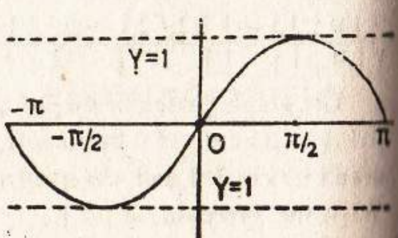


Fig. 27

from 0 to—1 and then increases from—1 to 0 again. The graph is shown in Fig. 27

(i) **Draw the graph of** $y=\sin^{-1}x$

Let us consider the equation $x=\sin y$

Since $\sin y$ takes all values from—1 to 1 for real values of y, the inverse function $y=\sin^{-1}x$ is defined over the principal part of the graph $x=\sin y$ given by

$y \in [-\frac{1}{2}\pi, \frac{1}{2}\pi]$

with $x \in [-1, 1]$

The graph of $y=\sin^{-1}x$ passes through points $(-1, -\frac{1}{2}\pi)$ (0, 0) and $(1, \frac{1}{2})$

The graph of $y=\sin^{-1}x$ can also be obtained by reflecting the graph of $y=\sin x$ drawn for $-\frac{1}{2}\pi \leq x \leq \frac{1}{2}\pi$ about the line $y=x$ [Fig 28]

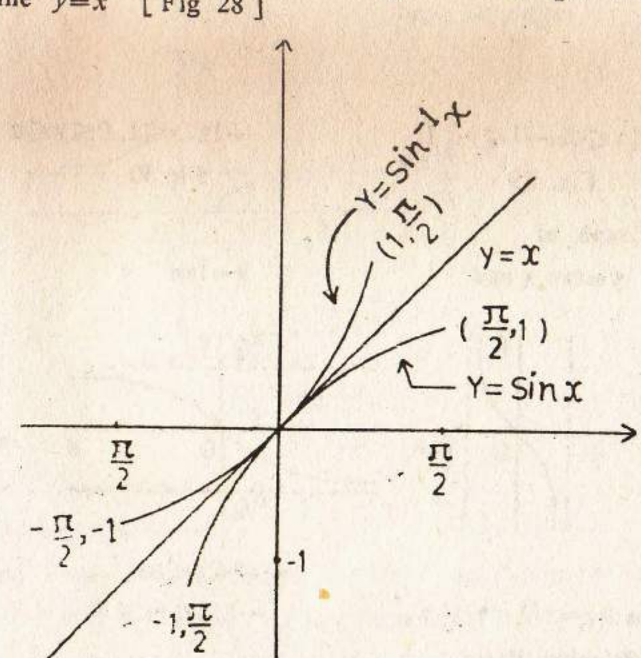
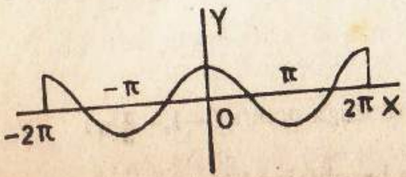


Fig 28

((j) **Graphs of**

$y=\cos x$ and $y=\cos^{-1} x$

$y=\cos x$

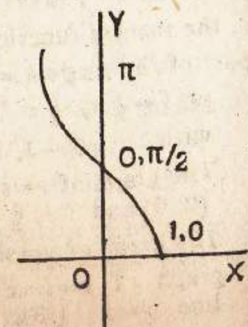


$-\infty\leqslant x\leqslant\infty, -1\leqslant y\leqslant 1.$

Fig. 29

$(-1, \pi)$

$y=\cos^{-1} x$

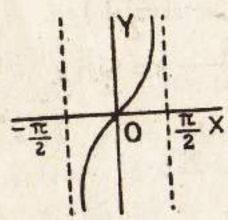


$-1\leq x\leqslant 1, 0\leq y\leqslant\pi$

Fig 30

(k) Graph of

$y=\tan x$ and

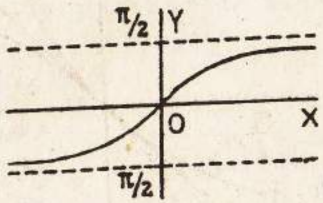


$y=\tan x$; $-\pi/2<x<\frac{1}{2}\pi$;

(Principal Part)

Fig. 31

$y=\tan^{-1} x$



$-\infty<x<\infty$

$-\pi/2<y<\pi/2$

Fig. 32

(l) Graphs of $y=\sec x$ and $y=\sec^{-1} x$.

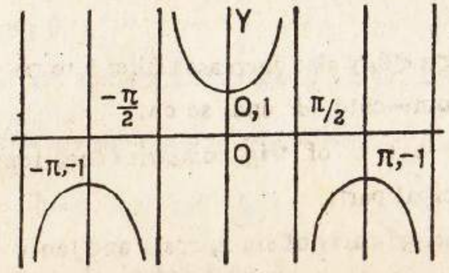


Fig 33

$y=\sec x;$ $-\infty<x<\infty$; $-\infty<y\leqslant-1$

or, $1\leqslant y<\infty$

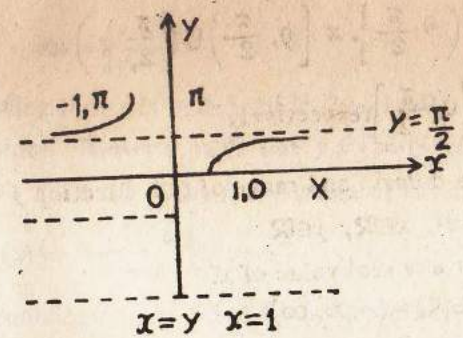


Fig. 34

$y=\sec^{-1} x,$

$-\infty<x\leqslant-1$ or $1\leqslant x<\infty$

and $0\leqslant y\leqslant\pi$ with $y\neq\frac{1}{2}\pi$

$y=\sec^{-1} x$; y increases from 0 to $\pi/2$ as x increases from 1 to ∞, y increases from $\pi/2$ to π as x increases from $-\infty$ to -1.

$y=\sec^{-1} x$ passes through $(1, 0)$ ond $(-1, \pi)$.

$y=\sec x$ the function is discontinuous at $x=\pi/2, 3\pi/2, \ldots \ldots$

$-\pi/2, -3\pi/2 \ldots\ldots$; y increases from 1 to ∞ for the values x in $0<x<\pi/2$.

For $-\pi/2<x<0$; y also increases from 1 to ∞. For $\frac{1}{2}\pi \leq x \leq \pi$ y increases from $-\infty$ to -1 and so on.

Note : The inverse of trigonometric functions one defined over their principal parts.

For the principal parts of $\sin x$, $\cos x$ and $\tan x$ x lies respectively in the intervals

$[-\frac{1}{2}\pi, \frac{1}{2}\pi]$, $[0, \pi]$ and $\left(-\frac{1}{2}\pi, \frac{1}{2}\pi\right)$

The principal parts of cosec$x$, sec$x$ and cot$x$ are defined

for $x \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$, $x \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$

and $x \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ respectively.

Ex. 1 : Find the domain and range of the function f where

$y=f(x)=1-x^2$, $x \in \mathbb{R}$, $y \in \mathbb{R}$.

Ans. y is real for any real value of x.

$D_f=\{x : x \in \mathbb{R}\}=(-\infty, \infty)$.

Since $x^2 \geq 0$, therefore the maximum value of y is 1. **Hence**

$R_f=$ range of $f=\{y : y \leq 1\}=(-\infty, 1]$.

Ex 2. Find the domain and range of f where

$y=\{f(x)=\sqrt{1-x^2}$; $x \in \mathbb{R}$, $y \in \mathbb{R}.\}$

Since $y \in \mathbb{R}$, we have $y^2 \geq 0$

or, $1-x^2 \geq 0$ or $x^2 \geq 1$ or $|x|^2 \leq 1$

or $|x| \leq 1$ $\Rightarrow -1 \leq x \leq 1$

$\therefore$ $D_f=[-1, 1]$.

For $x \in D_f$, $0 \leq y \leq 1$ and so

$R_f=[0, 1]$

Ex. 3. Find the domain and range of f given by

$y=f(x)=\dfrac{x-1}{2x-3}$ where $x \in \mathbb{R}$, $y \in \mathbb{R}$

Ans. If $2x-3=0$ or $x=3/2$

then $y=\dfrac{3/2}{0}$ is not defined.

$\therefore$ $D_f=x : \{x \in \mathbb{R}$ but $x \neq 3/2)=(-\infty, 3/2) \cup (3/2, \infty)$

Again solving for x,

$y(2x-3)=x-1$

$\Rightarrow x=\dfrac{3y-1}{2y-1}$

Showing that x is undefined if $2y-1=0$ or $y=\frac{1}{2}$

Hence $R_f=\{y : y \in \mathbb{R}$ but $y \neq \frac{1}{2}\}=(-\infty, \frac{1}{2}) \cup (\frac{1}{2}, \infty)$.

• **Ex. 4** Find the domain and ranga of the function

$f(x)=\dfrac{x^2+1}{x^2-5x+6}$

The denominator $x^2-5x+6=(x-2)(x-3)$

which is zero when $x=2$ or $x=3$

Therefore $f(x)$ is not defined for $x=2$ and $x=3$.

Hence $D_f=\mathbb{R}-\{2, 3\}=(-\infty, 2) \cup (2, 3) \cup (3, \infty)$.

Let $y=f(x)=\dfrac{x^2+1}{x^2-5x+6}$

$\Rightarrow$ $(y-1)x^2-5y.x+(6y-1)=0$.

Treating this as a quadratic equation in x for a given y, we [illegible] real solutions for x, if the discriminant

$(-5y)^2-4(y-1)(6y-1)\geq 0$

or, $y^2+28y-4\geq 0$

Now the roots of $y^2-28y-4=0$

are given by $y=\dfrac{-28\pm\sqrt{28^2+16}}{2}$

or $y=-14\pm 10\sqrt{2}$.

Hence $y^2+28y-4\geq 0$

When $y\leqslant -14-10\sqrt{2}$ or $y\geq -14+10\sqrt{2}$.

$\therefore$ $R_f=(-\infty,-14-10\sqrt{2}\,]\cup[\,-14+10\sqrt{2},\infty)$.

Ex 5 Find the domain and range of the function

$$f(x)=\begin{cases}\dfrac{|x|^2}{e}, & -1<x<0\\ =x^2, & 0\leqslant x<2\end{cases}$$ (D. U, 1987)

If $-1<x<0$, x is negative, $|x|=-x$

$\therefore$ $f(x)=y=-e^{-x}/_2=e^{-x}/_2$

when $x=0$, then $y=1$, and $x=-1$, then $y=1/\sqrt{e}$

Domain is the subset of D_f i. e. $(-1,0)\subset D_f$ ...(1)

and range, $(1,1/\sqrt{e})\subset R_f$... ...(2)

For $y=x^2$, $0\leqslant x<2$

x	0	1	2
y	0	1	4

Domain, $[0,2)\subset D_f$, range, $[0,4)\subset R_f$.

Hence the domain of $f(x)$, $D_f=[-1,0)\cup[0,2)=[-1,2)$

Range of $f(x)$, $R_f=(1,1/\sqrt{e})\cup[0,4)=[0,4)$

Ex 6 Find the domain and range of $f(x)=\dfrac{x^2-9}{x-3}$

Draw the graph of the function.

[f (x) এর চারণ ও ব্যপ্তি নির্ণয় কর এবং লেখ চিত্রটি অঙ্কন কর।]

Let $f(x)=\dfrac{x^2-9}{x-3}=x+3$

(If $x-3\neq 0$, or, $x\neq 3$)

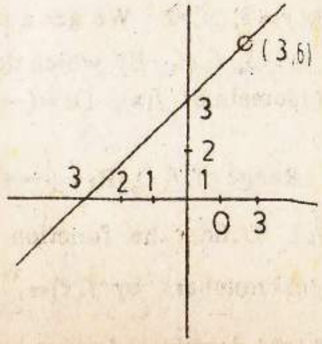


Fig 35

The function is undefined for $x=3$, i. e, the point $(3,6)$ is missing in the graph.

Domain, $D_f=(-\infty,3)\cup(3,\infty)$

Range, $R_f=(-\infty,6)\cup(6,\infty)=\mathbb{R}-\{6\}$

Ex 7 Find the domain and range of

$$f(x)=\begin{cases}-4; & x<-2\\ -1; & -2\leq x\leq 2\\ 3; & x>2\end{cases}$$

Draw the graph.

when $y=-4$, $x<-2$

$\dfrac{|\,x\,|-3\,|-2}{|\,y\,|-1\,|-4}$

The straight line AC which does not contain $(-2,-4)$

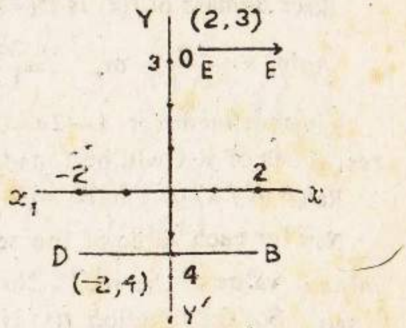


Fig 36

For. $y=-1$, $-2\leq x\leq 2$, BD is the straight line $y=-1$ including the point B(2,−1), D(−2,−1)

For $y=3$, $x>2$ We get a straight line,

$y=3$, *i. e.*, EF which doess not contain the point E (2,3).

The domain of $f(x)$, $D_f=(-\infty,-2)\cup[-2,2]\cup(2,\infty)$
$=(-\infty,\infty)$

Range of $f(x)$, $R_f=\{-4\}\cup\{-1\}\cup\{3\}=\{-4,-1,3\}$

Ex. 8 Define the function f : A→B where A, B⊆ℝ, the set of real numbers by $f(x)=\frac{x-3}{2x+1}$

Find the domain and range of $f(x)$. Show that f is one-one and onto. Find a formula for f^{-1} (f : A→B) A,B⊆ℝ (বাস্তব সংখ্যার জন্য একটি ফাংশান ইহার চারণ স্থল ও ব্যপ্তি নির্ণয় কর। দেখাও যে, ফাংশানটি এক-এক এবং on to f^{-1} এর জন্য একটি সূত্র লিখ।]

(D. U. 1988)

Let $y=\frac{x-3}{2x+1}$ for $2x+1=0$ or, $x=-\frac{1}{2}$, there is no value of y, so y is undefined for $x=-\frac{1}{2}$.

Hence domain of $f(x)$ is $D_f=(-\infty,\infty)-\{-\frac{1}{2}\}=\mathbb{R}-\{-\frac{1}{2}\}$

Again, $y=\frac{x-3}{2x+1}$ or, $x=\frac{3+y}{1-2y}$ or, $f^{-1}: x=\frac{3+y}{1-2y}$

x is undefined for $1-2y=0$ or $y=\frac{1}{2}$. So x is true for all real values of y, x will be found.

Range of $f(x)$ or y is $R_f=(-\infty,\infty)-\{\frac{1}{2}\}=\mathbb{R}\neq\{\frac{1}{2}\}$

Now for each value of the set A has a corresponding one and only one value of the set B. No member is unrepresented present in sets. So, the function $f(x)$ is one-one and onto. (এক-এক এবং সার্বিক) The inverse formula, f^{-1}: $x=\frac{3+y}{1-2y}$

Ex 9. (a) Find the domain and range of $f(x)=|x|+|x+1|$ and draw the graph. [$f(x)$ এর ফাংশানের চারণগুলি ও ব্যপ্তি বা বিস্তার নির্ণয় কর। চিত্রটি অংকন কর।]

(b) Show that $f(x)=|x|+|x+1|$ may be expressed as

$$f(x)=\begin{cases}-2x-1\,; & x<0\\ 1\,; & -1<x\leq 0\\ 2x+1\,; & x\geq 0\end{cases}$$

Sol. (a) For $x<0$, $|x|=-x$, $|x+1|=-x-1$ Then

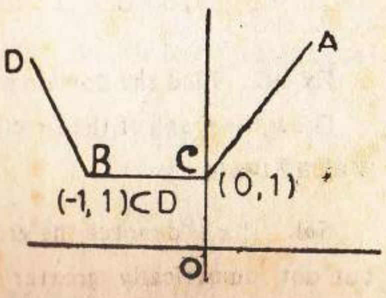


Fig 38

$y=|x|+|x+1|=-x-x-1=-2x-1$

Domain is the subset of D_f

x	...	-2	-1
y	...	3	1

(i) ∴ $[-\infty,-1)\subset D_f$ and $(1,\infty)\subset R_f$, range is the subset of R_f. The graph is CD. C (0, 1),

For $-1<x\leq 0$; $x>-1$; $x+1>0$ and $|x|=-x$

Then $y=|x|+|x+1|=-x+x+1=1$.

(2) ∴ Domain $[-1,0)\subset D_f$, $\{1\}\subset R_f$

x	-1	$\frac{1}{3}$	$\frac{1}{2}$	0
y	1	1	1	1

The graph $y=1$ is a straight line CB. ; C (0, 1) B (−1, 1)

For $x\geq 0$, $|x+1|\geq 1$ ∴ $|x+1|=x+1$, $|x|=x$

Then $y=|x|+x+1|=x+x+1=2x+1$

(3) Domain $[0,\infty)\subset D_f$,
Range $[1,\infty)\subset R_f$

x	0	1	2	... ...
y	1	3	5	... ...

The graph is the straight line

$y=2x+1$ represents a part B A. ; B$(-1, 1)$

From (1), (2) and (3)

$D_f=(-\infty,-1)\cup[-1,0)\cup[0,\infty]=(-\infty,\infty)$

$R_f=[1,\infty)\cup\{1\}\cup[1,\infty)=[1,\infty)$

Ex. 10. Find the domain and range of $f(x)=x-[\,|x|\,]$

Draw the graph of the function [$f(x)$ এর চারণস্থল ব্যাপ্তি নির্ণয় কর লেখচিত্রটি দেখাও।]

Sol. $|x|$ denotes the greatest integer positive or negative but not numerically greater than x.

If $0<x<1, |x|=x, [\,|x|\,]=0.$

$\therefore\ y=x-0=x.$

when $x=1$, then $y=0$.

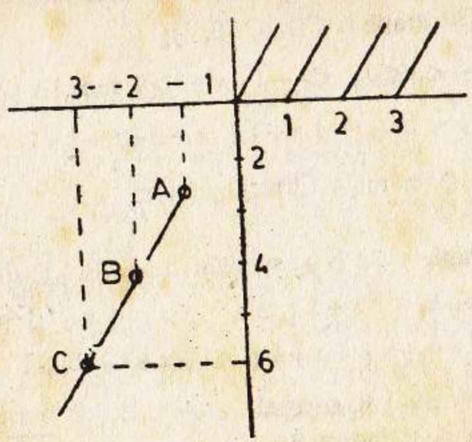


Fig 39

x	0	$0<x<1$	1	$1<x<2$	2	$2<x<3$	3	...
y	0	x	0	$x-1$	0	$x-2$	0	...

x	$-1<x<0$	$-2<x<-1$	$-3<x<-2$	...
y	$x-1$	$x-2$	$x-3$	...

The graph is shown where A$(-1,-2)$, B$(-2,-4)$, C$(-3,-6)$ etc are excluded.

where $[\,|x|\,]$ is the largest integer in $|x|$ less than or equal to $|x|$.

Ans. Clearly, $f(x)$ is defined for all real values of x. Hence

$D_f=\mathbb{R}=(-\infty,\infty)$.

$\cdots\cup(-5,-4]\cup(-3,-2]\cup(-1,0]\subset R_f$

Alternative Method

Consider the values of $f(x)$ for $x\geq 0$.

We have $f(0)=0$.

When $0<x<1$, $[\,|x|\,]=0$ and

$f(x)=x-0=x$

$\Rightarrow 0<f(x)<1$ when $0<x<1$.

For $1\leqslant x<2$ $[\,|x|\,]=1$ with

$f(x)=x-1$

$\Rightarrow 0\leqslant f(x)<1$,

Since $f(1)=1-1=0$ and $f(2^-)=2^--1=1^-$.

In general, when N is zero a positive integer

and $N\leqslant x<N+1$.

$f(x)=x-N$; $f(N)=N-N=0$,

$\lim\limits_{x\to(N+1)^-} f(x)=N+1-N=1$;

So $0\leqslant f(x)<1$ for $N\leqslant x<N+1$.

Hence considering $x \geq 0$, we find that

$$[0, 1] \subset R_f$$

Now consider that $x<0$.

when $-1<x<0$, $[|x|]=0$.

$$f(x)=x-0=x \Rightarrow -1<f(x)<0;$$

When $-2<x\leqslant -1$, $[|x|]=1$,

$$f(x)=x-1 \Rightarrow -3<f(x)\leqslant -2;$$

In general, if $-(N+1)<x\leqslant -N$, where N is zero or a positive integer, $[|x|]=N$,

$$f(x)=x-N \Rightarrow -(2N+1)<f(x)\leqslant -2N$$

Hence considering $x\leqslant 0$, we see that

$$\dots \dots \cup (-5,-4] \cup (-3,-2] \cup (-1,0] \subset R_f$$

Combining the results for $x\geq 0$ and $x\leqslant 0$, we have,

$R_f=\{y : -(2N+1)<y\leqslant -N\} \cup [0, 1)$

Where N is zero or a positive integer.

Ex 11. Sketch the graph of the piecewise defined function

$$f(x)=\begin{cases} -1, & x<0 \\ 0, & x=0 \\ x+2, & x>0 \end{cases}$$ (D. H. 1987)

Let $y=f(x)=-1$, when $x<0$

x	...-2	-1	0
y	...-1	-1	-1

For $x<0$, i. e. domain of x is $(-\infty, 0)$; $y=-1$, is a infinite straight line AB (part is shown).

For $x=0$, $y=f(x)=0$.

The origin of the axis is O (0, 0) a point.

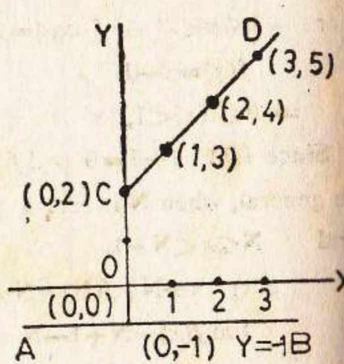


Fig 40

For $x>0$, $y=f(x)=x+2$

The domain of x is $(0, \infty)$ and the range is $(2, \infty)$

x	0	1	2	3... ...
y	2	3	4	5... ...

Therefore we get a straight line (infinite, part is shown) CD, where C(0, 2) is excluded.

Ex. 12. If R be the set of real numbers and the function $f: R\rightarrow R$ given by $f(x)=x^2-x-2$, find $f([1,2])$, $f^{-1}([-2,0])$ and $f^{-1}(\{0\})$ D. U. 1983

Sol. $f(x)=x^2-x-2$ or, $y=x^2-x-2=(x-\frac{1}{2})^2-9/4$

or, $y+9/4=(x-\frac{1}{2})^2$

It is a parabola whose vertex is at $(\frac{1}{2},-9/4)$ [$y+9/4=0$ or, $b=-9/4$, and $x-\frac{1}{2}=0$ or, $x=\frac{1}{2}$]

The graph is shown below

$f([1,2])=\{f(x) \mid 1\leqslant x\leqslant 2$, $f(x)=x^2-x-2\}$

Here domain of f is R.

for each $x\in$ R.

$f(1)=1^2-1-2=-2$

$f(2)=2^2-2-2=0$

$f([1,2])=\{f(x) \mid 1\leqslant x\leqslant 2\}$
$=[-2, 0]$

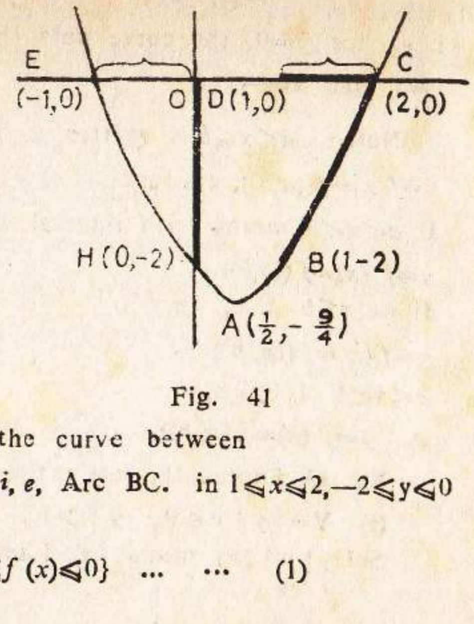


Fig. 41

i. e. the portion of the curve between B $(1,-2)$ to C $(2, 0)$, i, e, Arc BC. in $1\leqslant x\leqslant 2, -2\leqslant y\leqslant 0$

To find $f^{-1}([-2, 0])$;

$$f^{-1}([-2, 0])=\{x \mid -2\leqslant f(x)\leqslant 0\} \quad \dots \quad \dots \quad (1)$$

Here $f(x)=-2=x^2-x-2$ or, $x(x-1)=0$ or, $x=1, 0$

Again $f(0)=0=x^2-x-2$ or, $(x-2)(x+1)=0$ or, $x=2,-1$

[The points on the graph are $(0,-2)$, $(1,-2)$, $(2,0)$, $(-1, 0)$ we get the arc between E $(-1, 0)$ and H $(0,-2)$ arc between B $(1,-2)$ to C $(2, 0)$ *i. e.* arc EH and arc BC This can be symbolically expressed below]

From the graph we see that

$f^{-1}([-2,0])=\{x \mid -2 \leqslant f(x) \leqslant 0\}$

$=\{x \mid -1 \leqslant x \leqslant 0$ and $1 \leqslant x \leqslant 2\}$

$=[-1, 0] \cup [1, 2]$

Sol : $f^{-1}(\{0\})=\{x \mid f(x)=0\}$

$=\{x \mid x^2-x-2=0\}=\{x \mid (x-2)(x+1)=0\}$

$=\{x \mid x=1,-2$ only two points $\}$

$=\{1,-2\}$

i. e. for $y=0$, the curve cuts the x—axis at two points $x=1$, and $x=-2$

Note : $a \leqslant x \leqslant b$ is written as $[a,b]$ called interval: $y=f(x)=f([a,b])$, y is function of x in the interval $x=a$ and $x=b$

If $a<x<b$ means open interval of x i. e. (a, b)

$y=f(x)=f((a,b))$.

If $a<x\leqq b$ i. e, $(a, b]$

$y=f(x)=f((a, b])$

$a \leqslant x<b$ i, e, $[a, b)$

$\therefore$ $y=f(x)=f([a, b))$,

Ex. 13. Express the sets as the union of interval

(i) $Y=\{y \mid y \in R, \mid y \mid >1\}$.

Sol. ; $\mid y \mid >1$ means, $y>1$ and $-y>1$ or, $y<-1$

Graph of $y>1$ is drawn. It is the Y-axis A to infinity.

This line is in the open interval 1 and ∞ i. e., $(1, \infty)$

Again $-y>1$ or, $y<-1$ i. e., the negative y—axis from -1 to $-\infty$ i. e., in $(-\infty,-1)$

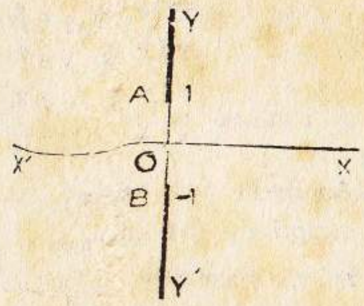


Fig. 42

Therefore, $Y=(-\infty,-1) \cup (1, \infty)$

= The two black lines AY and BY'.

$y>1$ means all valus of y bigger than 1 *i. e.*, upto $+\infty$ *i. e.* all values of y between 1 and $+\infty$. It is an open interval and is $(1, \infty)$ ... ... (1)

Again—$y>1$ or, $y<-1$ means all values of y, less than -1. These values are between $-\infty$ to -1. It is an open interval and is $(-\infty, -1)$...... (2) The graph of (1) and (2) is the two portions of y—axis expressed by two intervals $(-\infty, -1)$, $(1, \infty)$. These are symbolically expressed as the union of two intervals. From the graph also the result may be obtained.

Ex. 14. Express the set $A=\{x \mid x \in R, \mid x \mid \leqslant 1\}$ as an interval and the set $B=\{y \mid y \in R, \mid y-2 \mid >1\}$ as the union of two intervals. Indicate the set $A \times B$ in the cartesian plane.

D. U. 1984.

Sol. : $A=\{x \mid x \in R, \mid x \mid \leqslant 1\}$

$=\{x \mid x \in R, -1 \leqslant x \leqslant 1\}$

$=[-1, 1]$

$B=\{y \mid y \in R \mid y-2 \mid >1\}$

$= \{y \mid y \in R,\ y-2>1 \text{ and} -(y-2)>1\}$
$= \{y \mid y \in R,\ y>3 \text{ and } y<1\}$
$= \{y \mid y \in R,\ 3<y<\infty \text{ and } -\infty<y<1\}$
$= (-\infty, 1) \cup (3, \infty)$

$A \times B$ = Product set comprising the area of the infinite planes by $x=-1$ and $x=1$, excluding the area LMNP, but including sides LM, PN; excluding sides LP, MN.

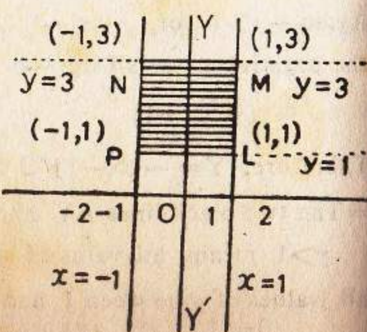


Fig. 43

EXERCISE—1

1, If $f(x)=3x^2-\dfrac{1}{x}+4x-7$

find $f(1), f(-2), f(h)$.

Ans. $-1, -5/2, 3h^2-1/h+4h-7$

2. Find the limit of the sequence (ধারার সীমা) $\{x_n\}$ For x_n as defined in each case.

(i) $2^{-1/\sqrt{n}}$ (ii) $\dfrac{1}{n}\sin\dfrac{n\pi}{2}$

(iii) $\left(100+\dfrac{1}{n}\right)^2\left(1+\dfrac{n-1}{n^2}\right)^{100}$ (iv) $\dfrac{1+(-1)^n}{n}$

Ans. (i) 1 (ii) 0 (iii) 10^4 (iv) 0.

3. Determine the domains (চারণক্ষেত্র) of definition of the following functions.

1. (i) $f(x)=\dfrac{|x|}{x}$ Ans. all values of x, except $x=0$

(ii) $f(x)=\dfrac{\sqrt{x}}{x+2}$ Ans. all +ve real values of x including zero

(iii) $f(x)=\dfrac{x^2+x}{x^2-x}=\dfrac{x+1}{x-1}$ Ans. all real values of x, except $x=0$ and 1.

(iv) $y^2=(x-x)^3$ Ans. $-\infty<x\leq-1$ and $0\leq x\leq 1$

(v) $y=\log\dfrac{2+x}{2-x}$ Ans. $-2<x<2$

(vi) $y=\cos^{-1}\dfrac{2x}{1+x}$ Ans. $-\dfrac{1}{3}\leq x\leq 1.$

(vii) $y\sqrt{(x^2-1)}=1$ Ans. $-\infty<x<-1,\ 1<x<\infty$

(viii) $y^2=\sin 2x$ Ans. $k\pi\leq x<k\pi+\frac{1}{2}\pi\ (k=0,\pm1,\pm2\ldots)$

4. Find the odd and even functions of the following functions

(i) $f(x)=\frac{1}{2}(2^x+2^{-x})$ Ans. even

(ii) $f(x)=\sqrt{(1+5x+7x^2)}-\sqrt{(1-5x+7x^2)}$ Ans. odd

(iii) $f(x)=\log\dfrac{2+x}{2-x}$ Ans. odd

(iv) $f(x)=\log\{x+\sqrt{(1+x^2)}\}$ Ans. odd

5. Show that $\dfrac{\sin x}{1+\cos x}$ is not defined for $x=\pi$

6. Show that $x^{5/2}$ is not defined for all negative values of x.

7. Show that the function.

$y=\sqrt{\{(x-1)(x-2)\}}$ is non existent for every value of x lying between 1 and 2. ($1\leq x\leq 2$-এর জন্য y বিশ্লেষণযোগ্য নয়)।

8. Show that $\sqrt{(x^3-3x^2+2x)}$ is defined for any value of x lying between 0 and 1 but undefined for any value of x, in $1<x<2$

9. Is $\cos^{-1} x$ defined for $3 \leqq x \leqq 4$ Ans. No.

10. Prove that $\frac{x^2+4x-1}{2x^2-3x-9}$ is not defined for $x=3$.

EXERCISE—1 (A)

(1) Express the sets as an interval. R is the set of real numbers. (D. U. 1984)

(i) $A=\{x \in R, |x| \leqq 1\}$

(ii) $A=\{x: x \in R, |x-3| \leqq 1\}$

(11) Express as the union of two intervals

$x=\{x|2 \in R, |x-3|>1\}$

(12) In each of the following cases, decide whether the given relation F and the inverse relation F^{-1} are functions. In case F or F^{-1} is a function, decide also whether it is one to one. (D. U. 1984)

(i) $F=\{x, y) \in R^2 \mid x+y=1\}$

(ii) $F=\{(x, y) \in R^2 \mid y^2=x\}$

(iii) $F=\{x, y) \in R^2 \mid y=x^2\}$

(iv) $F=\{x, y) \in R^2 \mid x^2+y^2=1\}$

Exercise 1 (b)—Domain and Range

1. Find the domain and Range of the following

(i) $y=\sin x$ (ii) $y=x+2$. (iii) $y=\sin {}^1/x$.

Ans. (i) $D_f=(-\pi/2, \pi/2)$, $R_f=(-1, 1)$

(ii) $D_1=R_f=(-\infty, \infty)$

2. If x is any real number and $f(x)=|x|-x$, find the domain and the range of $f(x)$ Draw the graph (যদি x একটি বাস্তব সংখ্যা হয়, তাহা হইলে দেখাও যে $f(x)=|x|-x$ এর চারণস্থল এবং ব্যপ্তি নির্ণয় কর। এবং লেখচিত্র অঙ্কন কর)। (D. U. 1987)

Ans. $D_f=[0, \infty)$, $R_f=\{-\infty, 0)$

2 a) Show that $f(x)=|x|-x$ may also be expressed as (দেখাও যে $f(x)=|x|-x$ কে লিখা যায়)

$$f(x)=\begin{cases} 0, & x \geq 0 \\ -2x, & x<0 \end{cases}$$

3. If the function $f(x)$ is defined by

$$f(x)=\begin{cases} 2x-3 ; & x<1 \\ -x^2 ; & x \geq 1 \end{cases}$$

Find the domain and range of the function $f(x)$. Draw the graph ($f(x)$ যদি উল্লিখিত ভাবে বর্ণিত হয়, তাহা হইলে $f(x)$ এর চারণস্থল ও ব্যাপ্তি নির্ণয় কর।)

Ans. $D_f=(-\infty, \infty)$, $R_f=(-\infty, -1)$

4. Find the domain and function defined by

$$f(x)=\begin{cases} x-1, & x<2 \\ 2x+1, & x \geq 2. \end{cases}$$ Ans. $D_f=[2, \infty)$, $R_f=[5, \infty)$

5. Find the domain and range of $f(x)=|x|+|x-1|$ and show that $f(x)$ may be expressed as C. H. 1992

Ans. $D_f=(-\infty, \infty)$, $R f=[1, \infty)$

$$f(x)=\begin{cases} -2x+1, & x<0 \\ 1, & 0 \leqslant x< \\ 2x-1, & x \geq 1 \end{cases}$$

Draw the graph of the function. [$f(x)=|x|+|x-1|$ এর চারণস্থল এবং ব্যাপ্তি নির্ণয় কর। দেখাও যে $f(x)$ কে উল্লিখিত আকারে প্রকাশ করা যায়। $f(x)$ এর লেখচিত্র অঙ্কন কর।]

6. Find the domain and range of $f(x)=x-[x]$ in $-3\leq x\leq 3$; $[x]$=greatest integer that is $\leq x$

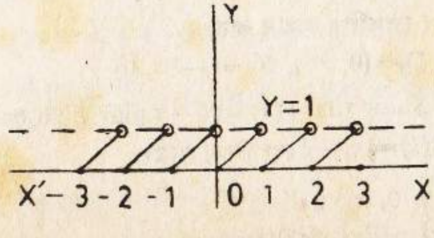


Fig 44

7. Draw the graph of the following function

$$f(x)=\begin{cases} x^2+1, & x<0 \\ x, & 0\leq x\leq 1 \\ \dfrac{1}{x}, & x>1 \end{cases}$$ D. U. 1986

Find the domain and range of $f(x)$. [$f(x)$ ফাংশানের লেখচিত্র অঙ্কন কর এবং $f(x)$ চারণক্ষেত্র ও ব্যাপ্তি নির্ণয় কর।

Ans. $D_f=(-\infty,0)\cup[0,1]\cup(1,\infty)$, $R_f=[1,\infty)\cup[0,1]\cup(1,0)=[0,\infty]$

8. Find the domain and range of the functions

(i) $f(x)=\sqrt{\dfrac{x+1}{x-1}}$ (ii) $f(x)=x,\ 0\leq x\leq \frac{1}{2}$ $=3-x,\ \frac{1}{2}<x<3$ (C. H. 1988)

Ans. $D_f=(-\infty,-1)\cup(1,\infty)$. $R_f=(-\infty,\infty)-(-1,1)$ $=\mathbb{R}-\{-1,1\}$

(ii) $D_f=(0,3)$, $R_f=[0,\frac{1}{2}]\cup[\frac{1}{2},3)=[0,3]$

9. Find the domain and range of f when

$y=f(x)=\dfrac{4x+3}{x^2+1}$ $x\in\mathbb{R}$ $y\in\mathbb{R}$

10. (i) Define the domain of a function. Find the domain of the function defined by $y=x^2$ (R. U. 1970)

(ii) Define the range of a function. Find the range of the function defined by the equation $y^2=(x-2)(x-5)$

(iii) Give an example of a function which has an inverse.

11. Find the domain and range of f when

$y=f(x)=\dfrac{4x+3}{x^2+1}$, $x\in\mathbb{R}$, $y\in\mathbb{R}$

Ans. $D_f=(-\infty,\infty)$ $R_f=[1,4]$

Ex. (12) In each of the following cases, decide whether the given relation F is a function. In each case F is a function, determine its domain and range and decide whether it is one-one. (D. U. 1984)

(i) $F=\{(x,y)\in R^2 \mid y=x\}$

Ans. function, $D_f=R$, $R_f=R$ 1−1 function

(ii) $F\{(x,y)\in R^2 \mid y=x^2$ *Ans.* function, $D_f=R$, $R_f=R^+$

(iii) $F=\{(x,y)\in R^2 \mid y^2=x\}$ *Ans.* No $D_f\neq R$

(iv) $F=\{(x,y)\in R^2 \mid y=\sqrt{x}\}$ Ans. No $D_f\neq R$

13. Find the domain and range of the following function

(i) $y=\sqrt{(x^2-2x+2)}$ (ii) $y=\dfrac{1}{x^2-1}$

Ags. (i) $0\leqslant y\leqslant\infty$, $1\leqslant x\leqslant 2$ (ii) $D_f=\mathbb{R}-\{-1,1)$, $R_f=[\infty,0)\cup[-\infty,-1$

14. Determine the domain and range of the following functions.

(a) $y=[x]^2$ (b) $y=[x]+\frac{1}{2}$ in $[-4,4]$

(c) $y=[\frac{1}{2}x]+1$ in $(-6,6)$ (d) $y=[x+\frac{1}{3}]$ in $[-4,4]$

Ans. $y=0$, $0\leq x<1$; $y=1$, $1\leqslant x<2$, and so on, $R_f=\{0, 1, 4, 9, \}$

(b) $y=\frac{1}{2}$, $0 \leqslant x < I$, $y=\frac{3}{2}$, $I \leqslant y < 2$ and so on

$R_f=\{^{-}\frac{9}{2}, ^{-}\frac{7}{2}, ^{-}\frac{3}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}\}$

(c) $y=I$, $0 \leq x < 2$, $2=2$, $2 \leq x < 4$ and so on,

$R_f=\{-2, -I, 0, 3, 2, I\}$

(d) $y=0$, $0 \leq x+\frac{1}{3} < 1$, $y=\frac{7}{3}$, $1 \leq x < 2$, and so on,

$R_f=\{-4, ^{-}\frac{7}{3}, -1, 0, 1, \frac{8}{3}, 3\}$

EXERCISE—1 (C)

If A={1, 2, 3, 4) and B={1, 2, 3, 4)

1. Which of the following collections of ordered pairs of numbers are functions. Find the domain and range of each it.

(i) (1, 1), (2, 1), (3, 1), (4, 1)

(ii) (1, 1), (2, 2), (3, 3), (4, 4)

(iii) (1, 1), (1, 2), (2, 1), (3, 2)

(iv) (0, 3), (1,—1), (2, 4), (3, 3), (4, 5)

2. *Find* the set of ordered pairs which is $\{f+g\}(x)$ and which is $\{f-g\}(x)$.

(i) $f(x)$ is ; (0, 2), (1, 1) and (2,—4) *Ans.* (0, 3), (1, 2) (2,—2)

$g(x)$ is : (0, 1)), (1, 1) and (2, 2) *Ans.* (0, 1), (1, 0), (2,—6)

(ii) $f(x)$: (—3, 1). (—2, 5), (1, 4) and (3, 6) *Ans.* (—2, 2)

$g(x)$ is : (—2,—3),(—1, 7)(0, 5) and (2, 1) *Ans* (—2, 8)

(iii) $f(x)$ is : (—10, 3), (—4, 8), (0, 1) and (15, 3)

$g(x)$ is (—5, 1), (1, 3), (2, 2) and (10, 10)

3. Find the ordered pairs of Ex—2 above.

4. Which of the following relations in I are functions ? Give the domain and range of each function.

$F=\{(x, y) \mid (x, y) \in I \times I, y^4=x\}$ Ans. not function.

$F=\{(x, y) \mid (x, y) \in I^2, y=x^3\}$ *Ans.* $D=I$; R=set of integers expressive as the cube of integer.

$F=\{(x, y) \mid (x, y) \in I^2, x<y\}$ *Ans.* not function.

$F=\{(x, y) \mid (x, y) \in I^2, x^2-y=16\}$ *Ans.* $D=I$. R=set of all in tegers expressible as 16 less than the square.

$F=\{(x, y) \mid (x, y) \in A \times B, y=2x^2+3\}$

$A=\{x \mid x \in I, 1 \leqslant x \leqslant 5$, $B=\{x \mid x \in I\}, 1 \leqslant \in \leqslant 100\}$

Ans. yes, D=A, Range={5, 11, 21, 35, 53}

$F=(1, 5)$ (2, 11), (3, 21), (4, 35), (5, 53)

$F=\{x, y) \mid (x, y \in R \times R, y=x^2\}$, $R=\{x \mid x \in I, |x| \leqslant 10\}$

Ans. yes, $F=(0,0),(1, 1),(2, 4), (3, 2,(-1, 1), (-2, 4). (-3, 9)$

$F=\{(x, y) \mid (x, y) \in R^2, x=y^2\}$, $R=\{x \mid x \in I, |x| \leqslant 10\}$

Ans No, F=(0, 0), (1, 1),(1—1), (4,2) (4,—2), (9,3), (9,—3)

The pairs (1, 1) and (1,—1) have same first component.

5. (i) If I is the set of all integers and $x \in I$, which of the following mapping of $I \times I$ are mapping of I onto I ? What are 1—1 of I onto I.

(a) $x \to x+3$ *i, e* $F=\{x, y) \in I^2 \mid y=x+3\}$

(b) $x \to x^2+x$, *i, e,* $F=\{(x, y) \in I^2 \mid y=x^2+x\}$

(c) $F=\{(x, y) \in I^2 \mid x=x^3\}$

(d) $F=\{x, y) \in I^2 \mid x=2x-1\}$

(e) $F=\{x, y) \in I^2 \mid x=x-4\}$

6. (i) Let $A=\{x \mid x \in R$ and $-5 \leqslant x \leqslant 5\}$

and $B=\{y \mid y \in R$ and $-2 \leqslant y \leqslant 2\}$

Find $A \times B$ and sketch $A \times B$ in certain plane

(ii) Let $A=\{x \mid x \in R$ and $|x-1| < 2\}$

and $B=\{y \mid y \in R$ and $|y+2| > 1\}$

Find A × B and sketch A × B in caertesian plane.

(iii) Let $A=\{x \mid x \in R, -6 \leqslant x \leqslant 6\}$

and define the relation R in A. Find A × A and sketch A × B in certesian plane.

7. Describe which of the relation below are functions, which into and which are 1—1 : also 1—1 correpondence,

(i) $f: \{(x, y) \in R \times R \mid y=2x+3\}$ *Ans* 1—1 Correspondence

(2) $f: \{(x, y) \in R \times R \mid y=\sqrt{x}$, R^+ be the set of all non-negative real numbers) $Df \neq R$, it is not function.

(3) $f: \{(x, y) \in R \times R \mid y=\sin x\}$ *Ans.* into function

(4) $f: \{(x, y) \in R \times R \mid y=x^3\}$ *Ans.* one—one function.

(5) $f: \{(x, y) \in R \times R \mid f(x)=x^2+1\}$ *Ans.* function onto.

(6) $f: \{(x, y) \in R^2 \mid f(x)=e^x\}$ *Ans* f is one—one function.

(7) $f: \{(x, y) \in R \times R \mid y=\tan x\}$ *Ans.* f is onto,

(8) $f: \{(x, y) \in R^2 \mid f(x)=\log x$, where x is real$\}$

8. Let A and B set of real numbers

$f=\{(x, y) \in A \times B \mid f(x)=6x-x^2)$

Find $f[0, 1]$, $f(1, 4)$ $f^{-1}([0, 5)]$, $f^{-1}(5, 9])$

Ans. $[0, 1]$, $[5, 9]]$ $[0, 5]U[5,6]$ $[1,5]$

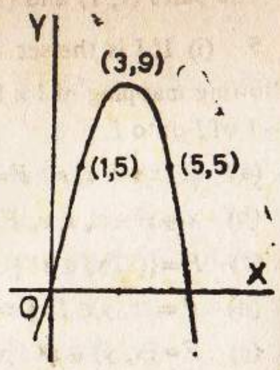


Fig 45

9. If $f(x)=\{(x, \ ) \in R^2 \mid f((x)=x^2-4x+4$ Find $f([0, 1)$, $f([3; 4])$, $f^{-1}([1, 2])$

Ans. $[1, 4]$, $[1, 4]$ $[2-\sqrt{2}, 1]$ $U[3, 2+\sqrt{2}]$

10. If $f: X \to Y$ is a function and $A \subset X$; then what is meant by $f^{-1}(A)$?

Ans. See Higher Algebra, set Theory Art. 9.6

Show that $f^{-1}(A \cup B)=f^{-1}(A) \cup f^{-1}(B)$

and $f^{-1}(A \cap B)=f^{-1}(A) \cap f^{-1}(B)$ (D. U. 1982)

See Higher Algebra, set Theory Art. 9. 7.

11. Show that the function

$y=\sqrt{\{(x-a)(x-b)(x-c)}$: $a<b<c$ is defined for all real values of x lying between a and b and is not defined for all real values of x lying between b and c.

12 Draw the graphs of the following function. Find also the domains of x and y.

(i) $f(x)=x, \quad 0 \leq x \leq 2$
$\quad =3-x \quad \frac{1}{2}<x<3$

(ii) $f(x)=x^2, \quad x \leq 0$
$\quad =\sqrt{x}, \quad x>0$

(iii) $f(x)=x, \quad 0 \leq x<\frac{1}{2}$
$\quad =1 \quad x=\frac{1}{2}$
$\quad =1-x, \quad \frac{1}{2}<x<1$

(iv) $f(x)=x \quad -2 \leq x \leq 2$
$\quad =4-x \quad 2<x \geqq 4$

(v) $f(x)=\dfrac{2x}{x-3}$, $-\infty<x<\infty$ for $x \neq 3$
$\quad =2 \quad x=3$

(vi) $f(x)=\dfrac{x^2-16}{x-4}$ for $x \neq 4$
$\quad =2$ for $x=4$

13. Find the inverse (বিপরীত) functions of y and the regions (এলাকা) where they are defined.

(i) $y=1-2^{-x}$ Ans. $x=-\log(1-y)/\log 2, -\infty<y<1$

(ii) $y=x^2-5$ Ans. $x=\sqrt{(y+5)}, -\sqrt{(y+5)}, (-5<y<\infty)$

(iii) $y=\log_{10}x$, Ans. $x=10^y, -\infty<y<\infty$

(iv) $y=x^2, x>0$ Ans. $x=\sqrt{y}, 0<y<\infty$
$\quad =x, x \leq 0 \quad y=x, -\infty<y \leq 0.$

14. Find the Domain and draw the graph.

(i) $f(x) = 1 + x; -1 \le x < 0$ Ans. (i) [-1,1]

$= 1 - x; 0 \le x < 1$ (0, 1) U (1,&)

$= 0 ; 1 < x$ N.U. 1994

(ii) $f(x) = |x + 1| + |x - 2|$ C.H. 1992

Ans. EDOBC .... (-&,-1) U [-1, 0] U [0, -20] U [2, &]

14. ডোমেন ক্ষেত্র ও লেখচিত্র অঙ্কন কর

(i) $f(x) = 1 + x; -1 \le x < 0$

$= 1 - x; 0 \le x < 1$

$= 0 ; 1 < x$ N.U.1994

C. H. 1992

14(ii) $= x$ যখন ও ≤ 0

$f(x) = |x + 1| + |x - 2|$

উত্তরমালা–১

1. $- 1, 5/2 \; 3h^2 - \frac{1}{h} + 4h - 7$

2. (i) 1 (ii) 0 (iii) 10^4 (iv) 0.

3. (a) (i) $x = 0$ ব্যতিত x-এর সকল মান।

(ii) শূন্য সহ x-এর সকল ধনাত্মক বাস্তব মান।

(iii) $x = 0$ ব্যতিত x-এর মান সকল বাস্তব মান।

(b) $-\infty < x \le -1$ এবং $0 \le x \le 1$

(c) $-2 < x < 2$ (d) $-\frac{1}{3} \le x \le 1$

(e) $-\infty < x < -1, \; 1 < x < \infty$

(f) $k\pi \le x < k\pi + \pi/2 \; (k = 0 \pm 1 \pm 2 \ldots \ldots \ldots)$

4. (i) জোর (ii) বিজোড় (iii) বিজোড় (iv) বিজোড়।

9. না,

13. (i) $x = -\log(1 - y) / \log 2; \; -\infty < y < 1$

(ii) $x = \sqrt{(y + 5)}, \; -\sqrt{y + 5}, \; (-5 < y < \infty)$

(iii) $x = 10^y, \; -\infty < y < \infty$

(iv) $x = \sqrt{y}, \; 0 < y < \infty$

$y = x, \; -\infty < y \le 0$ 14(i) $[-1, 1); \; [0, 1) \cup (1, \infty)$

14(ii) EDOBC .... $[-\infty, -1] \cup [-1, 0] \cup [0, -20] \cup [2, \infty)$

ধারণক্ষেত্র $(\infty, 3) \cup \{3\} \cup [3, \infty]$ ব্যবধি

EXERCISE—1 (A)

(1) Express the sets as an interval. R is the set of real numbers

(i) $A=\{x \in R, | x | \leqslant 1\}$ (D.U. 19..

(ii) $A=(x | x \in P, | x-3 | \leqslant 1\}$

(11) Express as the union of two intervals (দুইটি ব্যবধি যোগফলে প্রকাশ কর)

$x=\{x | x \in R, | x-3 | >1\}$

(12) In each of the following cases, decide whether the given relation F and the inverse relation F^{-1} are functions. In case F or F^{-1} is a function, decide also whether it is one to one. (প্রশ্ন F এবং F^{-1} ফাংশান কি? যদি ফাংশান হয় তাহারা এক-এক কি ?)
(D.U. 19..

(i) $F=\{(x, y) \in R^2 | x+y=1\}$

(ii) $F=\{(x, y) \in R^2 | y^2=x\}$

(iii) $F=\{(x, y) \in R^2 | y=x^2\}$

(iv) $F=\{(x, y) \in R^2 | x^2+y^2=1\}$

Exercise 1 (b)—Domain and Range (চারণক্ষেত্র ও ব্যাপ্তি বা বিস্তার)

1. Find the domain and Range of the following

(i) $y=\sin x$ (ii) $y=x+2$. (iii) $y=\sin {}^1/x$.

Ans. (i) $D_f=(-\pi/_2, \pi/_2)$, $R_f=(-1, 1)$

(ii) $D_f=R_f=(-\infty, \infty)$

2. If x is any real number and $f(x)=| x | -x$, find the domain and the range of $f(x)$ Draw the graph (যদি x একটি বাস্তব সংখ্যা হয়, তাহা হইলে দেখাও যে $f(x)=| x | -x$ এর চারণক্ষেত্র এবং ব্যাপ্তি নির্ণয় কর। এবং লেখচিত্র অঙ্কন কর)। (D. U. 19..

Ans. $D_f=[0, \infty)$, $R_f=(-\infty, 0)$

2. (a) Show that $f(x)=| x | -x$ may also be expressed as (দেখাও যে $f(x)=| x | -x$ কে লিখা যায়)

$$f(x)=\begin{cases} 0, & x\geqslant 0 \\ -2x, & x<0 \end{cases}$$

3. If the function $f(x)$ is defined by

$$f(x)=\begin{cases} 2x-3 ; & x<1 \\ -x^2 ; & x\geqslant 1 \end{cases}$$

Find the domain and range of the function $f(x)$ Draw the graph ($f(x)$ যদি উল্লিখিতভাবে বর্ণিত হয়, তাহা হইলে $f(x)$ এর চারণক্ষেত্র ও ব্যাপ্তি নির্ণয় কর।)

Ans. $D_f=(-\infty, \infty)$, $R_f=(-\infty, -1]$

4. Find the domain and function defined by

$$f(x)=\begin{cases} x-1 , & x<2 \\ 2x+1 , & x\geqslant 2 \end{cases}$$ Ans. $D_f=[2, \infty)$, $R_f=[5, \infty)$

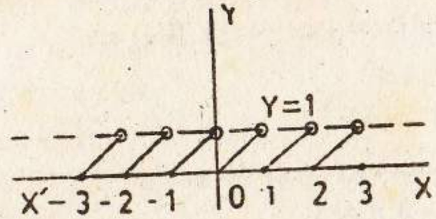
5. Find the domain and range of $f(x)=| x | + | x-1 |$ and show that $f(x)$ may be expressed as

$$f(x)=\begin{cases} -2x+1, & x<0 \\ 1 , & 0\leqslant x<1 \\ 2x-1 , & x\geqslant 1 \end{cases}$$

Ans. $D_f=(-\infty, \infty)$, $R_f=[1, \infty)$

Draw the graph of the function [$f(x)=| x | + | x-1 |$ এর চারণক্ষেত্র এবং ব্যাপ্তি নির্ণয় কর। দেখাও যে $f(x)$ কে উল্লিখিত আকারে প্রকাশ করা যায়। $f(x)$ এর লেখচিত্র অঙ্কন কর।]

6. Find the domain and range of $f(x)=x-[x]$ in $-3\leqslant x\leqslant 3$; greatest integer that is $\leqslant x$



চিত্র—৬৪

7. Draw the graph of the following function

$$f(x)=\begin{cases} x^2+1, & x<0 \\ x, & 0\leqslant x<1 \\ \dfrac{1}{x}, & x>1 \end{cases}$$ *D. U. 19*

Find the domain and range of $f(x)$. [$f(x)$ ফাংশানের লেখ অঙ্কন কর এবং $f(x)$ চারণক্ষেত্র ও ব্যাপ্তি নির্ণয় কর।

Ans. $D_f=(-\infty, 0)\cup[0, 1]\cup(1, \infty), R_f=[1,\infty)\cup[0, 1]$
$\cup(1, 0)=[0,$

8. Find the domain and range of the functions

(i) $f(x)=\sqrt{\dfrac{x+1}{x-1}}$

Ans. $D_f=(-\infty,-1)\cup(1, \infty)$, $R_f=(-\infty, \infty)-(-1, 1)$
$=\mathrm{IR}-\{-1,$

9. Find the domain and range of f when

$$y=f(x)=\frac{4x+3}{x^2+1} \quad \begin{matrix} x \in \mathrm{IR} \\ y \in \mathrm{IR} \end{matrix}$$

10. (i) Define the domain of a function. Find the dom of the function defined by $y=x^2$ *(R. U. 19*

(ii) Define the range of a function. Find the range of function defined by the equation $y^2=(x-2)(x-5)$

(iii) Give an example of a function which has an inverse.

11. Find the domain and range of f when

$y = f(x) = \dfrac{5x+3}{x^2+1}$, $x \in \mathrm{IR}; y \in \mathrm{IR}$

Ans. $D_f = (-\infty, \infty)$; $R_f = [1, 4]$

Ex. (12) In each of the following cases, decide whether the given relation F is a function, In each case F is unction. Determine its domain and range and decide whether it is one-one. *(D. U. 198)*

(i) $F = \{(x, y) \in R^2 \mid y = x\}$

Ans. function, $D_f = R$, $R_f = R$, *1 - 1 function*

(ii) $F\ \{(x, y) \in R^2 \mid y = x^2$ **Ans.** *function*, $D_f = R, R_f = R^+$

(iii) $F = \{(x, y) \in R^2 \mid y^2 = x\}$ **Ans.** *No.* $D_f \neq R$

(iv) $F = \{(x, y) \in R^2 \mid y = \sqrt{x}\}$ **Ans.** *No.* $D_f \neq R$

13. Find the domain and range of the following function

(i) $y = \sqrt{(x^2 - 2x + 2)}$ (ii) $y = \dfrac{1}{x^2-1}$

(iii) $f(x) = \sqrt{(x^2 - 4x + 1)}$ *N. U. 1993*

Ans. *(i)* $0 \leq y \leq \infty$, $1 \leq x \leq 2$ *(ii)* $D_f = \mathrm{IR} - \{-1, 1\}$,

$R_f = [\infty, 0) \cup [-\infty, -1]$

(iii) $2 - \sqrt{3} \leq x \leq 2 + \sqrt{3}$, $\alpha\,(-\infty - \sqrt{3}\,] \cup$

$\sqrt{3}, -\infty\,]$ $R_f = [\infty, 0) \cup [-\infty, -1]$

14. Determine the domain and range of the follwoing functions with diagram (নিম্নলিখিত ফাংশন গুলির চারণক্ষেত্র ও বিস্তার নির্ণয় কর। চিত্রগুলি অঙ্কন কর।)

(a) $y = [x]^2$ *in* $[2, 4]$ Ans. $D_f = 0 \leq x < 1$, $1 \leq x < 2$, $2 \leq x < 3$

$3 \leq x < 4$ ……… $R_f = \{0, 1, 4, 9, 16 \ldots\ldots\ldots\}$

(b) $y = [x] + \dfrac{1}{2}$ *in* $[-4, 4]$

Ans. D_f প্রশ্ন *(a)* এর D_f

$R_f = \{-9/2, -7/2, -3/2, \frac{1}{2}, 3/2, 5/2, 7/2\}$

(c) $y=[\frac{1}{2}x]+1$ in $[-6, 6]$

Ans. D_f *Ans.* (*a*) এর D_f

$R_f=\{-4,-3,-2, 0,-1,-2\}$

(d) $y=[x+\frac{1}{3}]$ in $[-4, 4]$

Ans. D_f *Ans.* (*a*) এর D_f

$R_f=\{-4,-7/3,-1, 0, 1, 8/3, 3\}$

(e) (i) $f(x)=\sqrt{\left(\frac{x+1}{x-1}\right)}$ (ii) $f(x)=x,\ 0\leqslant x\leqslant\frac{1}{2}$

C. H. 1988 $\quad=3-x,\ \frac{1}{2}<x>3$ *C. H.* 1988

Ans. $D_f=[-\infty,\infty]-(-1, 1)$ *Ans.* $D_f=[0, 3]$, $R_f=[0, 5/2)$

$R_f=(-1, \infty]-(-1, 1)$ লেখচিত্রটি OA এবং BC ; $B(3, 0)$ এবং $C(\frac{1}{2}, 5/2)$ বিন্দু দুইটি বহিভূত।

EXERCISE—1 (C)

If $A=\{1, 2, 3, 4\}$ and $B=\{1, 2, 3, 4\}$

1. Which of the following collections of ordered pairs of numbers are functions. Find the domain and range of each it. [সজ্জিত জোড়ের সমাবেশ গুলির মধ্যে কোনটি ফাংশান? ফাংশানের চারণক্ষেত্র ও বিস্তার নির্ণয় কর]

(i) (1, 1), (2, 1), (3, 1), (4, 1)

(ii) (1, 1), (2, 2), (3, 3), (4, 4)

(iii) (1, 1), (1, 2), (2, 1), (3, 2)

(iv) (0, 3), (1, −1), (2, 4), (3, 3), (4, 5)

2. Find the set of ordered pairs which is $\{f+g\}(x)$ and which is $\{f-g\}(x)$. [সজ্জিত জোড়ের সেট নির্ণয় কর]

(i) $f(x)$ is : (0, 2), (1, 1) and (2, −4) *Ans.* (0, 3), (1, 2) (2, −3)

$g(x)$ is : (0, 1), (1, 1) and (2, 2) *Ans.* (0, 1), (1, 0), (2, −6)

(ii) $f(x)$: (−3, 1), (−2, 5), (1, 4) and (3, 6) *Ans.* (−2, 2)

$g(x)$ is : (−2, −3), (−1, 7), (0, 5) and (2, 1) *Ans.* (−2, 8)

(iii) $f(x)$ is : (−10, 3), (−4, 8), (0, 1) and (15, 3)

$g(x)$ is (−5, 1), (1, 3), (2, 2) and (10, 10)

3. Find the ordered pairs of Ex−2 above.

4. Which of the following relations in I are functions ? Give the domain and range of each function. [সম্পর্ক I এর মধ্যে কোনটি ফাংশান। ইহাদের ডোমেন ও রেঞ্জ নির্ণয় কর।]

$F=\{(x, y) \mid (x, y) \in I\times I,\ y^4=x\}$ Ans. not function.

$F=\{(x, y) \mid (x, y) \in I^2, y=x^3\}$ *Ans.* $D=I$; $R=$set of integers expressive as the cube of integer.

$F=\{(x, y) \mid (x, y) \in I^2,\ x<y\}$ *Ans.* not function.

$F=\{(x, y) \mid (x, y) \in I^2, x^2-y=16\}$ *Ans.* $D=I$. $R=$set of all integers expressible as 16 less than the square.

$F=\{(x, y) \mid (x, y) \in A\times B, y=2x^2+3\}$

$A=\{x \mid x\in I,\ 1\leqslant x\leqslant 5,\quad B=\{x \mid x\in I;\ 1\leqslant x\leqslant 100\}$

Ans. yes, $D=A$, Range$=\{5, 11, 21, 35, 53\}$

$F=(1, 5)\ (2, 11),\ (3, 21),\ (4, 35),\ (5, 53)$

$F=\{x, y) \mid (x, y) \in R\times R,\ y=x^2\}$, $R=\{x \mid x\in I, |x| \leqslant 10\}$

Ans. yes, $F=(0, 0), (1, 1), (2, 4), (3, 2), (-1, 1), (-2, 4), (-3, 9)$

$F=\{(x, y) \mid (x, y) \in R^2,\ x=y^2\}$ $R=\{x \mid x\in I, |x| \leqslant 10\}$

Ans. No. $F=(0, 0), (1, 1), (1,-1), (4, 2) (4,-2), (9, 3), (9,-3)$

The pairs (1, 1) and (1, −1) have same first component.

5. (i) if I is the set of all integers and $x \in I$, which of the following mapping of $I \times I$ are mapping of I onto I ? What are $1-1$ of I onto I. [I একটি পূর্ণ সংখ্যার সেট $x \in I$. $I \times I$ চিত্রের কোনটি I এর উপর I চিত্রণ? I এর উপর I হইলে কোনটি এক-এক চিত্রণ?]

(a) $x \to x+3$ *i, e* $F=\{x, y) \in I^2 \mid y=x+3\}$

(b) $x \to x^2+x$, *i, e*, $F=\{(x, y) \in I^2 \mid y=x^2+x\}$

(c) $F=\{(x, y) \in I^2 \mid y=x^3\}$

(d) $F=\{x, y) \in I^2 \mid y=2x-1\}$

(e) $F=\{x, y) \in I^2 \mid y=x-4\}$

6. (i) Let $A=\{x \mid x \in R \text{ and} -5 \leq x \leq 5\}$

and $B=\{y \mid y \in R \text{ and} -2 \leq y \leq 2\}$

Find $A \times B$ and sketch $A \times 3$ in certain plane

(ii) Let $A=\{x \mid x \in R$ and $\mid x-1 \mid < 2\}$

and $B=\{y \mid y \in R$ and $\mid y+2 \mid > 1\}$

Find $A \times B$ and sketch $A \times B$ in cartesian plane.

(iii) Let $A=\{x \mid x \in R, -6 \leq x \leq 6\}$

and define the relation R in A. Find $A \times A$ and sketch $A \times B$ in certesian plane.

7. Describe which of the relation below are functions, which are into and which are $1-1$: also $1-1$ correpondence,

(i) $f: \{(x, y) \in R \times R \mid y=2x+3\}$ *Ans.* $1-1$ Correspondence

(2) $f: \{(x, y) \in R \times R \mid y=\sqrt{x}$, R^+ be the set of all non-negative real numbers) $D_f \neq R$, it is not function.

(3) $f: \{(x, y) \in R \times R \mid y=\sin x\}$ *Ans.* into function.

(4) $f: \{(x, y) \in R \times R \mid y=x^3\}$ *Ans.* one-one function

(5) $f: \{(x, y) \in R \times R \mid f(x)=x^2+1\}$ *Ans.* function onto.

(6) $f: \{(x, y) \in R^2 \mid f(x)=e^x\}$ *Ans.* f is one-one function.

(7) $f: \{(x, y) \in R \times R \mid y = \tan x\}$ Ans. f *is onto.*

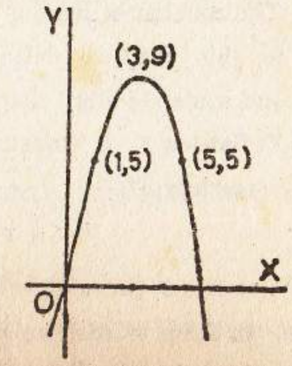
(8) $f: \{(x, y) \in R^2 \mid f(x) = \log x$. where x is real$\}$

8. Let A and B set of real numbers

$f = \{(x, y) \in A \times B \mid$
$f(x) = 6x - x^2\}$

find f $[0, 1]$, $f(1, 4)$, f^{-1} $([0, 5])$,
f^{-1} $([5, 9])$

Ans. $[0, 5]$, $[5, 9]$ $[0, 8]$ $\cup$ $[5, 6]$, $[1, 5]$



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9. If $f(x) = \{(x, y \in R^2 \mid f(x) = x^2 - 4x + 4$ find $f([0, 1])$, $f([3; 4])$,
$f^{-1}([1, 2])$

Ans. $[1, 9]$, $[1, 4]$, $[2 - \sqrt{2}, 1]$
$\cup$ $[3, 2 + \sqrt{2}]$

10. If $f: x \to y$ is a function and A X, then what is meant by $f^{-1}(A)$?

Ans. See Higher Algebra, set Theory Art. 9.6

Show that $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

and $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

see Higher Algebra, set theory Art. 9.7

11. find the domain of the following (চারণ ক্ষেত্র নির্ণয় কর।

D. U. 1991

$f(x) = x^2 - 1$, $g(x) = 3x + 1$

(i) $\frac{f}{g}(x)$ (ii) $\frac{g}{f}(x)$ (iii) $f(g(x))$ (v) $g(f(x))$

Ans. $(-\infty, \infty) - \{^1/_3\}$, $(-\infty, \infty) - \{-1, 1\}$, $(-\infty, \infty)$, $(-\infty)$, $(-\infty, \infty)$

CHAPTER II
LIMITS

2.1. The concept of limit is the most basic concept of Calculus. In this chapter, definition of a limit, properties of limits and some standard theorems on limits will be discussed.

2.2. **Definition :** A constant a is said to be a limit of the variable x, if

$$0 < | x - a | < \delta$$

Where δ is a pre-assigned positive quantity as small as we please. In other words, we say that "**x approaches the constant a**, or '**x tends to a**'', Symbollically, it is denoted by

$$x \to a \quad \text{or} \quad \lim x = a,$$

Note that $x \to a$ never implies that $x = a$

When x approaches a but always remains less than a, we say that x approaches a from the left on the real axis and we write. $x \to a^-$

Similarly, when x tends to a with values always greater than a' x approaches a from the right on the real axis and this is expressed as $x \to a^+$

For example, when x takes successive numerical values, 3·9, 3.99, 3·999, 3·999...... x tends to 4 from the left or $x \to 4^-$. Smilarly, if x assumes the successive values

4·1, 4·01, 4·001, 4.0001, ... x approaches 4 from the right or $x \to 4^+$.

Considering the two sets of values

{3·9, 3·99, 3·999, 3·999.........}

and {4·1, 4·01, 4·001, 4·001...........}

assumed by x, we say that $x \to 4$

or $0 < | x - 4 | < \delta$ where $\delta = \frac{1}{10^n}$, $n = 1, 2, 3, 4, ...$

Art 2·3 Limits of a function :

Definition : $\lim\limits_{x \to a} f(x) = l$,

In common language '$\lim\limits_{x \to a} f(x) = l$, means that $f(x)$ is very close to the fixed number l *whenever x is very close to a.*

In terms of mathematical analysis, we give the meaning of $\lim\limits_{x \to a} f(x) = l$, as follows :

Let S be a set of numbers and let $f(x)$ be defined for all numbers in S (that is, S is a subset of D_f, the domain of f). We assume that S is arbitrarily close to a ; i. e. given $\delta > 0$, there exists an element x of S such that $0 < | x - a | < \delta$, we shall say that $f(x)$ **approaches the limit l as x tends to a** if the following condition is satisfied :

Given a number $\epsilon > 0$, however small, there exists a positive number δ such that for all x in S satisfying

$$0 < | x - a | < \delta$$

we have

$$| f(x) - l | < \epsilon$$

If this is the case, we write

$$\lim_{x \to a} f(x) = l.$$

Note that *a may or may not belong to* D_f.

We can also define

$$\lim_{x \to a} f(x) = l$$

In the following way :

We write

$$\lim_{h \to 0} f(a+h) = l$$

and say that the limit of $f(a+h)$ is l if $h \to 0$, provided given $\epsilon > 0$, however small, there exists $\delta > 0$ such that whenever $0 < |h| < \delta$ and $(a+h) \in S$, then

$$|f(a+h) - l| < \epsilon$$

Cor : If the set S is such that any number $x \in S$ is less than a and for any $\epsilon > 0$, however small, there exists a positive number δ and fixed number l_1 satisfying.

$$0 < |x - a| < \delta \text{ with } |f(x) - l_1| < \epsilon,$$

Left hand Limit : We say that l_1 is the **left hand limit of** $f(x)$ as $x \to a'$ and we write

$$\lim_{x \to a^-} f(x) = l_1$$

Similarly, if each number x of S is greater than a and there is a fixed number l_2 such that whenever

$$0 < |x - a| < \delta$$

We have $|f(x) - l_2| < \epsilon$,

Righ hand Limit : Where ϵ and δ have the same meaning as before, we say that l_2 is **the right hand limit** of $f(x)$. This is expressed as $\lim_{x \to a^+} f(x) = l_2$

Note : When the set S consists of numbers less than as well as greater than a, we say that $\lim_{x \to a} f(x)$ exists,

if $l_1 = l_2 = l$ and it is equal to l.

Ex 1. Prove that C. H. 199?

$\lim_{x \to 2} (3x + 4) = 10$ by (δ, ϵ) definition of a function

Let us consider an arbitrary positive number $\delta > 0$ however small such that

$$|3x + 4 - 10| < \epsilon \quad \text{or} \quad |3x - 6| < \epsilon$$

$$\text{i. e.,} \quad |x - 2| < \tfrac{1}{3}\epsilon,$$

$$|x - 2| < \delta \quad \text{when } \delta = \tfrac{1}{3}\epsilon > 0$$

This means that, $\lim_{x \to 2} (3x + 4) = 10$

Ex. 2. Prove that $\lim_{x \to a} \dfrac{x^2 - a^2}{x - a} = 2a$

Let us consider an arbitrary positive number $\epsilon > 0$, however small. Then

$$\left|\frac{x^2 - a^2}{x - a} - 2a\right| < \epsilon. \quad \dots \quad \dots \quad \dots \quad (1)$$

$$\text{or} \left|\frac{(x+a)(x-a)}{x-a} - 2a\right| = |x + a - 2a| < \epsilon \quad (\because x \neq a)$$

$$\text{or,} \quad |x - a| < \epsilon$$

$$\text{or,} \quad |x - a| < \delta \quad \text{where } \delta = \epsilon \dots \dots (2)$$

Hence $\lim\limits_{x\to a} \dfrac{x^2-a^2}{x-a}=2a$ [From (1) and (2)]

Ex. 3. From the definition of (δ, ϵ) show that

$$\lim_{x\to 3}(2x^3-3x^2-18x+29)=2$$

Let for a given $\epsilon>0$, however small,

$$0<|x-3|<\lambda<1 \ldots\ldots (1)$$

$\therefore \quad |2x^3-3x^2-18x+29-2|=|2x^3-3x^2-18+27|$

$=|2(x-3)^3+15(x-3)^2+18(x-3)|$

$\leqq 2|x-3|^3+15|x-3|^2+18|x-3|\leqslant 2\lambda^3+15\lambda^2+18\lambda<35\lambda$

$$(\because \lambda<1, \therefore \lambda^3<\lambda^2<\lambda)$$

Now $|(2x^3-3x^2-18x+29)-2|<\epsilon$, where $\epsilon=35\lambda$ (2)

or, $\lambda=\epsilon/35$ therefore, we can determine a small positive number δ depending on ϵ such that the limit is established. Here $\delta=\epsilon/35$

Hence $\lim\limits_{x\to 3}(2x^3-3x^2-18x+29)=2$ [From (1) and (2)]

Ex. 4. For $f(x)=x^2-3x+5$, find $\delta>0$ such that whenever $0<|x-2|<\delta$ then $|f(x)-3|<\epsilon$, when (a) $\epsilon=\frac{1}{3}$, (b) $\epsilon=0.07$

Let ϵ be given; we are to find $\delta>0$ such that

$0<|x-2|<\lambda<1$ implies that

$|f(x)-3|=|(x^2-3x+5)-3|$

$=|(x-2)^2+(x-2)|\leqslant|x-2|^2+|x-2|<\lambda^2+\lambda<2\lambda$

$$[\because \lambda^2<\lambda<1]$$

So $|f(x)-3|<\epsilon$, if $\lambda=\epsilon/2$. Therefore we can determine a positive number δ smaller than 1 and equal to $\epsilon/2$ so that the limit exists.

Hence $\delta=\epsilon/2$

(a) If $\epsilon=\frac{1}{3}$ then $\epsilon=\dfrac{\epsilon}{2}=\dfrac{1}{3\cdot2}=\dfrac{1}{6}$

(b) If $\epsilon=0.07$, then $\delta=\dfrac{\epsilon}{2}=\dfrac{0.07}{2}=0.035$

2:4 Distinction Between $\lim\limits_{x\to a} f(x)$ and f (a)

$f(a)$ means that the value of $f(x)$ when $x=a$.

or, the value of $f(x)$ at $x=a$ is $f(a)$; clearly $a\in D_f$.

$\lim\limits_{x\to a} f(x)$ is a statement about the values of $f(x)$ when x assumes all values of x in the neighbourhood of a except $x=a$.

When $\lim\limits_{x\to a} f(x)=f(a)$

The function $f(x)$ is said to be **continuous** at $x=a$.

Ex. 5. A function $f(x)$ is defined in the following way

$$\begin{aligned} f(x)&=1+2x \quad \text{for} -\tfrac{1}{2}\leq x<0\\ &=1-2x \quad \text{for } 0\leqslant x<\tfrac{1}{2}\\ &=-1+2x \quad \text{for } x>\tfrac{1}{2}\end{aligned}$$

Investigate the function at $x=0$ and $x=\frac{1}{2}$

For $x=0$

$\lim\limits_{h\to0^-} f(0+h)=\lim\limits_{h\to0^-}\{1+2(0+h)\}=1$; take $f(x)=1+2x$ as $x<0$

$\lim\limits_{h\to0^+} f(0+h)=\lim\limits_{h\to0^+}\{1-2(0+h)\}=1$; take $f(x)=1-2x$. as $x>0$

and $f(0)=1-2.0=1$. take $f(x)=1-2x$, as $x=0$

Thus we get, $\lim\limits_{x\to0} f(x)=1=f(0)$.

i. e., the value of the function $f(x)$ at $x=0$ is equal to the limit of the function $f(x)$ when $x\to 0$. Hence $f(x)$ is continuous at $x=0$

For $x=\frac{1}{2}$

$\lim\limits_{h\to 0^+} f(\frac{1}{2}+h)=\lim\limits_{h\to 0^-} [-1+2(\frac{1}{2}+h)]=-1+1=0$, take $f(x)=-1+2x$

$\lim\limits_{h\to 0^-} f(\frac{1}{2}+h)=\lim\limits_{h\to 0^+} \{1-2(\frac{1}{2}-h)\}=-1+1=0$, take $f(x)=-1+2x$

$\lim\limits_{h\to 0^-} f(\frac{1}{2}+h)=\lim\limits_{h\to 0^+} f(\frac{1}{2}+h)=0$

Hence the limit of the function $f(x)$ exists and is equal to zero

2. 5. Meaning of the symbols $+\infty$ and $-\infty$ (সূচকদ্বয়ের অর্থ)

If a variable x assumes all positive values and increases without limit such that it is greater than any positive number, however big which we may imagine, x is said to tend to infinity and symbolically it is written as $x\to\infty$.

Similarly if the variable possessing negative values only decreases without any limit and less than any negative number which we may imagine, x said to tend to minus infinity ($-\infty$) and it is denoted by $x\to-\infty$.

These symbols $+\infty$ and $-\infty$, by themselves do not possese any meaning and the phrases in which they occur do not take any meaning from the symbols.

The meaning of the phrase $x\to\infty$ is not found from the statement $x\to a$ by substituting $a=\infty$.

2.6 Meaning of

(i) $\lim\limits_{x\to a} f(x)=\infty$ (ii) $\lim\limits_{x\to a} f(x)=-\infty$

(i) A function $f(x)$ is said to tend to $+\infty$ when x approaches a, if for any pressigned positive number N, however large, we can determine another positive number δ such that $f(x)>N$ for all values of x satisfying the inequality :

$$0<|x-a|\leqq\delta \text{ with } x>a, \text{ or } a<x\leqq a+\delta$$

(ii) A function $f(x)$ is said to tend to $-\infty$ when x approaches a, if for any given positive number N, however large, we can determine a positive number δ such that

$$-f(x)>N, \quad \text{or,} \quad f(x)<-N$$

for all values of x satisfying the inequality $0<x-a\leqslant\delta$

Ex. 6. Evaluate $\lim\limits_{x\to 2} \dfrac{3}{(x-2)^2}$

Let $f(x)=\dfrac{3}{(x-2)^2}$

The limit of $f(x)$ is $+\infty$ at $x\to 2$ if both $f(a+h)$ and $f(a-h)$ become greater and greater when h approaches zero with $h>0$

$$\lim_{h\to 0^+} f(2+h)=\lim_{h\to 0} f(2+h)=\lim_{h\to 0}\frac{3}{(2+h-2)^2}=\lim_{h\to 0}\frac{3}{h^2}=\infty$$

$$\lim_{h\to 0^+} f(2-h)=\lim_{h\to 0^+}\frac{3}{(2-h-2)^2}=\lim_{h\to 0^+}\frac{3}{h^2}=\infty$$

$$\lim_{h\to 0^+} f(2+h)=\lim_{h\to 0^+} f(a-h)=\infty$$

Therefore limit exists and $\lim\limits_{x\to 2}\dfrac{3}{(x-2)^2}=\infty$

Ex 7. Evaluate $\lim\limits_{x\to 2} \dfrac{-1}{(x-2)^2}$

Let $f(x)=-\dfrac{1}{(x-2)^2}$

We have

$$\lim_{h\to 0^+} -f(2+h) = \lim_{h\to 0^+} \frac{-1}{(2+h-2)^2} = -\infty$$

$$\therefore \lim_{h\to 0} f(2+h) = -\infty$$

Ex. 8. **Show that**

$$\lim_{x\to 0+} \frac{1}{x} = \infty, \text{ and } \lim_{x\to 0^-} \frac{1}{x} = -\infty$$

Let $f(x)=1/x$

Let $x>0$, so that $f(x)$ is positive. If x diminishes gradually then $f(x)$ or $1/x$ increases gradually. If x tends to zero then $f(x)$ or $\frac{1}{x}$ tends to be infinite.

If we take N any given positive number, however large then $1/x>N$ when $x<1/N$

Hence $\lim_{x\to 0^+} (1/x)=\infty$

Let $x<0$, so that $f(x)$ is negative.

Let N be any given positive number then for $x<0$,

$-f(x)>N$ or, $-(1/x)>N$. if $x<-1/N$,

As N increases, $\frac{1}{N}$ decreases

Hence $\lim_{x\to 0^-} (1/x)=-\infty$

Art 2·6 Meaning of $\lim_{x\to\infty} f(x)=l$.

Let a be any positive numbers, that $f(x)$ is defined for all numbers $x \geq a$. We say that $f(x)$ *approaches l as x tends to infinity*, and we write $\lim_{x\to\infty} f(x)=l$

If the following condition is satisfied; Given any $\epsilon>0$, there exists a positive number A, such that whenever $x>A.$, we have

$$|f(x)-l|<\epsilon.$$

A similar meaning is given to $\lim_{x\to -\infty} f(x)=l$.

That is, for a given $\epsilon>0$, there exists a positive number A, such that whenever

$$x<-A, \text{ we get } |f(x)-l|<\epsilon$$

[where $(-\infty,-A)$ is a subset of D_f]

(i) when $x>1$. Let $x=1+h$, $0<h<1$

By Binomial theorem,

$$x^n=(1+h)^n=1+nh+\frac{n(n-1)h^2}{\underline{2}}+\ldots$$

$$\therefore x^n>1+nh>nh$$

Let N be any positive number, such that $nh>N$ or, $n>N/h$.

Therefore $x^n>N$ where N is any large number for all $n>N/h$.

Hence $\lim_{n\to\infty} x^n=\infty$ when $x>1$

(ii) When $-1<x<1$ or $|x|<1$,

let $|x|=\frac{1}{1+h}$, where $0<h<1$.

$$\therefore |x|^n=\frac{1}{(1+h)^n}=\frac{1}{1+nh+\frac{n(n-1)h^2}{2}+\ldots}<\frac{1}{1+nh}<\frac{1}{nh}$$

If $n \to \infty$, $nh \to \infty$ and so $\frac{1}{nh} \to 0$.

Hence $\lim\limits_{n\to\infty} | x |^n = 0$

$\Rightarrow \lim\limits_{n\to\infty} x^n = 0$ when $-1 < x < 1$.

(iii) When $x = -1$ or $+1$ according as n is an odd or even integer. Therefore when $n \to \infty$, x^n oscillates between -1 and $+1$ when $x = -1$.

Hence $\lim\limits_{n\to\infty} (-1)^n$ does not exist.

(iv) When $x < -1$,

or $-x > 1$,

we have from case (ii),

$$\lim_{n\to\infty} (-x)^n = \infty$$

or $\lim\limits_{n\to\infty} x^n = (-1)^n \infty = -\infty$ or $+\infty$ according as n is odd or even.

Hence $\lim\limits_{n\to\infty} x^n$ does not exist for $x < -1$.

(v) The trivial case :

When $x = 0$, $x^n = 0$ for all positive integral values at n

$\therefore \lim\limits_{n\to\infty} x^n = 0$ when $x = 0$.

2. 7. Find $\lim\limits_{n\to\infty} \dfrac{x^n}{\lfloor n}$,

Where n is a positive integer and x is any real number.

Let x be any positive number which lies betwen two consecutive integers m and $m+1$, that is, $m < x < m+1$, where m is a positive integer,

We have,

$$\frac{x^n}{\lfloor n} = \frac{x}{1} \cdot \frac{x}{2} \cdot \frac{x}{3} \cdots \frac{x}{m} \cdot \frac{x}{m+1} \cdots \frac{x}{n}$$

$$= \frac{x^m}{\lfloor m} \cdot \frac{x}{m+1} \cdot \frac{x}{m+2} \cdots \frac{x}{n}$$

Now $\frac{x}{m+2} < \frac{x}{m+1}$, $\frac{x}{m+3} < \frac{x}{m+1}, \ldots, \frac{x}{n} < \frac{x}{m+1}$.

$$\therefore \quad \frac{x^n}{\lfloor n} < \frac{x^m}{\lfloor m} \left[\frac{x}{m+1} \cdot \frac{x}{m+1} \cdot \frac{x}{m+1} \cdots \frac{x}{m+1} \right]$$

or $\dfrac{x^n}{\lfloor n} < \dfrac{x^m}{\lfloor m} \cdot \left(\dfrac{x}{m+1} \right)^{n-m}$ (since in the square bracket, there are $(n-m)$ factors').

or $\dfrac{x^n}{\lfloor n} < \dfrac{(m+1)^m}{\lfloor m} \left(\dfrac{x}{m+1} \right)^n$

Hence $0 < \dfrac{x^n}{\lfloor n} < L . \left(\dfrac{x}{m+1} \right)^n$

Where $L = \dfrac{(m+1)^m}{\lfloor m}$ is a constant free from n.

As $\frac{x}{m+1}$ is positive and less than 1, that is, $0 < \frac{x}{m+1} < 1$,

therefore $\lim\limits_{n\to\infty} \left(\dfrac{x}{m+1} \right)^n = 0$.

Hence $\lim\limits_{n\to\infty} \dfrac{x^n}{\lfloor n} = 0$

Let x be any negative number say $x=-a$. so that a is a positive number.

$$\left|\frac{x^n}{\text{L}n}\right|=\left|\frac{(-1)^n a^n}{\text{L}n}\right|=\frac{a^n}{\text{L}n}$$

By the previous result, $\lim_{n\to\infty}\frac{a^n}{\text{L}n}=0$

Hence $\lim_{n\to\infty}\frac{x^n}{\text{L}n}=0$ whatever be the value of x

2.8. Fundamental Theorems on limits (সীমার যোগ উপপাদ্য)

Theorem 1. The limit of a sum of any definite number of function is equal to the algebraic sum of the limits of these functions.

Let us first consider two functions of x

If $\lim_{x\to a} f(x)=l$, $\lim_{x\to a} \phi(x)=m$

then $\lim_{x\to a} \{f(x)\pm\phi(x)\}=l\pm m$

If $\in$ be a given arbitrary small positive number, we can choose another two positive smaller numbers δ_1 and δ_2 depending on $\in$ such that

$|f(x)-l|<\in/_2$ for $0<|x-a|<\delta_1$......(1)

$|\varphi(x)-m|<\in/_2$ for $0<|x-a|<\delta_2$......(2)

Let us consider another positive number δ which is less than both δ_1 and δ_2. Then

$|f(x)-l|<\in/_2|$, $|\varphi(x)-m|<\in/_2$ for $0<|x-a|<\delta$

Thus

$|f(x)+\varphi(x)-(l+m)|=|(f(x)-l)+|\varphi(x)-m)|$

$<|f(x)-l|+|\varphi(x)-m|$

$<\frac{\in}{2}+\frac{\in}{2}=\in$ when $0<|x-a|<\delta$.

Hence $\lim_{x\to a}\{f(x)+\varphi(x)\}=l+m$......(3)

Similarly we can prove that

$\lim_{x\to a}\{f(x)-\varphi(x)\}=l-m$,......(4)

Now combining (3) and (4), we have

$\lim_{x\to a}\{f(x)\pm\phi(x)\}=l\pm m$

Similarly we can extend it to any number of functions

$\lim_{x\to a}\{f(x)\pm\phi(x)\pm\psi(x)\pm......\}$

$=\lim_{x\to a} f(x)\pm\lim_{x\to a}\phi(x)\pm\lim_{x\to a}\psi(x)\pm$ ....

$=l_1\pm l_2\pm l_3\pm$ ....

Theorem 2 The limit of the product of any finite number of functions is equal to the product of the limits of these functions :

If $\lim_{x\to a} f_1(x)=l_1$, $\lim_{x\to a} f_2(x)=l_2$,.........

then $\lim_{x\to a}[f_1(x)\,f_2(x)\,......]=l_1\,l_2$.........

Proof : We prove that result for two functions first.

Let $\lim_{x\to a} f(x)=l$ and $\lim_{x\to a} g(x)=m$.

Then for a given number $\in>0$, however small, there exists a positive number δ such that

$$\left.\begin{array}{ll} |f(x)-l| < \dfrac{\epsilon}{2|m|}, & \text{if } m \neq 0 \\ |f(x)-l| < \dfrac{\epsilon}{2}, & \text{if } m = 0 \end{array}\right\}$$

$|g(x)-m| < \frac{\epsilon}{2}$ $\quad \frac{1}{|l|+1} \leqslant \frac{\epsilon}{2}$, (the equality holds when $l=0$),

for $0 < |x-a| < \delta$.

From $|f(x)-l| < \frac{\epsilon}{2|m|}$ or $|f(x)-l| < \frac{\epsilon}{2}$

We have, $|f(x)| < |l|+1$

Hence, for $0 < |x-a| < \delta$.

$$\begin{aligned} |f(x)\, g(x)-lm| &= |\{f(x)g(x)-f(x)m\}+\{f(x)m-lm\}| \\ &= |f(x)(g(x)-m)+m(f(x)-l)| \\ &\leqslant |f(x)|\ |g(x)-m| + |m|\ |f(x)-l| \\ &< (|l|+1)\frac{\epsilon}{2}\frac{1}{|l|+1} + |m| \cdot \frac{\epsilon}{2|m|} \\ &= \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon \end{aligned}$$

Therefore $\lim_{x\to a} f(x)\, g(x) = lm = \lim_{x\to a} f(x).\ \lim_{x\to a} g(x)$.

Cor : It c be a constant, them

$\lim_{x\to a} cf(x) = cl$ where $\lim_{x\to a} f(x) = l$.

The theorem can be extented to any number of functions.

Theorem 3. If $\lim_{x\to a} f(x) = l$, then $\lim_{x\to a} \frac{1}{f(x)} = \frac{1}{l}$.

providid $l \neq 0$,

Proof : Given $\epsilon > 0$ let ϵ_1 be the smaller of $\frac{|l|}{2}$ and ϵ Then there exists $\delta > 0$ such that whenever

$0 < |x-a| < \delta$, we have

$$|f(x)-l| < \epsilon_1 \quad \text{(i)}$$

and also

$$|f(x)-l| < \frac{\epsilon |l|^2}{2} \quad \text{(ii)}$$

From (i).

$$|f(x)| > |l| - \epsilon_1 \geq |l| - \frac{|l|}{2} = \frac{|l|}{2} \quad \text{... (iii)}$$

when $0 < |x-a| < \delta$. For such x,

$$\left|\frac{1}{f(x)} - \frac{1}{l}\right| = \frac{|l-f(x)|}{|f(x)||l|} < \frac{2}{|l|} \cdot \frac{1}{|l|} \cdot \frac{\epsilon |l|^2}{2}$$

[by (ii) and (iii)]

or $\left|\frac{1}{f(x)} - \frac{1}{l}\right| < \epsilon$

Hence $\lim_{x\to a} \frac{1}{f(x)} = \frac{1}{l}$.

Cor : If $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$

then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} f(x).\ \lim_{x\to a} \frac{1}{g(x)} = \frac{l}{m}$,

provided $m \neq 0$

The theorem can be easily extended for any definite number of functions

$\operatorname{Lt}_{x\to a} f(x)\varphi(x)\psi(x) \ldots \ldots = lmn \ldots \ldots$

Theorem 4 : Limit of a function of a function

(ফাংশানের ফাংশন সীমা)

If $\lim\limits_{x\to a} f(x)=t$ and $\lim\limits_{u\to t} \varphi(u)=\varphi(t)$

is in other words, if φ (u) is continuous at u = t, then

$$\lim_{x\to a} \varphi\{f(x)\} = \varphi\{\lim_{x\to a} (x)\}$$

As $\lim\limits_{u\to t} \varphi(u)=\varphi(t)$, we can select a number δ, depending upon ϕ

to a given small positive number $\in$, such that

$$|\varphi(f(x)-t)| < \in, \text{ for } |f(x)-t| \leqslant \delta_1, \quad \ldots\ (1)$$

Since $\lim\limits_{x\to a} f(x)=t$, a number δ_2 can be selected depending on δ_1

such that

$$|f(x)-t| < \delta_1 \text{ for } |x-a| < \delta_2 \ \ldots\ldots\ldots (2)$$

Now from [1] and [2] we have

$$|\varphi\{f(x)-t\}| < \in \text{ for } 0 < |x-a| < \delta_2$$

i e $\lim\limits_{x\to a} \varphi\{f(x)\} = \varphi(t) = \phi\{\lim\limits_{x\to a} f(x)\}$

Ex 9. If $\lim\limits_{x\to a} f(x)=l$

[i] $\lim\limits_{x\to a} \sin f(x) = \sin\left(\lim\limits_{x\to a} f(x)\right) = \sin l$

[ii] $\lim\limits_{x\to a} e^{f(x)} = e^{\lim\limits_{x\to a} f(x)} = e^{l}$

[iii] $\lim\limits_{x\to a} (f(x))^n = \left\{\lim\limits_{x\to a} f(x)\right\}^n = l^n$

Summary on the fundamental Theorems on limits, (সীমার মৌলিক উপপাদ্য)

If $\lim\limits_{x\to a} f(x)=l$. and $\lim\limits_{x\to a} \phi(x)=m$

where l and m are finite quantities then

(i) $\lim\limits_{x\to a} \{f(x)\pm f(x)\}=l\pm m$

(ii) $\lim\limits_{x\to a} \{f(x)\phi(x)\}=lm$

(iii) $\lim\limits_{x\to a} \left\{\dfrac{f(x)}{\phi(x)}\right\}=\dfrac{l}{m}$ if $\lim\limits_{x\to a} \phi(x)\neq 0$

(iv) $\lim\limits_{x\to a} \phi\{f(x)\}=\phi\left(\lim\limits_{x\to a} f(x)\right)$

Limits of a sequence :—(See fuuction) (ধারার সীমা).

A number l is the limit of a sequence $a_1\ a_2 \ldots \ldots a_n \ldots\ldots$

or, $\lim\limits_{n\to\infty} a_n = l$

If for every positive value of $\in$ ($\in > 0$) however small, there is a number N such that

$$|a_n - l| < \in \text{ when } n \geqq N.$$

we can also express limit of a sequence as

$$\{a_n\} \to l \text{ when } n \to \infty,$$

Ex, 10. Show that

$$\lim_{n\to\infty} \frac{3n+2}{n+3} = 3.$$

Here

$$\left|\frac{3n+2}{n+3} - 3\right| = \left|\frac{-7}{n+3}\right| = \frac{7}{n+3} < \in \ \ldots\ldots (1)$$

if $n+3 > 7/\in$ or, if $n > 7/\in - 3 = N$

Thus for every positive value of ϵ, there will be a number $N = 7/\epsilon - 3$ such that for $n > N$. ; (i) holds

$$\therefore \lim_{n \to \infty} \frac{3n+2}{n+3} = 3.$$

2. 13. Some Important Limits (কতিপয় গুরুত্বপুর্ণ সীমা)

(a) Prove that

$$\lim_{x \to 0} \frac{\sin x}{x} = 1. \quad \text{when } x \text{ is measured in radian.}$$

Proof : This function is not continuous at $x=0$. Let us consider a circle of unit radius, Let A, B, C be three points on the circle such that

$\angle AOB = \angle AOC = x$

radians, where

$0 > x < \pi/2$

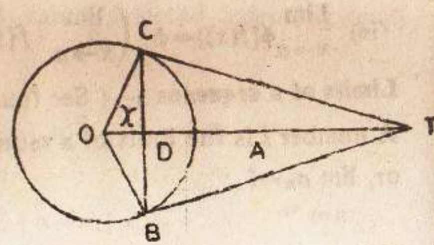

Fig. 33

Let the tangents at B and C meet at T on OT. Since BC is a chord of the circle, we have.

Chord $BC <$ arc $BAC < TC + TB$

or, $2\,CD < 2$ arc $AC < 2CT$,

or, $\dfrac{CD}{OC} < \dfrac{\text{arc } AC}{OC} < \dfrac{CT}{OC}$ or, $\sin x < x < \tan x$.

or, $1 < \dfrac{x}{\sin x} < \dfrac{1}{\cos x}$ or, $1 > \dfrac{\sin x}{x} > \cos x$ ... ... (I)

This inequality is assumes that $x > 0$, It is also true if $x > 0$

Now $\lim\limits_{x \to 0} \cos x = 1$

Therefore from (1) we notice that in the limits both sides of $\dfrac{\sin x}{x}$ are the same and equal to unity,

Hence $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$

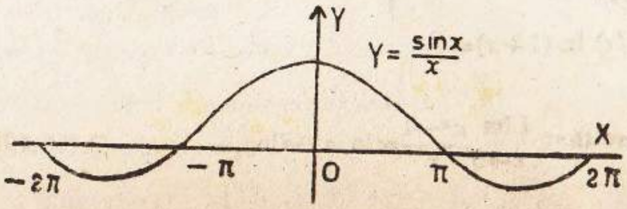

fig. 47

(b) Prove that $\lim\limits_{x \to 0} (1+x)^{1/x} = e$

Proof :—Expanding binomially.

$$(1+x)^{\frac{1}{x}} = \left\{ 1 + \frac{1}{x}.x + \frac{(1/x)(1/x-1)}{\lfloor 2} x^2 + \dots \right\}$$

$$= \left\{ 1 + 1 + \frac{(1-x)}{\lfloor 2} + \frac{(1-x)(1-2x)}{\lfloor 3} + \dots\dots \right\}$$

Now $\lim\limits_{x \to 0} (1-x) = 1$, $\lim\limits_{x \to 0} (1-2x) = 1$, etc,

Hence

$$\lim_{x \to 0} (1+x)^{1/x} = 1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \dots \dots = e$$

(c) **Proof** $\lim\limits_{x \to 0} (1 + 1/x)^x = e$,

The result follows on expanding $(1 + 1/x)^x$ binomially and then taking the limit.

94-98

Thus for every positive value of $\in$, there will be a number $N = 7/\in - 3$ such that for $n > N$. ; (i) holds

$$\therefore \lim_{n\to\infty} \frac{3n+2}{n+3} = 3.$$

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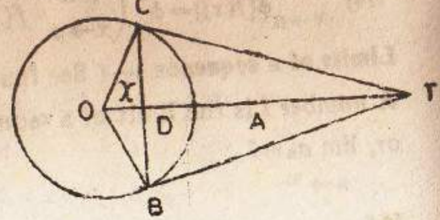

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or, $1 < \frac{x}{\sin x} < \frac{1}{\cos x}$ or, $1 > \frac{\sin x}{x} > \cos x$ ... ... (I)

This inequality is assumes that $x > 0$, It is also true if $x > 0$

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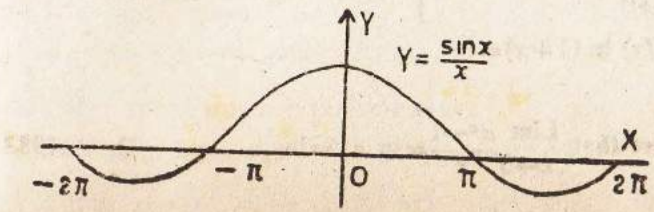


fig. 47

(b) Prove that $\lim_{x\to 0} (1+x)^{1/x} = e$

Proof :—Expanding binomially.

$$(1+x)^{\frac{1}{x}} = \left\{ 1 + \frac{1}{x}.x + \frac{(1/x)(1/x-1)}{\lfloor 2}x^2 + \ldots \right\}$$

$$= \left\{ 1 + 1 + \frac{(1-x)}{\lfloor 2} + \frac{(1-x)(1-2x)}{\lfloor 3} + \ldots\ldots \right\}$$

Now $\lim_{x\to 0} (1-x) = 1$, $\lim_{x\to 0} (1-2x) = 1$, etc,

Hence

$$\lim_{x\to 0} (1+x)^{1/x} = 1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \ldots \ldots = e$$

(c) **Proof** $\lim_{x\to 0} (1 + 1/x)^x = e,$

The result follows on expanding $(1+1/x)^x$ binomially and then taking the limit.

(d) prove that $\lim_{x\to 0}\left\{\frac{1}{x}\log(1+x)\right\}=1$

$$\lim_{x\to 0}(1/x)\ln(1+x)=\lim_{x\to 0}\ln(1+x)^{1/x}$$

$$=\ln\left\{\lim_{x\to 0}(1+x)^{1/x}\right\}=\ln e=\log_e e$$

$$\Rightarrow \lim_{x\to 0}(1/x)\ln(1+x)=1.$$

(e) prove that $\lim_{x\to 0}\frac{a^x-1}{x}=\ln a=\log_e a$ D. U. 1983

Proof :

Put $a^x-1=y$ or, $a^x=1+y$

$\therefore\ x\ln a=\ln(1+y)$ or, $x=\frac{\ln(1+y)}{\ln a}$,

$$\text{Now } \frac{a^x-1}{x}=\frac{y\ln a}{\ln(1+y)}=\frac{\ln a}{(1/y)\ln(1+y)}$$

$$\therefore \lim_{x\to 0}\frac{a^x-1}{x}=\lim_{y\to 0}\frac{\ln a}{(1/y)\ln(1+y)}$$

$$=\lim_{y\to 0}\frac{\ln a}{\ln(1+y)^{1/y}}=\frac{\log a}{\log e}=\ln a=\log a \ldots\ \ldots \text{ by (c)}$$

$$\lim_{x\to 0}\frac{a^x-1}{x}=\log a$$

(f) show that $\lim_{x\to 0}\frac{e^x-1}{x}=1$

Proof :—Put $e^x-1=y$ or, $e^x=1+y$.

$\therefore\ x=\log_e(1+y)=\ln(1+y)$

Since $\ln 1=0$ therefore $y\to 0$ as $x\to 0$.

$$\text{Now } \lim_{x\to 0}\frac{e^x-1}{x}=\lim_{x\to 0}\frac{y}{\ln(1+y)}=\lim_{y\to 0}\frac{1}{(1/y)\ln(1+y)}$$

$$=\frac{1}{\ln e}=1\quad [\because \log_e e=\ln_e=1]$$

(g) **Prove that** $\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}$

for all rational values of n provided a is positive.

Case. I. When n is a positive integer.

$$\lim_{x\to a}\frac{x^n-a^n}{x-a}=\lim_{x\to a}\{x^{n-1}+x^{n-2}a+\ldots\ldots+a^{n-1}\}$$

$$=a^{n-1}+a.\,a^{n-2}+\ldots+xa^{n-1}=na^{n-1}$$

Case II. When n is a negative integer.

Let $n=-m$, m being a positive integer and $a\neq 0$

$$\therefore \lim_{x\to a}\frac{x^n-a^n}{x-a}=\lim_{x\to a}\frac{x^{-m}-a^{-m}}{x-a}=\lim_{x\to a}\left(\frac{1}{x^m}-\frac{1}{a^m}\right)$$

$$=\lim_{x\to 1}\frac{-1}{a^m x^m}\left(\frac{x^m-a^m}{x-a}\right)$$

$$=-\frac{1}{a^m}\lim_{x\to a}\frac{1}{x^m}\lim_{x\to a}\left(\frac{x^m-a^m}{x-a}\right)=-\frac{1}{a^m}\cdot\frac{1}{a^m}ma^{m-1}$$

[by case I]

$$=(-m)\,a^{(-m)-1}$$

$$=na^{n-1}\qquad [\because -m=n].$$

(h) Prove that $\lim_{x\to a}\frac{(1+x)^n-1}{x}=n$ D. U. 1983.

Proof :—

$$\lim_{x\to 0}\frac{(1+x)^n-1}{x}=\lim_{x\to 0}\left[(1+nx+\frac{n(n-1)x^2}{\lfloor\underline{2}}+\ldots\ldots)-1\right]\Big/x$$

$$=\lim_{x\to 0}\left\{nx+\frac{n(n-1)x^2}{\lfloor\underline{2}}+\frac{n(n-1)(n-2)^2}{\lfloor\underline{2}}+\ \ldots\ \ldots\right\}\Big/x$$

$$=\lim_{x\to 0}\left\{n+\frac{n(n-1)x}{\lfloor\underline{2}}+\frac{n(n-1)(n-2)x^2}{\lfloor\underline{3}}+\ \ldots\ \ldots\right\}$$

$= n$, since each of the remaining terms $\to 0$ as $x \to 0$

$\therefore \lim_{x \to 0} \frac{(1+x)^n - 1}{x} = n$

The f llowing limits are very useful in solving problems.

(কয়েকটি প্রয়োজনীয় সীমা)

(a) $\lim_{x \to 0} \frac{\sin x}{x} = 1$ (b) $\lim_{x \to 0} (1+x)^{1/x} = e$

(c) $\lim_{x \to \infty} (1+1/x)^x = e$ (d) $\lim_{x \to 0} \frac{1}{x} \log_e (1+x) = 1$

(e) $\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a$ (f) $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

(g) $\lim_{x \to 1} \frac{x^n - a^n}{x - a} = na^{n-1}$ (h) $\lim_{x \to 0} \frac{(1+x)^n - 1}{x} = n$

Proofs of these are given in Art. 2.13

Ex. 10 (*a*) Prove that, $\lim_{x \to \infty} \frac{2x+3}{2x} = 1$

We can write the ab ve functi n also in the form

$\lim_{x \to \infty} \left(1 + \frac{3}{2x}\right) = 1$

Now for an arbitrary ε, the inequality is satisfied

$|(1+3/2x) - 1| < \varepsilon,\ \varepsilon \to 0$

provided $x > N$. w ere N is determined by choice of ε

or, $3/2x < \varepsilon$ or, $\frac{1}{x} < \frac{2\varepsilon}{3}$ or. $x/>3/2\varepsilon = N$

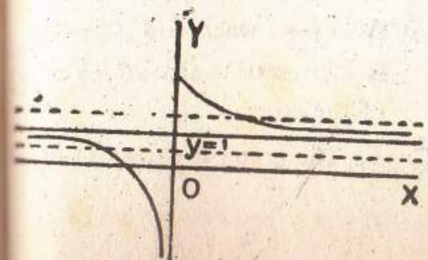
which means that

$\lim_{x \to \infty} \left| 1 + \frac{3}{2x} \right| = \lim_{x \to \infty} \frac{2x+3}{2x} = 1$

$\therefore \lim_{x \to \infty} \frac{2x+3}{2x} = 1$

It can be shown similarly that $\lim_{x \to \infty} \frac{2x+3}{2x} = 1$

This graph is not defined for $x = 0$ *i.e*, y-axis is the asymptote of the curve. Again $y \to 1$ if $x \to \infty$ *i. e.*, the straight line $y = 1$ is another asymptote of the curve.



The graph $y = 1 + \frac{3}{2x}$ is shown

Fig 48

Ex. 11. Prove that $\lim_{x \to 2} \frac{1}{(2-x)^2} = +\infty$

For any positive number $N > 0$, however large, we have

$\frac{1}{(2-x)^2} < N$ *i.e.*, $(2-x)^2 < \frac{1}{N}$

or, $|2-x| < \frac{1}{\sqrt{(N)}} = \delta$(say)

which depends upon N such that $|\delta| \geq 0$

The function $\frac{1}{(2-x)^2}$ assumes only positive values.

and it is greater than any large positive number

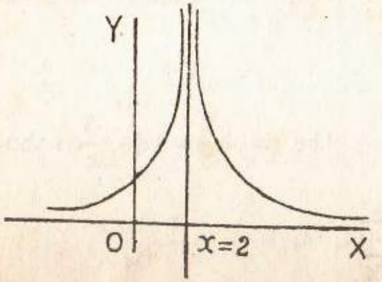
whenever $0 < |x-2| < \frac{1}{\sqrt{N}}$

Hence $\lim_{x \to 2} \frac{1}{(2-x)^2} = +\infty$

Let $y = \frac{1}{(2-x)^2}$

For all values of x, y will be positive and for $x=2$, y is undifined. The line $x=2$ is an Asymptote of the graph ;

the graph is symmetrical about the line $x=2$.



The graph passes through $(0,2.5)$;$(1,1)$;$(-1,0.1)$ etc.

Also $y\to 0$ when $x\to\infty$ or $-\infty$

Hence x-axis is also an asymptote of the curve.,

The graph is shown in fig 49

Fig. 49

x	-2	-1	0	1	3/2	$2\pm$	2·5	3	4	5	etc
y	0·0625	0·11	.25	1	4	∞	4	1	·25	0·11	etc

Ex. 12 Show that $\lim\limits_{x\to\infty} \dfrac{\sin x}{x}=0$……

Since $|\sin x| \leqslant 1$ for all x,

therefore $\left|\dfrac{\sin x}{x}\right| \leqslant \dfrac{1}{|x|} = \dfrac{1}{x}$

which tends to 0 as $x\to\infty$

$$\therefore \lim_{x\to\infty}\left(\frac{\sin x}{x}\right)=0$$

Ex. 13. Examine the limit of $f(x)=\dfrac{1}{x}\sin\dfrac{1}{x}$

as x tends to zero.

Since $\left|\sin\dfrac{1}{x}\right| \leqslant 1$ for all real values of x except $x=0$,

therefore $\sin\dfrac{1}{x}$ oscillates between -1 and $+1$ as $x\to 0$ or

$\dfrac{1}{|x|}\to\infty$. Hence $\dfrac{1}{x}\sin\dfrac{1}{x}$ oscillates between $-\infty$ and

$+\infty$ as $x\to 0$ So $\lim\limits_{x\to 0}\left(\dfrac{1}{x}\sin\dfrac{1}{x}\right)$ does not exist.

Ex. 14. Show that $\lim\limits_{x\to 0} x\sin(1/x)=0$

For non zero values of x

$|x\sin(1/x)| = |x|\,.\,|\sin(1/x)| \leqslant |x|$ as $\sin(\sin(1/x)| \leqslant 1$

Thus $|x\sin(1/x)-0| = |x\sin(1/x)| \leqslant |x| < \epsilon$.

such that for all values x in,

$0<|x-0|<\epsilon$ i. e. $|x|<\epsilon$ or, $-\epsilon<x<\epsilon$ with $x\neq 0$.

Thus we notice that if ϵ is a positive number however small there is an interval $(-\epsilon, \epsilon)$ around 0 such that for all values of x, $x\neq 0$, the difference between $\sin(1/x)$ and 0 is less than a positive number ϵ, $\epsilon>0$

Hence $\lim\limits_{x\to 0} x\sin(1/x)=0$

Let $y=\sin(1/x)$; $|\sin(1/x)| \leq 1$

Thus $x\sin(1/x)$ oscillates between $y=x$ and $y=-x$, as x tends to zero.

Ex. 15. Find the limits of $\lim\limits_{x\to 0}\dfrac{1+2^{1/x}}{3+2^{1/x}}$

If $x=0+h$, where $h>0$

$\therefore \quad \dfrac{1}{x} = \dfrac{1}{h}\to\infty$ as $h\to 0$

Now $\lim_{h\to 0} f(0+h) = \lim_{h\to 0} \frac{1+2^{1/h}}{3+2^{1/h}} = \lim_{h\to 0}\frac{2^{-1/h}+1}{3.2^{-1/h}+1} = \frac{0+1}{0+1}=1$

Again. $\lim_{h\to 0} f(0-h) = \lim_{h\to 0}\frac{1+2^{-1/h}}{3+2^{-1/h}} = \frac{1+0}{3+0}=\frac{1}{3}$

$$[\therefore\ 2^{-1/h}\to 0 \text{ as } \frac{1}{h}\to\infty]$$

$\lim_{h\to 0} f(0+h) \neq \lim_{h\to 0} f(0-h)$

Hence the limit does not exist.,

Note : The notation '$\lim_{x\to a} f(x)$' will mean the same statement as, $\lim_{x\to a} f(x)$

16. A real function f is defined by $f(x)=\frac{2x}{1-x}$

(i) Determine the domain and the range of f

(ii) Find $\lim_{x\to 1+} f(x)$ and $\lim_{x\to 1-} f(x)$

(iii) Find $\lim_{x\to +\infty} f(x)$ and $\lim_{x\to -\infty} f(x)$ R. H 1988

(iv) Find $f'(x)$ and the interval where f is increasing.

(v) Draw a sketch of the graph of f

Sol : (i) $f(x)=y=\frac{2x}{1-x}$ ... ... (1)

or, $y(1-x)=2x$ or, $x=\frac{y}{y+2}$ ... ... (2)

If x is real then from (1), $\frac{2x}{1-x}$ is also real,

$\frac{2x}{1-x}$ is undefined at $x=1$.

So the domain of $f=(-\infty,\infty)-\{1\}$

$=(-\infty, 0] \cup [0, 1) \cup (1, \infty)$ from (2)

For range of f, $x=\frac{y}{y+2}$ form (2)

For any real y, $\frac{y}{y+2}$ is also real but f^{-1} is undefined

for $y=-2$

Hence range of $f=(-\infty,\infty)-\{-2\}=(\infty,-2)\cup(-2, 0]$
$\cup [0, \infty]$

(ii) $\lim_{x\to 1+} f(x) = \lim_{h\to 0}\frac{2(1+h)}{1-(1+h)} = \lim_{h\to 0}\frac{2-2h}{-h} = -\infty$

$\lim_{x\to 1-} f(x) = \lim_{h\to 0}\frac{2(1-h)}{1-(1-h} = \lim_{h\to 0}\frac{2-2h}{h} = \infty$

(iii) $f(x)=\frac{2x}{1-x}=\frac{2x}{x(1/x-1)}=\frac{2}{1/x-1}$

$\lim_{x\to\infty} f(x)=\frac{2}{0-1}=-2$

$\lim_{x\to -\infty} f(x)=\frac{2}{-0-1}=-2$

(iv) $f(x)=\frac{2x}{1-x}$: $f'(x)=\frac{(1-x).2-(-1)2x}{(1-x)^2}=\frac{2-2x+2x}{(1-x)^2}$

$$=\frac{2}{(1-x)^2}$$

$\therefore$ $f'(x)>0$ for all x but $x\neq 1$

When $x=0, f(0)=y=0$. From (ii), (iii), (iv), $f(x)=0$ to ∞ i. e, increasing from $x=0$ to 1 i. e. in $[0, 1)$ Also f is increasing from $-\infty$ to -2 in $(-\infty, -2)$ and $[-\infty, 0]$

(v) From the above points. the graph is $f(x)$, and passes $(0, 0)$ through increases to ∞ at $x=1$, then $f(x)$ decreases to $y=-2$ when $x=0$ to $-\infty$.

Again $f(x)$ increases from $-\infty$ to -2 when $x=1+$ to ∞.

The graph is shown

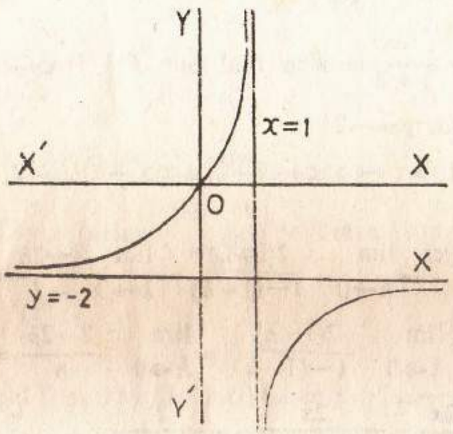


Fig 50

Exercise 11

Show that,

1. $\lim_{x\to 0} \frac{\sqrt{(1+x)}-1}{\sqrt{x}}=0$ (i) $\lim_{n\to\infty} \frac{n^2+n-1}{3n^2+1}=\frac{1}{3}$

2. $\lim_{x\to\infty} \{\sqrt{(x+1)}-\sqrt{x}\}=0.$ 3. $\lim_{n\to\infty} (2^n+3^n)^{1/n}=3$

4. $\lim_{x\to 1} \frac{x-\sqrt{(2-x^2)}}{2x-\sqrt{(2+2x^2)}}=2$ 4(a) $\lim_{x\to 4} \frac{\sqrt{x}-2}{\sqrt{x+\sqrt{7}}-3}=6$

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5. $\lim_{x\to 1} \frac{\sqrt{x}-1}{\sqrt[3]{x}-1}=3/2$ 6. $\lim_{x\to a} \frac{x^4-a^4}{x-a}=4a^3$

7. $\lim_{x\to 0} \frac{1-\cos x}{x}=\frac{1}{2}$ 8. $\lim_{x\to a} \frac{\tan x-\sin x}{x^3}=\frac{1}{2}$

9. $\lim_{x\to 0} \frac{\sin 5x}{\sin 6x}=5/6$ 10. $\lim_{x\to\infty} \frac{5x+2}{3x+7}=5/3$

11. $\lim_{x\to\infty} \frac{x^2-2x+3}{5x^2+7x-5}=7/5$ 12. $\lim_{x\to 0} \frac{\tan x}{x}=1$

13. $\lim_{x\to 0} \frac{\sqrt{(x+1)}-1}{x}=\frac{1}{2}$ 14. $\lim_{x\to 0} \frac{1-\cos x}{x}=0$

15. $\lim_{x\to 3} \frac{x^2-4x+3}{x^2-2x-3}=\frac{1}{2}$ 15(a) $\lim_{x\to\pi/2} \frac{e^{\tan x}-1}{e^{\tan x}+1}=1$

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16. Find the following limits

(i) $\lim_{x\to\infty} \frac{\sin x}{x}$ *Ans.* 0

(ii) $\lim_{x\to 0} \frac{1}{x} e^{1/x}$ *Ans.* not exist.

(iii) $\lim_{x\to a} \frac{1}{1-e^{1/x-a}}$ *Ans.* does not exist

(iv) $\lim_{x\to 0} \cos \frac{1}{x}$ *Ans.* ,,

(v) $\lim_{x\to a} \cos \frac{1}{x-a}$ *Ans.* ,,

(vi) $\lim_{x\to 0} \frac{1+5x^2}{x}$ *Ans.* ,,

(vii) $\lim_{x\to 0} e^{-1/x}$ *Ans.* ,,

(viii) $\lim_{x\to 0} \frac{1}{1-e^{1/x}}$ *Ans.* ,,

(ix) $\lim_{x\to\infty} \frac{3^x-3^{-x}}{3^x+3^{-x}}$ *Ans.* 1

17. Show that $\lim_{x\to 0} \frac{1}{x}$ does not exist

18. Show that

$\lim_{x\to\infty} 2^x = \infty$ and draw the graph of the function.

19. Show that $\lim_{x\to -\infty} 2^x = 0$

20. By (δ, ω) definition prove that

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = 2$$

21. Use (δ, $\in$) definition to show that

$$\lim_{x\to 2} x^3-3x+7=9$$

22. For $f(x)=x^2-2x-8$, find a $\delta>0$.

such that whenever $0<|x-3|<\delta$

then $|f(x)+5|<\in$ when (a) $\in=1/5$ (b) $\in=0.001$.

Ans. 1/25. 0.0006

23. Find the limit of the sequence $\{a_n\}$ for a_n as defined in such case.

(i) $\frac{n^2+n-1}{4n^2-5}$ Ans. $\frac{1}{4}$

(ii) $a_n=\frac{a}{n}$ Ans. 0

(iii) $a_n=1+1/2^n$ Ans. 1

(iv) $a_n=\frac{2n^3-n}{n^4+n}$ Ans. 0

(v) If $0<r<1$, then $\lim_{n\to\infty} r^n=0$

(vi) $a^n=\frac{1}{\sqrt{n}}\cos\frac{n\pi}{2}$ Ans 0

24. Test whether a limit exists of the followings equences.

(a) 1, 0, 1, 0, 1 .... $\frac{1-(-1)^n}{2}$ Ans. does not exist.

(b) 0.9, 0.0, 0.998, ...$1-\frac{2}{10^n}$ Ans. 1

(c) 5, 4, $\frac{11}{3}$, $\frac{7}{2}$, $\frac{17}{5}$ ... ... $3+2/n$ Ans, 34

(d) 2, 5/2, 8/3, 1/8, 14/5 ... ... Ans 3

(e) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}$, 1/16, 1/32 ... ... Ans. 0

25 Consider the real function defined by

$$f(x)=\frac{x}{1-x^2}$$

a. Find $\lim_{x\to +1} f(x)$ and $\lim_{x\to -1} f(x)$

b. Find $\lim_{x\to -1+} f(x)$ and $\lim_{x\to -1-} f(x)$ D. U. 1984

c. Find $\lim_{x\to +\infty} f(x)$ and $\lim_{x\to -\infty} f(x)$

d. Find $f(x)$ and the interval, where $f(x)$ increasing.

e. sketch the graph of f

26. $\lim_{x\to a} f(x)=l \rightarrow \lim |f(x)|=|l|$

The converse will not be true. Establish it with an example.

27. If $f(x)=x$, x is rational

$=-x$ x is irrational

Show that $\lim_{x\to a} f(x)$ exists only when $a=0$

Continuity

CHAPTER III

3. 1 : Continuity : In ordinary sense continuity means something is done without any break. For example if a boy runs for half an hour without stopping anywhere, we say that the boy has run *continuously* for half an hour. If, however, in course of half an hour running, the boy meets a ditch (say), then he has neither *to stop* or *to jump* the ditch to continue his running ; In both the cases his running is *discontinuous over this interval of half on hour.*

Mathematical definitions of continuity and discontinuity are as follows :

Continuity : A function $f(x)$ is said to be **continuous** at $x=a$, if the following conditions are satisfied

(i) $a \in D_f$ **(that is, f is defined at $x=a$)**

and (ii) $\lim\limits_{x \to a} f(x) = f(a)$.

If any of the above two conditions is not satisfied, then $f(x)$ is said to have a **discontinuity** at $x=a$.

3. 2. Cauchy's Definition of continuity :

A function $f(x)$ is continuous at $x=a$ if $f(a)$ is defined and for a small positive mumber $\in$, a mumber $\delta > 0$ can always be determined such that

$$|f(x)-f(a)| < \in \text{ whenever } |x-a| \leqq \delta \text{ or } a-\delta \leqq x \leqq a+\delta$$

3. 3. Discontinuous Functions. (বিচ্ছিন্ন ফাংশন)

A function $f(x)$ is said to be discotinuous for $x=a$ if $f(x)$ does not satisfy any one of the conditions of continuity in Art. 3.2. That is if $f(x)$ is not finite at $x=a$ or, $\lim\limits_{x \to 0} f(x)$ does not exist

or, $\lim\limits_{x \to 0} f(x) \neq f(a)$ then the function $f(x)$ is said to be discontinuous at $x=a$

Different classes of discontinuities have been discussed in the next Article.

3.4. Classification of discontinuites (বিচ্ছিন্নতার প্রকার)

The discontinuities are (1) Ordinary discontinuities (2) Mixed discontinuities, (3) Removal discontinuities. (4) Infinite and (5) Oscillatory discontinuities.

(A) Ordinary discontinuity or, discontinuity of the first kind (সাধারণ ছেদযুক্ততা)

If the function $f(x)$ has finite limit but

$$\lim_{h \to o} f(a+h) \neq \lim_{h \to 0} f(a-h) \neq f(a)$$

the function is said to have ordinary discontinuity or discontinuity of the first kind at $x=a$

(B) Discontinuity of the second kind.

If the limits of $f(x)$.

$$\lim_{h \to 0} f(a+h) \text{ and } \lim_{h \to 0} f(a-h)$$

do not exist for $x=a$ then $f(x)$ has a discontinuity of the second kind at $x=a$.

(C) Mixed discontinuity : (মিশ্র ছেদযুক্ততা)

If one of the limits of $f(x)$ exists then the discontinuities of the function $f(x)$ at $x=a$ is called a mixed discontinuity.

That is if $\lim\limits_{h \to 0} f(a+h) = f_L(a)$ but $\lim\limits_{h \to 0} f(a-h) \neq f(a)$, then

$f(x)$ is continuous on the right but it has a ordinary discontinuity on the left for $x=a$. Similarly if $\lim\limits_{h\to0} f(a-h)=f(a)$ but $\lim\limits_{h\to0} f(a+h)\neq f(a)$, then $f(x)$ is continuous on the right at $x=a$

Such types of discontinuities are called **mixed discontinuities**

(D) Removal discontinuity (অপসারণীয় ছেদযুক্ততা)

If $\lim\limits_{h\to0} f(a+h)=\lim\limits_{h\to0} f(a-h)\neq f(a)$

then the function $f(x)$ is said to have a **removal discontinuity** for $x=a$.

(E) Infinite discontinuity (অসীম ছেদযুক্ততা)

If both the limits of $f(x)$ are infinite, the function $f(a)$ has an infinite discontinuity. That is

if $\lim\limits_{h\to0} f(a+h)$ and $\lim\limits_{h\to0} f(a-h)$ tend to $+\infty$ or $-\infty$ then

$f(x)$ is said to have an infinite discontinuity at $x=a$.

Ex. 1. Prove that the function $f(x)=\sin x$ is continuous for every value of x.

$\sin x$ is defined for all values of x.

Therefore $D_f=\mathbb{R}=(-\infty,\infty)$.

For any $x\in\mathbb{R}$,

$$\lim_{h\to0} f(x+h)=\lim_{h\to0}\sin(x+h)=\lim_{h\to0}(\sin x\cos^h+\cos x\sin^h)$$

$$=\sin x.\lim_{h\to0}(\cos^h)+\cos x.\lim_{h\to0}(\sin^h)$$

$$=\sin x.1+\cos x\times0=\sin x=f(x).$$

Hence $f(x)=\sin x$ is continuous for all real valus of x.

Ex. 2. Discuss the continuity of the function

$$f(x)=\frac{1}{3-e^{1/x}} \text{ at } x=0.$$

Since $\frac{1}{0}$ is undefined, therefore $f(0)$ is undefined.

Hence $f(x)$ is discontinous at $x=0$.

Ex. 3. Show that $f(x)=\cos\frac{\pi}{x}$ has a discontinuity at $x=0$.

$0\notin D_f$, since $\frac{\pi}{0}$ is undefined.

Hence $f(x)$ has a discontinuity at $x=0$.

Ex. 4. Examine the continuity of the function $f(x)$ at $x=\frac{3}{2}$ where

$$f(x)=3-2x \quad \text{for } 0\leqslant x<\tfrac{3}{2},$$
$$f(x)=-3-2x \quad \text{for } x\geqslant\tfrac{3}{2}.$$

Clearly $\frac{3}{2}\in Df$ and $f(\frac{3}{2})=-3-2(\frac{3}{2})=-6$.

Again $\lim\limits_{\substack{h\to0\\h>0}} f(\frac{3}{2}-h)=\lim\limits_{h\to0}[3-2(\frac{3}{2}-h)]=0,$

$\lim\limits_{\substack{h\to0\\h>0}} f(\frac{3}{2}+h)=\lim\limits_{h\to0}[-3-2(\frac{3}{2}+h)]=-6.$

$\therefore \lim\limits_{h\to0} f(\frac{3}{2}+h)=f(\frac{3}{2})=-6\neq\lim\limits_{h\to0} f(\frac{3}{2}-h)$, where $h>0$.

Although $\frac{3}{2}\in D_f$, the limit $\lim\limits_{x\to\frac{3}{2}} f(x)$ does not exist. Hence $f(x)$ is not continuous at $x=\frac{3}{2}$.

Ex. 5. Show that the function

$$f(x)=\frac{x^2-a^2}{x-a} \text{ for } x\neq a,$$
$$f(x)=0 \quad \text{for } x=a,$$

has a removable discontinuity at $x=a$.

By **removable discontinuity** of $f(x)$ at $x=a$

We means that $f(x)$ is discontinuous at $x=a$ and continuous at any other values of $x\in D_f$ with $\lim\limits_{h\to0^+}[f(a+h)-f(a-h)]$ finite.

The number a may or may not belong to D_f.

We have, $f(a)=0 \Rightarrow a \in D_f$.

Again $\lim_{x\to a} f(x)=\lim_{x\to a} \frac{x^2-a^2}{x-a}=\lim_{x\to a}(x+a)=2a$.

Hence $\lim_{x\to a} f(x)$ exists.

But $\lim_{x\to a} f(x) \neq f(a)$.

$\therefore$ $f(x)$ is discontinuous at $x=a$.

Again $\lim_{\substack{h\to 0\\(h>0)}} [f(a+h)-f(a-h)]=\lim_{h\to 0} [(2a+h)-(2a-h)]=0$

which is finite. Hence the discontinuity at $x=a$ is removable.

Ex. 6. Show that the function

$$f(x)=\frac{x-2}{x-1}$$

has an infinite discontinuity at $x=1$.

If $f(x)$ is discontinous at $x=a$.

and $\lim_{h\to 0} [f(a+h)-f(a-h)]=\infty$,

Lim is $|l_2-l_1|=\infty$ where l_2 is the right hand limit and l_1 is the left hand limit of $f(x)$ as $x\to a$, then $f(x)$ is said to have an **infinite discontinuity** at $x=a$.

Since $f(1)=\frac{-1}{0}$ is undifined,

$1 \notin D_f$ (1 is not included in the domain)

$\therefore$ $f(x)$ is discontinuous at $x=1$.

Now $\lim_{\substack{h\to 0\\(h>0)}} |f(1+h)-f(1-h)|$

$$=\lim_{h\to 0}\left|\frac{(-1+h)}{h}-\frac{(-1-h)}{-h}\right| =\lim_{h\to 0}\left|-\frac{2}{h}\right|=\infty.$$

Hence the discontiniuty at $x=1$ is infinite.

Ex 7. show that the functim

$$f(x)=e^{-1/x}$$

has an infinite discontinuity at $x=0$.

since $f(o)=e^{-\frac{1}{0}}$ is undefined, so $f(x)$ is discontinuous at $x=0$.

Now $\lim_{\substack{h\to 0\\(h>0)}} \left|f(0+h)-f(0-h)\right| =\lim_{h\to 0}\left|e^{-1/h}-e^{1/h}\right|$

$$=\left|e^{-\infty}-e^{\infty}\right|=|0-\infty|=\infty$$

Hence $f(x)$ has an infinite discontinuity at $x=0$.

(F) Oscillatory discontinuities (দোদুল্যমান ছেদযুক্ততা)

A function $f(x)$ having a discontinuity at a point $x=a$ may oscillate finitely or does not tend to a finite limit or to ∞ or $-\infty$ as x tends to infinity. In such a case, $f(x)$ has an oscillatory discontinuity at $x=a$.

Ex. 8. $f(x)=\sin\frac{1}{x}$ oscillates between -1 and 1 and more rapidly as x approaches zero from either sides. $f(x)$ oscillates finitely at $x=0$

Ex. 9. $f(x)=\frac{1}{x-a}\sin\frac{1}{x-a}$ the function $f(x)$ oscillates infinitely as $x\to a$.

3.5. Properties of continuous Functions (অবিচ্ছিন্ন ফাংশানের গুণাবলী।)

(i) The sum of difference of two continuous functions is a continuous function over the intersection at their domains.

Let $f(x)$ and $\varphi(x)$ be two functions of x each being continuous at $x=a$. Then

$$\lim_{x\to a} f(x)=f(a)\ ,\ \lim_{x\to a}\varphi(x)=\phi(a)$$

Thus $\lim_{x\to a}\{f(x)\pm\varphi(x)\}=\lim_{x\to a} f(x)\pm\lim_{x\to a}\varphi(x)=f(a)\pm\varphi(a)$

Hence $f(x)\pm\varphi(x)$ is continuous at $x=a$

We can extend the theorem for any finite number of functions.

(ii) The product of two continuous functions is a continuous function over the intersection of their domains.

$$\lim_{x\to a}\{f(x)\,\varphi(x)\}=\lim_{x\to a} f(x)\lim_{x\to a}\varphi(x)=f(a)\,\varphi(a)$$

Where $f(x)$ and $\varphi(x)$ are two continuous functions at $x=a$ and $a\in D_f\cap D_\varphi$.

(iii) The quotient of two continuous functions in some common domain is also a continuous function in the same domain if the denominator is not zero anywhere in it.

Let $f(x)$ and $\varphi(x)$ be two continuous functions at $x=a$ where $a\in D_f\cap D_\varphi$.

$$\text{Now}\quad \lim_{x\to a}\frac{f(x)}{\varphi(x)}=\frac{\left(\lim_{x\to a} f(x)\right)}{\left(\lim_{x\to a}\varphi(x)\right)}=\frac{f(a)}{\varphi(a)},\quad \text{if } \varphi(a)\neq 0.$$

Hence $\dfrac{f(x)}{\varphi(x)}$ is continuous at $x=a$.

(iv) If a function is continuous in a closed interval, it is bounded in that interval.

(v) A function which is cotinuous in a closed interval attains at least once its least upper and greatest lower bounds.

(vi) A continuous function which has opposite sign at two points meets its domain vanishes at least once between these points.

(vii) A continuous function $f(x)$, in the interval (a,b), assumes at least once every value between $f(a)$ and $f(b)$, it being supposed that $f(a)\neq f(b)$.

(vii) The converse of this theorem is not true i, e, a function $f(x)$ which takes all values between $f(a)$ and $f(b)$ is not necessarily continuous in the interval (a,b).

3.6. Continuity of some elementery functions.

(i) The function $f(x)=x^n$ is continuous for all values of x when n is any rational number, except at $x=0$ when n is negative.

Let us investigate the continuity of the function at $x=a$.

$$\underset{(h>0)}{\lim_{h\to 0}} f(a+h)=\lim_{h\to 0}(a+h)^n=\lim_{h\to 0} a^n(1+h/a)^n$$

$$=\lim_{h\to 0} a^n\left\{1+nh/a+\frac{n(n-1)h^2/a^2}{\underline{|1}}+\dots\right\}$$

$$\underset{(h>0)}{\lim_{h\to 0}} f(a-h)=\lim_{h\to 0}(a-h)^n=\lim_{h\to 0} a^n(1-h/a)^n$$

$$=\lim_{h\to 0} a^n(1-nh/a+\dots)$$

Also $f(a)=a^n$.

Hence $\lim_{x\to a} f(x)=a^n=f(a)$ for all values of n and a except $a=0$ when n is negative, When n is negative say $n=-m$, where m is positive. Then $x^n=x^{-m}=(1/x^m)$ which is undefined when $x=0$

Hence $f(x)=x^n$ is continuous for all values of x, except at $x=0$ when n is negative.

(ii) Polynomials are Continuous (অবিচ্ছিন্ন বহুপদী)

Let $f(x)=a_0x^n+a_1x^{n-1}+\dots+a_{n-1}x+a_n$ be a polynomial in x

The polynomial $f(x)$ is the sum of a finite number of terms containing only positive integral powers of x. In the article 3.6. (i) we see that $\lim\limits_{x\to 0} x^n$ is continuous. Thus by repeated application of Art. 3.6. (i) for the terms of $f(x)$ here we see that each term is continuous.

Hence the polynoimal is itself continuous for all values of x.

(iii) Rational (আনুপাতিক) Algebraic functions are Continuous.

Let $f(x)$ and $\phi(x)$ be two polynomials which have no common factor and $\phi(x)\neq 0$

The rational algebraic function

$$R(x)=\frac{f(x)}{\phi(x)};\ \phi(x)\neq 0$$

If $f(x)$ and $\phi(x)$ are continuous for all values of x then $R(x)$ is also continuous for all values of x except for those values of x which make $\phi(x)=0$ *i. e.*, for those values for which the denominator becomes zero.

(iv) Exponential (সূচীয়) functions $f(x)=e^x$

Let us investigate its continuity for any value of x say a. Then

$$\lim_{h\to 0} f(a+h)=\lim_{h\to 0} e^{a+h}=e^a\lim_{h\to 0} e^h$$

$$=e^a\lim_{h\to 0}\left(1+h+\frac{h^2}{\lfloor 2}+\ \dots\ \dots\right)=e^a$$

Also $f(a)=e^a$

Hence $\lim\limits_{h\to 0} f(a+h)=f(a)=e^a$

Therefore e^x is continuous for all values of x.

(v) Logarithmic Function

Let us consider the function $f(x)=\log x=\ln x,\ x>0$

Now we are to investigate its continuity for any vlaue of x $(x>0)$

Let $\log x=\ln x=y$, then $\ln(x+h)=y+k$.

or, $x=e^y$ and $x+h=e^{y+k}$ $\quad\therefore\ h=(x+h)-x=e^{y+k}-e^y$

If $h\to 0$ then $k\to 0$. But e^y is continuous as in Art. 3.6 (v). So by definition of continuity 3.2 (6) we have:

$$\lim_{h\to 0}\left|\ln(x+h)-\ln x\right|=\lim_{k\to 0}\left|y+k-y\right|=\lim_{k\to 0}|k|=0$$

Hence $\log x=\ln x$ is continuous for all positive values of x.

When $x\leqq 0$. the function $\log x=\ln x$ is no' defined.

3.7. Infinitely small quantities or Infinitesimals. (অনু)

Definition :—An infinitesimal is a variable quantity whose limit is zero.

Comparison of Infinitesimals.

Let α aud β be the two infinitesimals

(a) Infinitesimal of the same order.

If $\lim\limits_{\alpha\to 0}\left(\dfrac{\beta}{\alpha}\right)=\text{constant}=k\neq 0$,

then α and β are called the infinitesimals of the same order.

Ex. 8. If $\alpha=x$, $\beta=\sin x$, then

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

(b) Infinitesimals of different orders

If $\lim_{\alpha\to 0}\left(\frac{\beta}{\alpha}\right)=0$; that infinitesimal β is called an infinitesimal of higher order than α

If $\lim_{\alpha\to 0}\left(\frac{\beta}{\alpha}\right)=\pm\infty$, β is an infinitesimal of lower order than α.

(c) Infinitesimal of the nth order.

An infinitesimal β is said to be an infinitesimal of order n with respect to the infinitesimal α.

If $\lim_{\alpha\to 0}\frac{\beta}{\alpha^n}=k\neq 0$ i. e., β and α^n are of the same order.

(d) Equivalent Infinitesimals.

If $\lim_{\alpha\to 0}\left(\frac{\beta}{\alpha}\right)=1$,

then α and β are called equivalent infinitesimals.

(e) If α and β are equivalent infinitesimals, their difference $(\alpha-\beta)$ is an infinitesimal of higher order than that of α or β,

Proof: $\lim_{\alpha\to 0}\left(\frac{\alpha-\beta}{\alpha}\right)=\lim_{\alpha\to 0}\left(1-\frac{\beta}{\alpha}\right)=1-1=0$

The result follows from result (b)

(f) A quantity which is the product of a finite quantity and an infinitesimal of any order is an infinitesimal of that order,

$\lim_{\alpha\to 0}\left(\frac{A\alpha}{\alpha}\right)=A$, a finite quantity.

Therefore $A\alpha$ is of the same order as α [by (a)]

(g) Principal part of the Infinitesimal.

An infinitesimals β of any order may be split up into two parts one of which is of the same order as the given infintesimal α and the other is of higher order. The part which is of the same order as α is called the principal part of the infinitesimal β.

If $\lim_{\alpha\to 0}\left(\frac{\beta}{\alpha}\right)=\lim_{\alpha\to 0}\frac{A(\alpha+k)}{\alpha}=A+\lim_{\alpha\to 0}\left(\frac{Ak}{\alpha}\right)$

$=A$ as k vanishes with α.

The Principal part of infinitesimal β is A. Thus the prineipal parts of β is obtained by multiplying the infinitesimal by the finite limit of the ratio it bears to the infinitesimal α.

Ex. 9 Show that $1+\sin^2\alpha-\cos\alpha$ is of the 2nd order and its principal part $(3/2)\alpha^2$.

$$\lim_{\alpha\to 0}\frac{1+\sin^2\alpha-\cos\alpha}{\alpha^2}=\lim_{\alpha\to 0}\frac{(1-\cos\alpha)+\sin^2\alpha}{\alpha^2}$$

$$=\lim_{\alpha\to 0}\left\{\frac{2\sin^2\alpha/_2}{\alpha^2}+\frac{\sin\alpha^2}{\alpha^2}\right\}$$

$$=\frac{1}{2}\lim_{\alpha\to 0}\left\{\frac{\sin\alpha/_2}{\alpha/_2}\right\}^2+\lim_{\alpha\to 0}\left(\frac{\sin\alpha}{\alpha}\right)^2=\frac{1}{2}+1=2/3$$

which is finite and non-zero.

Therefore $1+\sin^2\alpha-\cos\alpha$ is of 2nd order w. r, α and the principal part is $\frac{3}{2}\alpha^2$,

3.8. Differentiablity of a function;

A function $f(x)$ is said to be Differetiable at $x=a$ if $a+h$ and a both belong to the domain of f as $h\to 0$

and $\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ exists.

we write.

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

provided the limit exists.

If $\lim\limits_{h\to0^+}\dfrac{f(a+h)-f(a)}{h}=\lim\limits_{h\to0^-}\dfrac{f(a-h)-f(a)}{h}$

or, $\lim\limits_{h\to0}\dfrac{f(a+h)-f(a)}{h}$ The fun ction $f(x)$ is differentiable at $x=a$

3.9. Every finitely derivable function is Continuous.

Let $f(x)$ be differentiable at $x=a$ i. e,

$$f'(a)=\lim_{h\to0}\frac{f(a+h)-f(a)}{h} \quad \text{exists.}$$

$\Rightarrow \lim\limits_{h\to0}[f(a+h)-f(a)]=\lim\limits_{h\to0} h\, f'(a)=0$

or $\lim\limits_{h\to0} f(a+h)=f(a)$.

Hence $f(x)$ *is* continuous at $x=a$,

Note : $\mathit{Lim}_{h\to0+} f(a+h)=\mathit{Lim}_{h\to0-} f(a+h)=f(a)$

or, $\mathit{Lim}_{h\to0} f(a+h)=f(a)$; $f(x)$ *at* $x=a$ is Continuous.

The converse of these theorems is not necessarily true i, e, a function may be continuous for a value of the variable in an interval but derivative at this point may not exist. This will be shown with an example. (উপরিউক্ত উপপাদ্যের বিপরীত কিন্তু সত্য নাও হতে পারে। একটি ফাংশান কোন চারণস্থলের কোন বিন্দুতে অবিচ্ছিন্ন হইলেও সেই বিন্দুতে ইহার ডিফারেন্সিয়েল সহগ নাও থাকিতে পারে।)

Ex. 10. Consider the function, D. U. 1983

$f(x)=x \quad ; \quad 0\leq x<\frac{1}{2}$ N.U. 1993

$\quad\;\; =1-x \; ; \quad \frac{1}{2}\leq x<1$

Is the function continuous at $x=\frac{1}{2}$ Is it defferentiable at $x=\frac{1}{2}$.

From the conditions.

$\mathit{Lim}_{h\to0\,(h>0)} f(\frac{1}{2}-h)=\mathit{Lim}_{h\to0} f(\frac{1}{2}-h)=\mathit{Lim}_{h\to0}(\frac{1}{2}-h)=\frac{1}{2}$

$\lim\limits_{h\to0\,(h>0)} f(\frac{1}{2}+h)=\lim\limits_{h\to0}[1-(\frac{1}{2}+h)]=\lim\limits_{h\to0}(\frac{1}{2}-h)=\frac{1}{2}$.

Also $f(\frac{1}{2})=1-\frac{1}{2}=\frac{1}{2}$.

Thus $\lim\limits_{h\to0+} f(\frac{1}{2}-h)=\mathit{Lim}_{h\to0+}(\frac{1}{2}+h)=f(\frac{1}{2})=\frac{1}{2}$

Hence the function $f(x)$ is continuous at $x=\frac{1}{2}$.

Now $\frac{1}{2}$ and $\frac{1}{2}+h$ as $h\to0+$ belong to D_f.

Again. $\lim\limits_{h\to0\,(h>0)}\dfrac{f(\frac{1}{2}+h)-f(\frac{1}{2})}{h}=\lim\limits_{h\to0}\dfrac{1-(\frac{1}{2}+h)-\frac{1}{2}}{h}=-1$ ...(a)

and $\lim\limits_{h\to0\,(h>0)}\dfrac{f(\frac{1}{2}-h)-(\frac{1}{2})}{h}=\lim\limits_{h\to0}\dfrac{\frac{1}{2}-h-\frac{1}{2}}{-h}=1$ (b)

Thus right hand limit (a) and the left hand limit (b) are not equal. Hence $f'(\frac{1}{2})$ does not exist *i. e.* $f(x)$ is not differentiable at $x=\frac{1}{2}$

Note : If $f(x)=x^n$, $(0<n<1)$ $f(x)$ is continuous at $x=0$, but $f'(x)$ does not exist at $x=0$.

Ex. 11. Test the continuity and differentiability $f(x)$ at $x=0$ when.

$f(x)=+\sqrt{|x|}\;, x\geq0$

$\quad\;\; =-\sqrt{|x|}\;; x<0$

We have $\mathit{Lim}_{h\to0\,(h>0)} f(0+h)=\lim\limits_{h\to0}\sqrt{|0+h|}=0$

$\mathit{Lim}_{h\to0\,(h>0)}$

$f(0-h)=\mathit{Lim}_{h\to0}-\sqrt{|-h|}=\lim\limits_{h\to0}\sqrt{|h|}=0$

Now $Rf'(0) = \lim_{\substack{h\to 0\\(h>0)}} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{\sqrt{|h|}-0}{h}$

$= \lim_{h\to 0} \frac{\sqrt{h}}{h} = \lim_{h\to 0} \frac{1}{\sqrt{h}} = \infty$

Again

$Lf'(0) = \underset{\substack{h\to 0\\(h>0)}}{Lim} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{-\sqrt{|-h|}-0}{-h}$

$= \lim_{h\to 0} \frac{\sqrt{h}}{h} = \infty$

$Rf'(0) = Lf'(0) = \infty$ Hence $f'(0)$ exis's .....(V)

Also $f(0) = 0$ from the given conditions

Thus $f(0+h) = f(0-h) = f(0) = 0$ ......(2)

This example shows that a function having an infinite derivative at a point may be continuous at that point.

Ex. 12 A function $f(x)$ is defined in the following way.

$f(x) = 0$ for $0 \leqq x \leqq 3$

$= 4$ for $x = 3$

$= 5$ for $3 < x \leqq 4$

Investigate the continuitv and differentiability at $x = 3$.

we have $\lim_{\substack{h\to 0\\(h>0)}} f(3+h) = \lim_{h\to 0} 5 = 5$;

$\lim_{\substack{h\to 0\\(h>0)}} f(3-h) = \lim_{h\to 0} 0 = 0$;

$f(3) = 4$

Thus $f(3+h) \neq f(3-h) \neq f(3)$.

Hence $f(x)$ is discontinuous at $x = 3$.

Now $\lim_{\substack{h\to 0\\(h>0)}} \frac{f(3+h)-f(3)}{h} = \underset{h\to 0}{Lim} \frac{5-4}{h} = \alpha$

$\lim_{\substack{h\to 0\\(h>0)}} \frac{f(3-h)-f(3)}{-h} = \lim_{h\to 0} \frac{0-4}{-h} = 0 \lim_{h\to 0} \frac{4}{h} = \infty$ (ii)

$Rf'(3) = Lf'(3) = \infty$ *i. e.* $f'(3)$ exists ......(3)

This is a falacy. '∞' is not a finite number. Had it been a a finite number then we would have, from (i) and (ii).

$Rf'(3) = \alpha$ and $Lf'(3) = 4\infty$

$\Rightarrow Rf'(3) \neq Lf'(3)$,

that is, $f'(3)$ does not exist.

Hence $f(x)$ is not differentiable at $x = 3$

Ex. 13. If $f(x) = 1$; $x < 0$

$= 1 + \sin x$, $0 < x < \frac{1}{2}\pi$

$= 2 + (x - \frac{1}{2}\pi)^2$; $\frac{1}{2}x \geqq \frac{1}{2}\pi$

Discuss the continuity and differentiability of the function at $x = \pi/2$.

Now $Rf(\frac{1}{2}\pi) = \lim_{\substack{h\to 0\\(h>0)}} f(\frac{1}{2}\pi + h) = \lim_{h\to 0} [2 + \{(\frac{1}{2}\pi + h) - \frac{1}{2}\pi\}^2] = 2$

$Lf(\frac{1}{2}\pi) = \lim_{\substack{h\to 0\\(h>0)}} f(\frac{1}{2}\pi - h) = \underset{h\to 0}{Lim} [1 + \sin(\frac{1}{2}\pi - h)]$

$= \lim_{h\to 0} (1 + \cos h) = 2$

Also $f(\frac{1}{2}\pi) = 2 + (\frac{1}{2}\pi - \frac{1}{2})^2 = 2$

$\therefore Rf(\frac{1}{2}\pi) = Lf(\frac{1}{2}\pi) = f(\frac{1}{2}\pi) = 2$.

The function $f(x)$ is continuous at $= \pi/2$.

For differentiablity.

$Rf'(\frac{1}{2}\pi) =$

$\underset{\substack{h\to 0\\(h>0)}}{Lim} \frac{(\frac{1}{2}\pi + h) - f(\frac{1}{2}\pi)}{h} = \underset{h\to 0}{Lim} \frac{\{2 + (\frac{1}{2}\pi + h - \frac{1}{2}\pi)^2\} - \{2 + (\frac{1}{2}\pi - \frac{1}{2}\pi)\}^2}{h}$

$$=\lim_{h\to 0}\frac{h^2}{h}=\lim_{h\to 0} h=0$$

$$Lf'(\tfrac{1}{2}\pi)=\lim_{\substack{h\to 0\\(h>0)}}\frac{f(\frac{1}{2}\pi-h)-f(\frac{1}{2}\pi)}{-h}=\lim_{h\to 0}\frac{1+\sin(\frac{1}{2}\pi-h)-(1+\sin\frac{1}{2}\pi)}{-h}$$

$$=\lim_{h\to 0}\frac{1+\cos h-2}{-h}=\lim_{h\to 0}\frac{(1-\cos h)}{h}=\lim_{h\to 0}\frac{2\sin^2 h/2}{h}=0$$

Thus $Rf'(\frac{1}{2}\pi)=Lf'(\frac{1}{2}\pi)=0$

Hence $f'(x)$ exists *i. e.*, $f(x)$ is differentiable at $x=\frac{1}{2}\pi$.

Ex. 14. Show that

$f(x)=x^2\sin 1/x$ when $x\neq 0$

$=0$, when $x=0$

is derivable at $x=0$ Also find $f'(0)$

Ex. 15, Draw the graph of function

$$y=\sqrt{\{(x-1)(x-2)(x-3)\}}$$

Sign of $(x-1)(x-2)(x-3)$:

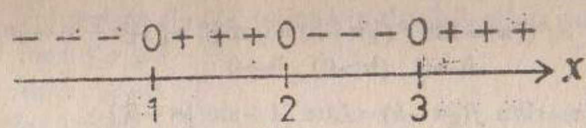


Fig. 50

From the above sign graph we see that

$y\geqslant 0$ for $1\leqslant x\leq 2$ and for $x\geqslant 3$.

y is imaginary for $x<1$ and for $2<x<3$.

Therefore the graph of

$y=\sqrt{(x-1)(x-2)(x-3)}$

has two branches—one betwen 1 and 2 and the other for $x\geqslant 3$.

we may form a table out of the values of x and y

x	1	1·1	1.2	1.5	1.7	1.9	2	3	4	5
y	0	+4	+5	.6	+0.5+	3	0	0	+2.4	+4.5

The graph may now be drawn

Ex. 15. (a) Draw the graph of the function

N.U.1995

$$f(x)=\frac{x}{|x|}$$

For $x<0$, $\frac{x}{|x|}=\frac{x}{(-x)}=-1$,

for $x>0$, $\frac{x}{|x|}=\frac{x}{x}=1$;

for $x=0, f(o)=\frac{0}{0}$ so the function is not defined at $x=0$.

Hence $f(x)$ is discontinuous at $x=0$

The graph of the fnnction is shown in fig. 51

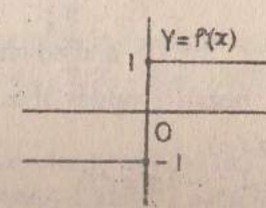


Fig. 51

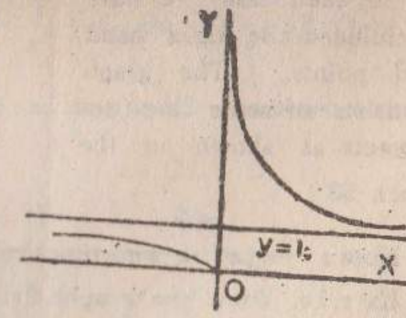
Ev. 16. Show that the $f(x)=3^{1/x}$ is discontinuous at $x=0$

$$Rf(0)=\lim_{\substack{h\to 0\\(h>0)}} f(0+h)$$

$$\lim_{h\to 0}\frac{1}{3^{0+h}}=\lim_{h\to 0} 3^{1/h}=3^{\infty}=\infty$$

$$Lf(0)=\lim_{h\to 0} f(0-h)$$

$$=\lim_{h\to 0} 3^{-1/h}=3^{-\infty}=0$$



Eig. 52

$Rf(0)\neq Lf(0)$

Now $f(0)=3^{\frac{1}{0}}$ which is undefined since $\frac{1}{0}$ is undefined.

So the function is decontinuous at $x=0$

Also note that $f(x)\to 1$ as $x\to\infty$ or $-\infty$.

The graph of function is shown in fig. 52

Ex 17. Draw the graph of $y=[x]$
where $[x]$ denotes the greatest integer positive or negative but not numerically greater than x.

The function $y=[x]$
is replaced by following statements

$$\begin{aligned} y=f(x)&=0\ ; & 0\leq x>1 \\ &=1\ ; & 1\leq x>2 \\ &=2\ ; & 2\leq x<3 \\ &=3\ ; & 3\leq x<4 \end{aligned}$$

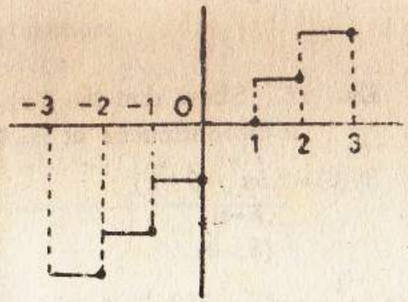
and so on.

For negative values of x.

$$\begin{aligned} y&=f(x) \\ &=-1\ ;\ -1\leq x<0 \\ &=-2\ ;\ -2\leq x<1 \\ &=-3\ ;\ -3\leq x<-2 \end{aligned}$$

and so on.

In each case we have excluded the right hand end points. The graph consists of some line segments as shown in the figur. 53



Eig. 53.

Note : $y=[x]$ is sometimes called a step function.

Ex : 18. Draw the graph of the function $y=f(x)=\sin 1/x$ and show that limit does not exist when x tends to zero.

$f(o)=\sin(1/o)$ is not defined.

For all other values of x, $\sin(1/x)$ exists. Let us investigate the nature of the graph.

For $x=\frac{2}{\pi},\ \frac{3}{\pi},\ \frac{4}{\pi},\ \frac{6}{\pi},$ ...... (1) } .... (1)

$y=1,\ \sqrt{3}/2,\ 1/\sqrt{2},\ 1/2$

i, e, $\sin(1/x)$ **decreases continuously from 1 to 0 with the increasing values of x form $2/\pi$ to ∞.**

For $x=\frac{2}{\pi},\ \frac{2}{2\pi},\ \frac{2}{3\pi},\ \frac{2}{4\pi},\ \frac{2}{5\pi},\ \frac{2}{6\pi},\ \frac{2}{7\pi},\ \frac{2}{8\pi}$

$y=1,\ 0,\ -1,\ 0,\ -1\ 0,\ -1,\ 1$..........

Thus $\sin\frac{1}{x}$ Oscilletes between

$(1,-1)$ and $(-1,1)$ for the values of x in the intervals $(2/\pi, 2/3\pi)$, $(2/3\pi, 2/5\pi)$, $(2/5\pi, 2/7\pi)$...$\{2/(2n+1)\pi,\ 2/(2n-1)\pi\}$ where n is positive integer.

If $n\to\infty$ then $x\to 0$, but y does not tend to any finite limit but oscillates more frequently between $+1$ and -1 as x is nearer to zero.

Now combining (1) and (2) we get the nature of the graph of. positive values of x only.

Again $f(-x)=\sin\left(-\frac{1}{x}\right)=-\sin\frac{1}{x}=-f(x)$ so

$\sin\left(\frac{1}{x}\right)$ is an odd function and its graph is symmetrical about the origin.

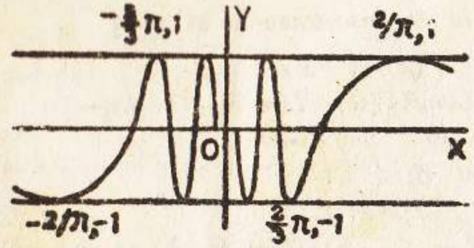


Fig. 54

Ex. 18. (a) Draw the graph of the function

$$y=\cos\frac{\pi}{x}$$

The graph has a discontinuity at $x=0$

x	∞	2	1	2/3	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	......
y	1	0	-1	0	1	-1	1	......

For $x=\dfrac{2}{(2n+1)}$; $y=0$ when $n=0, 1, 2,$

For the values of x in the interval (0, 2/3) y oscillates frequently between 1 and -1 but never becomes zero.

For the values of x in the intervel ($\frac{2}{3}$, 2), the values of y-gradually decreases to $y=-1$ and then increases to 0 when $x=2$.

The graph is continually increasing for the values of x in $(2, \infty)$ while y increases from 0 to 1. $\cos\frac{\pi}{x}$ is an even function. So similar graph will be obtained for the negative values of x. The graph has a discontinuity at $x=0$.

Ex. 19. A function $f(x)$ is defined as follows.

$f(x)=x, 0\leqq x<\frac{1}{2}$

$=1, x=\frac{1}{2}$

$=1-x, \frac{1}{2}<x<1$

Show that $f(x)$ is discontinuous at $x=\frac{1}{2}$ R, U. 1964

For $\frac{1}{2}<x<1, f(x)=1-x$;

$\therefore\ Rf(\frac{1}{2})=\lim\limits_{\substack{h\to 0\\(h>0)}} f(\frac{1}{2}+h)=\lim\limits_{h\to 0}[1-(\frac{1}{2}+h)]=\frac{1}{2}$;

For $x<\frac{1}{2}, f(x)=x$.

$\therefore\ Lf(\frac{1}{2})=\lim\limits_{\substack{h\to 0\\(h>0)}} f(\frac{1}{2}-h)=\lim\limits_{h\to 0}(\frac{1}{2}-h)=\frac{1}{2}$;

also $f(\frac{1}{2})=1$ (given);

$\therefore\ Rf(\frac{1}{2})=\lim (\frac{1}{2})\neq f(\frac{1}{2})$

Hence $f(x)$ is discontinuous at $x=\frac{1}{2}$.

Ex. 20. If each of $f(x)$ and $g(x)$ is continuous at $x=a$, show that $2f(x)+3g(x)$ is continuous at $x=a$.

As $f(x)$ and $g(x)$ are continuous at $x=0$, then

$|f(x)-f(a)| < \epsilon_1 \cdots \ldots (1)$

and $|g(x)-g(a)| \leqq \epsilon_2 \ldots\ldots 2$

for $0\leqq |x-a| \leqq\delta$

Now, $|\{2f(x)+3g(x)\}-\{2f(a)+3g(a)\}|$

$\leqq 2|f(x)-f(a)| - 3|g(a)-g(a)|$

$<2\epsilon_1-3\epsilon_2=\epsilon$ for $0\leqq |x-a| \leqq\delta$

Hence

$\lim\limits_{x\to a}\{2f(x)+3g(x)\}=2f(a)+3g(a)$

Therefore the function is continuous at $x=a$.

Ex. 21. A functions $f(x)$ is defined in an interval [0, 2] by formulae $f(x)=x$ when $0\leqq x\leqq 1$; $f(x)=2x-1$ when $1<x\leqq 2$

Show that $f(x)$ is continuous at $x=1$ but $f'(1)$ does not exist. Draw a graph of the function.

For $x=1$

$Rf(1)=\lim\limits_{h\to 0^+} f(1+h)\ \lim\limits_{h\to 0^+}[2(1+h)-1]=1$

$Lf(1)=\lim\limits_{h\to 0^+} f(1-h)=\lim\limits_{h\to 0^+}(1-h)=1$

Again $f(1)=1$

$$\lim_{h\to 0} f(0+h) = \lim_{h\to 0} |0+h| = 0$$

$$\lim_{h\to 0} f(0-h) = \lim_{h\to 0} |0-h| = 0$$

Thus $Lf(x) = Rf(x) = f(0) = 0$

Hence $f(x)$ is continuous at $x = 0$

For differentiability,

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{|h|-1}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$

And $$\lim_{h\to 0} \frac{f(0-h)-f(0)}{-h} = \lim_{h\to 0} \frac{|-h|-0}{-h} = \lim_{h\to 0} \frac{h}{-h} = -1$$

Thus $Lf(x) \neq Rf(x)$ at $x = 0$

Hence $F(x)$ is not differentiable at $x = 0$

24. Find the differential coefficient of

$f(x) = |x|+1$ at $x = -1$

What is the domain of f' ? C.H. 1995

25. Check the continuity of the function

$$f(x) = \begin{cases} 0, & -\& < x \le 0 \\ \frac{1}{x}, & 0 < x < 1 \\ x & 1 \le x < \& \end{cases}$$

C.H. 1995

26. Show that $f(x) = |x| + |x-1|$ is continuous but not differentiable at $x = 0, 1$ (দেখাও যে, $f(x) = |x| + |x-1|$, $x = 0, 1$ বিন্দুতে অবিচ্ছিন্ন কিন্তু অন্তরীকরণযোগ্য নহে)

27. Show that $f(x) = |x| + |x-1| + |x-2|$ is continuous at $x = 0, 1, 2$ but not differentiable at $x = 0, 1, 2$

(দেখাও যে, $f(x) = |x| + |x-1| + |x-2|$, অবিচ্ছিন্ন $x = 0, 1, 2$ বিন্দুতে কিন্তু অন্তরীকরণযোগ্য নহে)

28. Show that the function
$$f(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$
is discontinous at $x=0$

See APPENDIX for Uniform Continuity

Exercise 1 (C)

1. What are the types of discontinuities, give examples of each.
2. Show that $f(x) = |x|$ is every where continuous.
3. Show that x^2+1 is continuous at $x=2$
4. A function is defined as follows
$f(x) = \cos x$ for $x \geq 0$
$= -\cos x$ for $x < 0$
Is $f(x)$ continuous at $x=0$
5. Examine the continuity of $f(x) = \dfrac{1}{x-3}$ at $x=3$

(i) Show that function $|x-a|$ is continuous but not differentiable at a.

6. Show that $f(x) = \dfrac{1}{1-e^{1/x}}$ has an ordinary discontinuity at $x=0$.
7. Show that $f(x) = \dfrac{2}{\pi} \lim_{n\to\infty} \tan^{-1} nx$ is discontinuous at $x=0$
8. $f(x) = e^{1/x}$ for $x \neq 0$ and $=1$ for $x=0$
Show that $f(x)$ is discontinuous at $x=0$.
9. $f(x) = (1+2)^{1/x}$ when $x \neq 0$
$= e^2$ when $x=0$
Is $f(x)$ continuous at $x=0$
10. Test the continuity of the function at $x=0$
$$f(x) = \begin{cases} \dfrac{e^{1/x^2}}{e^{1/x^2}-1} & \text{when } x \neq 0 \\ 1 & \text{when } x=0 \end{cases}$$
11. Show that $f(x) = x^2$, $x \neq 1$
$= 2$, $x = 1$ is discontinuous for $x=1$.

See APPENDIX—Uniform Continuity

12. Prove that $f(x)=\dfrac{\sin^2(ax)}{x^2}$ for $x\neq 0$

$=1$ for $x=0$

is discontinuous at $x=0$ unless $a=1$

13. Examin the continuity of R. U, 1965

$f(x)=\sin(1/x)$ at $x=0$.

14. Discuss whether the function defined as

$f(x)=1+x,\quad x\geqq 0$

$=0,\quad x<0$ is continuous at $x=0$. Ans. no

15. $f(x)$ is defined as follows N.U. 1995

$f(x)=0,\quad x=0$

$=x,\quad x>0$

$=-x\quad x<0$

Is $f(x)$ continuous at $=0$. Does $f'(x)$ exist at $x=0$ *i. e.* is $f(x)$ differentiable at $x=0$ Ans. no

16. Test the continuity of

$f(x)=\sin(\pi/x)$ for $x\neq 0$

$=1,$ for $x=0$

at the point $x=0$

17. Is the following functions continuous at $x=4$

$f(x)=4x+3$ for $x>4$ and for $x<4$

$=3x+7$ for $x=4$

18. Examine $f(x)$ posesses first derivative $[f(x)]$ at $x=0$

when $f(x)=0, x=0$

$=\dfrac{1}{1+e^{1/x}}, x\neq 0$

19. Examine whether or not $e^{-1/x}$ is continuous at $x=0$

20. A function $f(x)$ defined as follows

$f(x)=x\sin(1/x);\quad x\neq 0$

$=0\quad ;\quad x=0$

Examine the continuity and differentiablity of $f(x)$ at $x=0$. Show that $f'(x)$ does not exist at $x=0$

21. Prove that $f(x)=x\sin\dfrac{1}{x}, x\neq 0$

$=5\quad x=0$

is not continuous at $x=0$, Can you redefine $f(0)$ so that $f(x)$ is continuous at $x=0$.

22. Let f be defined by $f(x)=\begin{cases}\dfrac{|x-3|}{x-3} & \text{if } x\neq 3\\ 0 & \text{if } x=3\end{cases}$

Discuss the continuity at $x=3$. Find the domain and range of $f(x)$. Draw the graph. C. H. 1989

$$\lim_{x\to 3+} f(x)=\lim_{x\to 3+}\frac{x-3}{x-3}=1;\ [\because (x-3)\geq 0]$$

$$\lim_{x\to 3-} f(x)=\lim_{x\to 3-}-\frac{(x-3)}{x-3}=-1;\ [\because (x-3)<0]$$

For $x=3$

$f(x)=0$

Thus, $\lim\limits_{x\to 3+} f(x)\neq\lim\limits_{x\to 3-} f(x)$

The function $f(x)$ is not continuous at $x=3$

For $|x-3|>0$, $|x-3|=x-3$

$$y=f(x)=\frac{|x-3|}{x-3}=\frac{x-3}{x-3}=1$$

$(3,\infty)\subset D_f$; $\{1\}\subset R_f$

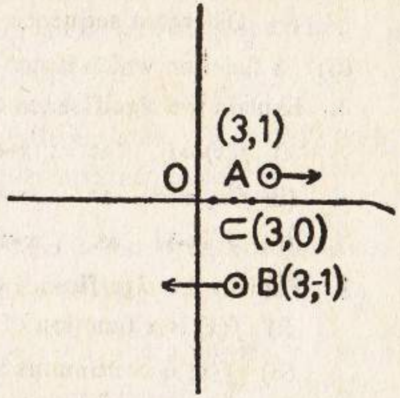


Fig 55

For $|x-3|<0$; $|x-3|=-(x-3)$

$$y=\frac{|x-3|}{x-3}=-1$$

$(-\infty,3,)\subset D_f$, $\{-1\}\subset R_f$

For $x=3$, $f(x)=0$.

Thus domain $D_f=(-\infty,3)\cup\{3\}\cup(3,\infty)=\mathbb{R}$

$$R_f=\{-1,0,1\}$$

The graph does not contains A (3, 1), B (3,—1). It contains two line $y=1$, $y=-1$ and an iscolated point (3, 0)

Miscellaneons Examples

(on functions, limits and continuties)

1. Give examples of a function which is not continuous at the origin.

(a) Let x be the name of a boy appearing in an examination and y be the roll number of boy x. Is the relation between y and x functional ?

(b) Give examples

(i) Divergent sequence of real numbers.

(ii) A function which is not defferentiable at the origin.

2. Explain the significance of the statements :—

(i) $f(x)\to l$ as $x\to a$

(ii($f(x)\to\infty$ as $x\to a$

(iii) $f(x)\to l$ as $x\to\infty$

3. Explain the significance of the statements :—

(i) $f(x)$ is a function of x in the interval (a, b)

(ii) $f(x)$ is continuous at $x=a$.

(iii) $f(x)$ has a differential Co-efficient at $x=a$

4. If $\lim\limits_{x\to a} f(x)=l$, $\lim\limits_{x\to a} g(x)=m$

Show that $\lim\limits_{x\to a}\{f(x)+g(x)=l+m$

5. $\lim\limits_{x\to a} f(x)=l$, $\lim\limits_{x\to a} g(x)=m$

Show that $\lim\limits_{x\to a} f(x)\,g(x)=lm$

6. Show that if $f(x)$ has a derivative at $x=a$, then $f(x)$ is continuous at $x=a$, but the converse is not always true.

R. H. 1964

7. Prove that continuity is a necessary condition for differentiability but not a sufficient one. Illustrate your answer with an example.

R. H. 1988

8. State any one of the fundamental properties on a continuous function and show that there is one and only one positive root of the equation $x^5=2$ by that property.

9. A function $f(x)$ is differentiable for every point of definition. What will you infer from this statement ?

10. State any one of the fundamental properties of a continuous function other than used in proving Rolle's Theorem.

13. Let $f(x)=\sin(1/x)$ when $x\neq0$ and $g(x)=x\sin(1/x)$ when $x\neq0$, prove that $\lim\limits_{x\to0} f(x)$ does not exist and $\lim\limits_{x\to0} g(x)=0$

12. Show that the limits of the following

(i) $\lim\limits_{x\to 0} \dfrac{x-\sin x \cos x}{3x} = 0$

(ii) $\lim\limits_{x\to 0} \dfrac{xe^x - \log(1+x)}{x^2} = 3/2$

(iii $\lim\limits_{x\to \frac{1}{4}\pi} \dfrac{1-\tan x}{1-\cot x} = -1$, (iv) $\lim\limits_{\theta\to 0} \dfrac{\sin\theta-\theta}{\theta^3} = -\dfrac{1}{6}$

R. U. 1966

(iii) $\lim\limits_{x\to\infty} \dfrac{1}{x} \sin x = 0$ (vi) $\lim\limits_{x\to 1} \dfrac{1}{1-x} - \dfrac{3}{1-x^3} = -1$

(vii) $\lim\limits_{x\to 4} \dfrac{\sqrt{(2x+1)}-3}{\sqrt{(x-2)}-\sqrt{2}} = \dfrac{2\sqrt{2}}{3}$

13. If $f(x) = \dfrac{x^2+1}{x-1}$, $x<3$

$= \dfrac{\sin x}{x-1}$, $x>3$

For what value of x the function is defined ?

14. Cite an example of a bounded function in $(1,-1)$ which is discontinuous at the single point $x=a$.

15. Show that $f(x)=(x-a)\sin\dfrac{1}{x-a}$ for $x\neq a$

$=0$ for $x=a$

is continuous and differentiable for $x=a$

16. For the function $f(x)=x\cos(1/x)$; $x\neq 0$

$=0$ $x=0$

Show that (i) the limit is zero when $x\to 0$

(ii) $f(x)$ is continuous at $x=0$

(iii) $f'(x)$ does not exist at $x=0$

17. Examine the continuity and differentiability of the function difined in the interval $(-\infty, \infty)$ such that C.U. 1992

$f(x)=1$; $-\infty<x<0$ C. H. 1989

$=1+\sin x$ $0\leq x<\frac{1}{2}\pi$

$=2+(x-\frac{1}{2}\pi)^2$, $\frac{1}{2}\pi\leq x<\infty$

and show that $f'(x)$ exists for $x=\frac{1}{2}\pi$ and does not exist for $x=0$.

18. A function is defined as follows.

$f(x)=1+x^2$, $0<x\leq 4$

$=4$, $-1\leq x\leq 0$

$=1+x$. $-4\leq x<-1$

show that $f(x)$ is continuous at $x=0$ but discontinuous at $x=-1$

19. A function is defined as fowllows D.U. 1990

$f(x)=x^2$, $x\leq 0$

$=1$, $0<x<1$ R. U, 1980

$=1/x$, $x>1$

Show that the function $f(x)$ is not differentiable at $x=0$. What is about $f'(x)$ at $x=1$?

20. Find where the function is discontinuous

$f(x)=x^2+1$, $0\leq x<\frac{1}{2}$

$=0$ $x=\frac{1}{2}$

$=x+3$, $\frac{1}{2}<x\leq 1$ R. U. 1960

(I) Find $\dfrac{dy}{dx}$ at $x=0$ for the following function

$y=x^2+1$; $x\geq 0$

$=\cos x$; $x\geq 0$ R. U. 1986

21. If $y = x^2$ when $x \leq 1$ N.U(C-2) 1994

$= x$, $1 < x \leq 2$ C. U. 1987 '89

$= \frac{1}{4}x^3$, $x > 2$

Show that y is continuous at $x = 1$, $x = 2$.

22. If $y = -x$, $x \leq 0$

$= x$, $0 < x \leqslant 1$ D. U. 1989

$= 2 - x$, $x > 1$

Show that y is continuous a $x = 0$ $x = 1$

23. If $f(x) = 3 + 2x$, $-3/2 \leq x < 0$ C.U. 1993

$= 3 - 2x$, $0 \leqslant x < 3/2$

$= 3 + 2x$, $x \geqq 3/2$

Show that $f(x)$ is continuous at $x = 0$ and discontinuous $x = 3/2$

24. If $f(x) = \frac{1}{2}(b^2 - a^2)$, $0 \leq x \leq a$

$= \frac{1}{2}b^2 - \frac{1}{6}x^2 - \frac{1}{3}(a^3/x)$, $a < x \leqq b$ R. U. 1988

$= \frac{1}{3}(b^3 - a^3)/x$, $x > b$

Show that $f(x)$ and $f'(x)$ are continuous for every positive value of x.

25. A given function $y = f(x)$ is defined as follows

$f(x) = 0$, $x^2 > 1$

$= 1$ $x^2 < 1$

$= \frac{1}{2}$, $x^2 = 1$

Show that $f(x)$ is discontinuous at $x = \pm 1$, Explain the discontinuous of the function although it has a value for every value of x.

26. Prove that

$\lim_{n \to \infty} y^n = 0$ if y is a proper function

and n is a positive integer

27. $$\lim_{n \to \infty} \left(\frac{1}{n^2} + \frac{2}{n^2} + \frac{3}{n^2} \cdots + \frac{n}{n^2} \right) = \frac{1}{2}$$

28. $$\lim_{n \to \infty} \left(1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^n} \right) = 2$$

29. $$\lim_{n \to \infty} \frac{1^2 + 2^2 + 3^2 \ldots + \ldots n^2}{n^3} = \frac{1}{3}$$

30. Define limit of a sequence, Has the sequence defined by $S_n = (-1)^n \frac{n}{n+1}$ a limit ? R. U. 1966

31. Define a function. Find the domain of the following real functions

(i) $f(x) = \frac{1}{x}$ Ans. $0 < x < \infty, -\infty < x < 0$.

(ii) $f(x) = \sqrt{(x+1)}$ Ans. $x \geqq -1$.

(iii) $f(x) = \frac{x^2 + 11}{x^2 - 5x + 6}$ Ans. all real values of x except $x = 2, 3$

C. U. 1986

(iv) $f(x) = \frac{x^3 - 1}{x - 1}$ Ans. all real values of x. except $x = 1$.

(v) $f(x) = \frac{x}{|x|}$ Ans. all real values of x except $x = 0$

(vi) $f(x) = \frac{\sqrt{(x-1)}}{(x^2 - 5x - 1)}$. Ans. except the values of x which make $x^2 - 5x + 1 = 0$ and $x < 1$.

(vii) $f(x) = \frac{x^2 - 4}{x - 2}$ (viii) $f(x) = \sqrt{(x^2 - 4x + 3)}$

(ix) $f(x) = \frac{\sqrt{(x-2)}}{5x^2 - 27x + 10}$ (x) $f(x) = \frac{x}{\sin (1/x)}$

R. U. 1967, C. U. 1969.

32. Determine the biggest domain and range for the following functions.

(i) $f(x)=-1$ when $x<0$ Ans, $|x|\geq 0$ Range $(-1,1)$

$=0$ when $x=0$

$=1$ when $x>0$

(ii) $f(x)=2$ when $-5<x<-1$

$=\sin x$ when $0<x<2$

(iii) $f(x)=x+5$; $-\infty<x<\infty$ Ans. $-\infty<y<\infty$

(iv) $f(x)=x^2+x+1$ Ans $x\geq 0, y\geq 0$

(v) $f(x)=x\sin(1/x)$ Ans. all values of x, except $x=0$,

Range. $(-1, 1)$

33. Investigate whether the following function tend to limit or not as $n\to\infty$.

(i) $f(n)=\dfrac{(-1)^n}{n}$ (ii) $f(n)=\dfrac{1}{n-(-1)^n}$

(iii) $f(n)=1+1/n$. (iv) $f(n)=n[1+(-1)^n]$ R. U. 1964.

34. Show that if α is an infinitesimal, $\sin\alpha$ and $\tan\alpha$ are infinitesimals of the same order as α and $1-\cos\alpha$ is an infinitesimal of the second order with respect to α.

we know $\lim_{\alpha\to 0}\dfrac{\sin\alpha}{\alpha}=1$, $\lim_{\alpha\to 0}\dfrac{\tan\alpha}{\alpha}=1$ *i. e.*, $\sin\alpha$, $\tan\alpha$, are of same order.

Also $\lim_{\alpha\to 0}\dfrac{1-\cos\alpha}{\alpha^2}=\frac{1}{2}$ which is finite other than zero.

Hence $1-\cos\alpha$ and α^2 are of same order, *i. e.*, 2nd order,

35. Show that $3\alpha+2\alpha^2$ is an infinitesimal of the same order.

36. Prove that $\sqrt{(\sin\alpha)}$ is of lower order than α.

37. Show that $\sqrt{\alpha}$ is an infinitesimal of lower than α.

38. Show that $\sin\alpha-\tan\alpha$ is of 3rd order and its principal part is $\frac{1}{2}\alpha^2$.

39. Show that $\sin\alpha(1-\cos\alpha)$ is an infinitesimal of the 3rd order and its principal part is $\frac{1}{2}\alpha^3$,

40. Demonastrate with an example that at a given point a function may be discontinuous but its limit may exist

R. U. 1986

Answers I (C)

4. Not-continuous 5. not continuous.

9. Continuous. 10. Continuous.

13. discontinuous. 14. not cont.

15. yes, not. 16. not cont. 17. cont

18. not. 19. not. 20. cont.

Miscellaneous Exercise

1. $y=1/x$, no.

9. Function is continuous in the interval.

13. at $x=1$ 14. $f(x)=3-x$, $-1\leq x<0$

$=0$, $x=0$

$=x-3$, $0<x\leq 1$

10. undefined at $x=1$ 20. $x=0, \frac{1}{2}$ 28. 2. 30. no.

31. (vii) $-2<x<2$ (iii) $3\leq x\leq 1$

(ix) $2\leq x<\infty$, $5<x<\infty$ (x) $0<x\leq\dfrac{1}{n\pi}$

33. (i) 0, (ii) 0, (iii) 1, (vi) 0, n is odd, no. n is even.

Ex. 1. Locate the discontinuities of the function

$$f(x)=\begin{cases} 1 & \text{if} \quad 0 \le x \le 3 \\ x+1 & \text{if} \quad 3 < x \le 6 \\ 7 & \text{if} \quad 6 < x < 9 \end{cases}$$

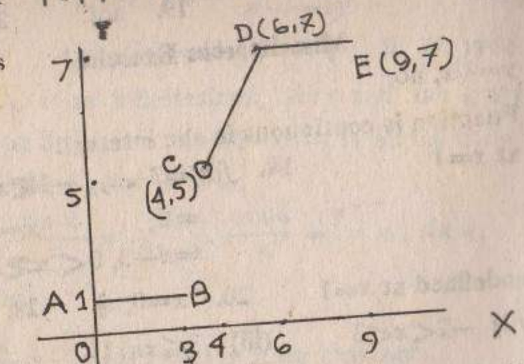
Find the domain and range of f(x)

The graph of $f(x)$ consists of the line segments such as $y=1$ for $0 \le x \le 3$, $y=x+1$ for $3< x \le 6$, $y=7$ if $6< x \le 9$ and draw the grap.

Solution : Put the points as below:

x	0	1	2	3	4	5	6	7	8	9
y	1	1	1	1	5	6	7	7	7	7

The graph is



From the figure we see that only discontinuity exists at $x=3$ for the values of x greater than 3 get closer and closer to 3, the corresponding function values $f(x)$ get closer and closer to 4. This is $\underset{x\to 3+}{\text{Lt}} f(x) = 4$.

The plus sign indicates that values of x under consideration are slightly greater than 3 i.e., $3+h$, $h \to o$

Again when the values of x are less than 3, then the values $f(x)$ gradualy approaches to 1. i.e $f(x) =1$ for all values of x, between 0. and 3. This is expressed as $(x \to 3-h,\ h \to o)$ i. e.

Thus only discontinuity exits at x=3 i.e. the line AB fails to connect C i.e., with the line segments CDE.

The domain of $f(x)$ is $[0, 3] \cup [3, 6] \cup [6, 9]$

Range of $f(x)$ is $\{1\} \cup [4, 7] \cup \{6, 7\}$

Ex.2. The $f(x)$ is defined in the following way.

$$\begin{aligned} f(x) &= x^2 && \text{if} \quad 0 \le x < 3 \\ &= 10 && \text{if} \quad 3 \le x < 6 \\ &= x-1 && \text{if} \quad 6 \le x < 10 \end{aligned}$$

Locate the discontinuties. Also find the domain and range of $f(x)$.

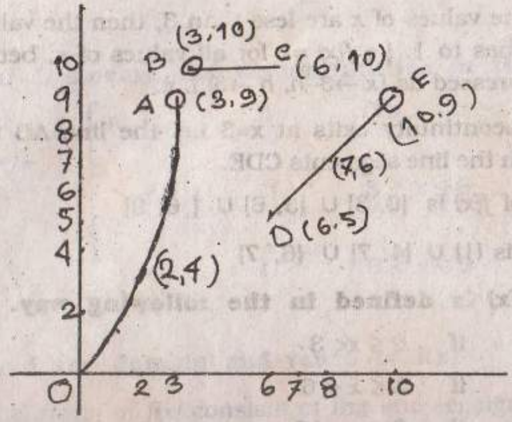
Ans : Let us draw the curves for $y=x^2$ in $0\le x < 4$, $y = 5$ in $4 \le x < 6$ and $y = x - 1$ in $6 \le x < 8$

x	0	1	2	3	for fig OA
y	0	1	4	9	

$y = x^2$

x	4	5	6
y	10	10	10

x	6	7	8	9	10	fig 3
y	5	6	7	8	9	



The graphs are OA, BC and DE

From the figure we see that the curve is discontinuous at A (3,9) and E (10,9). By the formula

$\underset{x\,3+0}{Lt}\ f(x) = 10$, $\underset{x\,3-0}{Lt}\ f(x) = \underset{x\,3+0}{Lt}\ x^2 = 9$. Limits are not equal

$f(3+0) = f(3) = 10 \neq f(3-0)$

Hence $f(x)$ is discontinuous at $x=3$

Again

$\underset{x\to 6-0}{Lt} f(x) = 10$ $\quad \underset{x\to 6+0}{Lt} f(x) = \underset{x\to 6+0}{Lt} x-1 = 5$; $Lt\ f(x) = 5$ when $x = 6$

$\therefore\ f(6-0) \neq f(6) = f(6) = 5$

Limits are not equal. Hence $f(x)$ is discontinuous at $x=6$.

Hence discontinuties of $f(x)$ are at $x = 3$ and 6

Domain of $f(x)$ is $[0,3) \cup [3,6) \cup [6,10) = [0,10)$

Range of $f(x)$ is $[0,9) \cup \{10\} \cup [5,9)- = 9]-\{3\}-\{6\}$

For Exercise

Ex.1 Show that for the function

$$f(x) = x^2 \quad \text{if } 0 \le x < 2$$
$$= 5 \quad \text{if } 2 \le x < 4$$
$$= x-1 \quad \text{if } 4 \le x < 6$$

The discontinuities are at $x=2$ and $x=4$

Also find the domain and Range of $f(x)$

Ans : Domain = $[0,6)$, Range = $[0,5) - \{2\}-\{4\}$

Ex.2 Find the discontinuities of

$$f(x) = 1 \quad \text{if } 0 \le x \le 2$$
$$= x+1 \quad \text{if } 2 < x \le 5$$
$$= 6 \quad \text{if } 5 < x \le 7$$

Also find the domain and range of $f(x)$

Ans : Disconduities are at $x = 2$

Domain. $[0, 2] \cup [2, 5] \cup [5,.7] = [0, 7]$

Range = $\{1) \cup (3, 6] \cup \{6\}$

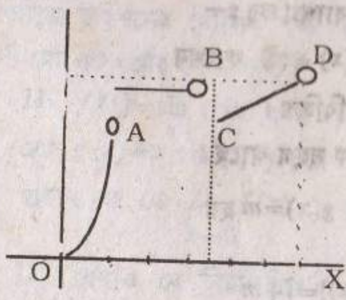


Fig. for Ex.1

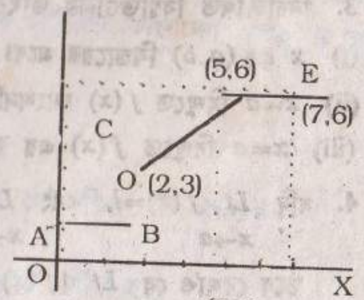


Fig 2.

বিবিধ প্রশ্নমালা

(Miscellaneous Examples)

ফাংশন, সীমা এবং অবিচ্ছিন্নতা

(on functions, limits and continuties)

1. এরূপ একটি ফাংশনের উদাহরণ দাও যাহা মূল বিন্দুতে অবিচ্ছিন্ন নয়।

(a) মনে কর কোন পরীক্ষার্থী বালকের নাম x এবং তার পরীক্ষার ক্রমিক নং y. এখন x ও y এর মধ্যে সম্পর্কটা কি ফাংশন পর্যায়ে পড়ে ?

(b) নিম্নলিখিতগুলির উদাহরণ দাও।

(i) বাস্তব রাশিগুলির অপসারী অনুক্রম,

(Divergent sequence of real numbers.)

(ii) একটি ফাংশন যাহা মূলবিন্দুতে অন্তরীকরণ যোগ্য নয়।

2. নিম্নলিখিত বিবৃতিগুলির তাৎপর্য্য ব্যাখ্যা কর :—

Explain the significance of the statements :—

(i) $f(x)\to l$ যখন $x\to a$

(ii) $f(x)\to\infty$ যখন $x\to a$

(iii) $f(x)\to l$ যখন $x\to\infty$

3. নিম্নলিখিত বিবৃতিগুলির তাৎপর্য্য ব্যাখ্যা কর :—

(i) x এর (a, b) বিস্তারের মধ্যে $f(x)$ একটি ফাংশন

(ii) $x=a$ বিন্দুতে $f(x)$ ফাংশনটি অবিচ্ছিন্ন।

(iii) $x=a$ বিন্দুতে $f(x)$ এর অন্তরক সহগ আছে।

4. যদি $\underset{x\to a}{Lt}\ f(x)=l$, এবং $\underset{x\to a}{Lt}\ g(x)=m$ হয়

তবে দেখাও যে $\underset{x\to a}{Lt}\ [f(x)+g(x)]=l+m$

5. যদি $\underset{x\to a}{Lt}\ f(x)=l$, এবং $\underset{x\to a}{Lt}\ g(x)=m$ হয়

তবে দেখাও যে $\underset{x\to a}{Lt}\ f(x)\,g(x)=lm$

6. দেখাও যে $x=a$ বিন্দুতে $f(x)$ ফাংশনের বৃদ্ধিহারের (derivative) অস্তিত্ব থাকলে $x=a$ বিন্দুতে $f(x)$ অবিচ্ছিন্ন হবে ; কিন্তু এর বিপরীত বিবৃতিটি সবসময় সত্য নাও হতে পারে। R. H. 1964

7. প্রমাণ কর যে অন্তরীকরণ যোগ্যতার জন্য কোন ফাংশনের অবিচ্ছিন্নতা একটি প্রয়োজনীয় শর্ত কিন্তু শর্তটি যথেষ্ট নয়। তোমার উত্তরের সাথে একটি উদাহরণ উল্লেখ কর।

8. অবিচ্ছিন্ন ফাংশনের যে কোন একটি মৌলিক ধর্মের উল্লেখ কর এবং ঐ ধর্মের সাহায্যে দেখাও যে $x^5=2$ সমীকরণের একটি এবং কেবলমাত্র একটি ধনাত্মক বীজ আছে।

9. প্রত্যেক সংজ্ঞায়িত বিন্দুতে একটি ফাংশান $f(x)$ অন্তরীকরণ যোগ্য। এ বক্তব্য থেকে কি বুঝা যায়।

(A function $f(x)$ is differentiable for every point of definition. What will you infer from this statement ?)

10. রোলের—উপপাদ্য (Rolle's Theorem.) প্রমাণের জন্য ব্যবহৃত অবিচ্ছিন্ন ফাংশনের মৌলিক ধর্মটি ব্যতিত অবিচ্ছিন্ন ফাংশনের অপর যে কোন একটি মৌলিক ধর্মের বর্ণনা দাও।

11. $f(x)=\sin(1/x)$ যখন $x\neq 0$

এবং $g(x)=x\sin(1/x)$ যখন $x\neq 0$

প্রমাণ কর যে $\underset{x\to 0}{Lt}\ f(x)$ এর অস্তিত্ব নেই কিন্তু $\underset{x\to 0}{Lt}\ g(x)=0$

12. দেখাও যে নিম্নলিখিত ফাংশনগুলির সীমা হবে

(i) $\underset{x\to 0}{Lt}\ \dfrac{x-\sin x\cos x}{3x}=0$

(ii) $\underset{x \to 0}{Lt} \dfrac{xe^x - \log(1+x)}{x^2} = 3/2$

(iii) $\underset{x \to \frac{1}{4}\pi}{Lt} \dfrac{1 - \tan x}{1 - \cot x} = -1$ (iv) $\underset{\theta \to 0}{Lt} \dfrac{\sin \theta - \theta}{\theta^3} = -\frac{1}{6}$

R. U. 1966

(v) $\underset{x \to \infty}{Lt} \dfrac{1}{x} \sin x = 0$ (vi) $\underset{x \to 1}{Lt} \left[\dfrac{1}{1-x} - \dfrac{3}{1-x^3}\right] = -1$

(vii) $\underset{x \to 4}{Lt} \dfrac{\sqrt{(2x+1)} - 3}{\sqrt{(x-2)} - \sqrt{2}} = \dfrac{2\sqrt{2}}{3}$

13. যদি $f(x) = \dfrac{x^2+1}{x-1}$ যখন $x < 3$

$= \dfrac{\sin x}{x-1}$ যখন $x > 3$ হয়, তবে

x এর কোন মানের জন্য ফাংশনটিকে সংজ্ঞায়িত করা যাবে?

14. $(1, -1)$ ব্যবধিতে একটি সীমাবদ্ধ ফাংশন নির্ণয় কর যাহা একটি একক বিন্দু $x = 0$ তে বিচ্ছিন্ন হয়। [Cite an example of a bounded function in $(1, -1)$ Which is discontinuous at the single point $x = 0$.

15. $f(x) = (x-a) \sin \dfrac{1}{x-a}$ যখন $x \neq a$

$= 0$ যখন $x = a$

দেখাও যে $x = a$ বিন্দুতে ফাংশনটি অবিচ্ছিন্ন এবং অন্তরীকরণ যোগ্য।

16. ফাংশন $f(x) = x \cos(1/x)$ যখন $x \neq 0$

$= 0$ যখন $x = 0$ হইলে

দেখাও যে (i) $x \to 0$ হয় সীমা (limit) শূন্য হয়।

(ii) $x = 0$ বিন্দুতে $f(x)$ অবিচ্ছিন্ন।

(iii) $x = 0$ বিন্দুতে $f'(x)$ এর অস্তিত্ব থাকে না।

17. $(-\infty, \infty)$ বিস্তারের মধ্যে সংজ্ঞায়িত নিম্নলিখিত ফাংশনটির অবিচ্ছিন্নতা এবং অন্তরীকরণ যোগ্যতার পরীক্ষা কর :— C. H. 1989 C. U. 1993

$f(x) = 1,$ যখন $-\infty < x < 0$

$= 1 + \sin x$ যখন $0 \leqslant x < \pi/2$

$= 2 + (x - \pi/2)^2$ যখন $\pi/2 \leqslant x < \infty$

আরো দেখাও যে $f(x)$ অস্তিত্ব $x = \pi/2$ বিন্দুতে থাকে কিন্তু $x = 0$ বিন্দুতে থাকেনা।

18. নিম্নলিখিত উপায়ে একটি ফাংশনকে সংজ্ঞায়িত করা হলো :—

$f(x) = 4 + x^2$ যখন $0 < x \leqslant 4$

$= 4,$ যখন $-1 \leqslant x \leqslant 0$

$= 1 + x,$ যখন $-4 \leqq x < -1$

দেখাও যে $f(x)$ ফাংশনটি $x = 0$ বিন্দুতে অবিচ্ছিন্ন কিন্তু $x = -1$ বিন্দুতে বিচ্ছিন্ন।

19 নিম্নে একটি ফাংশন সংজ্ঞায়িত করা হলো D.U. 1990

$f(x) = x^2;$ যখন $x \leqslant 0$

$= 1;$ যখন $0 < x < 1$ R.U. 1980

$= 1/x;$ যখন $x > 1$

দেখাও যে $x = 0$ বিন্দুতে $f(x)$ ফাংশনটি অন্তরীকরণের অযোগ্য। $x = 1$ বিন্দুতে $f'(x)$-এর অবস্থা কি হবে?

20. কোথায় ফাংশনটি অবিচ্ছিন্ন (discontinuous) তা নির্ণয় কর :—

$f(x) = x^2 + 1,$ যখন $0 \leqslant x < \frac{1}{2}$

$= 0$ যখন $x = \frac{1}{2}$ R.U. 1960

$= x + 3,$ যখন $\frac{1}{2} < x \leqq 1$

21. যদি $y = x^2$ যখন $x \leqslant 1$ C.U. 1987, '89

$= x,$ যখন $1 < x \leqslant 2$

$= \frac{1}{4}x^3,$ যখন $x > 2$ হয় তবে

দেখাও যে $x = 1$, এবং $x = 2$ বিন্দুদ্বয়ে y অবিচ্ছিন্ন।

(i) $x = 0$ বিন্দুতে $\dfrac{dy}{dx}$ এর মান নির্ণয় কর

$y = x^2 + 1;\ x \geqq 0$

$= \cos x,\ x \leqq 0$ R.U. 1986

22. যদি $y=-x$, যখন $x\leqslant 0$

$= x$, যখন $0<x\leqslant 1$

$=2-x$, যখন $x>1$ হয়, তবে D.U. 1989

দেখাও যে $x=0$ এবং $x=1$ বিন্দুদ্বয়ে y অবিচ্ছিন্ন।

23. যদি $f(x)=3+2x$, যখন $-3/2\leqslant x<0$

$=3-2x$, যখন $0\leqq x<3/2$ C.U. 1993

$=-3+2x$, যখন $x\geqslant 3/2$ হয়, তবে

দেখাও যে $x=0$ বিন্দুতে $f(x)$ অবিচ্ছিন্ন এবং $x=3/2$ বিন্দুতে $f(x)$ বিচ্ছিন্ন।

24. যদি $f(x)=\frac{1}{2}(b^2-a^2)$, যখন $0\leqslant x\leqslant a$ R. U. 1988

$=\frac{1}{2}b^2-\frac{1}{6}x^2-\frac{1}{3}(a^3/x)$, যখন $a<x\leqslant b$

$=\frac{1}{3}(b^3-a^3)/x$, যখন $x>b$

দেখাও যে x-এর সকল ধনাত্মক মানের জন্য $f(x)$ এবং $f'(x)$ অবিচ্ছিন্ন।

25. প্রদত্ত ফাংশন $y=f(x)$ কে নিয়ে সংজ্ঞায়িত করা হলো

$f(x)=0$, যখন $x^2>1$

$=1$, যখন $x^2<1$

$=\frac{1}{2}$, যখন $x^2=1$

দেখাও যে $x=\pm 1$, বিন্দুতে $f(x)$ বিচ্ছিন্ন। x-এর সকল মানের জন্য ফাংশনটির একটি মান থাকা সত্ত্বেও ইহা বিচ্ছিন্ন কেন তাহা ব্যাখ্যা কর।

26. প্রমাণ কর যে $\underset{x\to\infty}{Lt}\; y^n=0$ হবে যদি y একটি প্রকৃত ভগ্নাংশ এবং n একটি ধনাত্মক পূর্ণ সংখ্যা হয়।

27. দেখাও যে

$$\underset{n\to\infty}{Lt}\left(\frac{1}{n^2}+\frac{2}{n^2}+\frac{3}{n^2}+\ldots+\frac{n}{n^2}\right)=\frac{1}{2}$$

28. দেখাও যে

$$\underset{n\to\infty}{Lt}\left(1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^n}\right)=2$$

29. দেখাও যে $\underset{n\to\infty}{Lt}\dfrac{1^2+2^2+3^2+\ldots\ldots+n^2}{n^3}=\frac{1}{3}$

30. কোন ধারার সীমার সংজ্ঞা লিখ। $S_n=(-1)^n\dfrac{n}{n+1}$ এভাবে সংজ্ঞায়িত একটি ধারার কি কোন সীমা আছে?

(Define limit of a sequence. Has the sequence defined by

$S_n=(-1)^n\dfrac{n}{n+1}$ a limit? R.H. 1966

31. ফাংশনের সংজ্ঞা দাও। নিম্নলিখিত বাস্তব ফাংশনগুলির চারণ ক্ষেত্র (domain) নির্ণয় কর।

(i) $f(x)=\dfrac{1}{x}$ উত্তর : $0<x<\infty, -\infty<x<0$

(ii) $f(x)=\sqrt{(x+1)}$ উত্তর : $x\geqslant -1$

(iii) $f(x)=\dfrac{x^2+1}{(x^2-5x+6)}$ C.U. 1986 উত্তর ; $x=2, 3$ ব্যতিত x-এর সকল বাস্তব মান।

(iv) $f(x)=\dfrac{x^3-1}{x-1}$ উত্তর : $x=1$, ব্যতিত x-এর সকল বাস্তব মান।

(v) $f(x)=\dfrac{x}{|x|}$ উত্তর : $x=0$ ব্যতিত x-এর সকল বাস্তব মান।

(vi) $f(x)=\dfrac{+\sqrt{(x-1)}}{(x^2-5x+1)}$, উঃ x-এর যে সব মান $x^2-5x+1=0$ সমীকরণ কে সিদ্ধ করে এবং $x<1$ এসব মান ব্যতিত x-এর সকল মানের জন্য

(vii) $f(x)=\dfrac{x^2-4}{x-2}$ উঃ $-2<x<2$

(viii) $f(x)=+\sqrt{(x^2-4x+3)}$ উঃ $3\leqslant x\leqslant 1$

(ix) $f(x)=\dfrac{\sqrt{(x-2)}}{5x^2-27x+10}$ উঃ $2\leqslant x<5,\ 5<x<\infty$

(x) $f(x)=\dfrac{x}{\sin(1/x)}$ উঃ $0<x\leqslant\dfrac{1}{n\pi}$

R. U. 1967 ; C. U. 1969

(xi) $f(x)=\dfrac{x^2}{(x-1)(x-2)}$ উঃ $(-\infty,\infty)-\{1,2\}$ C.U.1992

32. নিম্নলিখিত ফাংশনগুলির বৃহত্তম চারণক্ষেত্র এবং বৃহত্তম ব্যাপ্তি নির্ণয় কর :—

(i) $f(x)=-1$ যখন $x<0$ উঃ চারণ ক্ষেত্র $|x|\geqq 0$ ব্যাপ্তি $(-1, 1)$
$\quad = 0$ যখন $x=0$
$\quad = 1$ যখন $x>0$

(ii) $f(x)=2$ যখন $-5<x<-1$
$\quad =\sin x$ যখন $0<x<1$

(iii) $f(x)=x+5$ যখন $-\infty<x<\infty$ উঃ $-\infty<f(x)<\infty$

(iv) $f(x)=x^2+x+1$ উঃ $x\geqq 0,\ y\geqq 0$

(v) $f(x)=x\sin(1/x)$ উঃ $x=0$, ব্যতিত x-এর সকল মান চারণক্ষেত্র। ব্যাপ্তি $(-1, 1)$

33. $n\to\infty$ অগ্রসর হলে নিম্নলিখিত ফাংশনগুলি সীমার দিকে অগ্রসর হয় কি না হয় তাহা পরীক্ষা কর :—

(i) $f(n)=\dfrac{(-1)^n}{n}$, (ii) $f(n)=\dfrac{1}{n-(-1)^n}$

(iii) $f(n)=1+1/n$, (iv) $f(n)=n\,[1+(-1)^n]$ R. U. 1964.

34 দেখাও যে যদি α একটি ক্ষুদ্রাতিক্ষুদ্র রাশি হয়, তবে x এর তুলনায় $\sin\alpha$ এবং $\tan\alpha$ ও কই মাত্রার ক্ষুদ্রাতিক্ষুদ্র রাশি হবে কিন্তু x এর তুলনায় $(1-\cos x)$ হবে দ্বিতীয় মাত্রার (second order) ক্ষুদ্রাতিক্ষুদ্র রাশি।

[আমরা জানি $\underset{\alpha\to0}{Lt}\dfrac{\sin x}{\alpha}=1$ এবং $\underset{\alpha\to0}{Lt}\dfrac{\tan\alpha}{\alpha}=1$ অর্থাৎ $\sin\alpha$, $\tan\alpha$ এবং α এর মাত্রা একই।

আরো $\underset{\alpha\to0}{Lt}\dfrac{1-\cos\alpha}{\alpha^2}=\dfrac{2\sin^2\alpha/2}{(\alpha/2)^2\times4}=\frac{1}{2}$ যাহা শূন্য নহে এবং সসীম।

সুতরাং $1-\cos\alpha$ এবং α^2 এর মাত্রা একই এবং ইহারা উভয়ে ২য় মাত্রার।].

35. দেখাও যে $3x+2x^2$ একটি একই মাত্রার ক্ষুদ্রাতিক্ষুদ্র রাশি।

36. দেখাও যে $\sqrt{(\sin\alpha)}$ এর মাত্রা α এর মাত্রার নীচে।

37. দেখাও যে $\sqrt{x}$ একটি ক্ষুদ্রাতিক্ষুদ্র রাশি যার মাত্রা α এর মাত্রার নীচে।

38. দেখাও যে $\sin x-\tan x$ এর মাত্রা তৃতীয় এবং ইহার প্রধান অংশ $\frac{1}{2}x^3$.

39. দেখাও যে $\sin\alpha\,(1-\cos\alpha)$ একটি তৃতীয় মাত্রার ক্ষুদ্রাতিক্ষুদ্র রাশি এবং ইহার প্রধান অংশ $\frac{1}{2}\alpha^3$,

40. উদাহরণের সাহায্যে দেখাও যে কোন বিন্দুতে একটি ফাংশান বিচ্ছিন্ন হইলে সীমা বিদ্যমান।

উত্তরমালা I (C)

4. অবিচ্ছিন্ন নয়।	5. অবিচ্ছিন্ন নয়।	9. অবিচ্ছিন্ন।
10. অবিচ্ছিন্ন।	13. বিচ্ছিন্ন।	14. অবিচ্ছিন্ন নয়।
15. হাঁ, না।	16. অবিচ্ছিন্ন নয়।	17. অবিচ্ছিন্ন।
18. না।	19 না।	20. অবিচ্ছিন্ন।

বিবিধ প্রশ্নের উত্তরমালা

1. $y=1/x$, না, 9. ঐ বিস্তারে ফাংশনটি অবিচ্ছিন্ন। 13. $x=1$ এ,

14. $f(x)=3-x$, যখন $-1\leqslant x<0$ 19. $x=1$ বিন্দুতে সংজ্ঞায়িত করা যায় না।
$\quad =0$, যখন $x=0$
$\quad =x-3$, যখন $0<x\leqslant1$

20. $x=0,\ \frac{1}{2}$ 28. 2. 30. না,

33. (i) 0, (ii) 0, (iii) 1, (iv) 0, n বিজোড় না, n জোড় সংখ্যা।

CHAPTER IV

Differential Co-efficient

4.1. Differential Co-efficient :—The differential Co-efficient $f'(a)$ of a function $f(x)$ at $x=a$ is defined as

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \quad \ldots\ldots\ldots(1)$$

provided the limit exists and finite. The differential co-efficient $f'(a)$ is also known as the first derivative or simply the derivative of $f(x)$ at $x=a$.

Note that both a and $a+h$ belong to domain of f.

For the function $f : x\to y$, we can interpret the differential Co-efficient at a point as the rate of change of y with respect to x at the point. Let x change small quantity δx or h and the corresponding change in y is δy or k. That is, $f : (x+\delta x)\to y+\delta y$.

Now $k=\delta y=(y+\delta y)-y=f(x+\delta x)-f(x)$

$$\frac{\delta y}{\delta x}=\frac{f(x+\delta x)-f(x)}{\delta x}$$

Hence $f'(x)=\lim_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\right)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}, \ldots\ldots\ldots (2)$

provided the limit exists.

We write $\frac{dy}{dx}=\lim_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\right)$, when the limit exists.

Thus for $y=f(x)$, $f'(x)=\frac{dy}{dx}$ is the rate of change of y with respect to x at the point x.

The process of finding the differential co-efficient or derivative of a functions is called differentiation.

It is often said that "differentiate $f(x)$ *w. r.* to x" means that differentiation is made *w.r.* to the independent variable x"

In (2) we do not consider the derivative of $f(x)$ for any particular value of x but it is considered at any $x\in D_f$

4. 2. Prove that every finitely derivable function is continuous

If $f(x)$ is differentiable at $x=a$, then

it is continuous at $x=a$.

For proof see Art-3·9

Cor. The converse of the theorem is not necessarily true *i.e*," a function may be continuous at a point $x=a$ but it is not true that finite derivative should exist for that value. An example is given below.

Ex. If $f(x)=x-a$ for $x>a$

$=a-x$ for $x<a$

$=0$ for $x=a$

Show that $f(x)$ is continuous at $x=a$ but has no differential co-efficient at $x=a$

$\lim_{x\to a^+} f(x)=\lim_{\substack{h\to 0\\(h>0)}} f(a+h)=\lim_{h\to 0}(a+h-a)=0$

$\lim_{x\to 0^-} (f(x)=\lim_{\substack{h\to 0\\(h>0)}} f(a-h)=\lim_{h\to 0}\{(a-(a-h)\}=0$

Also $f(a)=0$

$\therefore \lim_{x\to a^+} f(x)=\lim_{x\to a^-} f(x)=f(a)$

Hence $f(x)$ is continuous at $x=a$,

Now

$$Rf'(a)=\lim_{h\to 0^+}\frac{(a+h)-f(a)}{h}=\lim_{h\to 0^+}\frac{(a+h-a)-0}{h}$$

$$=\lim_{h\to 0^+}\left(\frac{h}{h}\right)=1$$

$$Lf'(a)=\lim_{h\to 0^-}\frac{f(a-h)-f(a)}{-h}=\lim_{h\to 0-}\frac{\{a-(a-h)\}-0}{-h}=-1$$

Thus $Rf'(a)\neq Lf'(a)$ *i. e.,* limit does not exist. Hence derivative of $f(x)$ at $x=a$ does not exist.

Some General Theorems on Differentiation

4.3. The differential co-efficient of any constant is zero.

Let $y=f(x)=c$ where c is a constant.

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{(x+h)-x}$$

$$=\lim_{h\to 0}\frac{c-c}{h}=\lim_{h\to 0}\frac{0}{h}=0$$

Hence $\frac{dy}{dx}(c)=0$. where c is a constant

4.4. Product of a constant and a Function

The differential co-efficient of a product of a constant and a function is equal to the product of the constant and differential co-efficient of the function

Let $y=f(x)=c\phi(x)$ then $f(x+h)=c\phi(x+h)$

$$\therefore \frac{dy}{dx}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{c\phi(x+h)-c\phi(x)}{h}=c\phi'(x)$$

i. e. $\frac{dy}{dx}\{c\phi(x)\}=c\,\frac{dy}{dx}\phi(x)$, where c is a constant

4.5. Differential Co-efficient of a sum or difference

The differential co-efficient of the sum or difference of a set of functions is the sum or difference of the differential co-efficients of that set of function.

Let $y=f(x)=\phi(x)\pm\psi(x)$

then $f(x+h)=\phi(x+h)\pm\psi(x+h)$

$$\therefore \frac{dy}{dx}=\lim_{h\to 0}\frac{\{\phi(x+h)\pm\psi(x+h)\}-\{\phi(x)+\psi(x)\}}{h}$$

$$=\lim_{h\to 0}\frac{\phi(x+h)-\phi(x)}{h}\pm\lim_{h\to 0}\frac{\psi(x+h)-\psi(x)}{h}=\phi'(x)\pm\psi'(x)$$

Hence $\frac{d}{dx}\{\phi(x)\pm\psi(x)\}=\phi'(x)\pm\psi'(x)$

Generalisation :—By repeated applications of the above result obtained it can be proved that if

$y=u_1\pm u_2\pm u_3\pm v_4\pm \quad \cdots \quad \cdots$

and $y\pm\delta y=u_1\pm\delta u_1\pm u_2\pm\delta u_2\pm(u_3+\delta u_3)\pm(u_4\pm\delta u_4)+\cdots$

then $\delta y=\delta u_1\pm\delta u_2\pm\delta u_3\pm\delta u_4\pm \quad \cdots \quad \cdots$

$$\text{or,}\quad \frac{\delta y}{\delta x}=\frac{\delta u_1}{\delta x}\pm\frac{\delta u_2}{\delta x}\pm\frac{\delta u_3}{\delta x}\pm\frac{\delta u_4}{\delta x}+\quad\cdots$$

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\lim_{\delta x\to 0}\left(\frac{\delta u_1}{\delta x}\pm\frac{\delta u_2}{\delta x}\pm\frac{\delta u_3}{\delta x}\pm\frac{\delta u_4}{\delta x}+\cdots\right)$$

$$\text{or,}\quad \frac{dy}{dx}=\frac{du_1}{dx}\pm\frac{du_2}{dx}\pm\frac{du_3}{dx}\pm\frac{du_4}{dx}+\cdots$$

4. 7. Differential Co-efficient of a product

Let $y=uv$

where u and v are two derivable functions of x.

Let u change to $u+\delta u$; v change to $v+\delta v$ when x changes to $x+\delta x$.

Let u change to $u+\delta u$ and v changes to $v+\delta v$ when x changes to $x+\delta x$.

$\delta y = u\delta v + v\delta u + \delta u\, \delta v$

Or, $\dfrac{\delta y}{\delta x} = u\dfrac{\delta v}{\delta x} + v\dfrac{\delta u}{\delta x} + \delta u\,\dfrac{\delta v}{\delta x}$

Let $\delta x \to 0$ then δu, δv also $\to 0$ for u and v which are derivable function of x are continuous.

$$\frac{dy}{dx} = \lim_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\right) = \lim_{\delta x\to 0}\left(u\frac{\delta v}{\delta x} + v\frac{\delta u}{\delta x} + \delta u\,\frac{\delta v}{\delta x}\right)$$

$$= u\frac{dv}{dx} + v\frac{du}{dx} + 0\,\frac{dv}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Hence $\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

The differential co-efficient of the product of the functions is equal to

First function × derivative of the second + second function × derivative of the first.

Cor. If $y = u.\,v$ then

$\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$ is divided by $u.v$

$\dfrac{1}{uv}\cdot\dfrac{dy}{dx} = \dfrac{1}{v}\dfrac{dv}{dx} + \dfrac{1}{u}\dfrac{du}{dx}$

or. $\dfrac{dy/dx}{y} = \dfrac{du/dx}{u} + \dfrac{dv/dx}{v}$

Now if we consider y as the function of several variables *i.e.*,

$y = u.\,v.\,w.\,t \ldots\ldots\ldots$ then

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx} + \frac{1}{t}\frac{dt}{dx} + \ldots$$

4, 8. Differential co-efficient of a quotient of the functions

Let $y = \dfrac{u}{v}$ where u and v are two derivable functions of x and $v \neq 0$

Let δy, δu, δv be the increments of y, u, v, respectively when x changes by δx. Then.

$$y + \delta y = \frac{u+\delta u}{v+\delta v}$$

$$\delta y = \frac{u+\delta u}{v+\delta v} - \frac{u}{v} = \frac{v\delta u - u\delta v}{v(v+\delta v)}. \text{ or, } \frac{\delta y}{\delta x} = \frac{v\dfrac{\delta u}{\delta x} - u\dfrac{\delta v}{\delta x}}{v(v+\delta v)}$$

If $\delta x \to 0$ then δv also $\to 0$

$$\therefore\ \frac{dy}{dx} = \lim_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

$$\text{Hence } \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} = \frac{u'v - uv'}{v^2}$$

The differential co-efficient of the quotient of two functions is equal to

$$\frac{\text{(Differential Co-efficient of Numerator)} \times \text{(Denominator)} - \text{Differential (Co-efficient of Denominator)} \times \text{Numerator.}}{\text{(Denominator)}^2}$$

4.9. Differential Co-efficient of x^n (সংখ্যার সাহায্যে ডিফা-রেন্সিয়েল সহগ নির্ণয়)

Let $y = f(x) = x^n$ where n is a positive integer.

Then $f(x+h) = (x+h)^n$

$$\therefore \quad f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h\to 0} x^n \frac{(1+h/x)^n - 1}{h} = \lim_{h\to 0} x^{n-1} \frac{(1+h/x)^n - 1}{h/x}$$

$$= \lim_{h\to 0} x^{n-1} \frac{(1+h/x)^n - 1}{(1+h/x)-1} \; if\ x \neq 0$$

Put $z = 1+h/x$. If $h\to 0$, then $z\to 1$.

$$\therefore \quad f'(x) = \lim_{z\to 1} x^{n-1} \frac{z^n - 1}{z-1}$$

$$= x^{n-1} \lim_{z\to 1} (z^{n-1} + z^{n-2} + \ldots + 1) = x^{n-1} n = n x^{n-1}$$
(n terms)

Hence $\frac{d}{dx} x^n = n x^{n-1}$ when n *is* a positive integer.

If n is not a positive integer,

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h} = \lim_{h\to 0} \frac{x^n\left(1+\frac{h}{x}\right)^n - 1}{h}$$

$$= x^n \lim_{h\to 0} \frac{1}{h}\left[\left\{1 + \frac{n}{1}\left(\frac{h}{x}\right) + \frac{n(n-1)}{1.2}\left(\frac{h}{x}\right)^2 + \ldots\right\} - 1\right]$$

$$= x^n \left[\frac{n}{1}\left(\frac{1}{x}\right)\right] \left[\text{using Binomial expansion with } \left|\frac{h}{x}\right| < 1\right]$$

or. $f'(x) = nx^{n-1}$

$$\therefore \quad \frac{d}{dx}(x^n) = n x^{n-1} \quad \text{for any real values of } n.$$

4.10 Differential co-efficient of a^x

Let $y = f(x) = a^x$, then $f(x+h) = a^{x+h}$

$$\therefore \quad f'(x) = \frac{dy}{dx} = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h\to 0} a^x \left(\frac{a^h - 1}{h}\right)$$

$$= a^x \lim_{h\to 0} \left(\frac{e^{h\,\log a} - 1}{h}\right)$$

$$= a^x \lim_{h\to 0} \frac{1}{h}\left[\left\{1 + \frac{h\,\log a}{\lfloor 1} + \frac{(h\,\log a)^2}{\lfloor 2} + \ldots\right\} - 1\right]$$

i e. $f'(x) = a^x \log a$

Hence $\frac{d}{dx} a^x = a^x \log_e a$

4.11. If $y = e^x$, $\frac{dy}{dx} = e^x \log_e e = e^x \quad \therefore \frac{d}{dx} e^x = e^x$

4.12 Differential Co-efficient of $\log_a x$. *w. r.* to x

Let $y = f(x) = \log_a x$. then $f(x+h) = \log_a (x+h)$

$$\therefore \quad \frac{dy}{dx} = \lim_{h\to 0} \frac{\log_a (x+h) - \log_a x}{h} = \lim_{h\to 0} \frac{1}{h} \log_a \left(1 + \frac{h}{x}\right)$$

Put $\frac{x}{h} = z$. If $h\to 0$, theh $z \to \infty$

$$\therefore \quad \frac{dy}{dx} = \lim_{z\to\infty} \frac{z}{x} \log_a\left(1 + \frac{1}{z}\right) = \frac{1}{x} \lim_{z\to\infty} \log_a\left(1 + \frac{1}{z}\right)^z$$

or, $\frac{dy}{dx} = \frac{1}{x} \log_a e \left[\therefore \lim_{z\to\infty} \left(1 + \frac{1}{z}\right)^z = e\right]$

Hence $\frac{dy}{dx}(\log_a x) = \frac{1}{x} \log_a e$

4.13 If $y = \log_e x$ then $\frac{dy}{dx} = \frac{1}{x} \log_e e = \frac{1}{x} .1 = \frac{1}{x}$

$$\therefore \quad \frac{d}{dx} \log_e x = \frac{1}{x}$$

4.14 Differential Co-efficient of $\sin x$ *w. r.* ro x

Let $y = f(x) = \sin x$ then $f(x+h) = \sin(x+h)$

$$\frac{dy}{dx} = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{2\cos(x + \frac{1}{2}h)\sin\frac{1}{2}h}{h} = \lim_{h\to 0} \cos(x + \tfrac{1}{2}h) \frac{\sin\frac{1}{2}h}{\frac{1}{2}h}$$

$= \cos x.\ 1$ by Art. 2. 13.

Hence $\frac{d}{dx}(\sin x)=\cos x$.

4.15 Similarly the differential co-efficient of cosx is $-\sin x$

i. e. $\frac{d}{dx}\cos x=-\sin x$

4.16 Differential Co-efficient of tan x w. r to x

Let $y=f(x)=\tan x$; then $f(x+h)=\tan(x+h)$

$$\therefore \frac{dy}{dx}=\lim_{h\to 0}\frac{\tan(x+h)-\tan x}{h}$$

$$=\lim_{h\to 0}\frac{\sin(x+h)\cos x-\cos(x+h)\sin x}{h\cos(x+h)\cos x}$$

$$=\lim_{h\to 0}\frac{\sin(x+h-x)}{h\cos(x+h)\cos x}=\lim_{h\to 0}\frac{\sin h}{h}\cdot\frac{1}{\cos(x+h)\cos x}$$

$$=1.\frac{1}{\cos x\cos x}=\frac{1}{\cos^2 x}=\sec^2 x \quad \text{by Art. 2. 13.}$$

Hence $\frac{d}{dx}(\tan x)=\sec^2 x$

4.17 Similarly the differential co-efficient of cotx is $-\text{cosec}^2\, x$

i. e. $\frac{d}{dx}\cot x=-\text{cosec}^2 x$

4. 18. Differential Co-efficient of sec x. w. r. to x

Let $y=f(x)=\sec x$, then $f(x+h)=\sec(x+h)$

$$\therefore \frac{dy}{dx}=\lim_{h\to 0}\frac{\sec(x+h)-\sec x}{h}=\lim_{h\to 0}\frac{\cos x-\cos(x+h)}{h\cos x\cos(x+h)}$$

$$=\lim_{h\to 0}\frac{\sin\frac{1}{2}h}{\frac{1}{2}h}\cdot\frac{\sin(x+\frac{1}{2}h)}{\cos x\cos(x+\frac{1}{2}h)}=\frac{\sin x}{\cos^2 x}=\sec x\tan x$$

Hence $\frac{d}{dx}(\sec x)=\sec x\tan x$

4. 19. Similarly the differential co-efficient of cosecx is $-$cosecx cot x i. e.,

$$\frac{d}{dx}(\text{cosec}\, x)=-\text{cosec}\, x\cot x.$$

4. 20. Differential Co-efficient of $\sin^{-1}x$. w. r to x

Let $y=f(x)=\sin^{-1}x$; then $f(x+h)=\sin^{-1}(x+h)=y+k$

Then $x=\sin y$ and $x+y=\sin(y+k)$

Therefore, $h=(x+h)-x=\sin(y+k)-\sin y$

and $k=\sin^{-1}(x+h)-\sin^{-1}x$

$$\therefore \frac{dy}{dx}=\lim_{h\to 0}\frac{\sin^{-1}(x+h)-\sin^{-1}x}{h}$$

$$=\lim_{h\to 0}\frac{k}{h}=\lim_{k\to 0}\frac{k}{\sin(y+k)-\sin y}$$

$$=\lim_{k\to 0}\frac{k}{2\cos(y+\frac{1}{2}k)\sin\frac{1}{2}k}=\lim_{k\to 0}\left(\frac{\frac{1}{2}k}{\sin\frac{1}{2}k}\right)\frac{1}{\cos(y+\frac{1}{2}k)}$$

$$=1.\frac{1}{\cos y}=\frac{1}{\sqrt{(1-\sin^2 y)}}=\frac{1}{\sqrt{(1-x^2)}}$$

[as $x=\sin y$ with $\frac{-\pi}{2}<y\le\frac{\pi}{2}$ and so $\cos y\geqslant 0$].

Hence $\frac{d}{dx}\sin^{-1}x=\frac{1}{\sqrt{(1-x^2)}}$

4.21. Similarly the differential Co-efficient of

$\cos^{-1}x$ is $\frac{-1}{\sqrt{(1-x^2)}}$ i. e., $\frac{d}{dx}\cos^{-1}x=-\frac{1}{\sqrt{(1-x^2)}}$

4 22. Differential Co efficient of $\tan^{-1}x$, w. r. to x

Let $y=f(x)=\tan^{-1}x$, then $f(x+h)=\tan^{-1}(x+h)=y+k$

[Note that $y\in(-\frac{\pi}{2},\frac{\pi}{2})$]

$\tan^{-1}\frac{x}{a}$
$\sec^{-1} x$

therefore, $x = \tan y$, $x+h = \tan(y+k)$

and $h = \tan(y+k) - \tan y$; $k = \tan^{-1}(x+h) - \tan^{-1} x$

$$\therefore \frac{dy}{dx} = \lim_{h \to 0} \frac{\tan^{-1}(x+h) - \tan^{-1} x}{h} = \lim_{k \to 0} \frac{1}{\tan(y-k) - \tan y}$$

$$= \lim_{k \to 0} \frac{k \cos(y+k) \cos y}{\sin(y+k) \cos y - \cos(y+k) \sin y}$$

$$= \lim_{k \to 0} \frac{k \cos(y+k) \cos y}{\sin k} = \lim_{k \to 0} \frac{k}{\sin k} \cos(y+k) \cos y = \cos^2 y$$

$$= \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

Hence $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

4.23. Similarly the Differential co-efficient of

$\cot^{-1} x$ is $-\frac{1}{1+x^2}$ w, r, to x

i. e., $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$

4.24. Differential co-efficient of $\sec^{-1} x$. w, r. to x.

Let $y = f(x) = \sec^{-1} x$, then $f(x+h) = \sec^{-1}(x+h) = y+k$

therefore, $x = \sec y$, $x+h = \sec(y+k)$

and $h = \sec(y+k) - \sec y$; $k = \sec^{-1}(x+h) - \sec^{-1} x$

$$\therefore \frac{dy}{dx} = \lim_{h \to 0} \frac{\sec^{-1}(x+h) - \sec^{-1} x}{h} = \lim_{h \to 0} \frac{k}{h}$$

$$= \lim_{k \to 0} \frac{k}{\sec(y+k) - \sec y} = \lim_{k \to 0} \frac{k \cos(y+k) \cos y}{\cos y - \cos(y+k)}$$

$$= \lim_{k \to 0} \frac{\frac{1}{2}k}{\sin \frac{1}{2}k} \cdot \frac{\cos(y+k) \cos y}{\sin(y+\frac{1}{2}k)} = 1 \cdot \frac{\cos^2 y}{\sin y}$$

$$= \cot y \cos y = \frac{1}{\sec y \tan y} = \frac{1}{x\sqrt{(\sec^2 y - 1)}} = \frac{1}{x\sqrt{(x^2-1)}}$$

Hence $\frac{d}{dx} \sec^{-1} x = \frac{1}{x\sqrt{(x^2-1)}}$

4.25. Similarly the Differential Co-efficient of $\operatorname{cosec}^{-1} x$ is

$-\frac{1}{x\sqrt{(x^2-1)}}$ i. e $\frac{d}{dx} \operatorname{cosec}^{-1} x = -\frac{1}{x\sqrt{(x^2-1)}}$

Examples

Ex. 1. Find from first principle the differential co-efficient of $\sqrt{(b-2ax)}$ w. r. to x

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{\{b-2a(x+h)\}} - \sqrt{(b-2ax)}}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \frac{\{b-2a(x+h)\} - (b-2ax)}{\sqrt{\{b-2a(x+h)\}} + (b-2ax)}$$

$$= \lim_{h \to 0} \frac{1}{h} \frac{-2ah}{\sqrt{\{b-2a(x+h)\}} + \sqrt{(b-2ax)}}$$

$$= \lim_{h \to 0} \frac{-2a}{\sqrt{\{b-2a(x+h)} + \sqrt{(b-2ax)}}$$

$$= \frac{-2a}{2\sqrt{(b-2ax)}} = -\frac{1}{\sqrt{(b-2ax)}}$$

$$\therefore \frac{d}{dx}\sqrt{(b-2ax)} = -\frac{a}{\sqrt{(b-2ax)}}$$

Ex. 2. Find from the definition the differential co-efficient of $\frac{2x-3}{2x+5}$ w. r. to x. [R. U. 1986]

Let $f(x) = \frac{2x-3}{3x+5}$

Let $f'(x = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{2(x+h) - 3}{3(x+h)+5} - \frac{2x-3}{3x+5}\right]$$

$$=\lim_{h\to 0}\frac{\{2(x+h)-3\}(3x+5)-\{3(x+h)+5\}(2x-3)\}}{h\cdot\{3(x+h)+5\}(3x+5)}$$

$$=\lim_{h\to 0}\frac{1}{h}\,\frac{19h}{\{3(x+h)+5\}(3x+5)}=\lim_{h\to 0}\frac{19}{\{3x+h)+5\}(3x+5)}$$

$$=\frac{19}{(3x+5)}$$

$$\therefore\ \frac{d}{dx}\left(\frac{2x-3}{3x+5}\right)=\frac{19}{(3x+5)^2}$$

Ex. 3. Find from the first principle the derivative of $\cos^2 x$. $w.\ r.$ to x. [D. U. 1965]

Let $f(x)=\cos x^2$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\cos(x+h)^2-\cos x^2}{h}$$

$$=\lim_{h\to 0}\frac{2\sin\frac{1}{2}\{(x+h)^2+x^2\}\sin\frac{1}{2}\{x^2-(x+h)^2\}}{h}$$

$$=\lim_{h\to 0}2\sin\tfrac{1}{2}\{(x+h)^2+x^2\}\,\frac{\sin\frac{1}{2}\{-2xh-h^2\}}{\frac{1}{2}\{-2xh-h^2\}}\,\frac{(-2xh-h^2)}{h}$$

$$=2\sin\tfrac{1}{2}(x^2+x^2).\ 1.\ \tfrac{1}{2}(-2x)=-2x\sin x^2$$

$$\therefore\ \frac{d}{dx}(\cos x^2)=-2x\sin x^2$$

Ex. 4. Differentiate $\tan^{-1}\frac{x}{a}$ from the first principle $w.$ r to x

[R. U. 1952]

Let $z=\tan^{-1}\frac{x}{a}$, and $z+k=\tan^{-1}\frac{x+h}{x}$

If $h\to 0$, then $k\to 0$. Now we have

$$\frac{x+h}{a}=\tan(z+k),\ \frac{x}{a}=\tan z \text{ and}$$

$$\frac{h}{a}=\frac{x+h}{a}-\frac{x}{a}=\tan(z+k)-\tan z$$

$$\therefore\ f'(x)=\lim_{h\to 0}\frac{\tan^{-1}(x+h)/a-\tan^{-1}(x/a)}{h}$$

$$=\lim_{k\to 0}\frac{k}{a\{\tan(z+k)-\tan z\}}$$

$$=\lim_{k\to 0}\frac{1}{a}\cdot\frac{k}{\sin k}\cos(z+k)\cos z=\frac{1}{a}\,1\cos z.\cos z$$

$$=\frac{\cos^2 z}{a}=\frac{1}{a}\,\frac{1}{\sec^2 z}=\frac{1}{a}\,\frac{1}{1+\tan^2 z}=\frac{1}{a}\,\frac{1}{1+(x/a)^2}$$

$$=\frac{1}{a}\,\frac{a^2}{x^2+a^2}=\frac{a}{a^2+x^2}$$

$$\frac{d}{dx}\left(\tan^{-1}\frac{x}{a}\right)=\frac{a}{a^2+x^2}$$

Ex. 5. Differentiate log (sin x) from the definition.

Let $z=\sin x$, $z+k=\sin(x+h)$

$k=(z+k)-z=\sin(x+h)-\sin x$

If $h\to 0$ then k also $\to 0$

$$f'(x)=\lim_{h\to 0}\frac{\log_e\sin(x+h)-\log_e\sin x}{h}$$

$$=\lim_{h\to 0,\,k\to 0}\frac{\log_e(z+k)-\log_e z}{k}\,\frac{k}{h}=\lim_{h\to 0,\,k\to 0}\frac{\log_e\{(z+k)/z\}}{k}\,\frac{k}{h}$$

$$=\lim_{k\to 0}\frac{\log(1+k/z)}{k/z}\,\frac{1}{z}\cdot\lim_{h\to 0}\frac{k}{h}$$

$$=\lim_{k\to 0}\frac{\log_e(1+k/z)}{k/z}\times\lim_{h\to 0}\frac{1}{z}\,\frac{\sin(x+h)-\sin x}{h}$$

$$=\lim_{h\to 0}\frac{1}{z}\,2\cos(x+\tfrac{1}{2}h)\,\frac{\sin\frac{1}{2}h}{\frac{1}{2}h}\,\tfrac{1}{2}$$

$$=1.\ 1/z.\ 2\cos x.\ \tfrac{1}{2}=\cos x/\sin x=\cot x.$$

$$\therefore\ d(\log_e\sin x)/dx=\cot x$$

Ex. 6. Differentiate x sin$x$ from the first principle $w.$ $r.$ to x [C. U. 1986]

Let $f(x)=x^{\sin x}=e^{\sin x\log x}$

$$f'(x)=\lim_{h\to0}\frac{e^{\sin(x+h)\log(x+h)}-e^{\sin x\log x}}{h}$$

$$=\lim_{h\to0}\frac{e^{\sin x\log x}\left[e^{\sin(x+h)\log(x+h)-\sin x\log x}-1\right]}{h}$$

$$=e^{\sin x\log x}\lim_{h\to0}\frac{e^z-1}{h}=e^{\sin x\log x}\lim_{h\to0}\frac{e^z-1}{z}\cdot\frac{z}{h}$$

where $z=\sin(x+h)\log(x+h)-\sin x\log x$

If $h\to0$, then z also$\to0$.

$$\therefore\ f'(x)=e^{\sin x\log x}\lim_{z\to0}\left(\frac{e^z-1}{z}\right)\lim_{h\to0}\left(\frac{z}{h}\right)$$

$$=x^{\sin x}\ 1.\ \lim_{h\to0}\left(\frac{z}{h}\right).$$

$$=x^{\sin x}\lim_{h\to0}\frac{\sin(x+h)\log(x+h)-\sin x\log x}{h}=x^{\sin x}\times$$

$$\lim_{h\to0}\left[\frac{\sin x\{\cos h\log(x+h)-\log x\}+\cos x\sin h\log(x+h)}{h}\right]$$

$$=\lim_{h\to0}x^{\sin x}\left[\sin x\left\{\frac{\log(1+h/x)}{h/x}\ \frac{1}{x}\right\}+\lim_{h\to0}\frac{\sin h}{h}\cos x.\right.$$

$\left.\log(x+h)\right]$ $\left[\because \lim_{h\to0}\cosh=1\right]$

$$=x^{\sin x}\left[\sin x\,\frac{1}{x}+1.\log x\cos x\right]=x^{\sin x}\left[\frac{\sin x}{x}+\log x\cos x\right]$$

$\left[\because \lim_{h\to0}\frac{\log(1+h/x)}{h/x}=1\right]$

Ex. 7. $f(x)=e^x\sin x$, find $f'(0)$ from definition.

$f(0)=e^0\sin 0=0.$

$$\therefore\ f'(0)=\lim_{h\to0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to0}\frac{e^{0+h}\sin(0+h)-e^0\sin0}{h}$$

$$=\lim_{h\to0}\frac{e^h\sin h}{h}=\lim_{h\to0}e^h\lim_{h\to0}\frac{\sin h}{h}=1.1$$

$\therefore\ f'(0)=1$

Ex. 8. Differentiate cot x from the first principle at $x=\frac14\pi.$ $w.$ $r,$ to x.

Let $f(x)=\cot x$

$$f'\left(\frac{\pi}{4}\right)=\lim_{h\to0}\frac{f(\frac14\pi+h)-f(\frac14\pi)}{h}=\lim_{h\to0}\frac{\cot(\frac14\pi+h)-\cot\frac14\pi}{h}$$

$$=\lim_{h\to0}\frac{\sin\frac14\pi\cos(\frac14\pi+h)-\cos\frac14\pi\sin(\frac14\pi+h)}{h\sin(\frac14\pi+h)\sin\frac14\pi}$$

$$=\lim_{h\to0}\frac{-\sin(\frac14\pi+h-\frac14\pi)}{h\sin(\frac14\pi+h)\sin\frac14\pi}$$

$$=\lim_{h\to0}\frac{-\sin h}{h}\cdot\frac{1}{\sin(\frac14\pi+h)\sin\frac14\pi}=-1.\frac{1}{\sin^2\frac14\pi}=-2.$$

$\therefore\ f'(\frac14\pi)=-2.$

Ex. 9. Find the differential co-efficient of $x^n\sin ax$ from the definition. [C. U. 1987]

Let $y=f(x)=x^n\sin ax$ N. U. (C-2) 1994

$$\therefore\ \frac{dy}{dx}=f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to0}\frac{(x+h)^n\sin\{a(x+h)-x^n\sin ax\}}{h}$$

$$=\lim_{h\to 0}\left[\left\{\frac{(x+h)^n-x^n}{h}\right\}\sin a(x+h)\right.$$

$$\left.+x^n\left\{\frac{\sin a(x+h)-\sin ax}{h}\right\}\right]$$

$$=\lim_{h\to 0}[nx^{n-1}+\text{terms containing higher power of } h]\times$$

$$\sin(ax+h)+x^n\lim_{h\to 0}\frac{2\cos(ax+ah/2)\sin ah/2}{h}$$

$$=nx^{n-1}\sin ax+x^n\lim_{h\to 0}\cos(ax+ah/2)\frac{\sin ah/2}{ah/2}\,a.$$

$$=nx^{n-1}\sin ax+x^n\cos ax.\,1a=nx^{n-1}\sin ax+ax^n\cos ax$$

$$\therefore\ \frac{d}{dx}(x^n\sin ax)=nx^{n-1}\sin ax+ax^n\cos ax.$$

Ex. 10. A function $f(x)$ is defined as follows :—

$$f(x)=x^2\sin\frac{1}{x}\text{ when } x\neq 0.$$

$$=0,\qquad\text{when } x=0$$

Show that $f(x)$ is continuous and differentiable at $x=0$.

Since $\left|x^2\sin\frac{1}{x}\right|\leqslant\left|x^2\right|1=x^2$

and $\lim_{h\to 0} x^2=0.$

$$\therefore\ \lim_{x\to 0}x^2\sin\frac{1}{x}=0.$$

Also $f(0)=0$

Hence $f(x)$ is continuous at $x=0$.

Differentiability at $x=0$

we have $\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{(0+h)^2\sin\left(\frac{1}{0+h}\right)-0}{h}$

$$=\lim_{h\to 0}\frac{h^2\sin(1/h)}{h}=\lim_{h\to 0}h\sin(1/h)=0$$

Hence $f(x)$ is differentiable at $x=0$

and $f'(0)=0$

Ex. 11. Determine whether $f(x)$ is continuous and has a derivative at the origin where

$$f(x)=a+x \text{ if } x>0$$

$$f(x)=a-x \text{ if } x<0$$

Let us consider

$\therefore f(0)=a+0=a$ ($\therefore$ $f(x)=a+x$ for $x\geqslant 0$)

Again

$$\lim_{x\to 0+}f(x)=\lim_{x\to 0+}(a+x)=a$$

$$\lim_{x\to 0-}f(x)=\lim_{x\to 0-}(a-x)=a$$

$$\therefore\ \lim_{x\to 0+}f(x)=\lim_{x\to 0-}f(x)=f(0)=a$$

Hence the function $f(x)$ is continuous at $x=0$

Differentiability at $x=0$

$$Rf'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{a+h-a}{h}=1\quad(h>0)$$

$$Lf'(0)=\lim_{h\to 0}\frac{f(0-h)-f(0)}{h}=\lim_{h\to 0}\frac{a+h-a}{-h}=-1\quad(h>0)$$

$$\therefore\ Rf'(0)\neq Lf'(0)$$

Hence the $f(x)$ is not differentiable at $x=0$

Fx. 12. A function $f(x)$ is defined in the following way.

$$f(x)=0,\ 0\leq x<\tfrac{1}{2}$$
$$=1,\ x=\tfrac{1}{2}$$
$$=2,\ \tfrac{1}{2}<x\leq 1$$

Show that $f(x)$ is discontinuous at $x=\frac{1}{2}$;does $f'(\frac{1}{2})$ exists ?

When $x=\frac{1}{2}$, then $f(x)=1$ i. e., $f(\frac{1}{2})=1$

$$f(\tfrac{1}{2}+0)=\lim_{h\to 0} f(\tfrac{1}{2}+h)=2, \quad \text{when } x>\tfrac{1}{2}$$

$$f(\tfrac{1}{2}-0)=\lim_{h\to 0} f(\tfrac{1}{2}-h)=0, \quad \text{when } x<\tfrac{1}{2}$$

Thus $f(\frac{1}{2}+0)\neq f(\frac{1}{2}-0)\neq f(\frac{1}{2})$

Hence $f(x)$ is discontinuous at $x=\frac{1}{2}$

Differentiability at $x=\frac{1}{2}$

$$Rf'(\tfrac{1}{2})=\lim_{h\to 0}\frac{f(\frac{1}{2}+h)-f(\frac{1}{2})}{h}=\lim_{h\to 0}\frac{2-1}{h}=\infty$$

$$\underset{h\to 0}{Lf'(\tfrac{1}{2})}=\frac{f(\frac{1}{2}-h)-f(\frac{1}{2})}{-h}=\lim_{h\to 0}\frac{0-1}{-h}=\infty$$

$$Rf'(\tfrac{1}{2})=Lf'(\tfrac{1}{2})=\infty$$

The limits are not finite.

Hence $f(\frac{1}{2})$ does not exist.

Ex.13. Discuss the Continuity of function $f(x)=[x]$ at $x={}^1/_n$ fractional, $h\neq 0$; where $[x]$ denotes the integral past of x i.e. $[x]$ denotes the greatest integer $\leq x$. Draw the graph.

Does $f(x)$ at $x={}^1/3$ or: $f({}^1/3)$ exists?

($x=1/n$ অথবা ভগ্নাংশের $f(x)=[x]$ এর জন্য ফাংশানটি অবিচ্ছিন্ন কিনা পর্য্যালোচনা কর। $[x]$, x এর বৃহত্তম পূর্ণ সংখ্যা নির্দেশ করে, $[x]\leq x$.)

Sol. At $x=1/n$, $n\neq 0$. an integer $f[({}^1/3)]=[({}^1/3)]=0$ ...(1)

Also we have

$$\underset{x\to {}^1/_3+0}{\mathrm{Lt}} f(x)=\underset{h\to o}{\mathrm{Lt}} f({}^1/_3+h)=\underset{h\to o}{\mathrm{Lt}}[{}^1/_3+h]=[{}^1/_3]=0 \ldots (2)$$

and $$\underset{x\to {}^1/_3-0}{\mathrm{Lt}} f(x)=\underset{h\to o}{\mathrm{Lt}} f({}^1/_3-h)=\underset{h\to o}{\mathrm{Lt}}[{}^1/_3-h]=[{}^1/_3]=0 \ldots (3)$$

From (1), (2), (3) we see that

$$\underset{x\to {}^1/3-0}{\mathrm{Lt}} f(x)=\underset{h\to\frac{1}{3}+o}{\mathrm{Lt}} f(x)=f({}^1/3)=0$$

The function in continuous at $x=1/3$ i. e; it will be continuous for fractional values of n.

For. Differentiabilty at $x=1/3$

$$Rf\left(\frac{1}{3}\right)=\mathrm{Lt}\frac{f(1/3+h)-f(1/3)}{h}=\mathrm{Lt}\frac{0-0}{h}=0/0$$

$$Lf\left(\frac{1}{3}\right)=\underset{h\to o}{\mathrm{Lt}}\frac{f(1/3-h)-f(1/3)}{-h}=\underset{h\to o}{\mathrm{Lt}}\frac{o-o}{-h}=-0/0$$

Limits are not finite

Hence $f(1/3)$ does not exist.

For graph see Ex17 chapter III(c)

Ex. 14. Discuss the continuity of the rational infegsal values of x for the function $f(x)=[x]$ or; integral part of x. Also draw the graph.

Sol. We know that $[x]$ denotes the greatest integer $\leq x$.

If n is an integer we conclude that

$f(x)=n-1$, for $n-1\leq x<n$...(1)

$=h$, for $n\leq x<n+1$ ...(2)

$=n+1$, for $n+1\leq x<n+2$ ....(3)

and so on.

Let us consider now $x=n$ ...(4)

Then $f(n)=n$ from (2)

$$\underset{x\to n-h}{\mathrm{Lt}}=\underset{h\to o}{\mathrm{Lt}} f(n+h)=\underset{h\to o}{\mathrm{Lt}}[n-h]=[n-o]=n-1 \ldots (5)$$

$$\underset{x\to n+h}{\mathrm{Lt}}=\underset{h\to o}{\mathrm{Lt}} f(n+h)=\underset{h\to o}{\mathrm{Lt}}[n+h]=[n+o]=n+1 \ldots (5)$$

From (4), (5) and (6) we see thet

$f(n-o)\neq f(n)\neq f(n+o)$

i.e.; the function in discontinuous for x=n

i.e; when x is an integer. the function is continuous when x is fractianal

For graph see Ex17 Chapter III(c)

Ex. 15. Determine the continuities and discontinuties of the following functions f(x). composite functions $f\{f(x)\}$ and $f[f\{f(x)\}]$ it $f(x) = 1/(1-2x)$.

[f(x), f{f(x)}, এবং f[f{f(x)}] ফাংশান এবং সংযুক্ত ফানশানগুলির অবিচ্ছিন্নতা ও বিচ্ছিন্নতা বিন্দুগুলি নির্ণয় কর, যদি $f(x)=1/(1-2x)$]

Sol. $f(x)=1/(1-2x)$ if $1-2x=0$ or, $x=1/2$

Then $x={}^1/_2$, f (x) becomes infinite So f(x) is discontinuous at $x={}^1/_2$.

If $x\neq 1/2$, let $F(x)=f\{f(x)\}=f\left(\dfrac{1}{1-2x}\right)$

$$=\frac{1}{1-\{1/(1-2x)\}}=\frac{1-2x}{1-2x-1}=\frac{1-2x}{-2x}=\frac{2x-1}{2x}$$

Hwnce F(x) is disconontinuous at $x=0$

If $x\neq 0$, $x\neq 1/2$, then

$$v(x) = f[f\{f(x)\}] = f[f\{1/(1-2x)\}] = f\left(\frac{2x-1}{2x}\right)$$

$$=\frac{1}{1-\dfrac{2x-1}{2x}}=\frac{2x}{2x-2x+1}=2x$$

Hence $v(x)=2x$ is a continuous straight line passing through the origin. $Lf(x) = Rf(x) = f(0)=0$. and limits exist for all values of x.

Thus the points of discontinuities are $x=1/2$ for $f(x)$, $x=0$, $1/2$ for $f[f\{f(x)\}]$ and $x=0$ for $f\{f(x)\}$

Exercise (IVA)

নিম্নলিখিত ফাংশনের x এর ভিত্তিতে সংজ্ঞার দ্বারা ডিফারেন্সিয়েল সহগ নির্ণয় কর।

Differentiation the following from the first principle $w.r.$ to x.

1. x^2+3x+5 2. $7x^2+5/x$ R. U. 1960

3. ${}^1/x$ D. U. 1957   4. $\dfrac{1}{\sqrt{(2+x)}}$

(4.1) $x+\sqrt{x^2+1)}$ R. U. 1988

5. $\sqrt{x}$ D.U, 1965   6. $e^{\tan - x}$

7. $e^{\sqrt{x}}$ 8. e^{5x+a} R.U. 1966

9. $\tan x/a$ 10. $\tan x^2$ D.U. 1966

11. $\cos(ax+b)$ R. U. 1958   12. $\sin(ax+b)$ D.U. 1955

13. $a \sin x/a$ 14. $\log_{10} x$

15 $\sqrt[n]{x}$ 16. $\sqrt{(\sin x)}$

17. $\log\cos x$ 18. $\tan^2 2x$ D,U. 1954

19. $\sin x^2$ D. U. 1964   20. $\log\sin x/a$ R. H 1988

21. x^x D. H. 1987   22. $x \sin x$

23. $\log\sin^{-1}x$ 24. $\cos\log x$

25. $\dfrac{\cos x}{\log x}$ 26. $\dfrac{e^{2x}}{\log x}$ (a) $\dfrac{x^2+3x+1}{x}$

27. $x^3 \sin x$ 28. $e^{\tan x}$

(i) $\sin^{-1} x$ at $x=0$ (i) $e^{\sin x}$ at $x=a$ C. U. 1984

29. If $f(x)=\sin x$ find $f'(\pi/2)$ from the definition

30. If $f(x)=\tan x$, find $f'(\pi/4)$ from the first principle

31. Show that the function $|x|$ is continuous at $x=0$ but is not differentiable at that point.

32. A function is defined as follows :

$$f(x) = -x, \quad x\leq 0$$
$$= x, \quad x\geq 0 \qquad i.e., f(x) = |x|$$

Show that $f(x)$ is continuous but not differentiable at $x=0$

33 If $f(x)=x\tan^{-1}\dfrac{1}{x}$ when $x\neq 0$

$=0$ when $x=0$

Show that $f(x)$ is continuous but not differentiable at $x=0$

34. Examine whether $f(x)$ possesses $f'(x)$ at $x=0$.

where $f(x)=0$, when $x=0$

$$f(x)=\frac{1}{1+e^{1/x}} \text{ when } x\neq 0$$

35. A function $f(x)$ is defined as follows :

$$f(x)=0, \quad x=0$$
$$=x, \quad x>0$$
$$=-x, \quad x<0$$

Does $f'(0)$ exist ?

36. A function is defined as follows :--

$$f(x)=1, \quad -\infty<x<0$$
$$=1+\sin x, \quad 0<x<\tfrac{1}{2}\pi$$
$$=2+(x-\tfrac{1}{2}\pi)^2, \quad \tfrac{1}{2}\pi\leq x<\infty$$

Show that $f(x)$ is continuous at $x=0$ and $x=\frac{1}{2}\pi$ but $f'(x)$ exists for $x=\frac{1}{2}\pi$ and does not exist for $x=0$

37. Discuss the continuity and differentiability of the following functions defined as

(i) $f(x)=x\sin\dfrac{1}{x}, \quad x\neq 0$ D. H. 1984

$=0, \quad x=0$

Investigate at $x=0$

(i) $f(x)=x\cos 1/x, x\neq 0$: and $f(0)=0$

Investigate at $x=0$

(ii) $f(x)=(x-a)\sin\dfrac{1}{x-a}$ for $x\neq a$

$=0$ for $x=0$

Investigate at $x=0$

38. Show that the function

$$f(x)=x\{1+\tfrac{1}{3}\sin(\log x^2)\}, \quad x\neq 0$$
$$=0 \text{ when } x=0$$

is continuous but has no derivative at $x=0$

39. A function $f(x)$ is defined as follows

$$f(x)=1+x\ ;\ x\leq 0$$
$$=x\ ;\ 0<x<1$$
$$=2-x\ ;\ 1\leq x\leq 2$$
$$=3x-x^2,\ x>2$$

C.H. 1992

Show that $f'(x)$ at $x=1$ and 2 does not exist, though $f(x)$ is continuous at these points.

39. (i) Sketch roughly the graph of the function (ফাংশানটির চিত্র অংকন কর)।

$$f(x)=\begin{cases} x^2+1, & x<0 \\ x, & 0\leq x\leq 1 \\ 1/x, & x>1 \end{cases}$$

D. U. 1986, 88

Discuss the continuity of the function at $x=0$ and $x=1$ ($x=0, x=1$ বিন্দুতে ফাংশানটির অবিচ্ছিন্নতা পর্যালোচনা কর)

39. (ii) যদি

$$f(x)=\begin{cases} x^2\cos\dfrac{1}{x}; & x\neq 0 \\ 0, & x=0 \end{cases}$$

D. U. 1986

$f'(0)$ নির্ণয় কর।

40. Prove that the product of two continuous functions is continuous. Prove that if $f'(x)<0$ in $a<x<b$, then $f(x)$ is steadily decreasing function in this interval.

Reduce that $2x/\pi<\sin x<x$. If $0<x\leq\pi/2$.

41. Distinguish between derivability and differentiability of a function at a given point and show that a necessary and sufficient condition for the differentiability of $f(x)$ at a given point is that it possesses a finite derivative at that point.

41 (a) The function f is defined thus:-

$f(x) = e^x$ when $x< 0$

$= x^2 + 1$ when $0 \le x \le 1$ N.U.1994

$= \dfrac{x}{x-1}$ when $x> 1$

Find

(a) $\lim\limits_{x\to\alpha} f(x)$, $\lim\limits_{x\to 0-} f(x)$. $\lim\limits_{x\to 0+} f(x)$

(a) $\lim\limits_{x\to\alpha} f(x)$, $\lim\limits_{x\to 0+} f(x)$, $\lim\limits_{x\to 0+\alpha} f(x)$

(b) Determine where f is not continuous and where it is not differentiable. [f কোথায় অবিচ্ছিন্ন নয় এবং কোথায় অন্তরীকরণযোগ্য নয় তাহা নির্ধারণ কর।]

(c) Draw a rough Sketch of the graph of f and the range of f

[মোটামোটিভাবে f–এর লেখচিত্র অঙ্কন কর এবং f এর রেঞ্জ উল্লেখ কর।]

42. Let $f(x)=\sqrt{x}$ in the interval $0\leqq x\leqq 4$. If $\in$ is a positive number, choose small at pleasure and find a $\delta>0$ depending on $\in$, such that $| f(x_1)+f(x_2) | <\in$ whenever $| x_1-x_2 | \leqq \delta$.

43. Show that $f(x)= | x |$ is continuous at $x=0$ but not differentiable at $x=0$

for derivability for $x=0$,

$$\frac{f(0+h)-f(0)}{h}=\frac{f(h)}{h}=\frac{|h|}{h}=\begin{cases}1, \text{ if } h>0\\-1, \text{ if } h<0\end{cases}$$

$\therefore Rf'(0)=1\neq Lf'(0)=-1$

Hence $f'(0)$ does not exist

For continuity at $x=0$

$| f(x)-f(0) | = | x | <\in$, when $| x-0 | \leq \delta$

δ is any positive number less than $\in$.

Hence $f(x)$ is continuous at $x=0$

44. Examine the continuity and differentiability of the function f defined by $f(x)= | x-2 |$ at $=2$

Draw the graph of the function in $-4\leq x\leq 4$. D. U. 1986

Ans. conti. at $x=2$ but not differentiable.

The graph consists of two straight lines inclined at 45^0 at $x=2$ with the $x-$axis, points (2, 0), (3, 1), (4, 2) ; (1, −3), (2, −4), (3, −5), (4, −6).

45. Show that $| x+1 | + | x | + | x-1 |$ is continuous but not differentiable at $x=-1, 0, 1$.

Sol : At $x=-1$,

$Rf'(-1)=$

$$\lim_{x\to -1+h} \frac{(-1+h+1-1+h-1+h-1)-0+1+1-h+1}{h}$$

$$=\lim_{h\to 0}\frac{3h}{h}=3$$

$$Lf'(-1)=\lim_{x\to -1-h}\frac{-1-h+(-1-h-1-h-1)-(-1-2)}{h}$$

$$=\lim_{h\to 0}\frac{-3h}{h}=-3$$

$\therefore Rf'(-1)\neq Lf'(-1)$. So it is not differentiable

46. (a) Sketch roughly the graph of the function $f(x)$

where $f(x)=\begin{cases}e^{-|x|/2}, & -1<x<0\\ x^2, & 0\leqq x<2\end{cases}$ D.U. 1987

Find the domain and range of the function.

(b) Discuss the existence of $\lim\limits_{x\to 0} f(x)$. where $f(x)$ is given in (a)

(c) Discuss the continuity of the above function.

$f(x)$ also at $x=0$

47. Let the function f be given by

$$f(x)=\begin{cases} x^3, & x\leq 1 \\ 3x-2, & x<1 \end{cases}$$

Does the function f have a continuous derivative at $x=0,\ 1$? Explain. D. U. 1987

48. With diagram discuss the continuity and differentiability of the following functions at the point indicated.

(নির্দিষ্ট বিন্দুতে $f(x)$ এর অবিচ্ছিন্নতা ও অন্তরীকরণযোগ্যতা চিত্র সহকারে বর্ণনা কর ।)

(i) $f(x)=(\sin x),\ 0<x\leqq 4\pi$, at $x=3\pi$

(ii) $f(x)=\begin{cases} 2x^{-1}, & 0<x\leqq 1 \\ x^2-x+1, & x>1 \end{cases}$ at $x=1$ C. H. 1985

49. A function $f:\ \mathbb{R}\to\mathbb{R}^{+}\cup\{0\}$ is defined by $f(x)=(x-1)$ Is it an onto function ? Sketch the graph of f.

Does the $\lim\limits_{h\to 0}\dfrac{f(1-h)-f(1)}{h}$ exists D. U. 1989

50. Let $f(x)=5x-4$ for $0<x\leqslant 1$

$\quad = 4x^2-3x$ for $1<x<2$

Discuss continuity of $f(x)$ at $x=1$ and existence of $f'(x)$ for this value. R. U. 1987

51 Examine whether $f'(x)$ exists at $x=0$ and $x=2$ where

$f(x) = x,\ 0\leq x<2$

$\quad = x-1,\ 2\leq x$ C.U. 1991

52. Draw the graph of the function defined by $y=[x]$ where $[x]$ denotes the integral part of x (largest integer not exceeding x) Discuss continuity and differentiabiltiy of the function at $x=1$ ($y=$

$[x]$ দ্বারা বর্ণিত ফাংশনটির লেখচিত্র অঙ্কন কর, যেখানে $[x]$ দ্বারা x-এর পূর্ণাংশ বুঝায় (x-এর অনুর্দ্ধ বৃহত্তম পূর্ণ সংখ্যা) $x=1$ বিন্দুতে ফাংশনটির অবিচ্ছিন্নতা ও অন্তরীকরণযোগ্যতা আলোচনা কর। D.U. 1991

53. Find the points of continuity and discontinuties of the function $f(x)=\dfrac{1}{1-x}$ of $f(x)$ and the composite functions $f[f(x)]$ and $f[f\{f(x)\}]$ Ans. At $x=1$, $f(x)$ is discontinuons.

$x=0$and 1 are discontinuties of $f[f\{f(x)\}]$; $x=1$ is the discontinuity of $f[f(x)$

[$f(x)={}^1/(1-x)$ ফাংশানের জন্য f(x) এর অবিচ্ছিন্নতা ও বিচ্ছিন্নতা বিন্দুগুলি নির্ণয় কর। সংযুক্ত ফাংশানের $f[(x)\}$, $f[f\{(x)\}]$ জন্য ও বিন্দুগুলি নির্ণয় কর]

54. $f(x+y)=f(x)f(y)$ for all x and y and $f(x)=1+xg(x)$ where $\lim\limits_{x\to 0} g(x)=1$.

Show that the derivative $f'(x)$ exists and $f'(x)=f(x)$ for all x.

[যদি $f(x+y)=f(x)f(y)$ সকল x এবং y এর জন্যে এবং $f(x)=1+xg(x)$ যেখানে $\lim\limits_{x\to 0} g(x)=1$, তখন দেখাও যে অন্তরগ সহজ $f'(x)$ বিদ্যমান এবং $f'(x)=f(x)$ সকল x এর জন্য।]

54. If f is continuous function of x satisfying the functional equation $f(x+y)=f(x)+f(y)$

Show that $f(x)=ax$, a is a constant.

55.(i) Show that f is defined by $f(x)=|x|+|x-1|$

is continuous but not derivable for $x=0$, $x=1$.

(ii) Show that f defined by $f(x)=|x|+|x-1|+|x-2|$ is continous and not differentiable at $x=0$, $x=1$.: $x=2$.

56. If $\phi''(x)\geq 0$ for every value of x, then

$\phi[\frac{1}{2}(x_1+x_2)]\leq\frac{1}{2}[\phi(x_1)+\phi(x_2)]$

$\phi[\frac{1}{n}(x_1+x_2+\ldots x_n)]\leq\frac{1}{n}[\phi(x_1)+\phi(x_2)+\ldots\phi(x_n)]$

ANSWER

1. $2x+3$ 2. $14x-(5/x^2)$ 3. $-(1/x^2)$

4. $-\frac{1}{2}(2+x)^{-3/2}$ 5. $1/2\sqrt{x}$ 6. $e^{\tan x}\sec^2 x$

7. $e^{\sqrt{x}}\frac{1}{2\sqrt{x}}$ 8. $5e^{5x+a}$ 9. $2x\sec^3 x^2$

10. $\frac{1}{a}\sec^2\frac{x}{a}$ 11. $-a\sin(ax+b)$ 12. $a\cos(ax+b)$

13. $\cos\frac{x}{a}$ 14. $\frac{1}{x}\log_{10}^{e}$ 15. $\frac{1}{x}x^{(1/n)-1}$

16. $\frac{1}{2}\frac{\cos x}{\sqrt{\sin x}}$ 17. $-\tan x$ 18. $4\tan 2x\sec^2 2x$

19. $2x\cos x^2$ 20. $-\frac{1}{a}\cot\frac{x}{a}$ 21. $x^x(1+\log x)$

22. $x\cos x+\sin x$ 23. $\frac{1}{\sin^{-1}x\sqrt{(1-x^2)}}$

24. $-\frac{1}{x}\sin\log x$ 25. $\left(-\sin x\log x-\frac{1}{x}\cos x\right)(\log x)^2$

26. $\left(2e^{2x}\log x-\frac{1}{x}e^{2x}\right)(\log x)^2$ 27. $3x^2\sin x+x^3\cos x$

28. $e^{\tan x}\sec^2 x$ (i) $e^{\sin a}\cos a$.

29. 0. 30. 2. 31. $f'(x)$ exists

37. (i) Continuous, $f(o)$ does not exist.

(ii) Continuous, (o) does not exist.

(iii) Continuous, $f(o)$ does not exist.

4.27. Derivative of a function of a function.

Let $y=f(v)$, where $v=\phi(x)$. so is the function of x, then

$$\frac{dy}{dx}=\frac{dy}{dv}\cdot\frac{dv}{dx}$$

where $f(v)$ and $\phi(x)$ are continuous.

Therefore y is also continuous function of x.

Let $v+\delta v=\phi(x+h)$ and $y+\delta y=f(v+\delta v)$

Let $h=\delta x\to 0$. then δv also tends to zero.

$$\therefore \frac{dy}{dx}=\lim_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\right)=\lim_{\delta x\to 0}\left(\frac{\delta y}{\delta v}\cdot\frac{\delta v}{\delta x}\right)$$

$$=\lim_{\delta v\to 0}\left(\frac{\delta y}{\delta v}\right)\cdot\lim_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\right)=\frac{dy}{dv}\cdot\frac{dv}{dx},$$

Hence $\dfrac{dy}{dx}=\dfrac{dy}{dv}\,\dfrac{dv}{dx}$ provided the limits exist.

Ex. Differentiate log(sin x) *w. r, to x*

Let $y=\log(\sin x)$; put $z=\sin x$

Then $y=\log z$, $\frac{dy}{dz}=\frac{1}{z}$, $\frac{dz}{\delta x}=\cos x$.

$$\therefore \frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=\frac{1}{z}\cdot\cos x=\frac{1}{\sin x}\cos x=\cot x$$

Cor. we can generalize the above differentiation

If $y=f(v)$, $v=f(t)$, $t=f(x)$,

be the three continuous functions, y is also continuous function of x. Then

$$\frac{dy}{dx}=\frac{dy}{dv}\cdot\frac{dv}{dt}\cdot\frac{dt}{dx}$$

This is known as the **chain rule of differentiation.**

In this way we can establish the theorem for any number of functions.

Ex. Differentiate log sec $(ax+b)^3$ $w.r.$ to x

$$\frac{dy}{dx}=\frac{1}{\sec(ax+b)^3}\sec(ax+b)^3\tan(ax+b)^3.\ 3(ax+b)^2.\ a$$

4.28 Logarithmic Differentiation.

• Differentiate $x^{\cos x}$ $w.r.$ to x.

Let $y=x^{\cos x}$

Take logarithm of both sides. Then

$\log y=\log x^{\cos x}=\cos x\log x.$

Differentiating both sides $w.r.$ to x,

$$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{d}{dx}(\cos x).\ \log x+\cos x.\ \frac{d}{dx}(\log x)$$

$$=-\sin x\log x+\cos x\frac{1}{x}$$

or, $\dfrac{dy}{dx}=y\left(-\sin x\log x+\dfrac{1}{x}\cos x\right)=x^{\cos x}\left(\sin x\log x+\dfrac{1}{x}\cos x\right)$

In order to differentiate a function of the form u^v. where u and v-are variable quantities. we are to take logarithm of the expression and then differentiate. The process which is shown above is called the **logarithmic differentiation.**

In another way we can differentiate the function like $y=x^{\cos x}$ The process is shown below.

$$y=x^{\cos x}=e^{\log x^{\cos x}}=e^{\cos x\log x}-e^v.\ \text{where } v=\cos x\log x$$

$$\therefore\ \frac{dy}{dv}=e^v=y=x^{\cos x};\ \frac{dv}{dx}=-\sin x\log x+\frac{1}{x}\cos x$$

$$\therefore\ \frac{dy}{dx}=\frac{dy}{dv}\frac{dv}{dx}=x^{\cos x}\{-\sin x\log x+(^1/x)\cos x\}$$

This process is not always helpful if the expression is the product of many functions. Such as $y=\sin e^x\log x\ x^x$.

In this case logarithmic differentiation is very helpful.

4.29 Derivatives of Hyperbolic function.

(পরাবৃত্তিক ফাংশনের ডিফারেন্সিয়েল সহগ)

Let $y=\sin h\,x=\frac{1}{2}(e^x-e^{-x})\ \therefore(dy/dx)=\frac{1}{2}(e^x+e^{-x})=\cos h\,x$

(i) **Hence** $\dfrac{d}{dx}(\sin h\,x)=\cos h\,x$

Let $y=\cos hx=\frac{1}{2}(e^x+e^{-x})\ \therefore\ (dy/dx)=\frac{1}{2}(e^x-e^{-x})=\sin h\,x$

(ii) **Hence** $\dfrac{d}{dx}(\cos h\,x)=\sinh x$

Let $y=\tanh x=\dfrac{\sinh x}{\cos h\,x}$

$$\therefore\ \frac{dy}{dx}=\frac{(\sin h\,x)'\cosh x-(\sinh x)(\cos h\,x)'}{(\cosh x)^2}$$

$$\Rightarrow\frac{dy}{dx}=\frac{\cos h^2x-\sin h^2x}{\cos h^2x}=\frac{1}{\cosh^2x}=\operatorname{sech}^2x$$

(iii) **Hence** $\dfrac{d}{dx}(\tan h\,x)=\operatorname{sech}^2x$

(iv) **Similarly.** $\dfrac{d}{dx}(\coth x)=-\operatorname{cosec} h^2x$

Let $y=\operatorname{sech}x=\dfrac{1}{\cosh x}$

$$\therefore\ \frac{dy}{dx}=\frac{(1)^1(\cosh x-(1)(\cosh x)^1}{\cosh^2x}=\frac{-\sinh x}{\cosh^2x}$$

$$\Rightarrow\frac{dy}{dx}=-\operatorname{sech}x\tan h\,x$$

Hence $\dfrac{d}{dx}(\operatorname{sech}x)=-\operatorname{sech}x\tanh x$

(v) **Similarly** $\dfrac{d}{dx}(\operatorname{cosech}x)=-\operatorname{cosech}x\coth x$

4.30 Derivative of $\sinh^{-1} x$ w r to x

(বিপরীত পরাবৃত্তিক ফাংশনের ডিফারেন্সিয়াল সহগ)

Let $y = \sinh^{-1} x \therefore x = \sinh y \ (x \geqslant 0)$

$\therefore (dx/dy) = (\cosh y) = \sqrt{(\cosh^2 y)} = \sqrt{(1+\sinh^2 y)} = \sqrt{(1+x^2)}$

or, $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{(1+x^2)}}$ Hence $\dfrac{d}{dx}(\sinh^{-1} x) = \dfrac{1}{\sqrt{(1+x^2)}}$

Similarly,

(i) $\dfrac{d}{dx}\left(\cosh^{-1} x\right) = \dfrac{1}{\sqrt{(x^2-1)}};\ (x>1)$

(ii) $\dfrac{d}{dx}(\tanh^{-1} x) = \dfrac{1}{1-x^2};\ (-1<x<1)$

(iii) $\dfrac{d}{dx}(\coth^{-1} x) = -\dfrac{1}{x^2-1},\ (|x|>1)$

(iv) $\dfrac{d}{dx}(\text{cosech}^{-1} x) = -\dfrac{1}{x\sqrt{(x^2+1)}}\ (x \neq 0)$

(v) $\dfrac{d}{dx}(\text{sech}^{-1} x) = \dfrac{1}{x\sqrt{(1-x^2)}},\ (x \neq 0 \text{ and } |x|<1)$

Cor : *Let* $y = \log\{x+\sqrt{(x^2+1)}\}$

$\therefore \dfrac{dy}{dx} = \dfrac{1}{x+\sqrt{(x^2+1)}}\left\{1+\dfrac{2x}{2x(x^2+1)}\right\} = \dfrac{1}{\sqrt{(x^2+1)}}$

Also $\dfrac{d}{dx}(\sinh^{-1} x) = \dfrac{1}{\sqrt{x^2+1}}$

$\therefore \sinh^{-1} x = \log(x+\sqrt{x^2+1}) + c \left[\begin{array}{l}\text{when } c \text{ is a} \quad \dfrac{dc}{dx}=0 \\ \text{constant and}\end{array}\right]$

when $x=0$, we have,

$\sinh^{-1} 0 = \log 1 + c \Rightarrow c = 0.$

Hence (i) $\sinh^{-1} x = \log(x+\sqrt{x^2+1}) \qquad (x \geqslant 0)$

Similarly, we can show that

(ii) $\cosh^{-1} x = \log(x+\sqrt{x^2-1})\ . \qquad x \geq 1$

(iii) $\tanh^{-1} x = \frac{1}{2}\log\left(\dfrac{1+x}{1-x}\right) \quad -1<x<1$

4.31. Differentiation of Parametric Equations

(প্যারামেট্রিক ফাংশনের ডিফারেন্সিয়াল সহগ)

Sometimes x and y are expressed in terms of a third variable usually called a **Parameter**. In such cases we can find $\dfrac{dy}{dx}$ without eliminating the parameter. The process of differentiation in such cases is shown below.

Let $x=f_1(t)$ and $y=f_2(t)$ then

$\dfrac{dy}{dx} = \dfrac{dy}{dt}\cdot\dfrac{dt}{dx} = \dfrac{dy}{dt}\Big/\dfrac{dx}{dt}$, where $\dfrac{dx}{dt} \neq 0$

Ex. If $x = a\cos t$ and $y = b\sin t$, then

$\dfrac{dx}{dt} = -a\sin t,\ \dfrac{dy}{dt} = b\cos t$

$\therefore \dfrac{dy}{dx} = \dfrac{dy}{dt}\cdot\dfrac{dt}{dx} = \dfrac{dy}{dt}\Big/\dfrac{dx}{dt} = b\cos t/(-a\sin t) = -(b/a)\cot t$

4.32. Differentiation of Implicit Functions (অব্যক্ত ফাংশন)

If in an equation involving x and y, the variable y is not given in terms of x or it is not suitable to express y in terms of x by solving the equation, then y is said to be an **implicit function of** x. In this case, we may get $\dfrac{dy}{dx}$ by differentiating every term in the equation with respect to x as shown below :

Ex. Differentiate $ax^2+2hxy+by^2+d=0$ *w. r.* to x.

Differentiating every term *w. r.* to x,

$$a\ 2x+\left(2hy+2hx\frac{dy}{dx}\right)+2by\frac{dy}{dx}+0=0$$

or $(2hx+2by)\dfrac{dy}{dx}=-2ax-2hy$

Note : Let $f(x, y)=ax^2+2hxy+by^2+d=0$, then

$$\frac{dy}{dx}=-\frac{\delta f/\delta x}{\delta f/\delta y}=-\frac{f_x}{f_y},\ (f_y\neq 0)$$

f_x = derivative of $f(x, y)$ *w. r.* to x treating y as constant,

f_y = derivative of $f(x, y)$ *w. r.* to y treating x as constant. For detail, consult Chapter IX.

4.33. Explicit (বাক্ত) Functions :— If we can express a variable y exclusively in terms of another variabale x, then y is called an **explicit function** of x. For example,

$y=\sin\log\cos x^2$, $y=f(x)$.

$$y=\frac{3x^2+5x+7}{7x^3+4x^2+3}\ \text{ete.}$$

4.34. Examples :—

Ex. 1. Find $\dfrac{dy}{dx}$ if $y=e^{\sin x}\sin a^x$ D. U. 1965

$y=e^{\sin x}\sin a^x$

$\therefore\ \log y=\log(e^{\sin x}\sin a^x)=\sin x+\log(\sin a^x)$

$$\therefore\ \frac{1}{y}\cdot\frac{dy}{dx}=\cos x+\frac{\cos a^x}{\sin a^x}a^x\log a$$

or, $\dfrac{dy}{dx}=y(\cos x+a^x\log a\cot a^x)=e^{\sin x}\sin a^x(\cos x+a^x\log a\cot a^x)$

Ex. 2. Find the differential co-efficient of

$\tan^{-1}\dfrac{a+x}{1-ax}$ *w. r.* to x. D. U. 1969 ; R U. 1959

Let $y=\tan^{-1}\left(\dfrac{a+x}{1-ax}\right)=\tan^{-1}a+\tan^{-1}x$

$$\therefore\ \frac{dy}{dx}=0+\frac{1}{1+x^2}=\frac{1}{1+x^2}$$

Ex. 3. Find $\dfrac{dy}{dx}$ if $y=\dfrac{e^{x^2}\tan^{-1}x}{\sqrt{(1+x^2)}}$ D. U. 1962 ; R. U. 1958 D. H. 1983

we have,

$$y=\frac{e^{x^2}\tan^{-1}x}{\sqrt{(1+x^2)}}$$

$\therefore\ \log y=\log e^{x^2}+\log\tan^{-1}x-\frac{1}{2}\log(1+x^2)$

$=x^2+\log\tan^{-1}x-\frac{1}{2}\log(1+x^2)$

$$\therefore\ \frac{1}{y}\frac{dy}{dx}=2x+\frac{1}{\tan^{-1}x(1+x^2)}-\frac{1}{2}\frac{2x}{1+x^2}$$

$$\text{or, }\frac{dy}{dx}=y\left(2x+\frac{1}{\tan^{-1}x(1+x^2)}-\frac{x}{1+x^2}\right)$$

Now put the value of y, the result will follow.

Ex. 4. Find (dy/dx) if $y=(\tan x)^{\cot x}+(\cot x)^{\tan x}$ D. U. 1983

Let $y=(\tan x)^{\cot x}+(\cot x)^{\tan x}=e^{\cot x\log\tan x}+e^{\tan x\log\cot x}$

$$\frac{dy}{dx}=e^{\cot x\log\tan x}\left(-\operatorname{cosec}^2x\log\tan x+\cot x\frac{\sec^2x}{\tan x}\right)$$

$$+e^{\tan x\log\cot x}\left(\sec^2x\log\cot x+\tan x\frac{-\operatorname{cosec}^2x}{\cot x}\right)$$

$=(\tan x)^{\cot x}(1-\log\tan x)\operatorname{cosec}^2x+(\cot x)^{\tan x}(\log\cot x-1)\sec^2x$

Ex. 5. Differentiate $\sqrt{(y/x)}+\sqrt{(x/y)}=0$ *w. r.* to x.

D. U. 1962

Now $\sqrt{(y/x)}+\sqrt{(x/y)}=0$

or, $y+x=3\sqrt{(xy)}$ or, $y+x=3\sqrt{x}\sqrt{y}$ [on simplification]

Differentiating both sides *w. r.* to x,

$$\frac{dy}{dx}+1=3\left(\frac{1}{2\sqrt{x}}\sqrt{y}+x\frac{1}{2\sqrt{y}}\cdot\frac{dy}{dx}\right)$$

$$\text{or, } \frac{dy}{dx}\left(1-\frac{3}{2}\sqrt{\frac{x}{y}}\right)=\frac{3}{2}\sqrt{\frac{y}{x}}-1$$

$$\Rightarrow \frac{dy}{dx}=\frac{\sqrt{y}(3\sqrt{y}-2\sqrt{x})}{\sqrt{x}(2\sqrt{y}-3\sqrt{x})}=\frac{3y-2\sqrt{(xy)}}{2\sqrt{(xy)}-3x}$$

Ex. 6. Find $\frac{dy}{dx}$, if $\log(xy)=x^2+y^2$ D. U. 1981, '86.

$\log(xy)=x^2+y^2$ or, $\log x+\log y=x^2+y^2$

$$\therefore \frac{1}{x}+\frac{1}{y}\frac{dy}{dx}=2x+2y\frac{dy}{dx}$$

$$\text{or, } \frac{dy}{dx}\left(\frac{1}{y}-2y\right)=2x-\frac{1}{x} \text{ or; } \frac{dy}{dx}=\frac{(2x^2-1)y}{x(1-2y^2)}$$

Ex 7. Find $\frac{dy}{dx}$ if $x=a\cos^3 t$, $y=a\sin^3 t$ R. U. 1965

$x=a\cos^3 t$, $y=a\sin^3 t$

$$\frac{dx}{dt}=-3a\cos^2 t\sin t.\ \frac{dy}{dt}=3a\sin^2 t\cos^2 t.$$

$$\therefore \frac{dy}{dx}=\frac{dy}{dt}\bigg/\frac{dx}{dt}=\frac{-3a\sin^2 t\cos t}{3a\cos^2 t\sin t}=-\tan t$$

Ex 8. Find $\frac{dy}{dx}$ if $y=\tan^{-1}\sqrt{\left\{\frac{1-\cos x}{1+\cos x}\right\}}$ D. U. 1952,'59

Let $y=\tan^{-1}\sqrt{\left(\frac{1-\cos x}{1+\cos x}\right)}=\tan^{-1}\sqrt{\left(\frac{2\sin^2 x/2}{2\cos^2 x/2}\right)}$

$=\tan^{-1}(\tan x/2)=\frac{1}{2}x$

$$\therefore \frac{dy}{dx}=\frac{1}{2}$$

Ex. 9. Differentiate $\frac{\tan}{x}\log\frac{e^x}{x^x}$ *w. r.* to x

Let $y=\frac{\tan x}{x}\log\frac{e^x}{x^x}=\frac{\tan x}{x}(x-x\log x)$

$=\tan x-\tan x\log x$

$\therefore (dy/dx)=\sec^2 x-\sec^2 x\log x-(1/x)\tan x$

$=\sec^2 x(1+\log x)-(1/x)\tan x$

Ex. 10. Differentiate $\tan^{-1}\frac{\sqrt{(1+x^2)}-1}{x}$ *w. r.* to $\tan^{-1}x$

R. U. 1981, 1987

Let $y=\tan^{-1}\frac{\sqrt{(1+x^2)}-1}{x}$ and $z=\tan^{-1}x$

we are to find dy/dz

In $y=\tan^{-1}\frac{\sqrt{(1+x^2)}-1}{x}$, put $x=\tan\theta$

$=\tan^{-1}\frac{\sqrt{(1+\tan^2\theta)}-1}{\tan\theta}$

$=\tan^{-1}\frac{\sec\theta-1}{\tan\theta}=\tan^{-1}\frac{1-\cos\theta}{\sin\theta}$

$=\tan^{-1}\left(\frac{2\sin^2\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}\right)=\tan^{-1}(\tan^{-1}\frac{1}{2}\theta)$

$=\frac{1}{2}\theta=\frac{1}{2}\tan^{-1}x=\frac{1}{2}z$ $\therefore$ $dy/dz=\frac{1}{2}$

Ex. 11. Differentiate $x^{\sin x}$ *w. r.* $(\sin x)^x$

Let $y=x^{\sin x}$ and $z=(\sin x)^x$

we are to find $\frac{dy}{dz}$

Now $y=x^{\sin x}$ or, $\log y=\sin x\log x$

$$\frac{1}{y}\frac{dy}{dx}=\cos x\log x+\frac{1}{x}\sin x$$

or, $\frac{dy}{dx}=x^{\sin x}\cos x\log x+\frac{1}{x}\sin x\, x^{\sin x}$

Again $z=(\sin x)^x=e^{x\log\sin x}$

$\frac{dz}{dx}=(\sin x)^x(\log\sin x+x\cot x)$

$\therefore \frac{dy}{dz}=\frac{dy}{dx}\frac{dx}{dz}=\frac{x^{\sin x}\cos x\log x+(1/x)\sin x\, x^{\sin x}}{(\sin x)^x(\log\sin x+x\cot x)}$

Ex. 12. If $\sin y=x\sin(a+y)$ prove that

$\frac{dy}{dx}=\frac{\sin^2(a+y)}{\sin a}$ C. U. 1987

Now $\sin y=x\sin(a+y)$ ... ... (1)

or $\log\sin y=\log x+\log\sin(a+y)$

$\therefore \frac{dy}{dx}\frac{\cos y}{\sin y}=1/x+\frac{\cos(a+y)}{\sin(a+y)}\frac{dy}{dx}$

or, $\frac{dy}{dx}\left(\frac{\cos y}{\sin y}-\frac{\cos(a+y)}{\sin(a+y)}\right)=1/x$

or, $\frac{dy}{dx}\frac{\sin(a+y-y)}{\sin y\sin(a+y)}=1/x$

or, $\frac{dy}{dx}=\frac{\sin y\sin(a+y)}{x\sin a}=\frac{\sin^2(a+y)}{\sin a}$ [by (1)]

Proved

Ex. 13. Find $\frac{dy}{dx}$ when $(\cos)^y+(\sin y)^x=0$ C. H. 1985, 89 D. H. 1987

$(\cos x)^y+(\sin y)^x=0$

$\therefore \frac{d}{dx}[(\cos x)^y+(\sin y)^x]=0$

or, $\frac{d}{dx}\left[e^{y\log\cos x}+e^{x\log\sin y}\right]=0$

or, $e^{y\log\cos x}\left\{\frac{dy}{dx}\log\cos x+y\frac{-\sin x}{\cos x}\right\}$

$+e^{x\log\sin y}\left\{\log\sin y+x\frac{\cos y}{\sin y}\frac{dy}{dx}\right\}=0$

or, $(\cos x)^y[(dy/dx)\log\cos x-y\tan x]$

$+(\sin y)^x\left\{\log\sin y+x\cot y\frac{dy}{dx}\right\}=0$

or, $\frac{dy}{dx}[(\cos x)^y\log\cos x+(\sin y)^x x\cot y]$

$=(\cos x)^y y\tan x-(\sin y)^x\log\sin y$

or, $\frac{dy}{dx}=\frac{(\cos x)^y y\tan x-(\sin y)^x\log\sin y}{(\cos x)^y\log\cos x+(\sin y)^x x\cot y}$

Ex. 14. $y=\sqrt{\{a^2-b^2\cos^2(\log x)\}}$ C. U. 1986

$\frac{dy}{dx}=\frac{d\sqrt{\{a^2-b^2\cos^2(\log x\}}}{d\{a^2-b^2\cos^2(\log x)\}}\times\frac{d\{a^2-b^2\cos^2(\log x)\}}{d\{\cos(\log x)\}}$

$\times\frac{d\{\cos\log x\}}{d\log x}\times\frac{d(\log x)\}}{dx}$

$=\frac{1}{2}\{a^2-b^2\cos^2(\log x)\}^{-1/2}\times\{-2b^2\cos(\log x)\}$

$\times\{-\sin\log x\}\times 1/x$

$=\frac{b^2\sin 2(\log x)}{3x\sqrt{\{a^2-b^2\cos^2(\log x)\}}}$

Ex. 15. Find if $\frac{dy}{dx}$ if $x^3+y^3=3axy$

$\frac{d}{dx}(x^3+y^3)=\frac{d}{dx}(3axy)$

or, $3x^2+3y^2\frac{dy}{dx}=3ay+3ax\frac{dy}{dx}$

or, $\frac{dy}{dx}(y^2-ax)=ay-x^2$

$\Rightarrow \frac{dy}{dx}=\frac{ay-x^2}{y^2-ax}$

Ex. 16. Differentiate $\log \dfrac{a+b\tan\frac{1}{2}x}{a-b\tan\frac{1}{2}x}$ *w. r* to

$$\frac{1}{a^2\cos^2\frac{1}{2}x-b^2\sin^2\frac{1}{2}x} \qquad \text{C. U. 1986}$$

Let $y=\log\dfrac{a+b\tan\frac{1}{2}x}{a-b\tan\frac{1}{2}x}=\log(a+b\tan\tfrac{1}{2}x)-\log(a-b\tan\tfrac{1}{2}x)$

$$\frac{dy}{dx}=\frac{\frac{1}{2}b\sec^2\frac{1}{2}x}{a+b\tan\frac{1}{2}x}-\frac{-\frac{1}{2}b\sec^2\frac{1}{2}x}{a-b\tan\frac{1}{2}x}=\frac{\frac{1}{2}b\sec^2\frac{1}{2}x\,(2a)}{a^2-b^2\tan^2\frac{1}{2}x}$$

$$=\frac{ab\sec^2\frac{1}{2}x\cos^2\frac{1}{2}x}{a^2\cos^2\frac{1}{2}x-b^2\sin^2\frac{1}{2}x}=\frac{ab}{a^2\cos^2\frac{1}{2}x-b^2\sin^2\frac{1}{2}x}$$

Let $z=\dfrac{1}{a^2\cos^2\frac{1}{2}x-b^2\sin^2\frac{1}{2}x}$

$$\frac{dz}{dx}=\frac{-\frac{1}{2}a^2\sin x-\frac{1}{2}b^2\sin x}{(a^2\cos^2\frac{1}{2}x-b^2\sin^2\frac{1}{2}x)^2}=\frac{(a^2+b^2)\sin\frac{1}{2}x\cos\frac{1}{2}x}{(a^2\cos^2\frac{1}{2}x-b^2\sin^2\frac{1}{2}x)^2}$$

$$\therefore \frac{dy}{dz}=\frac{dy}{dx}\Big/\frac{dz}{dz}=\frac{ab(a^2\cos^2\frac{1}{2}x-b^2\sin^2\frac{1}{2}x)^2}{(a^2\cos^2\frac{1}{2}x-b^2\sin^2\frac{1}{2}x)(a^2+b^2)\sin\frac{1}{2}\cos\frac{1}{2}x}$$

$$=\frac{ab}{a^2+b^2}\left\{\frac{a^2\cos^2\frac{1}{2}x-b^2\sin^2\frac{1}{2}x}{\sin\frac{1}{2}x\cos\frac{1}{2}x}\right\}$$

$$=\frac{ab}{a^2+b^2}(a^2\cot\tfrac{1}{2}x-b^2\tan\tfrac{1}{2}x)$$

Ex. 17. Prove that

$$\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\frac{8x^7}{1+x^8}+\cdots\infty=\frac{1}{1-x}$$

if $1<x$

We know that

$(1-x)(1+x)=1-x^2,\ (1-x)(1+x)(1+x^2)=1-x^4,$

$(1-x)(1+x)(1+x^2)(1+x^4)=1-x^8,$

$\therefore (1-x)(1+x)(1+x^2)(1+x^4)(1+x^8)\ldots\infty=1-x^{\infty}=1\ldots(1)$

when $|x|<1$

and $\lim\limits_{x\to\infty} x^n=0$

Taking logarithm of both sides of (i), we have,

$\log(1-x)+\log(1+x)+\log(1+x^2)+(1+x^4)\ldots\infty=\log 1=0$

Differentating both sides *w. r.* to x,

$$\frac{-1}{1-x}+\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\ldots\ldots\infty=0$$

$$\text{or } \frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\frac{8x^7}{1+x^8}+\ldots\ldots\infty=\frac{1}{1-x}$$

Proved.

Exercise IV (B)

Find the differential co-efficients of the following with respect to x.

(x এর ভিত্তিতে ডিফারেন্সিয়েল সহগ নির্ণয় কর)

1. $2x+\dfrac{1}{x}+\dfrac{2}{x^3}$

2. $\log(1+\sin x)+2\log\{\sec(\frac{1}{2}\pi-x)\}$

3. $\log\dfrac{\sqrt{(4x+3)}}{2x+1}$ 4. $\log\{\sqrt{(x-a)}+\sqrt{(x-b)}\}$

5. $\log\sqrt{\left(\dfrac{2+3x}{2-3x}\right)}$ 6. $\log\dfrac{\sqrt{(x^2+1)}-x}{\sqrt{(x^2+1)}+x}$

7. $(\log\sin x)^2$ 8. $\log\sec(ax+b)^3$

9. $\log\{x+\sqrt{(x^2+2)}\}+\sec^{-1}(x^2)+\sqrt{5}$

10. $\log(\sec x+\tan x)$

11. $\log\dfrac{a+b\tan x}{a-b\tan x}$ 12. $\log_a x+\log_x a$

13. $\dfrac{x^n}{\log_a x}$ 14. $\dfrac{e^x+\tan x}{\cot x-x^n}$

15. $\sec x^0$ 16. $\dfrac{\sin x+\cos x}{\sqrt{(1+\sin 2x)}}$

17. $(a^{2/3}-x^{2/3})^{3/2}$ (ii) $\dfrac{(x^2+1)^3}{(x^3-1)^2},\ x\neq 1$ D.U. 1988

18. $\sqrt{\left(\dfrac{a^2+ax+x^2}{a^2-ax+x^2}\right)}$ 18 (i) $\tan^{-1}\dfrac{a+bx}{b-ax}$ D. U. 1984

See APPENDIX=Extra sums-142-155

19. $x(a^2+x^2)\sqrt{(a^2-x^2)}$ 20. $\frac{1-x}{\sqrt{(1+x^2)}}$ (i) $\sqrt[3]{x^2+\sqrt{3}}$ C.U.1993

21. $\sqrt{(1+\log x \log \sin x)}$ 22. $\log\{(2x-1)+2\sqrt{(x^2-x-1)}\}$

23. $\frac{\sqrt{x}}{1+x}$ (i) $\tan^{-1}\frac{x}{a}$ R. U. 1982

24. $e^{\sin x^2}-(\sin x^2)^2$ 25. $\frac{x^3\sqrt{(x^2+4)}}{\sqrt{(x^2+3)}}$

26. $\left\{\frac{1}{1+\sqrt{(1-x^2)}}\right\}^n$ 27 $\frac{1-x}{\sqrt{(1-x^3)}}$

28. $\log\frac{\sqrt{(x+1)}-1}{\sqrt{(x+1)}+1}$ 29. $\log(\sqrt{1+\log x})-\sin x\}$

30. $\log\frac{1}{\sqrt{x}}$ 31. $\frac{1-x}{\sqrt{(1-x^2)}}$

32. $\sqrt{(a^2\cos^2 x+b^2\sin^2 x)}$ (a) $(\sin x)^{\log x}+\cot\{e^x(a+bx)\}+a^b$

33. $\sqrt{\left(\frac{3-x^2}{x^2+2}\right)}$ 34. $\frac{1}{\sqrt[3]{(3x^2-x+1)}}$

35. $\frac{x^{3/2}(1+x^2)^{3/2}}{(1-x^2)^{3/2}}$ 36. $\log\frac{x^2\sqrt{(1-x^2)}}{\sqrt{(1+x^2)}}$

37. $(\cos hx)^x$ 38. $x^{2/3}\sqrt{\frac{x-1}{x+1}}$

39. x^{x^2} 40. $x^{\cot^{-1}x}$

40. x^{x^x} 42. x^x (i) $x^x+x^{1/x}$ D.U.1991

43. $x^{\cos^{-1}}$ 44. $(1+x)^x$ 45. $x^{2\sin x}$

46. 3.2^{x^2} 47. $(\sin x)^x$ 48. x^{e^x}

49. $\sqrt{(x+1)}^{\log(x+1)}$ 50. $(\sin x)^{\log x}$

51. $(\sin^{-1}x)^{\log x}$ 52. $(\tan^{-1}x)^{(\cos x+\sin x)}$

53. $x^x+x^{1/x}$ 54. $10^{\log \sin x}$

55. $\left(1+\frac{1}{x}\right)^x+x^{1/x}$ (i) $\sin\left(x^{\log x}\right)+\sin^2(\cos^{-1}x)$

R. U. 1982, D. U. 1981

56. $\sqrt{(\cot^{-1}e^x)}$ 57. $(\sin x)^{\cos x}+(\cos x)^{\sin x}$ C.U.1993

58. $e^{\sin^{-1}x}$ 59. $\sin^{-1}\frac{a+b\cos x}{b+a\cos x}$ R.H. 1987 C.H. 1988

D. U. 1962, R. U. 1990

59. (i) $\sin^{-1}\frac{1-x^2}{1+x^2}$ D. U. 1986

60. $\tan^{-1}\frac{\sqrt{(1-\cos x)}}{\sqrt{(1+\cos x)}}$ (i) $\sin^{-1}\frac{x+\sqrt{(1-x^2)}}{\sqrt{2}}$ R.U.1982 D.U.1992

61. $\tan^{-1}\frac{1+\tan x}{1-\tan x}$ N.U.1993 62. $x^n e^{\cos x}$

63. $\tan^{-1}\frac{a^{1/3}+x^{1/3}}{1-a^{1/3}x^{1/3}}$ (i) $\tan^{-1}\left(\frac{a-b}{a+b}\tan\frac{x}{2}\right)$

D. U. 1981

64. $\tan^{-1}(m\tan x)$ 65. $\tan(\sin^{-1}x)$

66. $\tan^{-1}\frac{x}{\sqrt{(1-x^2)}}$ 67. $\tan^{-1}\frac{\cos x}{1+\sin x}$ R. U. 1983

68. $\cos^{-1}(1-2x^2)$ (i) $x^{\cos^{-1}x}$ C. U. 1984

69. $\tan^{-1}\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)$ 70. $\tan^{-1}\frac{2x}{1+x^2}$

71. $\sin\left\{2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right\}$ D.H. 1987 72. $\tan^{-1}\frac{1}{\sqrt{(x^2-1)}}$

73. $\sin^{-1}\sqrt{(1-x^2)}$ D.H. 1987 74. $\tan(\log x^2)$

75. $\frac{3+5\cos x}{5+3\cos x}$ 76. $\sqrt{\left(\frac{\sec x+\tan x}{\sec x-\tan x}\right)}$

77. $\frac{\tan x}{x+e^x}$ 78. $\tan x+\frac{1}{3}\tan^3 x$

79. $x\cos e^x$ 80. $e^{ax}\sin^m rx$

81. $\sec\{\frac{1}{2}\log(x^2+a^2)\}$ 82. $\sin x^2\cos hx$

83. $e^{\sqrt{(\cot x)}}$ 84. $\sec^{-1}\frac{x^2+1}{x^2-1}$

85. $\cos^{-1}\{2x\sqrt{(1-x^2)}\}$ 86. $e^{(\sin^{-1}x)^2}$ D. U. 1986

87. $2\tan^{-1}\sqrt{\left\{\frac{x-a}{b-x}\right\}}$ 88 $3^{\sqrt{(1+x+x^2)}}$

89. $\sin^2(\log x^2)$

90. $x^x+(\sin x)^{\log x}$

91. $\tan^{-1}\dfrac{\sqrt{(1+x^2)}-1}{x}$

92. $\left\{\sin^{-1}x\right\}^{x/a}$

93. $\left\{\dfrac{x}{a}\right\}^{\sin^{-1} x/a}$

94. $\dfrac{x^4}{\sin x}$

95. $\tan x+\sec x$

96. $\sec(\tan^{-1} x)$

97. $e^{x^2\sqrt{(1-x^2)}}$

98. $(2x-3)^{2x-3}$

99. (i) $(1+x^2)\tan x + (2-\sin x)\log x$ D.H. 1983

(ii) $(x^2+1)^{\sin^{-1}} + e^{\sqrt{(1+x^2)}}$ D.H. 1984

(iii) $(1+x)\tan^{-1}\sqrt{x}-\sqrt{x},\ x>0$ D.H. 1984

(iv) $(x^2+1)\sqrt{(1-x^2)}+(\sin^{-1}x)^2,\ |x|<1$

(v) x^{yx} 100 $y^2-4ax=0$

101. $x\sqrt{(1+\cos y)}$ .102. $\sqrt{(y/x)}+\sqrt{(x/y)} = 6$

(i) $y=x\log\dfrac{y}{a+bx}$ C.U 1986 (i) $y=\left(\dfrac{n}{x}\right)^{nx}\left(1+\log\dfrac{x}{n}\right)$. C.H 1992

103. $x^6+x^4y-y^3=0$ 104. $x^y=y^x$ 105. $x^yy^x=1$

106. $3x^4-x^2y+2y^3=0$ 107. $x^n+u^n=a^n$

108. $y=x^y$ D.U. 1986 109. $x^4+x^2y^2+y^4=0$

108 (i) $y\ x^{\log x}+x^{\cos^{-1}}/x$ 109 (i) $(\sin x)^{\cos y}+(\cos x)^{\sin y}=a$

(ii) $x\tan x+(\sin x)^{\cos x}$ C.H. 1992; D.U. 1990

110. $x^y+y^x=a^b$ R.U 83 N.U 93 111. $y=x\log y$

112. $x+y=\sin^{-1}(ay/x)$ 113(a) $\{\sin\sqrt{(1+x^2)}\}^{\log\cos y}$

112. (i) $\sin y=x(2+y)$ D.U. 1988 D.H.1984

113. $y=\sin^{-1}(ax/y)$ 114. $\sin(x+y)=a$

(i) $\log(x+y)=xy$ D.U. 1991

115. $y=\sin^n(ax/y)$ N.H.94 116. $x=a(\theta+\sin\theta)$, $y=(1+\cos\theta)$ C.U.81

117. $y=\dfrac{3at^2}{1+t^2}$; $x=\dfrac{3at}{1+t^3}$ (1) $(\cos x)^y=(\sin x)^x$ C. U. 1984

118. $\tan y=\dfrac{2t}{1-t^2}$, $\sin x=\dfrac{2t}{1+t^2}$ D.U.1989

119. $x=\log t+\sin t$, $y=e^t+\cos t$

120. $x=\sin^2\theta, y=\tan\theta$

$=0$, $(x,y)=(0,0)$ at the

Origin $f_{xy}\neq f_{yx}$

121. Differentiate $\sin x^2$. w. r. to x^2.

121. x^2- এর সাপেক্ষে $\sin x^2$-এর অন্তরক সহগ নির্ণয় কর।

(i) $\cos 3x$ এর সাপেক্ষে $e^{\sin^{-1}x}$এর অন্তরক সহগ নির্ণয় করা। N.U. 1994

Differentiate $e^{x\sin^{-1}x}$ w.r.to $\cos 3x$. C.U.1991

122. Differentiate $\tan^{-1}\dfrac{2x}{1-x^2}$ w.r. to $\sin^{-1}\dfrac{2x}{1+x^2}$

$\sin^{-1}\dfrac{2x}{1-x}$ এর সাপেক্ষে $\tan^{-1}\dfrac{2x}{1+x}$ কে অন্তরীকরণ কর।

C.H. 1989; D.H. 1984, 87

123. Differentiate $\log_{10} x$ w. r. to x^3

x^3-এর সাপেক্ষে $\log_{10} x$-এর অন্তরীকরণ কর।

124. Differentiate $x^{\sin^{-1}x}$ w.r. to $\sin^{-1} x$

$\sin^{-1} x$ -এর সাপেক্ষে $x^{\sin^{-1}}x$-এর অন্তরীকরণ করা।

124. (i) $\log(2+x)$-এর তুলনায় $2x^2$-এর অন্তরক সহগ নির্ণয় কর।

(i) Differentiate $\cos 2x^2$ w.r.to $\log(2+x)$ C.U.1984

125. Differentiate e^t w.r. to $\sqrt{t}$.

$\sqrt{t}$ এর সাপেক্ষে e^t এর অন্তরক সহগ নির্ণয় কর।

125. (i) Find $\dfrac{dy}{dx}$ in trems of x and z where

$y=e^{-x}\sec^{-1}(x\sqrt{z})$ and $x^4+x^2z=x^5$ C.H. 1985

125. (ii) Differentiate $x^n\log\tan^{-1}x$ with respect to $\dfrac{\sin\sqrt{x}}{x^{3/2}}$

C.H. 1985

125. (iii) Differentiate $\tan^{-1}\dfrac{x}{\sqrt{(1-x^2)}}$ w.r to $\sec^{-1}\dfrac{1}{2x^2-1}$

C.H. 1988

125. (iv) Differentiate $\dfrac{\sqrt{(1+x^2)}+\sqrt{(1-x^2)}}{\sqrt{(1+x^2)}-\sqrt{(1-x^2)}}$ w. r. to $\sqrt{(1-x^2)}$

125. (v) Differentiate

$(x^2+ax+a^2)^n \log \cot \frac{1}{2} x \tan^{-1}$ (a cos bx) $w.\ r.$ to (acosbx)

(vi) Differentiate $\log \frac{1+\sqrt{x}}{1-\sqrt{x}}$ $w.\ r.$ to $\sqrt{x^3}$ D.H. 1989

126. $y = u^{u}$-এর অন্তরীকরণ কর যেখানে u এবং v উভয়েই x-এর ফাংশন।
Differentiate $y= u^{v}$ where u, v are the functions of x)

127. $(x+y)^{m+n}=x^m y^n$)

128. if $y = \sin^{-1} \frac{1}{\sqrt{(1+x^2)}}$ show that $\frac{dy}{dx} = \frac{1}{1+x^2}$

129. If $y=x\sqrt{x^2+a^2})+a^2\log\{x+\sqrt{(a^2+x^2)}\}$. prove that $(dy/dx) = 2\sqrt{(a^2+x^2)}$

130. If $y=\cot(\cos^{-1} x)$, from that $\frac{dy}{dx}=\frac{1}{(1-x^2)^{3/2}}$

131. If $P(x)=ax^2+bx+c$, $y=\sqrt{\{P(x)\}}$

show that $4y^3\frac{d^2y}{dx^2}=4ac-b^2$

132. If $f=\sqrt{\left\{\frac{x^2(3-x)}{x+1}\right\}}$ Find the domain of f and f' D. H. 1983

133. If $y=\frac{\sin x}{1+} \frac{\cos x}{1+} \frac{\sin x}{1+}. 1 \ldots \ldots \infty$

Prove that $\frac{dy}{dx}=\frac{(1+y)\cos x+y\sin x}{1+2y+\cos x-\sin x}$

134. If $c=1+r\cos\theta+\frac{r^2\cos 2\theta}{\lfloor 2}+\frac{r^3\cos 3\theta}{\lfloor 3}+\ldots\ldots$

Show that $s=r\sin\theta+\frac{r^2\sin 2\theta}{\lfloor 2}+\frac{r^3\sin 3\theta}{\lfloor 3}+\ldots\ldots$

$c\frac{dc}{dr}+s\frac{ds}{dr}=(c^2+s^2)\cos\theta$ and

$c\frac{ds}{dr}-s\frac{ds}{dr}=(c^2+s^2)\sin\theta$

134. (a) If $f(x)=\left(\frac{a+x}{b+x}\right)^{a+b+2x}$ show that

$f'(0)=2\left(\log\frac{a}{b}+\frac{b^2-a^2)}{ab}\right)\left(\frac{a}{b}\right)^{a+b}$ R. U. 1987

135. If $y=\sqrt{\sin x+\sqrt{\sin x+\sqrt{\sin x+\sqrt{\text{etc to }\infty}}}}$

Prove that $\frac{dy}{dx}=\frac{\cos x}{2y-1}$

(i) $y=\sqrt{4+\sqrt{4-x}}$, $x\leqslant 4$

136. If $y=\sec 4x$, prove that

$\frac{dy}{dt}=\frac{16t(1-t^4)}{(1-6t^2+t^4)^2}$, $t=\tan x$.

137. If S_n= the sum of a G. P. to n terms of which r is the common ratio,

Prove that

$(r-1)\frac{dS_n}{dr}=(n-1)S_n-nS_n-1$

138. If $x^2y+y^2x+\sqrt{(xy)}=1$, then show that

$\frac{dy}{dx}=-\frac{y(3x^2y+xy^2+1)}{x(x^2y+3xy^2+1)}$ D. U. 1987

139. Prove that

$\frac{1-2x}{1-x+x^2}+\frac{2x-4x^3}{1+x^2+x^4}+\frac{4x^3-8x^7}{1-x^4+x^8}+\ldots\ldots\infty=\frac{1+2x}{1+x+x^2}$

if $x<1$.

140. If $\cos\frac{1}{2}x\cos\frac{x}{2^2}\cos\frac{x}{2^3}\ldots\cos\frac{x}{2^n}=\frac{\sin x}{2^n\sin(x^n/2^n)}$ R. U. 1988

then show that

$\frac{1}{2}\tan\frac{x}{2}+\frac{1}{2^2}\tan\frac{x}{2^2}+\cdots+\frac{1}{2^n}\tan\frac{x}{2^n}=\frac{1}{2^n}\cot\frac{1}{2^n}-\cot x$

and $\frac{1}{2^2}\sec^2\frac{x}{2}+\frac{1}{2^4}\sec^2\frac{x}{2^2}+\cdots+\frac{1}{2^{2n}}\sec^2\frac{x}{2^n}$

$=-\frac{1}{2^{2n}}\text{cosec}^2\frac{x}{2^n}+\text{cosec}^2 x$.

What happens if $n\to\infty$

141. If $(1+x)^n=c_0+c_1x+c_2x^2\ldots\ldots c_nx^n$

show that $c_1+2c_2+3c_3+\cdots nc_n=n2^{n-1}$ R. U. 1987

See APPENDIX= Extra Sums= 142 → 155

31. $-\frac{1}{(1+x)\sqrt{(1-x^2)}}$ 32. $\frac{(b^2-a^2)\sin 2x}{2y}$

33. $\frac{-5x}{\sqrt{(x^2+2)^2}\sqrt{(3-x^2)}}$ 34. $\frac{-(6x-1)}{3\sqrt[3]{(4x^2-x+1)^4}}$

35. $3y\left[\frac{3}{4}x+\frac{2}{1-x^4}\right]$ 36. $y\,\frac{2(1-x^2-x^4)}{x-x^5}$

37. $y(x\tanh x+\log\cosh x)$ 38. $y\,\frac{2x^2+3x-2}{3x(x^2-1)}$

39. $x^{x^2}x(1+2\log x)$ 40. $y\left(\frac{\cot^{-1}x}{x}-\frac{\log x}{1+x^2}\right)$

41. $x^{x^x}\log x\left[1+\log x+\frac{1}{x\log x}\right]$

42. $x^x(1+\log x)$ 43. $x^{\cos x-1/x}\left(\frac{\cos^{-1}x}{x}-\frac{\log x}{\sqrt{(1-x^2)}}\right)$

44. $(1+x)^x\left[\log(1+x)+\frac{x}{1+x}\right]$

45. $2x^{2\sin x}\left[\cos x\log x+\frac{\sin x}{x}\right]$ 46. $3.2x^2\times 2x\log 2$

47. $(\sin x)^x[x\cot x+\log\sin x]$ 48. $x^{e^x}e^x[1/x+\log x]$

49. $y\,\frac{\log(x+1)}{(x+1)}$ 50. $(\sin x)^{\log x}\left[\log x\cot x+\frac{1}{x}\log\sin x\right]$

51. $(\sin^{-1}x)^{\log x}\left[\frac{\log x}{\sin^{-1}x\sqrt{(1-x^2)}}+\frac{\log\sin^{-1}x}{x}\right]$

52. $y\left\{\frac{\cos x+\sin x}{(1+x^2)\tan^{-1}x}+(\cos x-\sin x)\log\tan^{-1}x\right\}$

54. $x^x(1+\log x)+x^{1/x}\frac{1-\log x}{x^2}$ 54. $10^{\log\sin x}\log 10\cot x$

ANSWERS

1. $2-\frac{1}{2}x^2-6/x^4$ 2. $\tan(\frac{1}{4}\pi-\frac{1}{2}x)-2\tan(\frac{1}{2}\pi-x)$

3. $\frac{-4(x+1)}{8x^2+10x+3}$ 4. $\frac{1}{2\sqrt{\{(x-a)(x-b)\}}}$

5. $6/(4-9x^2)$ 6. $-2/\sqrt{(x^2+1)}$ 7. $2\cot x\log\sin x$

8. $3a(ax+b)^2\tan(ax+b)^3$ 9. $1/\sqrt{(x^2+2)}+\frac{2}{x\sqrt{(x^4-1)}}$

10. $\sec x$ 11. $\frac{2ab\sec^2x}{a^2-b^2\tan^2x}$ 12. $\frac{1}{x}\log_a e-\frac{\log a}{x(\log x)^2}$

13. $(nx^{n-1}\log_a x-x^{n-1}\log^a e)/(\log_a x)^2$

14. $(e^x+\sec^2x)(\cot x-x^n)+(\operatorname{cosec}^2x+nx^{n-1})(e^x+\tan x)/(\cot x-x^n)^2$

15. $\frac{\pi}{180^\circ}\,\frac{\sin x^\circ}{\cot x^\circ}$ 16. 0 17. $-\frac{\sqrt{(a^{2/3}-x^{2/3})}}{x^{1/3}}$

18. $\frac{a(a^2-x^2)}{(a^2-ax+x^2)^{3/2}\sqrt{(a^2+ax+x^2)}}$ 19. $\frac{a^4+a^2x^2-5x^2}{\sqrt{(a^2-x^2)}}$

(i) $1/(1+x^2)$

20. $-\frac{1+x}{(1+x^2)^{3/2}}$ 21. $\frac{\log\sin x+x\log x\cot x}{2x\sqrt{(1+\log x\log\sin x)}}$

22. $\frac{1}{(x^2-x-1)^{1/2}}$ 23. $\frac{1-x}{2\sqrt{x}(1+x)^2}$ (i) $\frac{a}{x^2+a^2}$

24. $2x\cos x^2(e^{\sin x^2}+2\sin x^2)$

25. $\frac{x^2\sqrt{(x^2+4)}}{\sqrt{(x+3)}}\left(3-\frac{x^2}{(x^2+3)(x^2+4)}\right)$ 26. $\frac{ny}{x\sqrt{(1-x^2)}}$

27. $\frac{(1-x^2)(x^2-2x-2)}{2(1-x^3)^{3/2}}$ 28. $\frac{1}{x\sqrt{(x+1)}}$

29. $\frac{1-2x\cos x\sqrt{(1+\log x)}}{\{\sqrt{(1+\log x)}(-\sin x)2x\sqrt{(1+\log x)}}$ 30. $-1/2x$

55. $(1+1/x)^x\left\{-\frac{1}{x+1}+\log(1+1/x)\right\}+x^{1+1/x}\left(\frac{x+1-\log x}{x^2}\right)$

56. $-\frac{e^x}{2\sqrt{(\cot^{-1}e^x)}(1+e^{2x})}$

57. $(\sin x)^{\cos x}\{\cot x\cos x-\sin x\log\sin x\}$
$+(\cos x)^{\sin x}(-\tan x\sin x+\cos x\log\cos x)$

58. $\frac{e^{\sin^{-1}x}}{\sqrt{(1-x^2)}}$ 59. $\frac{-\sqrt{(b^2-a^2)}}{b+a\cos x}$ 60. $\frac{1}{3}$ 61. 1.

62 $\cos_x x^n(n/x-\sin x)$ 63. $\frac{1}{3x^{2/3}(1+x^{2/3})}$

64. $\frac{m}{\cos^2 x+m^2\sin^2 x}$ 65. $\frac{1}{\sqrt{(1-x^2)^3}}$ 66. $\frac{1}{\sqrt{(1-x^2)}}$

67. $-\frac{1}{2}$ 68. $\frac{2}{\sqrt{(1-x^2)}}$ 68. (i) $x\cos^{-1}x\left[\frac{\log x}{\sqrt{(1-x^2)}}+\frac{\cos^{-1}x}{x}\right\}$

69. -1 70. $\frac{2}{1+x^2}$

71. $-\frac{x}{\sqrt{(1-x^2)}}$ 72. $\frac{-1}{x\cdot(x^2-1)}$ 73. $\frac{1}{\sqrt{(1-x^2)}}$

74. $\frac{2}{x}\sec^2\log x^2$ 75. $\frac{-16\sin x}{(5+3\cos x)^2}$ 76. $-\frac{1}{2}\sec^2(\pi/4-x/2)$

77. $\frac{\sec^2 x}{x+e^x}+\frac{\tan x(1+e^x)}{(x+e^x)^2}$ 78. $\sec^4 x$

79. $-xe^x\sin e^x+\cos e^x$. 80. $e^{ax}\sin^m rx(a+rm\cot rx)$

81. $\frac{x}{x^2+a^2}\sec\{\frac{1}{2}\log(x^2+a^2)\}\tan\{\frac{1}{2}\log(x^2+a^2)\}$

82. $\sin x^2\sin hx+2x\cos x^2\cos hx$. 83. $\frac{e^{\sqrt{\cot x}}}{\sin^2 x\,2\sqrt{\cot x}}$

84. $-\frac{2}{1+x^2}$ 85. $\frac{-1}{\sqrt{(1-x^2)}}$

86. $\frac{2\sin^{-1}x\,e^{(\sin^{-1}x)^2}}{\sqrt{(1-x^2)}}$ 87. $\frac{8}{\sqrt{\{(x-a)(b-x)\}}}$

88. $\frac{(2x+1)\log 3}{2\sqrt{(1+x+x^2)}}3^{\sqrt{(1+x+x^2)}}$ 89. $\frac{2}{x}\sin(2\log x^2)$

90. $x^x(1+\log x)+(\sin x)^{\log x}\left(\cot x\log x+\frac{\log\sin x}{x}\right)$

91. $\frac{1}{2(1+x^2)}$ 92. $\frac{1}{a}\left(\sin^{-1}\frac{x}{a}\right)^{x/a}\left(\frac{x}{\sin^{-1}x/a\sqrt{(a_1-x^2)}}+\log\sin^{-1}\frac{x}{a}\right)$

93. $\left(\frac{x}{a}\right)^{\sin^{-1}x/a}\left[\frac{\sin^{-1}\frac{x}{a}}{x}+\frac{\log\frac{x}{a}}{\sqrt{(a^2-x^2)}}\right]$

94. $x^3\operatorname{cosec}x(3-x\cot x)$

95. $\sec^2 x+\sec x\tan x$. 95. $\frac{x}{\sqrt{(1+x^2)}}$

97. $\frac{x^2e^{x\sqrt{/(1-x^2)}}}{\sqrt{(1-x^2)}}(2-3x^2)$ 98. $2(2x-3)^{2x-3}\{1+\log(2x-3)\}$

99. $\frac{y^5(\log x\log y+1/x)}{1/y-y^{x-1}x\log x}$ 100. $\sqrt{(a/x)}$

101. $-2x\operatorname{cosec}y$ 102. $\frac{x-17y}{17x-y}$ 103. $\frac{5x^4+4x^5y}{x^4+3y^2}$

104. $\frac{y}{x}\;\frac{x\log y-y}{y\log x-x}$ 105. $\frac{y^2}{x^2}\frac{1-\log x}{1-\log y}$

106. $\frac{2x(y-6x^2)}{6y^2-x^2}$ 107. $-(x/y)^{x-1}$

108. $\dfrac{y^2}{x(1-y\log x)}$ 109. $-\dfrac{2x^3+y^2x}{2y^3+x^2y}$

110. $-\dfrac{y^x \log y+yx^{y-1}}{x^y \log x+xy^{x-1}}$ 111. $\dfrac{y\log y}{y-x}$

112. $\dfrac{ay+x^2\cos(x+y)}{ax-x^2\cos(x+y)}$ 113. $\dfrac{ay}{xa+y\sqrt{(y^2-a^2x^2)}}$

114.—1 115. $\dfrac{2\cos(x+y)^2(x+y)}{1-2(x+y)\cos(x+y)}$

116. $\tan\frac{1}{2}\theta$ 117. $\dfrac{t(2-t^3)}{1-2t^3}$ 118. 1

119 $\dfrac{t(e^t-\sin t)}{1+t\cos t}$ 120. $\cos^2\theta \operatorname{cosec} 2\theta$

121. $\dfrac{\pi\cos x}{180^\circ \times 2x}$ 122. 1.

123. $\frac{1}{3}x^{-3}\log_{10}e$

124. $x^{\sin^{-1}x}\left(\sin^{-1}x\ \dfrac{\sqrt{(1-x^2)}}{x}+\log x\right)$(i)—$(4x^2+8x)\sin 2x^2$

125. $2\sqrt{t}\,e^t$ (i) $\dfrac{e^{-x^2}}{4z^3+x^2}\left[\dfrac{8z^4+5x^3}{2xz\sqrt{(x^2z-1)}}\right.$

$\left.-(5z^4+5x^5)\sec^{-1}x\sqrt{z}\right]$

(ii) $2\ \dfrac{n(1+x^2)\tan^{-1}x\log\tan^{-1}x+x}{(1+x^2)\tan^{-1}x(\sqrt{x}\cos\sqrt{x}-3\sin\sqrt{x})}x^{\frac{2n+3}{2}}$

(iii) $-\frac{1}{2}$ (iv) $\dfrac{1}{x^4}\ \dfrac{\sqrt{(1+x^2)}+\sqrt{(1-x^2)}}{\sqrt{(1+x^2)}-\sqrt{(1-x^2)}}$

(v) $\dfrac{(1+a^2\cos^2 bx)(x^2xax+a^2)^{n-1}[n(2x+a)\log\cot\frac{1}{2}x}{-ab\sin bx}$

$\dfrac{-\operatorname{cosec} x(x^2+ax+a^2)]}{}$

126. $u^v\left[\dfrac{v}{u}\dfrac{du}{dx}+\log v\dfrac{dv}{dx}\right]$ 127. y/x 127. $\dfrac{1}{1+x^2}$

CHAPTER—V

APPLICATIONS OF DERIVATIVES

The concept of derivatives comes from the intutive ideas of (1) finding the velocity of a particle at an instant and (2) constructing tangent to a curve at a point.

Let us first consider the case of velocity. Let algebraic distances of a particle moving on a straight line be S and S_1 at times t and t_1 respectively where the distances one measured from a fixed point on the line. Then

$$v=\frac{S_1-S}{t_1-t}$$

is defined as the average velocity of the particle over the time interval $[t, t_1]$. Writing $t_1=t+\Delta t$, $S_1=S+\Delta S$, we have,

$$v=\frac{(S+\Delta S)-S}{(t+\Delta t)-t}=\frac{\Delta S}{\Delta t}$$

When Δt becomes infinitesimally small the length of interval $[t, t+\Delta t]$ becomes almost zero and in these case, v can be termed as the velocity of the particle of time t. Thus

$$v=\text{velocity at time } t=\lim_{\Delta t\to 0}\left(\frac{\Delta S}{\Delta t}\right)=\frac{ds}{dt}$$

For example, if $S=f(t)=t^2$

$$v=\frac{ds}{dt}=f^1(t)=2t$$

If t is in seconds and S is in feet, then the velocity at $t=1$, 2, 3, sec., are respectively 2×1 or 2 ft/sec, 2×2 or 4ft/sec, 2×3 or 6 ft/sec.

108. $\frac{y^2}{x(1-y\log x)}$ 109. $-\frac{2x^3+y^2x}{2y^3+x^2y}$

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124. $x^{\sin^{-1}x}\left(\sin^{-1}x\ \frac{\sqrt{(1-x^2)}}{x}+\log x\right)(i)-(4x^2+8x)\sin 2x^2$

125. $2\sqrt{t}\ e^t$ (i) $\frac{e^{-x^2}}{4z^3+x^2}\left[\frac{8z^4+5x^3}{2xz\sqrt{(x^2z-1)}}\right.$

$$\left.-(5z^4+5x^5)\sec^{-1}x\sqrt{z}\right]$$

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If t is in seconds and S is in feet, then the velocity at $t=1$, 2, 3, sec., are respectively 2×1 or 2 ft/sec, 2×2 or 4ft/sec, 2×3 or 6 ft/sec.

Similarly the acceleration of a particle is the derivative of velocity ; that is,

f=acceleration of time $t=\frac{dv}{dt}$

5. 2. Geometrical Interpretation of the derivative $\frac{dy}{dx}$

Let $y=f(x)$ represent the curve APQ. Let $P\ (x, y)$ be any point on the curve, and $Q\ (x+\Delta x, y+\Delta y)$. be any point in the neighbourhood of P. Thus

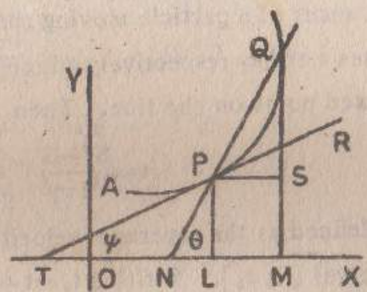
$OL=x$, $PL=SM=y$

$OM=x+\Delta x$, $QM=y+\Delta y$

Let $\angle QPS=\angle QNM=\theta$

$\angle XTR=\psi$

Now from $\triangle PQS$,



Fig

$$\tan QPS=\frac{QS}{PS}=\frac{QM-SM}{LM}=\frac{QM-PL}{OM-OL}$$

$$\text{or, } \tan\theta=\frac{y+\Delta y-y}{x+\Delta x-x}=\frac{\Delta y}{\Delta x}$$

If Q approaches P along the curve, Δx tends to zero. The straight line QPN becomes the straight line TPR in the limit as $Q\to P$ or $\Delta x\to 0$. In this case, $\theta\to\psi$, which is the inclination of the tangent at P with the positive direction of x—axis.

Therefore

$$\lim_{\theta\to\psi}\tan\theta=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}\Rightarrow \tan\psi=\frac{dy}{dx} \text{ if the limit exist.}$$

5.9 Differentials

If $f'(x)$ is the derivative or differential co-efficient of $f(x)$ and Δx is an increment of x, the product $f'(x)\,\Delta x$ is denoted by $df(x)$ and is called the differential of $f(x)$. Symbolically it is written as

$df(x)=f'(x)\,\Delta x$......(1)

If x is an independent variable, $f(x)=x$, then $f'(x)=1$

$\therefore\ dx=\Delta x$ ... ... (2)

If $y=f(x)$ then (1) reduces to

$dy=f'(x)\,dx$ ... ... (3)

which is the differential of the function $y=f(x)$

Proof. From the limit of a function, we have

$$\left|\frac{f(x+\Delta x)-f(x)}{\Delta x}-f'(x)\right|<\epsilon,\ 0<|\Delta x|<0$$

where ϵ_1 is arbitrarily small and Δ depends on ϵ and x, From this we have.

$\Delta f(x)=f(x+\Delta x)-f(x)=\{f'(x)+\epsilon_1\}\,\Delta x$

when $\epsilon_1\to 0$ as $\Delta x\to 0$ ... ...(2)

$\Delta f(x)$ is almost equal to $f(x)\,\Delta x$ since $\epsilon_1\to 0$. We represent the product $f'(x)\Delta x$ by $df(x)$ and the differential is $df(x)=f'(x)\,\Delta x$ when $f(x)=x$, then $f'(x)=1$.

5.5. Geometrical representation of the Differential of a function. (ডিফারেন্সিয়ালের জ্যামিতিক ব্যাখ্যা)

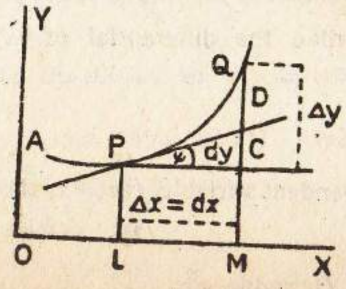


Fig 54

Let $y=f(x)$ represent the curve APQ and the derivatives $f'(x)$ exists at every point of the curve. Let the co-ordinates of P and θ be (x, y) and $(x+\Delta x, \ y+\Delta y)$ respectively, PD is the tangent at P.

Let $\angle CPD=\psi, PC=\Delta x=dx, QC=\Delta y, \ DC=dy$

Thus $\tan \psi=\dfrac{CD}{PC}=\dfrac{CD}{\Delta x}=f'(x)$ ... ... [4]

Since the value of the derivative at a point on a curve equal to the slope of the tangent at that point.

From (4) we have $CD=f'(x)\ \Delta(x)=df(x)=dy$

or, $dy=f'(x)dx$

The symbols dx and dy are also called differentials (differential x and differential y)

If $dx\neq 0$; then $\dfrac{dy}{dx}=f'(x)$ or $\dfrac{\text{differential } y}{\text{differential } x}$ =derivatives of $f(x)$ at x.

Examples

Ex. 1. $d(u+v+w)=du+dv+dw$

Ex. 2. $u=xy$

$du=d(xy)=ydx+xdy$

Ex. 3 $d(uv)=udv+vdu$

Ex. 4. $A=\pi r^2$

$\therefore \ dA=2\pi r dr$

Ex. 5. $x=\cos\theta, \ y=\sin\theta$

$\therefore \ dx=-\sin\theta\, d\theta, \ dy=\cos\theta\, d\theta$

Note : In the above theorem we notice that the increment Δx of x is equal to the differential dx but is not generally the case with the dependent variable i. e. the increment Δy of y is not equal to dy i. e., $\Delta y\neq dy$.

If $y=f(x)=x^4-3x^2$, then

$\Delta y=f(x+\Delta x)-f(x)=(x+\Delta x)^4-3(x+\Delta x)^2-(x^4-3x^2)$

$=x^4+4x^3\Delta x+6x^2(\Delta x)^2+4x(\Delta x)^3+(\Delta x^4-3x^2-6x\Delta x+3$

$(\Delta x)^2-x^4+3x^2=(4x^3-6x)2x+(16x^2-3)(\Delta x)^2+4x(\Delta x)^3$

$+\Delta(x)^3$

dy=Principal part of $\Delta y=(4x^3-6x)\Delta x=(4x^3-6x)dx$, since $\Delta x=dx$.

$\therefore \ f(x)=\dfrac{dy}{dx}=4x^3-6x$ and $dy=(4x^3-6x)dx$ i. e ,

$dy/dx=4x^3-6x$ which shows that dy and dx are not necessarily small.

5. 6. Increasing and decreasing functions (ঊর্ধ এবং অধঃক্রমিক ফাংশন)

Theorem : If y is a function of x, then y increases as x

increases if (dy/dx) is positive, and y decreases as x increases if $\frac{dy}{dx}$ is negative.

For an small increment of x, say Δx, let Δy be the corresponding small increment in y.

As x change by Δx and y by Δy and if both of them have the same sign (i. e., if x and y both increase or decrease together) $\frac{\Delta y}{\Delta x}$ is positive. On the other hand if Δx and Δy have opposite signs (if one increases and other decreases) $\frac{\Delta y}{\Delta x}$ is negative.

$$\text{Now } \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

If we ignore the limit then

$\frac{\Delta y}{\Delta x} = \frac{dy}{dx} + \alpha$ where α is an infinitesimal which tends to zero as $\Delta x \to 0$.

Thus the sign of (dy/dx) is the same as the sign of $(\Delta y/\Delta x)$. Hence y increases with increase of x if (dy/dx) is positive and y decreases with the increase of x if (dy/dx) is negative.

Therefore $y=f(x)$ is called an *increasing function* of x for an interval of x if y increases for increasing values of x in that interval. The function $y=f(x)$ is called a *decreasing function* of x in an interval if the value of v decreases with the increasing values of x in the interval.

Increasing and Decreasing Functions

Art. 5.7 Prove that it $f'(a) < 0$ in $a < x < b$, then $f(x)$ is steadilly decreasing in this interval.

Ans. $f'(a) = \underset{h\to 0}{\text{Lt}} \frac{f(a+h)-f(a)}{h}$, $f'(a) = \underset{-h\to 0}{\text{Lt}} \frac{f(a-h)-f(a)}{-h}$

But given $f'(a) < 0$ $\therefore$ $f(a+h)-f(a) < 0$ or $f(a+h)$ ...(1

Also $-f(a-h) + f(a) < 0$ Or, $f(a-h) > f(a)$ ... (2

From (1) and (2), $f(a-h) > f(a) > f(a+h)$

The function is steadily decreasing in the nbd of

Alternative Method :—

Let a be a point in (c, d) such that $c < a < d$.

Consider two points x_1, x_2 in the interval such that ... d. By the Mean value Theorem, we have.

$f(x_2)-f(x_1) = (x_2-x_1) f'(a)$ where $x_1 < a < x_2$ i.e. ... But $f'(a) < 0$ in (c, d) i.e.: $f(x_2)-f(x_1) < 0$ or, $f(x_2)$ ... steadily decreases in (c, d)

Art. 5.8 State the meaning of derivative at ... signify the increasing or decreasing of a function?

Ans. Let $f(x)$ be differentiable in (a, b). Let c ... point of (a, b). Then

$$f'(c) = \underset{h\to 0}{\text{Lt}} \frac{f(c+h)-f(c)}{h}$$

$$\therefore \left| \frac{f(c+h)-f(c)}{h} - f'(c) \right| < \epsilon > 0 \text{ Where } 0 < |$$

$$\text{Or, } f'(c) - \epsilon < \frac{f(c+h)-f(c)}{h} < f'(c) + \epsilon$$

Let $f'(c) > 0$, Choose $\epsilon < f'(c)$. Then $f'(c)-\epsilon$ α f... positive. Thus $\frac{f(c+g)-f(c)}{h} > 0$ When $0 < |h| \le \delta$

The function $f(x)$ will then be called an increasing ... the interval $(c-\delta, c+\delta)$

Next, let $f'(c) < 0$, chose $\epsilon < |f'(c)|$. Then $f'(c)$ — ... are both negative

Now $\frac{f(c+h)-f'(c)}{h} < 0$ so that $f(c+h)-f(c)$ ... apposite signs.

If $c + h = x$, then $c-\delta \le x \le c + \delta$ and hence we ...

$f(c)$, it $x > c$ and $f(x) > f(c)$ it $x < c$.

Thus, it $f'(c) < 0$, we can find a nbd. of c such that $f(x_2) < f(x'_1)$ when $c-\delta < x_1 < x_2 < c + \delta_2$

The function $f(x)$ will, be called a decreasing function in $(c-\delta, c + \delta)$.

For the end points a and b, we only restrict the intervals about them.

Or, $f(c+h) > f(c)$ it $0 < h \leq \delta$

and $f(c+h) < f(c)$ it $-\delta \leq h < 0$

If $c+h = x \therefore c-\delta \leq x \leq c + \delta$ it $x < c$

Hence we see that $f(x) > f(c)$ it $x > c$ and $f(x) < f(c)$ it $x < c$

From the above discussions, we have come to the conclusion that $f'(x)$ is positive or negative, $f(x)$ is an increasing or, decreasing function in a suitably restlicted ndb. of c.

Ex. 1 Show that $y=x^3-6x^2+15x+3$ is an increasing function of x.

Let $y=x^3-6x^2+15x+3, \frac{dy}{dx}=3x^2-12x+15$

$$=3(x^3-4x+5)=3\{x-2)^2+1\}$$

$\Rightarrow(dy/dx)$ is positive for all real values of x. Hence y is an increasing function of x.

Ex. 2 If $y=2x^3-9x^2+12x-6$, find the range of values of x for which y is increasing and that in which y is decreasing.

$$\frac{dy}{dx}=6x^2-18x+12$$

$$=6(x^2-3x+2)=6(x-1)(x-2)$$

$\therefore \frac{dy}{dx}$ is positive for $x<1$ or $x>2$.

So, y is an increasing function of x as x increases from $-\infty$ to $+1$ or from $+2$ to ∞.

Again dy/dx is negative for $1<x<2$, so y is a decreasing function of x as x increases from $+1$ to $+2$.

Hence we have a tangent at a point, not parallel to y—axis, if and only if the derivative (dy/dx) or, $f'(x)$ exists, in that case, the derivative is the slope or, gardient of the tangent line).

If ψ be the inclination of the tangent to the curve $y=f(x)$ then $\tan\psi= \frac{dy}{dx}$. So ψ is acute if $\frac{dy}{dx}>0$ or if y increases with x or the curve is *rising* with x and ψ is obtuse, if $\frac{dy}{dx}<0$ or y decreases as x increase implying that the curve is *falling* with increasing values of x.

If $\frac{dy}{dx}=0$. then $\psi=0$ and the tangent is parallel to the x-axis.

If $\tan\psi=\frac{dy}{dx}=\lim_{\Delta x\to 0}\left(\frac{\Delta y}{\Delta x}\right)=\pm\infty$, then $\cos\psi=0$ or $\psi=90$ and so the tangent prependicular to the x-axis.

Ex. 3. If $0\leqslant x<\frac{1}{2}\pi$, prove that $\frac{2}{\pi}\leqslant\frac{\text{sni } x}{x}<1$

Let $f(x)=\frac{\sin x}{x}$, then

$$f'(x)=\frac{x\cos x-\sin x}{x^2}=\frac{(x-\tan x)\cos x}{x^2}$$

When x is a positive acute angle, we know that $x<\tan x$, Also in the given range $\cos x$ and x^2 are both positive.

Hence $f'(x)\leqslant 0$ when $0\leqslant x\leqslant\frac{1}{2}\pi$. Thus $f(x)$ is a decreasing function of x throughout the interval $(0, \frac{1}{2}\pi)$ Hence its greatest value occurs at $x=0^+$ and least at $x=\frac{1}{2}\pi$. When

$$x\to 0,\ f(x)=\frac{\sin x}{x}\to 1 \text{ and } f(\tfrac{1}{2}\pi)=2/\pi$$

Thus we have $\frac{2}{\pi}\leqslant\frac{\sin x}{x}\leqslant 1$ for $0\leqslant x\leqslant\frac{1}{2}/\pi$

Ex. 4 Use differential calculus, calculate the value *of* $\sqrt{48}$

Let $y = \sqrt{x}$, $\delta y = \frac{1}{2\sqrt{x}} \delta x$

See APPENDIX
Worked out Examples
NO5 NO.6; NO.7

If $x = 49$. and $\delta x = -1$. Then

$y = \sqrt{49} = 7$, $\delta y = -\frac{1}{2.7} = -.071$

$\therefore \sqrt{49} = 7$, $\sqrt{(x + \delta x)} = y + \delta y = 7 - .071 = 6.929$

প্রশ্নমালা V

1. দেখাও যে, $y = x^2 - 10x + 3$ হলে y কমবে যদি $x < 5$ হয় এবং y বাড়বে যদি $x > 5$ হয়।

2. দেখাও যে $y = (x-2)e^x + x + 2$ এর মান x এর সকল ধনাত্বক মানের জন্য সর্বদা ধনাত্বক হবে।

3. দেখাও যে $-x^3 + 9x^2 - 30$ এর মান x এর সকল ধনাত্বক মানের জন্য সর্বদা ঋনাত্মক হবে।

4. দেওয়া আছে $y = 2x - \tan^{-1} x - \log\{x + \sqrt{(1 + x^2)}\}$; x-এর কম বর্ধমান মানের জন্য y সর্বদা ধনাত্মক হবে x-এর সেই বিস্তারটি নির্ণয় কর।

C. H. 1972 উঃ $(0,$ থেকে $\infty)$

5. If $f(x) = (x-1)e^x + 1$, show that $f(x)$ is positive for all positive values x.

6. Prove that $x - \frac{1}{2}x^2 < \log(1+x) < x - \frac{x^2}{2(1+x)}$ for $x > 0$

7. If x is not equal to zero, prove that
(i) $\frac{x}{1+x} < 1 - e^{-x} < x$ for $x > -1$ (ii) $x < e^x - 1 < \frac{x}{1-x}$ for $x < 1$

8. Use differentials to calculate approximately $\sqrt{99}$, $\sqrt[3]{533}$

Ans. 9.95, 8.005

9. In what interval is the function $f(x) = 3x^5 - 25x^3 + 60x$ increasing and decreasing? Sketch ($f(x)$ ফাংশনটি কোন্ ব্যবধিতে বৃদ্ধিমান আর কোন্ ব্যবধিতে হ্রাসমান তাহা নির্ণয় কর। চিত্রটি অঙ্কন কর

10. Determine the gradient of the tangent to the curve $\sqrt{(1-x^2)} - \sin y = 0$ at the point whose coordinate is $1/\sqrt{2}$ N.U. 1991

CHAPTER—VI

SUCCESSIVE DIFFERENTIATION

(ক্রমিক ডিফারেন্সিয়েশন)

6.1. Definition :—If $y=f(x)$, its differential co-efficient $f'(x)=dy/dx$ will be in general a function of x, $f'(x)$ is called the **first derivative** of $f(x)$. The differential co-efficient of $f'(x)$ is called the **2nd derivative** of $f(x)$ i, e.

$$f''(x) = \lim_{h\to 0} \frac{f'(x+h)-f'(x)}{h}$$ provided the limits exist.

Similarly the derivative of $f''(x)$ is called the 3rd derivative of $f(x)$ and denoted by $f'''(x)$ and so on. If $f(x)$ is differentiated n times with respect to x then we get what is called the nth derivative of $f(x)$ and this is denoted by $f^n(x)$.

The successive derivative of $y=f(x)$ are denoted by

$f'(x), f''(x), f'''(x), \ldots\ldots f^n(x), \ldots\ldots$

6. 2. Notation :—The first derivative of $y=f(x)$ is denoted by $f'(x)=\frac{dy}{dx}=y_1$

The first derivative of $f'(x)=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2}=\left(\frac{d}{dx}\right)^2 y$

$$f''(x)=\frac{d^2}{dx^2}(y)=\frac{d^2y}{dx^2}=y_2$$

It is convenient to denote the operator

$\frac{d}{dx}$ by D i. e. $D=\frac{d}{dx}$

Therefore $\frac{d^2}{dx^2}=\frac{d}{dx}\left(\frac{d}{dx}\right)=D.\,D=D^2$

If the operator $\frac{d}{dx}$ is applied n times on $y=f(x)$ then n th derivative of $y=f(x)$ is denoted by

$$y_n=\left(\frac{d}{dx}\right)^n y=D^n y=y_n.$$

Thus for $y=$ the successive derivatives are denoted by

$$f'(x), f''(x), f'''(x), \cdots \quad \cdots \quad \cdots f^{(n)}(x),$$

$$y', y'', y''', \quad \cdots \quad \cdots \quad \cdots y^{(n)}$$

or, $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3} \quad \cdots \quad \cdots \quad \cdots \frac{d^ny}{dx^n}$

or, $Dy, D^2y, D^3y, \quad \cdots \quad \cdots \quad \cdots D^n y$

or, $y_1, y_2, y_3, \quad \cdots \quad \cdots \quad \cdots y_n$

Ex. Find the 3rd derivative of $y=x^7$.

Let $y=x^7$

$\therefore \frac{dy}{dx}=y_1=Dy=7x^6 \quad \therefore \frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2}=y_2=D^2y=7.6x^5$

$\therefore \frac{d^3y}{dx^3}=D^3y=7.6.5x^4$

6. 3. Standard Results (প্রামাণ্য ফল)

(1) Find the nth derivative of $y=(ax+b)^m$.

$y=(ax+b)^m$

$\therefore y_1=m.a(ax+b)^{m-1}, \; y_2=m(m-1)a^2(ax+b)^{m-2}$

$y_3=m(m-1)(m-2)a^3(ax+b)^{m-3}$ and so on.

Hence,

$D^n(ax+b)^m=m(m-1)(m-2)\cdots\cdots(m-n+1)a^n(ax+b)^{m-n}$

$\therefore y_n=D^n(ax+b)^m=\frac{\lfloor m}{\lfloor (m-n)} a^n(ax+b)^{m-n}$

$\Rightarrow$ **Cor : (A)** Let m be a positive integer.

(i) If $n<m$, then

$y_n=m(m-1)(m-2)\cdots\cdots(m-n+1)a^n(ax+b)^{m-n}$

or $y_n=\frac{\lfloor m}{\lfloor (m-n)} a^n(ax+b)^{m-n}$

(ii) If $n=m$, then

$y_n=D^n(ax+b)^n=n(n-1)(n-2)\cdots(n-n+1)a^n(ax+b)^{n-m}$
$=\lfloor n \, a^n$.

(iii) If $n>m$ or $n=m+r$, then

$y_n=D^{m+R}(ax+b)^m=D^R\{D^m(ax+b)^m\}$
$=D^R(\lfloor m.a^m)=0$.

(iv) If m is a negative integer,

Let $m=-r$, where R is a positive integer, Then

$y_n=D^n(ax+b)^m=D^n\left(\frac{1}{(ax+b)^r}\right)$

$=(-r)(-r-1)(-r-2)\cdots(r-n+1)a^n(ax+b)^{-r-n}$

$=(-1)^n r(r+1)(r+2)\cdots(r+n-1) \quad \frac{a^n}{(ax+b)^{r+n}}$

$\Rightarrow D^n\left(\frac{1}{(ax+b)^r}\right)=(-1)^n \quad \frac{\lfloor (r+n-1)}{\lfloor (r-1)} \quad \frac{a^n}{(ax+b)^{r+n}}$

When $r=1$,

$D^n\left(\frac{1}{ax+b}\right)=(-1)^n\lfloor n \frac{a^n}{(ax+b)^{n+1}}$

(v) Find the nth differential of $y=e^{ax}$

$\therefore y_1=ae^{ax}, \; y_2=a^2e^{ax}, \; y_3=a^3e^{ax}$ so on.

Hence $y_n=D^n(e^{ax})=a^ne^{ax}$

Cor :— If $a=1$, then

$y_n=D^n(e^x)=e^x$

Cor :—If $y=a^x=e^{x\log a}$

∴ $y_1=(\log a)\,e^{x\log a}$, $y_2=(\log a)^2e^{x\log a}$ and so on

Hence $y_n=D^n(a^x)=(\log a)^n\,e^{x\log a}=(\log a)^n\,a^x$

∴ $y_n=D^n(a^x)=(\log a)^n a^x$

(vi) Find the nth derivative of $y=\log(ax+b)$

∴ $y_1=\dfrac{a}{ax+b}=a(ax+b)^{-1}$, $y_2=a^2(-1)(ax+b)^{-2}$

$$y_3=(-1)(-2)(a^3)(ax+b)^{-3}=(-1)^2\lfloor\underline{2}\;a^3(ax+b)^{-3}$$

$$=(-1)^{3-1}\lfloor\underline{(3-1)}\;a^3(ax+b)^{-3}$$

$$y_4=(-1)^{4-1}\lfloor\underline{(4-1)}\,a^4(ax+b)^{-4}=\frac{(-1)^{4-1}\lfloor\underline{(4-1)}\,a^4}{(ax+b)^4}$$

and so on,

$$y_n=D^n\{\log(ax+b)\}=\frac{(-1)^{n-1}\lfloor\underline{(n-1)}\,a^n}{(ax+b)^n}$$

Cor : Put $b=0$ and $a=1$ in $\log(ax+b)$ then $y=\log_e x$.

The nth derivative is

$$y_n=D^n(\log x)=\frac{(-1)^{n-1}\lfloor\underline{(n-1)}}{x^n}$$

(vii) Find the nth derivative of $y=\sin(ax+b)$ C. U. 1983

∴ $y_1=a\cos(ax+b)=a\sin\{\frac{1}{2}\pi+(ax+b)\}$

$y_2=a^2\cos\{\frac{1}{2}\pi+(ax+b)\}=a^2\sin\{\frac{1}{2}\pi+\frac{1}{2}\pi+(ax+b)\}$

$=a^2\sin\{2.\frac{1}{2}\pi+(ax+b)\}$

$$y_3=a^3\cos\{2.\tfrac{1}{2}\pi+(ax+b)\}=a^3\sin\left\{\frac{\pi}{2}+2.\,\frac{\pi}{2}+(ax+b)\right\}$$

$=a^3\sin\{3.\frac{\pi}{2}+(ax+b)\}$, and so on,

Hence $y_n=D^n\{\sin(ax+b)\}=a^n\sin\{\frac{1}{2}n\pi+(ax+b)\}$

(viii) Find the derivative of $y=\cos(ax+b)$

Differentiation

Proceeding as in (iv), we get

$y_n=D^n\{\cos(ax+b)\}=a^n\cos\{\frac{1}{2}n\pi+(ax+b)\}$

Cor : If $a=1$, $b=0$, then

$D^n(\sin x)=\sin(\frac{1}{2}n\pi+x)$,

$D^n(\cos x)=\cos(\frac{1}{2}n\pi+x)$.

6. 4. (1) Find the nth differential co-efficient of

$$y=e^{ax}\sin(bx+c)$$

∴ $y_1=ae^{ax}\sin(bx+c)+e^{ax}\,b\cos(bx+c)$

Put $a=r\cos\phi$ and $b=r\sin\phi$ then

$r^2=a^2+b^2$ and $\tan\phi=b/a$ or $\phi=\tan^{-1}\left(\dfrac{b}{a}\right)$

Now y_1 becomes

$y_1=re^{ax}\sin(bx+c)\cos\phi+e^{ax}r\cos(bx+c)\sin\phi$

$=re^{ax}\sin(bx+c+\phi)$

Similarly

$y_2=r^2e^{ax}\sin(bx+c+2\phi)$ and so on.

$$y_n=r^ne^{ax}\sin(bx+c+n\phi)=(a^2+b^2)^{n/2}\sin\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$$

If $y=e^{ax}\sin bx$, then

$y_n=D^n(e^{ax}\sin bx)=(a^2+b^2)^{n/2}\sin(bx+n\tan^{-1}b/a)$

Similarly.

$y_n=D^n(e^{ax}\cos bx)=(a^2+b^2)^{n/2}\cos(bx+n\tan^{-1}b/a)$

6.5 Partial Fractions. Expression of the form

$\dfrac{\phi(x)}{f(x)}$, where $\phi(x)$ and $f(x)$ are both rational integral algebraic functions, can be differentiated easily if the expression is first resolved into partial fractions. This is explained with examples below.

Ex. Differentiate x times the expression

$$y=\frac{x^2-6x+1}{(x-1)(3x-2)(2x+3)}=\frac{A}{x-1}+\frac{B}{3x-2}+\frac{C}{2x+3}$$

$$\Rightarrow \quad y=\frac{-4/5}{x-1}-\frac{23/13}{3x-2}+\frac{49/65}{2x+3}$$

Therefore the nth derivative of y is

$$\therefore \quad y_n=\frac{4}{5}\frac{(-1)^n\lfloor n}{(x-1)^{n+1}}-\frac{23(-1)^n3^n\lfloor n}{13(3x-2)^{n+1}}+\frac{49(-1)^n\lfloor n\, 2^{n+1}}{65(2x+3)^{n+1}}$$

[See Ex. 6.3(1)]

Use of De Moivre's Theorem

6.6 Find the nth derivative of $y=\dfrac{1}{x^2+a^2}$ C. U. 1986

$$\therefore \quad y=\frac{1}{x^2+a^2}=\frac{1}{(x+ai)(x-ai)}=\frac{1}{2ia}\left(\frac{1}{x-ia}-\frac{1}{x+ia}\right)$$

$$\text{or,}\quad y=\frac{1}{2ia}(x-ia)^{-1}-\frac{1}{2ai}(x+ia)^{-1}$$

$$y_1=\frac{1}{2ia}(-1)(x-ia)^{-2}-\frac{1}{2ia}(-1)(x+ia)^{-2}$$

$$y_2=\frac{1}{2ia}(-1)(-2)(x-ia)^{-3}-\frac{1}{2ia}(-1)(-2)(x+ia)^{-3}$$

and so on

$$y_n=\frac{1}{2ia}(-1)^n n!\,(x-ia)^{-n-1}-\frac{1}{2ia}(-1)^n n!\,(x+ia)^{-n-1}$$

Foot Note : See Higher Algebra and Trigonometry by Shahidulla & Bhattacharjee

$$=\frac{(-1)^n\lfloor n}{2ia}\left\{\frac{1}{(x-ia)^{n+1}}-\frac{1}{(x+ia)^{n+1}}\right\}$$

Put $x=r\cos\theta$, $a=r\sin\theta$, $r^2=x^2+a^2$, $\theta=\tan^{-1}\left(\dfrac{a}{x}\right)$

$$y_n=\frac{(-1)^n n)}{2ia}\frac{1}{r^{n+1}}\left\{\frac{1}{(\cos\theta-i\sin\theta)^{n+1}}-\frac{1}{(\cos\theta+\sin\theta)^{n+1}}\right\}$$

$$=\frac{(-1)^n\lfloor n}{2iar^{n+1}}\{\cos\theta-i\sin\theta)^{-n-1}-(\cos\theta+\sin\theta)^{-n-1}$$

$$=\frac{(-1)^n\lfloor n}{2iar^{n+1}}\,i2\sin(n+1)\theta=\frac{(-1)^n n!}{2_ia^{n+1}a}\sin^{n+1}\theta,\ \sin(n+1)\theta$$

$$\therefore \quad D^n\left(\frac{1}{x^2+a^2}\right)=\frac{(-1)^n n!}{a^{n+2}}\sin(n+1)\theta\,\sin^{n+1}\theta,\ \theta=\tan^{-1}a/x$$

6.7 Trigonometrical Transformation.

In finding nth derivative of expressions like $\sin^p\theta\ \cos^q\theta$, $\sin^q\theta$ $\sin^n\theta$, $\cos^n\theta$ etc, we are to transform the expression into a sum by Trigonometry. In such cases the substitutions are

$z=\cos\theta+i\sin\theta$, then $z^{-1}=\cos\theta-i\sin\theta$

$\therefore$ $2\cos\theta=z+1/z$, $2i\sin\theta=z-1/z$

The method is explained below with an example.

Ex. Find the nth derivative of $\cos^3x\sin^2x$

Let $z=\cos x+i\sin x$, then $z^{-1}=\cos x-i\sin x$

$\therefore$ $2\cos x=z+z^{-1}$, $2i\sin x=z-z^{-1}$......(1)

Also $z^n=(\cos x+i\sin x)^n=\cos nx+i\sin nx$

$z^{-n}=(\cos x+i\sin x)^{-n}=\cos nx-i\sin nx$

$\therefore$ $2\cos nx=z^n+z^{-n}$, $2i\sin nx=z^n-z^{-n}$... ... [2]

Thus $2^2 \cos^2 x \; 2^3 i^3 \sin^3 x = (z+z^{-1})^2 (z-z^{-1})^3$

or, $2^5 i^3 \cos^2 x \sin^3 x = \left(z^5 - \frac{1}{z^5}\right) - \left(z^3 - \frac{1}{z^3}\right) - 2\left(z - \frac{1}{z}\right)$

$$= 2i \sin 5x - 2i \sin 3x - 2.2i \sin x$$

or, $-2^5 \cos^2 x \sin^3 x = -2(\sin 5x - \sin 3x - 2 \sin x)$

∴ $D^n(\cos^2 x \sin^3 x) = -2^{-5}. 2\{5^n \sin(\tfrac{1}{2}n\pi + 5x)$

$$-3^n \sin(\tfrac{1}{2}n\pi + 3x) - 2\sin(\tfrac{1}{2}\pi + x)\}$$

$= (1/16)[2 \sin(\tfrac{1}{2}n\pi + x) + 3^n \sin(\tfrac{1}{2}n\pi + 3x) - 5^n \sin(\tfrac{1}{2}n\pi + 5x)]$

6. 8. Leibnitz's Theorem (লিবনীটজের উপপাদ্য)

With the help of this theorem we can find the nth differential co-efficient of a product of two functions.

The theorem states that if u and v are two function of x and each has derivatives upto order n, then the nth derivatives of the product of the functions is given by

$$D^n(uv) = (D^n u)v + {}^nC_1 D^{n-1} u Dv + {}^nC_2 D^{n-2} u D^2 v + \dots\dots + {}^nC_r D^{n-r} u D^r v + \dots + u. D^n v$$

or, $(uv)_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \dots\dots + {}^nC_r u_{n-r} v_r + \dots\dots + uv_n$

Where the suffixes of u and v indicate the order of derivatives with respect to x.

Let $y = uv$. Differentiate it $w.\ r.$ to x

$$\frac{dy}{dx} = \frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx} = vu_1 + uv_1 \text{ or. } y_1 = u_1 v + uv_1$$

$$\therefore \quad y_2 = \frac{dy_1}{dx} = \frac{d}{dx}(u_1 v) + \frac{d}{dx}(uv_1)$$

or, $y_2 = (u_2 v + u_1 v_1) + (u_1 v_1 + uv_2) = u_2 v + 2u_1 v_1 + uv_2$

$$= u_2 v + {}^2C_1 u_{2-1} v_1 + uv_2$$

Similarly,

$y_3 = u_3 v + 3u_2 v_1 + 3u_1 v_2 + uv_3 = u_3 v + {}^3C_1 u_{3-1} v_1 + {}^3C_2 u_{3-2} v_2 + v_3$

Because ${}^3C_1 = {}^3C_2 = 3$.

Thus we see that the theorem holds good for $n = 1$, 2 and 3.

Let us assume that the theorem is true for $n = m$ (a positive integer). Then

$$y_m = u_m v + {}^mC_1 u_{m-1} v_1 + {}^mC_2 u_{m-2} v_2 + \dots \dots + {}^mC_r u_{m-r} v_r + \dots \dots\dots + uv_m \dots \dots \dots \quad (1)$$

Differentiate (1) with respect to x :

$y_{m+1} = u_{m+1} v + (u_m v_1 + {}^mc_1 u_m v_1) + ({}^mc_1 u_{m-1} v_2 + {}^mc_2 u_{m-1} v_2)$

$+ \dots \dots + {}^mc_{r-1} u_{m-r+1} v_r + {}^mc_r u_{m-r+1} v_r + \dots \dots$

$= u_{m+1} v + ({}^mc_1 + 1) u_m v_1 + ({}^mc_2 + {}^mc_1) u_{m-1} v_2 + \dots \dots$

$+ ({}^mc_r + {}^mc_{r-1}) u_{m-r+1} v_r + \dots \dots + uv_{m+1}$

But ${}^mc_r + {}^mc_{r-1} = {}^{m+1}c_r$ and ${}^mc_1 + 1 = {}^{m+1}c_1$

Therefore

$$y_{m+1} = u_{m+1} v + {}^{m+1}c_1 u_m v_1 + {}^{m+1}c_2 u_{m-1} v_2 + \dots \dots + {}^{m+1}c_r u_{m-r+1} v_r + \dots \dots uv_{m+1} \dots \dots (2)$$

If the theorem holds good for $n = m$, then, it, will also hold good for $n = m+1$. But it was proved that the theorem is true for $n = 1, 2$ and 3......When the theorem is true for $n = 3$, it is also true for $n = 4$. When it is true for $n = 4$, it is true for $n = 5$ and so on. Hence the theorem must be true for every positive integral value of n.

Ex. Find the nth derivative of $x^2 e^{2x}$

Let $y = uv = x^2 e^{2x}$

Where $u = e^{2x}$, $v = x^2$.

$(uv)_n$ . . . $- + uv_n$

Then $u_1=2e^{2x},\ u_2=2^2e^{2x},\ \ldots\ldots,\ u_r=2^re^{2x}\ldots\ldots$

$v_1=2x,\ v_2=2,\ v_3=v_4=\ldots\cdots 0$

$$\therefore\ y_n=u_nv+{}^nc_1u_{n-1}v_1+{}^nC_2u_{n-2}v_2+{}^nC_3u_{n-3}v_3+\cdots\cdots$$
$$+{}^nC_ru_{n-r}v_r+\ldots\ldots+uv_n$$

$$=2^ne^{2x}x^2+\frac{n}{1}2^{n-1}.\,e^{2x}\,2x+\frac{n(n-1)}{1.2}2^{n-2}e^{2x}.2+0$$

$$=2^ne^{2x}\left[x^2+nx+\frac{n(n-1)}{4}\right]$$

6.9. Some Important Symbolic Operators.

$F(D)=A_nD^n+A_{n-1}D^{n-1}+\ldots\ldots\ldots+A_1D+A.$

$=\Sigma A_rD^r$, where A_r is independent of D

$F(D)$ is any rational integral algebraic function of D or $\frac{d}{dx}$

The following result may be obtained from the above function

(i) $F(D)e^{ax}=F(a)e^{ax}$

(ii) $F(D)e^{ax}V=e^{ax}\,F(D+a)\,V$, where V is the function of a

(iii) $F(D^2)\sin(ax+b)=F(-a^2)\sin(ax+b)$

(iv) $F(D^2)\cos(ax+b)=F(-a^2)\cos(ax+b)$

Proof : (1) $F(D)e^{ax}=(A_nD^n+A_{n-1}D^{n-1}+\cdots+A_1D+A_0)e^{ax}$

We know that $D^re^{ax}=a^re^{ax}$

$\therefore F(D)e^{ax}=e^{ax}(a^nA_n+\ldots+a^{n-1}A_{n-1}\ldots+aA_1+A_0)=F(a)e^{ax}$

Proof. (ii) By Leibnitz's Theorem we have

$$D^n(e^{ax}V)=VD^ne^{ax}+{}^nC_1D^{n-1}e^{ax}DV+{}^nC_2D^{n-2}e^{ax}D^2V+\ldots\ldots+e^{ax}D^nV$$

$$=e^{ax}(a^nV+{}^nC_1a^{n-1}DV+{}^nC_2a^{n-2}D^2V+\ldots\ldots+D^nV)$$

By $D^re^{ax}=a^re^{ax}$

$$=e^{ax}(a+D)^nV$$

V কে ডানদিকে রাখা হইয়াছে কারণ ইহা দ্বারা D, D^2, D^3 ইত্যাদির সংগে এর অপারেশন হইবে

$$D^r(e^{ax}V)=e^{ax}(a+D)^rV$$

এখন $F(D)\{e^{ax}V\}=\Sigma A_rD(e^{ax}V)=e^{ax}\Sigma A(a+D)^rV$

$$=e^{ax}F(D+a)V$$

Proof. (iii) $F(D^2)$ ফাংশানটির চলক D^2

মনে করি $F(D^2)=\sum_{r=0}^{n}A_rD^{2r}$

এখন $D\sin(ax+b)=a\cos(ax+b)$

$D^2\sin(ax+b)=(-a^2)\sin(ax+b)=(-a^2)\sin(ax+b)$

$D^3\sin(ax+b)=-a^3\cos(ax+b)$

$D^4\sin(ax+b)=(-a^2)^2\sin(ax+b)$

... ...

$D^{2r}\sin(ax+b)=(-a^2)^r\sin(ax+b)$

$\therefore\ F(D)^2\sin(ax+b)=\Sigma A_r(-a^2)^r\sin(ax+b)$

$$=F(-a^2)\sin(ax+b)$$

অনুরূপ ভাবে প্রমাণ করা যায় যে

(iv) $F(D^2)\cos(ax+b)=F(-a^2)\cos(ax+b)$

Examples

Ex. Find nth differential co-efficient of $\sin^3x$. D. U. 1954

Let $y=\sin^3x=\frac{1}{4}(3\sin x-\sin 3x)$

$\therefore\ y_n=\frac{3}{4}\sin(\frac{1}{2}n\pi+x)-\frac{1}{4}.3^n\sin(\frac{1}{2}n\pi+3x)$

Ex. 2. Find the nth derivative of $\frac{1}{(x-1)^2(x-2)}$

Let $y=\frac{1}{(x-1)^2(x-2)}$

Put $x-1=z$ or, $x=1+z$; $x-2=1+z-2=z-1$

$$y=\frac{1}{z^2(z-1)}=\frac{A}{z}+\frac{B}{z^2}+\frac{C}{z-1}$$

or, $1=Az(z-1)+B(z-1)+Cz^2$

Equate co-efficients of z^2, z and constant terms from both sides. Then $A=-1$, $B=-1$, $C=1$,

$$\text{Now } y=\frac{-1}{z}-\frac{1}{z^2}+\frac{1}{z-1}=-\frac{1}{x-1}-\frac{1}{(x-1)^2}+\frac{1}{x-2}$$

Differentiating n times we have

$$y_n=-\frac{(-1)^n n!}{(x-1)^{n+1}}-\frac{(-1)^n n!}{(x-1)^{n+3}}+\frac{(-1)^n n!}{(x-2)^{n+1}}$$

$$=(-1)^{n+1} n!\left[\frac{1}{(x-1)^{n+1}}+\frac{1}{(x-1)^{n+3}}-\frac{1}{(x-2)^{n+1}}\right]$$

Ex. 3, Find the nth differential co-efficient of $y=\sin^4 x \cos^3 x$

Let $\cos x+i\sin x=z$, then $\cos x-i\sin x=z^{-1}$

$\cos rx+i\sin rx=z^r$, $\cos rx-i\sin rx=z^{-r}$

$\therefore$ $2\cos x=z+z^{-1}$ $\quad 2i\sin x=z-z^{-1}$

$2\cos rx=z^r+z^{-r}$ $\quad 2i\sin rx=z^r-z^{-r}$

Now $(2i\sin x)^4(2\cos x)^3=(z-z^{-1})^4(z+z^{-1})^3$

or, $2^7 i^4 \sin^4 x\cos^3 x=(z-4z^2+6-4/z^2+1/z^4)(z^3+3z+3/z+1/z^3)$

or, $2^7\sin^4 x\cos^3 x=(2^7+1/z^7)+(-4+3)(z^5+1/z^5)$

$+(6+3-12)(z^3+1/z^3)+(18-12-4+1)(z+1/z)$

or, $2^7\sin^4 x\cos^3 x=2\cos 7x-2\cos 5x-3.2\cos 3x+3.2\cos x$

or, $\sin^4 x\cos^3 x=(\frac{1}{2})^6(\cos 7x-\cos 5x-3\cos 3x+3\cos x)$

Hence nth derivative of y is

$y^n=(\frac{1}{2})^6\{7^n\cos(\frac{1}{2}n\pi+7x)-5^n\cos(\frac{1}{2}n\pi+5x)$

$-3.3^n\cos(\frac{1}{2}n\pi+3x)+3\cos(\frac{1}{2}n\pi+x)\}]$

Ex. 4. Find nth derivative of $\sin^7 x$.

Let $z=\cos x+i\sin x$, $1/z=\cos x-i\sin x$

$\therefore$ $2\cos x=z+z^{-1}$, $2i\sin x=z-z^{-1}$

$\therefore$ $2^7 i^7\sin^7 x=(z-z^{-1})^7=(z^7-1/z^7)-7(z^5-1/z^5)$

$+21(z^3-1/x^3)-35(z-1/z)$

$=2i\sin 7x-7.2i\sin 5x+2.21i\sin 3x-35.2i\sin x$

or, $\sin^7 x=(\frac{1}{2})^6(\sin 7x-7\sin 5x+21\sin 3x-35\sin x)$

$\therefore$ $y_n=(\frac{1}{2})^6(7^n\sin\{\frac{1}{2}n\pi+7x)-7.5^n\sin\frac{1}{2}(n\pi+5)$

$+21.3^n\sin(\frac{1}{2}n\pi+3x)-35\sin(\frac{1}{2}n\pi+x)\}$

Ex. 5. If $y=\tan^{-1}x$, show that $y_3=2\cos(\pi+3y)\cos^3 y$

we have $x=\tan y$ ... ... (1)

Differentiate it w. r. to x :

$$1=\sec^2 y\frac{dy}{dx}\Rightarrow y_1=\cos^2 y \quad \cdots\ \cdots\ \ldots\ \ldots \ (2)$$

$$=-2\cos y\sin y\frac{dy}{dx}=-2\cos^3 y\sin y=y_2 \quad [\text{ by } (2)]$$

$$\therefore\ y_3=-\left(2\cos^4 y\frac{dy}{dx}-6\cos^2 y\sin^2 y\frac{dy}{dx}\right)$$

$=-2\cos^3 y(\cos^2 y-3\sin^2 y)\cos^2 y$ $\quad$ [by (2)]

$=-2\cos^3 y\{\cos^3 y-3\cos y(1-\cos^2 y)\}$

$=-2\cos^3 y(4\cos^3 y-3\cos y)=-2\cos^3 y\cos 3y$

$=2\cos(\pi+3y)\cos^3 y$ (Proved.)

Ex. 6. If $y=e^{ax}\cos bx$, prove that $y_2-2ay_1+(a^2+b^2)y=0$

Let $y=e^{ax}\cos bx$ $\quad$ (1)

$y_1 = ae^{ax}\cos bx - be^{ax}\sin bx = ay - be^{ax}\sin bx$ by (1)

$\therefore\ y_2 = ay_1 - abe^{ax}\sin bx - b^2 e^{ax}\cos bx$ ... (2)

$= ay_1 - a(be^{ax}\sin bx) - b^2 y$ [by (1)]

$= ay_1 - a(ay - y_1) - b^2 y$ ... [by (2)]

or, $y_2 - 2ay_1 + (a^2+b^2)y = 0$ (Proved)

Ex. 7. Find the nth derivative of $x^3 \sin x$

Let $y = x^3 \sin x$, Then

$y_n = D^n(\sin x.\ x^3)$

$= D^n(\sin x).\ x^3 + {}^nC_1\, D^{n-1}(\sin x).(3x^2)$

$+ {}^nC_2\, D^{n-2}(\sin x).\ 6x + {}^nC_3\, D^{n-3}(\sin x)6$

$= \sin(\tfrac{1}{2}n\pi + x)x^3 + {}^nC_1 \sin\{\tfrac{1}{2}(n-1)\pi + x\}3x^2$

$+ {}^nC_2 \sin\{\tfrac{1}{2}(n-2)\pi + x\}6x + {}^nC_3 \sin\{\tfrac{1}{2}(n-3)\pi + x\}6$

or, $y_n = x^3 \sin\{\tfrac{1}{2}n\pi + x\} + 3nx^2 \sin\{\tfrac{1}{2}(n-1)\pi + x\} +$

$3xn(n-1)\sin\{\tfrac{1}{2}(n-2)\pi + x\} + n(n-1)(n-2)\sin\{\tfrac{1}{2}(n-3)\pi + x\}$

Ex. 8; If $y = x^2 e^x$ show that

$y_n = \tfrac{1}{2}n(n-1)y_2 - n(n-2)y_1 + \tfrac{1}{2}(n-1)(n-2)y$

Differentiate $y = x^2 e^x$ *w. r.* to x

$\therefore\ y_1 = e^x x^2 + e^x 2x = e^x(x^2 + 2x)$

$y_2 = e^x(x^2 + 2x) + e^x(2x+2) = e^x(x^2 + 4x + 2)$

$\therefore\ y_n = e^x x + ne^x 2x + \dfrac{n(n-1)}{2}e^x 2 = e^x\{x^2 + 2nx + n(n-1)\}$ ...(4)

Now $e^x\{x^2 + 2nx + n)(n-1)\} = e^x(x^2 + 4x + 2)\tfrac{1}{2}(n^2 - n) -$

$e^x(x^2 + 2x)(n^2 - 2n) + e^x\ \dfrac{n^2 - 3n + 2}{2} n^2$

$= \tfrac{1}{2}n(n-1)y_2 - n(n-2)y_1 + \tfrac{1}{2}(n-1)(n-2)y$ by [1], [2], [3]

$y_n = \tfrac{1}{2}n(n-1)y_2 - n(n-2)y_1 + \tfrac{1}{2}(n-1)(n-2)y.$ Proved.

Ex. 9. Differentiate n times the equation

$(1-x^2)y_2 - xy_1 + a^2 y = 0$ D. U. 1982

$D^n[(1-x^2)y_2] = (1-x^2)y_{n+2} - ny_{n+1}2x - n(n-1)y_n1;$

$D^n xy_1 = y_{n+1}x + ny_n$ $\therefore\ D_n a^2 y = a^2 y_n$

Therefore $D^n(1-x^2)y_2 - D_1 xy_1 + D^n a^2 y = (1-x^2)y_{n+2}$

$-(2n+1)xy_{n+1} - (n^2 - a^2)y_n = 0$

or, $(1-x^2)y_{n+2} - \{2n+1)xy_{n+1} - (n^2 - a^2)y_n = 0$

Ex. 10. If $y = \sin(m \sin^{-1}x)$, show that

(i) $(1-x^2)y_2 = xy_1 - m^2 y$ D. U. 1983

(ii) $(1-x^2)y_{n+2} = (2n+1)xy_{n+1} + (n^2 - m^2)y_n$ D. H. '62, R. U. 67

$y = \sin(m \sin^{-1}x)$ ... ... (1)

$y_1 = \cos(m \sin^{-1}x)\ \dfrac{m}{\sqrt{(1-x^2)}}$ ... ... (2)

or, $y_1\sqrt{(1-x^2)} = m\cos(m \sin^{-1}x)$

or, Squaring,

$y_1^2(1-x^2) = m^2\cos^2(m\sin^{-1}x) = m^2[1 - \sin^2(m\sin^{-1}x)]$

or, $y_1^2(1-x^2) = m^2(1-y^2)$

Differentiating both sides *w. r.* to x

$2y_1y_2(1-x^2) + y_1^2(-2x) = m^2(-2yy_1)$

or, dividing throughout by $2y_1$,

$y_2(1-x^2) - xy_1 = -m^2 y$

$\Rightarrow (1-x^2)y_2 = xy_1 - m^2 y.$ (proved)

Differentiate it n times :

$(1-x^2)y_{n+2} - n(2x)y_{n+1} - \dfrac{n(n-1)}{2}(2)y_n = y_{n+1}x + ny_n(1) - m^2 y_n$

or, $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (n^2 - m^2)y_n = 0$ (proved).

Ex. 11. If $y = e^{a\sin^{-1}x}$ prove that C. U, 1984

(i) $(1-x^2)y_2 - xy_1 - a^2 y = 0$ C. H. 1992

(ii) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (x^2 + a^2)y_n = 0$

and hence find the value of y_n where $x = 0$

R. U. 1964, '66 ; D. U. 1952

Differentiate $y=e^{a\sin^{-1}x}$ ... ... (1)

∴ $y_1=e^{a\sin^{-1}x}\dfrac{a}{\sqrt{(1-x^2)}}=\dfrac{ay}{\sqrt{(1-x^2)}}$ ... ... (2)

or, $y_1^2(1-x^2)=a^2y^2$

∴ $2y_1y_2(1-x^2)-2xy_1^2=2y_1a^2$ [Differentiating both sides w. r. to x]

$(1-x^2)y_2-xy_1-ya^2=0$ ... (proved)

Differentiate it n times :

∴ $(1-x^2)y_{n+2}-n.\,2x\,y_{n+1}-\dfrac{n(n-1)}{2}.\,2y_n$

$-xy_{n+1}-ny_n-a^2y_n=0$

or, $(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2+a^2)y_n=0$ (4) proved

Put $x=0$ in (1), (2), (3), (4), then

$(y)_0=1$, $(y_1)_0=a$ and $(y_{n+2})_0=(n^2+a^2)(y_n)_0$ ... ... (5)

Putting $n=n-2$ in (5),

$(y_n)_0=\{(n-2)^2+a^2\}(y_{n-2})_0$ ... ... (5)

$=\{(n-2)^2+a^2\}\{(n-4)^2+a^2\}(y_{n-4})_0$ etc.

But $(y)_0=1$, $(y_2)_0=a^2$, $(y_4)_0=(2^2+a^2)a^2$

when n is even.

∴ $(y_n)_0=\{(n-2)^2+a^2\}\{(n-4)^2+a^2\}\ldots(4^2+a^2)(2^2+a^2)a^2$

(when n is even)

When n is odd, $(y_1)_0=a$, $(y_3)_0=(1^2+a^2)a$

∴ $(y_n)_0=\{(n-2)^2+a^2\}\{(n-4)^2+a^2\{\ldots(3^2+a^2)(1^2+a^2)(y_1)_0$

$=\{(n-2)^2+a^2\}\{(n-4)^2)+a^2\}\cdots\cdots(3^2+a^2)(1^2+a^2)a$

Ex. 12. If u, v, w be functions of t, and if suffixes denote differentiations with regard to t, prove that.

$$\frac{d}{dt}\begin{vmatrix}u_1&v_1&w_1\\u_2&v_2&w_2\\u_3&v_3&w_3\end{vmatrix}=\begin{vmatrix}u_1&v_1&w_1\\u_2&v_2&w_2\\u_4&v_4&w_4\end{vmatrix}$$

[For the differentiation of Determinant see authors's Higher Algebra.

When a determinant is differentiated, there will be as many determinants as the order of the determinant and each determinant contains only one row or one column differentiated once only.]

$$\frac{d}{dt}\begin{vmatrix}u_1&v_1&w_1\\u_2&v_2&w_2\\u_3&v_3&w_3\end{vmatrix}=\begin{vmatrix}\frac{d}{dt}u_1&\frac{d}{dt}v_1&\frac{d}{dt}w_1\\u_2&v_2&w_2\\u_3&v_3&w_3\end{vmatrix}$$

$$+\begin{vmatrix}u_1&v_1&w_1\\\frac{d}{dt}u_2&\frac{d}{dt}v_2&\frac{d}{dt}w_2\\u_3&v_3&w_3\end{vmatrix}+\begin{vmatrix}u_1&v_1&w_1\\u_2&v_2&w_2\\\frac{d}{dt}u_3&\frac{d}{dt}v_3&\frac{d}{dt}w_3\end{vmatrix}$$

$$=\begin{vmatrix}u_2&v_2&w_2\\u_2&v_2&w_2\\u_3&v_3&w_3\end{vmatrix}+\begin{vmatrix}u_1&v_1&w_1\\u_3&v_3&w_3\\u_3&v_3&w_3\end{vmatrix}+\begin{vmatrix}u_1&v_1&w_1\\u_2&v_2&w_2\\u_4&v_4&w_4\end{vmatrix}$$

$$=\begin{vmatrix}u_1&v_1&w_1\\u_2&v_2&w_2\\u_4&v_4&w_4\end{vmatrix}$$ proved.

Ex. 13. If $y=px$ and $z=qx$, all the variables are the funcions of t, then prove that

$$\begin{vmatrix}x&y&z\\x_1&y_1&z_1\\x_2&y_2&z_2\end{vmatrix}=x^3\begin{vmatrix}p_1&q_1\\p_2&q_2\end{vmatrix}$$: p_1, p_2 etc. indicate the successive differentiation.

Now $y=px$ ∴ $y_1=p_1x+x_1p$, $y_2=px_2+2p_1x_1+p_2x$

$$z=qx \quad \therefore \quad z_1=qx_1+q_1x,\ z_2=qx_2+2q_1x_1+q_2x$$

Here $\begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \begin{vmatrix} x & px & qx \\ x_1 & px_1+p_1x & qx_1+q_1x \\ x_2 & px_2+2p_1x_1+p_2x_1 & qx_2+2q_1x_1+q_2x \end{vmatrix}$

$c_2-pc_1,\ c_3-qc_1$

$$=\begin{vmatrix} x & 0 & 0 \\ x_1 & p_1x & q_1x \\ x_2 & 2p_1x_1+p_2x & 2q_1x_1+q_2x \end{vmatrix} = x\begin{vmatrix} p_1x & q_1x \\ 2p_1x_1+p_2x & 2q_1x_1+p_2x \end{vmatrix}$$

$$=x^2\begin{vmatrix} p_1 & q_1 \\ 2p_1x_1+p_2x & 2q_1x_1+q_2x \end{vmatrix} = x^2\begin{vmatrix} p_1 & q_1 \\ p_2x & q_2x \end{vmatrix};\quad R_1-2x_1R_1$$

$$=x^3\begin{vmatrix} p_1 & q_1 \\ p_2 & q_2 \end{vmatrix}$$

Exercise

Find y_n in the following cases.

1. $y=(2x+3)^m$
2. $y=\dfrac{x^3}{(x-1)(x-2)}$
3. $y=\dfrac{c}{(3-5x)}$
4. $y=\dfrac{x^2}{(x-1)^3(x-2)}$
5. $y=\dfrac{x^2}{(x-1)^2(x+2)}$
6. $y=\dfrac{1}{1-5x+6x^2}$
7. $y=\sin^5x\cos^8x$
8. $y=\sin 2x \sin 3x$
9. $y=e^{ax+b}$
10. $y=\sin^2x\sin 2x$
11. $y=\sin^6x$
12. $y=\cos^5x$
13. $y=e^{2x}\cos^2x$
14. $(i)\ y=e^{3x}\sin 4x$

13(*i*) If $y=\dfrac{ax^2+bx+c}{1-x}$ (*ii*) $y=e^{ax}\sin(p-qx)$

Then prove that $(1-x)y_3=3y_2$ R. U. 1988

15. $y=\dfrac{x}{x^7+a^2}$
16. $y=\tan^{-1}\dfrac{x}{a}$ R.H. 1988

17. (*i*) $y=\tan^{-1}\dfrac{1+x}{1-x}$ (*ii*) $y=\dfrac{1-x}{1+x}$
18. $y=\tan^{-1}\dfrac{2x}{1-x^2}$
19. $y=\sin^{-1}\dfrac{2x}{1+x^2}$
20. $y=e^x\log x$
21. $y=e^x(ax+b)^3$
22. (i) $y=x^2\cos x$
 (ii) $y=x^n\log bx$ D. U. 1954
23. If $y=\sin x$, prove that $4\dfrac{d^3}{dy^3}\cos^7 x=105\sin 4x$
24. If $y=\sin nx+\cos nx$, show that R.U. 19988
 $y_r=n^r\sqrt{\{1+(-1)^r\sin 2nx\}}$
25. If $y=ax^{n+1}+bx^{-n}$, prove that $x^2y_2=n(n+1)y$
26. If $y=\tan^{-1}x$, show that $(y_{10})=0$
27 If $y=x^4\log x$, prove that $y=24\log x+50$
(i) If $y=x^{n-1}\log x$. prove that $y_n=\lfloor(n-1)/x$ R. U. 1987
28. Find $(y_n)_0$ if $y=e^{m\cos^{-1}x}$
29. **Find the derivative at $x=0$. if**
 $y=\{x+\sqrt{(1+x^2)}\}^m$
30. Find $(y_n)_n$ when $y=\sin(a\sin^{-1}x)$
31. If $y=x^{2n}$ where n is a positive integer, show that
$y_n=2^n\{1.3\,5\ldots\ldots\ldots(2n-1)\}x^n$ R. U. 1987
31. (i) Show that $y=e^x\sin x$ is the solution of $y_4+4y=0$
 (ii) Show that $y=\sin h\,(m\sin h^{-1}x)$ satisfies the equation $(1+x)^2y_2+xn_1=m^2y$
32, If $y=\sin^{-1}x$, show that $(y_n)_0=0$. or,
 $(n-2)^2(n-4)^2\ldots5^2.3^2.1^2$ when n is even or odd
33. If $y=A\tan\frac{1}{2}\theta+B(2+\theta\tan\frac{1}{2}\theta)$, when A and B are any

constants, prove that

$(1+\cos\theta)y_2=y$ R. U. (H) 1962

34. Prove that $\frac{1}{n!}\left(\frac{d}{dx}\right)^n \frac{1}{x(1-x)}=\frac{(-1)^n}{x^{n+1}}+\frac{1}{(1+x)^{n+2}}$

R. U. (H) 1962

35. Show that $\frac{d^{n+1}}{dx^{n+1}}(x^n\log x)=\frac{n!}{x}$ D. U. 1957, D. U. 1969

36. If $y=\sin(\sin y)$, prove that

$\frac{d^2y}{dx^2}+\tan x\frac{dy}{dx}+y\cos^2x=0$ D. U. 1954

37. If $y=m\cos^{-1}x$, show that

$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2+m^2)y_n=0$

38. If $y=ax\sin x$, prove that $x^2y_2-2xy_1+(x^2+2)y=0$

39. If $y=\sin\{a\log(x+b)\}$ prove that

(i) $(x+b)^2y_2+(x+b)y_1+a^2y=0$ N. U. 1995, 94

(ii) $(x+b)^2y_{n+2}+(2n+1)(x+b)y_{n+1}+(n^2+a^2)y_n=0$

C. U 1980, D. U. 1954

40. If $y=A\cos\{m\sin^{-1}(ax+b)\}$ prove that

$\{1-(ax+b)^2\}y_{n+2}-(2n+1)a(ax+b)y_{n+1}+(m^2-n^2)a^2y_n=0$

D. U. 1964

41. If $y=\tan^{-1}x$, prove that

$(1+x^2)y_{n+2}+2(n+1)xy_{n+1}+n(n+1)y_n=0$

42. If $y=a\cos(\log x)+b\sin(\log x)$, show that C. U. 1993

$x^2y_{n+2}+(2n+1)xy_{n+1}+(n^2+1)y_n=0$ D. H. 1961

D. U. 1991

43. If $y=e^{x\cos a}\sin(x\sin a)$ prove that

$y_{n+2}-2y_{n+1}\cos a+y_n=0$ D. H. 1955

44 If $y=(\sinh^{-1}x)^2$, prove that

$(1+x^2)y_{n+2}+(2n+1)xy_{n+2}+x^2y_n=0$ D. H. 1959

45. If $y\sqrt{(1-x^2)}=\sin^{-1}x$, prove that

$(1-x^2)y_{n+1}-(2n+3)xy_n-n^2y_{n-1}=0$ D. U. 1958

46. If $y=\cos\{\log(1+x)^2\}$ prove that N. U. 1994

$(1+x)^2y_{n+2}+(2n+1)(1+x)y_{n+1}+(n^2+1)y_n=0$

D. U. 1951

47. Show that the nth derivative of the differential equation.

$x^2y_2+xy_1+(a^2-m^2)y=0$ is

$x^2y_{n+2}+(2n+1)xy_{n+1}+(n^2-a^2-m^2)y_n=0$

(i) If $\log y=\tan^{-1}x$, show that

$(1+x^2)y_{n+2}+(2nx+2x-1)y_{n+1}+n(n+1)y=0$ C.U. 1980, '84

48. If $y=x^2\sin x$, prove that N. U. 1994

$y_n=(x^2-n^2+n)\sin(x+n\pi/2)-2nx\cos(x+n\pi/2)$

49. Differentiate n times the equation.

[i] $x^2(d^2y/dx^2)+x(dy/dx)+y=0$ [ii] $(1+x^2)y_2+(2n-1)y_1=0$

50. If $x=\sin\left(\frac{1}{m}\log y\right)$ or, $y=e^{m\sin^{-1}x}$

Show that

$(1-x^2)y_{n+2}-(2n+1)y_{n+1}-(n^2+m^2)y_n=0$ D. U. 1986

Show that the value of y_n at $x=0$ i. e., D. H. 87

$(y_n)_0=m(m^2+2^2)(m^2+3^2)\ldots\{m^2+(n-2)^2\}$ when n is odd

or, $m^2(m^2+2^2)(m^2+4^2)\ldots(m^2+\{n-2)^2\}$ when n is even

50. If $y=e^{2\sin^{-1}x}$ prove that C. H. 1987

(i) $(1-x^2)y_2-xy_1-4y=0$

(ii) $(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2+4)y=0$

51. If $y=(x^2-1)^n$ prove that

$$(y^2-1)y_{n+2}+2xy_{n+1}-n(n+1)y_n=0$$

Hence if $u=\frac{d^n}{dx^n}(y^2-1)^n$

Show that $\frac{d}{dx}\left\{(1-x^2)\frac{du}{dx}\right\}+n(n+1)u=0$

52. If $y^{1/m}+y^{-1/m}=2x$, prove that D. H. 1989, D.U. 1990

$$(x^2-1)y_{n+2}+(2n+1)yx_{n+1}+(n^2-m^2)y_n=0$$ R.U. 1960

53. If $x=(A+Bt)e^{-nt}$, prove that D.U. 1984

$$\frac{d^2x}{dt^2}+2n\frac{dx}{dt}+n^2x=0$$ D.U. 1962

54. If $y=\sin^{-1}x$, then prove that

$$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-n^2y_n=0$$

(i) If $y=(a\sin^{-1}bx)^2$, obtain an equation connecting y, y_1, and y_2

Apply Leibnitz Theorem on this equation and a relation connecting y_n, y_{n+1}, y_{n+2}.

55. If $n=\tan^{-1}x$, prove that

$$(1+x^2)\frac{d^2u}{dx^2}+2x\frac{du}{dx}=0$$ C. U. 1992

Hence determine the nth derivatives of u, with respect to x when $x=0$ R.U. 1958

56. If $y=x^m\log x$, show that $xy_1=my+x^m$ where $y_1=dy/dx$

Differentiate this equation n times where $n>m$

57. If $y=\{\log\{x+\sqrt{(1+x^2)}\}^2$ Prove that

[i] $(1+x^2)y_2+xy_1=2$

[ii] $(1+x^2)y_{n+2}+(2n+1)xy_{n+1}+n^2y_n=0$

58. If $y=x^2+\frac{1}{x^2}$ show that $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-4y=0$ R.U. 1962

59. Show that $\frac{d^n}{dx^n}\left(e^{\frac{1}{2}x^2}\right)=u_n(x)e^{\frac{1}{2}x^2}$ where $u_n(x)$ is a polynomial of degree n. Establish the recurrence relation. D.H. 1962

$u_{n+1}=xu_n+nu_{n-1}$ and hence obtain the differential equation $u''_n+xu'_n-nu_n=0$ satisfied by $u_n(x)$

(i) If $x=\tan(\log y)$, prove that

$$(1+x^2)y_{n+1}+(2nx-1)y_n+n(n-1)y_{n-1}=0$$ C.H. 1977

60. If $y=x/(x^2+a^2)$ and $x=a\cot\theta$. show that

$$y_n=(-1)^n\frac{\lfloor n}{a^{n+1}}(\sin\theta)^{n+1}\cos(n+1)\theta$$ R.H. 1967

61. Show that

$$\frac{d^n}{dx^n}\left(\frac{\sin x}{x}\right)=\frac{P\sin(x+\frac{1}{2}n\pi)+Q\cos(x+\frac{1}{2}n\pi)}{x^{n+1}}$$

$$\frac{d}{dy^n}\left(\frac{\cos x}{y}\right)=\frac{P\cos(x+\frac{1}{2}n\pi)-Q\sin(y+\frac{1}{2}n\pi)}{y^{n+1}}$$

where $P=y^n-n(n-1)y^{n-2}+n(n-1)(n-2)(n-3)y^{n-4}$ ...

$Q=ny^{n-1}-n(n-1)(n-2)y^{n-3}+\cdots$

62. If $y=e^{\tan^{-1}x}=a_0+a_1x+a_2x^2+$ ... ...

Show that

(i) $(1+x^2)y_2+(2x-1)y_1=0$

(ii) $(1+x^2)y_{n+2}+\{2(n+1)x-1\}y_{n+1}+n(n+1)y_n=0$

(ii) $(n+2)a_{n+2}+na_n=a_{n+1}$

Sol :—See worked out example 14 of chapter VII General Theorems.

63. If $y=\sin(m\sin^{-1}x)=a_0+a_1x+a_2x^2+\cdots\ \cdots$

Show that *D. H.* 1986 N. H. 1994

(i) $(1-x^2)y_2=xy_1-m^2y$

(ii) $(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2-m^2)y_n=0$

(iii) $(n+1)(n+2)a_{n+2}=(n^2-m^2)a_n$

Sol :— see worked out Ex. 10 for (i) and (ii) and for (iii) differentiate y, then put the values of y_1 and y_2 in (i), equate the co-efficients of x^n from both sides. The result will follow.

64. If $y=e^{a\sin^{-1}x}=a_n+a_1x+a_2x^2+\cdots\ \cdots$

Show that

$(n+1)(n+2)\,a_{n+2}=(n^2+a^2)a_n$

Hints : see worked out Ex. 11, then equate co-efficients of x^n from both sides after putting the values of y_1 and y_2 in (1)

65. If $y=(\sin^{-1}x)^2=a_0+a_1x+a_2x^2+\cdots\ \cdots$

Show that $(n+1)(n+2)\,a_{n+2}=n^2a_n$

66. Prove that

$$\left(\frac{d}{dx}\right)^r e^{ax}x^n=a^{r-n}x^{n-r}\left(\frac{d}{dx}\right)^n e^{ax}x^r$$

67. Prove that if $x+y=1$.

$$\frac{d^n}{dx^n}(x^ny^n)=n!\,(y^n-{}^nc_1{}^2\,y^{n-1}x+{}^nc_2{}^2y^{n-2}x^2+\cdots\ \ldots)$$

98. Prove that if $y=\frac{\log x}{y}$, then

$$y_n=\frac{(-1)^n\,n!}{x^{n+1}}(\log x-1-\tfrac{1}{2}-\tfrac{1}{3}\cdots\ \cdots\ \cdots 1/n)$$

69. By forming in two different ways the nth derivative of x^{2n}, prove that

$$1+\frac{n^2}{1^2}+\frac{n^2(n-1)^2}{1^2.2^2}+\frac{n^2(n-1)^2(n-2)^2}{1^2.2^2 3^2}+\cdots\cdots=\frac{(2n)!}{n!^2}$$

Hint : If $y=x^{2n}$, $y_n=\frac{(2n)!}{n!}x^n$

Again $y=x^n.\,x^n$

$D^n(x^{2n})=D^n(x^n.\,x^n)=x^n\,n!$ [given series] $=\frac{(2n!)^2}{n!}$

70. If $\cot y=x$ then show that

$$\frac{d^n}{dx^n}\left\{\frac{x^n}{1+x^2}\right\}=n!\sin y\,[\sin y-{}^nc_1\cos y\sin 2y+{}^nc_2\cos^2 y\sin^3 y\cdots]$$

প্রশ্নমালা VI

নিম্নলিখিত ফাংশনগুলির y_n নির্ণয় কর :

1. $y=(2x+3)^m$
2. $y=\frac{x^3}{(x-1)(x-2)}$
3. $y=\frac{c}{(4-5x)}$
4. $y=\frac{x^2}{(x-1)^3(x-2)}$
5. $y=\frac{x^2}{(x-1)^2(x+2)}$
6. $y=\frac{1}{1-5x+6x^2}$
7. $y=\sin^5x\cos^3x$
8. $y=\sin 2x\sin 3x$

9. $y=e^{ax+b}$

10. $y=\sin^2 x \sin 2x$

11. $y=\sin^6 x$

12. $y=\cos^5 x$

13. $y=e^{2x}\cos^2 x$

14. (i) $=e^{3x}\sin 4x$

13. (i) যদি $y=\dfrac{ax^2-bx+c}{1-x}$ (ii) $y=e^{ax}\sin(p-qx)$

প্রমাণ কর যে $(1-x)y_3-3y_2$ R. U. 1933 D. U. 1954

15. $y=\dfrac{x}{x^2+a^2}$

16. $y=\tan^{-1}\dfrac{x}{a}$ R. H. 1988

17. (i) $y=\tan^{-1}\dfrac{1+x}{1-x}$ (ii) $y=\dfrac{1-x}{1+x}$

18. $y=\tan^{-1}\dfrac{2x}{1-x^2}$

19. $y=\sin^{-1}\dfrac{2x}{1+x^2}$

20. $y=e^x\log x$

21. $y=e^x(ax+b)^3$

22. (i) $y=x^2\cos x$

(ii) $y=x^5\log bx$ D. U. 1954

23. যদি $y=\sin x$, হয় তবে প্রমাণ কর যে $4\ \dfrac{d^3}{dy^3}\ \cos^7 x=105\ \sin 4x$

24. যদি $y=\sin nx+\cos nx$ হয় তবে দেখাও যে

$y_r=n^r\sqrt{\{1+(-1)^r\sin 2nx\}}$ C.H.1993 R. U. 1988

25. যদি $y=ax^{n+1}+bx^{-n}$ হয় তবে প্রমাণ কর যে $x^2y_2=n(n+1)y$

26. যদি $y=\tan^{-1}x$ হয় তবে দেখাও যে $(y_{10})=0$

27. যদি $y=x^4\log x$ হয় তবে প্রমাণ কর যে $y=24\ \log\ x+50$

28. যদি $y=e^{m\cos^{-1}x}$ হয় তবে $(y_n)_0$ এর মান নির্ণয় কর।

29. যদি $y=\{x+\sqrt{(1+x^2)}\}^m$ হয়, তবে $x=0$ বিন্দুতে n তম অন্তরক সহগ নির্ণয় কর।

30. $y=\sin(a\sin^{-1}x)$ হইলে $(y_n)_0$ নির্ণয় কর।

31. (i) দেখাও যে $y_4+4y=0$ এর সমাধান হল $y=e^x\sin x$

(ii) দেখাও যে $(1+x^2)\,y_2+xy_1=m^2y$ সমীকরণকে $y=\sin h\,(m\sin h^{-1}x)$ সিদ্ধ করে। R. U. 19

32. যদি $y=\sin^{-1}x$ হয়, তবে দেখাও যে $(y_n)_0=0$ বা, $(n-2)^2(n-4)^2\ldots5^2.3^2.1^2$ যখন n জোর অথবা বিজোড় হয়।

33. যদি $y=A\tan\frac{1}{2}\theta+B(2+\theta\ \tan\frac{1}{2}\theta)$ হয়, যেখানে A এবং B দুইটি ধ্রুবক, তবে প্রমাণ কর যে $(1+\cos\theta)y_2=y$ R.U. (H) 1962

34. প্রমাণ কর যে $\dfrac{1}{n!}\left(\dfrac{d}{dx}\right)^n\dfrac{1}{x(1-x)}=\dfrac{(-1)^n}{x^{n+1}}+\dfrac{1}{(1+x)^{n+2}}$

R. U. (H) 1962

35. দেখাও যে $\dfrac{d^{n+1}}{dx^{n+1}}(x^n\log x)=\dfrac{n!}{x}$ D. U. 1957 D. U. 1969

36. যদি $y=\sin(\sin y)$ হয় তবে প্রমাণ কর যে

$\dfrac{d^2y}{dx^2}+\tan x\ \dfrac{dy}{dx}+y\cos^2x=0$ D. U. 1954

37. যদি $y=e^{\cos^{-1}x}$, হয় তবে দেখাও যে

$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2+m^2)y_n=0$

38. যদি $y=ax\ \sin\ x$, হয় তবে প্রমাণ কর যে $x^2y_2-2xy_1+(x^2+2)y=0$

39. যদি $y=\sin\{a\log(x+b)\}$ হয় তবে প্রমাণ কর যে

(i) $(x+b)^2y_2+(x+b)y_1+a^2y=0$

(ii) $(x+b)^2y_{n+2}+(2n+1)(x+b)y_{n+1}+(n^2+a^2)y_n=0$

N.U.1994. C. U. 1980. D. U. 1953

40. যদি $y=A\cos\{n\sin^{-1}(ax+b)\}$ হয় তবে প্রমাণ কর যে

$\{1-(ax+b)^2\}y_{n+2}-(2n+1)a(ax+b)y_{n+1}+(m^2-n^2)a^2y_n=0$

D. U. 1964

41. যদি $y=\tan^{-1}x$ হয় তবে প্রমাণ কর যে

$(1+x^2)y_{n+2}+2(n+1)xy_{n+1}+n(n+1)y_n=0$

42. যদি $y=a\cos(\log x)+b\sin(\log x)$, হয় তবে দেখাও যে

$x^2y_{n+2}+(2n+1)xy_{n+1}+(n^2+1)y_n=0$ D. H. 1961

43. যদি $y = e^{x\cos a} \sin(x \sin a)$ হয়, তবে প্রমাণ কর যে D. H. 1950

$y_{n+2} - 2y_{n+1} \cos a + y_n = 0$

44. যদি $y = (\sin h^{-1} x)^2$ হয় তবে প্রমাণ কর যে D. H. 1959

$(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + x^2 y_n = 0$

45. যদি $y\sqrt{(1-x^2)} = \sin^{-1}x$ হয় তবে প্রমাণ কর যে D. U. 195

$(1-x^2)y_{n+1} - (2n+1)xy_n - n^2 y_{n-1} = 0$

46. যদি $y = \cos\{\log(1+x)\}$ হলে প্রমাণ কর যে

$(1+x)^2 y_{n+2} + (2n+1)(1+x)\, y_{n+1} + (n^2+1)y_n = 0$ D. U. 195

N.U. 1994.

47. দেখাও যে অন্তরক সমীকরণ $x^2y_2 + xy_1 + (a^2 - m^2)y = 0$ এর n-তম বৃদ্ধিহার হবে,

$x^2 y_{n+2} + (2n+1)xy_{n+1} + (n^2 - a^2 - m^2)y_n = 0$

(i) যদি $\log y = \tan^{-1} x$ হয়, তবে দেখাও যে C. U. 1980,

$(1+x^2)y_{n+2} + (2nx + 2x - 1)y_{n+1} + n(n+1)y_n = 0$

48. যদি $y = x^2 \sin x$ হয়, তবে প্রমাণ কর যে

$y_n = (x^2 - n^2 + n) \sin(x + n\pi/2) - 2nx \cos(x + n\pi/2)$

49. নিম্নলিখিত সমীকরণগুলিকে n বার অন্তরীকরণ কর,

[i] $x^2(d^2y/dx^2) + x(dy/dx) + y = 0$ [ii] $(1+x^2)y_2 + (2n-1)y_1 = 0$

50. $x = \sin\left(\frac{1}{m} \log y\right)$ হলে দেখাও যে D. U. 19

$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2 + m^2)y_n = 0$ D. H. 19

আরো দেখাও যে $x = 0$ বিন্দুতে y_n এর মান হবে

$(y_n)_0 = m(m^2+1^2)(m^2+3^2)\ldots\{m^2+(n-2)^2\}$ যখন n বিজোড়

বা, $m^2(m^2+2^2)(m^2+4^2)\ldots(m^2+\{n-2\}^2)$ যখন n জোড়

51. (i) If $y = e^{2\sin^{-1}x}$ prove that C. H. 1987

(i) $(1-x^2)y_2 - xy_1 - 4y = 0$

(ii) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+4)y = 0$

52. যদি $y = (x^2-1)^n$ হয়, তবে প্রমাণ কর যে

$(x^2-1)y_{n+2} + 2xy_{n+1} - n(n+1)y_n = 0$

তাহাতে যদি $u = \frac{d^n}{dx^n}(x^2-1)^n$ হয়, তবে দেখাও যে

$\frac{d}{dx}\left\{(1-x^2)\frac{du}{dx}\right\} + n(n+1)u = 0$

53. $y^{1/m} + y^{-1/m} = 2x$, হলে প্রমাণ কর যে D.U. 1990, D.H. 1989.

$(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2 - m^2)y_n = 0$ R. U. 1960

54. যদি $x = (A+B)e^{-nt}$ হয়, তবে প্রমাণ কর যে D. U. 1984

$\frac{d^2x}{dt^2} + 2n\frac{dx}{dt} + n^2x = 0$ D. U. 1962

55. যদি $y = \sin^{-1}x$ হয় তবে প্রমাণ কর যে

$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$

(i) If $y = (a \sin^{-1} bx)^2$, obtain an equation connecting y, y_1, and y_2

Apply Leibnitz Theorem on this equation and find a relation connecting y_n, y_{n+1}, y_{n+2},

56. যদি $u = \tan^{-1}x$ হয়, তবে প্রমাণ কর যে

$(1+x^2)\frac{d^2u}{dx^2} + 2x\frac{du}{dx} = 0$ C.U. 1992.

এবং $x = 0$ হলে n তম বৃদ্ধিহার নির্ণয় কর। R. U. 1958

57. যদি $y = x^m \log x$, হয় তবে দেখাও যে

$xy_1 = my + x^m$ যেখানে $y_1 = dy/dx$

এ সমীকরণের n-তম বৃদ্ধিহার নির্ণয় কর যখন $n > m$ হয়।

57. যদি $y=\{\log\{x+\sqrt{(1+x^2)}\}^2$ হয়, তবে প্রমাণ কর যে

[i] $(1+x^2)y_2+xy_1=2$

[ii] $(1+x^2)y_{n+2}+(2n+1)xy_{n+1}+n^2y_n=0$

58. যদি $y=x^2+\frac{1}{x^2}$ হয় তবে দেখাও যে

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-4y=0$$ R. U. 1962

59. দেখাও যে $\frac{d^n}{dx^n}\left(e^{\frac{1}{2}x^2}\right)=u_n(x)e^{\frac{1}{2}x^2}$ এখানে $u_n(x)$ হল n মাত্রার একটি বহুপদী রাশি। পৌনঃপুনিক সম্পর্কটি প্রতিষ্ঠা কর।

D. H. 19

$u_{n+1}=xu_n+nu_{n-1}$ এবং ইহা হতে $u_n(x)$ দ্বারা সিদ্ধ অন্তরক সমীক

$u''_n+xu'_n-nu_n=0$ প্রতিষ্ঠা কর।

(i) যদি $x=\tan(\log y)$ হয় তবে প্রমাণ কর যে

$(1+x^2)y_{n+1}+(2nx-1)y_n+n(n-1)y_n=0$ C. H. 19

60. যদি $y=x/(x^2+a^2)$ এবং $x=a\cot\theta$. হয় তবে দেখাও যে

$$y_n=(-1)^n\frac{\lfloor n}{a^{n+1}}(\sin\theta)^{n+1}\cos(n+1)\theta$$ R. H. 19

61. দেখাও যে

$$\frac{d^n}{dx^n}\left(\frac{\sin x}{x}\right)=\frac{P\sin(x+\frac{1}{2}n\pi)+Q\cos(x+\frac{1}{2}n\pi)}{x^{n+1}}$$

$$\frac{d}{dy^n}\left(\frac{\cos x}{y}\right)=\frac{P\cos(x+\frac{1}{2}n\pi)-Q\sin(y+\frac{1}{2}n\pi)}{x^{n+1}}$$

যেখানে $P=y^n-n(n-1)y^{n-2}+n(n-1)(n-2)(n-3)y^{n-4}\ \ldots$

$Q=ny^{n-1}-n(n-1)(n-2)y^{n-3}+\ldots$

62. If $y=e^{\tan^{-1}x}=a_0+a_1x+a_2x^2+\ \ldots\ \ldots$

Show that

(i) $(1+x^2)y_2+(2x-1)y_1=0$

(ii) $(1+x^2)y_{n+2}+\{2(n+1)x-1\}y_{n+1}+n(n+1)y_n=0$

(iii) $(n+2)a_{n+2}+na_n=y_{n+1}$

Sol : see worked out example 14 of chapter VII. General Theorems.

63. If $y=\sin(m\sin^{-1}x)=a_0+a_1x+a_2x^2+\ \ldots\ \ldots$

N.H. 1994 *D. H. 1986*

Show that

(i) $(1-x^2)y_2=xy_1-m^2y$

(ii) $(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2-m^2)y_n=0$

(iii) $(n+1)(n+2)a_{n+2}=(n^2-m^2)a_n$

Sol :—see workedout Ex. 10 for (i) and (ii) and for (iii) differentiate y, then put the values of y_1 and y_2 in (i), equate the coefficients of x^n from both sides. The result will follow.

64. If $y=e^{a\sin^{-1}x}=a_0+a_1x+a_2x^2+\ \ldots\ \ldots$

Show that

$(n+1)(n+2)a_{n+2}=(n^2+a^2)a_n$

Hints : see worked out Ex. 11, then aquate co-efficients of x^n from both sides after putting the values of y_1 and y_2 in (1).

65. If $y=(\sin^{-1}x)^2=a_0+a_1x+a_2x^2+\ldots\ \ldots$

Show that $(n+1)(n+2)a_{n+2}=n^2a_n$

66. Prove that

$$\left(\frac{d}{dx}\right)^r e^{ax}x^n=a^{r-n}x^{n-r}\left(\frac{d}{dx}\right)^n e^{ax}x^r$$

67. Prove that if $x+y=1$,

$$\frac{d^n}{dx^n}(x^ny^n)=n!\,(y^n-{}^nc_1^2y^{n-1}x+{}^nc_2^2y^{n-2}x^2-\ \ldots\ \ldots)$$

68. Prove that *if* $y = \dfrac{\log x}{y}$, then

$$y_n = \frac{(-1)^n\, n!}{x^{n+1}}\ (\log x - 1 - \frac{1}{2} - \frac{1}{3} \dots\dots - 1/n)$$

69. By forming in two different ways the nth derivative of x^{2n}, prove that

$$1 + \frac{n^2}{1^2} + \frac{n^2\,(n-1)^2}{1^2.\,2^2} + \frac{n^2\,(n-1)^2\,(n-2)^2}{1^2\,.\,2^2\,.\,3^2} + \dots\dots\dots = \frac{(2n)!}{(n!)^2}$$

Hint : If $y = x^{2n}$, $y_n = \dfrac{(2n)}{n!}\, x^n$

Again $y = x^n.\, x^n$

$D^n\,(x^{2n}) = D^n\,(x^n\,.\,x^n) = x^n\; n!$ [Given series]= $\dfrac{(2n\;!)^2}{n!}$

70. If $x = \text{coty}$, then show that

$\dfrac{d^n}{dx^n}\left(\dfrac{x^n}{1+x^2}\right) = n!\ \text{siny}\ [\ \sin y - {}^nc_1 \cos y \sin 2y + {}^nc_2 \cos^2 y \sin 3y \dots\dots\dots]$

উত্তরমালা VI

1. $2^n\, m!/\,(m-n)\,!\,(2x+3)^{\,m-n}$

2. $(-1)^{n-1}\; n!\;\{(x-1)^{-n-1} - (8x-x)^{-n-1}\}$

3. $c \lfloor n\; 5^n\,(4-5x)^{-n-1}$

4. $(-1)^{n-1}\, n!\ \left[\dfrac{(n+2)\,(n+1)}{2\,(x-1)^{n+3}} + \dfrac{3(+1)}{(x-1)^{n+1}} + \dfrac{4}{(x-1)^{n-1}} + \dfrac{4}{(x-2)^{n+1}}\right.$

5. $\dfrac{(n+1)\,!\,(-1)^n}{3\,(x-1)^{n+1}} + \dfrac{5n\,!\,(-1)^n}{9\,(\,x-1)^{\,n+1}} + \dfrac{4n\,!\,(-\,1)^n}{9\,(x+2)^{n+1}}$

6. $n!\ \left[\dfrac{3^{n+1}}{(1-3x)^{n+1}} + \dfrac{3n+1}{(1-2x)^{n+1}}\right]$

7. $2^{-7}\,\{8^n \sin(8x+\frac{1}{2}n\pi) - 2.6^n \sin(6x+\frac{1}{2}n\pi)$
$\quad - 2.4^n \sin(\frac{1}{2}x+\frac{1}{2}n\pi) + 6.2^n \sin(2x+\frac{1}{2}n\pi)\}$

8. $y\{\cos(x+\frac{1}{2}n\pi) - 5^n \cos(5x+\frac{1}{2}n\pi)\}$ 9. $a^n e^{\,ax+b}$

10. $2^{n-1}\sin(2x+\frac{1}{2}n\pi) - 4^{n-1}\sin(4x+\frac{1}{2}n\pi)$

11. $-(\frac{1}{2})^5\,\{6^n\cos(6x+\frac{1}{2}n\pi) - 6.4^n\cos(4x+\frac{1}{2}n\pi)$
$\quad + 15.2^n \cos(\frac{1}{2}n\pi+2x)\}$

12. $(\frac{1}{2})^4\,\{5^n\cos(5x+\frac{1}{2}n\pi) + 5.3^n\cos(3x+\frac{1}{2}n\pi)$
$\quad + 10\cos(x+\frac{1}{2}n\pi)\}$

13. $2^{n-1}\, e^{2x} + (\frac{1}{2})^{n/2}\cos(\frac{1}{2}n\pi+2x)$

14. (i) $5^n\, e^{3x}\sin(4x+n\tan^{-1}4/3)$

(ii) $(a^2+q^2)^{n/2}\, e^{\,ax}\sin(p-qx-n\tan^{-1}q/a)$

15. $\{(-1)^n\, n\,!\cos(n+1)\,\theta\,\sin^{n+1}\theta\}/a^{n+1}$, $x = a\cos\theta$

16. $\dfrac{(-1)^{n-1}(n-1)\,!}{a^n}\sin n\theta\,\sin^n\theta$, যেখানে $x = a\cot\theta$

17. (i) $(-1)^{n-1}\lfloor(n-1)\,!\ \sin^n\theta\,\sin n\theta$, এখানে $x = \cot\theta$

(ii) $(-1)^n\dfrac{2\lfloor n}{(1+x)^{n+1}}$

18. 17 নং এর দুইগুণ   19. 18 নং এর মত।

20. $e^x\,\{\log x + {}^nc_1 x^{-1} + (-1)\,{}^nc_2 x^{-2} + \dots (-1)^{n-1}\lfloor(n-1)\,x^{-n}\}$

21. $y_n = e^x\{(ax+b)^3 + 3\,{}^nc_1\, a(ax+b)^2 + 6a^2\,{}^nc_2(ax+b) + 6a^3\}$

22. (i) $(x^2+n-n^2)\cos(\frac{1}{2}n\pi+x) + 2ax\sin(\frac{1}{2}n\pi+x)$

(ii) $y_n = \dfrac{(-1)^{n-1}}{x^{n-1}}\,\{-(n-1)\,! + 5\,{}^nc_1(n-2)\,!\dots + {}^nc_3\lfloor 5(n-6)\,!$

28. $\{(n-2)^2+m^2\}\{(n-4)^2+m^2\}\dots\ \dots$

$\dots(4^2+m^2)(2^2+m^2)m^2 e^{m\frac{1}{2}\pi}$ যখন n জোড় সংখ্যা হয়।

বা, $\{(n-2)^2+m\}^2\{(n-4)^2+m^2\} \ldots \ldots$ 249

$\ldots(3^2+m^2)(1^2+m^2)me^{m\frac{1}{2}\pi}$ যখন n বিজোর সংখ্যা হয়।

29\. $\{m^2-(n-2)^2\{m^2-(n-4)^2\}\ldots(m^2-1)\ 1^2$ যখন n জোড় হয়।

বা, $\{m^2(n-2)^2\}\{m^2-(n-4)^2\}\ldots(m^2-1^2)\ m$ যখন n বিজোড় সংখ্যা হয়।

49\. (i) $x^2y_{n+2}+(2n+1)xy_{n+1}+(n^2+1)y_n=0$

(ii) $(1+x)y_{n+2}+\{2x(n+1)-1\}y_{n+1}+n(n+1)y_n=0$

54\. (i) $y_2(1-b^2x^2)-y_1 b^2x=2a^2b^2$

$(1-b^2x^2)y_{n+2}-(2n+1)b^2x\,y_{n+1}-b^2(n^2-1)y_n=0$

55\. $(u_n)_0=0$ or, $\frac{n-1}{(-1)^2}(n-1)\,!$ যখন n জোড় বা বিজোড়

56\. $xy_{n+1}+(n-m)y_n=0$

— o —

CHHPTER VII

GENERAL THEOREMS AND EXPANSIONS

Art. 7.1. Introduction :—That a function $f(x)$ is continuous in the closed interval $[a,b]$ means that the function $f(x)$ is continuous at every point in the interval $[a,b]$ including the end point a and b and the interval is given by $a \leqslant x \leqslant b$.

The function $f'(x)$ is contiपuous in the open interval (a, b) means that $f(x)$ is derivable at every point in the open interval (a, b) satisfying the condition $a<x<b$.

7.2. Rolle's Theorem

If a function $f(x)$ is contiouous in the closed interval $a \leqslant x \leqslant b$, $f'(x)$ exists in the open interval $a<x<b$, and $f(b)=f(a)$, then there exist at least one point, say $x=c$, $a<c<b$ at which the derivative $f'(x)$ vanishes i. e., $f'(c)=0$

Proof : As the function $f(x)$ is continuous is the interval $a \leqslant x \leqslant b$, and $f(a)=f(b)$, $f(x)$ has at least a maximum or a minimum or both in the interval. Either $f(x)$ has minimum value M at $x=c$ i. e., $f(c)=M$, or a minimum value m at $x=c$ i. e., $f(c)=m$, c lying in the interval (a, b)

There are two cases

[i] $M=m$ [ii] $M \neq m$.

[i] When $M=m$

let $f(x)$ be constant in the interval [a, b] with

$f(x)=m$, The derivative of $f(x)$ is zero for every point in the interval. So $f'(x)$ in also zero for $x=c$, $a<c<b$,

Hence $f'(c)=0$

(ii) When $M \neq m$, then at least one value of $f(x)$ is different from $f(b)$ and $f(a)$ in the interval. Let $f(c)=M$ be different from them, where c lies within the interval.

Since $f(c)$ is the maximum value of the function $f(x)$ then $f(c+h)-f(c) \leqslant 0$

whether h is positive or negative

Thus we have

$$\frac{f(c+h)-f(c)}{h} \leqslant 0 \text{ when } h>0,$$

$$\text{and} \quad \frac{f(c+h)-f(c)}{h} \geq 0, \text{ when } h<0$$

But from the statement of the theorem we see that $f'(x)$ exist at $x=c$

$$\text{Now } \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} \leqslant 0 \text{ and } \lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h} \geq 0$$

$$\Rightarrow f'(c) \geq 0 \quad \ldots \ldots \ldots \ldots \quad (1)$$

$$\text{and } f'(c) \leqslant 0 \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

The relation (1) and (2) are true if and only if $f'(c)=0$

Consequently there is a point c inside the interval $a<c<b$, at which the derivative $f'(x)$ is equal to zero *i. e.* $f'(c)=0$.

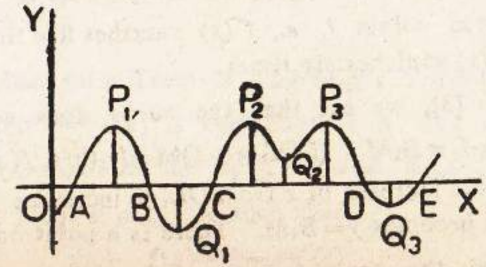
We can reach the same conclusion if $f(x)$ has a minimum value which differs from $f(a)$ and $f(b)$.

Hence the theorem is established.

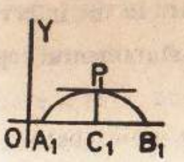
73 Geometric Interpretation of Rolle's Theorem

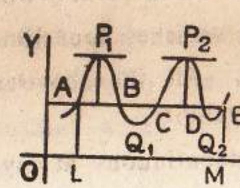
Statement :— *If $f(x)$ be a continuous curve which has tangent at every point in an interval (a, b) and the ordinates of*

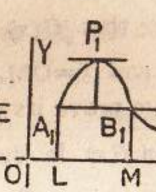
the extremeties of it are equal, i. e. $f(a)=f(b)$, then there exists at least one point c, $a<c<b$. at which the tangent to the curve is parallel to the x-axis.



Fig—2







Fig—1 Fig—4 Fig—3

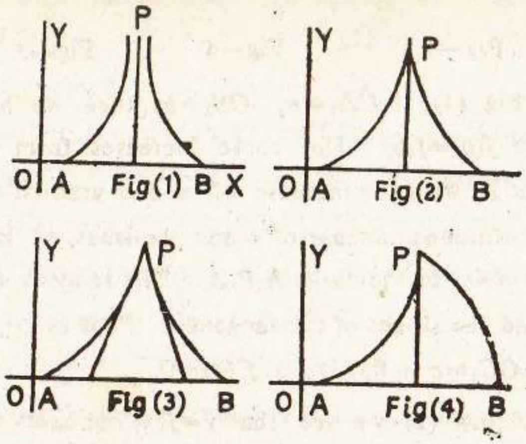
In the Fig (1), if $OA_1=a$, $OB_1=b$, then we have $f(a)=0$, $f(b)=0$, So $f(a)=f(b)$. The curve increases from A_1 becomes maximum at P_1 with the increase of x and gradually diminishes with the continuous increase of x and vanishes at B_1. Thus P_1 is a highest point in the curve $A_1P_1B_1$. The tangent P_1 is parallel to x-axis and the slopes of the tangent at P_1 is zero. *i. e.* $f'(x)=0$ for $x=OC_1=c$ in fig (1) *i. e,* $f'(c)=0$.

In the figure (2) we see that $y=f(x)$ increases with the increase of x, attains its maximum. then diminishes and becomes zero at B. For further increase, of x from OB to OC, y diminishes first and then increase, and at some point Q_1, y is minimum and $f'(x)$ vanishes at Q_1 i. e., $f'(x)=0$.

If we consider the interval OD, we get three maxima, and two minima at these points tangents are parallel to x-axis i. e. $f'(x)=0$ at these points *i, e.*, $f'(x)$ vanishes five times. In the interval OE, $f(x)$ vanishes six times.

Is the figure [3], we see that the curve does not meet the x—axis, but $A_1L=B_1M$. If $OL=a$, $OM=b$ then $f(a)=f(b)$. In this case with the increase of x from OL, y increases first, then diminishes and becomes $y=B_1M$. There is a point on the curve. $A_1 P_1 B_1$ where the tangent is parallel to x—axis *i, e.* $f'(x)$ vanishes. Hence $f'(x)=0$ at a point x where $OL<x<OM$. In fig. (4), we see that $f'(x)$ vanishes four times in the interval between $x=OL$ and $x=OM$. All the above statements regarding Roll's theorem prove its validity.

If function $f(x)$ is discontinuous at any point between $x=a$



and $x=b$ or, $f'(x)$ does not exist at any point within the interval, then the theorem does not hold good.

In the above figures we see that $f'(x)$ is discontinuous at P in fig. (1) while $f'(x)$ does not exist at P in fig, (2), (3) and (3), and (4) ; $f'(x)$ does not vanish at the point P in the above figures. There is no point in the interval (a, b), where the tangent is parallel to x-axis.

7. 5. Mean value Theorem (*Lagrange's Theorem*).

If a function $f(x)$ is continuous in the closed interval $a\leqslant x\leqslant b$, $f'(x)$ exists in the open interval $a<x<b$, there exists at least one point c, $a<c<b$ such that

$$f(b)-f(a)=(b-a)f'(c)$$

To prove this theorem let us consider another function $\phi(x)$ involving $f(x)$ such that $\phi(x)$ satisfies all the conditions of Rolle's Theorem. Let $\phi(x)=f(x)+Ax$ ... ... (1)

Where A is a constant which is to be determined by the condition $\phi(b)=\phi(a)$.

From (1),

$$\phi(b)=f(b)+Ab \text{ and } \phi(a)=f(a)+Aa$$

Now $\phi(b)=\phi(a)$

$$\therefore\quad f(b)+Ab=f(a)+Aa \Rightarrow A=-\frac{f(b)-f(a)}{b-a} \quad \dots \quad \dots \quad (2)$$

Since $f(x)$ and x are continuous in the closed interval and derivable in the open interval, then

$\phi(x)$ is also continuous in the interval $a\leqslant x\leqslant b$, and derivable in $a<x<b$ and also $\phi(a)=\phi(b)$.

Thus $\phi(x)$ satisfies all the condition of Rolle's Theorem,

Then there is at least one point c in the interval $a<c<b$ such that $\phi'(c)=0$ ... ... (3)

Now differentiate (1) w. r. to x :

$\phi'(x)=f'(x)+A$

$\therefore\ \phi'(c)=f'(c)+A.$ or, $0=f'(c)+A$ by (5)

or, $0=f'(c)-\dfrac{f(b)-f(a)}{b-a}$ by (2)

or. $f(b)-f(a)=(b-a)\ f'(c)$ [Proved]

or, $f'(c)=\dfrac{f(b)-f(a)}{b-a}$ ... ... (4)

This is called First mean value Theorem.

Cor. 1. If we put $b-a=h$ or, $b=a+h$, then the relation [4] becomes,

$f(a+h)-f(a)=hf(a+\theta h),\ c=a+\theta h$

where θ is a number such that $0<\theta<1$, so that $a<c<a+h$.

So the statement of mean value Theorem also runs as follows.

If a function $f(x)$ is continuons in the closed interval $a\leqq x\leqq a+h$, $f'(x)$ exists in the open interval $a<x<a+h$, then there exists at least one number θ lying between 0 and 1 such that

$f(a+h)-f(a)=hf'(a+\theta h),\quad 0<\theta<1,$

or, $f(a+h)=f(a)+hf'(a+\theta h)$

Note : The Theorem proved in the Article 7·5 is called First mean value theorem The order of the Theorem depends on the highest order of the derivative of $f(x)$ involved.

Cor. 2. If $f'(x)=0$ for all points in (a, b), then $f(x)$ is constant throughout (a, b),

For any interval (a, x) lying in (a, b), we have by Mean value theorem,

$f(x)=f(a)+(x-a)f'\{a+\theta(x-a)\},\ 0<\theta<1,$

According to the given condition

$f(x)$ is zero throughout the interval (a, b) so $f\{a+\theta(x-a)\}=0$.

Hence. $f(x)=f(a)=$ constant

If $f(x)$ and $g(x)$ are two functions such that $f'(x)=g'(x)$ for all points in (a, b), then $f(x)$ and $g(x)$ differ by a constant in (a, b).

For $f(x)-g(x)=$constant in (a, b)

Differentiate it w. r. to x,

$f'(x)-g'(x)=0$ or, $f'(x)=g'(x)$ in (a, b)

Cor 3 If $f'(x)=0$ in the closed interval $a\leqq x\leqq b$, show that $f(x)$ is steadily increasing in $a\leqslant x\leqslant b$,

Let us consider an internal $[c,d]$ of $[a,b]$ such that $a\leqq c<d\leqq b$ Then $f(x)$ exists in the interval $c\leqslant x\leqslant d$ and is greater than zero i. e., $f'(x)$ exists and remains greater than zero in $a\leqslant x\leqslant b$.

As $f'(x)$ exists in $c\leqslant x\leqslant d$, so $f(x)$ is continuous in $c\leqslant x\leqslant d$ then by Mean value Theorem, we have.

$f(d)-f(c)=(d-c)\,f'(\xi),\ c<\xi<d$

But $f'(x)>0$ and $d>c$. Therefore $f(d)-f(c)$ is positive i. e, $f(d)-f(c)>0$ for $d>c$

or $f(d)>f(c)$ for any interval $[c.d]$ with $a\leqq c<\xi\leqslant d$.

Thus $f(x)$ is steadily (monotonically) increasing in the given interval, $a\leqslant x\leqslant b$

Similarly $f(x)$ is steadily (or monotonically) decreasing if $f'(x)$ is negative in the interval.

Cor. 4. If the function $f(x)$ and $g(x)$ have the same derivative at all points in an interval $a\leqslant x\leqslant b$, show that $f(x)-g(x)$ is constant in $a\leqslant x\leqslant b$.

Let $\phi(x)=f(x)-g(x)$ ... ... (i)

$\therefore \quad \phi'(x)=f'(x)-g'(x)$ ... ... (ii)

According to the given condition $f'(x)-g'(x)$ for all value of x in $a\leq x\leq b$.

Hence from [ii], $\phi'(x)=0$.

or, $\phi(x)=\text{constant}$ i. e., $f(x)-g(x)=\text{constant}$.

7.6. Second Mean Value Theorem (দ্বিতীয় গড়মান উপপাদ্য)

If $f(x)$ and $f'(x)$ are continuous in the closed interval $a\leq x\leq a+h$, $f''(a)$ exists in the open interval $a<x<a+h$, then

$f(a+h)=f(a)+hf'(a)+\frac{1}{2}h^2f''(a+\theta h)$.

where θ lies between 0 and 1 i. e., $0<\theta<1$.

Let us consider the function.

$\phi(x)=f(x)-f(a)-(x-a)f'(a)+A(x-a)^2$ ... ... [1]

where A is a constant the constant A is chosen in such a way that

$\phi(a+x)=\phi(a)$ ... ... (2)

or, $f(a+h)-f(a)-hf'(a)+Ah^2=\phi(a)=0$ from (1)

or, $A=\frac{\{f(a+h)-f(a)-hf'(a)\}}{h^2}$ ... ... (3)

Now $\phi(x)$ satisfies the conditions of Rolie's Theorem in the interval $(a, a+h)$ Then x is a point in the interval such that $a<x_1<a+h$,

where $\phi'(x_1)=0$ ... ... (4)

Now differentiate (1) w. r. to x,

$\phi'(x)=f'(x)-f'(a)+2A(x-a)$ ... (5)

$\phi''(x)=f''(x)+2A$ ... (6)

when $x=a$ in (5), we get

$\phi'(a)=f'(a)-f'(a)+2A.\ 0=0$

or, $\phi'(a)=0$ ... ... (7)

Therefore, we get another interval (a,x_1) such that $\phi'(x)$ is continuous, in $a\leq x_1\leq x$, $\phi'(x)$ exists in $a<x_1<x$

and moreover $\phi'(a)=\phi'(x_1)=0$

Thus $\phi'(x)$ satisfied all the conditions of Rolle's theorem So, there is at least one point x_1 lying between a and x_2 such that

$\phi''(x_1)=0,\ a<x_2<x_1$

or, $\phi''(x_2)=f''(x_2)+2A=0$ from (7)

or. $f''(x_2)=-2A$

or, $f''(x_2)=\frac{f(a+h)-f(a)-hf'(a)}{h^2}.\ 2$ from (3)

where $a<x_2<a+h$

or, $\frac{h^2}{2}f''(a+\theta h)=f(a+h)-f(a)-hf'(a)$

where $0<\theta<1$, $x_2=a+\theta h$

or, $f(a+h)=f(a)+hf'(a)+\frac{h^2}{2}f''(a+\theta h)$

Hence the theorem is established

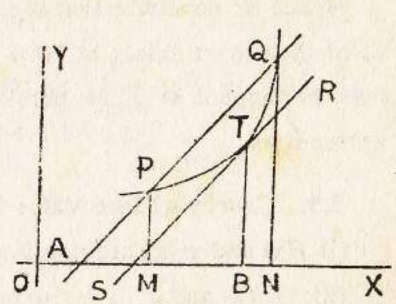
7.7. Geometrical Interpretation of Mean value Theorem

Let $y=f(x)$ represent the curve PTQ in the interval (a, b) Then

$OM=a,\ PM=f(a)$

$ON=b\ \ QN=f(b)$

Thus the equation of chord PQ passing through $P\{a, f(a)\}$ and $Q\{b, f(b)\}$ is



Fig—9

$$\frac{y-f(b)}{f(b)-f(a)}=\frac{x-b}{b-a} \text{ or, } y=\frac{f(b)-f(a)}{b-a}(x-b)$$

Let PQ make angle ψ with the x-axis, then

$$\tan\psi=\frac{f(b)-f(a)}{b-a} \quad \ldots \quad (1)$$

But the curve $y=f(x)$ is continuous in the closed interval $a \leqslant x \leqslant b$ and $f(x)$ is differentiable in the open interval $a<x<b$ i.e., $f'(x)$ exists in the interval $a<x<b$. It means that the curve has a tangent at every point between P and Q. Then there is a point T in the curve where the tangent STR is parallel to the chord PQ. Let the point T corresponds to $x=c$. The tangent STR at $T\{c, f(c)\}$ makes angle ψ with the x-axis where

$$\tan\psi=\frac{dy}{dx}=f'(x) \text{ at } x=c$$

that is, $\tan\psi=f'(c)$ (1)

From [1] and [2] we have

$$f'(c)=\frac{f(b)-f(a)}{b-a} \quad \text{or, } f(b)-f(a)=(b-a)f'(c),\ a<c<b$$

Hence we conclude that if a curve has tangent at each of its point then there exists at least one point T on the curve such that the tangent at T is parallel to the chord PQ joining its extremities.

7.8. Cauchy's Mean Value Theorem

(i) $f(x)$ and $g(x)$ are continuous in the closed interval $a \leqslant x \leqslant b$

(ii) $f'(x)$ and $g'(x)$ exist in the open interval $a<x<b$,

(iii) $g(b) \neq g(a)$

and (iv) $f'(x)$ and $g'(x)$ do not vanish for the same value of x then there exists at least one point c in the interval (a, b) such that

$$\frac{f'(c)}{g'(c)}=\frac{f(b)-f(a)}{g(b)-g(a)}$$

Let us consider a function $\phi(x)$ defined in the following way

$$\phi(x)=f(x)+Ag(x) \quad \ldots \quad \ldots \quad (1)$$

where A is a constant. We select A in such a way that

$$\varphi(a)=\varphi(b)$$

Then from (1).

$$f(a)+Ag(a)=f(b)+Ag(b)$$

$$\Rightarrow A=-\frac{f(b)-f(a)}{g(b)-g(a)} \quad \ldots \quad \ldots \quad (2)$$

Again

$$\phi'(x)=f'(x)+Ag'(x) \quad \ldots \quad \ldots \quad (3)$$

Now we see that $\phi(x)$ is continuous in the closed interval $a \leqslant x \leqslant b$, $\phi'(x)$ exists in the open interval $a<x<b$ as $f'(x)$ and $g'(x)$ exist also $\varphi(a)=\varphi(b)$. Hence by Rolle's theorem, there exists at least one point c in the interval (a, b) such that

$$\phi'(c)=0$$

or, $\phi'(c)=f'(c)+Ag'(c)=0$ from [3]

or, $f'(c)+Ag'(c)=0$

But $g'(c) \neq 0$, otherwise : $f'(c)$ would also be zero which is contrary to the assumption (iv) of the hypothesis.

$$\text{Thus } A=-\frac{f'(c)}{g'(c)} \quad \ldots \quad \ldots (4)$$

From (2) and (4), we have

$$\frac{f'(c)}{g'(c)}=\frac{f(b)-f(a)}{g(b)-f(a)} \quad ; (a<c<b)$$

Hence the theorem is established.

7. 9. Taylor's Theorem with Remainder.

If a function $f(x)$ and all of its derivatives upto $(n-1)$th order [i. e., $f^{n-1}(x)$] are continuous in the closed interval $a \leqslant x \leqslant a+h$ and the nth derivative $f^n(x)$ exists in the open interval $a<x<a+h$, then

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{\lfloor 2}f''(a)+\ldots+\frac{h^{n-1}}{\lfloor n-1)}f^{n-1}(x)+R_n$$

where R_n is the remainder after n terms.

Let us consider a function $\varphi(x)$ in the interval $[a, a+h]$ such that $\varphi(x)=f(x)+(a+h-x)f'(x)-\frac{(a+h-x)^2}{\lfloor 2}f''(x)+$

$$\ldots \ldots +\frac{(a+h-x)^{n-1}}{(n-1)!}f^{n-1}(x)+(a+h-x)^m A$$

where A is a constant and $m>0$ ... (1)

We are to select A in such a way that

$$\phi(a)=\varphi(a+h) \quad \ldots \ldots \quad (2)$$

Now

$$\varphi(a)=f(a)+hf'(a)+\frac{h^2}{\lfloor 2}f''(a)+\frac{h^3}{\lfloor 3}f'''(a)+$$

$$\ldots \ldots \ldots +\frac{h^{n-1}}{\lfloor (n-1)}f^{n-1}(a)+h^m A \ldots \ldots (3)$$

$$\varphi(a+h)=f(a+h) \cdots \quad \ldots \quad \ldots \quad (4)$$

Putting (3) and (4) in (2),

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{\lfloor 2}f''(a)+\ldots+\frac{h^{n-1}(a)}{\lfloor (n-1)}f^{n-1}(a)$$
$$+h^m A \quad \ldots \quad \ldots \quad \ldots (5)$$

Again

$$\varphi'(x)=f'(x)-f'(x)+(a+h-x)f''(x)-\frac{2}{\lfloor 2}(a+h-x)f''(x)$$
$$+\frac{(a+h-x)^2}{\lfloor 2}f'''(x)\ldots+\frac{(a+h-x)^{n-1}}{\lfloor (n-1)}f^n(x)-m(a+h-x)^{m-1}A$$

$$\text{or, } \phi'(x)=\frac{(a+h-x)^{n-1}}{\lfloor (n-1)}f^n(x)-m(a+h-x)^{m-1}A \;\ldots\ldots (6)$$

Since $\varphi(x)$ is the sum of $(n+1)$ continuous terms in the interval $[a, b]$. Therefore $\varphi(x)$ is continuous in $[a, b]$; as $f^n(x)$ and $(a+h-x)^{m-1}$ are defined for $a<x<a+h$, so $\varphi'(x)$ exists in the open interval (a, b). Also $\varphi(a)=\varphi(b)$ Hence all the conditions of Rolle's theorem are satisfied and so there exists a point c such that

$$\varphi'(c)=0 \quad \text{where} \quad a<c<a+h$$

or, $\varphi'(a+\theta h)=0$, $c=a+\theta h$ and, $0<\theta<1$

Now from (6)

$$\phi'(a+\theta h)=\frac{(a+h-a-\theta h)^{n-1}}{\lfloor (n-1)}f^n(a+\theta h)$$
$$-m(a+h-a-\theta h)^{m-1}A=0$$

$$\text{or, } \frac{h^{n-1}(1-\theta)^{n-1}}{\lfloor (n-1)}f^n(a+\theta h)=mh^{m-1}(1-\theta h)^{m-1}A$$

$$\text{or, } A=\frac{(1-\theta)^{n-m}h^{n-m}}{\lfloor m(n-1)}f^n(a+\theta h)$$

Put the value of A in (5) then

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\ldots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)$$
$$+\frac{h^m h^{n-m}(1-\theta)^{n-m}}{m(n-1)!}f^n(a+\theta h)$$

$$=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\quad\dots\quad\dots$$

$$+\frac{h^{n-1}f^{n-1}(a)}{(n-1)!}+\frac{h^n(1-\theta)^{n-m}}{m(n-1)!}f^n(a+\theta h)\dots\dots(7)$$

or, $f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\dots\dots$

$$+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+R_n\dots\dots(8)$$

where R_n is called the Remainder after n terms in the expansion of $f(a+h)$.

$$R_n=\frac{h^n(1-\theta)^{m-n}}{m(n-1)!}f^n(a+\theta h)\dots\ \dots\ \dots(9)$$

which is called **Schomilche and Roche's Remainder.**

When $n=m$ in [9]

$$R_n=\frac{h^n}{n(n-1)!}f^n(a+\theta h)=\frac{h^n}{n!}f^n(a+\theta h)\ \dots\ \dots\ (10)$$

which is called **Lagrange's Remainder.**

When $m=1$ in [9].

$$R_n=\frac{h^n(1-\theta)^{n-1}}{(n-1)!}f^n(a+\theta h)\dots\dots(11)$$

which is called **Cauchy's form of Remainder.**

Changing a to x we have, from (9), (10), (11) respectively.

(i) **Taylor's series with Schlomiche and Roche's remainder :**

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\dots\dots\frac{h^{n-1}}{(n-1)!}f^{n-1}(x)+$$

$$\frac{h^n(1-\theta)^{n-m}}{m(n-1)!}f^n(x+\theta h)\dots(12)$$

(ii) **Taylors Series with Lagranges Remainder**

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\dots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(x)+$$

$$\frac{h^n}{n!}f^n(x+\theta h)\ \cdots\ \dots\ (13)$$

(iii) **Taylors Series with Cauchy's Form of Remainder**

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)+\ \cdots\ \cdots$$

$$+\frac{h^{n-1}}{(n-1)!}f^{n-1}(x)+\frac{h^n(1-\theta)^{n-1}}{(n-1)!}f^n(x+\theta h)\ \dots\cdots\ (14)$$

Cor. In the Taylors Expansion with Lagranges Remainder, when $n=1$,

$$f(x+h)=f(x)+hf'(x+\theta h)\quad\dots\quad\dots\quad\dots\quad(15)$$

which is called Lagrange's Mean value Theorem of the First Mean Value Theorem, See Art. 7·5

If in the expansion, we have $n=2$, then

$$f(x+h)=f(x)+f'(x)+\frac{h^2}{2!}f''(x+\theta h)\quad\cdots\quad\cdot\ (16)$$

which is called the Second Mean Value Theorem. See Art. 7.7

7.10. Taylor Series (infinite form)

If $f(x)$ and $f^n(x)$ be finite and continuous in the interval $[a, a+h]$ for every positive integral value of n and if the remainder R_n tends to zero when n tends to infinity, *i. e.*, $\lim\limits_{n\to 0} R_n=0$, then

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)\ \dots\ +\ \frac{h^n}{n!}f^n(a)\ +\dots\cdots$$

If $a=x$, then the Taylor series becomes.

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x)\dots+\frac{h^nf^n(x)}{n!}+\cdots.$$

The necessary and sufficient condition that $f(x+h)$ can be expanded in an infinite series is $\lim_{n \to \infty} R_n = 0$

R_n may be any one of the form (9), (10), (11) in Art. 7.9.

2.11. Failure of Taylor's theorem

In the following cases, Taylor's theorem fails.

(i) If $f(x)$ or one of its derivatives becomes undefined in the given interval.

(ii) If $f(x)$ or one of its derivatives becomes discontinuous in the same interval

(iii) If the remainder R_n cannot be made to vanish in the limit when n is taken sufficiently large so that R_n does not tend to any finite limit for, a given n.

7.12. Maclaurin's series with Remainder after n terms ;

If we put $x=0$ and $h=x$ in Taylor's seris (12)

$$f(x)=f(o)+xf'(o)+\frac{x^2}{\lfloor 2}f''(o)+\ldots+x^{n-1}\frac{f^{n-1}(o)}{\lfloor (n-1)}$$

$$+\frac{x^n(1-\theta)^{n-m}}{\lfloor m(n-1)}f^n(\theta h) \text{ where } 0<\theta<1 \ldots\ldots (17)$$

$$\text{or, } f(x)=f(o)+xf'(o)+\frac{x^2}{\lfloor 2}f''(o)\ldots\ldots+\frac{x^{n-1}f^{n-1}(o)}{\lfloor (n-1)}+R_n \quad \ldots \quad \ldots \quad (18)$$

where R_n is the remainder after n terms in the Maclaurins expansion. Now

(i) $$R_n=\frac{x^n(1-\theta)^{n-m}}{\lfloor m(n-1)}f^n(\theta x) \ldots \ldots \quad (19)$$

is called Schlomilch and Roche form of Remainder.

(ii) when $n=m$ in (19) then

$$R_n=\frac{x^n}{\lfloor n(n-1)}f^n(\theta x)=\frac{x^n}{\lfloor n}f^n(\theta x)$$

$$\text{or, } R_n=\frac{x^n}{\lfloor n}f^n(\theta x) \quad \ldots \quad \ldots \ldots (20)$$

which is called Lagranges Remainder.

(iii) When $m=1$ in (19), then

$$R_n=\frac{x^n}{\lfloor n(n-1)}(1-\theta)^{n-1}f^n(\theta x) \ldots \quad \ldots \quad (21)$$

which is called Cauchys form of Remainder.

Cor. when $n=1$, the Maclaurin series with Lagranges Remainder is

$$f(x)=f(o)+xf'(\theta x) \quad \ldots \quad \ldots \quad \ldots (22)$$

when $n=2$ in the above expansion then

$$f(x)=f(o)+xf'(o)+\frac{x^2}{\lfloor 2}f''(\theta x) \ldots \quad \ldots (23)$$

7 13. Maclaurins Series (*infinite form*)

If $f(x)$ and $f^n(x)$ are continuous and limit at $x=0$ for every positive integral value of n and the remainder R_n tends to zero as n tends to infinity then,

$$f(x)=f(o)+xf'(o)+\frac{x^2}{\lfloor 2}f''(o)\ldots+\frac{x^n}{\lfloor n}f^n(o)+\ldots\ldots\ldots$$

Where R_n may be any one of the forms (17), (19), (20) of Art. 7.12

7 14. Failure of Maclaurin's theorem.

In the following cases Maclaurin's theorem fails.

(i) If any of the expressions $f'(o), f''(o)\ldots f^n(o)$ becomes undefined

(ii) If $f(x)$ or any one of its derivative is discontinuous at $x=0$

(iii) If $\lim_{n\to\infty} \frac{x^n}{\lfloor n} f^n(\theta x) \neq 0$ *i. e.*,

If R_n does not tend to a limit when n tends to infinity

7.15. Expansions

Students are already familiar with the expansions of given explicit functions in ascending powers of a variable. For example the expansions of $(a+x)^n$, $\log(1+x)$, e^{ax}, $\tan^{-1}x$ etc. are already known to them in Algebra and Trigonometry. These expansions are generally made with the following Principal Methods.

(i) By purely Algebaric and Trigonometric Methods See. Ex. 7

(ii) By Taylor's and Maclaurin's Theorems See Ex. 8, Ex. 2, Ex. 10

(iii) By differentiation or, Integration of known series See Ex. 11. Ex. 12, Ex. 13

(iv) By the use of differential equations. See Ex. 14.

The above methods will be explained with examples.

7. 16. Determination of the co-efficients in the expansion of $f(x)$ and $f(x+h)$.

Taylor's Theorem (infinite form)

(A) If $f(x+h)$ can be expanded in a convergent series of positive integral powers of h in an interval, prove that

$$(x+h)=f(x)+hf'(x)+\frac{h^2}{\lfloor 2} f''(x)+\text{to infinity}$$

Let $f(x+h)=a_0+a_1h+a_2h^2+a_3h^3+ \quad \dots \quad \dots (1)$

Where a_0, c_1, a_2, a_3 etc are all free from h, but are functions of x only.

Now differentiate (1) successively $w.\ r.$ to h, treating x as constants and hence a_0. a_1 a_2, etc are all constants.... ... Then

$$f'(x+h)=a_1+2a_2h+3.a_3h^2+4.a_4h^3+ \quad \dots \dots (2)$$

$$f''(x+h)=2a_2+2.3a_3h+3.4a_4h^2+ \quad \dots \dots (3)$$

$$f'''(x+h)=3.2a_3+2.34a_4h+ \quad \dots \dots (4)$$

and so on

Putting $h=0$ in (1), (2), (3), (4), we have,

$f(x)=a_0, f'(x)=a_1, f''(x)=2.1\ a_2, f'''(x)=\lfloor 3 a_3$

or, $a_0=f(x), a_1=f'(x), a_2=f''(x)/2\ !$

$a_3=f'''(x)/3\ !$

Now from (1)

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{\lfloor 2}f''(x)+\frac{h^3}{\lfloor 3}f'''(x)+$$

$$+\frac{h^r}{\lfloor r}f^r(x) \quad \dots \dots (5)$$

or, $$f(x+h)=f(x)+h\frac{d}{dx}f(x)+\frac{h^2}{\lfloor 2}\frac{d^2}{dx^2}f(x)$$

$$+\frac{h^3}{\lfloor 3}\frac{d^3}{dx^3}f(x)+$$

$$=\left\{1+h\frac{d}{dx}+\frac{h^2}{\lfloor 2}\left(\frac{d}{dx}\right)^2+\frac{h^3}{\lfloor 3}\left(\frac{d}{dc}\right)^3+\dots\right\}f(x)$$

$$\Rightarrow f(x+h)=e^{hd/dx} f(x)$$

Note :

Put $x=a$, then we get a series of $f(a)$.

If $h=x-a$ in (5), then

We have,

$F(a)=0=F(b)$.

Then Rolle's theorem, there is a value of $x=\xi$, $a<\xi<b$ such that $F'(\xi)=0$.

$$\text{Now } F'(x)=\begin{vmatrix} f(a) & f'(x) \\ \varphi(a) & \varphi'(x) \end{vmatrix}-\frac{1}{(b-a)}\begin{vmatrix} f(a) & f(b) \\ \varphi(a) & \varphi(b) \end{vmatrix}$$

Hence

$$F'(\xi)=0=\begin{vmatrix} f'(a) & f'(\xi) \\ \varphi'(a) & \varphi'(a) \end{vmatrix}-\frac{1}{b-a}\begin{vmatrix} f(a) & f(b) \\ \varphi(a) & \varphi(b) \end{vmatrix}$$

$$\begin{vmatrix} f'(a) & f(b) \\ \varphi(a) & \varphi(b) \end{vmatrix}=(b-a)\begin{vmatrix} f(a) & f'(\xi) \\ \varphi(a) & \varphi'(\xi) \end{vmatrix}$$

Art 17.19. If $f(x)$, $\varphi(x)$, $\psi(x)$ have derivatives when $a\leqslant x\leqslant b$, show shat there is a value ξ of x lying between a and b such that

$$\begin{vmatrix} f(a) & \varphi(a) & \psi(a) \\ f(b) & \varphi(b) & \psi(b) \\ f'(\xi) & \varphi'(\xi) & \psi'(\xi) \end{vmatrix}=0$$

and deduce $\dfrac{f(b)-f(a)}{\varphi(b)-\varphi(a)}=\dfrac{f'(\xi)}{\varphi'(\xi)}$

Art. 17. 20. If α, β lie between the least and the greatest of the numbers a, b and c; show that

$$\begin{vmatrix} f(a) & f(b) & f(c) \\ \varphi(a) & \varphi(b) & \varphi(c) \\ \psi(a) & \psi(b) & \psi(c) \end{vmatrix}=k\begin{vmatrix} f(a) & f'(\alpha) & f''(\beta) \\ \phi(a) & \varphi'(\alpha) & \varphi''(a) \\ \psi(a) & \psi'(\alpha) & \psi''(\beta) \end{vmatrix}$$

Where $k=\frac{1}{2}(b-c)(c-a)(a-b)$

Sol Let $f(x)$, $\varphi(x)$, $\psi(x)$ be continuous in the closed interval (a, c) and twice differentiable in the open interval (a, c) $a<b<c$.

Let us consider the function $F(x)=$

$$\begin{vmatrix} f(a) & f(b) & f(x) \\ \varphi(a) & \varphi(b) & \varphi(x) \\ \psi(a) & \psi(b) & \psi(x) \end{vmatrix}-\frac{(x-a)(x-b)}{(c-a)(c-b)}\begin{vmatrix} f(a) & f(b) & f(c) \\ \varphi(a) & \varphi(b) & \varphi(c) \\ \psi(a) & \psi(b) & \psi(c) \end{vmatrix}$$

Here $F(a)=F(b)=F(c)=0$. By Rolle's theorem

$$F'(x_1)=F'(x_2)=0,\ a<x_1<b,\ b<x_2<c$$

Using Rolle's Theorem once again on $F'(x)$,

$F''(\beta)=0$ where $x_1<\beta<x_2$

$$\therefore\begin{vmatrix} f(a) & f(b) & f''(\beta) \\ \varphi(a) & \varphi(b) & \varphi''(\beta) \\ \psi(a) & \psi(b) & \psi''(\beta) \end{vmatrix}=\frac{2}{(c-a)(c-b)}\begin{vmatrix} f(a) & f(b) & f(c) \\ \varphi(b) & \varphi(b) & \varphi(c) \\ \psi(a) & \psi(b) & \psi(c) \end{vmatrix}\quad \dots (1)$$

Again consider the function

$$F_1(x)=\begin{vmatrix} f(a) & f(x) & f''(\beta) \\ \varphi(a) & \varphi(x) & \varphi''(\beta) \\ \psi(a) & \psi(x) & \psi''(\beta) \end{vmatrix}-\frac{2(x-a)}{(c-a)(c-b)(b-a)}\begin{vmatrix} f(a) & f(b) & f(c) \\ \varphi(a) & \varphi(b) & \varphi(c) \\ \psi(a) & \psi(b) & \psi(c) \end{vmatrix}$$

$F_1(a)=0$ from (2) $F_1(b)=0$ [by(1)] ......(2)

From Rolle's Theorem, $F_1'(\alpha)=0$, $a<\alpha<b$

$$\text{or, }\begin{vmatrix} f(a) & f'(\alpha) & f''(\beta) \\ \varphi(a) & \varphi'(\alpha) & \varphi''(\beta) \\ \psi(a) & \psi'(\alpha) & \psi''(\beta) \end{vmatrix}=\frac{2}{(c-a)(c-b)(b-a)}\begin{vmatrix} f(a) & f(b) & f(c) \\ \varphi(a) & \varphi(b) & \varphi(c) \\ \psi(a) & \psi(b) & \psi(c) \end{vmatrix}$$

Art 7. 21. If $\varphi(x)=f(x)+f(1-x)$ and $f''(x)>0$ in $0\leqslant x\leqslant 1$, Show that $\varphi(x)$ increases in $0\leqslant x\leqslant \frac{1}{2}$ and decreases in $\frac{1}{2}\leqslant x\leqslant 1$ and then prove that

$$\pi<\frac{\sin \pi x}{x(1-x)}<4,\ 0<x<1.$$

$$\text{Given}\quad \left.\begin{aligned} \varphi(x)&=f(x)+f(1-x) \\ \varphi'(x)&=f'(x)-f'(1-x) \\ \varphi''(x)&=f''(x)+f''(1-x) \end{aligned}\right\}\quad \dots\ \dots (1)$$

If x varies from 0 to 1, then $1-x$ varies from 1 to 0, so $f''(1-x)$ is also negative as $f''(x)$ is negative so $\varphi''(x)$ is negative throughout the interval (0, 1) by hypothesis.

∴ $\varphi''(x)$ is monotonically decreasing in (0, 1)

Again $\varphi'(0)=f'(0)-f'(1)$, $\varphi'(\frac{1}{2})=f'(\frac{1}{2})-f'(\frac{1}{2})=0$

$\varphi'(1)=f'(1)-f'(0)$ ... ... (2)

Since $f''(x)<0$ throughout the interval $0\leq x\leq 1$ and on that account $f'(x)$ is monotone decreasing in (0, 1), then $f'(0)>f'(1)$

So $\varphi'(x)$ is positive in $(0, \frac{1}{2})$ and negative in $(\frac{1}{2}, 1)$ by (2)

∴ $\varphi(x)$ is monotone increasing in $(0, \frac{1}{2})$ and monotone decreasing in $(\frac{1}{2}, 1)$

Let $\varphi(x)=\dfrac{\sin \pi x}{x(1-x)}=\dfrac{2\sin \pi x/2 \sin \pi(1-x)/2}{x(1-x)}$ ... ...(3)

∴ $\log\frac{1}{2}\varphi(x)=\log\dfrac{\sin \pi x/2}{x}+\log\dfrac{\sin \pi(1-x)/2}{1-x}$

$=f(x)+f(1-x)$

where $f(x)=\log\dfrac{\sin \pi x/2}{x}$

But $f'(x)=\dfrac{\cos \pi x/2}{\sin \pi x/2}\dfrac{\pi}{2}-\dfrac{1}{x}$

$f''(x)=-\dfrac{\pi^2}{4}\operatorname{cosec}^2 \pi x/2+\dfrac{1}{x^2}$

$=\dfrac{(-\pi^2x^2)/4+\sin^2 \pi x/2}{x^2\sin^2(\pi x/2)}<0,\ 0<x<1$

∴ $\log\dfrac{\phi(x)}{2}$ or, $\phi(x)$ increases in $(0, \frac{1}{2})$

and decreases in $(\frac{1}{2}, 1)$

As $\phi(x)=\dfrac{\sin \pi x}{x(1-x)}$ ... ... (4)

As $x\to+0+$, $\phi(x)\to\pi$

As $x\to 1-0$, $\phi(x)\to\pi$ as $\sin \pi(1-h)\phi=\sin \pi h$

Also $x=\frac{1}{2}$, $\phi(x)=4$, from (4)

∴ $\pi<\phi(x)<4$ in $(0, \frac{1}{2})$

and $4<\phi(x)<\pi$ in $(\frac{1}{2}, 1)$

Hence $\pi<\dfrac{\sin \pi x}{x(1-x)}\leq 4$

Art. 7. 22. A Function is twice differentiable and satisfies the inequalities.

$|f(x)|<A$, $|f''(x)|<B$, in the range $x>a$;

Where A and B are constants Prove that $|f(x)|<2\sqrt{(AB)}$.

Ans. For positive number h, and $x>a$;

$f(x+h)=f(x)+hf'(x)+\dfrac{h^2}{\lfloor 2}f''(x+\theta h),\ 0<\theta<1$

∴ $|hf'(x)|=\left|f(x+h)-f(x)-\dfrac{h^2}{\lfloor 2}f''(x+\theta h)\right|$

$\leq|f(x+h)|+|f(x)|+\left|\dfrac{h^2}{\lfloor 2}f''(x+\theta h)\right|$

$\leq A+A+Bh^2/2$

or, $|f'(x)|<\dfrac{2A}{h}+Bh/2$; h is +ve

Thus $|f'(x)|$ must be less than the least value of $(2A/h+Bh/2)$

Thus $(2A/h+Bh/2)=\{\sqrt{(2A/h)}-\sqrt{(Bh/2)}\}^2+2\sqrt{(AB)}\geq 2\sqrt{AB}$

Square of a quantity is positive.

(∵ $[\sqrt{(2A/h)}-\sqrt{2Bh/2)}]^2>0$)

Hence $|f(x)|<(2A/h+Bh/2)$ then

$\Rightarrow |f'(x)|<2\sqrt{(AB)}$

Examples

Ex. 1. Verify the truth of Rolle's theorem for the function $f(x)=x^2-3x+2$ in the interval (1, 2).

when $x=1$, then $f(1)=1-3+2=0$

when $x=2$, then $f(2)=4-6+2=0$

$\therefore\ f(1)=f(2)$.

Now $f'(x)=2x-3$

If $f'(x)=0$ then $2x-3=0$ or, $x=3/2=1.5$.

Thus we see that $f(x)$ is continuous in $1\leqslant x\leqslant 2$.

$f'(x)$ exists in $1<x<2$

and $f(1)=f(2)$ There exists a point $x=1.5$.

within the interval (1, 2) such that, $1<1.5<2$

where $f'(x)=0$ i.e., $f(1.5)=0$

Hence Rolle's theorem is verified.

Ex. 2. Verify Rolle's theorem for the function.

$f(x)=x^3-x^2-4x+4$

$f(x)=x^3-x^2-4x+4$

If $f(x)=0$, then $x^3-x^2+4x+4=0$ or, $(x-1)(x+2)(x-2)=0$

$\therefore\ x=1, 2, -2$

So $f(x)=0$ for $x=1$, $x=2$, and $x=-2$

i. e. $f(1)=0$ $f(2)=0$, $f(-2)=0$

Now $Rf'(x)=\lim\limits_{h\to0+}\dfrac{f(x+h)-f(x)}{h}$

$$=\lim_{h\to0}\frac{(x+h)^3-(x+h)^2-4(x+h)+4-x^3+x^2+4x-4}{h}$$

$=3x^2-2x-4$

$Lf'(x)=\lim\limits_{h\to0}\dfrac{f(x-h)-f(x)}{h}=3x^2-2x-4,\quad (h>0)$

Thus $f'(x)$ exists as a finite differential co-efficient in the interval $-2<x<2$.

$\therefore\ f'(x)=3x^2-2x-4$

$f(x)$ is continuous in the interval $-2\leqslant x\leqslant 2$.

Now $f(x)$ is also continuous in $-2\leqslant x\leqslant 1$ and $1\leqslant x\leqslant 2$

$f(x)$ is differentiable in $2<x<1$ and $1<x<2$

Also $f(-2)=f(1)=f(2)=0$

Hence all conditions of Rolle's theorem are satisfied.

Therefore $f'(x)=0$ at a point in $(-2, 1)$ and also at a point in $(1, 2)$

Now $f'(x)=0\Rightarrow 3x^2-2x-4=0$ giving

$$x=\frac{2\pm\sqrt{(4+4.4.3)}}{6}=\frac{2\pm2\sqrt{(13)}}{6}=\frac{1\pm3.6}{2}=0.87,\ 1.55$$

Thus $f'(x)=0$ for $x=-0.87$ which lies between -2 and 1 and $f'(x)=0$ again for $x=1.55$ which lies between 1 and 2,

Hence Rolle's theorem is verified.

Ex 3. Verify the truth of Rolle's theorem for the function.

$f(x)=1-\sqrt[5]{x^2}$

The function $f(x)$ is continuous in the interval $-1\leqslant x\leqslant 1$.

$f(1)=0$ and $f(-1)=0$

Again $f'(x)=-\dfrac{2}{5}x^{-3/5}=-\dfrac{2}{5}\dfrac{1}{\sqrt[5]{x^3}}$

But $f'(x)$ does not vanish in the interval $-1<x<1$, besides $f'(x)$ does not exist at $x=0$

Hence Rolle's Theorem does not hold good.

Ex. 4. Verify Mean Value Theorem for the function.

$f(x)=2x-x^2$ in the interval (0, 1).

The function $f(x)=2x-x^2$ ... ... [1]

is continuous in the intervl $[0,1]$ and differentiable for all values of x in the interval (a, b).

$$f'(x) = 2-2x \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Clearly $f'(x)$ is continuous for $o<x<1$

By the Mean value theorem, we have

$f(1)-f(o)=(1-o)f'(c)$ where c is a point such that $0<c<1$

or, $(2-1)-0=f'(c)$ or, $f'(c)=1$

or, $2-2c=1$ [from (2)] or. $c=\frac{1}{2}$

Since $0<\frac{1}{2}<1$.

Hence the Mean value Theorem verified.

Ex. 5. At what point is the tangent by the curve $y=x^3$ parallel to the chord joining the points (1,1) to (2, 8,)

we have $f(x)=x^3$ ... ... (1)

The function $f(x)$ is continuous in $[1,2]$ and differentiable in the interval $(1, 2)$.

Now $f'(x)=3x^2$ ... ... (2)

exists at every point in the interval $(1, 2)$

By Mean value theorem, we have

$f(2)-f(1)=(2-1)f'(c)$ we have

c is a point in $1<c<2$.

or, $8-1=3c^2$ or, $c=1.15$

Now $f(1.15)=(1.15)^3=2.68$

The tangent at (1.15 ; 2.68) is parallel to the chord passing through the points (1,1) and (2,8)

Ex. 5. (a) show that $(x-\sin x)$ is a steadily increasing function in $0\leqslant x\leqslant\frac{1}{2}\pi$.

Let $y=x-\sin x$ ... ... (1)

$\therefore$ $dy/dx=f'(x)=1-\cos x$ ... (2)

Now for values of x in $0\leqslant x\leqslant\frac{1}{2}\pi$, the values of $\cos x$ change from 1 to 0 i.e., $0\leqslant\cos x\leqslant 1$.

Therefore $1-\cos x$ is always positive

Hence $f'(x)$ is always positive i. e., $f'(x)>0$ for $0\leqslant x\leqslant\pi/2$

Then by cor Art. 7.5 we see that $f(x)$ is a steadily (monotonically) increasing function of x in the given interval.

Ex. 6. If $f(x+h)=f(x)+hf'(x)+\dfrac{h^2}{2!}f''(x+\theta h)$

where $f(x)=(x-a)^{5/2}$ then

show that $\theta = \dfrac{64}{225}$ for $x=a$

$$f(x+h)=f(x)+hf'(x)+\frac{h}{2!}f''(x+\theta h) \quad \ldots \quad \ldots \quad (1)$$

and $f(x)=(x-a)^{5/2}$ ... ... (2)

$\therefore$ $f'(x)=5(x-a)^{3/2}/2$

$f''(x)=(5/2)(3/2)\sqrt{(x-a)}=15\sqrt{(x-a)}/4$

when $x=a$, then $f(a)=0$, $f(a+h)=h^{5/2}$

$f''(a+\theta h)=15\sqrt{(\theta h)}/4$; $f'(a)=0$

In (1), put $x=a$; then

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a+\theta h)$$

$$\therefore \quad (h)^{5/2}=\frac{h^2}{2!}\cdot\frac{15}{4}\theta^{1/2}h^{1/2} \text{ or, } \sqrt{\theta}=8/15 \text{ or, } \theta=64/225$$

Ex. 7. Expand $x\operatorname{cosec}x$ in a series of ascending powers up to the fourth power of x inclusive. D. U, 1966

$\dfrac{x}{\sin x}$ is not defiened at $x=0$, Hence

$\left(\dfrac{x}{\sin x}\right)$ does not possess a Taylor siries expansion.

However, $\left(\frac{x}{\sin x}\right)\to 1$ as $x\to 0$, Hence a sine expansion of $\left(\frac{x}{\sin x}\right)$ can be obtained by Binomial theorem :

$$\frac{x}{\sin x}=\frac{x}{x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}\cdots\quad\cdots}$$

$$=\left\{1-\left(\frac{x^2}{3!}-\frac{x^4}{5!}+\frac{x^6}{7!}\cdots\quad\cdots\right)\right\}^{-1}$$

$$=1+\left(\frac{x^2}{3!}-\frac{x^4}{5!}+\cdots\cdots\right)+\left(\frac{x^2}{3!}-\frac{x^4}{5!}+\cdots\cdots\right)$$

$$+\quad\cdots\quad\cdots\quad\cdots$$

$$=1+\frac{x^2}{6}-\frac{x^4}{120}+\frac{x^4}{30}+\text{terms}$$

containing x^5 and higher powers of x

or, $x\operatorname{cosec}x=1+\frac{x^2}{6}+\frac{7x^4}{360}+\quad\cdots\quad\cdots\quad\cdots$

Ex. 8. Expand $e^x\cos x$ in a finite series in powers of x with Lagrange's form of Remainder. R.U. 1950

Let $f(x)=e^x\cos x=f(o)+xf'(o)+\frac{x^2}{2!}f''(o)+\quad\cdots\quad\cdots$

$$+\frac{x^n}{n!}f^n(\theta x)\quad\cdots\quad\cdots\quad\cdots\quad(1)$$

$f(x)=e^x\cos x\quad\cdots\quad\cdots\quad\cdots\quad(2)$

$f^n(x)=2^n/^2\, e^x\cos(x+\frac{1}{4}n\pi)\ldots\quad\cdots\quad(3)$

$f^n(\theta x)=2^n/^2\, e^{\theta x}\cos(\theta x+\frac{1}{4}n\pi)$

From [1] and [2] when $x=0$

$f(o)=1,\ f'(o)=\sqrt{2}\cos\pi/4=1$

$f''(o)=2\cos 2\pi/4=0.\quad f'''(o)=-2^3/^2\frac{1}{\sqrt{2}}=-2$

$f^{iv}(o)=2^4/^2\cos\pi=-4$ and so on.

Now from (1), we have

$e^x\cos x=1+x-\frac{x^3}{3}-\frac{x^6}{6}\ldots-\frac{x^n}{n!}f^n(\theta x)$ where $0<\theta<1$

Lagrange's Remainder is $\frac{x^n}{n!}f^n(\theta x)=\frac{x^n}{n!}2^n/^2 e\theta^x\cos\ (\frac{1}{2}n\pi+\theta x)$

Ex. 9. Expand $\log(1+x)$ in powers of x by Maclaurin's Theorem R. U. 1962

Let $f(x)=\log(1+x)=f(o)+xf'(o)+\frac{x^2}{2!}f''(o)\ldots\ldots$

$$+\frac{x^n}{n!}f^n(\theta x),\ 0<\theta<1$$

Here $f(x)=\log\ (1+x),\ f'(x)=\frac{1}{1+x}=(1+x)^{-1}$,

$f^n(x)=\frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$ which exists for all values of n for $x\neq-1$

$f^n(o)=(-1)^{n-1}(n-1)!$ when $x=0$

Therefore $f(o)=0,\ f'(o)=1,\ f''(o)=-1$

$f'''(o)=2,\ f^{iv}(o)=-3$, and so on

Now $R_n=\frac{x^n}{n!}f^n(\theta x)=\frac{x^n}{n!}(-1)^{n-1}(n-1)!\ \frac{1}{(1+\theta x)^n}$

$$=(-1)^{n-1}\frac{1}{n}\left(\frac{x}{1+\theta x}\right)^n$$

$\therefore\ \log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots+(-1)^{n-1}\frac{1}{n}\left(\frac{x}{1+\theta x}\right)^n$

(i) If $0 < x < 1$ then

$$\lim_{n \to \infty} \left(\frac{x}{1+\theta x}\right)^n = 0 \text{ as } \left|\frac{x}{1+\theta x}\right| < 1$$

Hence $R_n = 0$ if $n \to \infty$;

(ii) If $-1 < x < 0$. in this case $\frac{x}{1+\theta x}$ may not be numerically less than 1

So $\lim_{n \to \infty} \left(\frac{x}{1+\theta x}\right)^n \neq 0$

Thus we fail to get the definite value of R_n from Lagrange's form of Remainder.

From Cauchy's form of Remainder.

$$R_n = \frac{x^n}{\lfloor (n-1)} (1-\theta)^{n-1} f^n(\theta x)$$

$$= \frac{x^n}{\lfloor (n-1)} (1-\theta)^{n-1} \lfloor (n-1) \frac{1}{(1+\theta x)^n}$$

$$= (-1)^{n-1} \frac{x^n}{1+\theta x} \left(\frac{1-\theta}{1+\theta x}\right)^{n-1}$$

As $\frac{1-\theta}{1+\theta x} < 1$ then $\lim_{n \to \infty} \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} = 0$

So $R_n = 0$ if $n \to \infty$ for $-1 < x < 1$

Hence the expansion of

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \dots (-1)^{n-1} \frac{x^n}{n} + \dots$$

Ex. 10. Use Maclaurin's theorem to expand $\tan^{-1} x$ into an infinite series of ascending powers of x. [R U. 58, 65]

Let $f(x) = \tan^{-1} x = f(o) + x f'(o) + \frac{x^2}{2!} f''(o) + \frac{x^3 f'''(o)}{3!} + \dots (1)$

Here $f(x) = \tan^{-1} x$, $f'(x) = \frac{1}{1+x^2}$

$f^n(x) = (-1)^{n-1}(n-1)!\, \sin^n \phi \sin n\phi$ see Art. 6.6

where $\tan\phi = \frac{1}{x}$. $\sin\phi = \frac{1}{\sqrt{(1+x^2)}}$

$$\therefore f^n(o) = (-1)^{n-1} (n-1)! \left(\frac{1}{1+x^2}\right)^{n/2} \sin\left(n \sin^{-1} \frac{1}{\sqrt{(1+x^2)}}\right)$$

$$f^n(o) = (-1)^{n-1}(n-1)! \sin(n\pi/2)$$

$\therefore f(o) = 0, f'(o) = 1, f''(o) = 0, f'''(o) = -(2!) = -2!$

$f^{iv}(o) = 0, f^{v}(o) = 4!$ and so on

Now from (1)

$\tan^{-1} x = x - \frac{1}{3}x^3 + x^5/5$

We may get the expansion of $\tan^{-1} x$ in an ascending powers of x by integrating a known series. (See below)

Ex. 11. Expand $\tan^{-1} x$ in an ascending powers of x by integrating a series.

Let $y = \tan^{-1} x$.

$$\therefore \frac{dy}{dx} = \frac{1}{1+x^2} = (1+x^2)^{-1}$$

or, $\frac{dy}{dx} = 1 - x^2 + x^4 - x^6 + x^8 - \dots + (-1)^n x^{2n} + \dots [\dots if |x^2| < 1)$

or, $dy = \{1 - x^2 + x^4 - x^6 + x^8 - \dots + (-1)^n x^{2n} + \dots\dots\} dx$

Now integrate both sides.

$$y = x - \frac{1}{3}x^3 + x^5/5 - x^7/7 + \dots + \frac{(-1)^n x^{2n+1}}{2n+1} + \dots\dots$$

(The constant of integration is zero since $\tan^{-1} 0 = 0$)

or, $\tan^{-1} x = x - \frac{1}{3}x^3 + x^5/5 - \dots + \frac{(-1)^n x^{2n+1}}{2n+1} + \dots \dots$

Ex. 12. If the expansion of

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots \dots$$

Show that the expansion of

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \dots$$

Since

$\cos x = 1 - x^2/2! + x^4/4! - x^6/6! + x^8/8! - \dots \dots$

Integrating both sides w. r to x.

$$\sin x = x - \frac{x^3}{2!\,3} + \frac{x^5}{5\,4!} - \frac{x^7}{7.6!} + \dots \dots$$

or, $\sin x = x - x^3/3! + x^5/5! - x^7/7! + \dots \dots$

[for all values of x, the series is convergent]

Ex. 13. Find the expansion of $\tan x$ if

$$\log \sec x = \frac{x^2}{2!} + \frac{x^4}{4!} + 16\frac{x^6}{6!} + \dots \dots$$

Since $\log \sec x = \dfrac{x^2}{2!} + \dfrac{2x^4}{4!} + 16\dfrac{x^6}{6!} + \dots \dots$

Differentiate both sides w. r. to x then.

$$\frac{\sec x \tan x}{\sec x} = \frac{2x}{2!} + \frac{2\,4x^3}{4!} + 16\,6\,\frac{x^5}{6!} + \dots \dots$$

or, $\tan x = x + \frac{1}{3}x^3 + (2/15)x^5 + \dots\dots$

Ex. 14. If $y = e^{\tan^{-1}x} = a_0 + a_1x + a_2x^2 + \dots\dots + a_nx^n + \dots\dots$

Show that (i) $(n+2)a_{n+2} + na_n = a_{n+1}$

(ii) $e^{\tan^{-1}x} = 1 + x + \frac{1}{2}x^2 - \frac{1}{6}x^3 \dots\dots$

Since

$$y_1 = e^{\tan^{-1}x} = a_0 + a_1x + a_2x^2 + \dots\dots + a_nx^n + a_{n+1}x^{n+1} + a_{n+2}x^{n+2} + \dots \quad \dots \quad (1)$$

$\therefore\ y_1 = e^{\tan^{-1}x}\dfrac{1}{1+x^2} = a_1 + 2a_2x + 3a_3x^2 + \dots\dots + na_nx^{n-1}$

$+ (n+1)a_{n+1}x^n + (n+2)a_{n+2}x^{n+1} + \dots\dots(2)$

$\therefore\ y_2 = 2a_2 + 6a_3x + \dots + n(n-1)a_nx^{n-2} + (n+1)na_{n+1}x^{n-1}$

$+ (n+1)(n+2)a_{n+2}x^n + \dots\dots(3)$

From (2).

$$y_1(1+x^2) = e^{\tan^{-1}x} = y$$

Differentiating

$y_2(1+x^2) + 2xy_1 = y_1$ or, $y_2(1+x^2) + (2x-1)y_1 = 0 \dots\dots(4)$

Put the values of y_1 and y from (2) and (3) in (4) then

$(1+x)^2\{2a_2 + \dots\dots n(n-1)a_nx^{n-1} + n(n+1)a_{n+1}x^{n+1} +$

$(n+1)(n+2)(a_{n+2}x^n) + \dots\dots\} + (2x-1)\{a_1 + \dots\dots$

$+ na_nx^{n-1} + (n+1)a_{n+1}x^n + (n+2)a_{n+2}x^{n+1} + \dots\dots\} = 0$

Equating the co-efficients of x^n from both sides,

$(n+1)(n+2)a_{n+2} + n(n-1)a_n + 2na_n - (n+1)a_{n+1} = 0$

or, $(n+1)(n+2)a_{n+2} + n(n+1)a_n = (n+1)a_{n+1}$

or, $(n+2)a_{n+2} + na_n = a_{n+1}$ ... ... (5)

when $x = 0$ in (1), and (2),

$1 = a_0, 1 = a_1$

Putting $n = 0, 1, 2, \dots$ ...successively in (5),

$2a_2 = a_1$ or, $2a_2 = 1$ or, $a_2 = \frac{1}{2}$

$3a_3 + 1a_1 = a_2$ or, $3a_3 + 1 = \frac{1}{2}$ or, $a_3 = -\frac{1}{6}$

Continuting this way we may get as many terms as we like

Thus

$$e^{\tan^{-1}x} = 1 + x + \tfrac{1}{2}x^2 - \tfrac{1}{6}x^3 \dots \dots$$

Ex. 15. If $\varphi(x)$ is continuous for $a \leqslant x \leqslant b$ and $\varphi''(x)$ exists and positive in $a < x < b$ then $\frac{\varphi(x)-\varphi(a)}{x-a}$ increases steadilly and strictly for $a<x<b$.

Hints. $f(x)=\frac{\varphi(x)-\varphi(a)}{x-a}$

Since $\varphi''(x)$ exists obviously as a finite differential co efficient $\varphi'(x)$ is continuous and therefore finite in $a<x<b$. Also $\varphi(x)$ is given as a continuous function. Hence by the second mean value theorem.

$$f(x)=\frac{\varphi(x)-\varphi(a)}{x-a}=\varphi'(a)+\frac{x-a}{\lfloor 2}\varphi''(\xi)$$

where $a<x<b$ and $a<\xi<b$

Hence $f'(x)=\frac{\varphi''(\xi)}{\lfloor 2}$ = a positive quantity by hypothesis throughout the interval $a<x<b$.

Now $f(x)=\frac{\varphi(x)-\varphi(a)}{x-a}$ increases steadily and strictly throughout the same interval $a<x<b$.

Ex. 16 If $\varphi(x)=\begin{vmatrix} f(a) & f(b) & f(x) \\ g(a) & g(b) & g(x) \\ h(a) & h(b) & h(x) \end{vmatrix}$

Where $a<c<b$ and $f(x)$, $g(x)$, $h(x)$ are continuous in the closed interval $(a\ b)$ and twice differentiable in the open interval (a,b) Prove that there exists a number ξ such that $a<\xi<b$ and $\varphi(c)=\frac{1}{2}(c-a)(c-b)\varphi''(\xi)$

Let $\psi(x)\begin{vmatrix} f(a) & f(b) & f(x) \\ g(a) & g(b) & g(x) \\ h(a) & h(b) & h(x) \end{vmatrix}-\frac{(x-a)(x-b)}{(c-a)(c-b)}\begin{vmatrix} f(a) & f(b) & f(c) \\ g(a) & g(b) & g(c) \\ h(a) & h(b) & h(c) \end{vmatrix}$

Here $\psi(a)=\psi(c)=\psi(b)=0$

By Rolle's Theorem

$\psi'(\xi_1)=0,\ a<\xi_1<c$

$\psi'(\xi_2)=0,\ c<\xi_2<b$

Now for (1)

$$\psi(x)=\varphi(x)-\frac{(x-a)(x-b)}{(c-a)(c-b)}\varphi(c)$$

$$\because\quad \psi'(x)=\varphi'(x)-\left\{\frac{(x-a)}{(c-a)(c-b)}+\frac{x-b}{(c-a)(c-b)}\right\}\varphi(c)$$

$$\text{and } \psi''(x)=\varphi''(x)-\frac{2}{(c-a)(c-b)}\varphi(c) \quad \ldots \quad \ldots \quad (2)$$

As $\psi'(\xi_1)=\psi'(\xi_2)=0$, then by Rolle's theorem,

$\psi''(\xi)=0,\ \xi_1<\xi<\xi_2$

or, $\varphi''(\xi)-\frac{2}{(c-a)(c-b)}\varphi(c)=0$

or, $\varphi(c)=\frac{1}{2}(c-a)(c-b)\varphi''(\xi)$

Ex. 17. Show that

$f(b)-f(a)=\xi f'(\xi)\log(b/a)$, where $f(x)$ is continuous and differentiable in (a,b) and $a<\xi<b$.

From Cauchy's Mean Value Theorem

$$\frac{f'(\xi)}{g'(\xi)}=\frac{f(b)-f(a)}{g(b)-g(a)} \quad \ldots \quad \ldots \quad (1)$$

If $g(x)=\log x$, $g'(x)=1/x \neq i.e$; $g'(\xi)=1/\xi$.

putting $g'(\xi)=1/\xi$ in (1)

$$\frac{f'(\xi)}{1/\xi}=\frac{f(b)-f(a)}{\log(b)-\log(a)}=\frac{f(b)-f(a)}{\log(b/a)}$$

or, $f(b)-f(a)=\xi f'(\xi)\log(b/a)$

Ex. 18. If $y=\sqrt{(1-x^2)}\sin^{-1}x$, then prove that

$(1-x^2)y_n=(2n-3)xy_{n-1}+(n-1)(n-3)y_{n-2}$

and find the co-efficient x^7 in the expansion. C. H, 1986

Sol. $y=\sqrt{(1-x^2)}\sin^{-1}x$ ... ... (1)

$$y_1=\sqrt{(1-x^2)}\times\frac{1}{\sqrt{(1-x^2)}}+\frac{\sin^{-1}x(-2x)}{2\sqrt{(1-x^2)}}$$

$$=1-\frac{x\sqrt{(1-x^2)}\sin^{-1}x}{(1-x^2)}$$

or $y_1(1-x^2)=(1-x^2)-xy$ [by (1)] ... (2)

Differentiating this,

$y_2(1-x^2)-2xy_1=-2x-xy_1-y$

or, $y_2(1-x^2)=-2x+xy_1-y$ ... ... (3)

$\therefore$ $y_3(1-x^2)-2xy_2=-2+xy_2+y_1-y_1$

or $y_3(1-x^2)=-2+3xy_2$ ... ... (4)

Differentiating both sides of (4) $(n-3)$ times w. r. to x,

$$y_n(1-x^2)+\frac{(n-3)}{1}y_{n-1}(-2x)+\frac{(n-3)(n-4)}{1.2}y_{n-2}(-2)$$

$$=3xy_{n-1}+3\frac{(n-3)}{1}y_{n-2} \quad (1)$$

$\Rightarrow(1-x^2)y_n=(2n-3)xy_{n-1}+(n-1)(n-3)y_{n-2}$...(5) (proved)

where $n-3\leq$ or $n\geq4$

From (1)—(4)

$y(o)=0$, $y_1(o)=1$, $y_2(o)=-y(o)=0$, $y_3(o)=-2$. ...(6)

Now from (5), putting $n=4,5,6,7$ successively,

$y_4(o)=(3)(1)y_2(o)=0$

$y_5(o)=(4)(2)y_3(o)=8(-2)=-16$

$y_6(o)=(5)(\ )y_4(o)=0$

$y_7(o)=(6)(4)y_5(o)=(24)(-16)=-384$,

Hence the coefficient of x^7 in the expansion of $\sqrt{1-x^2}\sin^{-1}x$ is

$$\frac{y_7(o)}{7!}=\frac{-384}{7!}=-\frac{8}{105} \quad \text{[by Maclaurin's theorem]}$$

Ex. 19. Expand $2x^3+7x^2+x-1$ in powers of $(x-2)$

(ii) log x in powers of (x−2) C.U.1992

We know that

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+...$$

Here $f(x)=2x^3+7x^2+x+1$ and $a=2$

$\therefore$ $f(a)=f(2)=2\,2^3+7.2^2+2-1=45$

$f'(x)=6x^2+14x+1$, $f'(a)=f'(2)=6.2^2+14.2+1=53$

$f''(x)=12x+14x$; $f''(a)=f''(2)=12.2+14=38$

$f'''(x)=12$; $f'''(a)=f'''(2)=12$

Hence $2x^3+7x^2+x-1$

$$=45+53(x-2)+\frac{38}{2!}(x-2)^2+\frac{12}{3!}(x-2)^3$$

$$=45+53(x-2)+19(x-2)^2+2(x-2)^3$$

Ex 20. Expand $\sin x$ in powers of $x-\frac{1}{2}\pi$.

We have

$f(x)=f(\frac{1}{2}\pi+x-\frac{1}{2}\pi)$ and $a=\frac{1}{2}\pi$

$$f(x)=f(a)+(x-a)f'(a)+\frac{(x-a)^2}{\lfloor 2}f''(a)+...$$

$$\Rightarrow f(x)=\sin x=1-\frac{1}{2!}(x-\tfrac{1}{2}\pi)^2+\frac{1}{4!}(x-\tfrac{1}{2}\pi)^4+...$$

Ex. 21. $a=-1$ $b>1$ and $f(x)=\frac{2}{|x|}$, show that the conditions of Lagrange's mean value theorem are not satisfied in the interval (a, b), but the conclusion of the theorem is true, if and only if $b>1+\sqrt{2}$.

Sol : For h +ve,

$$f(0+h)=\frac{1}{|0+h|}=\frac{1}{h}\ ;\ f(0-h)=\frac{1}{|0-h|}=\frac{1}{h}......(1)$$

$f(0)$ is not defined. Let $f(0)=A$, according to the definition of the function A is definite finite number.

Hence, $\lim\limits_{h\to 0}\dfrac{f(0+h)-f(0)}{h}=\lim\limits_{h\to 0}\dfrac{1/h-A}{h}=\lim\limits_{h\to 0}\dfrac{1}{h}$

$$\times\lim_{h\to 0}\left(\frac{1}{h}-A\right)=\infty\times\infty=\infty \quad\ldots\quad\ldots\quad (2$$

Aso $\lim\dfrac{(0-h)-f(0)}{-h}=\lim\dfrac{1/h-A}{-h}=\lim\limits_{h\to 0}\dfrac{1}{-h}\times\lim\limits_{h\to 0}\left(\dfrac{1}{h}-A\right)$

$$=-\infty\times\infty=\infty \quad\ldots\quad\ldots\quad (3)$$

From (2) and (3), it is noticed that the function is not differentiable at $x=0$

The conditions of $M.V.$ Theorem are not satisfied (a, b) which includes 0 also. Hence the conclusion is

$$\frac{f(b)-f(a)}{b-a}=f'(c),\ a<c<b \quad\ldots\quad\ldots\quad (4)$$

It is true for $x=c$, $a<c<b$,

$$\frac{\dfrac{1}{|b|}-\dfrac{1}{|a|}}{b-a}=\frac{d}{dc}\left(\frac{1}{|c|}\right)=-\frac{1}{|c|^2}=-\frac{1}{c^2}$$

or, $\dfrac{1/b-1/1}{b-(-1)}=\dfrac{1}{c^2}$ or, $\dfrac{1-b}{b(b+1)}=-\dfrac{1}{c^2}$ or, $c^2=\dfrac{b^2+b}{b-1}$

or, $\dfrac{b^2+b}{b-1}<b^2$ or, $b^2<c^2$

or, $\dfrac{b+1}{b-1}<b$ or, b is +ve, which can be divided by b,

or, $b^2-2b>1$ or, $b^2-2b+1>2$

or, $(b-1)^2>2$ or, $b=1+\sqrt{2}$

Under this condition the conlusion of $M.\ V.T.$ is true although the conditions for the validity of the theorem are not fulfilled.

See APPENDIX— Art. 17.23, Art. 17.24, Art. 17.25

Exercise VII

1. Verify Rolle's Theorem for the following forms

[i] $f(x)=x^2+6x-6$ in the interval $[-6, 1]$

[ii] $f(x)=x^2$ in $[-2, 2]$

[iii] $f(x)=\sin x/e^x$ in $[0\ \pi]$

[iv] $f(x)=(x-2)(x-3)(x-4)$ in $[2\ \ 3]$

2. Verify Rolle's theorem for the function

$f(x)=x(x+5)\,e^{-x/2}$

3. Verify Rolle's theorem for the function

$f(x)=2x^3+x^2-4x-2$

4. Verify Rolle's theorem for the function

$f(x)=3x^3+7x^2-11x-15$

5. Verify the truth of Rolle's theorem for the function

(i) $f(x)=x^3-7x^2+36$ (ii) $f(x)=\log\dfrac{x^3+ab}{a+b}$

6. Verify Rolle's theorem for the function in $(-\pi/4, \pi/4)$

(i) $f(x)=\cos^2 x$ (ii) $f(x)=e^x(\sin x-\cos x)$

7. Verify Rolle's Theorem for the following functions.

(i) $f(x)=1-x^{4/5}$ in $[0, 2]$

(ii) $f(x)=\sqrt[3]{(x^2-5x+6)}$ in $[2, 3]$

(iii) $f(x)=2+(x-1)^{2/3}$

8. Verify the Mean Value Theórem for the function

(i) $f(x)=x-x^3$ in the interval $[-2, 1]$

(ii) $f(x)=3+2x-x^2$ in $[0,1]$ D. U. 1988

9. Using Rolle's Theorem, show that $f(x)$ cannot have equal values for two distinct values of x for the function

$f(x)=2x^3+x^2+6x$.

10. Verify the Mean Value Theorem for the function

$f(x)=lx^2+mx+n$ in the interval $[a, b]$

11. At what point is the tangent to the curve $y=\log x$ parall[illegible] to the chord passing through the points $(1,0)$ and (c,o).

Ans. $c=e$[illegible]

12. Does the Mean Value Theorem hold good for the functi[illegible] $f(x)=1/x, (x\neq 0), f(o)=0$ in the interval $[-1,1]$?

13. AB is a chord of the parabola $y=x^2$ passing through t[illegible] points $(1, 1)$ and $(3, 9)$; Show that tangent at $(2, 4)$ to the cu[illegible] is parallel to the chord AB.

14. Find the value of c in the Mean Value Theorem

$f(b)-f(a)=(b-a) f'(c)$

(i) if $f(x)=x^{4/3}$ in $(-1, 1)$ (ii) If $f(x)=e^x$ in $(0, 1)$

(iii) if $f(x)=x^3-2x^2+3x-2$ in $[0, 2]$

(iv) if $f(x)=x^4-2x^3+x^2-2x$ in $[-1, 2]$

(v) if $f(x)=x^2$ in $(1,2)$ (vi) if $f(x)=\sin \pi x/2$ in $[0,1]$

(vii) if $f(x)=\sqrt{(x^2-4)}$ in $[5, 3]$

(viii) if $f(x)=\dfrac{-2x+3}{3x-2}$ in $[1,4]$ (ix) If $f(x)=(x-1)(x$[illegible] in $[0,4]$

15. Verify Cauchy's Mean Value Theorem for the functi[illegible] $f(x)=x^2+2$ and $g(x)=x^3-1$ in the interval $[1, 2]$

16. Verify that Cauchy's Mean Value Theorem fails for [illegible] functions, $f(x)=x^2$ and $g(x)=x^3$ in the interval $[-1, 1]$

16. (i) State and prove Cauchy's Mean Value Theorem [illegible] deduce that

$f(b)-f(a)=cf'(c) \log(b/a)$ where $f(x)$ is continuous [illegible] differentiable in (a,b) and $a<x<b$

Hints : Put $g(x)=\log x$.

17. If $f'(x)=0$ in $1\leq x\leq 3$ and if $f(2)=2$, show that f[illegible] throughout the interval $1\leq x\leq 3$

[illegible] If $y=2x-\tan^{-1}x-\log\{x+\sqrt{(1+x^2)}\}$, show that y [illegible] with x in $0\leq x\leq\infty$

[illegible] In the Mean value Theorem.

$f(a+h)=f(a)+hf'(a+\theta h)$, $0<\theta<1$,

If $f(x)=\frac{1}{3}x^3-\frac{3}{2}x^2$ find the value of θ in the interval $[0,3]$

[illegible] (i) Applying mean value Theorem show that

$$\frac{a-b}{1+a^2}<\tan^{-1}a-\tan^{-1}b<\frac{a-b}{1+b^2}$$ D. H. 1986

[illegible] Show that

$$(x+h)^{3/2}=x^{3/2}+\tfrac{3}{2}x^{1/2}h+\frac{3.1\,h^2}{2.2.2!}\frac{1}{\sqrt{(x+\theta h)}},\ 0<\theta<1,$$

and find the value of θ when $x=0$.

[illegible] (i) If $f''(x)$ exists then show that

$$f''(x)=\lim_{h\to 0}\frac{f(x+h)+f(x-h)-f(x)}{h^2}$$ D. H. 1984

[illegible] Let $f(x)=\dfrac{2x+3}{3x+2}$ in $[1, 0]$

Show that there is no number c between -1 and 0.

[illegible] (a) Expand the following functions in powers of h with [illegible]'s (Schlomiltch, Lagrange and Cauchy's form)

(i) $\sin(x+h)$ (ii) e^{x+h} (iii) $\log(x+h)$ (vi) $(x+h)^m$ (v) $e^{m\tan^{-1}(x+h)}$

[illegible] (b) Expand the following functions in powers of x with [illegible]'s (Schlomiltch Lagrange and Cauchy's form)

(i) [illegible]x, (ii) $\cos^{-1}x$ (iii) e^x (iv) $e^{2x}\cos bx$

[illegible] Show that expansion of $x\cot x$ as far as the term [illegible] x^4 is $1-x^2/3-x^4/45+\ldots\ \ldots\ \ldots$

[illegible] Prove that $e^x \log(1+x)=x+x^2/2!+2x^3/3!+9x^5/5!+\ldots$

[illegible] Prove that the expansion of $e^{\sin x}$ in a series up to the [illegible] is $1+x+\frac{1}{2}x^2-\frac{1}{8}x^4-\ldots\ \ldots$

[illegible] Prove that $\log\sec x=\frac{1}{2}x^2+\frac{1}{12}x^4+\frac{1}{45}x^6+\ldots$; $\log\cos x$

N.U. 1994.

26. Show that $\cos^2 x = 1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6 + \ldots\ldots$

27. Prove that

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \frac{x^2}{6.2\,!} - \frac{1}{30}\frac{x^4}{4\,!} + \ldots\ldots +$$ D. U. 195

28. Show that

$\sin^{-1} x = x + \frac{1}{2}\cdot\frac{1}{3}x^3 + \frac{1}{2}\cdot\frac{3}{4}\frac{1}{5}x^5 + \ldots\ldots$

29. Prove that

$e^x \cos x = 1 + x + x^2/2 - x^3/3 - 11x^4/24 - \quad \ldots \quad \ldots$

30. Prove that $(1+x)^x = 1 + x^2 - \frac{1}{2}x^3 + \frac{5}{6}x^4 - \frac{3}{4}x^5 \ldots\ldots$

31. Prove that $\log(1 + \sin x) = x - x^2/2 + x^3/6 - x^4/12 + \ldots$ N. U.

32. Find the expansion of $\sin(e^x - 1)$ upto and including the term x^4. *Ans.* $x + \frac{1}{2}x^2 - (\frac{5}{24})x^4 + \ldots\ldots$

32. (i) If $\tan^{-1} x$ in powers of $x - \frac{1}{4}\pi$ expanded, then

$$\tan^{-1} x = \tan^{-1}\frac{\pi}{4} + \left(x - \frac{\pi}{4}\right)\frac{1}{(1+\pi^2/16)} - \frac{(x_1 - \frac{1}{4}\pi)^2 \pi}{4(1+\pi^2/16)^2} + \ldots$$

(ii) Prove by Taylor's Theorem

(a) $\log(x+h) = \log x + \frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} - \ldots$

(b) $\frac{1}{x+h} = \frac{1}{x} - \frac{h}{x^2} - \frac{h^2}{x^3} - \frac{h^3}{x^4} \quad \ldots \ldots$

(c) $f\left(\frac{x}{1+x}\right) = f(x) - \frac{x}{1+x}f'(x) + \frac{x^2}{(1+x^2)}\frac{f''(x)}{2\,!} + \ldots$

(d) $\log \sin x = \log \sin 2 + (x-2)\cot 2 - \frac{1}{2}(x-2)^2 \operatorname{cosec}^2 2 + \frac{1}{3}(x-2)^3 \operatorname{cosec}^2 2 \cot 2 + \ldots \ldots \ldots$

33. Show that Maclaurin's theorem fails to expand sin ...

34. Show that θ which occurs in the Lagrange's form of remainder, $R_n = \frac{h^n}{n\,!} f^n(a + \theta h)$ tends to the limits $\frac{1}{n+1}$ as $h \to 0$ Provided that $f^{n+1}(x)$ is continuous at a and $f^{n+1}(a) \neq 0$

... If $f''(x)$ exists for all points in (a, b) and $\frac{f(c) - f(a)}{c - a} = \frac{f(b) - f(c)}{b - c}$ ... $a < c < b$ then there is a number such that $a < \xi < b$ and $f''(\xi) = 0$

... If $y = \sin \log(x^2 + 2x + 1)$, prove that

$(x+1)^2 y_{n+2} + (2n+1)(x+1)y_{n+1} + (n^2+4)y_n = 0$

... expand y in ascending powers of x as far as x^4

... If $y = \tan(m \tan^{-1} x)$ show that first three terms in the ...aurin's series for y are

$mx + m(m^2 - 1)x^3/3 + m(m^2 - 1)(2m^2 - 3)x^5/15$

... If $y = \sin(m \sin^{-1} x)$ show that

$(1 - x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2 - n^2)y_n = 0$ and hence find ... of (i) $\sin(m \sin^{-1} x)$ (ii) $\sin^{-1} x$.

... If $a_n x^n + a_{n+1}x^{n+1} + a_{n+2}x^{n+2}$ be three consecutive terms ... expansion of $\sqrt{(1 - x^2)}\sin^{-1} x$ in powers of x,

... that $a_{n+2} = \frac{n-1}{n+2} a_n$

... show that all even terms vanish, and that the expansion is

$x - \frac{1}{3}x^3 - \frac{2}{3.5}x^5 - \frac{2.4}{3.5.7}x^7 - \ldots\ldots$ R. H. 1955, C. H. 1986

... If $y = e^{a \sin^{-1} x} = a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_n x^n + \ldots$ ... that

(i) $(1 - x^2)y_2 - xy_1 + a^2 y$ (ii) $(n+1)(n+2)a_{n+2} = (n^2 + a^2)a_n$

... State any one of the fundamental properties of a continuous ... and show that there is one and only one positive root of ... $x^6 = 2$ by the property.

... State any one of the fundamental properties of a continu... ... other than one used in proving Rolle's Theorem,

43. The function u, v (of x) and their derivatives u', v' are continuous in $a \leqslant x \leqslant b$ and $uv' - u'v \neq 0$ in $a \leqslant x \leq b$. Show that between any two roots of $u=0$ lies one of v.

44. If $f'(x) > 0$ in $a \leqslant x \leqslant b$ then show that $f(x)$ is strictly increasing in $a \leqslant x \leqslant b$. Deduce $a^x > x^a$ if $x > a \geqslant e$.

45. If $\log_e y = \tan^{-1} x$, prove that

$$(1+x^2)y_n = \{1 - 2(n-1)x\}y_{n-1} - (n-1)(n-2)y_{n-2}$$ and

hence find the co-efficients of x^5 in the expansion of y by Maclaurin's Theorem.

46. If $\dfrac{(\tan^{-1}x)^2}{\lfloor 2} = \dfrac{a_2 x^2}{2} - \dfrac{a_4 x^4}{4} + \dfrac{a_6 x^6}{6} \ldots \ldots$

Prove that $a_{2n} - a_{2n-2} = \dfrac{1}{2n-1}$

47. If $\sin^{-1}x = \sum_{n=1}^{\infty} \dfrac{b_n x^n}{n!}$ and $\dfrac{(\sin^{-1}x)^3}{3!} = \sum_{n=1}^{\infty} \dfrac{a_n x^n}{n!}$

Show that $a_{n+1} = n^2 a_n + b_n$

48. If $x^2 + 2x = 2\log\left(c \dfrac{dx}{dy}\right)$

and $y = a_0 + a_1 x + \dfrac{a_2}{\lfloor 2} x^2 + \cdots \cdots$

Show that $a_{n+2} + a_{n+1} + n^2 a_n = 0$

49. If $f^n(x)$ is continuous in the given equation

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \ldots\ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(x + \theta h)$$

then prove that $\underset{h \to 0}{\text{Lt}}\ \theta = \dfrac{1}{n}$

50. Deduce from Cauchy's mean value theorem that

$f(b) - f(a) = g\, f'(g) \log\left(\dfrac{b}{a}\right)$ where $f(x)$ is continuous and differentiable in (a, b) and $a < g < b$.

51. If $\varphi''(x) > 0$ for all values of x, prove that

$$\varphi\left(\frac{x_1+x_2}{2}\right) < \frac{1}{2}\varphi(x_1) + \frac{1}{2}\varphi(x_2)$$

SOL. If we consider $\phi''(x)$ to be finite so that $\phi'(x)$ and $\phi(x)$ are both continuous and finite for all values of x. Now

$\phi(x_1) = \phi\,\{(x_1+x_2)/2 + (x_1-x_2)/2$ of the form

$$\phi(x_1+h) = \phi\{(x_1+x_2)/2\} + \{(x_1-x_2)/2\}\,\phi'\{(x_1+x_2)/2\} + \frac{1}{2!}\{(x_1-x_2)/2\}^2\,\phi''\{(x_1+x_2)/2 + \theta_1(x_1-x_2)/2\} + \ldots (1);\ 0 < \theta_1 < 1$$

Similarly

$$\phi(x_2) = \phi\{(x_1+x_2)/2\} + \{(x_2-x_1)/2\}\,\phi'\{(x_1+x_2)/2\} + \frac{1}{2!}\left(\frac{x_2-x_1}{2}\right)^2 \phi''\left\{\frac{x_1+x_2}{2} + \theta_2\frac{x_2-x_1}{2}\right\} + \ldots\ldots (2),\ 0 < \theta_2 < 1$$

Adding (1) and (2)

$$\phi(x_1) + \phi(x_2) = 2\phi\left\{\frac{1}{2}(x_1+x_2)\right\} + \frac{1}{2!}(x_1-x_2)^2\left[\phi''\left\{\frac{1}{2}(x_1+x_2) + \theta_1\frac{1}{2}(x_1-x_2)\right\} + \phi''\left[\frac{1}{2}(x_1+x_2) + \theta_2\frac{1}{2}(x_2-x_1)\right]\right]$$

But $\dfrac{1}{2}(x_1+x_2)$ is always positive. Also $\phi''(x)$ is positive for all values of x as a given condition.

$$\therefore\ \phi(x_1) + \phi(x_2) = 2\phi\left\{\frac{1}{2}(x_1+x_2)\right\} + \text{a positive quantity}$$

i.e. $2\phi\left\{\dfrac{1}{2}(x_1+x_2)\right\} < \phi(x_1) + \phi(x_2)$ or; $\phi\dfrac{(x_1+x_2)}{2} < \dfrac{1}{2}\{\phi(x_1) + \phi(x_2)\}$

See APPENDIX ~ Extra Sums
NO.52, NO.53, NO.54, NO.55, NO.56, NO.57
Art. 17.22; Art. 17.24. Art. 17.25.

প্রশ্নমালা VII

1. নিম্নলিখিত ফাংশনগুলির উপর রোলের উপপাদ্য পরীক্ষা কর।

(i) $[-6, 1]$ বিস্তারে ফাংশন $f(x)=x^2+6x-6$

(ii) $[-2, 2]$ বিস্তারে ফাংশন $f(x)=x^2$

(iii) $[0, \pi]$ বিস্তারে ফাংশন $f(x)=\sin x/e^x$

(iv) $[2, 3]$ বিস্তারে ফাংশন $f(x)=(x-2)(x-3)(x-4)$

2. $f(x)=x(x+5)\, e^{-x/2}$ এর জন্য রোলের উপপাদ্য প্রতিপাদন কর।

3. $f(x)=2x^3+x^2-4x-2$-এর জন্য রোলের উপপাদ্য প্রতিপাদন কর

4. $f(x)=3x^3+7x^2-11x-15$ ফাংশনটির জন্য রোলের উপপাদ্য প্রতিপাদন কর।

(Verify Rolle's theorem for the function)

$f(x)=3x^3+7x^2-11x-15$.)

5. (i) $f(x)=x^3-7x^2+36$ (ii) $f(x)=\log \dfrac{x^2+ab}{a+b}$

ফাংশনগুলির জন্য রোলের উপপাদ্যটির সত্যতা প্রতিপাদন কর।

6. $(-\pi/4, \pi/4)$ বিস্তারে ফাংশনদ্বয়

(i) $f(x)=\cos^2 x$ এবং (ii) $f(x)=e^x (\sin x-\cos x)$-এর জন্য রোলের উপপাদ্যের সত্যতা প্রতিপাদন কর।

7. নিম্নলিখিত ফাংশনগুলির জন্য রোলের উপপাদ্যের সত্যতা যাচাই কর।

(i) $[0, 2]$ বিস্তারে $f(x)=1-x^{4/5}$

(ii) $[2, 3]$ বিস্তারে ফাংশন $f(x)=\sqrt[3]{(x^2-5x+6)}$

(iii) $f(x)=2+(x-1)^{2/3}$

8. $[-2, 1]$ বিস্তারে ফাংশন $f(x)=x-x^3$ এর জন্য গড় মান উপপাদ্যের সত্যতা প্রতিপাদন কর।

(i) $[0, 1]$ বিস্তারে $f(x)=3+2x-x^2$ এর জন্য গড়মান উপপাদ্যের সত্যতা প্রমাণ কর। D. U.

9. রোলের উপপাদ্যটির ব্যবহার করে দেখাও যে, x-এর দুটি নির্দিষ্ট মানের জন্য $f(x)=2x^3+x^2+6x$ ফাংশনের দুটি সমান মান থাকতে পারেনা।

[Using Rolle's Theorem, show that $f(x)$ cannot have equal values for two distinct values of x for the function $f(x)=2x^3+x^2+6x$]

10. (a, b) বিস্তারে $f(x)=lx^2+mx+n$ ফাংশনের জন্য গড়মান উপপাদ্যের সত্যতা প্রতিপাদন কর। [Verify the Mean Value Theorem for the function $f(x)=lx^2+mx+n$ in the interval (a, b).]

11. $y=\log x$ এই বক্ররেখার কোন বিন্দুতে স্পর্শক $(1, o)$ এবং (c, o) গামী জ্যা-এর সহিত সমান্তরাল হবে ?

12. $(-1, 1)$ বিস্তারে $f(x)=1/x$, $(x\neq 0)$ এবং $f(o)=0$-এর জন্য গড়মান উপপাদ্যটি সত্য হবে কি ?

13. $(1, 1)$ এবং $(3, 9)$ বিন্দুগামী AB একটি $y=x^2$ পরাবৃত্তের জ্যা। দেখাও যে $(2, 4)$ বিন্দুগামী স্পর্শক AB জ্যা এর সহিত সমান্তরাল।

14. গড়মান উপপাদ্যে $f(b)-f(a)=(b-a)f'(c)$ হতে c-এর মান নির্ণয় কর যখন (a) $[0,4]$ বিস্তারে $f(x) = (x-1)(x-2)(x-3)$ N.U. 1994.

(i) $(-1, 1)$ বিস্তারে $f(x)=x^{4/3}$ হয়।

(ii) $(0, 1)$ বিস্তারে $f(x)=e^x$ হয়।

(iii) $(0, 2)$ বিস্তারে $f(x)=x^3-2x^2+3x-2$ হয়।

(iv) $(-1, 2)$ বিস্তারে $f(x)=x^4-2x^3+x^2-2x$ হয়।

(v) $(1, 2)$ বিস্তারে $f(x)=x^2$ হয়।

(vi) $(0, 1)$ বিস্তারে $f(x)=\sin \pi x/2$ হয়।

(vii) $(2, 3)$ বিস্তারে $f(x)=\sqrt{(x^2-4)}$ হয়।

(viii) $(1, 4)$ বিস্তারে $f(x)=\dfrac{-2x+3}{5x-2}$ হয়।

15. $(1, 2)$ বিস্তারে $f(x)=x^2+2$ এবং $g(x)=x^3-1$ ফাংশনদ্বয়ের জন্য কাওচির (cauchy) গড়মান উপপাদ্যটির সত্যতা প্রতিপাদন কর।

16. $(-1, 1)$ বিস্তারে $f(x)=x^2$ এবং $g(x)=x^3$ ফাংশনদ্বয়ের জন্য কাওচির গড়মান উপপাদ্য সত্য নহে—ইহা প্রমাণ কর।

16. (i) কাওচির গড়মান উপপাদ্যটি বর্ণনা কর এবং প্রমাণ কর। (a, b) এবং $a<x<b$ বিস্তারে $f(x)$ অবিচ্ছিন্ন ও অন্তরীকরণ যোগ্য হলে প্রমাণ কর যে

$$f(b)-f(a)=cf'(c)\log(b/a)$$

[ইঙ্গিত $g(x)=\log x$ বসাতে হবে।]

17. $1\leqslant x\leqslant 3$ বদ্ধবিস্তারে যদি $f'(x)=0$ হয় এবং $f(2)=2$, তবে দেখাও যে $1\leqslant x\leqq 3$ বিস্তারের সর্বত্রই $f(x)=1$ হবে।

18. যদি $y=2x-\tan^{-1}x-\log\{x+\sqrt{(1+x^2)}\}$ হয় তবে দেখাও যে $0\leqslant x\leqslant\infty$ বিস্তারে x বৃদ্ধির সাথে সাথে y ও বৃদ্ধি পাবে।

19. গড়মান উপপাদ্যে $f(a+h)=f(a)+hf'(a+\theta h)$, $0<\theta<1$, যদি $f(x)=\frac{1}{3}x^3-\frac{3}{2}x^2$ হয় তবে $(0, 3)$ বিস্তারে θ-এর মান নির্ণয় কর।

19. (i) গড়মান উপপাদ্য প্রয়োগ করে দেখাও যে

$$\frac{a-b}{1+a^2}<\tan^{-1}a-\tan^{-1}b<\frac{a-b}{1+b^2}$$

20. দেখাও যে *D. H. 1986*

$$(x+h)^{3/2}=x^{3/2}+3/2x^{1/2}h+\frac{3.1.h^2}{2.2.2!}\frac{1}{\sqrt{(x+\theta h)}},\ 0<\theta<1,$$

এবং $x=0$ যখন তখন θ-এর মান নির্ণয় কর।

20. (i) If $f''(x)$ exists then show that

$$f''(x)=\underset{h\to 0}{Lt}\ \frac{f(x+h)+f(x-h)-f(x)}{h^2}$$ *D. H. 1984*

21. $(-1, 0)$ বিস্তারে মনে কর $f(x)=\frac{2x+3}{3x+2}$ দেখাও যে -1 এবং 0 এর মধ্যে কোন সংখ্যা c নেই।

21. (a) ম্যোকলিনের, টেইলরের এবং কাওচির অবশিষ্ট-সহ নিম্নলিখিত ফাংশনগুলিকে h-এর শক্তিতে সম্প্রসারণ কর?

(i) $\sin(x+h)$ (ii) e^{x+h} (iii) $\log(x+h)$, (iv) $(x+k)^m$

(V) $e^{m\tan^{-1}(x+h)}$

21. (b) ম্যোকলিনের, টেইলরের এবং কাওচির অবশিষ্টসহ নিম্নলিখিত ফাংশনগুলিকে x-এর শক্তিতে সম্প্রসারণ কর:—

(i) $\cos x$, (ii) $\cos^{-1}x$ (iii) e^x (iv) $e^{ax}\cos bx$

22. দেখাও যে $x\cot x$ কে x^4 সম্বলিত পদ পর্যন্ত বিস্তৃত করলে পাওয়া যাবে $1-x^2/3-x^4/45+\ \dots\ \dots\ \dots$

23. প্রমাণ কর যে

$$e^x\log(1+x)=x+x^2/2!+2x^3/3!+9x^5/5!+\dots\ \dots$$

24. প্রমাণ কর যে $e^{\sin x}$ কে x^4 সম্বলিত পদ পর্যন্ত বিস্তৃত করলে পাওয়া যাবে $1+x+\frac{1}{2}x^2-\frac{1}{8}x^4-\ \dots\ \dots\ \dots$

25. প্রমাণ কর যে $\log\sec x=\frac{1}{2}x^2+1/12x^4+1/45x^6+\dots$ *D. U. 1964*

26. দেখাও যে $\cos^2 x=1-x^2+\frac{1}{3}x^4-2/45x^6+\dots\ \dots\ \dots$

27. প্রমাণ কর যে

$$\frac{x}{e^x-1}=1-\frac{x}{2}+\frac{x^2}{6.2!}-\frac{1}{30}\cdot\frac{x^4}{4!}+\dots\ \dots+$$ *D. U. 1955*

28. দেখাও যে $\sin^{-1}x=x+\frac{1}{2}\cdot\frac{1}{3}x^3+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{5}x^5+\ \dots\ \dots$

29. প্রমাণ কর যে

$$e^x\cos x=1+x+x^2/2-x^3/3-11x^4/24-x^5/5\ \dots\ \dots$$

30. প্রমাণ কর যে $(1+x)^x=1+x^2-\frac{1}{2}x^3+5/6x^4-\frac{3}{4}x^5\ \dots\ \dots$

31. প্রমাণ কর যে $\log(1+\sin x)=x-x^2/2+x^3/6-x^4/12+\ \dots$

32. $\sin(e^x-1)$ কে x^4 সম্বলিত পদ পর্যন্ত বিস্তৃত কর।

32. (i) If $\tan^{-1}x$ in powers of $x-\frac{1}{4}\pi$ expanded, then

$$\tan^{-1}x=\tan^{-1}\frac{\pi}{4}+\left(x-\frac{\pi}{4}\right)\frac{1}{(1+\pi^2/16)}-\frac{(x-\frac{1}{4}\pi)^2\pi}{4(1+\pi^2/16)^2}+\dots\ \dots$$

(ii) Prove by Taylor's Theorem

(a) $\log(x+h)=\log x+\frac{h}{x}-\frac{h^2}{2x^2}+\frac{h^3}{3x^3}-\ \ldots$

(b) $\frac{1}{x+h}=\frac{1}{x}-\frac{h}{x^2}+\frac{h^2}{x^3}-\frac{h^3}{x^4}\ \ldots\ \ldots$

(c) $f\left(\frac{x^2}{1+x}\right)=f(x)-\frac{x}{1+x}f'(x)+\frac{x^2}{(1+x^2)}\frac{f''(x)}{\lfloor 2}+\ \ldots\ \ldots$

(d) $\log \sin x=\log \sin 2+(x-2)\cot 2-\frac{1}{2}(x-2)^2\operatorname{cosec}^2 2+\frac{1}{3}(x-2)^3\operatorname{cosec}^2 2\cot 2+\ldots\ldots$

33. দেখাও যে ম্যাকলরিনের উপপাদ্য $\sin(x\sqrt{\pi})$ কে বিস্তৃত করণে অপারগ।

34. দেখাও যে ল্যাগ্রেঞ্জের আকারের অবশিষ্ট $R_n=\frac{h^n}{\lfloor n}f^n(a+\theta h)$-এ যে θ আছে তাহা সীমা $\frac{1}{n+1}$-এর দিকে অগ্রসর হবে যখন $h\to 0$, a বিন্দুতে $f^{n+1}(x)$ অবিচ্ছিন্ন এবং $f^{n+1}(a)\neq 0$

35. (a,b) বিস্তারে যদি $f''(x)$-এর অস্তিত্ব থাকে এবং $\frac{f(c)-f(a)}{c-a}=\frac{f(b)-f(c)}{b-c}$ যেখানে $a<c<b$ তবে এমন একটি সংখ্যা পাওয়া যাবে যেন $a<\xi<b$ এবং $f''(\xi)=0$ হয়।

36. যদি $y=\sin\log(x^2+2x+1)$ হয় তবে প্রমাণ কর যে

$(x+1)^2y_{n+2}+(2n+1)(x+1)y_{n+1}+(n^2+4)y_n=0$

এ হতে y-কে x-এর চতুর্থ শক্তি (x^4) পর্যন্ত বিস্তৃত কর।

37. যদি $y=\tan(m\tan^{-1}x)$ হয়, তবে ম্যাকলরিনের ধারায় ইহার তিনটি পদ হবে

$$mx+m(m^2-1)x^3/3+m(m^2-1)(2m^2-3)x^5/15$$

38. যদি $y=\sin(m\sin^{-1}x)$ হয় তবে দেখাও যে

$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}+(m^2-n^2)y_n=0$ এবং ইহা হতে (i) $\sin(m\sin^{-1}x)$ ও (ii) $\sin^{-1}x$-এর বিস্তৃতি নির্ণয় কর।

39. x-এর শক্তিতে এবং $\sqrt{(1-x^2)}\sin^{-1}x$-এর বিস্তৃতিতে পরপর তিনটি পদ যদি $a_nx^n+a_{n+1}x^{n+1}+a_{n+2}x^{n+2}$ হয়

তবে প্রমাণ কর যে $a_{n+2}=\frac{n-1}{n+2}a_n$

আরো দেখাও যে সকল জোড় পদগুলি শূন্য হয় এবং ধারাটি হয়

$x-\frac{1}{3}x^3-\frac{2}{3.5}x^5-\frac{2.4}{3.5.7}x^7-\ \ldots\ \ldots$ [R. U. 1964] C. H. 1986

40. যদি $y=e^{a\sin^{-1}x}=a_0+a_1x+a_2x^2+\ldots\ldots+a_nx^n+\ldots$ হয় তবে প্রমাণ কর যে

$(1-x^2)y_2=xy_1+a^2y$ (ii) $(n+1)(n+2)a_{n+2}=(n^2+a^2)a_n$

41. অবিচ্ছিন্ন ফাংশনের একটি মৌলিক ধর্মের উল্লেখ কর এবং দেখাও যে এই ধর্মের জন্য $x^5=2$ এই সমীকরণের একটি এবং কেবলমাত্র একটি ধনাত্মক বীজ পাওয়া যাবে।

42. রোলের উপপাদ্য প্রমাণের জন্য যে ধর্ম ব্যবহৃত হয় তাহা ব্যতিত অবিচ্ছিন্ন ফাংশনের অপর একটি মৌলিক ধর্মের উল্লেখ কর।

43. $u(x)$, $v(x)$, $u'(x)$, $v'(x)$ এবং $uv'-u'v\neq 0$ আবদ্ধ ব্যবধি (a,b) তে অবিচ্ছিন্ন। (a,b) ব্যবধিতে মনে করি p_1, p_2 দুইটি $u(x)=0$ এর মূল যাহার (a,b) ব্যবধির মধ্যে অবস্থান। $u(p_1)=0$ এবং $u(p_2)=0$

দেখাতে হইবে যে (p_1,p_2) ব্যবধিতে $v(x)=0$ এর একটি মূল বিদ্যমান, মনে করি $v(x)=0$-এর কোন মূল (p_1,p_2) ব্যবধিতে নাই।

অর্থাৎ (p_1,p_2)-এর মধ্যে কোন মানের জন্য $v(x)\neq 0$ ... ... (1)

প্রদত্ত সম্পর্ক হইতে $u(x)v'(x)-v(x)u'(x)\neq 0$ ... ... (2)

মনে করি $F(x)=u(x)/v(x)$

(i) হইতে (p_1,p_2) ব্যবধিতে $F(x)$ অবিচ্ছিন্ন।

$F'(x)=\frac{u'(x)v(x)-v'(x)u(x)}{[v(x)]^2}$ বিদ্যমান (1) এবং (2) হইতে

আরও $F(p_1)=F(p_2)=0$.

$\therefore$ রোলের উপপাদ্য হইতে, x-এর একটি মান ξ, $p_1<\xi<p_2$ এমন যে $F'(\xi)=0$

or, $d/dx\,[F(x)]=0$, $x=\xi$ বিন্দুতে

or, $\dfrac{u'(\xi)v(\xi)-u'(\xi)v'(\xi)}{[v(\xi)]^2}=0$, ইহা কল্পনা বিরুদ্ধে (a contradiction)

সুতরাং আমাদের সিদ্ধান্ত ভুল। $v(x)=0$-এর একটি মূল (p_1, p_2) ব্যবধিতে থাকিবে।

44. If $f'(x)>0$, in $a\leqslant x\leqq b$, then show that $f(x)$ is strictly increasing in $a\leqq x\leqslant b$, Deduce $a^x>x^a$ in $x>a>e$.

Mean Value Theorem হইতে

$f(d)-f(c)=(d-c)f'(\xi)$, $c>\xi<d$.

কিন্তু $d-c>0$ এবং $f'(\xi)=0$ $[\therefore\ a<\xi<b]$

$\therefore$ $f(d)-f(c)>0$ or, $f(d)>f(c)$ or

$\therefore$ $f(x)$ ফাংশনটি $[a, b]$ ব্যবধিতে একটি অভিসত্য বর্দ্ধমান (Strictly increasing) ফাংশন.

$a^x=x^a$ বা, $x\log a-a\log x=0$

মনে করি $f(x)=x\log a-a\log x$

$\therefore$ $f'(x)=\log a-a/x=(x\log a-a)/x>0$

$\therefore$ $f(x)>0$ বা, $x\log a>a\log x$ বা, $a^x>x^a$

45. If $\log_e y=\tan^{-1}x$, prove that

$(1+x^2)y_n=\{1-2(n-1)x\}y_{n-1}-(n-1)(n-2)\,y_{n-2}$ and hence find the co-efficients of x^5 in the expansion of y by Machlalwin's Theorem.

46. If $\dfrac{(\tan^{-1}x)^2}{\lfloor 2}=\dfrac{a_2x^2}{2}-\dfrac{a_4x^4}{4}+\dfrac{a_6x^6}{6}\ldots\ldots\ldots$

Prove that $a_{2n}-a_{2n-2}=\dfrac{1}{2n-1}$

ANSWERS VII

1. (i) yes (ii) yes (iii) yes (iv) yes.

2. yes 3. yes 4. yes for $(-1,-3)$

5. (i) true for $-2<0<3$; $3<14/3<6$

(ii) yes, $-\sqrt{(a+b-ab)}<0<\sqrt{(a+b-ab)}$

6. (i) yes, $-\pi/4\leq\theta\leqslant\pi/4$ (ii) no

7. (i) no, (ii) yes $2<5/2<3$ (iii) no.

8. $-2<-1<1$. 9. fails, 10 $a<\dfrac{a+b}{2}<b$

11. $e-1$, $\log(e-1)$, 12. no 13. (i) 0, (ii) $\log(e-1)$

(iii) $4/3$, (iv) $0, \frac{1}{2}, 1$, (v) $3/2$ (vi) $\dfrac{2}{\pi}\cos^{-1}\dfrac{2}{\pi}$

(vii) $\sqrt{5}$ (viii) $(2+\sqrt{10})/3$. 14. $1<4/9<2$

18. failed as $g'(0)=f'(0)=0$ 19. $\frac{1}{6}(3\pm\sqrt{3})$ 20. $9/64$

17. (a) (i) $\sin x+h\sin(\pi/2+x)+\dfrac{h^{n-1}}{\lfloor(n-1)}\sin\{(n-1)\pi/2+x\}+R_n$

$R_n=\dfrac{h^n(1-\theta)^{n-m}}{m\lfloor(n-1)}\sin(n\pi/2+x+\theta h)$ then put $m=n$

then put $m=1$ giving successively Lagrange's and Cauchy's Remainders.

(ii) $e^x+he^x+h^2e^x/2\ \ldots\ \ldots+\dfrac{h^{n-1}}{(n-1)!}e^x+R_n$

$R_n=\dfrac{h^n(1-\theta)^{n-m}}{m(n-1)!}e^{x+\theta h}$

Put $m=n$, and $m=1$. Lagrange's and Cauchy's Reminders

(iii) $\log x+h/x+(-1)\left(\dfrac{h^2}{\lfloor 2}(1/x^2)+(-1)^2\dfrac{h^3}{\lfloor 3}1/x^3\right)+\ldots$

$\ldots+(-1)^{n-2}\dfrac{h^{n-1}}{\lfloor(n-1)}\dfrac{1}{x^{n-1}}+R_n$

$$R_n = \frac{h^n(1-\theta)^{n-m}}{m\lfloor(n-1)}(-1)^{n-1}\frac{1}{x^n} \therefore \quad 0<\theta<1$$

Lagranges and Cauchy's Remainders by putting $m=n$, $n=1$ respectively.

(iv) . $x^n + mhx^{m-1} \ldots + \quad + \frac{h^{n-1}}{\lfloor(n-1)}\{m(m-1)\ldots\ldots(m-n+2)\}$

$x^{m-n+1} + R_n$

$$R_n = \frac{h^n(1-\theta)^{n-p}}{p\lfloor(n-1)} m(m-1)\ldots\ldots(m-n+1)(x+\theta h)^{m-n}$$

$p=n$, Lagrange's Remainder.

$p=1$, Cauchy's Remainder.

21. (b) (I) $\cos x = 1 - x^2/\lfloor 2 + x^4/\lfloor 4 \ldots\ldots + \frac{x^{n-1}}{\lfloor(n-1)}\cos\{(n-1)\pi/2\} +$

Schlemiltch Remainder R_n,

$$R_n = \frac{x^n(1-\theta)^{n-m}}{m\lfloor(n-1)}\ ; f^n(\theta x) = \frac{x^n(1-\theta)^{n-m}}{m\lfloor(n-1)}\cos\{\tfrac{1}{2}n\pi+\theta\}$$

$R_n = \frac{x^n}{n!}\cos(\frac{1}{2}n\pi+\theta x)$ Lagranges's $m=n$

$R_n = \frac{x^n(1-\theta)^{n-1}}{\lfloor(n-1)}\cos\{\frac{1}{2}n\pi+\theta x\}$ $\quad m=1$, Cauchy's

$\cos^{-1} x = 1 - x - x^3$ and so on.

(iii) $e^x = 1 + x + \frac{x^2}{\lfloor 2} + \ldots\ldots + \frac{x^{n-1}}{\lfloor(n-1)} + R_n$

$$R_n = \frac{x^n(1-\theta)^{n-m}}{m\lfloor(n-1)} e^{\theta x}$$

$n=m$, Lagrange's $m=1$, Cauchy's Remainder.

(iv) $e^{ax}\cos bx = 1 + x(a^2+b^2)^{1/2}\cos(\tan^{-1} b/a) + \ldots \ldots$

$+ \frac{x^n-1}{\lfloor(n-1)}\sqrt{(a^2+b^2)^{n-1}}\cos(n-1)\tan^{-1}b/a + R_n$

36. $2x - x^2 - \frac{2}{3}x^3 + \frac{2}{3}x^4 \quad \ldots \quad \ldots$

38. $\sin(m\sin^{-1}x) = x - m(m^2-1)\frac{x^3}{3!} + m(m^2-1^2)(m^2-2^2)$

$(m^2-3^2)x^5/\lfloor 5 + \ldots \ldots$

$\sin^{-1}x = x + \frac{1}{2}\cdot\frac{1}{2}x^3 + \frac{1}{2}\cdot\frac{3}{4}x^5/5 + \ldots \ldots$

CHAPTER VIII

INDETERMINATE FORMS

8. 1. In Chapter II we have already discussed limits in some cases of functions It is shown that limit of a certain function (continuous) can be determined by direct substitution of that value of the independent variable. But there are cases in which direct substitutions reduce the function forms like $\frac{0}{0}$, $\frac{\infty}{\infty}$ etc. which are meaningless. In this chapter attempts have been made to evaluate limits of such meaningless forms. Examples of meaningless form are shown below.

$\lim_{x\to 0}\frac{\sin x}{x}$ the expression becomes $\frac{0}{0}$ for $x=0$

$\lim_{x\to 3}\frac{x-3}{x^2-9}$. When $x=3$, the expression becomes $\frac{0}{0}$

$\lim_{x\to\infty}\frac{e^x}{x^n}$, $n>0$; When $x=\infty$, the expression becomes $\frac{\infty}{\infty}$

If $F(x)=\frac{f(x)}{\phi(x)}$, when $f(a)=0$, $g(a)=0$

Then $F(a) = \frac{0}{0}$, which is meaningless. The form $\frac{0}{0}$, $\frac{\infty}{\infty}$ etc. are called **Indeterminate forms.**

In some books these forms are also called **Singular form, Undetermined form, or Illusory forms.**

8. 2. List of Indeterminate forms.

There are many indeterminate forms of limit ; which after proper substitution reduce to any one of the forms given below,

$$\frac{0}{0},\ \frac{\infty}{\infty},\ 0\times\infty,\ \infty-\infty,\ 0^0,\ \infty^0,\ 1^\infty$$

8. 3. Method of determining the indeterminate forms.

There are two methods for determing indeterminate forms of limits.

(1) Algebraical Method

(2) Application of Differential Calculus.

The algebraical method has been displayed with an example and the application of Differential Calculus will need the establishment of some Theorems. The second method will be applied after the proof of these Theorems in Art. 8.4.

Ex. 1. Evaluate $\lim\limits_{x\to 0} \dfrac{\log(1+kx^2)}{1-\cos x}$

Let $y=\lim\limits_{x\to 0}\dfrac{\log(1+kx^2)}{1-\cos x}$ ($\because$ when $x=0$, y is of the form 0/0)

$$=\lim_{x\to 0}\frac{kx^2-\frac{1}{2}k^2x^4+\quad\ldots}{1-(1-x^2/2\,!+x^4/4\,!\ldots)}\text{ by expansion}$$

$$=\lim_{x\to 0}\frac{kx^2-\frac{1}{2}k^2x^4+\ldots}{x^2/2\,!-x^4/4\,!+\ldots}$$

$$=\lim_{x\to 0}\frac{k-\frac{1}{2}k^2x^2+\ldots}{1/2\,!-x^2/4\,!+\ldots\ldots}=\frac{k}{\frac{1}{2}}=k.$$

Hence the limit is $2k$

This limit can also be evaluated by the 2nd method.

8.4. Form $\frac{0}{0}$ (L. Hopsitals Theorem) R. H. 198

Let $\phi(x)$, $\psi(x)$ and their derivatives $\phi'(x)$ are all continuous at $x=a$, and also $\phi(a)=\psi(a)=0$, and $\psi'(a)\neq 0$, then

$$\lim_{x\to a}\frac{\phi(x)}{\psi(x)}=\lim_{x\to a}\frac{\phi'(x)}{\psi'(x)}=\frac{\phi'(a)}{\psi'(a)}$$

Sine $\varphi(a)=0$, $\psi(a)=0$, we have

$$\lim_{x\to a}\frac{\varphi(x)}{\psi(x)}=\lim_{x\to a}\frac{\varphi(x)-\varphi(a)}{\psi(x)-\psi(a)}$$

$$=\lim_{h\to 0}\frac{\varphi(a+h)-\varphi(a)}{\psi(a+h)-\psi(a)}\text{ — (taking } x=a+h)$$

$$=\lim_{h\to 0}\frac{\left(\dfrac{\varphi(a+h)-\varphi(a)}{h}\right)}{\left(\dfrac{\psi(a+h)-\psi(a)}{h}\right)}=\frac{\varphi'(a)}{\psi'(a)}$$

$$\lim_{x\to a}\frac{\varphi(x)}{\psi(x)}=\frac{\varphi'(a)}{\psi'(a)}\text{(proved)}$$

where $\psi'(a)\neq 0$

If $\varphi'(a)=0$, $\psi'(a)=0$, but $\psi''(a)\neq 0$ then proceeding as before with φ' and ψ' in place of φ and ψ we get

$$\lim_{x\to a}\frac{\varphi(x)}{\psi(x)}=\lim_{x\to a}\frac{\varphi'(x)}{\psi'(x)}=\frac{\varphi''(a)}{\psi''(a)}\text{ when }\psi''(a)\neq 0$$

Proceed in this way untill the given expression is free from the 0/0 form.

Mathematically we explain the above statement in this form.

If $\phi'(a)=\phi''(a)\ldots\ldots=\phi^{n-1}(a)=0$ and

$$\psi'(a)=\psi''(a)\ldots=\phi^{n-1}(a)=0$$

but $\psi^n(a)\neq 0$, then

$$\lim_{x\to 0}\frac{\varphi(x)}{\psi(x)}=\lim_{x\to 0}\frac{\phi^n(x)}{\psi^n(x)}=\frac{\varphi^n(a)}{\psi^n(a)}$$

Note : The proposition of Art. 8.4. is even true when $x\to\infty$ instead of $x\to a$. In this case we are to put $x=1/t$.

Then

$\lim\limits_{x\to\infty} \phi(x)=\lim\limits_{t\to 0} \phi(1/t)$ and $\lim\limits_{x\to\infty} \psi(x)=\lim\limits_{t\to 0} \psi(1/t)$

$$\therefore \quad \lim_{x\to\infty}\frac{\phi(x)}{\psi(x)}=\lim_{t\to 0}\frac{\phi(1/t)}{\psi(1/t)}=\lim_{t\to 0}\frac{\phi'(1/t)(-1/t^2)}{\psi'(1/t)(-1/t^2)}$$

$$=\lim_{t\to 0}\frac{\phi'(1/t)}{\psi'(1/t)}=\lim_{x\to\infty}\frac{\phi'(x)}{\psi'(x)}$$

Note : Students are advised to note the fact that the differentiation of the numerator and denominator made separately. They should not be confused by seeing the form which shows generally a fraction, the differentiation of which is made by the rule of differentiating the quotient of two functions.

Alternative proof : Let us consider a curve which passes through the origin and be defined by the equations.

$$\left.\begin{aligned} x&=\psi(t)\\ y&=\phi(t)\end{aligned}\right\}$$

Let, $P\,(x, y)$ be point very near to the origin O. Suppose when $t=a$, $x=0$ and $y=0$; *i. e.* $\phi(a)=0$, $\psi(a)=0$

$$\lim_{x\to 0}\left(\frac{y}{x}\right)=\lim_{\theta\to 0}\left(\tan\theta\right)=\lim_{x\to 0}\left(\frac{dy}{dx}\right)=\lim_{t\to a}\frac{\phi'(t)}{\psi'(t)}$$

$$\therefore \quad \lim_{t\to a}\frac{\phi(t)}{\psi(x)}=\lim_{t\to a}\frac{\phi'(t)}{\phi'(t)}=\frac{\phi'(a)}{\psi'(a)}, \quad \text{if } \psi'(a)\neq 0$$

Note : In the above theorem the functions and their derivative are continuous. If the functions are not continuous at the point concerned still the theorem holds good.

In fact, if

$\lim\limits_{x\to a}\phi(x)=\lim\limits_{x\to a}\psi(x)=0$ and $\lim\limits_{x\to a}\dfrac{\phi'(x)}{\psi'(x)}$

exists and equals L, then

$$\lim_{x\to a}\frac{\phi(x)}{\psi(x)}=L$$

Ex. 2. Evaluate $\lim\limits_{x\to 0}\dfrac{x-\tan x}{x^3}$

$$\lim_{x\to 0}\frac{x-\tan x}{x^3}\ \left[\text{form } \frac{0}{0}\right]$$

$$=\lim_{x\to 0}\frac{1-\sec^2 x}{3x^2}\left[\text{from } \frac{0}{0},\ \text{differentiating both numerator and denominator separately in each case.}\right]$$

$$=\lim_{x\to 0}\frac{-2\sec^2 x\tan x}{6x}=\lim_{x\to 0}\frac{-1}{3}\sec^2 x\left(\frac{\tan x}{x}\right)$$

$$=\lim_{x\to 0}\left(-\frac{1}{3}\sec^2 x\right)\lim_{x\to 0}\left(\frac{\tan x}{x}\right)=-\tfrac{1}{3}(1)=-\tfrac{1}{3}$$

§ 5 Form $\dfrac{\infty}{\infty}$.

Let $\lim\limits_{x\to a}\phi(x)=\infty$ and $\lim\limits_{x\to a}\psi(x)=\infty$. Then

$\lim\limits_{x\to a}\dfrac{\phi(x)}{\psi(x)}$ takes the form $\dfrac{\infty}{\infty}$. If limit exists, then

$$\lim_{x\to a}\left(\frac{\phi(x)}{\psi(x)}\right)=\lim_{x\to a}\frac{\phi'(x)}{\psi'(x)}=\frac{\phi'(a)}{\psi'(a)}$$

Proof : We can write the expression

$$=\lim_{x\to a}\frac{\phi(x)}{\psi(x)}=\lim_{x\to a}\frac{\left(\dfrac{1}{\psi(x)}\right)}{\left(\dfrac{1}{\phi(x)}\right)}\left[\text{form } \frac{0}{0}\right]$$

$$= \lim_{x\to a} \frac{-\frac{1}{\{\psi(x)\}^2}\psi'(x)}{-\frac{1}{\{\phi(x)\}^2}\phi'(x)} \quad [\text{by Art. 8.4.}]$$

$$= \lim_{x\to a} \frac{\psi'(x)}{\phi'(x)}\left[\frac{\phi(x)}{\psi(x)}\right]^2$$

$$= \lim_{x\to a}\frac{\phi(x)}{\phi(x)} = \lim_{x\to a}\frac{\psi'(x)}{\phi'(x)} \lim_{x\to a}\left[\frac{\phi(x)}{\psi(x)}\right]^2$$

If $\lim_{x\to a}\frac{\phi(x)}{\psi(x)} = l$ then

$$l = \lim_{x\to a}\frac{\psi'(x)}{\phi'(x)}\, l^2 \Rightarrow l = \lim_{x\to a}\frac{\phi'(x)}{\psi'(x)} \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Case 1. If $l \neq 0$ $l \neq \infty$, then

$$= l \lim_{x\to a}\frac{\phi'(x)}{\psi'(x)}\ l\,e,\ \lim_{x\to a}\frac{\phi(x)}{\psi(x)} = \lim_{x\to a}\frac{\phi'(x)}{\psi'(x)} \quad \ldots \quad \ldots \quad (2)$$

Case II. If $l = 0$, then adding 1 to each side of equation (2) we have.

$$l = \lim_{x\to a}\frac{\phi(x)}{\psi(x)} \text{ or, } l+1 = \lim_{x\to a}\frac{\phi(x)}{\psi(x)} + 1 = \lim_{x\to a}\frac{\phi(x)+\psi(x)}{\phi(x)}$$

$$\left(\text{Form } \frac{\infty}{\infty}\right)$$

$$= \lim_{x\to a}\frac{\phi'(x)+\psi'(x)}{\psi'(x)} = \lim_{x\to a}\frac{\phi'(x)}{\psi'(x)} + 1 \quad [\text{ by (1) as } l+1 \neq 0\,]$$

$$l = \lim_{x\to a}\frac{\phi(x)}{\psi x} = \operatorname{Lim}_{x\to a}\frac{\phi'(x)}{\psi'(x)}$$

Case III. If $l = \infty$. then

$$\therefore \quad \frac{1}{l} = \lim_{x\to a}\frac{1}{\frac{\phi(x)}{\psi(x)}} = \lim_{x\to a}\frac{\psi'x}{\phi(x)} = \lim_{x\to a}\frac{\psi'(x)}{\phi'(x)} \quad \left(\begin{matrix}\text{by case II} \\ 1/l = 0\end{matrix}\right)$$

$$\text{or, } \lim_{x\to a}\frac{\phi(x)}{\psi(x)} = \lim_{x\to a}\frac{\phi'(x)}{\phi'(x)}$$

From cases I, II & III, it is seen that the theorem is ture of all cases. Thus, if $\lim_{x\to a}\phi(x) = \infty$ ' $\lim_{x\to a}\psi'(x) = \infty$

$$\lim_{x\to a}\frac{\phi(x)}{\psi(x)} = \lim_{x\to a}\frac{\phi'(x)}{\psi'(x)}$$

Hence the theorem is proved for all cases.

Ex. 3. Evaluate

$$\lim_{x\to 1}\frac{\log(1-x)}{\cot \pi x}$$

$$y = \lim_{x\to 1}\frac{\log(1-x)}{\cot \pi x} \quad \left[\text{Form } \frac{\infty}{\infty}\right]$$

$$= \lim_{x\to 1}\frac{-\frac{1}{1-x}}{-\pi \operatorname{cosec}^2 \pi x} = \lim_{x\to 1}\frac{\sin^2 \pi x}{\pi(1-x)} [\text{form } 0/0]$$

$$= \lim_{x\to 1}\frac{2\sin \pi x \cos \pi x}{-\pi}\pi = 0$$

8. 6. Form $0\times\infty$

This form can be converted into the form $\frac{0}{0}$ or, $\frac{\infty}{\infty}$

Ex. 4. Evaluate $\lim_{x\to 0} \log(1+x)\frac{1}{\sin x}$.

$$y = \lim_{x\to 0} \log(1+x).\frac{1}{\sin x} \quad (\text{form } 0\times\infty)$$

$$= \lim_{x\to 0}\frac{\log(1+x)}{\sin x} \quad \left(\text{form } \frac{0}{0}\right)$$

$$= \operatorname{Lim}_{x\to 0}\frac{1/(1+x)}{\cos x} = \lim_{x\to 0}\frac{1}{(1+x)\cos x} = \frac{1}{(1)(1)} = 1$$

8. 7. Form $0^0, \infty^0, 1^\infty$

These forms can be converted to the form 0/0 or ∞/∞ with the help of logarithm.

The process is better to be demonstrated with some examples given below.

Ex. 5. Evaluate $\lim_{x\to 0} (\sin x)^x$ if $x>0$

Let $y=\lim_{x\to 0} (\sin x)^x$ [form 0^0]

$\because \log y=\lim_{x\to 0} \log (\sin x)^x = \lim_{x\to 0} x \log (\sin x)$ (form $0\times\infty$)

$$=\lim_{x\to 0} \frac{\log \sin x}{1/x}=\lim_{x\to 0} \frac{\cot x}{-1/x^2} \quad \left[\text{form } \frac{\infty}{\infty}\right]$$

$$=\lim_{x\to 0} \frac{-x^2}{\tan x}=\lim_{x\to 0} \left(\frac{x}{\tan x}\right) \lim_{x\to 0}(-x)=1.\ 0=0$$

or, $\log y=0=\log 1$ or, $y=1$ that is,

$\lim_{x\to 0} (\sin x)^x=1$

Ex 6. Evaluate $\lim_{x\to\frac{1}{2}\pi} (\sin x)^{\tan x}$

Let $y=\lim_{x\to\frac{1}{2}\pi} (\sin x)^{\tan x}$ $\therefore$ $\left(\text{form } 1^\infty\right)$

$\therefore \log y=\lim_{x\to\frac{1}{2}\pi} \log (\sin x)^{\tan x}$

$=\lim_{x\to\frac{1}{2}\pi} \tan x \log \sin x$ (form $\infty\times 0$)

$$=\lim_{x\to\frac{1}{2}\pi} \frac{\log \sin x}{\cot x} \quad \left(\text{form } \frac{0}{0}\right)$$

$$\log y=\lim_{x\to\frac{1}{2}\pi} \frac{\cot x}{-\operatorname{cosec}^2 x}=\lim_{x\to\frac{1}{2}\pi} -\cos x \sin x=0$$

$\Rightarrow y=e^0=1$

or $\lim_{x\to\frac{1}{2}\pi} (\sin x)^{\tan x}=1$

Ex. 7. Evaluate $\lim_{x\to 1} x^{1/(1-x)}$

Let $y=\lim_{x\to 1} x^{1/(1-x)}$ (form 1^∞)

Then. $\log y=\lim_{x\to 1} \log x^{\frac{1}{1-x}}$

$$=\lim_{x\to 1} \frac{1}{(1-x)} \log x \quad (\text{form } \therefore \infty\times 0)$$

$$=\lim_{x\to 1} \frac{\log x}{1-x}\left(\text{form } \frac{0}{0}\right)=\lim_{x\to 1}\left(\frac{1}{-x}\right)= 1$$

$$\Rightarrow y=e^{-1}=\frac{1}{e}$$

$$\therefore \lim_{x\to 1} x^{\frac{1}{1-x}}=\frac{1}{e}$$

Ex. 8. Find the limit of the expression

$$\lim_{x\to 0} \frac{x-\sin x}{x^3}$$

$$\text{Let } y=\lim_{x\to 0}\frac{x-\sin x}{x^3}\left(\text{form } \frac{0}{0}\right)$$

$$y=\lim_{x\to 0}\frac{1-\cos x}{3x^2}\left(\text{form}\frac{0}{0}\right)=\lim_{x\to 0}\frac{2\sin^2 x/2}{3x^2}$$

$$=\lim_{x\to 0}\left(\frac{\sin x/2}{x/2}\right)^2=\frac{1}{6}.$$

Ex. 9. Fvaluate $\lim_{x\to 0}\frac{\sin x-\log(e^x\cos x)}{x\sin x}$ C. U. 1984

Let $y=\lim_{x\to 0}\frac{\sin x\log(e^x\cos x)}{x\sin x}$

$$=\lim_{x\to 0}\frac{\sin x-x-\log\cos x}{x\sin x} \quad \therefore \left(\text{form}\frac{0}{0}\right)$$

$[\therefore \log e^x=x\log_e=x]$

$$=\lim_{x\to 0}\frac{\cos x-1+\tan x}{\sin x+\cos x} \quad \left(\text{form}\frac{0}{0}\right)$$

$$=\lim_{x\to 0}\frac{-\sin x+\sec^2 x}{\cos x+\cos x-x\sin x}=\tfrac{1}{2}$$

Ex. 10. Evaluate $\lim_{x\to 1}\left(\frac{x}{x-1}-\frac{1}{\log x}\right)$ D. U. 1966

$$\lim_{x\to 1}\left\{\frac{x}{x-1}-\frac{1}{\log x}\right\} \quad (\tan\infty-\infty)$$

$$=\lim_{x\to 1}\frac{x\log x-x+1}{(x-1)\log x}\left(\text{form}\frac{0}{0}\right)=\lim_{x\to 1}\frac{1+\log x-1}{\log x+(x-1)/x}\left(\text{form}\frac{0}{0}\right)$$

$$=\lim_{x\to 1}\frac{\log x}{\log x+1-1/x}=\lim_{x\to 1}\frac{1/x}{1/x+1/x^2}=\tfrac{1}{2}$$

Ex. 11 Find the values of a, b and c, if

$$\lim_{x\to 0}\frac{ae^x-b\cos x+ce^{-x}}{x\sin x}=2$$

Ans. Here $\varphi(x)=a\,e^x-b\cos x+c\,e^{-x}$

$\psi(x)=x\sin x$

Since $\psi(0)=0$,

therefore $\varphi(0)=0\Rightarrow a-b+c=0$ (i)

The given limit $=\lim_{x\to 0}\frac{\varphi(x)}{\psi(x)}=\lim_{x\to 0}\frac{\varphi'(x)}{\psi'(x)}$

Now $\psi'(x)=x\cos x+\sin x$

$\Rightarrow\psi'(0)=0.$

$\therefore\ \varphi'(0)=0$

But $\varphi'(x)=ae^x+b\sin x-ce^{-x}$

$\varphi'(0)=0\Rightarrow a-c=0$ (2)

By L' Hospitals Rule, the given limit is then

$$\lim_{x\to 0}\frac{\varphi''(x)}{\psi''(x)}$$

We have, $\psi''(x)=-x\sin x+\cos x+\cos x$

$\psi''(0)=0+1+1=2\neq 0$

Hence $\lim_{x\to 0}\frac{\varphi''(x)}{\psi''(x)}=2$ (given)

$\Rightarrow \frac{\varphi''(0)}{\psi''(0)}=2$

or $\varphi''(0)=2\times\psi''(0)=2\times 2=4$

Now $\varphi''(x)=ae^x+b\cos x+ce^x$

$\therefore\ a+b+c=4$ (3)

Solving (1), (2), and (3), we get

$a=1,\ b=2,\ c=1.$

Second Method :

Let $y=\lim_{x\to 0}\frac{ae^x-b\cos x+ce^{-x}}{x\sin x}=2.$

$$\text{or } \lim_{x\to 0} \frac{a\left(1+x+\frac{x^2}{2!}\cdots\right)-b\left(1+\frac{x^2}{2}+\dots\right)+c\left(1-x.+\frac{x^2}{2!}+\dots\right)}{x\left(x-\frac{x^3}{3!}+\dots\right)}$$

$$\text{or, } \lim_{x\to 0} \frac{(a-b+c)+x(a-c)+\frac{1}{2}x^2(a+b+c)+\dots\dots}{x^2\left(1-\frac{1}{6}x^2+\dots\dots\right)}=2$$

As the limit is 2, so

$$a-b+c=0,\ a-c=0 \text{ and } \tfrac{1}{2}(a+b+c)=2$$

Now solve for a, b, c.

Hence $a=1$, $b=2$, $c=1$.

Ex. 12. Evaluate $\displaystyle\lim_{x\to 0} \frac{\sin x-\tan^{-1}x}{x^2 \log(1+x)}$.

The form is $\dfrac{0}{0}$.

We get the result by using L' Hospital's Rule several times. So, expansion by taylor's Theorem is helpful.

$$\lim_{x\to 0} \frac{\sin x-\tan^{-1}x}{x^2\log(1+x)}$$

$$=\lim_{x\to 0} \frac{x-x^3/\lfloor 3+x^5/\lfloor 5-\ -\left(x-\frac{1}{3}x^3+\frac{1}{15}x^5\ \dots\right)}{x^2(x-x^2/2\ \ x^3/3-\ )}$$

$$\lim_{x\to 0} \frac{\left(\frac{1}{3}-1/\lfloor 3\right)x^3+x^5\left(1/\lfloor 5-1/15+\ \dots\right.}{x^3(1-x/2+x^2/3-\dots)}$$

$$=\lim_{x\to 0} \frac{1/6+x^2(1/\lfloor 5-1/15)+}{1+x/2+x^2/3-\dots}=\frac{1}{6}$$

Ex. 13. (i) Find $\displaystyle\lim_{x\to 0} x^{x^k}$ when $k>0$.

let $u=\displaystyle\lim_{x\to 0} x^{x^k}$.

Now $\log u=\displaystyle\lim_{x\to 0} x^k \log x$ (form $0\times\infty$; $x>0$)

$$=\lim_{x\to 0} \frac{\log x}{x^{-k}} \quad \because \text{ form } \infty/\infty$$

$$=\lim_{x\to 0} \frac{1/x}{-kx^{-k-1}}=\lim_{x\to 0}\frac{x^k}{-k}=0 \Rightarrow u=e^0=1 \qquad \text{(Ans)}$$

13. (ii) show that $\displaystyle\lim_{x\to\infty}\left(x^{1/x^k}\right)=1$; $k>0$

Let $u=\lim\left(x^{1/x^k}\right)$

Then $\log u=\displaystyle\lim_{x\to\infty}\frac{1}{x^k}\log x$

$$=\lim_{x\to\infty}\frac{\log x}{x^k}\left(\text{form}\frac{\infty}{\infty}\right)$$

$$=\lim_{x\to\infty}\frac{1/x}{kx^{k-1}}=\lim_{x\to\infty}\frac{1}{kx^k}=0$$

$\Rightarrow u=e^0=1$ (Ans)

Exercise VIII

Evaluate each of the following limits.

1. $\lim_{x \to a} \dfrac{x-a}{x^5-a^5}$ 2. $\lim_{x \to 1} \dfrac{\sin \pi x}{\tan 4\pi x}$

1. (i) $\lim_{x \to 0} \dfrac{\tan^{-1} x}{\sin^{-1} x}$ D. U. 1960. 2. (i) $\lim_{x \to 0} \dfrac{x - \sin x}{x}$

3. $\operatorname{Lim}_{x \to 0} \dfrac{3 \tan x - 3x - x^3}{x^5}$ 4. $\operatorname{Lim}_{x \to 0} \dfrac{\sqrt{x} \tan x}{(e^x-1)^{3/2}}$

5. $\operatorname{Lim}_{x \to 0} \dfrac{e^x - e^{-x} + 2 \sin x - 4x}{x^3}$ 6. $\operatorname{Lim}_{x \to 1} \dfrac{1 + \cos \pi x}{\tan^2 \pi x}$ R. H. '66

7. $\operatorname{Lim}_{x \to 0} \dfrac{\tan x - x}{x^2 \tan x}$ 8. $\operatorname{Lim}_{x \to 0} \dfrac{e^x - e^{\sin x}}{x - \sin x}$

9. $\operatorname{Lim}_{x \to 0} \dfrac{e^x - e^{-x} - 2 \log (1+x)}{x \sin x}$ 10. $\operatorname{Lim}_{x \to \frac{1}{2}} \dfrac{\cos^2 \pi x}{e^{2x} - 2e^x}$

11. $\operatorname{Lim}_{x \to \infty} \dfrac{x + \cos x}{1+x}$ R.H. '67 12. $\operatorname{Lim}_{x \to 0} \dfrac{\log \sin 3x}{\log \sin x}$ D. U. 1967

13. $\operatorname{Lim}_{x \to \infty} x^4/e^x$ D. U. 1964 (a) $\operatorname{Lt}_{x \to \pi/2} \dfrac{e^{\tan x} - 1}{e^{\tan x} + 1} = 1$ C.H. 1993

(i) $\operatorname{Lim}_{x \to 0} \dfrac{\tan x - \sin x}{\tan x - x}$ D. U. '60 (ii) $\operatorname{Lim}_{x \to 0} \dfrac{\sin x - \tan^{-1} x}{x^2 \log (1+x)}$

(14) $\operatorname{Lim}_{x \to 0} \dfrac{\log \tan x}{\log x}$ (15) $\operatorname{Lim}_{x \to \pi/2} \dfrac{\log (x - \pi/2)}{\tan x}$ D.U.'61

16. $\operatorname{Lim}_{x \to 0} \dfrac{\log \tan 2x}{\log \tan x}$ 17. $\dfrac{\log (x-1) + \tan \frac{1}{2} \pi x}{\cot \pi x}$

18. $\operatorname{Lim}_{x \to 0} x^m (\log x)^n \; m, n > 0$ 19. $\operatorname{Lim}_{x \to 0} \dfrac{\log (1-x^2)}{\log \cos x}$ D.U. '61

(i) $\operatorname{Lim}_{x \to 0} + x^2 \log x$ (ii) $\operatorname{Lim}_{x \to \frac{1}{2}\pi} (1 - \sin x) \tan x$ C. U. '67

20 $\operatorname{Lt}_{x \to \alpha} \dfrac{x^n}{e^x}$, $x > 0$ (i) $\operatorname{Lt}_{x \to 0} \dfrac{\log x^2}{\log \cot^2 x}$ (ii) $\operatorname{Lt}_{x \to 1} x(2^{1/x} - 1)$ N.U. 1994

21. $\operatorname{Lim}_{x \to 0} \dfrac{e^x - 1 - \log (1+x)}{x^2}$ 22. $\operatorname{Lim}_{x \to \pi} \left(\dfrac{\pi}{x - \pi} - \dfrac{x - \sin x}{x - \pi} \right)$

23 $\operatorname{Lt}_{x \to 0} \left(\dfrac{1}{x^2} - \dfrac{1}{\sin^2 x} \right)$ (i) $\operatorname{Lt}_{x \to 1} \dfrac{\sqrt{x} - 1}{\sqrt{x-1}} = \dfrac{3}{2}$ (ii) $\operatorname{Lt}_{x \to 0} \left(\dfrac{1}{x^2} - \dfrac{1}{\tan^2 x} \right)$

C.U. 1993

24. $\lim_{x \to 0} x \log \sin x$ 25. $\lim_{x \to \infty} 2^x \sin (a/2^x)$

26. $\lim_{x \to 1} \left(\dfrac{1}{\log x} - \dfrac{x}{\log x} \right)$ 27. $\lim_{x \to 0} \left(\dfrac{1}{x^2} - \cot^2 x \right)$ D.U. 67

28. $\lim_{x \to \frac{1}{2}\pi} \sec x \, (x \sin x - \frac{1}{2}\pi)$ 28. (a) $\lim_{x \to 0} \left\{ \dfrac{1}{x} - \dfrac{1}{\tan^{-1} x} \right\}$

28(a) $\lim_{x \to 4} \dfrac{\sqrt{x} - 2}{\sqrt{7 + \sqrt{x}} - 3} = 6$ C.U. 1993

D. U. 1968

29. $\lim_{x \to 0} \cot x \log \dfrac{1-x}{1+x}$ D.H.'62, C.H.'77'89 30. $\lim_{x \to 0} \tan x \log x$

31. $\lim_{x \to 0} \dfrac{2 \tan^{-1} x - x}{2x - \sin^{-1} x}$ 31. (a) $\lim_{x \to 0+} x^{1/x}$ D. U. '83

32. $\lim_{x \to a} (a - x) \tan (x\pi/2a)$ 33. $\lim_{x \to 0} \left(\dfrac{1}{x} \right)^{\sin x}$

34. $\lim_{x \to 0} (\sec x)^{\tan x}$ 35. $\lim_{x \to \pi} \left(\dfrac{2\pi}{x} - 1 \right)^{\tan x/2}$

36. $\lim_{x \to e} (\log x)^{1/(x-e)}$ 37. $\lim_{x \to 0} \dfrac{\sqrt[n]{(1+x)} - 1}{x}$ R. H. '66

38. $\lim_{x \to 0} \log (\cos x)^{1/x^2}$ R.H. '62, 39. $\lim_{x \to 0} \left(\dfrac{\tan x}{x} \right)^{1/x^2}$ D.H. '62

40. $\lim_{x \to 0} (\cos x)^{\operatorname{cosec}^2 x}$ 40. (i) $\lim_{x \to 1} \dfrac{x - \sqrt{(2 - x^2)}}{2x - \sqrt{(2 + 2x^2)}} = 2$

41 $\operatorname{Lt}_{x \to 0} \dfrac{\log (1+x^2)}{\sin x^2}$ C.U. 1993 42 $\operatorname{Lt}_{x \to 0} x^x$ (i) $\operatorname{Lt}_{x \to 0} (\cos x)^{1/x}$

43. $\underset{x\to 0}{\mathrm{Lt}} \left(\frac{\tan x}{x}\right)^{1/x}$ D.U. '68 43 (a) $\underset{x\to 1}{\mathrm{Lt}} (1+x)^{1/x}$ D.U.1991; C.U.1992

44. $\lim\limits_{x\to 0} (\cos x)^{1/x^2}$ 45 $\lim\limits_{x\to 0} (1+\sin x)^{\cot x}$

46. $\lim\limits_{x\to 1} (2-x)^{\tan\frac{1}{2}\pi x}$

47. $\lim\limits_{x\to \pi/2} (\tan x)^{\tan 2x}$ 48. $\lim\limits_{x\to 0} (\cos x)^{\cot^2 x}$ C. U. 1980

49. $\lim\limits_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x^2}$ D. U. 1963 D. U. 1987 49 (a) $\lim\limits_{x\to 0} \frac{\cosh x-\cos x}{x \sin x}$

50. $\lim\limits_{x\to e} (\log x)^{\frac{1}{1-\log x}}$ 51. $\lim\limits_{x\to 0} \frac{\sin x \sin^{-1} x}{x^2}$ D. U '69

52. $\lim\limits_{x\to 0} \frac{\log (0+x^3)}{\sin^2 x}$ 53 $\lim\limits_{x\to 0} (\coth x)^{\sinh x}$

54. $\lim\limits_{x\to 0} (\cot^2 x)^{\sin x}$ 55. $\lim\limits_{x\to 0} \frac{x-\sin^{-1} x}{\sin^3 x}$

56. $\lim\limits_{x\to 0} \frac{\tan x \tan^{-1} x - x^2}{x^6}$ 56 (a) $\lim\limits_{x\to 0} \frac{4^x-2^x}{x}$ R. U. 1961

57. $\lim\limits_{x\to 0} \frac{e^x \sin x - x - x^2}{x^2 + x \log (1-x)}$ 58. $\lim\limits_{x\to 0} \frac{e^x - e^{x \cos x}}{x-\sin x}$

59. $\lim\limits_{x\to \pi/4} \frac{\sec^2 x - 2\tan x}{1+\cos 4x}$ 60. $\lim\limits_{x\to \pi/2} (\cos x)^{\cos x}$

61. $\lim\limits_{x\to 0} \frac{\sin 2x + 2\sin^2 x - 2\sin x}{\cos x - \cos^2 x}$ 62. $\lim\limits_{x\to 0} \frac{x\cos x - \log (1+x)}{x^2}$

63. $\lim\limits_{x\to \infty} \left[x\left(1+\frac{1}{x}\right)^x - ex^2 \log\left(1+\frac{1}{x}\right)\right]$

64. Find the values of a, b, c; $\underset{x\to 0}{\mathrm{Lt}} \frac{x(a+b\cos x)+c\sin x}{x^5} = 1$

(a) $\underset{n\to\alpha}{\mathrm{Lt}} \frac{1}{n} \overset{n}{\underset{i=1}{w}} e^{i/n}$ C.H. 1992 (b) $\underset{x\to 0}{\mathrm{Lt}} \frac{e^{-1/x^2}}{x^n}$ C.H. 1992

$\lim\limits_{x\to 2} \frac{x^2-4}{\sqrt{(x+2)}-\sqrt{(3x+2)}} = 1$

Find the values of a and b so that

$\lim\limits_{x\to 0} \frac{x+ax\cos x - b\sin x}{x^3} = 1$

Find the value of m if $\lim\limits_{x\to 0} \frac{\sin x(2\cos x+m)}{x^2}$ is finite

Does limit exist for

$\lim\limits_{x\to 1} \frac{\log (x-1) - \tan\frac{1}{2}\pi x}{\cot \pi x}$ 68. $\lim\limits_{x\to 0+} x^{1/x} = 0$

Prove that

$\lim\limits_{n\to\infty} \frac{1^m+2^m+3^m \ldots +4n^m}{n^{m+1}} = \frac{1}{m+1}, m>0$

Does limit exist for

$\lim\limits_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x^3}$ R. H. 1988 71. $\lim\limits_{x\to 0} +x^m(\log x)^n = 0$ D. U. 1984

If $u=\frac{\cos xy}{\cos y}$ and $x=\sin y$ 71 (a) $\underset{x\to 0}{\mathrm{Lt}} \frac{x^2\sin(x/4)}{\sin x}$ N.U. (C-2) 1994

Prove that (b) $\underset{x\to 0}{\mathrm{Lt}} \frac{e^x - e^{-x} - 2x}{x - \sin x}$ D.U. 1991

$\lim\limits_{x\to 0} \frac{\frac{d^{n+1}u}{dx^{n+1}}}{\frac{d^{n-1}u}{dx^{n-1}}} = n^2 - m^2$

$y=(\sin^{-1}x)^2$, Prove that

$\lim\limits_{x\to 0} \frac{\frac{d^{n+2}y}{dx^{n+2}}}{\frac{d^{n}y}{dx^{n}}} = x^2$

72. If $u = \dfrac{\text{Cos}xy}{\cos y}$ and $x = \sin y$

Prove that $\underset{x\to 0}{\text{Lt}} \dfrac{d^{n+1}u}{dx^{n+1}} \Big/ \dfrac{d^{n-1}u}{dx^{x-1}} = n^2 - m^2$

(a) If $y = (\sin^{-1}x)^2$, Prove that

$\underset{x\to 0}{\text{Lt}} \dfrac{d^{x+2}y}{dx^{n+2}} \Big/ \dfrac{d^{n}y}{dx^{n}} = x^2$

73. Examine whether the function $f(x)$ is continuous at $x =$
Where $f(x) = x^{2x}$, $x \neq 0$, $f(0) = 1$

Sol. প্রদত্ত রাশি, $f(x) = x^{2x}$

$\therefore \log f(x) = 2x \log x = 2\dfrac{\log x}{1/x}$

$\therefore \lim\limits_{x\to 0} \log f(x) = 2 \lim\limits_{x\to 0} \dfrac{\log x}{1/x}$ $\therefore$ form $\dfrac{\infty}{\infty}$

$= -2 \lim\limits_{x\to 0} \dfrac{1/x}{1/x^2} = 0$

এখন $\lim\limits_{x\to 0} \log f(x) = \log \lim\limits_{x\to 0} f(x)$

অতএব $\log \lim\limits_{x\to 0} \log f(x) = 0$, ফলে

$\lim\limits_{x\to 0} f(x) = e^0 = 1$

দেওয়া আছে $f(0) = 1$. ফলে $\lim\limits_{x\to 0} f(x) = f(0)$

অতএব $x = 0$ বিন্দুতে $f(x)$ অবিচ্ছিন্ন।

73. Evaluate $\lim\limits_{a\to b} \dfrac{a^x \sin bx - b^x \sin ax}{\tan bx - \tan ax}$ R. U. 19

ANSWERS

1. $1/5a^4$ (i) 1	2. $-\frac{1}{4}$ (i) 0	3. 2/5
4. 1.	5. 0	6. $\frac{1}{2}$
7. $\frac{1}{3}$	8. 1	9. 1
10. $\pi^2/2e$	11. 1	12, 1
13. 0, (i) 3/2, (ii) 1/6	14. 1	15. 0
16. 1.	17. -2.	18. 0 (i) 0
19. 2. (i) 0	20. 0. (i) -1	21. 1
22. -2.	23. $-\frac{1}{3}$ (i) 0	24. 0
25. a	26. -1	27. 2/3
28. -1, (a) 0	29. -2	30. 1
31. 1	32. $2a/\pi$	33. 1
34. 1	35. 1	36. c
37. $1/n$	38. $-\frac{1}{2}$.	39. $e^{1/3}$
40. $1/\sqrt{e}$	41. $1/c$	42. 1. (i) $1/\sqrt{e}$
43. 1.	44. $e^{-1/2}$	45. e
46. $\dfrac{2/\pi}{e}$	47. 1	48. $\dfrac{-1/2}{e}$
49. $e^{-1/6}$ 49. (a) 1	50. $1/e$	51. 1
52. 1	53. 1	54. 1
55. $-\frac{1}{6}$	56. 2/9 (a) log2	57. $-2/3$
58. 3	59. $-\frac{1}{2}$	60. 1
61. 4	62. $\frac{1}{2}$	63. 0

64. $a = 120$, $b = 60$, $c = 180$. 65—8

66. $a = -5/2$, $b = -3/2$

67. $m = -2$, limit $= -1$

68. 2

69. Limit does not exist. 70. $b^x (b \cos bx - \sin bx) \cos^2 bx$

CHAPTER IX

PARTIAL DIFFERENTIATION

9.1. Functions of Several Variables

In previous chapters, we have discussed the functions of ... independent variable. In this chapter, we now turn to func... of more than one independent variable.

Let us give two examples which are familiar to us.

The volume, v of a right circular cylinder of radius x ... height y is given by $v = \pi x^2 y$ ... ... (1)

Thus we see that v depends upon two variables x and y. ... $v=(x, y)$; v is the function of x and y.

The volume of a Rectangular parallelopiped whose len... breadth and height are respectively x, y and z is $v = xyz$......(2)

Thus in this case v is the functions of three variables x, y ... z, and can be represented as $v = f(x, y, z)$.

Similarly if u is the functions of $x_1\, x_2, x_3$ ....x_n variabl... then we represent this function u in the form

$$u = f(x_1, x_2, x_3, \cdots x_n)$$

Art 9.1 (a) Domain

Let $u = f(x, y)$ ..... (1)

x, y are independent variables and u depends upon them ... the function of x and y. The ordered pair is called a point ... aggregate of the pair of numbers (x, y) is said to be dom... region of a detinition of the function. If the domain is closed ... curve C, then f is said to be closed when it is defined for the ... within and on the curve. It the function u or f is defined for ... within C not on the points on the boundary, then the dom... called open.

Art 9.1 (b) Neighbourhood (nbd.) of a point (a, b)

A non empty subset N of R is called a nbd. of a point $x_0 \in R$ if ... exists an open interval I such that $x_0 \in I \subseteq N$. Again we express ... $(x_0 - \epsilon, x_0 + \epsilon) \subseteq N$, $\epsilon > 0$. If x_0 be real number (a, b) such that ...b, then $(x_0 - \epsilon, x_0 + \epsilon)$ is a nbd. of $x_0 \in$ (a, b). This nbd. is called ...ighbourhood (nbd.) of x_0 and is denoted by $N(x_0, \epsilon)$.

... Nbd. of $x_0 \in$ (a, b) is expressed as $\{x \in R : |x_1 - x_2| < \epsilon\}$

For the case $u = f(x, y)$ at (a, b), we have the set of values x_1, y_1 ... than a, b that satisfy $|x_1 - a| < \epsilon$, $|y - b| < \epsilon$ where $d > 0$ but ... is said to form a nbd. of the point (a, b). Thus x may take any ... from $(a - \delta, a + \delta;\ b - \epsilon, b + \epsilon)$ such that a takes values from ... $a + \epsilon$ except a and is takes values from $b - \epsilon$ to $b + \epsilon$.

In metric space the set $N(p, r) = \{x \in X;\ d(p, x) < r$ is called a ... of a point p. The number r is called the sadius of N(p, r)

... $r(\theta^2 - 1) = a^r \theta^2$

... $N(p,r) = \{x \in R : |x - p| < r\}$, R is the set of rational numbers

... $x \in R : r - p < x < r + p\} = (p-r, p+r)$, open interval.

Thus nbd. of p is an open interval with p as centre of the circle. ... points inside the circle $x^2 + y^2 = \epsilon^2$ may be taken as a nbd of the ... (0,0)

A nonempty subset N of R is called a nbd. of a point x if there ... $\epsilon > 0$ such that

... $(x - \epsilon, x + \epsilon) \subseteq N \subset R$

$N(x, \epsilon) = \{x \in R : |x - x| < \epsilon\}$

Art 9.1 (c) Limit Point

A real number x is called a limit point of the set $A \subset R$ if every ... of x_0 contains infinitely many points of A

A point (x_0, y_0) is called a limit point or cluster point or ...ensation or accumulation point of A if every nbd of (x_0, y_0) ... an infinite number of points of A.

The limit point itself may or may not be a point of the set.

Ex. The point (x_0, y_0) is a limit point of the set $\{\frac{1}{m}, \frac{1}{n} : m$ ... N$\}$. Limit point is not in the set.

Ex. Prove that by using $(\delta, \in)$

$$\underset{(x,\, y) \to (x,3)}{\text{Lt}} (2x^2 - 3y) = -2$$

Sol. Let us consider a small poisitive number $\in > 0$ depen... upon $\delta > 0$ such that

$|2x^2 - 3y + 1| < \in$ When $|x-(-2)| < \delta$, $|y-3| < \delta$

Now $|x-(-1)| < \delta|$, $|y-3| < \delta$

i. e. $-2-\delta < x < -2+\delta$, $3-\delta < y < 3+\delta$

Where $x \neq -2$, $y \neq 3$

Again $(-2-\delta)^2 < x^2 < (-2+\delta)^2$, $9-3\delta < 3y < 9+3\delta$

or, $4+4\delta+\delta < x^2 < 4-4\delta+\delta^2$; $9-3\delta < 3y < 9+3\delta$

or, $8+8\delta+2\delta^2+ > 2x^2 < 8-8\delta+2\delta^2$, $9-3\delta < 3y < 9+3\delta$

$\therefore$ $8+8\delta+2\delta^2-9+3\delta < x^2-3y < 8-8\delta+2\delta^2-9-3\delta$

Or, $-1+11\delta+2\delta^2 < x^2-3y < -1-11d+2\delta^2$

Or, $11\delta < x^2-3y+1 < -11\delta$

$\therefore$ $|x^2-3y+1| < 11\delta$, it $\delta = \in/11$, then

$x^2-3y+1 < 11 \cdot \in/11 = \in$

Hence $x^2-3y = -1$.

9. 2. Continuity of a function of two variables.

The function $z = f(x, y)$ is said to be continuous for $x = a$, $y = b$ when

$$\lim_{(x,\, y) \to (a,\, b)} f(x, y) = f(a, b)$$

no matter how x and y approach their limits a and b respectively in the domain. We may explain the continuity of function in the following way also.

The function $f(x, y)$ is said to be continuous at (a, b) if and only if for each $\in > 0$, there exists a number $\delta > 0$ such that $|f(x, y) - f(a, b)| < \in$

for all points $(x, y) \neq (a, b)$

for which $|x-a| < \delta$ and $|y-b| < \delta$

i, e., $a-\delta \leqq x \leqq a+\delta$, and $b-\delta \leqq y \leqq b+\delta$

For the continuity of $f(x, y)$ at (a,b) there is a in a region R in the $xy-$plane bounded by the lines $x = a-\delta$, $x = a+\delta$, $y = b-\delta$, $y = b+\delta$ such that for any point (x, y) in R $f(x, y)$ lies between $f(a, b) - \in$ and $f(a, b) + \in$ where $\in$ is any positive number, however small.

Limit of a function of two variables

A function $z = f(x,y)$ is said to tend to the limit l as (x,y) tends to (a, b) if and only if for each positive number $\in > 0$ however small, there exists a number $\delta > 0$ such that.

$$|f(x, y) - l| < \in$$

for all $(x, y) \neq (a, b)$ lying in the region R given by

$$|x-a| < \delta \text{ and } |y-b| < \delta$$

i. e, $a-\delta \leqq x \leqq a+\delta$ and $b-\delta \leqq y \leqq b+\delta$.

We express this in as

$$\lim_{(x,\, y)\to(a,\, b)} f(x, y)=1$$

no matter how x and y approach their limits a and b in the domain.

9. 3. Geometrical Representation of functions of two variables.

Let us consider function of two variables x and y

$$z=f(x,y). \qquad \text{(i)}$$

Let us consider three perpendicular axis such as x—axis y— axis and z—axis meeting at point. O.

For each pair of values of x and y, there corresponds a point P on the xy plane and we can draw a perpendicular PQ to the xy—plane and (x,y) such that

$$PQ=z=f(x, y),$$

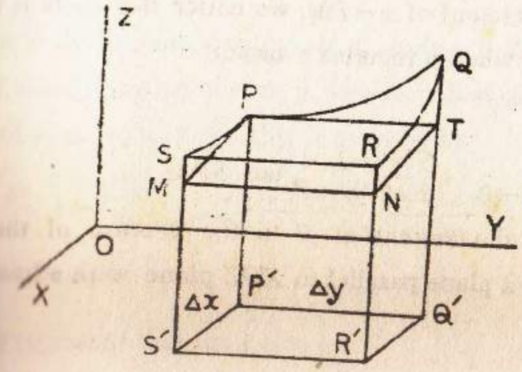
$(z>0,=a>0)$according as Q is above on or below the xy plane

Similary, lengths of the perpendiculars drawn at $P_2(x_2,y_2)$ $P_3(x_3, y_3)$......$P_n(x_n, y_n)$ are respectively

$$P_2Q_2=z_2=f(x_2,y_2)$$
$$P_3Q_3=z_3=f(x_3, y_3)$$
$$\dots \quad \dots \quad \dots$$
$$P_nQ_n=z_n=f(x_n,y_n)$$

All points Q_1, Q_2, Q_3.....Q_n are in space and they all lie in a surface. Thus $z=f(x, y)$ will represent a surface,

9. 4. Geometrical Representation of Partial Derivatives.



Let $z=f(x, y)$ represents a surface $PQRS$, a portion of the surface is cut off by the planes parallel to XOZ and YOZ.

Let the co-ordinates of P be (x, y, z) and these of

Q and $\{x, y+\Delta y\ f(x, y+\Delta y)\}$

R and $\{x+\Delta x, y+\Delta y, f(x+\Delta x, y+\Delta y)\}$

S and $\{x+\Delta x, y\ ;\ f(x+\Delta x, y)\}$

From the figure we notice that TQ is the increment of z and PT is the increment of y, while x remains constant. Thus from definition, we have

$$\frac{\partial z}{\partial y}=\lim_{\Delta y\to 0} \frac{TQ}{PT}=\lim_{Q\to P} \tan \angle TPQ \quad (TQ=\delta z,\ PT=\delta y)$$

= Slope of the tangent at P to section of the surface $z = f(x, y)$ by a plane throug P parallel to the plane YOZ with a line parallel to y—axis.

For the increment of $x = PM$, we notice that there is an increment SM of z, when y remains constant.

Then

$$\frac{\delta z}{\delta x} = \lim_{\Delta x \to 0} \frac{SM}{PM} = \lim_{S \to P} \tan SPM$$

= Slope of the tangent at P to the section of the surface $z = f(x, y)$ by a plane parallel to XOZ plane with a line parallel to x—axis.

9. 5. Partial Derivatives

Let $z = f(x, y)$ be a function of two indepent variables defined in a domain. If y remains constant, the point (x, y) will move along a line parallel to x—axis. In this case any variation in z depends on the variable of the single variable x only and the derivate of z, w, r, to x is called the derivative af z. with r. to x provid if such derivate exists.

Symbollically we represent it by

$$\left(\frac{\delta z}{\delta x}\right) \text{ or, simply by } \frac{\delta z}{\delta x}$$

Thus

$$\frac{\delta z}{\delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} \text{ if limit exists.}$$

The partial derivative is represented by the following symblos also

$$\frac{\delta z}{\delta x} \text{ or, } f_x \text{ or, } \frac{\delta f(x, y)}{\delta x} \text{ or, } D_x z$$

Similarly if x remains constant during dfferentiation and y varies, the derivative of z, w, r, to y is called the partial derivative of z, w, r, to y and this is denoted by

$$\left(\frac{\delta x}{\delta y}\right) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} \text{ ; if limit exists}$$

x = constant

We represent this limit also by

$$\frac{\delta z}{\delta y} \text{ or, } f_y \text{ or, } \frac{\delta f(x, y)}{\delta y} \text{ or, } D_y z$$

In case of a function of several variables the partial derivative of the function w, r. to a particular variable is obtained by treating all other variables as constants and differentiating the function with respect to the variable under consideration.

ART 9.5 (A) MEAN VALUE THROREM

If f_x (a,b) exists throughout a neighbourhood of a point $P(a,b)$ and $f_y(a, b)$ exists for any point $Q(a+h, b+k)$ of this neighbourbood (nbd.),

$f(a+h, b+k) - f(a, b) = h f_x(a + \theta h, b+k) + k\,[f_y(a, b) + \eta]$

Where $0 < \theta < 1$, x is the function of k if $k \to 0$, there $\pi \to 0$

Proof. We have

$f(a+h, b+k) - f(a,b) = f(a+h, b+k) - f(a, b+k) + f(a, b+k) - f(a, b)$

Since f_x exists in a nbd. of (a, b), therefore by...(1)

Lagrange's Mean Value Throrem, We have

$f(a+h, b+k)-f(a, b+k)$

$= (a + h-a) f_x(a+\theta_1 h, b+k) = hf_x(a+\theta_1 h, b+k),\ 0 < \theta_1 < 1$ ... (2)

Also if $f_y(a,b)$ exists in the same nbd. of (a,b), then

$$\underset{k\to 0}{\text{Lt}} \frac{f(a,b+k)-f(a,b)}{k} = f_y(a,b); \text{ (implies that)}$$

$\Rightarrow f(a, b+k)-f(a,b) = k f_y(a,b) + \eta k$, where x is the function of k; if $k\to 0$, then $\eta\to 0$ .... (3)

From (1), (2) and (3), we have

$f(a+h, b+k)-f(a,b) = hf_x(a+\theta h, b+k) + k[f_y(a, b)+\eta k]$ ... (4)

Where $0<\theta<1$, $h\to 0$, $k\to 0$, also $\eta\to 0$, η is the function of k.

Sccessive Partial Derivatives of some symbols.

$\frac{\delta^2 u}{\delta x^2}=f_{xx}(x,y)$ may be written as $u_{xx}=f_{xx}$

$\frac{\delta^2 u}{\delta y^2}=f_{yy}'x, y)$ is replaced by $u_{yy}=f_{yy}$

$\frac{\delta^2 u}{\delta y\delta x}=f_{yx}$ or, $u_{yx}=f_{yx}$ and $\frac{\delta^2 u}{\delta x\delta y}=f_{xy}$ or, u_{xy}

How to use the above symbol u_{yx} or, f_{yx}

Differentiate the function $u=f(x, y)$ *w. r.* to only x first then differentiate the differentiated function again *w. r.* to y; u_{yx} or f_{yx}

Differentiate the function $u=f(x, y)$ *w. r.* to only y first and differentiate the differentiated function *w. r.* to only x.

In Partial Derivative of higher order say u_{yzx}, differentiation is made *w. r.* to the outer most variable first, (here, *w. r.* to y) then differentiation is made *w. r.* to a variable just before it (i, e. z) and so on.

Ex. 1. If $u=xyz+x^3-z^3+2z^2y$

then

$$\frac{\delta u}{\delta x}= yz+3x^2,\ \frac{\delta u}{\delta y}= xz+3z^2,\ \frac{\delta u}{\delta z}=xy-3z^2+4yz$$

Ex. 2. If $u=ax^2y+by^2+czx$. find $u_{xx}, u_{yx}, u_{xy}, u_{yy}$

$$u=ax^2y+by^2+czx$$

$\therefore\ u_x=2axy+cz$; $\qquad u_{xx}=2ay$

$u_y=ax^2+2by$; $\qquad u_{xy}=2ax$

$u_x=2axy+cz$; $\qquad u_{yx}=ax$

$u_y=ax^2+2by$; $\qquad u_{yy}=2b$

9.6. If the derivatives $\frac{\delta f}{\delta x}=f_x$ and $\frac{\delta f}{\delta y}=f_y$ exist in the neigbourhood of the point (x, y) and are differentiable at that point then at (x, y)

i. e., if $f_x, f_y f_{xy}, f_{yx}$ all exist at a point (x, y) and f_{xy} or f_{yx} is continuous at the point, then $f_{xy}=f_{yx}$

We know $\frac{\delta f}{\delta x}=f_x(x, y)=\lim_{h\to 0}\frac{f(x+h, y)-f(x, y)}{h}$

Again $\frac{\delta^2 f}{\delta y\delta x}=f_{yx}(x, y)=\lim_{k\to 0}\frac{f_x(x, y+k)-f_x(x,y)}{k}$

$$=\lim_{k\to 0}\lim_{h\to 0}\frac{[f(x+h, y+k)-f(x, y+k)+f(x+h, y)-f(x,y)]}{hk}$$

$$=\lim_{k\to 0}\lim_{h\to 0}\frac{\varphi(y+k)-\varphi(y)}{hk} \quad \text{...(1)}$$

where $\phi(y)=f(x+h, y)-f(x, y)$

and $\phi(y+k)=f(x+h, y+k)-f(x, y+k)$

$\therefore\ \phi(y+k)-\phi(y)=(y+k-y)\phi_y(\xi)$; $0<\theta_1<1$ and $\xi=y+\theta_1 k$

$=k\phi_y(y+\theta_1 k)$ [*by Mean value theorem.*]

$=k\{f_y(x+h, y+\theta_1 k)-f_y(x, y+\theta_1 k\}$

Put $F(x)=f_y(x,y+\theta_1 k)$

then $F(x+h)=f_y(x+h, y+\theta_1 k)$

$\therefore\ \phi(y+k)-\phi(y)=k\{F(x+h)-F(x)\}$

$=kh\ F_x(\eta)$; $\eta=x+\theta_2 h$ and $0<\theta_2<1$ [by Mean value theorem]

$=kh\ F_x(x+\theta_2 h)=kh\, f_{xy}(x+\theta_2 h, y+\theta_1 h)$

Now from (1) we have,

$$f_{yx}(x, y)=\lim_{k\to 0}\ \lim_{h\to 0}\ \frac{khf_{xy}(x+\theta_2 h, y+\theta_1 k)}{kh}$$

$$=f_{xy}(x,y) \text{ or, } \frac{\delta^2 f}{\delta y\delta x}=\frac{\delta^2 f}{\delta x\delta y} \text{ (proved.)}$$

Ex. 3. Find the values of f_{xy} and f_{yx} when

$$f(x, y)=e^x \cos y$$

$f_x=e^x \cos y$

$\therefore\ f_{yx}=\frac{\delta}{\delta y}(f_x)=-e^x \sin y$

Again $f_y=-e^x \sin y$

So $f_{xy}=\frac{\delta}{\delta x}(f_y)=-e^x \sin y$

$\therefore\ f_{xy}=f_{yx}$

Note :—It is not always true that commutativity of f_{xy} and f_{yx} will always hold good. The commutativity depends upon the commutativity of the two limits which are not always true *e. g.*

$$\lim_{x\to 0}\ \lim_{y\to 0}\ \frac{x+y}{x-y}=\lim_{x\to 0}\frac{x}{x}=1$$

$$\lim_{y\to 0}\ \lim_{x\to 0}\ \frac{x+y}{x-y}=\lim_{y\to 0}\frac{y}{-y}=-1$$

$\therefore$ Hence $\lim_{x\to 0}\ \lim_{y\to 0}\frac{x+y}{x-y}\neq\lim_{y\to 0}\ \lim_{x\to 0}\frac{x+y}{x-y}$

9.7. The Total Differential

Let us consider the function

$$u=f(x, y) \quad \ldots \quad \ldots \quad (1)$$

Let u change to $u+\triangle u$, when x changes to $x+\triangle x$ and y changes to $y+\triangle y$. That is

$$u+\triangle u=f(x+\triangle x, y+\triangle y) \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

or, $\triangle u=f(x+\triangle x, y+\triangle y)-f(x, y)$

$=f(x+\triangle x, y+\triangle y)-f(x,y+\triangle y)+f(x,y+\triangle y)-f(x,y)$

$=\triangle x f_x(x+\theta_1\triangle x,y+\triangle y)+\triangle y f_y(x,y+\theta_2\triangle y)$

where $0<\theta_1<1$, $0<\theta_2<1$. [by Mean value Theorem]

If $\triangle x$ and $\triangle y$ are very small, then

$$f_x(x+\theta_1\triangle x,y+\triangle y)=f_x(x,y)+\epsilon_1 \ldots \quad \ldots \quad \ldots \quad (3)$$

where $\lim\limits_{(\triangle x,\triangle y)\to(0,0)}\epsilon_1=0$. also $\lim\limits_{(\triangle x, \triangle y)\to(0,0)} f(x+\theta_1\triangle x, y+\triangle y)=f_x(x, y)$

Similarly, $f_y(x, y+\theta_2\triangle y)=f_y(x, y)+\epsilon_2 \quad \ldots \quad \ldots \quad (4)$

where $\epsilon_2\to 0$ as $(\triangle x, \triangle y)\to(0, 0)$

$\therefore\ \triangle u=f_x(x, y)\triangle x+\epsilon_1\triangle x+f_y(x, y)\triangle y+\epsilon_2\triangle y$

$$=\frac{\delta f}{\delta x}\triangle x+\frac{\delta f}{\delta y}\triangle y+\epsilon_1\triangle x+\epsilon_2\triangle y \quad \ldots \quad \ldots \quad (5)$$

Where ϵ_1 and ϵ_2 are infinitesimals which will be zero if $\triangle x$ and $\triangle y$ both tend to zero.

The increment of u consists of two parts, first part contains the partial derivatives and the 2nd part contains the infinitesimals with increments of x and y.

The princpial part of $\triangle u$ that is,

$$\frac{\delta f}{\delta x}\Delta x+\frac{\delta f}{\delta y}\Delta y.$$

Is called the total differential of u and it is denoted by.

$$du=\frac{\delta f}{\delta x}\Delta x+\frac{\delta f}{\delta y}\Delta y=\frac{\delta u}{\delta x}\Delta x+\frac{\delta u}{\delta y}\Delta dy \quad \ldots\ldots(6)$$

This definition is true for all functions of u.

When $u=x$, (6) gives

$$dx=1.\ \Delta x+0=\Delta x \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

Similarly, then we take $u=y$ in (6), we get

$$dy=0+1.\ \Delta y=\Delta y \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

Hence by (7) and (8), (6) becomes

$$\left.du=\frac{\delta f}{\delta x}dx+\frac{\delta f}{\delta y}dy=\frac{\delta u}{\delta x}dx+\frac{\delta u}{\delta y}dy \quad \ldots\right\} \quad (9)$$

or, $du=f_x dx+f_y dy=u_x dx+u_y dy$

which is the total differential of u,

Cor. If $u=f(x, y, z)$ then by definition of total differential we have

$$du=\frac{\delta u}{\delta x}\Delta x+\frac{\delta u}{\delta y}\Delta y+\frac{\delta u}{\delta z}\Delta z$$

which, after putting

$u=x$, $u=y$ and $u=z$ successively, will give

$$du=\frac{\delta u}{\delta x}dx+\frac{\delta u}{\delta y}dy+\frac{\delta u}{\delta z}dz$$

$$\text{or, } du=f_x dx+f_y\,dy+f_z dz \quad \ldots \quad \ldots \quad (10)$$

If $u=f(x_1, x_2, \quad \ldots \quad \ldots \quad x_n)$ then

$$du=\frac{\delta f}{\delta x_1}dx_1+\frac{\delta f}{\delta x_2}dx_2++\cdots+\frac{\delta f}{\delta x_n}dx_n \cdots (11)$$

Total Differential Co-efficient

9.8. If $u=f(x, y)$ and $x=\phi(t)$ and $y=\psi(t)$ and $\phi'(t)$ and $\psi'(t)$ exist, then

$$\frac{du}{dt}=\frac{\delta f}{\delta x}\cdot\frac{dx}{dt}+\frac{\delta f}{\delta y}\cdot\frac{dy}{dt} \quad \ldots \quad \ldots \quad (13)$$

Proof : u is the function of x and y : x and y are functions of t.

Hence u is the function of t only

Now from (5), **Art. 9. 7.** we have

$$\Delta u=\frac{\delta f}{\delta x}\Delta x+\frac{\delta f}{\delta y}\Delta y+(\epsilon_1\Delta x+\epsilon_2\Delta y)\ ;\ \epsilon_1, \epsilon_2>0$$

Divide both sides by Δt,

$$\text{or, } \frac{\Delta u}{\Delta t}=\frac{\delta f}{\delta x}\frac{\Delta x}{\Delta t}+\frac{\delta f}{\delta y}\frac{\Delta y}{\Delta t}+\left(\epsilon_1\frac{\Delta x}{\Delta t}+\epsilon_2\frac{\Delta y}{\Delta t}\right)$$

$$\therefore \lim_{\Delta t\to 0}\frac{\Delta u}{\Delta t}=\frac{\delta f}{\delta x}\lim_{\Delta t\to 0}\frac{\Delta x}{\Delta t}+\frac{\delta f}{\delta y}\lim_{\Delta t\to 0}\frac{\Delta x}{\Delta t}$$

$$+\lim_{\Delta t\to 0}\left(\epsilon_1\frac{\Delta x}{\Delta t}+\epsilon_2\frac{\Delta y}{\Delta t}\right)$$

$$\text{or, } \frac{du}{dt}=\frac{\delta f}{\delta x}\frac{dx}{\delta t}+\frac{\delta f}{\delta y}\frac{dy}{dt}\ :\ \epsilon_1, \epsilon_2\to 0 \text{ as } \Delta t\to 0\ ]$$

$$\therefore \frac{du}{dt}=\frac{\delta f}{\delta x}\frac{dx}{dt}+\frac{\delta f}{\delta y}\frac{dy}{dt} \quad \ldots \quad \ldots \quad (14)$$

Which is called the Total differential co-efficient of u.

Cor. If $u\,f(x,y)$ and $y=\varphi(x)$, then

$$\frac{du}{dx}=\frac{\delta u}{\delta x}\cdot\frac{dx}{dx}+\frac{\delta u}{\delta y}\frac{dy}{dx}$$

$$\text{or, } \frac{du}{dx}=\frac{\delta u}{\delta x}+\frac{\delta u}{\delta y}\frac{dy}{dx}=\frac{\delta u}{\delta x}+\frac{\delta u}{\delta y}\cdots\varphi'(x).$$

Cor. If $u=f(x_1, x_2, x_3\cdots x_n)$ and

$x_1\, x_2 \ldots\ldots x_n$ are all functions of t only, and if all the derivatives are continuous, then

$$\frac{\delta u}{\delta t}=\frac{\delta u}{\delta x_1}\frac{dx_1}{dt}+\frac{\delta u}{\delta x_2}\frac{dx_2}{dt}\cdots+\frac{\delta u}{\delta x_n}\frac{dx_n}{dt}$$

9.9. If $u=f(x,y)$ and $x=\phi(r, s, t)$ and $y=\psi(r, s, t)$ then prove that

$$\frac{\delta u}{\delta r}=\frac{\delta u}{\delta x}\cdot\frac{\delta x}{\delta r}+\frac{\delta u}{\delta y}\cdot\frac{\delta y}{\delta r}$$

$$\frac{\delta u}{\delta r}=\frac{\delta x}{\delta x_1}\cdot\frac{\delta x}{\delta s}+\frac{\delta x}{\delta y}\cdot\frac{\delta y}{\delta s}$$

$$\frac{\delta u}{\delta t}=\frac{\delta u}{\delta x}\cdot\frac{\delta x}{\delta t}+\frac{\delta u}{\delta y}\cdot\frac{\delta y}{\delta t}$$

Proof : $x=\phi(r, s, t)$ and $y=\psi(r, s, t)$,

$u=f(x, y)=f\{(\phi(r, s, t), \psi(r, s, t)\}$

or, $u=F(r, s, t)$, (suppose)

The total differential of u is

$$du=\frac{\delta u}{\delta r}\,dr+\frac{\delta u}{\delta s}\,ds+\frac{\delta u}{\delta t}\,dt \quad \cdots \quad \cdots \quad (1)$$

Again $x=\phi(r, s, t)$, and $y=\psi(r, s, t)$

Then $$dx=\frac{\delta x}{\delta r}\,dr+\frac{\delta x}{\delta s}\,ds+\frac{\delta x}{\delta t}\,dt \quad \cdots \quad (2)$$

$$dy=\frac{\delta y}{\delta r}\,dr+\frac{\delta y}{\delta s}\,ds+\frac{\delta y}{\delta t}\,dt \quad \cdots \quad (3)$$

But if we consider u as the function of x and y i. e.

$u=f(x, y)$, then

$$du=\frac{\delta u}{\delta x}dx+\frac{\delta u}{\delta y}dy$$

$$=\frac{\delta u}{\delta x}\left(\frac{\delta x}{\delta r}\,dr+\frac{\delta x}{\delta s}\,ds+\frac{\delta x}{\delta t}\,dt\right)+\frac{\delta u}{\delta y}\left(\frac{\delta y}{\delta r}\,dr+\frac{\delta y}{\delta s}\,ds+\frac{\delta y}{\delta t}dt\right)$$

by (2) and (3)

$$=\left(\frac{\delta u}{\delta x}\cdot\frac{\delta x}{\delta r}+\frac{\delta u}{\delta y}\cdot\frac{\delta y}{\delta r}\right)dr+\left(\frac{\delta u}{\delta x}\cdot\frac{\delta x}{\delta s}+\frac{\delta u}{\delta y}\cdot\frac{\delta y}{\delta s}\right)ds$$
$$+\left(\frac{\delta u}{\delta x}\cdot\frac{\delta x}{\delta t}+\frac{\delta u}{\delta y}\cdot\frac{\delta y}{\delta t}\right)dt\ldots\ldots(4)$$

Now compains the co-efficient of dr, ds, dt from (1) and (4),

$$\frac{\delta u}{\delta r}=\frac{\delta u}{\delta x}\cdot\frac{\delta x}{\delta r}+\frac{\delta u}{\delta x}\cdot\frac{\delta y}{\delta r}$$

$$\frac{\delta u}{\delta s}=\frac{\delta u}{\delta x}\cdot\frac{\delta x}{\delta s}+\frac{\delta u}{\delta y}\cdot\frac{\delta y}{\delta s}$$

$$\frac{\delta u}{\delta t}=\frac{\delta u}{\delta x}\cdot\frac{\delta u}{\delta t}+\frac{\delta u}{\delta y}\cdot\frac{\delta y}{\delta t}$$

Cor. If $u=f(x_1, x_2, x_3\cdots\cdots x_n)$ and $x_1=(t_1, t_2, t_3\cdots\cdots\cdots t_n)$

$x_2=\psi(t_1, t_2, t_3\ldots\ldots\ldots t_n)$ etc. then

$$\frac{\delta u}{\delta t_1}=\frac{\delta u}{\delta x_1}\cdot\frac{\delta x_1}{\delta t_1}+\frac{\delta u}{\delta x_2}\cdot\frac{\delta x_2}{\delta t_1}+\cdots\cdots+\frac{\delta u}{\delta x_n}\cdot\frac{\delta x_n}{\delta t_1}$$

$$\frac{\delta u}{\delta t_2}=\frac{\delta u}{\delta x_1}\cdot\frac{\delta x_1}{\delta t_2}+\frac{\delta u}{\delta x_2}\cdot\frac{\delta x_2}{\delta t_2}+\cdots\cdots+\frac{\delta u}{\delta x_n}\cdot\frac{\delta x_n}{\delta t_2}$$

... ...

... ...

... ...

$$\frac{\delta u}{\delta t_n}=\frac{\delta u}{\delta x_1}\cdot\frac{\delta x_1}{\delta t_n}+\frac{\delta u}{\delta x_2}\cdot\frac{\delta x_2}{\delta t_n}+\cdots\cdots\frac{\delta u}{\delta x_n}\cdot\frac{\delta x_n}{\delta t_n}$$

9.10. Partial Differentiation of Implicit functions

Let y defind implicitly as a function of $u=f(x, y)=c$, when c is a constant. Then $du=0\Rightarrow$

$$0 = \frac{\delta f}{\delta x}dx + \frac{\delta f}{\delta y}dy \text{ or, } \frac{\delta f}{\delta x} + \frac{\delta f}{\delta x} \cdot \frac{\delta y}{\delta x} = 0$$

or. $\frac{dy}{dx} = -\frac{(\delta f/\delta x)}{(\delta f/\delta y)} = -\frac{f_x}{f_y}$: if $f_y \neq 0$

$$\therefore \quad \frac{dy}{dx} = -\frac{f_x}{f_y} \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Ex. 5. If $f(x, y) = 12x^3 - 2xy + 3y^2, = 0$ find $\frac{dy}{dx}$.

We have, $f_x = 36x^2 - 2y$; $f_y = -2x + 6y$

$$\therefore \quad \frac{dy}{dx} = -f_x/f_y = -\frac{(36x^2 - 2y)}{-2x + 6y} = \frac{18x^2 - y}{3y - x}$$

9.11. Perfect or Exact Differential

Find the condition that the expression

$$P\,dx + Q\,dy \quad \ldots \quad \ldots \quad (1)$$

will be a perfect differential, when P and Q are functions of x and y.

Let $u = f(x, y)$. If f_x and f_y are continuous, the total differential of u is

$$du = \frac{\delta f}{\delta x}dx + \frac{\delta f}{\delta y}dy \ldots \quad \ldots \quad (2)$$

If $\frac{\delta f}{\delta x}dx + \frac{\delta f}{\delta y}dy$ is equal to $Pdx + Qdy$ for all values of dx and dy, then the expression $Pdx + Qdy$ will always be a perfect differential.

Now comparing (1) and (2)

$$P = \frac{\delta f}{\delta y} \text{ and } Q = \frac{\delta f}{\delta y}$$

$$\therefore \quad \frac{\delta P}{\delta y} = \frac{\delta^2 f}{\delta y \delta x} \text{ and } \frac{\delta Q}{\delta x} = \frac{\delta^2 f}{\delta x \delta y}$$

As $\frac{\delta^2 f}{\delta y \delta x} = \frac{\delta^2 f}{\delta x \delta y}$ is always true for ordinary cases, so the expression $Pdx + Qdy$ will be an exact differential if

$$\frac{\delta P}{\delta y} = \frac{\delta Q}{\delta x}$$

Cor. The necessary and sufficient condition for the expression $Pdx + Qdy + Rdz$ to be an exact is

$$\frac{\delta P}{\delta y} = \frac{\delta Q}{\delta x}, \quad \frac{\delta Q}{\delta z} = \frac{\delta R}{\delta y}, \quad \frac{\delta R}{\delta x} = \frac{\delta P}{\delta z}$$

where the partial derivatives are continuous.

9.12. Homogeneous Functions

A function $f(x_1, x_2, \ldots \quad \ldots, x_k)$ is a homogeneous function of degree n in $x_1, x_2 \ldots, x_k$

If $f(tx_1, tx_2, \ldots, tx_k) = t^n f(x_1, x_2, \ldots, x_k)$.

For example,

$$f(x, y) = ax^2 + 2hxy + by^2$$

is a homogeneous function of degree two in x, y, since

$$f(tx, ty) = a(tx)^2 + 2h(tx)(ty) + b(ty)^2$$
$$= t^2(ax^2 + 2hxy + by^2) = t^2 f(x, y).$$

The function $g(x, y) = \frac{2x - 3y}{x + y}$ is homogeneous function of degree **Zero** in x, y since

$$g(tx, ty) = \frac{2(tx) - 3(ty)}{tx + ty} = \frac{t(2x - 3y)}{t(x + y)} = \frac{2x - 3y}{x + y} = t^0 g(x, y).$$

The functions

$$x^2 \tan^{-1}(y/x) + 2yz + z^2, \ \tfrac{1}{2}\log(x^2 + y^2) - \log x,$$

$\frac{x}{\sqrt[4]{x^2 + y^2}}$ are homogeneous ; theire degrees are

respectively 2, 0, $\frac{1}{2}$.

The homogenelity of $f(x_1, x_2, \ldots, x_k)$ can also be tested by writing,

$$v_1 = \frac{x_2}{x_1}, \quad v_2 = \frac{x_3}{x_1}, \dots \dots, v_{k-1} = \frac{x_k}{x_1}.$$

If $f(x_1, x_2, \dots, x_k)$ is homogeneous and degree n, then we should have,

$$f(x_1, x_2, \dots, x_k) = x_1^n \phi(v_1, v_2, \dots, v_{k-1}).$$

9.13. Euler's Theorem on Homogeneous Function.

If u be a homogeneous function of degree n in x any y, then

$$x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = uu$$

Proof : Since u is a homogeneous function of degree n in x y, we can write

$u = x^n f(y/x) = x^n f(v)$, when $v = y/x$.

$$\frac{\delta u}{\delta x} = \frac{\delta(x^n)}{\delta x} f(v) + x^n f'(v)\frac{\delta v}{\delta x} = nx^{n-}(v) + x^n f(v)\left(-\frac{y}{x^2}\right)$$

$$\text{or,} \quad x\frac{\delta u}{\delta x} = nx^n f(v) - yx^{n-1} f'(v) \qquad (1)$$

$$\frac{\delta u}{\delta y} = x^n f'(v)\frac{\delta v}{\delta y} = x^n f'(v)\frac{1}{x}$$

$$\text{or,} \quad y\frac{\delta u}{\delta y} = x^{n-1} yf'(v) \quad \dots \quad \dots \quad \dots \quad (2)$$

Now adding (1) and (2), we have

$$x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = nx^n f(v) - yx^{n-1} f'(v) + x^{n-1} yf'(v)$$

$$= nx^n f(v) = nu$$

$$\therefore \quad x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = nu \quad \text{Proved.}$$

Note : u has continuous first partial derivatives.

i. e. $\frac{\delta u}{\delta x}$ and $\frac{\delta u}{\delta y}$ exist

• 9. 14. Generalisation of Euler's Theorem on Homogeneous functions.

If $f(x_1, x_2, \dots \dots x_n)$ be a homogeneous function of $x_1\, x_2 \dots x_n$ of degree m and if all their first partial derivatives are continuous,

than

$$\left(x_1\frac{\delta f}{\delta x_1} + x_2\frac{\delta f}{\delta x_2} + \dots \dots \dots + x_n\frac{\delta f}{\delta x_n}\right)^n = m\,(m-1)\,(m-2) \dots \dots \dots (m-n+1) f(x_1, x_2 \dots x_n)$$

As $f(x_1, x_2, x_3, \dots x_n)$ is a homogeneous function of degree m, then

$$f(tx_1, tx_2 \dots \dots tx_n) = t^m f(x_1, x_2, \dots \dots x_n) \quad \dots \quad \dots \quad (1)$$

If $u_1 = tx_1, u_2 = tx_2 \quad \dots, u_n = tx_n \quad \dots \quad \dots \quad (2)$

then $f(u_1, u_2 \dots u_n) = t^m f(x_1, x_2 \dots \dots x_n) \quad \dots \quad (3)$

Also $du_1 = x_1 dt, \; du_2 = x_2\, dt, \dots du_n = x_n dt \quad \dots \quad (4)$

as $u_1, u_2\, u_3 \dots, u_n$ etc are linear functions of t

Differentiate (3) w. r. to to t then

$$\left(x_1\frac{\delta f}{\delta u_1} + x_2\frac{\delta f}{\delta u_2} + \dots \dots + x_n\frac{\delta f}{\delta u_n}\right) = mt^{m-1} f(x_1, x_2 \dots x_n)$$

Differentiate it again w. r. to t then

$$x_1^2\frac{\delta^2 f}{\delta u_1^2} + x_1 x_2\frac{\delta^2 f}{\delta u_1 \delta u_2} + x_1 x_2\frac{\delta^2 f}{\delta u_1 \delta u_2} + x_2^2\frac{\delta^2 f}{\delta u_2^2} + \dots \dots \dots$$

$$\dots \dots + x^2_n\frac{\delta^2 f}{\delta u^2_n} = m(m-1)t^{m-2} f(x_1, x_2 \cdot x_n)$$

$$\text{or}\left(x_1\frac{\delta f}{\delta u_1}+x_2\frac{\delta f}{\delta u_2}+\cdots+x_n\frac{\delta f}{\delta u_n}\right)^2=m(m-1)t^{m-2}f(x_1,x_2\cdots x_n)$$

Differentiate it upto n times $w.$ $r.$ to t then

$$\left(x_1\frac{\delta f}{\delta u_1}+x_2\frac{\delta f}{\delta u_2}+\cdots\cdots+x_n\frac{\delta f}{\delta u_n}\right)^n$$

$$=m(m-1)\ldots(m-n+1)\,t^{m-n}f(x_1,x_2\cdots x_n)$$

If $t=1$, then

$$\left(x_1\frac{\delta f}{\delta x_1}+x_2\frac{\delta f}{\delta x_2}+\cdots\cdots+x_n\frac{\delta f}{\delta x_n}\right)^n=m(m-1)(m-2)\cdots$$

$$(m-n+1)f(x_1,x_2\cdots x_n)$$ Proved.

Cor 1. $u=1$. then

$$x_1\frac{\delta f}{\delta x_1}+x_2\frac{\delta f}{\delta x_2}+\cdots\cdots\cdots+x_n\frac{\delta f}{\delta x_n}=m\,f(x_1\;x_2\cdots x_n)=mf$$

Cor. 2. If $u=f(x,y,z)$ is a homogeneous function of degree n in x, y and z then

$$x\frac{\delta u}{\delta x}+y\frac{\delta u}{\delta y}+z\frac{\delta u}{\delta z}=nf(x,y,z)=nu.$$ D. H. '86

CONVERSE OF EULER'S THEOREM

If u be a differential function of x, y, z and for all x, y, z let

$$u\frac{\delta u}{\delta x}+y\frac{\delta u}{\delta y}+z\frac{\delta u}{\delta z}=nu$$

Then show that u is a homogeneous function of degree n in x, y, z,

Proof : Let $r=\lambda x$, $s=\lambda y$, $t=\lambda z$. If we consider x, y, are constant, then

$$\frac{d}{d\lambda}f(r,s\,t)=x\frac{\delta f}{\delta r}+y\frac{\delta f}{\delta s}+z\frac{\delta f}{\delta t}$$

$$=\frac{1}{\lambda}\left(r\frac{\delta f}{\delta r}+s\frac{\delta f}{\delta s}+t\frac{\delta f}{\delta t}\right)\cdots\cdots\quad(1)$$

$$\text{Since } x\frac{\delta u}{\delta x}+y\frac{\delta u}{\delta y}+z\frac{\delta u}{\delta z}=nu\cdots\cdots\quad(2)$$

is true for all x, y, z, then

$$\frac{d}{d\lambda}f(r,s,t)=\frac{n}{\lambda}f(r,s,t)\cdots\cdots\quad(3)$$

Let $v=f(r,s,t)$, then for all valuss of λ ... ... (4)

$$\frac{dv}{\lambda}=\frac{n}{\lambda}v \text{ from (3) or, } \frac{1}{v}\cdot\frac{dv}{\lambda}=\frac{n}{\lambda}$$

Integrating

$\log v=n\log\lambda+c$,

or, $v=\lambda^n e^c=A\lambda^n$, $A=e^c$ is independent of λ.

or, $f(r,s,t)=A\lambda^n$ from (4)

or, $f(\lambda x,\lambda y,\lambda z)=A\lambda^n$

When $\lambda=1$, than $f(x,y,z)=A$

Henee $f(\lambda x,\lambda y,\lambda z)=\lambda^n f(x,y,z)$

for all values of x,y,z and λ and $f(x,y,z)$ is a homogeneous function of degree n in x, y, z.

Note : The function $\frac{\delta}{\delta\lambda}f(r, s, t)$ is some function of r, s, t ;

let it be $f(r, s, t)$. Then

$\frac{\delta}{\delta\lambda}f(x, y, z)$ is $F(x, y, z)$

Art. 9.15(i) Generalised Taylor'sTheorem.

Let $f(x, y, z)$ and all its partial derivatives upto n are continuous in the domain

$a \le x \le a + h,\ b \le y \le b+k,\ c \le z \le c+l$; then $f(a + h, b+k, c + l)$

$$= f(a, b, c) + \left(h\frac{\delta}{\delta a} + k\frac{\delta}{\delta b} + l\frac{\delta}{\delta c}\right)^{n-1} f(a, b, c)$$

$$+ \frac{1}{(n-1)!}\left\{\left(h\frac{\delta}{\delta x} + k\frac{\delta}{\delta y} + l\frac{\delta}{\delta z}\right)^{n} f(x, y, z)\right\}_{\substack{x=a+\theta h\\ y=b+\theta k\\ z=c+\theta l}}$$

Where $0 < \theta < 1$

Let us suppose that $x = a + ht,\ y = b + k^t,\ z = c+lt$

So that $f(x,y, z) = f(a + ht, b+kt, c+lt) = F(t)$ say.

$$F'(t) = \frac{\delta f}{\delta x}\cdot\frac{\delta x}{\delta t} + \frac{\delta f}{\delta y}\cdot\frac{\delta y}{\delta t} + \frac{\delta f}{\delta z}\cdot\frac{\delta t}{\delta t}$$

$$= h\frac{\delta f}{\delta x} + k\frac{\delta f}{\delta y} + l\frac{\delta f}{\delta z} = \left(h\frac{\delta}{\delta x} + k\frac{\delta}{\delta y} + l\frac{\delta}{\delta z}\right) f(x, y, z) \ \ldots\ldots (1)$$

Again differentiating since Right land side is a function of x, y and z, we have

$$F''(t) = \left(h\frac{\delta}{\delta x} + k\frac{\delta}{\delta y} + l\frac{\delta}{\delta z}\right)^2 f(x, y, z) \ \ldots\ldots\ldots\ldots (2)$$

....

$$F^{n-1}(t) = \left(h\frac{\delta}{\delta x} + k\frac{\delta}{\delta z} + l\frac{\delta}{\delta z}\right)^{n-1} f(x, y, z) \ \ldots\ldots (n-1)$$

$$F^{n}(t) = \left(h\frac{\delta}{\delta x} + k\frac{\delta}{\delta y} + l\frac{\delta}{\delta z}\right)^{n} f(x, y, z) \ \ldots\ldots\ (n)$$

We see from the above relations, it is clear that $f(t)$ and the first

n derivatives are continuous fuinctions in the interval (o,1) and by Maclaurin's theorem, we have,

$$F(t) = f(0) + tF'(0) + \frac{t^2}{\underline{|2}} F''(0) + \ldots\ldots$$

$$+ \frac{t^{n-1}}{(n-1)!}F^{n-1}(0) + \frac{t^n}{n!}F^n(\theta t), \ \ldots\ldots (3)$$

$0 < \theta < 1$

When $t = 0$, x, y, z become, a, b, c respectively and hence

$$\left(h\frac{\delta}{\delta x} + k\frac{\delta}{\delta y} + l\frac{\delta}{\delta z}\right)^{m} f(x, y, z) = \left(h\frac{\delta}{\delta a} + k\frac{\delta}{\delta b} + l\frac{\delta}{\delta c}\right)^{m} f(a, b, c)$$

Now applying the above relations (1), (2), ... (n) we get

$$F'(0) = \left(h\frac{\delta}{\delta x} + k\frac{\delta}{\delta b} + l\frac{\delta}{\delta c}\right) f(a,b,c)$$

$$F''(0) = \left(h\frac{\delta}{\delta a} + k\frac{\delta}{\delta b} + l\frac{\delta}{\delta c}\right)^2 f(a,b,c)$$

....

$$F^{n-1}(0) = \left(h\frac{\delta}{\delta a} + k\frac{\delta}{\delta b} + l\frac{\delta}{\delta c}\right)^{n-1} f(a,b,c)$$

$$\text{Also, } F^{n}(\theta t) = \left(h\frac{\delta}{\delta x} + k\frac{\delta}{\delta y} + l\frac{\delta}{\delta z}\right)^{n} f(x,y,z)$$

Where $x = a + h\theta t,\ y = b + k\theta t,\ z = c + l\theta t$

$\therefore o < \theta < 1$

Now putting these relatons in (3), we have

$F(t) = f(a + ht, b + kt, c+lt)$: if $t=1$, then

$f(a+h, b+k, c+l)$

$$= f(a,b,c) + \left(h\frac{\delta}{\delta a} + k\frac{\delta}{\delta b} + l\frac{\delta}{\delta c}\right) f(a,b,c)$$

$$+ \frac{1}{(n-1)!}\left(h\frac{\delta}{\delta a} + k\frac{\delta}{\delta b} + l\frac{\delta}{\delta c}\right)^{n-1} f(a,b,c)$$

$$+ \frac{1}{n!}\left(h\frac{\delta}{\delta x} + k\frac{\delta}{\delta y} + l\frac{\delta}{\delta z}\right)^{n} f(x,y,z) \ \ldots\ldots (4)$$

Where $x = a+\theta h,\ y=b+\theta k,\ z=c+\theta l,\ t=1$.
The Theorem is established.
The theorem is symbolically expressd as

$$f(a+h,\ b+k,\ c+l) = \left[e^{\left(h\frac{\delta}{\delta a}+k\frac{\delta}{\delta b}+l\frac{\delta}{\delta c}\right)}\right] f(a,b,c)$$

$$\text{up to n terms} + \frac{1}{n!}\left(h\frac{\delta}{\delta x}+k\frac{\delta}{\delta y}+l\frac{\delta}{\delta z}\right) f(x,y,z) \ \ldots\ldots (5)$$

$$\text{or; } f(x+h,\ y+k,\ z+l) = \left[e^{h\frac{\delta}{\delta x}+k\frac{\delta}{\delta y}+l\frac{\delta}{\delta z}}\right] f(x,y,z) \ldots\ldots (6)$$

Art 9.15 (d) Taylor's Theorem can be stated in another form.

$$f(x,y) = f(a,b) + \left[(x-a)\frac{\delta}{\delta x}+(y-b)\frac{\delta}{\delta y}\right] f(a,b)$$

$$+\frac{1}{2!}\left[(x-a)\frac{\delta}{\delta x}+(x-b)\frac{\delta}{\delta y}\right]^2 f(a,b) + \ldots\ldots\ldots$$

$$+\frac{1}{(n-1)!}\left[(x-a)\frac{\delta}{\delta x}+(x-b)\frac{\delta}{\delta y}\right]^{n-1} f(a,b) + R_n$$

$$\text{Where } R_n = \frac{1}{n!}\left[(x-a)\frac{\delta}{\delta x}+(y-b)\frac{\delta}{\delta y}\right]^n f\{a+(x-a)\theta\}.$$

$$\text{Where } R_n = \frac{1}{n!}\left[(x-a)\frac{\delta}{\delta x}+(y-b)\frac{\delta}{\delta y}\right]^n f\{a+(x-a)\theta,\ b+(y-b)\theta\}$$

$0<\theta<1$ called the Tayloy's expansion of $f(x,\ y)$ about the point $(a,\ b)$ in powers of $x-a$ and $y-b$

Alternative Form

$f(x+h,\ y+k)$

$$= f(x,\ y) + \left(h\frac{\delta f}{\delta x}+k\frac{\delta f}{\delta y}\right) + \frac{1}{2!}\left(h^2\frac{\delta^2 f}{\delta x^2}+2hk\frac{\delta^2 f}{\delta x\delta y}+k^2\frac{\delta^2 f}{\delta y^2}\right)$$

Art. 9.15 (*b*) Maclaurin's Theorem for three variables.

Putting $a=0,\ b=0,\ c=0$ and $h=x,\ k=y,\ l=z$ respectively in (4), we have,

$$f(x,\ y,\ z) = f(0,\ 0,\ 0) + \left(x\frac{\delta}{\delta u}+y\frac{\delta}{\delta v}+z\frac{\delta}{\delta w}\right) f(u,v,w)$$

$$\ldots\ldots + \frac{1}{(n-1)!}\left(x\frac{\delta}{\delta u}+y\frac{\delta}{\delta v}+z\frac{\delta}{\delta w}\right)^{n-1} f(u,v,w)$$

$$+\frac{1}{n!}\left(x\frac{\delta}{\delta u}+y\frac{\delta}{\delta v}+z\frac{\delta}{\delta w}\right)^n f(u,v,w)$$

u,v,w are to be replaced by 0,0,0 in all the terms except the last term which is to be replced by $u=\theta x,\ v=\theta y,\ w=\theta z,\ 0<\theta<1$

NOTE. Maclaurin's Theorem for two variables.

$$f(x,y) = f(0,0) + \left(x\frac{\delta}{\delta u}+y\frac{\delta}{\delta v}\right) f(u,\ v)$$

$$+\frac{1}{2!}\left(x\frac{\delta}{\delta u}+y\frac{\delta}{\delta v}\right)^2 f(u,\ v) + \ldots\ldots$$

$$+\frac{1}{(n-1)!}\left(x\frac{\delta}{\delta u}+y\frac{\delta}{\delta v}\right)^{n-1} f(u,v) + \frac{1}{n!}\left(x\frac{\delta}{\delta u}+y\frac{\delta}{\delta v}\right)^n f(u,\ v)$$

$$= \left(e^{x\frac{\delta}{\delta u}+y\frac{\delta}{\delta v}}\right) f(u,v)$$

Putting $u=0,\ v=0$, upto n terns and for the remainder or $(n+1)$th term $u=\theta x,\ v=\theta y$, the above theorem will be obtained.

NOTE. If there are n variables, then

$$f(x+h,\ y+k,\ z+l,\ \ldots..) = \left[e^{h\frac{\delta}{\delta x}+k\frac{\delta}{\delta y}+l\frac{\delta}{\delta z}}\right] + \ldots\ldots\ )\ f(x,y,z,\ldots\ldots..)$$

NOTE. For infinite Series,

$$f(x,\ y) = \left[e^{x\frac{\delta}{\delta u}+y\frac{\delta}{\delta v}}\right] .f(u,\ v), \text{ after expansion}$$

put $u=0,\ v=0$.

$$f(x,\ y,\ z) = \left[e^{x\frac{\delta}{\delta u}+y\frac{\delta}{\delta v}+z\frac{\delta}{\delta w}}\right] f(u,\ v,\ w).$$

after expansion, put $u=0,\ v=0,\ w=0$

Art. 9.15 (*e*) DIFFERENTIABILITY.

Show that a function differentiable at a point is necessarily continuous and possesses partial derivatives thereat.

Sol. Let us consider two neighbouring points $(x,\ y)$ and $(x+\delta x,\ y+\delta y)$ in the domain of definition of a function $f(x,y)$. The change df is given by $\delta f = f(x+\delta x,\ y+\delta y) - f(x,\ y)$

The function $f(x,y)$ is differentiable at (x,y) if the change df can be expressed as

$\delta f = A\delta x + B\delta y + \delta x\phi(\delta x, \delta y) + \delta y\psi(\delta x, \delta y)$

Where A and b are free from δx and δy and also constants. The functions ϕ and ψ are the finctions of δx and δy and $\phi(\delta x, \delta y)\to 0$. ψ $(\delta x, \delta y)\to 0$ simultaneously.

We may call $A\delta x + B\delta y$ is a differential of $f(x, y)$ at (x, y) and is denoted by $df(x, y)$ Or, df. Hence $df = A\delta x + B\delta y$

From (1) We see that $(\delta x, \delta y)\to 0$. then

$f(x+\delta x, y+\delta y) - f(x,y)\to 0$

Or: $f(x+\delta x, y+\delta y)\to f(x,y)$

We conclude that the function $f(x,y)$ is continuous at (x, y)

Thus every differentiable function is continuous.

If we put $dy = 0$ for y as constant in

(1), then $\delta\phi = A\,\delta x + B.o + \delta x\,\phi(\delta x, 0)$

Or; $\dfrac{\delta f}{\delta x} = A$ if $\lim \delta x\to 0$

Similarly $\dfrac{\delta f}{\delta y} = B$.

Hence the constants A and b are respectively the partial derivatives of f with respect of x and y.

Thus **a function which is differentiable at a point possesses the first order partial derivatives at that point.**

Converse of this theorem is not always true.

Functions which are continuous an may even possess partial derivatives at a point but are not differentiable at that point.

From (1), $\delta f = A\,\delta x + B\delta y = \dfrac{\delta f}{\delta x}\delta x + \dfrac{\delta f}{\delta y}\delta y$ .... (2)

If we consider $f = y$, then $dy = \delta y$

Similarly if $f = x$, then $dx = \delta x$

From (2)

$df = \dfrac{\delta f}{\delta x}dx + \dfrac{\delta f}{\delta y}dy = f_x dx + f_y dy$ .... (3)

is the differential of f at (x, y)

NOTE. 1 If the function f and its partial derivatives f_x, f_y are continuous at a point (x, y) in the domain of f, then

$\delta f = f(x+\delta x, y+\delta y) - f(x,y)$

$= f(x+\delta x, y+\delta y) - f(x, y+\delta y) + f(x, y+\delta y) - f(x, y)$

By Lagrange's Mean Value Theorem of one variable, we have

$\delta f = \delta x f_x(x+\theta\,\delta x, y+\delta y) + \delta y f_y, (x, y+\theta_2\delta y)$

Where $0 < \theta. < 1$, $0 < \theta_2 < 1$

Since f_x and f_y are continuous at (x,y) therefore when $(\delta x, \delta y)\to(0,0)$, we have $\delta\psi = (f_x + \phi)\delta x + (f_y + \psi)\,\delta y$

Where ϕ and ψ tend to zero as $(\delta x, \delta y)\to(0,0)$

$\therefore \delta f = f_x\delta x + f_y\delta y + \delta x\,\phi(\delta x, \delta y) + \delta y\psi\,(\delta x, \delta y)$

$= f_x\delta x + f_y dy + \phi\delta x + \psi\delta y$ ... (4)

NOTE-2. If $\delta x = h$, $\delta y = k$ in (1) then

$df = f(a+h, b+k) - f(a, b)$

$= Ah + Bk + h\phi\,(h,k) + k\psi\,(h, k)$ .... (5)

Where $A = f_x$, $B = I_y$, and ϕ, ψ are the functions of h, k and $\phi(h,k)\to 0$, $\psi(h,k)\to 0$ if $h\to 0$, $k\to 0$

NOTE-3 Theorem of 9 15(c), which is not true always is shown below with an example.

Example. Show that the given function has partial derivatives and is contunuous at (0, 0) still the functuon is not differential at the point.

$$f(x, y) = x\frac{x^3-y^3}{x^2+y^2},\ (x, y) \neq (0, 0)$$

$$= 0. \qquad (x, y) = (0, 0)$$

Sol. Let $x = r\cos\theta$, $y = r\sin\theta$ in

$$\left|\frac{x^3-y^3}{x^2+y^2}\right| = \left|\frac{r^3\cos^3\theta - r^3\sin^3\theta}{r^2(\cos^2\theta+\sin^2\theta)}\right| = |r(\cos^3\theta - \sin^3\theta|$$

$\leq 2|r| = 2\sqrt{(x^2+y^2)} < \epsilon$

if $x^2 < \epsilon^2/8$, $y^2 < \epsilon^2/8$

Or: if $x < \epsilon/2\sqrt{2}$, $y < \epsilon/2\sqrt{2}$

$$\therefore \left|\frac{x^3-y^3}{x^2+y^2}-0\right| < 2\sqrt{(x^2+y^2)} < 2.\ \epsilon/2 = \epsilon$$

i.e.: $\lim_{(x,y)\to 0} \frac{x^3-y^3}{x^2+y^2} = 0$

i.e.: $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$

Hence the finction is continuous at (0,0).

For partial derivatives at (0,0).

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{h-0}{h} = 1.$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0} \frac{-k}{k} = -1$$

Thus the function has partial derivatives at (0,0). i.e: f_x and f_y exist at (0,0)

If the function f is differentiable at (0,0).

Then

$df = f(h,k)-f(0,0) = Ah+Bk+h\phi+k\psi$ .... (1)

Where $a = f_x(0,0) = 1$. $b = f_y(0,0) = -1$. i. 2; A and b are constants and $\phi(h,k)$, $\psi(h,k)\to 0$ as $(h,k)\to(0,0)$

Now if we put $h = r_1 \cos\theta$. $k = r_1 \sin\theta$ in (1).

$f(h,k)-f(0,0) = 1.h+(-1)k+h\phi(h,k) + k\psi(h,k)$

Or: $\frac{r_1{}^3(\cos^3\theta - \sin^3\theta)}{\rho_1{}^2(\cos^2\theta+\sin^2\theta)} = r_1\cos_1 - r_1\sin\theta + r_1\phi\text{Cos}\theta + r_1\psi\sin\theta$

Or: $\cos^3\theta-\sin^3\theta=\cos\theta-\sin\theta+\phi\cos\theta+\psi\sin\theta$ ...... (2)

For. $\theta = \tan^{-1}(h/k)$. $r_1\to 0$ implies that $(h, k)\to(0,0)$. Therefore in limit, we have $\cos^3\theta-\sin^3\theta = \cos\theta-\sin\theta$

Or; $(\cos\theta-\sin\theta)(\cos^2\theta+\sin^2\theta+\cos\theta\sin\theta-1) = 0$

Or; $\cos\theta\sin\theta\,(\cos\theta-\sin\theta) = 0$

Which is impossible for arbitrary θ.

Hence the function f is not differentiable at the origin.

Taylor's Theorem for two variables.

9. 15(g) To expand ϕ (x+h, y+k) in powers of h and k, and show that

$$\phi(x+h, y+k)=e^{h\frac{\delta}{\delta x}+k\frac{\delta}{\delta y}}\ \phi(x, y)$$

By Taylor's Theorem, we have,

$$\phi(x+h, y+k)=\phi(x, y+k)+h\frac{\delta}{\delta x}\phi(x, y+k)+\frac{h^2}{2\,!}\frac{\delta^2}{\delta x^2}\phi(x,y+k)$$

$$+\frac{h^3}{3\,!}\frac{\delta^3}{\delta x^3}\phi(x, y+k)+\cdots$$

$$=\phi(x, y)+k\frac{\delta}{\delta y}\phi(x, y)+\frac{k^2}{2\,!}\frac{\delta^2}{\delta y^2}\phi(x, y)+\frac{k^3}{3\,!}\frac{\delta^3}{\delta y^3}\phi(x, y)+$$

$$+h\frac{\delta}{\delta x}\left(\phi(x, y)+k\frac{\delta}{\delta y}\phi(x, y)+\frac{k^2}{2\,!}\frac{\delta^2}{\delta y^2}\phi(x, y)+\ \ldots\right)$$

$$+\frac{h^2}{2\,!}\frac{\delta^2}{\delta x^2}\left(\phi(x, y)+k\frac{\delta}{\delta y}\phi(x, y)+\ \cdots\right)+\frac{h^3}{3\,!}\frac{\delta^3}{\delta x^3}(\phi(x, y)+\ldots$$

$$=\phi(x, y)+\left(h\frac{\delta}{\delta x}+k\frac{\delta}{\delta y}\right)\phi(x, y)+\left(\frac{h^2}{2\,!}\frac{\delta^2}{\delta x^2}\right.$$

$$\left.+hk\frac{\delta^2}{\delta x\delta y}+\frac{k^2}{2\,!}\frac{\delta^2}{\delta y^2}\right)\phi(x, y)$$

$$+\left(\frac{h^3}{3\,!}\frac{\delta^3}{\delta x^3}+\frac{h^2k}{2\,!}\frac{\delta^3}{\delta x^2\delta y}+\frac{hk^2}{2\,!}\frac{\delta^3}{\delta x\delta y^2}+\frac{k^3}{3\,!}\frac{\delta^3}{\delta y^3}\right)\phi(x, y)+\ \ldots$$

or, $\phi(x+h, y+k)=\phi(x, y)+\left(h\frac{\delta}{\delta x}+k\frac{\delta}{\delta y}\right)\phi(x, y)+$

$$\frac{1}{2\,!}\left(h^2\frac{\delta^2}{\delta x^2}+2hk\frac{\delta^2}{\delta x\delta y}+k^2\frac{\delta^2}{\delta y^2}\right)\phi(x,y)$$

$$+\frac{1}{3!}\left(h^3\frac{\delta^3}{\delta x^3}+3h^2k\frac{\delta^3}{\delta x^2\delta y}+3hk^2\frac{\delta^3}{\delta x\delta y^2}+k^3\frac{\delta^3}{\delta y^3}\right)\phi(x,y)+$$

$$=\left[1+\left(h\frac{\delta}{\delta x}+k\frac{\delta}{\delta y}\right)+\frac{1}{2!}\left(h\frac{\delta}{\delta x}+k\frac{\delta}{\delta y}\right)^2+\frac{1}{3!}\left(h\frac{\delta}{\delta x}+k\frac{\delta}{\delta y}\right)^3+\ldots\right]\phi(x,y)$$

$$\phi(x+h,y+k)=e^{h\frac{\delta}{\delta x}+k\frac{\delta}{\delta y}}\phi(x,y)$$

If there are n variables, then

$$\phi(x+h,y+k,z+l\ldots\ldots)=e^{h\frac{\delta}{\delta x}+k\frac{\delta}{\delta y}+l\frac{\delta}{\delta z}+\ldots\ldots}\phi(x,y,z,\ldots)\ldots$$

Art. 9.15(f) SUFFICIENT CONDTION FOR DIFFENTIABILITY

If $f(a,b)$ be a point in the domain of a function f such that

(i) f_x is continuous at (a,b)

(ii) f_y exists at (a,b)

then f is differentiable at (a,b)

Sol. f_x exists in a certain neighbourhood of $(a-\delta,\ a+\delta;\ b-\delta,\ b+\delta)$ of (a,b) since f_x is continuous by (i)

Let $(a+h,\ b+k)$ be a point of this neighbourhood. So

$df = f(a+h,\ b+k)-f(a,b)$

$= f(a+h,b+k)-f(a,b+k)+f(a,\ b+k)-f(a,b)$....(1)

As f_x exists in $(a-\delta,\ a+\delta,\ b-\delta,\ b+\delta)$. Then by Lagrange's Mean Value Theorem,

We have

$f(a+h,\ b+k)-f(a,\ b+k)=h\,f_x(a+\theta h,\ b+k)$ ..... (2)

Where $0<\theta<1$ and depends on h and k.

$\therefore \lim\limits_{h(h,k)\to(0,0)} f_x(a+\theta h,\ b+k)=f_x(a,b)$, so that we have now

$f_x(a+\theta h,b+k)=f_x(a,b)+\phi(h,k)$ ...... (3)

Where $\phi(h,k)\to 0$ as $(h,k)\to(0,0)$

By the second condition, $f_y(a,b)$ exists,

$\lim\limits_{k\to 0}\dfrac{f(a,b+k)-f(a,b)}{k}=f_y(a,b)$. So that

We can write

$\dfrac{f(a,\ b+k)-f(a,b)}{k}=f_y(a,\ b)+\psi(k)$ .... (4)

Where $\psi(k)\to 0$ as $k\to 0$

From (1), (2), (3) and (4), We have

$df = hf_x(a,b)+kf_y(a,b)+h\phi(h,k)+h\psi(h,k)$ implies that f is differentiable at (a,b).

NOTE. 4. In the same way we can show that f is differentiable at $(a,\ b)$, if f_x exists and f_y is continuous at $(a,\ b)$

Art. 9.15(g) SUFFICIFNT CONDITION FOR CONTINUIUTY

A sufficient condition for a function $f(x,y)$ be continuous at $(a,\ b)$ is that one of the partial derivatives exists and is bounded in a nbd of (a, b) and that the other exists

Proof. Let f_x exists and be bounded in a nbd of (a,b) and let f_y (a,b) exist, then for any point (a+h, b+k) of the nbd. of (a,b), we have

$f(a+h,b+k)-f(a,\ b+k)=(a+h-a)\,f_x(a+\theta h,\ b+k),\ 0<\theta<1$

Since $f_y(a,\ b)$ exists, so we have

$\mathop{Lt}\limits_{k\to 0}\dfrac{f(a,\ b+k)-f(a,b)}{k}=f_y(a,\ b)$

Or, $f(a,\ b+k)-f(a,b)=k\,[f_y(a,b)+\eta]$

Where η is a functon of k and if $k\to 0$, then $\eta\to 0$

$\therefore f(a+h,b+k)-f(a,b)=hf_x(a+\theta h,\ b+k)+k\,[f_y(a,b)+\eta]$

If $(h,k)\to 0$. Since $f_x(a+\theta h,\ b+k)$ is bcunded, thén

$\mathop{Lt}\limits_{(h,k)\to(0,0)}(a+h,b+k)=f(a,b)$

i. e; Left hand limit = Right Land limit = functional value.

Hence $f(x,\ y)$ is continuous.

Conclusion : A sufficient condition that a function will be continuous in a closed region is that both the partial derivatives exist and are bounded throughout the region.

Art 9.15 (h) SCHWARZ'S THEOREM.

It f_y, exists in a certain nbd. of a point (a,b) of the domain at definition of a function f and f_y is continuous at (a, b), then

f_{xy} (a,b) exists and is equal to f_{yx} (a, b) i.e: $f_{xy} = f_{yx}$.

Proof. Under the given conditions. f_x, f_y, f_{yx} all exist in a certain nbd. of (a, b). Let (a+h, b+k) be a point of this nbd. Let

$\phi(h, k) = f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a, b)$ ..... (1)

$G(x) = f(x, b+k) - f(x, b)$; $G(a+h) = f(a+h, b+k) - f(a+h, b)$

∴ [G(x) means inside the bracket the function remains the same in operation]

$\phi(h,k) = G(a+h) - G(a)$ .... (2)

$G(a) = f(a, b+k) - f(a, b)$

Since fx exists in a nbd of (a,b), the function G(x) is derivable in the open interval $(a, a+h)$ and therefore, by Lagrange's Mean Value Theoren from (2)

$\phi(h,k) = (a+h-a)\, G'(a+\theta h)$, $0 < \theta < 1$

$= h\{f_x(a+\theta h, b+k) - f_x(a+\theta h, b)\}$ ...... (3); ∴ Change for x

Again since f_{yx} exists in a nbd. of (a, b), the function f_x is derivable with resp ect to y in $(b\ b+k)$, open interval and then by Lagrange's M.V. Theorem, we get from (3)

$\phi(h,k) = h\,(b+k-b)\, f_{yx}(a+\theta h, b+\theta_1 k)$ $0 < \theta_1 < 1$

Or. $\dfrac{\phi(h,k)}{hk} = f_{yx}(a+\theta h, b+\theta_1 k)$

$$\therefore \lim_{h\to 0\,.\,k\to 0} \frac{\phi(h,k)}{hk} = \lim_{h\to 0\,.\,k\to 0} f_{yx}(a+\theta h, b+\theta_1 k) = f_{yx}(a.b)$$

Again

$\phi(h,k) = G(b+k) - (b) = G(b+k-b)\, G_y(b+\theta_1 k)$, $0 < \theta_1 < 1$

(By M.V. Theorem)

$= k\,\{f_y(a+h, b+\theta_1 k) - f_y(a, b+\theta_1 k)\}$

$= k(a+h-a)\,\{f_{xy}(a+\theta h, b+\theta_1 k)$, $0 < \theta_1 < 1$

$$\therefore \lim_{h\to 0\,.\,k\to 0} \frac{\phi(h,k)}{hk} = \lim_{h\to 0\,.\,k\to 0} f_{xy}(a+\theta h, b+\theta_1 k) = f_{xy}(a, b)$$

∴ Hence $f_{xy} = f_{yx}$ Proved.

★ 9. 16 JACOBIANS

If If $u_1, u_2, \ldots u_n$ are functions of variables $x_1, x_2, \cdots x_n$ then the determinant,

$$\begin{vmatrix} \dfrac{\delta u_1}{\delta x_1} & \dfrac{\delta u_1}{\delta x_2} & \cdots & \dfrac{\delta u_1}{\delta x_n} \\ \dfrac{\delta u_2}{\delta x_1} & \dfrac{\delta u_2}{\delta x_2} & \cdots & \dfrac{\delta u_2}{\delta x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\delta u_n}{\delta x_1} & \dfrac{\delta u_n}{\delta x_2} & \cdots & \dfrac{\delta u_n}{\delta x_n} \end{vmatrix}$$

is called the **Jacobian** of $u_1, u_2, \cdots u_n$ with r. to $x_1, x_2 \ldots\ldots x_n$

It is denoted by $\dfrac{\delta(u_1, u_2 \ldots u_n)}{\delta(x_1, x_2 \cdots x_n)}$ or, $J(u_1, u_2, \ldots u_n)$.

Ex. If $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r\cos\theta$.

find $\dfrac{\delta(x, y, z)}{\delta(r, \theta\ \phi)}$

$$\frac{\delta(x, y, z)}{\delta(r, \theta, \phi)} = \begin{vmatrix} \frac{\delta x}{\delta r} & \frac{\delta x}{\delta \theta} & \frac{\delta x}{\delta \phi} \\ \frac{\delta y}{\delta r} & \frac{\delta y}{\delta \theta} & \frac{\delta y}{\delta \phi} \\ \frac{\delta z}{\delta r} & \frac{\delta z}{\delta \theta} & \frac{\delta z}{\delta \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta \cos\phi & r \cos\theta \cos\phi & -r \sin\theta \sin\phi \\ \sin\theta \sin\phi & r \cos\theta \sin\phi & r \sin\theta \cos\phi \\ \cos\theta & -r \sin\theta & 0 \end{vmatrix}$$

$$= \cos\theta\, (r^2 \sin\theta \cos\theta \cos^2\phi + r^2 \sin\theta \cos\theta \sin^2\phi) +$$

$$r \sin\theta\, (r \sin^2\theta \cos^2\phi + r \sin^2\theta \sin^2\phi)$$

$$= r^2 \sin\theta \cos^2\theta + r^2 \sin^2\theta = r^2 \sin\theta.$$

★**9.17.** Let $u_1, u_2 \ldots\ldots n_n$ be functions of independent variables $x_1, x_2, \ldots x_n$. If there exists a relation between these n functions,

$$F(u_1, u_2, \ldots u_n) = 0$$

It is necessary and sufficient condition that the Jacobian should vanish.

$$\text{i e., } \frac{\delta(u_1, u_2, \cdots u_n)}{\delta(x_1, x_2, \ldots x_n)} = 0$$

Properties of Jacobians

Some properties of Jacobians are given below without proof. For proofs Edward's Calculus may be consulted.

9.18. Jacobian of function of function :—If $u_1, u_2 \ldots\ldots u_n$ are functions of $y_1, y_2, \ldots y_n$ and $y_1, y_2 \ldots y_n$ are functions of $x_1, x_2, \ldots x_n$.

$$\frac{\delta(u_1, u_2, \ldots\ldots\ldots u_n)}{\delta(x_1, x_2, \ldots\ldots\ldots x_n)} = \frac{\delta(u_1, u_2, \ldots\ldots\ldots u_n)}{\delta(y_1, y_2, \ldots\ldots\ldots y_n)} \times \frac{\delta(y_1, y_2, \ldots\ldots\ldots y_n)}{\delta(x_1, x_2, \ldots\ldots\ldots x_n)}$$

9 19. Jacobian of Implicit Functions :—If $u_1, u_2, \ldots u_n$ be connected implicity with the independent varirables $x_1, x_2, \ldots x_n$ by the relations.

$$f_1(u_1, u_2, \ldots u_n, \quad x_1, x_2 \ldots x_n) = 0$$

$$f_2(u_1, u_2 \ldots u_n, \quad x_1, x_2 \ldots x_n) = 0$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$f_n(u_1, u_2 \ldots u_n, \quad x_1, x_2 \ldots x_n) = 0$$

$$\text{then } \frac{\delta(f_1, f_2, \cdots\cdots f_n)}{\delta(u_1, u_2 \ldots\cdots u_n)} \times \frac{\delta(u_1, u_2 \ldots\cdots u_n)}{\delta(x_1, x_2, \cdots\cdots x_n)} = (-1)^n \frac{\delta(f_1, f_2 \cdots\cdots f_n)}{\delta(x_1, x_2, \cdots\cdots x_n)}$$

Note : The above result in 9.19 be regarded as generalisation of

$$\frac{dy}{dx} = -\frac{\delta f/\delta x}{\delta f/\delta y}$$

where x and y are connected by $f(x,y)=0$

9. 20 If J be the Jacobians of the system $u_1, u_2 \cdots u_n$ with regards to $x_1, x_2, \ldots x_n$ and J' the Jacobian of $x_1, x_2 \ldots x_n$ with regard to $u_1, u_2 \cdots u_n$ then $JJ' = 1$

$$\frac{\delta(u_1, u_2, \ldots\ldots u_n)}{\delta(x_1, x_2 \cdots\cdots x_n)} \times \frac{\delta(x_1, x_2 \ldots\ldots x_n)}{\delta(u_1, u_2, \cdots \ldots u_n)} = 1$$

9.21. If any set of homogeneous equations be satisfied by a common system of variables, the equation $J=0$ is also satisfied by the same system, and if the degrees are the same, the equations.

$$\frac{\delta J}{\delta x} = 0, \frac{\delta J}{\delta y} = 0, \frac{\delta J}{\delta z} = 0 \quad \ldots\ldots\ldots\ldots \text{etc.}$$

will also be satisfied by the same system

Cor. If $u=0$, $v=0$, $w=0$ have a common point, the curve $J=0$ will go through that point, and further, if the curves be of like degree, we shall have

$$\frac{\delta J}{\delta x}=0,\ \frac{\delta J}{\delta y}=0,\ \frac{\delta J}{\delta z}=0.$$

so that $J=0$ will have a double point there,

9.22. Covariant and Invariant.

Let u be any quantic i. e, a homogeneous function of any number of variables and of any degree. Another function ϕ is derived from u in any manner such that ϕ contains the constants and variables of u. Let u and ϕ be changed into U and ϕ respectively by any linear transformation. Then ϕ is said to be covarient of u provided the function derive from U by the same process by which the function ϕ was derived from u is merely ϕ multiplied by some powers of modulus of the transformation.

If ϕ does not contain any variables u or, ϕ contains cnly teh co-efficient of u, ϕ is called an invarient.

9. 32. The Jacobian of a system of function u, v, w, is a covariant of the system.

Let the transformation system be shown below, so that

$x=l_1x_1+m_1y_1+n_1\ z_1$

$y=l_2x_1+m_2y_1+n_2\ z_1$

$z=l_3x_1+m_3y_1+n_3\ z_1$

	x_1	y_1	z_1
x	l_1	m_1	n_1
y	l_2	m_2	n_2
z	l_3	m_3	n_3

$$\text{Now } \frac{\delta u}{\delta x_1}=\frac{\delta u}{\delta x}\cdot\frac{\delta x}{\delta x_1}+\frac{\delta u}{\delta y}\cdot\frac{\delta y}{\delta x_1}+\frac{\delta u}{\delta z}\cdot\frac{\delta z}{\delta x_1}$$

$$=\frac{\delta u}{\delta x}\ l_1+\frac{\delta u}{\delta y}\ l_2+\frac{\delta u}{\delta z}\ l_3$$

or, $u_{x1}=u_x l_1+u_y\ l_2+u_z\ l_3$

Similarly, u_{y1}, u_{z1}, v_{x1}, v_{y1}, v_{z1}, w_{x1}, w_{y1}, and w_{z1}

Now we have

$$\frac{\delta(u,v,w)}{\delta(x_1,y_1,z_1)}=\frac{\delta(u,v,w)}{\delta(x,y,z)}\cdot\frac{\delta(x,y,z)}{\delta(x_1,y_1,z_1)}$$

$$J_1=\begin{vmatrix} u_{x1} & u_{y1} & u_{z1} \\ v_{x1} & v_{y1} & v_{z1} \\ w_{x1} & w_{y1} & w_{z1}\end{vmatrix}=\begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z\end{vmatrix}\times\begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3\end{vmatrix}$$

or, $J_1=J\times\mu$, where $J=\dfrac{\delta(u,v,w)}{\delta(x,y,z)}$ is the jacobian of the original system and $\mu=\dfrac{\delta(x,y,z)}{\delta(x_1,y_1,z_1)}$ is the co-efficient of transformation.

9.24. The Hessian

The Jacobian of the first differential co-efficients u_x, u_y, u_z of any function u is

$$\begin{vmatrix} u_{xx} & u_{xy} & u_{xz} \\ u_{xy} & u_{yy} & u_{yz} \\ u_{xz} & u_{yz} & u_{zz}\end{vmatrix}$$

and is called the Hessian.

Ex. 6. (a) Expand f(x, y) = log (1+xy) by Taylors se-ies if 1+xy>0 at (a, b).

Ans. $f(x,y)=\log(1+xy)$, $\quad f(a,b)=\log(1+ab)$

$$f_x(x,y)=\frac{y}{1+xy} \Rightarrow f_x(a,b)=\frac{b}{1+ab}$$

$$f_y(x,y)=\frac{x}{1+xy} \Rightarrow f_y(a,b)=\frac{a}{1+ab}$$

$$f_{xx}(x,y)=-y^2/(1+xy)^2 \Rightarrow f_{xx}(a,b)=-b^2/(1+ab)^2$$

$$f_{xy}(x,y)=\{(1+xy)-xy\}/(1+xy)^2 \Rightarrow f_{xy}(a,b)=1/(1+ab)^2$$

$$f_{yy}(x,y)=-x^2/(1+xy)^2 \Rightarrow f_{yy}(a,b)=-a^2/(1+ab)^2$$

$$f(x,y)=f(a,b)+(x-a)f_x(a,b)+(y-b)f_y(a,b)+\frac{1}{\lfloor 2}(x-a)^2$$

$$f_{xx}(a,b)+2(x-a)(y-b)f_{xy}(a,b)+(y-b)^2 f_{yy}(a,b)] + \ldots\ldots$$

$$\therefore \quad \log(1+xy)=\log(1+ab)+(x-a)\frac{b}{1+ab}+(y-b)\frac{a}{(1+ab)^2}$$

$$+\frac{1}{\lfloor 2}\left[-(x-a)^2\frac{b^2}{(1+ab)^2}+2(x-a)(y-b)\frac{1}{(1+ab)^2}+(y-b)^2\right.$$

$$\left.\frac{(-a^2)}{(1+ab)^2}\right] + \cdots\cdots$$

$$\text{or, } \log(1+xy)=\log(1+ab)+\frac{b}{1+ab}(x-a)+\frac{a(y-b)}{1+ab}$$

$$+\frac{1}{\lfloor 2}\left[\frac{-b^2(x-a)^2}{(1+ab)^2}+\frac{2(x-a)(y-b)}{(1+ab)^2}-\frac{a^2}{(1+ab)^2}\right]+\cdots\cdots\cdots$$

Ex. 6. (b) Show that the functions

$u=x+y-z$, $v=x-y+z$, $w=x^2+y^2+z^2-2yz$ are not independent of one another. Show that $u^2+v^2=2w$.

$$\frac{\delta(u,v,w)}{\delta(x,y,z)}=\begin{vmatrix} \frac{\delta u}{\delta x} & \frac{\delta u}{\delta y} & \frac{\delta u}{\delta z} \\ \frac{\delta v}{\delta x} & \frac{\delta v}{\delta y} & \frac{\delta v}{\delta z} \\ \frac{\delta w}{\delta x} & \frac{\delta w}{\delta y} & \frac{\delta w}{\delta z} \end{vmatrix}=\begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 2x & 2y-2z & 2z-2y \end{vmatrix}$$

$$=\begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 2x & 2(y-z) & 0 \end{vmatrix}=0$$

As the Jacobian is zero, so the functions are not independent.

Again $u+v=2x$, $u-v=2(y-z)$

Now $w=x^2+y^2+z^2-2yz=\frac{1}{4}(2x)^2+\frac{1}{4}\{2(y-z)\}^2$

$=\frac{1}{4}(u+v)^2+\frac{1}{4}(u-v)^2$

or, $4w=2u^2+2v^2$ or, $u^2+v^2=2w$

Ex. 7. If $z^2=x^2+y^2+1$, Prove that

$$\frac{\delta^2 z}{\delta x\delta y}=\frac{\delta^2 z}{\delta y\delta x} \quad \ldots \quad \ldots \quad [\text{D. U. 1966}]$$

Now $z^2=x^2+y^2+1$ or, $z=\sqrt{(x^2+y^2+1)}=(x^2+y^2+1)^{1/2}$

$$\therefore \quad \frac{\delta z}{\delta y}=\frac{\delta}{\delta y}(x^2+y^2+1)^{1/2}=\tfrac{1}{2}(x^2+y^2+1)^{1/2-1}+\frac{\delta(y^2)}{\delta y}$$

$$=\frac{2y}{2\sqrt{x^2+y^2+1}}=\frac{y}{\sqrt{(x^2+y^2+1)}}$$

$$\frac{\delta^2 z}{\delta x\,\delta y}=\frac{\delta}{\delta x}\left\{\frac{y}{\sqrt{(x^2+y^2+1)}}\right\}=-\tfrac{1}{2}\cdot\frac{2xy}{\sqrt{(x^2+y^2+1)^3}}$$

Again $\dfrac{\delta z}{\delta y}=\dfrac{2x}{2\sqrt{(x^2+y^2+1)}}=\dfrac{x}{\sqrt{(x^2+y^2+1)}}$

$\therefore \quad \dfrac{\delta^2 z}{\delta y\delta x}=\dfrac{-2xy}{2\sqrt{(x^2+y^2+1)^3}} \quad \therefore \quad \dfrac{\delta^2 z}{\delta x\delta y}=\dfrac{\delta^2 z}{\delta y\delta x}$ Proved.

Ex. 8. $v=\dfrac{1}{\sqrt{(x^2+y^2+z^2)}}$ Show that

$\dfrac{\delta^2 v}{\delta x^2}+\dfrac{\delta^2 v}{\delta y^2}+\dfrac{\delta^2 v}{\delta z^2}=0$ [R. U. '54, D. U. '60, '83]

$$v=\frac{1}{\sqrt{(x^2+y^2+z^2)}}$$

$\therefore \quad \dfrac{\delta v}{\delta x}=\left[-\dfrac{2x}{2(x^2+y^2+z^2)^{3/2}}\right]=-\dfrac{x}{(x^2+y^2+z^2)^{3/2}}$

Similarly $\dfrac{\delta v}{\delta y}=\dfrac{y}{\sqrt{(\Sigma x^2)^3}},\ \dfrac{\delta v}{\delta z}=-\dfrac{z}{\sqrt{(x^2+y^2+z^2)^3}}$

Again $\dfrac{\delta^2 v}{\delta x^2}=\dfrac{\delta}{\delta x}\left(\dfrac{\delta v}{\delta x}\right)=\dfrac{\delta}{\delta x}\left\{\dfrac{-x}{\sqrt{(x^2+y^2+z^2)^3}}\right\}$

$$=\left\{\frac{\sqrt{(x^2+y^2+z^2)^3}.1-3/2x\sqrt{(x^2+y^2+z^2)}.2x}{(x^2+y^2+z^2)^3}\right\}$$

$$=-\frac{\sqrt{(x^2+y^2+z^2)}}{(x^2+y^2+z^2)^3}\{x^2+y^2+z^2-3x^2\}=-\frac{y^2+z^2-2x^2}{\sqrt{(x^2+y^2+z^2)^5}}$$

Similarly

$$\frac{\delta^2 v}{\delta y^2}=-\frac{z^2+x^2-2y^2}{\sqrt{(x^2+y^2+z^2)^5}},\quad \frac{\delta^2 v}{\delta z^2}=-\frac{x^2+y^2-2z^2}{\sqrt{(x^2+y^2+z^2)^5}}$$

$$\frac{\delta^2 v}{\delta x^2}+\frac{\delta^2 v}{\delta y^2}+\frac{\delta^2 v}{\delta z^2}=-\frac{y^2+z^2-2x^2+z^2+x^2-2y^2+x^2+y^2-2z^2}{\sqrt{(x^2+y^2+z^2)^5}}=0$$

Hence $\dfrac{\delta^2 v}{\delta x^2}+\dfrac{\delta^2 v}{\delta y^2}+\dfrac{\delta^2 v}{\delta z^2}=0$ Proved.

Ex. 9. Show that $x\dfrac{\delta u}{\delta x}+y\dfrac{\delta u}{\delta y}=0.$

if $u=\sin^{-1}\dfrac{x}{y}+\tan^{-1}\dfrac{y}{x}$ [D. U. 1961, '84]

$\dfrac{\delta u}{\delta x}=\dfrac{1}{\sqrt{(1-x^2/y^2)}}\ \dfrac{1}{y}+\dfrac{1}{1+y^2/x^2}\cdot\dfrac{-y}{x^2}$

$\therefore \quad x\dfrac{\delta u}{\delta y}=\dfrac{x}{\sqrt{(y^2-x^2)}}-\dfrac{xy}{x^2+y^2}$ ... ... (1)

Again $\dfrac{\delta u}{\delta x}=\dfrac{1}{\sqrt{(1-x^2/y^2)}}\cdot\dfrac{-x}{y^2}+\dfrac{1}{1+y^2/x^2}\cdot\dfrac{1}{x}$

$\therefore \quad y\dfrac{\delta u}{\delta x}=\dfrac{-x}{\sqrt{(y^2-x^2)}}+\dfrac{xy}{x^2+y^2}$ ... ... (2)

Therefore adding (1) and (2) we have

$x\dfrac{\delta u}{\delta x}+y\dfrac{\delta u}{\delta y}=0$

Alternative Method

since $\dfrac{y}{x}$ and $\dfrac{x}{y}$ are each homogeneous functions of degree zero, therefore u is a homogeneous function of degree zero. Hence by Euler's theorem

$$x\frac{\delta u}{\delta x}+y\frac{\delta u}{\delta y}=0\times u=0.$$

Ex. 10. What is the order of u, if

$$u=\frac{x+y}{x^2+y^2}$$

Verify Euler's therem for u.

We have,

$$u(tx,\ ty)=\frac{tx+ty}{(tx)^2+(ty)^2}=\frac{t(x+y)}{t^2(x^2+y^2)}=t^{-1}u(x,y)$$

Hence u is a homogeneous function of degree -1.

Now $\frac{\delta u}{\delta x}=\frac{1.(x^2+y^2)-2x(x+y)}{(x^2+y^2)^2}=\frac{y^2-2xy-x^2}{(x^2+x^2)^2}$

Similarly, $\frac{\delta u}{\delta y}=\frac{x^2-2xy-y^2}{(x^2+y^2)^2}$

$$\therefore\ x\frac{\delta u}{\delta x}+y\frac{\delta u}{\delta y}=\frac{x(y^2-2xy-x^2)+y(x^2-2xy-y^2)}{(x^2+y^2)^2}$$

$$=\frac{-xy^2-xy^2-x^3-y^3}{(x^2+y^2)^2}=-\frac{(x+y)(x^2+y^2)}{(x^2+y^2)^2}$$

$$\Rightarrow x\frac{\delta u}{\delta x}+y\frac{\delta u}{\delta y}=(-1)\frac{x+y}{x^2+y^2}=(-1)u$$

which verifies Euler's theorem.

Ex, 11. If $z=\tan^{-1}\frac{x^3+y^3}{x-y}$ then

Prove that $x\frac{\delta z}{\delta x}+y\frac{\delta z}{\delta y}=\sin 2z$ [R, U. 1966. D.U. 1964]

Now $z=\tan^{-1}\frac{x^3+y^3}{x-y}$ i. e., $\tan z=\frac{x^3+y^3}{x-y}$

Let $v=\tan z$ then

$$v(tx,\ ty)=\frac{t^3x^3+t^3y^3}{tx-ty}=t^2\frac{(x^3+y^3)}{(x-y)}=t^2v$$

Therefore v is a homogeneous function of degree 2.

By Euler's theorem,

$$x\frac{\delta v}{\delta x}+y\frac{\delta v}{\delta y}=2v$$

or $x\left(\sec^2 z\frac{\delta z}{\delta x}\right)+y\left(\sec^2 z\frac{\delta z}{\delta y}\right)=2\tan z\ \left[\because v=\tan z\right]$

or $x\frac{\delta z}{\delta x}+y\frac{\delta z}{\delta y}=2\tan z.\cos^2 z=2\sin z\cos z$

$\Rightarrow x\frac{\delta z}{\delta x}+y\frac{\delta z}{\delta y}=\sin 2z$ (proved)

Ex. 12. If $\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}=1$

prove that $u^2_x+u_y^2+u^2_z=2(xu_x+yu_y+zu_z)$

[R. U. 1964]

Ans. Differentiating both sides of

$$\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}=1$$

Partially with respect to x, we get

$$\frac{2x}{a^2+u}-\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]u_x=0$$

$$u_x=\frac{2x}{(x^2+u)F}$$

where $F=\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}$

Similarly partial differentiation w, r to y and z give

$$u_y=\frac{2y}{(b^2+u)F},\ u_z=\frac{2z}{(c^2+u)F}$$

$$\therefore\ u_x^2+u_y^2+u_z^2=\frac{4}{F^2}\left[\frac{x^2}{(a^2+u)^2}+\frac{y^2}{(b^2+u)^2}+\frac{z^2}{(c^2+u)^2}\right]$$

$$=\frac{4}{F^2}.F=\frac{4}{F}\ \cdots\cdots\ \text{(i)}$$

Also $xu_x+yu_y+zu_z=\frac{2}{F}\left[\frac{x^2}{(a^2+u)}+\frac{y^2}{(b^2+u)}+\frac{z^2}{(c^2+u)}\right]$

$$=\frac{2}{F}\ 1=\frac{2}{F}\ \cdots\ \cdots\ (2)$$

From (1) and (2), it follows that

$u_x^2+u_y^2+u_z^2=2(xu_x+yu_y+zu_z)$. (proved)

Ex. 13. If v be a function of x and y, prove that

$$\frac{\delta^2 v}{\delta x^2}+\frac{\delta^2 v}{\delta y^2}=\frac{\delta^2 v}{\delta r^2}+\frac{1}{r}\frac{\delta v}{\delta r}+\frac{1}{r^2}\frac{\delta^2 v}{\delta \theta^2}$$ D. U. 1955, 65 R. U. 1954, 65

where $x=r\cos\theta$ and $y=r\sin\theta$

Ans. We have,

$$\left.\begin{aligned} &x^2+y^2=r^2(\cos^2\theta+\sin^2\theta)=r^2\\ &\text{and } \tan\theta=\frac{y}{x} \text{ or } \theta=\tan^{-1}\left(\frac{y}{x}\right)\end{aligned}\right\}\quad(1)$$

hence

$$\left.\begin{aligned}&\frac{\delta r}{\delta x}=\frac{x}{r}=\cos\theta,\quad \frac{\delta r}{\delta y}=\frac{y}{r}=\sin\theta\ ;\\ &\frac{\delta\theta}{\delta x}=\frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right)=-\frac{y}{x^2+y^2}=-\frac{r\sin\theta}{r^2}=-\frac{\sin\theta}{r}\\ &\frac{\delta\theta}{\delta y}=\frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right)=\frac{x}{x^2+y^2}=\frac{r\cos\theta}{r^2}=\frac{\cos\theta}{r}\end{aligned}\right\}\ 2$$

$$\frac{\delta v}{\delta x}=\frac{\delta v}{\delta r}\cdot\frac{\delta r}{\delta x}+\frac{\delta v}{\delta\theta}\frac{\delta\theta}{\delta x}=\frac{\delta v}{\delta r}\cos\theta+\frac{\delta v}{\delta\theta}\left(-\frac{\sin\theta}{r}\right)$$

$$=\left(\cos\theta\frac{\delta}{\delta r}-\frac{\sin\theta}{r}\frac{\delta}{\delta\theta}\right)v \qquad \text{[by (2)]}$$

$$\therefore\ \frac{\delta^2 v}{\delta x^2}=\frac{\delta}{\delta x}\left(\frac{\delta v}{\delta x}\right)=\left(\cos\theta\frac{\delta}{\delta r}-\frac{\sin\theta}{r}\frac{\delta}{\delta\theta}\right)\left(\cos\theta\frac{\delta v}{\delta r}-\frac{\sin\theta}{r}\frac{\delta v}{\delta\theta}\right)$$

$$=\cos\theta\frac{\delta}{\delta r}\left(\cos\theta\frac{\delta v}{\delta r}-\frac{\sin\theta}{r}\frac{\delta v}{\delta\theta}\right)-\frac{\sin\theta}{r}\frac{\delta}{\delta\theta}\left(\cos\theta\frac{\delta v}{\delta r}-\frac{\sin\theta}{r}\frac{\delta v}{\delta\theta}\right)$$

$$\text{or, }\frac{\delta^2 v}{\delta x^2}=\cos^2\theta\frac{\delta^2 v}{\delta r^2}-\cos\theta\sin\theta\left(-\frac{1}{r^2}\frac{\delta v}{\delta\theta}+\frac{1}{r}\frac{\delta^2 v}{\delta r\delta\theta}\right)$$

$$-\frac{\sin\theta}{r}\left(-\sin\theta\frac{\delta v}{\delta r}+\cos\theta\frac{\delta^2 v}{\delta\theta\delta r}\right)$$

$$+\frac{\sin\theta}{r^2}\left(\cos\theta\frac{\delta v}{\delta\theta}+\sin\theta\frac{\delta^2 v}{\delta\theta^2}\right)$$

$$\text{or, }\frac{\delta^2 v}{\delta x^2}=\cos^2\theta\frac{\delta^2 v}{\delta r^2}+\frac{2\cos\theta\sin\theta}{r^2}\frac{\delta v}{\delta\theta}-\frac{2\cos\theta\sin\theta}{r}\frac{\delta^2 v}{\delta r\delta\theta}$$

$$+\frac{\sin^2\theta}{r}\frac{\delta v}{\delta r}+\frac{\sin^2\theta}{r^2}\frac{\delta^2 v}{\delta\theta^2}\cdots\text{(I)}$$

$$\frac{\delta v}{\delta y}=\frac{\delta v}{\delta r}\frac{\delta r}{\delta y}+\frac{\delta v}{\delta\theta}\frac{\delta\theta}{\delta y}=\frac{\delta v}{\delta r}\sin\theta+\frac{\delta v}{\delta\theta}\frac{\cos\theta}{r}$$

$$=\left(\sin\theta\frac{\delta}{\delta r}+\frac{\cos\theta}{r}\frac{\delta}{\delta\theta}\right)v$$

$$\therefore\ \frac{\delta^2 v}{\delta y^2}=\frac{\delta}{\delta y}\left(\frac{\delta v}{\delta y}\right)=\left(\sin\theta\frac{\delta}{\delta r}+\frac{\cos\theta}{r}\frac{\delta}{\delta\theta}\right)\left(\sin\theta\frac{\delta v}{\delta r}+\frac{\cos\theta}{r}\frac{\delta v}{\delta\theta}\right)$$

$$=\sin\theta\left[\sin\theta\frac{\delta^2 v}{\delta r^2}+\cos\theta\left(-\frac{1}{r^2}\frac{\delta v}{\delta\theta}+\frac{1}{r}\frac{\delta^2 v}{\delta r\delta\theta}\right)\right]$$

$$+\frac{\cos\theta}{r}\left[\cos\theta\frac{\delta v}{\delta r}+\sin\theta\frac{\delta^2 v}{\delta r\delta\theta}+\frac{1}{r}\left(-\sin\theta\frac{\delta v}{\delta\theta}+\cos\theta\frac{\delta^2 v}{\delta\theta^2}\right)\right]$$

$$\text{or, }\frac{\delta^2 v}{\delta y^2}=\sin^2\theta\frac{\delta^2 v}{\delta r^2}-\frac{2\sin\theta\cos\theta}{r^2}\frac{\delta v}{\delta\theta}+\frac{2\sin\theta\cos\theta}{r}\frac{\delta^2 v}{\delta r\delta\theta}$$

$$+\frac{\cos^2\theta}{r}\frac{\delta v}{\delta r}+\frac{\cos^2\theta}{r^2}\frac{\delta^2 v}{\delta\theta^2}\cdot\text{(II)}$$

Adding (I) and (II) and using $\cos^2\theta+\sin^2\theta=1$,

$$\frac{\delta^2 v}{\delta x^2}+\frac{\delta^2 v}{\delta y^2}=\frac{\delta^2 v}{\delta r^2}+\frac{1}{r}\frac{\delta v}{\delta r}+\frac{1}{r^2}\frac{\delta^2 v}{\delta\theta^2}\quad\text{[proved]}$$

Ex. 14. If $z=f(u, v)$, $u=x^2-2xy-y^2$ and $v=y$, show that

$(x+y)\dfrac{\delta z}{\delta x}+(x-y)\dfrac{\delta z}{\delta y}=0$ is equivalent to $\dfrac{\delta x}{\delta v}=0$.

we can write $\dfrac{\delta z}{\delta x}=\dfrac{\delta z}{\delta u}\dfrac{\delta u}{\delta x}+\dfrac{\delta z}{\delta v}\cdot\dfrac{\delta v}{\delta x}$

But $u=x^2-2xy-y^2$; $v=y$

$\therefore \quad \frac{\delta u}{\delta x}=2x-2y\ ;\ \frac{\delta v}{\delta x}=0\ ;\ \frac{\delta u}{\delta y}=-2x-2y\ ;\ \frac{\delta v}{\delta y}=1$

$\therefore \quad \frac{\delta z}{\delta x}=\frac{\delta z}{\delta u}\,2(x-y)+\frac{\delta z}{\delta y}\,0.=2\,(x-y)\frac{\delta z}{\delta u} \quad \ldots \ldots \quad (1)$

and $\frac{\delta z}{\delta y}=\frac{\delta z}{\delta u}\cdot\frac{\delta u}{\delta y}+\frac{\delta z}{\delta v}\cdot\frac{\delta v}{\delta y}=-2\,(x+y)\,\frac{\delta z}{\delta u}+\frac{\delta z}{\delta v} \quad (2)$

Multiply (1) by $(x+y)$ and (2) by $(x-y)$ and add. The result is

$(x+y)\,\frac{\delta z}{\delta x}+(x-y)\,\frac{\delta z}{\delta y}=2\,(x+y)(x-y)\frac{\delta z}{\delta u}$

$\quad -2(x-y)(x+y)\,\frac{\delta z}{\delta u}+(x-y)\,\frac{\delta z}{\delta v}=(x-y)\,\frac{\delta z}{\delta v}$

Given that $(x+y)\,\frac{\delta z}{\delta x}+(x-y)\,\frac{\delta z}{\delta y}=0$

$\therefore \quad 0=(x-y)\frac{\delta z}{\delta v} \Rightarrow \frac{\delta z}{\delta v}=0 \quad (x\neq y \text{ always})$

অর্থাৎ $(x+y)\,\frac{\delta z}{\delta x}+(x-y)\,\frac{\delta z}{\delta y}=0$ এবং $\frac{\delta z}{\delta v}=0$ সমতুল্য প্রমাণিত।

Ex. 15. If $v=f\,(x-y,\ y-z,\ z-x)$, then prove that

$$\frac{\delta v}{\delta x}+\frac{\delta v}{\delta y}+\frac{\delta v}{\delta z}=0$$

Let $X=y-z,\quad Y=z-x,\quad Z=x-y$

$\frac{\delta X}{\delta x}=0,\ \frac{\delta X}{\delta y}=1,\ \frac{\delta X}{\delta z}=-1$

$\frac{\delta Y}{\delta x}=-1,\ \frac{\delta Y}{\delta y}=0,\ \frac{\delta Y}{\delta z}=1$

$\frac{\delta Z}{\delta x}=1,\ \frac{\delta Z}{\delta y}=-1,\ \frac{\delta Z}{\delta z}=0$

Now $v=f$ (X, Y, Z) ; X, Y, Z, are functions of x, y, z

$\therefore \quad \frac{\delta v}{\delta x}=\frac{\delta v}{\delta X}\cdot\frac{\delta X}{\delta x}+\frac{\delta v}{\delta Y}\cdot\frac{\delta Y}{\delta x}+\frac{\delta v}{\delta Z}\cdot\frac{\delta Z}{\delta x}$

$=\frac{\delta v}{\delta X}\times 0+\frac{\delta v}{\delta Y}\,(-1)+\frac{\delta v}{\delta Z}\cdot 1=-\frac{\delta v}{\delta Y}+\frac{\delta v}{\delta Z}$

$\therefore \quad \frac{\delta v}{\delta x}=-\frac{\delta v}{\delta Y}+\frac{\delta u}{\delta Z} \ldots \ldots \ldots (1)$

Similarly $\frac{\delta v}{\delta y}=\frac{\delta v}{\delta X}\,\frac{\delta X}{\delta y}+\frac{\delta v}{\delta Y}\cdot\frac{\delta Y}{\delta y}+\frac{\delta v}{\delta Z}\cdot\frac{\delta Z}{\delta y}$

$\Rightarrow \frac{\delta v}{\delta y}=\frac{\delta v}{\delta X}-\frac{\delta v}{\delta Z} \ldots \ldots (2)$

and $\frac{\delta v}{\delta z}=-\frac{\delta v}{\delta X}+\frac{\delta v}{\delta Y} \ldots \ldots (3)$

Adding, (1), (2) and (3) we have

$\frac{\delta v}{\delta x}+\frac{\delta v}{\delta y}+\frac{\delta v}{\delta z}=0$

Ex. 16. Find

(ii) $\lim\limits_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$

(ii) Show that $\lim\limits_{y\to 0,x\to 0} f(x,y)\neq\lim\limits_{x\to 0,\ y\to 0} f(x,y)$

where $f(x,y)=\frac{x^2-y^2}{x^2+y^2}$ D. U. 1986

Ans. let $(x,y)\to(0,0)$ along a line $y=mx$ in the xy plane ; then along the line

$\lim\limits_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}=\lim\limits_{x\to 0} \frac{m^2x^3}{x^2+m^4x^4}=\lim\limits_{x\to 0} \frac{m^2x}{1+m^4x^2}=0.$

which is independent of m.

(ii) $$\lim_{y\to 0}\left\{\lim_{x\to 0}\frac{x^2-y^2}{x+y^2}\right\}=\lim_{y\to 0}-\frac{y^2}{y^2}=-1$$

$$\lim_{x\to 0}\left\{\lim_{y\to 0}\frac{x^2-y^2}{x^2+y^2}\right\}=\lim_{x\to 0}\frac{x^2}{x^2}=1$$

$$\therefore \lim_{y\to 0,\ x\to 0} f(x,y) \neq \lim_{x\to 0,\ y\to 0} f(x,y)$$

Ex. 17. Examine whether $f(x,y)=\begin{cases} xy & \text{if } |x|\geqslant |y| \\ -xy & \text{if } |x|<|y| \end{cases}$ is continuous at the origin.

Is $\dfrac{\delta^2 f}{\delta x\delta y}=\dfrac{\delta^2 f}{\delta y\delta x}$ for this function continuous at the origin.

Give reason for your answer.

Sol. $|xy|=|x|\,|y|\leqslant |x^2|$ for $|x|\geq |y|$

$\to 0$ as $(x,y)\to(0,0)$

Also $|-xy|=|x|\,|y|<|y|^2$ for $|x|<|y|$

$\to 0$ as $(x,y)\to(0,0)$

Also $f(0,0)=0$

Hence $f(x,y)$ is continuous at the origin.

Again $$f_x(0,y)=\lim_{h\to 0}\frac{f(h,y)-f(0,y)}{h}=\lim_{h\to 0}\frac{-hy}{h}=-y$$

when $|h|<|y|$.

$$\therefore f_{xy}(0,0)=\lim_{k\to 0}\frac{f_x(0,k)-f_x(0,0)}{k}=\lim_{k\to 0}\frac{-k}{k}=-1.$$

Also $$f_y(x,0)=\lim_{k\to 0}\frac{f(x,k)-f(x,0)}{k}=\lim_{k\to 0}\frac{kx}{k}=x$$

$\therefore\ |x|>|k|$; $f(x,0)=0$

$$\therefore f_{yx}(0,0)=\lim_{h\to 0}\frac{f_y(h,0)-f_y(0,0)}{h}=\lim_{h\to 0}\frac{h}{h}=1$$

Hence $f_{xy}(0,0)\neq f_{yx}(0,0)$

It is not continuous at the origin.

Ex. 18. If $u=\begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1\end{vmatrix}$

prove that $\dfrac{\delta u}{\delta x}+\dfrac{\delta u}{\delta y}+\dfrac{\delta u}{\delta z}=0$

Ans. $u=\begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1\end{vmatrix}=x^2(y-z)-y^2(x-z)+z^2(x-y)$

$$\therefore \frac{\delta u}{\delta x}=2x(y-z)-y^2+z^2$$

$$\frac{\delta u}{\delta y}=x^2-2y(x-z)-z^2$$

$$\frac{\delta u}{\delta z}=-x^2+y^2+2z(x-y)$$

Adding these, $\dfrac{\delta u}{\delta x}+\dfrac{\delta u}{\delta y}+\dfrac{\delta u}{\delta z}=0$

Ex. 19. **Directional Derivatives.**

Let $F(x,y,z)$ be defined at a point $P(x,y,z)$ on a space curve C. For a point near to P is Q where the value of the function at Q is given by $F(x+\Delta x,\ y+\Delta y,\ z+\Delta z)$. Let $PQ=\text{arc}\,\Delta S$:

Then $$\lim_{\Delta S\to 0}\frac{\Delta F}{\Delta S}=\lim_{\Delta S\to 0}\frac{F(x+\Delta x,\ y+\Delta y,\ z+\Delta z)-F(x,y,z)}{\Delta S}$$

is called the **directional derivative** of F at the point $P(x, y, z)$ along he curve C and is denoted by

$$\frac{dF}{dS}=\frac{\delta F}{\delta x}\frac{dx}{dS}+\frac{\delta F}{\delta y}\frac{dy}{dS}+\frac{\delta F}{\delta F}\frac{dz}{dS}$$

$$=\left(\frac{\delta F}{\delta x}i+\frac{\delta F}{\delta y}j+\frac{\delta F}{\delta z}k\right)\left(\frac{dx}{dS}i+\frac{dy}{dS}j+\frac{dz}{dS}k\right)$$

$$=\nabla F.\frac{dR}{dS}=\nabla F.T$$ where T is unit tangent. Thus the direction derivative is the component of ∇F along the unit tangent to P on C.

The maximum value of ∇F is given by $|\nabla F|$.

Ex. 19. (a) Find the directional derivative of $f(x, y)=\tan^{-1}y/x$ at (1,−1) towards (3, 0) D. H. 1986

[$f(x, x)=\tan^{-1}y/x$ ফাংশনটির (1,−1) বিন্দুতে (3, 0) বিন্দুর প্রতি ডাইরেকশনাল ডেরিভেটিভ নির্ণয় কর।]

Let $f(x, y)=\tan^{-1}y/x$ at $P(1,-1)$

$$\nabla f=\frac{\delta f}{\delta x}i+\frac{\delta f}{\delta y}j=\frac{x^2}{x^2+y^2}\left(-\frac{y}{x^2}\right)i+\frac{x^2}{x^2+y^2}\left(\frac{1}{x}\right)j$$

$$=\tfrac{1}{2}(1)\,i+\tfrac{1}{2}\,j \text{ at } P(1,-1)$$

The vector along $P(1,-1)$ and $Q(3, 0)$

$$PQ=r=(3-1)\,i+(0+1)\,j=2i+j$$

$\frac{dr}{dt}$ unit Tangent in the direction, $T=\frac{2i+j}{\sqrt{(4+1)}}=\frac{2i+j}{\sqrt{5}}$

Directional Derivative of $f=\nabla f.T=(\tfrac{1}{2}i+\tfrac{1}{2}j).\frac{(2i\pm j)}{\sqrt{5}}$

Since ∇f is positive, f is increasing $=(1+\tfrac{1}{2})/\sqrt{5}=\frac{3}{2\sqrt{5}}$

Note : [For details, please see Art 4.6 to 4. 9. vector Analyses of A Text Book on Co-ordinate Geometry and Vector Analysis. By Rahman and Bhattacharjee.]

Ex. 19 (b) Find the directional derivative of $F=2xy-z^2$ at (2,−1, 1) in a direction towards (3, 1,−1). Find the maximum directional derivative, also its magnitude [$F(x, y)=2xy-z^2$]

[ফাংশনটির (2,−1, 1) বিন্দুতে (3, 1,−1) বিন্দুর প্রতি ডাইরেকশনাল ডেরিভেটিভ নির্ণয় কর ইহার সর্বোচ্চ দিক কোনদিকে এবং মান কি ?]

Ans, $10/3.\ -2i+4j-2k,\ 2\sqrt{6}$

Ex. 20 Find the Jacobian of the transformation

u = r sins cost, v = r sin s sint, w = r cos s

Sol. The Jacobian is

$$|J|=\begin{vmatrix}\frac{\delta u}{\delta r} & \frac{\delta u}{\delta s} & \frac{\delta u}{\delta t}\\ \frac{\delta u}{\delta r} & \frac{\delta u}{\delta s} & \frac{\delta u}{\delta t}\\ \frac{\delta w}{\delta r} & \frac{\delta w}{\delta s} & \frac{\delta w}{\delta t}\end{vmatrix}$$

$$=\begin{vmatrix}\sin s\cos t & r\cos t\cos t & -r\sin s\sin t\\ \sin s\sin t & r\cos s\sin t & r\sin s\cos t\\ \cos s & 0 & 0\end{vmatrix}$$

$$=\begin{vmatrix}\cos s & 0 & 0\\ \sin s\cos t & r\cos s\cos t & -r\sin s\sin t\\ \sin s\sin t & r\cos s\sin t & r\sin s\cos t\end{vmatrix}$$

$= \cos s\,(r^2\cos s\sin s\cos^2 t + r^2\cos s\sin s\sin^2 t)$

$= r^2\cos^2 s\sin s\,(\cos^2 t + \sin^2 t)$

$= r^2\cos^2 s\sin s$

EXERCISE—IX

See APPENDIX: Ex. 97; Ex. 98., Ex. 99, Ex. 100.

1. Find $\delta f/\delta x$ when (i) $f=xy$ (ii) $f=x^y$ (iii) $f=\log(x^2+y^2)$ (iv) $f=\sin^{-1}(y/x)$ (v) $f=\tan^{-1}(x+y)$

2. Verify that $\dfrac{\delta^2 u}{\delta x\delta y}=\dfrac{\delta^2 u}{\delta y\delta x}$ where

(i) $u=\log(y\sin x+x\sin y)$ (ii $u=\dfrac{xy}{x^2+y^2}$

(ii) $u=ax^2+2hxy+by^2$ (iv) $u=\log\tan(y/x)$

2 (a) Prove that $f(x, y)=\dfrac{xy(x^2-y^2)}{x^2+y^2}$, $(x, y)\neq(0, 0)$

$f(0, 0)=0$ then $f_{xy}=f_{yx}$

(b) $\underset{x\to0,\ y\to0}{\text{Lt}}\dfrac{x^2-y2}{x^3+y^3}\neq\underset{y\to0,\ x\to0}{\text{Lt}}\dfrac{x^2-y^2}{x^3+y^3}$

3. If $u=\log\dfrac{(1+x^2)(1+y^2)}{xy}$, find $\dfrac{\delta^2 u}{\delta x\delta y}$

4 If $f(x, y)=xy\ (x^2-y^2)/(x^2+y^2)$; when $(x, y)\neq(0, 0)$ and $f(0, 0)=0$, show that $f_{xy}(0, 0)\neq f_{yx}(0, 0)$.

5. If $(x, y)=e^{xy}\cos x\sin x$, find

$f_x, f_y, f_{xx}, f_{xy}, f_{yx}$ R. U 1965; C. U. 1969

6. If $u=\log(x^2+y^2)$, show that C. U. 1979, 80

$\dfrac{\delta^2 u}{\delta x^2}+\dfrac{\delta^2 u}{\delta y^2}=0$ D. U. 1986.

7. If $u=(x, y)=e^y\cos x$, show that

$\dfrac{\delta^2 u}{\delta x^2}+\dfrac{\delta^2 u}{\delta y^2}=0$

8. If u and v are functions of x and y defined by

$x=u+e^{-v}\sin u$ and $y=v+e^{-v}\cos u$, prove that

$\dfrac{\delta u}{\delta y}=\dfrac{\delta v}{\delta x}$ D. U. H. 1962

9. If $u=\sqrt{(x^2+y^2+z^2)}$, prove that D. U. 1984,

$u_{xx}+u_{yy}+u_{zz}=2/u$ D. U. H. 1960

10. If $u=\log(x^2+y^2+z^2)$, Prove that

$x\dfrac{\delta^2 u}{\delta y\delta z}=y\dfrac{\delta^2 u}{\delta z\delta u}=z\dfrac{\delta^2 u}{\delta x\delta y}$

11. If $u=2(ax+by)^2-(x^2+y^2)$ and $a^2+b^2=1$.

Prove that $\dfrac{\delta^2 u}{\delta x^2}+\dfrac{\delta^2 u}{\delta y^2}=0$

12. If $u=\sin^{-1}\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$ show that R. U. 1982

$\dfrac{\delta u}{\delta x}=-\dfrac{y}{x}\dfrac{\delta u}{\delta y}$ or, $x\dfrac{\delta u}{\delta x}+y\dfrac{\delta u}{\delta y}=0$

13. If $u=f(r)$ and $r^2=x^2+y^2$ prove that

$\dfrac{\delta^2 u}{\delta x^2}+\dfrac{\delta^2 u}{\delta y^2}=\dfrac{d^2u}{dr^2}+\dfrac{1}{r}\dfrac{du}{dr}$

14. If $u=e^{xyz}$ show that

$\dfrac{\delta^3 u}{\delta x\delta y\delta z}=(1+3xyz+x^2y^2z^2)\,e^{xyz}$ R. U. 1988

15- Find $\dfrac{\delta z}{\delta u}$ and $\dfrac{\delta z}{\delta v}$ if $z=x^2+y^2+xy$ R. U. 1967

and $x=u\cos v$, $y=u\sin v$,

15. a) Verify Euler's Theorem for the following curves

(i) $u=\dfrac{1}{x^2+xy+y^2}$ (ii) $u=x^n \tan(y/x)$ (iii) $u=x^2 \log y/x$

(iv) $u=e^{x/y}+e^{y/x}$ (v) if $u=\sin(\sqrt{x}+\sqrt{y})$, prove that

$$x\frac{\delta u}{\delta x}+y\frac{\delta u}{\delta y}=\tfrac{1}{2}(\sqrt{x}+\sqrt{y})\cos(\sqrt{x}+\sqrt{y}).$$

16. If $z=xyf(y/x)$. Show that $x\dfrac{\delta u}{\delta x}+y\dfrac{\delta v}{\delta y}=nu\dfrac{\delta v}{\delta u}=2z$

16 (a) If $z=\tan^{-1}\dfrac{x^2+y^2}{x-y}$ the $x\dfrac{\delta u}{\delta x}+y\dfrac{\delta v}{\delta y}=\dfrac{1}{2}=\sin 2z$ D.U. 1989

17. If $u=\sin^{-1}\dfrac{x^2+y^2}{x+y}$, show that N. U. 1994

$$x\frac{\delta u}{\delta x}+y\frac{\delta u}{\delta y}=\tan u,$$ R. U. 1978, '83 1986

18. If $u=\sin^{-1}\dfrac{x-y}{\sqrt{x}+\sqrt{y}}$ Show that

$$x\frac{\delta u}{\delta x}+y\frac{\delta u}{\delta y}=\tfrac{1}{2}\tan u$$ C. H. 1977; D. U. 1988

19. If $v=f(u)$, being homogeneous function of degree n in x, y, then $x\dfrac{\delta v}{\delta x}+y\dfrac{\delta v}{\delta y}=nu\dfrac{dv}{du}$

20. Prove Euler's Theorem when $u=\dfrac{x-y}{x+y}$

21. If $u=x^3+y^3+z^3$, then prove that

$$x\frac{\delta u}{\delta x}+y\frac{\delta u}{\delta y}+z\frac{\delta u}{\delta z}=3u$$

22. If $u=\log(x^3+y^3+z^3-3xyz)$ then show that C H. 1989

$$\left(\frac{\delta}{\delta x}+\frac{\delta}{\delta y}+\frac{\delta}{\delta z}\right)^2 u=-\frac{9}{(x+y+z)^2}$$ N H 1993

23. If $u=x^2-y^2-2yx+y+z$. Show that

$$(x+y)\frac{\delta u}{\delta x}+(x-y)\frac{\delta u}{\delta y}+(y-x)\frac{\delta u}{\delta z}=0$$

24. Prove that $f_{xx}+f_{yy}=f_{zz}+f_{tt}$

where $u=f(x, y)$ and $x=z\cos\alpha-t\sin\alpha$, $y=z\sin\alpha+t\cos\alpha$

25. If $u=y^2+\tan(ye^{1/x})$, Show that

$$x^2\frac{\delta u}{\delta x}+y\frac{\delta u}{\delta y}=2y^2$$ D. U. 1956

26. If $z=\dfrac{x^2y^2}{x+y}$ then prove that

$$x\frac{\delta^2 z}{\delta x^2}+y\frac{\delta^2 z}{\delta x^2}=2\frac{\delta z}{\delta x}$$ D U. 1967

27. If $u=3(lx+my+nz)^2-(x^2+y^2+z^2)$ and $l^2+m^2+n^2=1$

Show that $\dfrac{\delta^2 u}{\delta x^2}+\dfrac{\delta^2 u}{\delta y^2}+\dfrac{\delta^2 u}{\delta z^2}=0$ R. U. 1987

28. If $F(x, y, z)=0$, show that

$$\frac{\delta x}{\delta y}\cdot\frac{\delta y}{\delta z}\cdot\frac{\delta z}{\delta x}=-1,$$

where each partial derivative is computed by holding the remaining variables constant. R. U. H. 1962

28. (a) If $u=\operatorname{cosec}^{-1}\sqrt{\dfrac{\{\sqrt{x}+\sqrt{y}\}}{\sqrt[3]{x}+\sqrt[3]{y}}}$ Show that

$$x^2\frac{\delta^2 u}{\delta x^2}+2xy\frac{\delta^2 u}{\delta x\delta y}+y^2\frac{\delta^2 u}{\delta y^2}=\frac{\tan u}{12}\left(\frac{10}{12}+\frac{\tan^2 u}{12}\right)$$

29. Prove that $xy(f_{xx}-f_{yy})-(x^2-y^2)f_{xy}=-r\dfrac{\delta^2 f}{\delta r\delta\theta}+\dfrac{\delta f}{\delta\theta}$ if $x=r\cos\theta$, $y=r\sin\theta$.

30. If $\phi(u, v)=f(x, y)$, $u=y^2-x^2$, $v=x^2+y^2$

Show that $\dfrac{1}{4xy}\cdot\dfrac{\delta^2 f}{\delta x\delta y}=\dfrac{\delta^2\phi}{\delta v^2}-\dfrac{\delta^2\phi}{\delta u^2}$ D. U. 1965 (s)

31.(i) If v is a differentiable function, the

$$x\frac{\delta v}{\delta x}+y\frac{\delta v}{\delta y}=nu\frac{dv}{du}$$

u is a differentiable homogeneous function of degree n.

(ii) If $u(x, y) = \log_e \frac{x^3 + y^3}{x + y}$, then $x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta u} = 2$. N.U. 1994

32. If $u = \frac{x^2y^2}{x^2+y^2}$ Show that

$$x^2 \frac{\delta^2 u}{\delta x^2} + 2xy\frac{\delta^2 u}{\delta y \delta x} + y^2 \frac{\delta^2 u}{\delta y^2} = 2u$$

33. If $u=f(x^2+2yz, y^2+2zx)$, show that

$$(y^2-zx) \frac{\delta u}{\delta x}+(x^2-yz)\frac{\delta u}{\delta y} +(z^2-xy) \frac{\delta u}{\delta z} =0$$

34. If $u=x^n F(y/x, z/x)$, Show that

$$x \frac{\delta u}{\delta x} +y \frac{\delta u}{\delta y} +z \frac{\delta u}{\delta z} =nu.$$ [R. U. 1964, C. U. 1985]

35. If $x \cos u+y \sin u=1, y=x \sin u-y \cos u$

prove that $v^2\frac{\delta^2 u}{\delta x\delta y} + v\frac{\delta u}{\delta x}. \frac{\delta u}{\delta y}=\cos 2u$ [D. U. H. 1960]

36. If $F(v^2-x^2, v^2-y^2, y^2-z^2)=0$, where $v=f(x, y, z)$

Show that $\frac{1}{x} \frac{\delta v}{\delta x}+\frac{1}{y} \frac{\delta v}{\delta y}+\frac{1}{z} \frac{\delta v}{\delta z}=\frac{1}{v}$

37. (i) If $u=x\,\phi(y/x)+\psi\,(y/x)$, prove that

Show that $x\left(\frac{\delta z}{\delta y}\right)-y\left(\frac{\delta z}{\delta x}\right)=x-y$ R. H. 1992

C. H. 1993

$$x^2 \frac{\delta^2 u}{\delta x^2}+2xy\frac{\delta^2 u}{\delta x\delta y}+y^2 \frac{\delta^2 u}{\delta y^2} =0$$ N. H. 1993

(ii) If $u=x\,\phi(x+y)+y\psi\,(x+y)$, show that

$$\frac{\delta^2 u}{\delta x^2} -2 \frac{\delta^2 u}{\delta x\delta y}+\frac{\delta^2 u}{\delta y^2}=0$$ [D. U. 1962]

38. If $u=f(x, y)$ and $x=r \cos \theta$, $y=r \sin \theta$. Prove that

(i) $\left(\frac{\delta r}{\delta x}\right)^2+\left(\frac{\delta r}{\delta y}\right)^2 =1$ (ii) $x \frac{\delta u}{\delta x}+y \frac{\delta u}{\delta y}=r \frac{\delta u}{\delta r}$

(iii) $\frac{\delta^2\theta}{\delta x^2}+\frac{\delta^2\theta}{\delta y^2}=0.$

*39. $x=(e^u+e^{-v})$ and $y=e^v+e^{-u}$, $z=f(x, y)$;

Prove that $\frac{\delta^2 z}{\delta u^2}- 2\frac{\delta^2 z}{\delta u\delta v}+\frac{\delta^2 z}{\delta v^2}$

$$=x^2 \frac{\delta^2 z}{\delta x^2}-2xy\frac{\delta^2 z}{\delta x\delta y}+y^2 \frac{\delta^2 z}{\delta y^2}+x\frac{\delta z}{\delta x}+y \frac{\delta z}{\delta y}$$

40. Denoting $\frac{\delta^2}{\delta x^2}+\frac{\delta^2}{\delta y^2}$ by ∇^2, prove that $\nabla^2 t=1/r$ and $\nabla^2 \tan^{-1} y/x=0$ if $r^2=x^2+y^2$ [D. U. H. 1960]

41. If v be function of r alone, where $r^2=x^2+y^2+z^2$

Show that $\frac{\delta^2 v}{\delta x^2}+\frac{\delta^2 v}{\delta y^2}+\frac{\delta^2 v}{\delta r^2}=\frac{d^2 v}{dz^2}+\frac{2dv}{rdr}$

*42. Prove that

$$\nabla^2 u=\frac{\delta^2 u}{\delta x^2}+\frac{\delta^2 u}{\delta y^2}+\frac{\delta^2 u}{\delta z^2}$$

$$=\frac{\delta^2 u}{\delta r^2}+\frac{2}{r}\frac{\delta u}{\delta r}+\frac{1}{r^2}\frac{\delta^2 u}{\delta \theta^2}+\frac{\cot\theta}{r^2}.\frac{\delta u}{\delta\theta}+\frac{1}{r^2\sin^2\theta}.\frac{\delta^2 u}{\delta\phi^2}$$

where r, θ, ϕ denotes the spherical polar co-ordinates of a point.

*43. If $u=\frac{(x^2+y^2)^m}{2m(m-1)}+x\phi\left(\frac{y}{x}\right)+\psi\left(\frac{y}{x}\right)$

Prove that

$$x^2 \frac{\delta^2 u}{\delta x^2}+2xy \frac{\delta^2 u}{\delta x\delta y}+y^2 \frac{\delta^2 u}{\delta y^2}=(x^2+y^2)^m$$

44. If $Pdx+Qdy+Rdz$ can be made a perfect differential of some function of x, y, z by multiplying each term by some factors, then

$$P\left(\frac{\delta Q}{\delta z}-\frac{\delta R}{\delta y}\right)+Q\left(\frac{\delta R}{\delta x}-\frac{\delta P}{\delta z}\right)+R\left(\frac{\delta P}{\delta y}-\frac{\delta Q}{\delta z}\right)=0.$$

45 If $u_1t_1=t_2t_3$, $u_2t_2=t_1t_3$, $u_3t_3=t_1t_2$ prove that

$$J(u_1, u_2, u_3)=\frac{\delta(u_1, u_2, u_3)}{\delta(t_1, t_2, t_3)}=4$$

46. Show that the function

$$u=x+3y+2z, \quad v=3x+4y-2z, \quad w=11x+18y-2z$$

are not independent and find a relation between them.

Ans. $J=0$; $w=2u+3v$ C. H. 1988

47. Show that $ax^2+by^2+cz^2+2fyz+2gzx+2hxy$ will resolve into linear factors if

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0$$

Hints $u=v\,w$, $v=l_1x+m_1y+n_1z$, $w=l_2x+m_2y+n_2z$,

In $\frac{\delta(u, v, w)}{\delta(x, y, z)}=0$, equate the co-efficients of x, y, z separately to zero

48. If $u=x+y+z$, $v=x-2y+3z$, $w=2xy-xz+4yz-2z^2$

Show that $J=\frac{\delta(u, v, w)}{\delta(x, y, z)}=0$ and find a relation between u, v and ω Ans. $u^2-v^2=4\omega$

48 (i) If $u=x/\sqrt{(1-r^2)}$, $v=y/\sqrt{(1-r^2)}$, $w=z/\sqrt{(1-r^2)}$ where $r^2=x^2+y^2+z^2$, prove that $J\left(\frac{u, v, w}{x, y, z}\right)=\frac{1}{\sqrt{(1-r^2)^5}}$

R. H. 1987

*49. Show that the function $u=3x+2y-z$, $v=x-2y+z$ and $\omega=x(x+2y-z)$ are not independent, and find the relation between them. Ans. $u^2-v^2=8\omega$

*50. If $u=x+y+z$, $v=xy+yz+zx$, $w=x^3+y^3+z^3-3xyz$ show that u, v, w are connected by a functional relation $w=u^2-3uv$

*51. If $\omega=f(x, y)$ and $x=u\cosh v$, $y=u\sinh v$, then show that

$$\left(\frac{\delta u}{\delta x}\right)^2-\left(\frac{\delta w}{\delta y}\right)^2=\left(\frac{\delta w}{\delta u}\right)^2-\frac{1}{u^2}\left(\frac{\delta w}{\delta v}\right)^2$$ [D. U. H. 1970]

*52. If $x=f(u, v)$, $y=g(u, v)$, $z=h(u, v)$ and $F(u, v, w)=0$.

prove that $\frac{\delta(y, z)}{\delta(u, v)}dx+\frac{\delta(z, u)}{\delta(u, v)}dy+\frac{\delta(x, y)}{\delta(u, v)}dz=0$

*53. If $x=f(u, v, \omega)$, $y=g(u, v, \omega)$, $z=h(u, v, \omega)$ prove that

$$\frac{\delta(x, y, z)}{\delta(u, v, w)}\cdot\frac{\delta(u, v, w)}{\delta(x, y, z)}=1 \text{ provided } \frac{\delta(x, y, z)}{\delta(u, v, w)}\neq 0$$

54. If $u=f(v)$ where u, v are functions of $f(x, y, z)$ prove tha

$$\frac{\delta(u, v)}{\delta(y, z)}\cdot\frac{\delta z}{\delta x}+\frac{\delta(u, v)}{\delta(z, x)}\cdot\frac{\delta z}{\delta y}=\frac{\delta(u, v)}{\delta(x, y)}$$

54. (i) If $U=f(x, y)$ and $x=e^u\cos v$, $y=e^u\sin v$. then prove that

$$\frac{\delta^2 U}{\delta x^2}+\frac{\delta^2 U}{\delta y^2}=e^{-2u}\left(\frac{\delta^2 U}{\delta u^2}+\frac{\delta^2 U}{\delta v^2}\right)$$

55. By the transformation $\xi = a + \alpha x + \beta y$, $\eta = b - \beta x + \alpha y$ in which a, b, α, β are constants and $\alpha^2 + \beta^2 = 1$. The function $u(x, y)$ is transformed into a function $U(\xi, \eta)$ of ξ and η. Prove that

$$U_{\xi\xi}\, U_{\eta\eta} - U^2_{\xi\eta} = u_{xx}\, u_{yy} - u^2_{xy}$$

56. If $v = 7x^2 + 8xy + 9y^2$, then show that

$$v^2_x\, v_{yy} - 2v_x\, v_y\, v_{xy} + v^2_y\, v_{xx} = 376\, v.$$ [D. U. 1982]

57. If $u = \phi(y + ax) + \psi(y - ax)$

Prove that $\dfrac{\delta^2 u}{\delta x^2} = a^2 \dfrac{\delta^2 u}{\delta y^2}$.

58. Given $xyz = a$, find all the differential coefficients of first and second order taking x and y are independent variables

59. Find the value of the expression

$\dfrac{\delta^2 z}{\delta x^2} + \dfrac{\delta^2 z}{\delta y^2}$, where $a^2x^2 + b^2y^2 - c^2z^2 = 0$

60. If $\left(\dfrac{x}{a}\right)^n + \left(\dfrac{y}{b}\right)^n + \left(\dfrac{z}{c}\right)^n = 1$, find $\dfrac{\delta z}{\delta x}$ and $\dfrac{\delta^2 x}{\delta y \delta z}$

Also, find $\dfrac{dy}{dx}$ when the variables are connected by the two variables

(i) $\left(\dfrac{z}{c}\right)^n = \left(\dfrac{x}{a}\right)^n - \left(\dfrac{y}{b}\right)^n$ (ii) $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$

61. If $v = (1 - 2xy + y^2)^{-1/2}$ prove that

$$x \frac{\delta v}{\delta x} - y \frac{\delta v}{\delta y} = y^2 v^3$$

Also show that

$$\frac{\delta}{\delta x}\left\{ (1 - x^2) \frac{\delta v}{dx} \right\} + \frac{\delta}{\delta y} (y^2) \frac{\delta v}{\delta y} \Big\} = 0$$

62. If $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$, $lx + my + nz = 0$

prove that

$$\frac{dx}{ny/b^2 - mz/c^2} = \frac{dy}{lz/c^2 - nx/a^2} = \frac{dz}{mx/a^2 - ly/b^2}$$

Hints :—Diff. the conditions w. r. to x and apply cross multiplication.

63. If $z = (x\, \delta/\delta x - 1)\, \{f(y + x) - \phi(y - x)\}$

prove that

$$x\left(\frac{\delta^2 z}{\delta x^2} - \frac{\delta^2 z}{\delta y^2}\right) = 2\, \frac{\delta z}{\delta x}$$

Hints $z = x(f' + \phi') - (f + \phi)$

$\delta z/\delta x = x(f'' + \phi'')$, $\delta z/\delta y = x(f'' + \phi'') - (f' + \phi')$

$\delta^2 z/\delta x^2 = x(f''' + \phi''') + (f'' + \phi'')$, $\delta^2 z/\delta y^2 = x(f''' + \phi''') - (f'' + \phi'')$

Multiply, the two equations by x and subtract the result will follow.

64. If $f(x, y, z, t) = \dfrac{f(t + r)}{r} + \dfrac{g(t - r)}{r}$, where

$r^2 = x^2 + y^2 + z^2$, prove that u satisfies the relation

$$u_{xx} + u_{yy} + u_{zz} = u_{tt}.$$

Ans. $\delta u/\delta x = f'(t + r)\, x/r^2 - f(t + r)x/r^3 + g'(t - r)(-x/r^2)$
$- g(t - r)x/r^3$

$u_{xx} = (x^2/r^3)f'' + (l/r^2 - 2x/r^4 - x^2/r^4)f' + (3x^2/r^5 - l/r^3)f$
$+ (x^2/r^3\, g'' + (2x^2/r^4 + x^2/r^4 - t/r^2)g' + (3x^2/r^5 - 1/r^3)g$

$\therefore\ u_{xx} + u_{yy} + u_{zz} = (1/r)\, (f'' + g'') = u_{tt}$

65. If $x^2 + y^2 + z^2 - 2xyz = 1$, Show that

$$\frac{dx}{\sqrt{(1 - x^2)}} + \frac{dy}{\sqrt{(1 - y^2)}} + \frac{dz}{\sqrt{(1 - z^2)}} = 0$$

66. If $u = F(x^2 + y^2 + z^2) f(xy + yz + zx)$, prove that
$(y-z) u_x + (z-x) u_y + (x-y) u_z = 0$

66 (a) If $u=x^2-y^2-2xy+y+z$, then $(x+y)u_x+(x-y)u_y+(y-x)u_z=0$

66 (b) Show that $xu_x+yu_y+zu_z=0$ if $u=y/z + z/x + x/y$ N.U. 1995

67. If $\dfrac{x^2}{a^2+u}+\dfrac{y^2}{b^2+u}+\dfrac{z^2}{c^2+u}=1$, u is the function of x, y, z, then prove that $(u_x)^2+(u_y)^2+(u_z)^2=2(xu_x+yu_y+zu_z)$

68. Find the directional derivative of $f(x, y)=\tan^{-1}y/x$ at $(1,-1)$ towards $(3, 0)$ Ans. $\dfrac{3}{2\sqrt{5}}$ D. H. 1986

69. Find the total differential co-efficients of $u=(x+y+z)e^{x}$ D. H. 1986

70. For $f(x, y)=\log(1+xy)$, find the Taylor's series expansion about any point (x, y) upto orders 2, where $1+xy>0$.

71. State and prove Euler's Theorem on homogeneous functions of three variables. R. H, 1917

72. If $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$, $\dfrac{x^2}{a^2+\lambda}+\dfrac{y^2}{b^2+\lambda}+\dfrac{z^2}{c^2+\lambda}=1$, there prove that

$$\frac{x(b^2-c^2)}{dx}+\frac{y(c^2-a^2)}{dy}+\frac{z(a^2-b^2)}{dz}=0.$$ R. H. 1987

73. Find the directional derivatives of $F=2x^3y-3y^2z$ at $P(1, 2,-1)$ in a direction towards $Q(3,-1, 5)$, find also the maximum directional derivative from P and its magnitude.

Ans. $90/7$, $12i+14j-12k$ at P, 22.

74. Test the continuity of the function

$$f(x, y)=\frac{2x^2y}{x^3+y^3}, \quad x+y\neq 0$$
$$=0, \quad x+y=0$$ D. H. 1986

75 Consider $f(x, y) = \dfrac{xy}{x^2+y^2}$, $(x,y)\neq(0, 0)$ and
$=0$ $f(0, 0)=0$

Prove that $f_x(0, 0)$ and $f_y(0, 0)$ both exist but $f(x, y)$ is discontinuous at $(0, 0)$.

76. When is a function $f(x, y, z)$ said to be continuous at a pt. (a, b, c)? Examine the continuity of the function

$$f(x, y)=\frac{x^3+y^3}{x^2+y^2}, \quad x^2+y^2\neq 0.$$

77. If the partial derivatives f_x, f_y of a function $f(x, y)$ exist and have the value zero at every point of R, prove that $f(x, y)$ is a constant.

78. Prove that the function $f(x, y)$ defined by

$$f(x, y)= -\frac{2xy}{x^2+y^2}, \quad x^2+y^2\neq 0$$ C.U. 1992
$$f(0, 0)=0$$

has partial derivatives everywhere. Is it continuous everywhere.

79 Examine whether $f(x, y)=\sqrt{|xy|}$ is totally differentiable at the origin. Ans. no

80. If $f(x, y)=\dfrac{x^2}{x^2+y^2-x}$, $x^2+y^2\neq 0$

$f(0, 0)=0$

81. Examine whether $f(x, y)$ is continuous at $(0, 0)$.

The partial derivatives f_x, f_y of a function $f(x, y)$ are continuous in R. prove that $f(x+h, y+k)-f(x, y)=hf_x(\xi,\eta)+kf_y f(\xi, \eta)$, where (x, y) and $(x+h, y+k)$ and the line segment joining them lie in R and (ξ, η) is an intermediate point on the line segment.

82. Find a function $f(x, y)$ which is a function of x^2+y^2 and is also a product of the form $\psi(x).\psi(y)$

Ans. $\psi(x)=a^{x^2}$, $\psi=a^{y^2}$.

83. If $f(x, y, z)$ be continuous together with derivatives of the first two orders, in the neighbourhood of the point (x_0, y_0, z_0) and if $f(x_0, y_0, z_0)=0$, and $f_\alpha(x_0, y_0, z_0)$ and

$\begin{vmatrix} f_x & f_y \\ f_{x\alpha} & f_{y\alpha} \end{vmatrix} \neq 0$, $f_{\alpha\alpha}\neq 0$ at this point, prove that the equations $f(x, y, \alpha)=0$, $f_\alpha(x, y, \alpha)=0$ define a curve which is tangent to each of the family.

84. If $f(x, y, z, t)$ has an extreme value at the point (p, q, r, s) subject to the subsidiary conditions $\phi(x, y, z, t)=0$, $\psi(x, y, z, t)=0$ and if at the point

$\dfrac{\delta(\phi, \psi)}{\delta(z, t)} \neq 0$ then two numbers λ, μ exist such that at the point (p, q, r, s) the equations.

$$f_x+\lambda\phi_x+\mu\psi_x=0, \quad f_y+\lambda\phi_y+\mu\psi_y=0$$

$$f_z+\lambda\phi_z+\mu\psi_z=0, \quad f_t+\lambda\phi_t+\mu\psi_t=0$$

and also the subsidiary conditions are satisfied.

85. যদি $x^3+y^3+z^3-3xyz=e^u$ হয়, তাহা হইলে দেখাও যে $x=1$, $y=1$, $z=1$ বিন্দুতে $u_{xx}+u_{yy}+u_{zz}=-\dfrac{1}{3}$ C. U. 1991

86. যদি $u=f(r)$ এবং $r^2=x^2+y^2$, প্রমাণ কর যে

$$\frac{\delta^2 u}{\delta x^2}+\frac{\delta^2 u}{\delta y^2}=f''(r)+\frac{1}{r}f'(r)$$ D. U. 1990

87. If z and lc are the functions of x and y defined by

$\{z-\phi(u)\}^2=x^2(y^2-u^2)$, $\{z-\phi(u)\}\phi'(u)=ux^2$

prove that $\dfrac{\delta z}{\delta x}, \dfrac{\delta z}{\delta y}=xy$ R. U. 1991, C. H. 1986

87 (a) If $u=f(ax^2+2hxy+by^2)$ R.H.1992

$v=\phi(ax^2+2hxy+by^2)$

Then show that $\dfrac{\delta}{\delta y}\left(u\dfrac{\delta v}{\delta x}\right)=\dfrac{\delta}{\delta x}\left(u\dfrac{\delta v}{\delta y}\right)$

88. Discuss the existence of $\underset{(x,y)\to(0,0)}{Lt} f(x,y)$

for $f(x,y)=\dfrac{x^3-y^3}{x^2+y^2}$, $(x, y)\neq(0,0)$ C.U. 1993

$=0$, $(x,y)=(0,0)$ nuous

89. If $u=\dfrac{x^2y^2}{x+y}$, then show that

(i) $x\dfrac{\delta u}{\delta x}+y\dfrac{\delta u}{\delta y}=3u$ C.U.1993

(ii) $x^2\dfrac{\delta^2 u}{\delta x^2}+2xy\dfrac{\delta^2 u}{\delta x\delta y}+y^2\dfrac{\delta^2 u}{\delta y^2}=2u$

90. If $Z=x\sin(x/y)+ye^{y/x}$, prove that $x\dfrac{\delta Z}{\delta x}+y\dfrac{\delta Z}{\delta y}=Z$ C.U. 1993

91. Show that the function f, where

$f(x,y)=\left\{\dfrac{xy}{\sqrt{(x^2+y^2)}}\right\}$; if $x^2+y^2\neq 0$,

$=0$; if $x=y=0$

is continuous, possesses partial derivatives but is not differentiable at the origin.

92. Prove that the function

$f(x,y)=\sqrt{|xy|}$

is not differentiable at the point $(0,0)$, but that f_x and f_y both exist at the origin and have the value 0, Hence deduce that these two partial derivatves are continous except at the origin

93. Show that for function

$f(x,y)=\dfrac{xy(x^2-y^2)}{x^2+y^2}$, $(x,y)\neq(0,0)$

94. Given that

$f(x, y) = \frac{xy^2}{x^2+y^4}$, $(x, y) \neq (0, 0)$

$= 0, x = y = 0.$

Then $f(x, y)$ is continuous at the origin. Give reason.

R.U. 1985; D.U. 1980

95. Given that

$f(n, y) = (x+y) \sin \frac{1}{x} \sin \frac{1}{y}$, $x \neq 0, y \neq 0$

$= 0, \quad x = y = 0$ C.H. 1991

Test Whether $\underset{x \to 0 \,.\, y \to 0}{\text{Lt}} f(x, y)$ exists or not Ans discontinuous.

96. For $f(x,y) = (x^2+y^2) \log(x^2+y^2)$, $x^2+y^2 \neq 0$

$= 0, \quad x = y = 0$

Prove that $f_{xy}(0, 0) = f_{yx}(0, 0)$

Show that at (0, 0) neither of the derivatives is continuous C.H. 1992

See APPENDIX
Ex. 97, 98, 99, 100

ANSWERS

1. (i) $f_x = y, f_y - x$ (ii) $f_x = yx^{y-1}, f_y = x^y \log x$

(iii) $f_x = \frac{2x}{x^2+y^2}, f_y = \frac{2y}{x^2+y^2}$,

(iv) $f_x = \frac{-y}{x\sqrt{x^2-y^2}}, f_y = \frac{1}{\sqrt{x^2-y^2}}$

(v) $f_x = \frac{1}{1+(x+y)^2}, f_y = \frac{1}{1+(x+y)^2}$

3. 0, 5, $f_x = e^{xy}\left(\frac{1}{2} y \sin 2x + \cos 2x\right)$,

$f_y = e^{xy}\left(\frac{1}{2} x \sin 2x\right)$, $f_{xx} = e^{xy}\left(\frac{1}{2} y^2 \sin 2x + 2y \cos x\right.$
$\left. - 2 \sin 2y\right)$, $f_{xy} = e^{xy}(xy+1) \sin 2x + e^{yx} x \cos 2x$,

$f_{yx} = e^{xy}\left(\frac{1}{2} xy + \frac{1}{2}\right) \sin 2x + e^{xy} x \cos 2x$,

$f_{yy} = \frac{1}{2} e^{xy} x^2 \sin 2x.$

15. $2u + u \sin 2v$, $u^2 \cos 2v$.

CHAPTER X (A)

TANGENTS AND NORMALS IN CARTESIAN CO-ORDINATES

Art 10. 1. Definition :—Let P be a given point on a curve and Q be any other point on it and let the point Q move along the curve nearer and nearer to the point P then the limiting position of the secant PQ, provided limit exists, when Q moves up to and ultimately coincide with P, is called the tangent to the curve at the point P.

The line through the point P perpendicular to the tangent is called the normal to the curve at the point P.

Art. 10. 2. Equation of Tangent.

To find the equation of the tangent at (x, y) of the curve $y=f(x)$.

(i) *Explicit Cartesian Equation.*

Let (x, y) be the co-ordinates of P on the curve $y=f(x)$ Let us take another point Q in the neighbourhood of P and let the co-ordinates of Q be $(x+\Delta x, y+\Delta y)$. Let (X, Y) be the current co-ordinates of a point.

The equation of the chord PQ is

$$Y-y=\frac{y+\Delta y-y}{x+\Delta x-x}(X-x)$$

or. $$Y-y=\frac{\Delta y}{\Delta x}(X-x) \quad \cdots \quad \cdots \quad (1)$$

Now, as $Q \to P$, Δx and Δy both tend to zero.

$\therefore$ from (i) the equation of the tangent at $P(x, y)$ is

$$Y-y=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}(X-x)=\frac{dy}{dx}(X-x)$$

if the limit exists.

Thus the tangent at P (x, y) on the curve $y=f(x)$ is

$$Y-y=\frac{dy}{dx}(X-x) \quad \ldots \quad \ldots \quad (1)$$

The equation of any line through (x, y) is

$$Y-y=m(X-x) \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

The line (2) is perpendicular to the tangent at (x, y) if

$$m\frac{dy}{dx}=-1 \quad \text{or,} \quad m=-1\Big/\frac{dy}{dx}$$

Therefore, the straight line (2) becomes the normal at (x, y) if

$$m=-\left(\frac{dx}{dy}\right)$$

Thus the equation of the normal at (x, y) is

$$Y-y=-\frac{dx}{dy}(X-x)$$

or, $(Y-y)\dfrac{dy}{dx}+(X-x)=0 \quad \ldots \quad \ldots \quad (3)$

(ii) Implicit cartesian equation.

Find the equation of the tangent at (x, y) to the curve $f(x, y)=0$.

In this curve we have

$$\frac{dy}{dx}=-\frac{\delta f/\delta x}{\delta f/\delta y}\ ;\ \frac{\delta f}{\delta y}\neq 0$$

or, $\dfrac{dy}{dx}=-\dfrac{f_x}{f_y},\ f_y\neq 0$

Thus the equation of the tangent at (x, y) is.

$$(Y-y)=\frac{dy}{dx}(X-x) \quad \text{or,} \quad (Y-y)=\frac{-f_x}{f_y}(X-x)$$

or, $(X-x)f_x+(Y-y)f_y=0 \quad (4)$

The equation of the normal at (x, y) on the curve $f(x, y)=0$ is

$$(Y-y)\frac{dy}{dx}+(X-x)=0 \quad \text{or,} \quad (Y-y)\frac{-f_x}{f_y}+(X-x)=0$$

or, $\dfrac{X-x}{f_x}=\dfrac{Y-y}{f_y} \quad (5)$

(iii) A symmetrical form of the equation of the tangent to a rational algebraic curve. Let $f(x, y)=0$ be a function of x and y of degree n. We can make the equation homogeneous by introducing another variable z. Then the equation $f(x, y)=0$ becomes a homogeneous equation in x, y, z and the new equation of the curve can be given as $f(x, y, z)=0$. Since $f(x, y, z)=0$ is a homogeneous equation in (x, y, z) of degree n, therefore, by Euler's Theorem, we have

$$x\frac{\delta f}{\delta x}+y\frac{\delta f}{\delta y}+z\frac{\delta f}{\delta z}=nf(x, y, z)=n.\ 0=0$$

or, $x\dfrac{\delta f}{\delta x}+y\dfrac{\delta f}{\delta y}+\dfrac{\delta f}{\delta z}z=0. \quad (6)$

Now the equation of the tangent (4)

$$(X-x)\frac{\delta f}{\delta x}+(Y-y)\frac{\delta f}{\delta y}=0 \text{ becomes}$$

$$X\frac{\delta f}{\delta x}+Y\frac{\delta f}{\delta y}=x\frac{\delta f}{\delta x}+y\frac{\delta f}{\delta y}=-z\frac{\delta f}{\delta z} \quad [by\ (6)]$$

or, $X\dfrac{\delta f}{\delta x}+Y\dfrac{\delta f}{\delta y}+z\dfrac{\delta f}{\delta z}=0 \quad \ldots \quad \ldots \quad \ldots \quad (7)$

$$\text{or,}\quad X\frac{\delta f}{\delta x}+Y\frac{\delta f}{\delta y}+Z\frac{\delta f}{\delta z}=0 \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

$$\text{or,}\quad Xf_x+Yf_y+Zf_z=0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

where the co-efficient of $\frac{\delta f}{\delta z}$ i. e. z is replaced by Z for the sake of symmetry.

After differentiation we are to put $Z=z=1$ in the equation.

This method is shown below with an example.

Ex Find the tangent to the curve.

$f(x, y)=ax^2+2hxy+by^2 \cdot 2gx+2fy+c=0$ at the point (x, y).

This equation can be written in a homogeneous form in x, y, z where $z=1$.

Thus $f(x, y, z)=ax^2+2hxy+by^2+2gxz+2fyz+cz^2$,

Now $f_x=\delta f/\delta x=2ax+2hy+2gz$

$f_y=\delta f/\delta y=2hx+2by+2fz$

$f_z=\delta f/\delta z=2gz+2fy+2cz$

The equation of the tangent is

$$Xf_x+Yf_y+Zf_z=0$$

or, $X(2ax+2hy+2gz)+Y(2hx+2by+2fz)+Z(2gx+2fy+2cz)=0$

or, Putting $Z=z=1$,

$$X(ax+hy+g)+Y(hx+by+f)+(gx+fy+c)=0.$$

Thus the tangent at (x, y) to the curve $f(x, y)=0$

is $X(ax+hy+g)+Y(hx+by+f)+gx+fy+c=0$

Note : The use of formula (8) or (9) in finding the equation of a tangent to a rational algebraic curve is advised rather than the use of formula (4) or (5)

(iv) Parametric Equation

Find the equation of the tangent at 't' the curve represented by $x=\phi(t)$ and $y=\psi(t)$

In this case we have

$$\frac{dy}{dx}=\frac{dy}{dt}\quad\frac{dt}{dx}=\frac{dy}{dt}\Big/\frac{dx}{dt}=\frac{\psi'(t)}{\phi'(t)}\ ;\ \phi'(t)\neq 0$$

The equation of the tangent at (x, y) is

$$(Y-y)=\frac{dy}{dx}(X-x)\ \text{ or, }\ \{Y-\psi(t)\}=\frac{\psi'(t)}{\phi'(t)}\{X-\phi(t)\}$$

$$\text{or,}\quad \{X-\phi(t)\}\psi'(t)=\{X-\psi(t)\}\ \phi'(t) \qquad (10)$$

The equation of the normal to the curve $x=\phi(t)$ and $y=\psi(t)$ at (x, y) is

$$(Y-y)\frac{dy}{dx}+(X-x)=0 \qquad \text{from (3)}$$

$$\text{or,}\quad \{Y-\psi(t)\}\ \frac{\psi'(t)}{\phi'(t)}+\{X-\phi(t)\}=0$$

$$\text{or,}\quad \{X-\phi(t)\}\ \phi'(t)+\{Y-\psi(t)\}\ \psi'(t)=0 \qquad (11)$$

Ex. . Find the equation of the tangent at 't' to the curve

$$x=t^2-a,\ y=t^3-b$$

Here $x=\phi(t)=t^2-a \qquad \phi'(t)=2t$

$y=\psi(t)=t^3-b, \quad \therefore\ \psi'(t)=3t^2$

Hence the equation of the tangent at 't' is

$$\{X-\phi(t)\}\ \psi'(t)=\{Y-\psi(t)\}\ \phi'(t)$$

or, $\{X-(t^2-a)\}3t^2-\{Y-(t^3-b)\}2t=0$

or, $3t\ X-2\ Y+t^3+3at-2b=0.$

10. 3. Tangent at the origin,

The equation of the tangent at (x, y) to any curve is

$$Y-y=(dy/dx)(X-x)$$

The equation of tangent at (0, 0)

$$Y=\left(\frac{dy}{dx}\right)_0 (X-0) \quad \text{or,} \quad Y=\left(\frac{dy}{dx}\right)_0 X \quad \ldots\ldots(1)$$

where $\left(\frac{dy}{dx}\right)_0$ the value of $\frac{dy}{dx}$ at $x=0, y=0$

Ex. Find the tangent at the origin to the curve

$$x^2+y^2+ax+by=0$$

$$\frac{dy}{dx}=-\frac{f_x}{f_y}=-\frac{2x+a}{2y+b}=-\frac{a}{b} \text{ when } x=0,\ y=0$$

Thus the equation of the tangent at (0, 0) is

$$Y-0=\left(\frac{dy}{dx}\right)_0 (X-0) \quad \text{or,} \quad \text{or,} \quad Y=-\frac{a}{b}X$$

or, $bY+aX=0$

Take x and y as current co-ordinates, then the equation of the tangent at the origin is $ax+by=0$

which is the lowest degree terms of the given equation.

Working Rule : If a rational algebraic equation passes through the origin the equation of the tangent or tangents can be obtained by equating to zero the lowest degree terms of the given equation. By inspection such tangents can be determined.

10. 4. Angle of Intersection of two curves.

The angle between the intersection of two curves is determined by the angle between the tangents drawn to each curve at the point of their intersection.

The angle between the two straight lines $y=m_1 x+c_1$ and $y=m_2x+c_2$ is $\tan\theta=\frac{m_1 \sim m_2}{1+m_1m_2}$

Let us consider two curve $f(x, y)=0$ and $\phi(x, y)=0$,

The equation of tangents at (x, y) are

$(X-x) f_x+(Y-y) f_y=0$ or, $Xf_x+Yf_y-xf_x-yf_y=0$

and $(X-x)\phi_x+(Y-y)\phi_y=0$ or, $X\phi_x+Y\phi_y-x\phi_x-y\phi_y=0$

The slopes are

$m_1=-f_x/f_y$ and $m_2=-\phi_x/\phi_y$

Let α be angle between the two tangents drawn at (x, y)

$$\therefore \quad \tan\alpha=\frac{-f_x/f_y+\phi_x/\phi_y}{1+f_x/f_y\phi_x/\phi_y}=\frac{f_x\phi_y-\phi_x f_y}{f_x\phi_x+f_y\phi_y}$$

Hence the angle α of intersection of two curves is given by

$$\tan\alpha=\frac{f_x\,\phi_y-\phi_x\,f_y}{f_x\,\phi_x+f_y\,\phi_y} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

Cor. 1. If the two curves touch at (x, y) then the angle between the curves is zero i. e. $\alpha=0$, is

$$f_x\phi_y \sim \phi_x f_y=0$$

$$\text{or,} \quad \frac{f_x}{\phi_x}=\frac{f_y}{\phi_y} \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

Cor. 2. If the angle between the two tangents is 90° the curves are said to intersect orthogonally.

If $\alpha=\frac{1}{2}\pi$ then

$$\tan\tfrac{1}{2}\pi=\frac{f_x\phi_y\sim\phi_x f_y}{f_x\phi_x+f_y\phi_y}=\infty$$

hence $f_x\phi_x+f_y\phi_y=0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (14)$

Cartesian Subtangent, Subnormal, Length of tangent and normal.

Let $y=f(x)$ be the curve and $P(x, y)$ be any point on the curve. Let the tangent PT and normal PN meet the x-axis respectively at T and N.

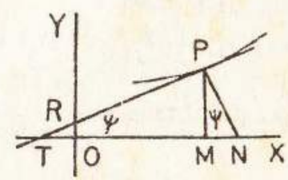


Fig 10 (a)

Draw PM perpendicular on x—axis.

The projection TM of the tangent PT on the x—axis is called **the subtangent.** While the projection, MN of the normal PN on the x—axis is called the **subnormal.** Let the tangent PT make an angle ψ with the x—axis ; then

$$\angle PTM = \psi = \angle NPM$$

$$\therefore \quad \tan \psi = \frac{dy}{dx} = y_1 \text{ and } MP = y.$$

(i) **Length of the Subtangent TM**

From the $\triangle TMP$, we have

$$TM = PM \cot\psi = y/\tan \psi = y/\ \frac{dy}{dx} = \frac{y}{y_1}$$

$\therefore$ The length of Subtangent $= y/y_1$ ... ... (16)

(ii) **Length of the Subnormal MN**

From the $\triangle MPN$ we have

$$MN = PM \tan \psi = y\frac{dy}{dx} = yy_1$$

$\therefore$ The length of the **Subnormal** $= \mathbf{y\frac{dy}{dx} = yy_1}$ ... ...(17)

(iii) **Length of the tangent PT.**

From the $\triangle TMP$,

$$TP^2 = TM^2 + MP^2$$

$$= (y/y_1)^2 + y^2 = (y/y_1)^2(1 + y_1^2) \quad \text{or,} \quad PT = (y/y_1)\sqrt{(1 + y_1^2)}$$

$\therefore$ **Length of tangent** $= \frac{y}{y_1}\sqrt{(1 + y_1^2)}$ ... ... (18)

(iv) **Length of the normal PN.**

From the $\triangle MPN$.

$$PN^2 = PM^2 + MN^2 = y^2 + (yy_1)^2 \quad \text{or.} \quad PN = y\sqrt{\{y(1 + y_1^2)\}}$$

$\therefore$ **Length of the normal** $= y\sqrt{(1 + y_1^2)}$ ... ... (19)

(v) **Points of Intersection –made by the tangent on axes.**

The equation to the tangent is as (x, y)

$$Y - y = (dy/dx)(X - x)$$

If the tangent meets the x—axis, then $Y = 0. \Rightarrow$

$$-y = (dy/dx)(X - x) = y_1 (X - x) \quad \text{or.} \quad X = x - y/y_1$$

(*a*) **The point of intersection with the x—axis is $(x - y/y_1, 0)$**

... ... (20)

If the tangent meets the Y—**axis, then** $X = 0$, *i. e.*,

$$Y - y = \frac{dy}{dx}(0 - x) = -y_1 x \quad \text{or,} \quad Y = y - y_1 x$$

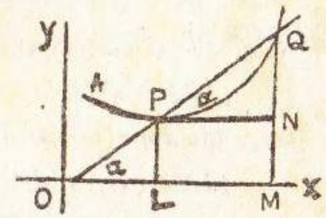
(*b*) $\therefore$ The point of intersection on Y—axis is $(0, y - y_1 x)$ ... ... (21)

10. 6. Arc Derivatives

To prove that for the curve $y = f(x)$:

$$\frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}$$

Let P (x, y) be any point on the curve and Q$(x + \Delta x,\ y + \Delta y)$ be a a neighbouring point on it.



Let A be a fixed point on the curve from which the arc length AP and AQ are measured.

Let arc AP $=$ S and arc AQ $=$ S $+ \Delta$S

So that arc PQ $= \Delta$S.

Draw PL, QM perpendiculars on QY. and PN perpendicular to QM.

$QN = QM - NM = QM - PL = y + \Delta y - y = \Delta y$

$PN = LM = OM - OL = x + \Delta x - x = \Delta x$

From the right angled triangle QPN

$(\text{Chord } PQ)^2 = PN^2 + NQ^2 = (\Delta x)^2 + (\Delta y)^2$

or, $\left(\frac{\text{chord } PQ}{\Delta x}\right)^2 = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2$

or, $\left(\frac{\text{chord } PQ}{\Delta x}\right)^2 \left(\frac{\text{arc } PQ}{\text{arc } PQ}\right)^2 = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2$

or $\left(\frac{\text{chord } PQ}{\text{arc } PQ}\right)^2 \left(\frac{\text{arc } PQ}{\Delta x}\right)^2 = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2$

or, $\left(\frac{\text{chord } PQ}{\text{arc } PQ}\right)^2 \left(\frac{\Delta s}{\Delta x}\right)^2 = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2$

If $Q \to P \frac{\text{chord } PQ}{\text{arc } PQ} \to 1$

$\therefore$ * $\lim_{Q \to P}\left(\frac{\text{chord } PQ}{\text{arc } PQ}\right)^2 \lim_{\Delta x \to 0}\left(\frac{\Delta s}{\Delta x}\right) = \lim_{\Delta x \to 0}\left\{1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right\}$

or, $1 \cdot (ds/dx)^2 = 1 + (dy/dx)^2$

or, $$\frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} \tag{22}$$

Cor. 1. If $x = f(y)$, then

$(ds/dy)^2 = 1(dx/dy)^2$ ... ... (23)

or, $(ds/dy) = \{1 + (dx/dy)^2\}$ ... ... (23i)

$(ds)^2 = (dx)^2 + (dy)^2$ ... ... (24)

Cor. 2. From the ΔPQN

$$\cos QPN = \frac{PN}{PQ} = \frac{\Delta x}{PQ} = \frac{\Delta x}{\Delta s} \cdot \frac{\Delta s}{PQ}$$

or, $\cos \alpha = \frac{\Delta x \text{ arc } PQ}{\Delta s \text{ chord } PQ}$

Let ψ be the angle which the tangent to the curve at P makes with the positive direction of axis of x.

if $Q \to P$, $\alpha \to \psi$ and $\Delta s \to 0$. Also $\lim_{Q \to P} \frac{\text{arc } PQ}{\text{chord } PQ} = 1$

Thus

$$\cos \psi = \lim_{\Delta s \to 0} \frac{\Delta x}{\Delta s} = \frac{dx}{ds} \quad \therefore \quad \cos \psi = \frac{dx}{ds} \tag{25}$$

Cor. 3. Similarly, $\sin PQN = \frac{QN}{PQ} = \frac{\Delta y}{\Delta s} \cdot \frac{\Delta s}{PQ}$

or, $\sin \alpha = \frac{\Delta y}{\Delta s} \cdot \frac{\text{arc } PQ}{\text{chord } PQ}$

If $P \to Q$, $\alpha \to \psi$, $\frac{\text{arc } PQ}{\text{chord } PQ} \to 1$ and hence

$$\sin \psi = \lim_{\Delta s \to 0}\left(\frac{\Delta y}{\Delta s}\right) 1 = \frac{dy}{ds} \quad \therefore \quad \sin \psi = \frac{dy}{ds} \tag{26}$$

Cor. 4. $\tan \psi = (dy/dx)$; $\cos \psi = (dx/dy)$

we have from (1)

$(ds/dx)^2 = 1 + (dy/dx)^2 = 1 + \tan^2\psi = \sec^2\psi$ or, $(ds/dx) = \sec \psi$

Similarly $\frac{ds}{dy} = \text{cosec } \psi$.

Also $(dx/ds)^2 + (dy/ds)^2 = \cos^2\psi + \sin^2\psi = 1$

$$\therefore \quad (dx/ds)^2 + (dy/ds)^2 = 1 \tag{27}$$

Cor. 5. Prove that for the parametric curve $x = \phi(t)$ and $y = \phi(t)$, the arc derivative is

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2$$

Now $\frac{ds}{dt} = \frac{dx}{ds} \cdot \frac{ds}{dt}$; $\frac{dy}{dt} = \frac{dy}{ds} \cdot \frac{ds}{dt}$

Therefore,

$$\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2=\left(\frac{dx}{ds}\right)^2\left(\frac{ds}{dt}\right)^2+\left(\frac{dy}{ds}\right)^2\left(\frac{ds}{dt}\right)^2$$

$$=\left\{\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2\right\}\left(\frac{ds}{dt}\right)^2$$

$$=1\left(\frac{ds}{dt}\right)^2 \text{ [by (27)]}$$

$$\therefore \quad \left(\frac{ds}{dt}\right)^2=\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2 \quad \ldots(28)$$

$$\text{or,} \quad \frac{ds}{dt}=\sqrt{\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right\}} \quad \ldots \quad \ldots \quad (29)$$

provided S increases with t.

Examples

10. 7.

Ex. 1. Find the equation to the tangent to the curve

$$(x/a)^m+(y/b)^m=1 \quad \text{at} \quad (x, y)$$

The equation can be written as $x^m/a^m+y^m/b^m-1=0$……(1)

$$\text{Now } f_x=\frac{m}{a^m}x^{m-1}=\frac{m}{a}\left(\frac{x}{a}\right)^{m-1} \text{ and } f_y=\frac{m}{b^m}y^{m-1}$$

$$=\frac{m}{b}\left(\frac{y}{b}\right)^{m-1}$$

The equation of the tangent at (x, y) is

$$(X-x)f_x+(Y-y)f_y=0,$$

$$\text{or,} \quad (X-x)\frac{m}{a}\left(\frac{x}{a}\right)^{m-1}+(Y-y)\frac{m}{b}\left(\frac{y}{b}\right)^{m-1}=0$$

$$\text{or,} \quad \frac{X}{a}\left(\frac{x}{a}\right)^{m-1}+\frac{Y}{b}\left(\frac{y}{b}\right)^{m-1}=\left(\frac{x}{a}\right)^m+\left(\frac{y}{b}\right)^m=1 \text{ by (1)}$$

$$(X/a)(x/a)^{m-1}+(Y/b)(y/b)^{m-1}=1.$$

Ex. 2. Find the angle of intersection of the curves

$$x^3+2xy^2-10a^2x+12a^2y+3a^3=0$$

and $y^3+2xy^2-5a^2x-a^3=0$ at $(3a -2a)$

Let $f(x, y)=x^3+2xy^2-10a^2x+12a^2y+3a^3=0$ … … (1)

$\phi(x, y)=y^3-2xy^2-5a^2x-a^2$ … … … … … (2)

Then $f_x=3x^2+2y^2-10a^2,\ f_y=4xy+12a^2$

$$\phi_x=2y^2-5a^2, \quad \phi_y=3y^2+4xy$$

At $(3a, -2a)$

$f_x=27a^2+8a^2-10a^2=25a^2,\ f_y=-24a^2+12a^2=-12a^2$

$\phi_x=8a^2-5a^2=3a^2,\ \phi_y=12a^2-24a^2=-12a^2$

Let α be the angle of their intersection. Then,

$$\tan\alpha=\left[\frac{f_x\phi_y-f_y\phi_x}{f_x\phi_x+f_y\phi_y}\right] \text{ When } x=3a,\ y=-2a$$

$$=\frac{25a^2(-12a^2)-(-12a^2 \cdot 3a^2)}{25a^2.3a^2+(-12a^2)(-12a^2)}=\frac{264a^4}{219a^4}=\frac{88}{73}$$

$$\therefore \quad \alpha=\tan^{-1}(88/73)$$

Ex. 3. Show that in curve $by^2=(x+a)^3$ the square of the subtangent varies as the subnormal. [R. U. 1961, '83]

$$\text{We have, } by^2=(x+a)^3 \quad \text{or,} \quad y^2=\frac{(x+a)^3}{b} \quad \ldots \quad \ldots \quad (1)$$

$$\therefore \quad 2y\frac{dy}{dx}=\frac{3(x+a)^2}{b} \quad \text{or,} \quad y_1=\frac{dy}{dx}=\frac{3(x+a)^2}{2yb}$$

Let m = length of subtangent $=y/y_1$, n = length of the subnormal $=yy_1$

$$\frac{m^2}{n}=\frac{(y/y_1)^2}{(yy_1)}=\frac{y}{y_1^3}=\frac{yy^3b^3}{27(x+a)^6}$$

$$=\frac{8b}{27}\frac{y^4}{\{(x+a)^2/b\}^2}=\frac{8b\,y^4}{27\,y^4}=\frac{8b}{27}=\text{constant}$$

$$\therefore \quad m^2 \propto n]$$

Hence square of the subtangent varies as subnormal.

Ex. 4. Show that the subnormal at any point of the curve $y^2x^2=a^2(x^2-y^2)$ varies inversely as the cube of the abscissa

[R. U, 1966 D. U. 1961]

$$y^2x^2=a^2(x^2-a^2) \quad \text{or,} \quad y^2=a^2-a^4/x^2$$

$$\therefore \quad 2y\frac{dy}{dx}=\frac{2a^4}{x^3} \text{ or } y_1=\frac{a^4}{yx^3}$$

$$n=\text{subnormal}=y\frac{dy}{dx}=yy_1=y\frac{a^4}{yx^3}=\frac{a^4}{x^3}$$

$$n \propto 1/x^3$$

Hence subnormal varies inversely as the cube of the abscissa.

Ex. 5. Prove that the segment (between the co-ordinates axes) of a tangent to astroid $x^{2/3}+y^{2/3}=a^{2/3}$ is of constant length.

C.U. 1992 [D. U 1962, 1969, 83, R. H, 1966]

From the equation $f(x,y)=x^{2/3}+y^{2/3}-a^{2/3}=0 \quad \cdots \; \cdots \; \cdots (1)$

$$f_x=\tfrac{2}{3}(1/x)^{-1/3},\ f_y=\tfrac{2}{3}(1/y)^{-1/3}$$

The equation of the tangent at (x, y) is

$$(X-x)f_x+(Y-y)f_y=0$$

or, $(X-x)\frac{2}{3}x^{-\frac{1}{3}}+(Y-y)\frac{2}{3}y^{-\frac{1}{3}}=0$

or $\dfrac{X}{x^{1/3}}+\dfrac{Y}{y^{1/3}}=(x)^{2/3}+(y)^{2/3}=a^{2/3}$ by (1)

Let the tangent line meet the axes at A and B. Then the co-ordinates of A and B are respectively,

$A(a^{2/3}.\ x^{1/3}, 0)$ and $B(0,\ a^{2/3}\ y^{1/3})$

Now $OA=a^{2/3}x^{1/3}$, $OB=a^{2/3}\ y^{1/3}$

Now $AB^2=OA^2+OB^2=a^{4/3}x^{2/3}+a^{4/3}y^{2/3}=a^{4/3}(x^{2/3}+y^{2/3})$

$=a^{4/3}a^{2/3}=a^3=$ constant.

Ex. 6. Show that in the curve $y=a\log(x^2-a^2)$, sum of the lengths of the tangent and the subtangent varies as the product of the co-ordinates of the Point of contact.

Let $m=$length of the tangent$=\dfrac{y\sqrt{(1+y_1^2)}}{y_1}$

$\triangle^2$ $n=$Length of the subtangent$=y/y_1$

Here $y=a\log(x^2-a^2)$ $\therefore$ $y_1=\dfrac{a.2x}{x^2-a^2}$

Thus $m+n=\left\{\dfrac{y\sqrt{(1+y_1^2)}}{y_1}+\dfrac{y}{y_1}\right\}=\dfrac{y}{y_1}\left\{\sqrt{(1+y_1^2)}+1\right\}$

But $\sqrt{(1+y_1^2)}=\sqrt{\left\{1+\dfrac{a^2x^2}{(x^2-a^2)^2}\right\}}=\dfrac{x^2+a^2}{x^2-a^2}$

$$\therefore \quad m+n=\frac{y}{2ax}(x^2-a^2)\left\{\frac{x^2+a^2}{x^2-a^2}+1\right\}=\frac{y}{2ax}2x^2=\frac{xy}{a}$$

$\therefore$ $(m+n)\propto xy$, as a is constant.

Ex. 7. Find the equations of the tangent at the origin to the curve $y^2(a+x)=x^2(3a-x)$.

Rewrite the equation $y^2(a+x)=x^2(3a-x)$

or, $x^3+xy^2-3ax^2+ay^2=0$ or, $-x^3-x^2y+a(3x^2-y^2)=0$

Equate to zero the lowest degree term in the above equation to get

$$3x^2-y^2=0 \quad \text{or,} \quad y=\pm\sqrt{3}x$$

Hence the tangents at the origin are $y=\sqrt{3}x$ and $y=-\sqrt{3}x$.

Ex. 8. Find the angle of intersection of the curves $2y^2=x^3$ and $y^2=32x$.

Let us find the points of intersection of the curves

$$2y^2=x^3 \quad \cdots \; \cdots \quad (1) \quad \text{and} \quad y^2=32x \quad \cdots \; \cdots \; (2)$$

From (1) & (2) $2 \cdot 32x = x^3$

or, $x^3 - 64x = 0$ or, $x(x^2-64)=0$ or. $x=0,\ 8, -8$

But $x=-8$ is inadmissable because that will give imaginary value of y.

From (2),

when $x=0$, $y=0$, and

when $x=8$, $y^2=32.8=(\pm 16)^2$ or, $y=\pm 16$

$\therefore$ The points of the intersection of the curve are

$O(0, 0)$, $P(8, 16)$, $Q(8, -16)$

Let $f(x, y) = 2y^2 - x^3$ and $\phi(x, y) = y^2 - 32x$

$\therefore$ $f_x = -3x^2, f_y = 4y$; $\phi_x = -32$ and $\phi_y = 2y$

Let θ be the angle of intersection, then

At $O(0, 0)$; $\tan\theta = \dfrac{f_x\phi_y - f_y\phi_x}{f_x\phi_x + f_y\phi_y} = \dfrac{(-3x^2)(2y)+4y(-32)}{(-3x^2)(-32)+4y \cdot 2y}$

$$= \frac{(-3\times 0)(2.0) - 4.0(-32)}{(-3)(-32)+4.0.2.0} = \frac{0}{0}$$

which is the indeterminate form $\therefore$ no tangent is possible,

At $P(8, 16)$, $\tan\theta = \dfrac{-6x^2y + 128y}{96x^2 + 8y^2} = \dfrac{-6\times 8^2 \times 16 + 128.16}{96.\ 8^2 + 8.\ 16^2}$

$$= \frac{8^2\times 16(-6+2)}{8^2 \times 16\,(6+2)} = \frac{-4}{8} = -\tfrac{1}{2}$$

$$\Rightarrow \theta = \tan^{-1}(-\tfrac{1}{2}) \quad \text{at } P(8, 16)$$

At $Q(8, -16)$, $\tan\theta = \dfrac{-6.8^2(-16)+12.8(-16)}{96.\ 8^2 + 8(-16)^2}$

$$= \frac{8^2\times 16(6-2)}{8^2\times 16(6+2)} = \frac{4}{8} = \tfrac{1}{2}$$

$$\Rightarrow \theta = \tan^{-1}(\tfrac{1}{2}) \text{ at } Q[(8, -16)]$$

Exercise X (A)

1. Find the equation to the tangent at the point (x_1, y_1) of the following curves.

(i) $x^2/a^2 + y^2/b^2 = 1$ at (x_1, y_1) (ii) $y = a \log \sin x$ at (x_1, y_1)

(iii) $(x^2+y^2)^2 = a^2(x^2-y^2)$ at (x_1, y_1)

1. (a) Prove that tangent at (a, b) to the curve $(x/a)^3 + (y/b)^3 = 2$ is $(x/a)+(y/b)=2$ [R. H. 1988, D. H. 1986]

2. Find the equations of the normals at the point (x_1, y_1) of the following curves.

(i) $y^2 = 4ax$ (ii) $x^3 - 3axy + y^3 = 1$

(iii) $x^2(x-y) + a^2(x+y) = 0$ at $(0, 0)$

3. Find the tangent and normol at the point determined by θ on.

(i) $x = a \cos\theta$, $y = b \sin\theta$

(ii) $x = a(\theta + \sin\theta)$, $y = a(1-\cos\theta)$

(iii) $x = a\cos^3\theta$, $y = b\sin^3\theta$

3 (a) If $f(x) = x^2 + x - 6$ find the equation to the tangent and the normal to the curve of $f(x)$ at the point $x=1$. Draw a rough sketch. D. U. 1987

(যদি $f(x) = x^2 + x - 6$ একটি বক্ররেখা হয় তবে $x=1$ বিন্দুতে উক্ত বক্র রেখার স্পর্শক ও অভিলম্বের সমীকরণ নির্ণয় কর। মোটামুটি লেখচিত্র অঙ্কন কর।) Ans. $3x - y - 7 = 0$, $x + 3y + 11 = 0$

At $x = 1$, $y = x^2 + x - 6 = -4$

we are to find tangent and normal at $(1, -4)$

$\dfrac{dy}{dx} = 2x + 1 = 2.1 + 1 = 3$ at $(1, -4)$

The eq of the tangent at $(1,-4)$ is

$y+4=(dy/dx)(x-1)$ or, $y+4=3(x-1)$

or, $3x-y-7=0$

Normal at $(1,-4)$ is

$(y+4)(dy/dx)+(x-1)=0$ or, $(y+4)3+(x-1)=0$

or, $3y+x+11=0$

3(b) Find the eq of the tangent and normal at $(2,-2)$ of the curve $y=x^3-3x+2$ (বক্ররেখাটির $(2,-2)$ বিন্দুতে স্পর্শক ও অভিলম্ব নির্ণয় কর।

C. U. 1981

Ans. $9x-y-20=0$, $x+9x+16=0$

4. Find the angle of intersection of the curves

(i) $x^2-y^2=a^2$ and $x^2+y^2=a^2\sqrt{2}$

(ii) $x^2=4ay$ and $2y^2=ax$

(iii) $y=4-x^2$ and $y=x^2$

(iv) $x^2-y^2=2a^2$, $x^2+y^2=4a^2$ D.U. 1991

5. For the curve $y=c\cosh\frac{x}{c}$ find the subtangent and the subnormal at any point. [R. U. 1962]

6. Show that the portion of the tangent at any point on the curve $x=27\cos^3\theta$, $y=27\sin^3\theta$ intercepted between the axis is of constant length.

7. Prove that the subtangent is of constant length at any pt. of the curve $\log y=x\log a$ and that the subnormal is constant at any point on the parabola $y^2=4ax$.

8. Show that the abscissa of the point on the curve. $\sqrt{(xy)}=a+x$, at which the normal makes equal intercepts from the co-ordinate axes is $a/\sqrt{2}$.

9. Find the condition that the conics shall cut orthogonally.

$ax^2+by^2=1$

$a_1x^2+b_1y^2=1$ [R. U 1964]

9. (i) Show that $xy=4$, $x^2-y^2=15$ cut each other orthogonally. [D. U. 1986]

(ii) prove that the curves $x^2/a+y^2/b=1$

$x^2/a_1+y^2/b_1=1$ cut orthogonally if $a-b=a_1-b_1$ R. U. 1987

(iii) Show that the curves $x^3-3xy^2+2=0$

$3x^2y-y^3=2$ cut orthgonally. D. U. 1989

10. Find the equation of the tangent and normal to the curve $x=e^{-t}\cos t$, $y=e^t\sin t$ at the point $t=\pi$.

10 (i) $x^2+2y^2=3$ at $(1,-1)$ D. U. 1980, '88

11. Show that the curve represented by

$$(x/a)^n+(y/b)^n=2$$

for different values of n have a common tangent at the point (a,b)

Hence show that the equation of the tangent is $\frac{x}{a}+\frac{y}{b}=2$

12. Show that all the points of the curve

$y^2=4a\left\{x+a\sin\frac{x}{a}\right\}$ [R. U. 1982]

at which the tangent is parallel to the axis of x lie on a parabola $y^2=4ax$.

13. Prove that $\frac{x}{a}+\frac{y}{b}=1$. touches the curve $y=be^{-x/a}$ at the point where the curve crosses the axis of y.

14. If $p=x\cos\alpha+y\sin\alpha$ touches the curves

$(x/a)^{n/n-1}+(y/b)^{n/n-1}=1$.

prove that $p^n=(a\cos\alpha)^n+(b\sin\alpha)^n$.

15. If $ax+by=1$ is a normal to the parabola $y^2=4cx$ then prove that $ca^3+2acb^2=b^2$.

16. Show that the condition that the curves $x^2/\alpha^2+y^2/\beta^2=c^2\beta$ and $x^2/a^2+y^2/b^2=1$ may touch is $c=a+b$.

17. For the catenary $y=c\cosh x/c$, prove that the length of the portion of the normal at (x, y) intercepted between the curve and the axis of x is y^2/c. [R. U. 1967, '78 ; C. H. 1969, '83]

18. If $p=x\cos\alpha+y\sin\alpha$ touch the curve,

$$\frac{x^m}{a^m}+\frac{y^m}{b^m}=1.$$ show that

$$p^{\frac{m}{m-1}}=(a\cos\alpha)^{\frac{m}{m-1}}+(b\sin\alpha)^{\frac{m}{m-1}}$$ [D. U. 1952]

19. Show that in the curve $a^2y^3=k(bx+e)^4$ the cube of the subtangent varies as the fifth power of the subnormal. D. U. 1962

20. Tangents are drawn from the origin to the curve $y=\sin x$. Prove that points of contract lie on $x^2y^2=x^2-y^2$.

21. Show that the length of the portion of the normal to the curve. $x=a(4\cos^3\theta-3\cos\theta)$; $y=a(4\sin^3\theta-9\sin\theta)$. intercepted between the co-ordinate axes is constant.

22. Find the condition that the line $p=x\cos\alpha+y\sin\alpha$ may be a tangent to the curve

$$x^m y^n=a^{m+n}$$

23. Show that the curves cut orthogonally.

(i) $x^3-3xy^2+2=0$ and $3x^2y-y^3=2$.

(ii) $y=x^2$ and $x^3+6y=7$

24. If ϕ be the angle between the tangent to a curve and the radius vectors drawn from the origin of co-ordinates to the point of contact, prove that

$$\tan\phi=\left(x\frac{dy}{dx}-y\right)\Big/\left(x+y\frac{dy}{dx}\right)$$

25. Show that for curve $by^2=(x+a)^2$, the square of the sub-tangent varies as the subnormal. D. H. 1986

26. Show that the value of $n=-2$, so that the subnormal of the curve $xy^n=a^{n+1}$ may be of constant length.

27. Prove that mth power of the subtangent varies as nth power of the sub-normal of the curve. C. H. 1988

$$x^{m+n}=a^{m+n}y^{2n}$$

28. Show that subtangent at any point of the curve.

$$x^m y^n=a^{m+n}$$

varies as the abscissa of the point

29. Show that for the curve $x=a+b\log[b^2+\sqrt{(b^2-y^2)}]-\sqrt{(b^2-y^2)}$ sum of subnormal and subtangent is constant.

30. Find $\frac{ds}{dx}$ for the following curve.

(i) $y^2=4ax$ (ii) $x^{2/3}+y^{2/3}=a^{2/3}$

(iii) $x=t^2$, $y=t-1$ (iv) $x=2\sin t$, $y=\cos 2t$.

31. Prove that in the curve $x=\frac{y^2-a^2}{4a}+\frac{a}{2}\log\frac{a}{y}$ the difference between the lengths of the tangent and subtangent is constant.

32. If the normal at any point to the curve $x^{2/3}+y^{2/3}=a^{2/3}$ makes an angle ϕ with the x-axis, show that its equation is $y\cos\phi-x\sin\phi=a\cos 2\phi$.

33. If x_1, y_1 be the portions of the axes of x and y intercepted by the tangent at any point (x,y) on the curve $(x/a)^{2/3}+(y/b)^{2/3}=1$.

show that $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$

34. Show that tangents at the origin of the following curve

(i) $x^3+y^3=3axy$ are $x=0$ and $y=0$

(ii) $a^2y^2=a^2x^2-x^4$ are $y=\pm x$

(iii) $y^3(x+3a)=x(x-a)(x-2a)$ is $x=0$

(iv) $x(y-x)^2=ay^2$ is $y=0$

(v) $x^2(x^2-4a^2)=y^2(x^2-a^2)$ are $y=\pm 2x$

(vi) $x^4-2x^2y^2+x^2+x-3y=0$ is $x=3y$

(vii) $x^5+y^5+x^3y+x^2y^2-6xy^2=0$

are $x=0$, $y=0$, $x=2y$, $x=-3y$

35. Find the area of the triangle formed by the axes and the tangent to the curve

$$x^{2/3}+y^{2/3}=a^{2/3}$$ **[R. U, 1966]**

36. Show that in the curve $ay^4=b^2(cx+f)^3$ the square of the subnormal varies as the subtangent. C. H. 1989

37. Find the tangent and normal to the curve

(a) $y(x-1)(2x-3)-x+4=0$ at the points where it cuts the x-axis N.H. 1994

(b) $y(x-2)(x-3)-x+7=0$ where it cuts the x-axis N.H. 1994

(c) $y(x^2+a^2)=ax^2$ at $y=a/4$ N.U. 1995

প্রশ্নমালা X (A)

1. নিম্নলিখিত বক্ররেখা সমূহের (x_1, y_1) বিন্দুতে স্পর্শকের সমীকরণ নির্ণয় কর।

(i) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ এর (x_1, y_1) বিন্দুতে। (ii) $y=a\log\sin x$-এর (x_1, y_1) বিন্দুতে। (iii) $(x^2+y^2)^2=a^2(x^2-a^2)$ এর (x_1, y_1) বিন্দুতে।

2. নিম্নলিখিত বক্ররেখা সমূহের (x_1, y_1) বিন্দুতে অভিলম্বের সমীকরণ নির্ণয় কর।

(i) $y^2=4ax$ (ii) $x^3-3axy+y^3=1$

(iii) $x^2(x-y)+a^2(x+y)=0$ এর $(0, 0)$ বিন্দুতে।

3. নিম্নলিখিত প্যারামিতিক সমীকরণগুলির 'θ' বিন্দুতে স্পর্শক এবং অভিলম্ব নির্ণয় কর।

(i) $x=a\cos\theta$, $y=b\sin\theta$.

(ii) $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$.

(iii) $x=a\cos^3\theta$, $y=b\sin^3\theta$.

3 (a) If $f(x)=x^2+x-6$ find the equation to the tangent and the normal to the curve of $f(x)$ at the point $x=1$. Draw a rough sketch. D. U. 1987

(যদি $f(x)=x^2+x-6$ একটি বক্ররেখা হয় তবে $x=1$ বিন্দুতে উক্ত বক্ররেখার স্পর্শক b অভিলম্বের সমীকরণ নির্ণয় কর। মোটামুটি লেখচিত্র অঙ্কন কর।) Ans. $3x-y-7=0$, $x+3y+11=0$

At $x=1$, $y=x^2+x-6=-4$

we are to find tangent and normal at $(1,-4)$

$$\frac{dy}{dx}=2x+1=2.1+1=3 \text{ at } (1,-4)$$

The eq of the tangent at $(1,-4)$ is

$y+4=(dy/dx)(x-1)$ or, $y+4=3(x-1)$

or, $3x - y - 7 = 0$

Normal at $(1, -4)$ is

$(y + 4)(dy/dx) + (x - 1) = 0$ or, $(y + 4)3 + (x - 1) = 0$

or, $3y + x + 11 = 0$

3 (b) Find the eq of the tangent and normal at $(2, -2)$ of the curve $y = x^3 - 3x + 2$ [বক্ররেখাটির $(2, -2)$ বিন্দুতে স্পর্শক ও অভিলম্ব নির্ণয় কর।] C. U. 1981

Ans. $9x - y + 20 = 0,\ x + 9x + 16 = 0$

4. নিম্নলিখিত বক্ররেখাগুলির ছেদক কোণ (Argle of intersection) নির্ণয় কর।

(i) $x^2 - y^2 = a^2$ এবং $x^2 + y^2 = a^2\ \sqrt{2}$. (ii) $x^2 = 4ay$ এবং $2y^2 = ax$

(iii) $y = 4 - x^2$ এবং $y = x^2$. (iv) $x^2 - y^2 = 2a^2,\ x^2 + y^2 = 4a^2$ D. U. 1991

5. সমীকরণ $y = c \cos h\ \frac{x}{c}$ এর জন্য যে কোন বিন্দুতে উপস্পর্শক ও উপলম্ব নির্ণয় কর। R. U. 1962

6. দেখাও যে $x = 27 \cos^3 \theta,\ y = 27 \sin^3\theta$ বক্ররেখার যে কোন বিন্দুতে অঙ্কিত স্পর্শক দুই অক্ষরেখাদ্বারা সীমাবদ্ধ স্পর্শকের দৈর্ঘ্য সর্বদা ধ্রুব।

7. দেখাও যে $\log y = x \log a$ বক্ররেখার যে কোন বিন্দুতে অঙ্কিত উপস্পর্শকের দৈর্ঘ্য ধ্রুব। এবং $y^2 = 4ax$ অধিবৃত্তে যে কোন বিন্দুতে অঙ্কিত উপলম্বের দৈর্ঘ্য ও ধ্রুব।

8. দেখাও যে $\sqrt{(xy)} = a + x$ বক্ররেখার যে বিন্দুতে অঙ্কিত অভিলম্ব অক্ষদ্বয় হইতে সমান সমান অংশ ছেদ করে ঐ বিন্দুর ভুজ হইবে $a/\sqrt{2}$.

9. $ax^2 + by^2 = 1$ এবং $a_1x^2 + b_1y^2 = 1$ কণিকদ্বয় পরস্পর লম্বভাবে ছেদ করিবার শর্ত নির্ণয় কর। R. U. 1964

9. (i) Show that $xy = 4,\ x^2 - y^2 = 15$ cut each other ortnogonally. [R. U. 1964]

(ii) prove that the curves $x^2/a + y^2/b = 1$

$x^2/a_1 + y^2/b_1 = 1$ cut orthogonally if $a - b = a_1 - b_1$ [R. U. 1987]

(iii) Show that the curves $x^3 - 3xy^2 + 2 = 0$

$3x^2y - y^3 = 2$ cut orthogonally. [D. U. 1989]

10. বক্ররেখা $x = e^{-t} \cos t,\ y = e^t \sin t$ এর $t = \pi$ বিন্দুতে স্পর্শক এবং অভিলম্বের সমীকরণ নির্ণয় কর।

10. (i) $x^2 + 2y^2 = 3$ at $(1, -1)$ [D. U. 1980, '88]

11. দেখাও যে n-এর বিভিন্ন মানের জন্য (a, b) বিন্দুতে

$$\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2.$$

বক্ররেখার উপর একটি সাধারণ স্পর্শক আঁকা যায়। ইহা হইতে দেখাও যে স্পর্শকের সমীকরণ $\frac{x}{a} + \frac{y}{b} = 2$.

12. দেখাও যে $y^2 = 4a\{x + a \sin x/a\}$ বক্ররেখার যে সকল বিন্দুতে স্পর্শকগুলি x- অক্ষের সমান্তরাল তাহার অধিবৃত্ত (Parabola) $y^2 = 4ax$ এর উপর থাকিবে।

13. বক্ররেখা $y = be^{-x/a}$ যে বিন্দুতে y-অক্ষকে অতিক্রম করে, দেখাও যে, সেই বিন্দুতে $x/a + y/b = 1$ সরলরেখাটি ঐ বক্ররেখাকে স্পর্শ করে।

14. যদি $P = x \cos \alpha + y \sin \alpha$ রেখাটি যদি বক্ররেখা

$$\left(\frac{x}{a}\right)^{n/n-1} + \left(\frac{y}{b}\right)^{n/n-1} = 1$$ কে স্পর্শ করে তবে দেখাও যে

$p = (a \cos \alpha)^n + (b \sin \alpha)^n$.

15. যদি $ax + by = 1$ রেখাটি অধিবৃত্ত (Parabola) $y^2 = 4cx$ এর উপর অভিলম্ব হয় তবে প্রমাণ কর যে $ca^3 + 2acb^2 = b^3$.

16. দেখাও যে $x^{2/3} + y^{2/3} = c^{2/3}$ এবং $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ বক্ররেখাদ্বয় পরস্পরকে স্পর্শ করার শর্ত হইল $c = a + b$.

17. কেটেনারী (Catenary) $y=c\cosh x/c$ এর যে কোন বিন্দু (x, y) এ যে অভিলম্ব অংকিত করা যায়, তাহার x-অক্ষ এবং ঐ বক্ররেখা দ্বারা কর্তিত অংশের দৈর্ঘ্য হইবে y^2/c. ইহা প্রমাণ কর।

[R. U. 1967, 78 C. U. H. 1969, 83]

18. যদি $P=x\cos\alpha+y\sin\alpha$ সরলরেখা $\left(\frac{x}{a}\right)^m+\left(\frac{y}{b}\right)^m=1$ বক্ররেখাকে স্পর্শ করে তবে দেখাও যে

$$P^{\frac{m}{m-1}}=(a\cos\alpha)^{\frac{m}{m-1}}+(b\sin\alpha)^{\frac{m}{m-1}}$$ [D. U. 1962]

19. দেখাও যে $a^2y^5=k(bx+c)^4$ বক্ররেখার যে কোন বিন্দুতে উপস্পর্শকের ঘন ঐ বিন্দুতে অঙ্কিত উপলম্বের পঞ্চম ঘাতের সমানুপাতিক।

[D. U. 1962]

20. $y=\sin x$ রেখার উপর মূল বিন্দু হইতে স্পর্শক অঙ্কিত করা হইলে, দেখাও যে স্পর্শ বিন্দুগুলি $x^2y^2=x^2-y^2$-এর উপর থাকিবে।

21. দেখাও যে বক্ররেখা $x=a(4\cos^3\theta-3\cos\theta)$; $y=a(4\sin^3\theta-9\sin\theta)$ এর যে কোন বিন্দুতে অঙ্কিত অভিলম্ব হইতে অক্ষ দ্বয় দ্বারা কর্তিত অংশের দৈর্ঘ্য ধ্রুব থাকে।

22. $P=x\cos\alpha+y\sin\alpha$ রেখাটি $x^my^n=a^{m+n}$ বক্ররেখাকে স্পর্শ করার শর্ত নির্ণয় কর।

23. দেখাও যে নিম্নলিখিত বক্ররেখাদ্বয় পরস্পরকে লম্বভাবে ছেদ করে। (cut orthogonally).

(i) $x^3-3xy^2+2=0$ এবং $3x^2y-y^3=2$.

(ii) $y=x^2$ এবং $x^3+6y=7$.

24. কোন বক্ররেখার কোন বিন্দুতে অঙ্কিত স্পর্শক এবং মূলবিন্দু এবং স্পর্শ বিন্দুর মধ্যে যোজিত ব্যাসার্ধ (Radius vector) ভেক্টরের মধ্যে কোণ ϕ হইলে প্রমাণ কর যে

$$\tan\phi=\left(x\frac{dy}{dx}-y\right)\Big/\left(x+y\frac{dy}{dx}\right).$$

25. $by^2=(x+a)^3$ বক্ররেখার জন্য দেখাও যে উপস্পর্শকের বর্গ উপলম্বের সমানুপাতিক।

26. দেখাও যে $n=-2$ হইলে বক্ররেখা $xy^n=a^{n+1}$ এর যে কোন বিন্দুতে উপলম্বের দৈর্ঘ্য ধ্রুব হইবে।

27. প্রমাণ কর যে $x^{m+n}=a^{m-n}y^{2n}$ বক্ররেখার যে কোন বিন্দুতে অঙ্কিত উপ-স্পর্শকের m-তম মাত্রা ঐ বিন্দুতে অংকিত উপলম্বের n-তম মাত্রার সমানুপাতিক। [C. H. 1988]

28. দেখাও যে $x^my^n=a^{m+n}$ বক্ররেখার যে কোন বিন্দুতে অঙ্কিত উপ-স্পর্শক ঐ বিন্দুর ভুজের সমানুপাতিক।

29. দেখাও যে $x=a+b\log[b^2+\sqrt{(b^2-y^2)}]-\sqrt{(b^2-y^2)}$ বক্ররেখার যে কোন বিন্দুতে অঙ্কিত উপস্পর্শক এবং উপলম্বের দৈর্ঘ্যের যোগফল ধ্রুব হইবে।

30. নিম্নলিখিত বক্ররেখাগুলির জন্য $\frac{ds}{dx}$ নির্ণয় কর।

(i) $y^2=4ax$ (ii) $x^{2/3}+y^{2/3}=a^{2/3}$

(iii) $x=t^2$, $y=t-1$ (iv) $x=2\sin t$, $y=\cos 2t$.

31. দেখাও যে, $x=\frac{y^2-a^2}{4a}+\frac{a}{2}\log\frac{a}{y}$ বক্ররেখার যে কোন বিন্দুতে অঙ্কিত স্পর্শকের এবং উপস্পর্শকের দৈর্ঘ্যের অন্তরফল ধ্রুব হইবে। D. U. H. 1960

32. যদি $x^{2/3}+y^{2/3}=a^{2/3}$ বক্ররেখার কোন বিন্দুতে অভিলম্ব অঙ্কিত করিলে ইহা x-অক্ষের সহিত ϕ কোণ উৎপন্ন করে তবে দেখাও যে উহার সমীকরণ হইবে $y\cos\phi-x\sin\phi=a\cos 2\phi$.

33. $\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}=1$ বক্ররেখার যে কোন বিন্দু (x, y) এ অংকিত স্পর্শক যদি x ও y অক্ষ হইতে যথাক্রমে x_1 ও y_1 অংশ কর্তন করে তবে প্রমাণ কর যে $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$.

34. দেখাও যে নিম্নলিখিত বক্ররেখাগুলির উপর মূলবিন্দুতে অংকিত স্পর্শক সমূহ হইবে

(i) $x^3 + y^3 = 3axy$ এর জন্য $x = 0$ এবং $y = 0$
(ii) $a^2y^2 = a^2 x^2 - x^4$ এর জন্য $y = \pm x$
(iii) $y^2 (x + 3a) = x (x - a) (x - 2a)$ এর জন্য $x = 0$
(iv) $x (y - x)^2 = ay^2$ এর জন্য $y = 0$.
(v) $x^2 (x^2 - 4a^2) = y^2 (x^2 - a^2)$ এর জন্য $y = \pm 2x$.
(vi) $x^4 - 2x^2 y^2 + x^2 + x - 3y = 0$ এর জন্য $x = 3y$.
(vii) $x^5 + y^5 + x^3y + x^2 y^3 - 6xy^2 = 0$ এর জন্য $x = 0, y = 0, x = 2y, x = -3y$.

35. বক্ররেখা $x^{2/3} + y^{2/3} = a^{2/3}$ এর যে কোন বিন্দুতে অংকিত স্পর্শক এবং অক্ষদ্বয়ের দ্বারা সৃষ্ট ত্রিভুজের ক্ষেত্রফল নির্ণয় কর। [R. U. H. 1966]

36. Show that in the curve $ay^4 = b^2 (cx + f)^3$ the square of the subnormal varies as the subtangent. C. H. 1989

37(a) $y (x - 1) (2x - 3) - x + 4 = 0$ এই রেখাটিতে স্পর্শক ও লম্ব বাহির কর যেখানে x-অক্ষ ছেদ করে (Find the tangent and normal to the curve $y (x - 1) (2x - 3) - x + 4 = 0$ at the point where it cuts the x- axis.) (b) $y(x-2)(x-3)-x+7=0$ N.U. 1994 C. U. 1991

উত্তরমালা

1. (i) $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$. (ii) $y - y_1 = a \cot x (x - x_1)$
(iii) $(2y_1 (x_1^2 + y_1^2) + a^2y_1)y + (2x_1 (x_1^2 + y_1^2) + a^2x_1) x = a^2 (x_1^2 - y_1^2 - y_1^2)$
2. (i) $(x - x_1) y_1 + 2a (y - y_1) = 0$
(ii) $x (ax_1 - y_1^2) + y (x_1^2 - ay_1) = (x_1 - y_1) (ax_1 + ay_1 + x_1y_1)$
(iii) $x - y = 0$.

3. (i) $(x/a) \cos \theta + (y/b) \sin \theta = 1$

by $\cos \theta$ - ax $\sin \theta + (a^2 - b^2) \sin \theta \cos \theta = 0$.

(ii) $y = (x - a\theta) \tan \theta/2; x + y \tan \theta/2 = a\theta + 2a \tan \theta/2$.

(ii) $y=(x-a\theta) \tan \theta/2$; $x+y \tan \theta/2=a\theta+2a \tan \theta/2$

(iii) $\frac{x}{a \cos \theta}+\frac{y}{b \sin \theta}=1$, $yb \sin\theta-xa \cos\theta=b^2 \sin^4\theta-a^2\cos^4\theta$

4. (i) $\pi/4$, (ii) $\pi/2$, $\tan^{-1} 3/5$
(iii) $\tan \alpha=-4\sqrt{2/7}$ (iv) $\tan \alpha=-\frac{1}{2}$

5. $cy/\sqrt{(y^2-c^2)}$; $(c/y)\sqrt{(y^2-c^2)}=9$, $aa_1x^2+bb_1y^2=0$

10. $ye^{3\pi\ \pi}=e(x+1)$; $y+xe^{2\pi\ \pi}+e=0$

22. $p^{m+n}m^mn^n=a^{m+n}(m+n)^{m+n} \cos^m\alpha \sin^n\alpha$

30. (i) $\sqrt{\frac{x+a}{x}}$ (iii) $(a/x)^{1/3}$ (iii) $\sqrt{(1+1/4x^2)}$ (iv) $y(1+x^2)$

35. $\Delta=\frac{1}{2}\alpha^{4/3}(xy)^{1/3}$

POLAR CO-ORDINATE X (B)

10. 8. Angle Between radius vector and Tangent.

Let P be any given point on the curve $r=f(\theta)$. Take any other Q on the curve very near to P.

Let the co-ordinates of P and Q be represented by (r, θ) and $(r+\Delta r, \theta+\Delta\theta)$ respectively.

Join PQ and produce. Then PQ is a secant of the curve $r=f(\theta)$ through P and Q. Draw PN perpendicular to OQ.

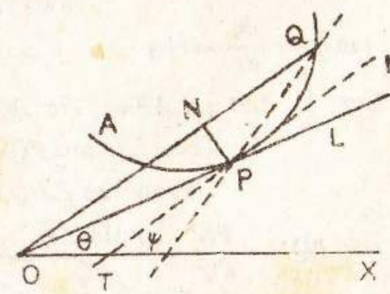


Fig No 13

If $Q\to P$, then $\angle OQP \to \angle OPT=\phi$, where ϕ is the angle between the radius vector OP and the tangent, PT at P.

We are to find ϕ. We have,

$OP=r,\ OQ=r+\Delta r$

$\angle XOQ=\theta+\Delta\theta,\quad \angle XOP=\theta;\ \angle POQ=\Delta\theta$

From $\triangle OPN$,

$PN=OP\sin\Delta\theta=r\sin\Delta\theta;\quad ON=r\cos\Delta\theta$

Now from the $\triangle PQN$,

$$\tan PQN=\frac{PN}{QN}=\frac{PN}{OQ-ON}=\frac{r\sin\Delta\theta}{r+\Delta r-r\cos\Delta\theta}$$

$$=\frac{r\sin\Delta\theta}{\Delta r+(1-\cos\Delta\theta)r}=\frac{r\sin\Delta\theta}{\Delta r+2r\sin^2\frac{1}{2}\Delta\theta}$$

or, $$\tan PQN=\frac{r(\sin\Delta\theta/\Delta\theta)}{\frac{1}{2}r\Delta\theta\left(\sin\frac{\Delta\theta}{2}\Big/\frac{\Delta\theta}{2}\right)^2+\Delta r/\Delta\theta}$$

If $Q\to P$, then $\Delta\theta\to 0$ and $\angle PQN\to\angle OPT=\phi$

$$\tan\phi=\lim_{\Delta\theta\to 0}\left[\frac{r(\sin\Delta\theta/\Delta\theta)}{\frac{1}{2}r\Delta\theta(\sin\frac{1}{2}\Delta\theta/\frac{1}{2}\Delta\theta)^2+\Delta r/\Delta\theta}\right]$$

$$=\lim_{\Delta\theta\to 0}\frac{r}{\frac{1}{2}\Delta\theta\ 1+r/\Delta\theta}=\frac{r}{dr/d\theta}=r/r_1$$

$$\tan\phi=r\frac{d\theta}{dr}=r/r_1\quad\ldots\quad\ldots\quad\ldots\quad(31)$$

Cor. 1. Let arc $AP=s$, arc $AQ=s+\Delta s$.

Therefore arc $PQ=\Delta s$

From the $\triangle PQN$,

$$\sin PQN=\frac{PN}{PQ}=\frac{r\sin\Delta\theta}{PQ}=r\frac{\sin\Delta\theta}{\Delta\theta}\cdot\frac{\Delta\theta}{\Delta s}\cdot\frac{\Delta s}{PQ}$$

If $Q\to P$, then $\Delta\theta\to 0$, $\Delta s\to 0$, $\angle PQN\to\phi$ and

$$\lim_{Q\to P}\frac{\Delta s}{PQ}=\lim_{Q\to P}\frac{\text{arc }PQ}{\text{chord }PQ}=1$$

Thus $$\sin\phi=\lim_{\Delta\theta\to 0} r\left(\frac{\sin\Delta\theta}{\Delta\theta}\right)\lim_{\Delta s\to 0}\frac{\Delta\theta}{\Delta s}$$

$$\Rightarrow\ \sin\phi=r\frac{d\theta}{ds}\quad(32)$$

Cor. 2 Similarly from the $\triangle PQN$

$$\cos PQN=\frac{QN}{PQ}=\frac{OQ-ON}{PQ}$$

$$=\frac{r+\Delta r-r\cos\Delta\theta}{PQ}=\frac{\Delta r+r\,2\sin^2\frac{1}{2}\Delta\theta}{PQ}$$

$$=\left[\frac{\Delta r}{\Delta s}+2r\cdot(\sin\tfrac{1}{2}\Delta\theta/\tfrac{1}{2}\Delta\theta)^2\frac{(\frac{1}{2}\Delta\theta)^2}{\Delta s}\right]\frac{\Delta s}{PQ}$$

In the limit, when $Q\to P$, then $\Delta s\to 0$, $\Delta\theta\to 0$ and so

$\cos\phi=[\,dr/ds+2r.1.0\,]=dr/ds$

$\cos\phi=(dr/ds)\quad\ldots\quad\ldots\quad\ldots\quad(33)$

We may get the value of $\cos\phi$ with the help of $\sin\phi$ and $\cot\phi$

$$\cos\phi=\frac{\cos\phi}{\sin\phi}\sin\phi=\cot\phi.\sin\phi$$

$$=\frac{dr}{rd\theta}r\frac{d\theta}{ds}\ \text{[by (31) and (32)}$$

$$=\frac{dr}{ds}$$

Cor. 3. We have

$$\cos\phi=\frac{dr}{ds},\ \sin\phi=r\frac{d\theta}{ds}$$

$$\therefore\ \left(\frac{dr}{ds}\right)^2+\left(r\frac{d\theta}{ds}\right)^2=\cos^2\phi+\sin^2\phi$$

or, $$\left(\frac{dr}{ds}\right)^2+\left(r\frac{d\theta}{ds}\right)^2=1$$

or, $$(dr)^2+(rd\theta)^2=(ds)^2\quad(34)$$

or, $(ds/d\theta)=\sqrt{\{r^2+(dr/d\theta)^2\}}$

provided s increases with θ.

Cor. 4. From the figure 13 we see that $\angle PTX=\psi$ is the angle made by the tangent PT with the positive direction of x axis. $\angle XOP=\theta$ is the angle by the radius vector OP with the (x-axis) initial line, $\angle TPO=\phi$ is the angle between radius vector and the tangent. We have from the figure, $\angle XTP=\angle TPO+\angle TOP.$

or, $\psi=\phi+\theta$ (35)

$\therefore$ $\tan\psi=\tan(\phi+\theta)$

or, $\tan\psi=\dfrac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}$ (36)

Alternative Method

10. 8. (a) Angle between radius vector and Tangent.

Let $P(r,\theta)$ be any point on the curve $r=f(\theta)$. Another point $Q(r+\Delta r, \theta+\Delta\theta)$ is taken on the curve very near to P.

Let PT be tangent at P which meets the initial line at T. Join PQ. Produce OP to L.

Let $\angle LPQ=\alpha$ and $\angle LPM=\phi$, the angle between the radius vector and tangent at P.

If $Q\to P$, then $\angle LPQ\to\angle LPM$ i. e, $\alpha\to\phi$.

Therefore $\angle OPQ=\pi-\alpha$

and $\angle OQP=\angle LPQ-\angle POQ=\alpha-\Delta\theta$

Now by sin Rule, from $\triangle OPQ$

$$\frac{OQ}{OP}=\frac{\sin OPQ}{\sin OQP}=\frac{\sin(\pi-\alpha)}{\sin(\alpha-\Delta\theta)}$$

or, $\dfrac{r+\Delta r}{r}=\dfrac{\sin\alpha}{\sin(\alpha-\Delta\theta)}$ or, $\dfrac{\Delta r}{r}=\dfrac{\sin\alpha}{\sin(\alpha\Delta\theta)}-1$

or, $\dfrac{r\Delta}{r}=\dfrac{\sin\alpha-\sin(\alpha-\Delta\theta)}{\sin(\alpha-\Delta\theta)}=\dfrac{2\cos(\alpha+\frac{1}{2}\Delta\theta)\sin\frac{1}{2}\Delta\theta}{\sin(\alpha-\Delta\theta)}$

or, $\dfrac{1}{r}\dfrac{\Delta r}{\Delta\theta}=\dfrac{\cos(\alpha+\frac{1}{2}\Delta\theta)}{\sin(\alpha-\Delta\theta)}\cdot\dfrac{\sin\frac{1}{2}\Delta\theta}{\frac{1}{2}(\Delta\theta)}$

If $Q\to P$, then $\alpha\to\phi$, $\Delta\theta\to 0$, thus

$$\frac{1}{r}\frac{dr}{d\theta}=\frac{\cos\phi}{\sin\phi}\cdot 1=\cot\phi.$$

or. $\tan\phi=r\dfrac{d\theta}{dr}$

10. 9. Angle of intersection of two curves

Let the two polar curves whose equations are $r=f(\theta)$, $r=\phi(\theta)$ intersect at P. Let α be the angle of their intersection. Find ϕ_1 the angle between the radius vector and the tangent of the first curve $r=f(\theta)$. Similarly ϕ_2 for the 2nd curve $r=\psi(\theta)$.

Thus the angle of intersection of two curves is given by

$\alpha=\phi_1\sim\phi_2$ (37)

If $\tan\phi_1=n_1$ and $\tan\phi_2=n_2$ then

$$\tan\alpha=\tan(\phi_1\sim\phi_2)=\frac{\tan\phi_1\sim\tan\phi_2}{1+\tan\phi_1\tan\phi_2}$$

$\therefore$ $\tan\alpha=\dfrac{n_1\sim n_2}{1+n_1n_2}$ (38)

Cor. 1. If the two curves intersect at right angles, then $\alpha=\pi/2$.

$\therefore$ $\phi_1\sim\phi_2=\frac{1}{2}\pi$ (39)

or, $n_1 n_2=-1$ (40)

10.10. Polar subtangent and Polar subnormal.

Let $r=f\ (\theta)$ be any Polar curve. O is the Pole and OX the initial line. Draw the tangent PT at P and the normal PN at P. Thus the tangent PT and normal PN meet the straight line TON through the pole at right angles to the radius vector OP.

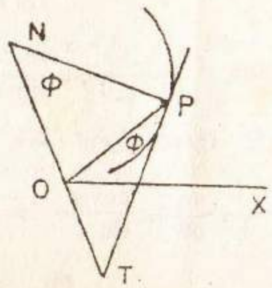


Fig. 15

Then

(i) **OT is called the polar subtangent and ON is called the polar subnormal.**

$\angle OPT=\phi$. the angle between the radius vector and the tangent at P, $OP=r$. From the $\triangle OPT$.

$$OT=OP\tan\phi=r\,\frac{r\,d\theta}{dr}\ [\text{ by (31)}] \quad \text{or,} \quad OT=r^2\,\frac{d\theta}{dr}$$

$\therefore$ **Polar subtangent** $=r^2\,\dfrac{d\theta}{dr}$ (41)

(ii) Again from the $\triangle OPN$, we have

$$ON=OP\cot\phi=r\,\frac{dr}{r\,d\theta}=\frac{dr}{d\theta}$$

$\therefore$ **Polar subnormal** $=\dfrac{dr}{d\theta}$ (42)

(iii) Length of the tangent.

$$PT^2=OT^2+OP^2=\left(r^2\frac{d\theta}{dr}\right)^2+r^2$$

or, $PT^2=(r^4/r_1^2+r^2)=\dfrac{r^2}{r_1^2}(r^2+r_1^2)$

$\therefore\ PT=\dfrac{r}{r_1}\sqrt{(r^2+r_1^2)}$ (43)

(iv) Length of the normal

$$PN^2=ON^2+OP^2=\left(\frac{dr}{d\theta}\right)^2+r^2=r_1^2+r^2$$

$\therefore\ PN=\sqrt{(r_1^2+r^2)}$ (44)

Examples

Ex. 1. Express ϕ in terms of θ, form the curve $r^2=a^2\cos 2\theta$

Take logarithm of both sides of

$$r^2=a^2\cos 2\theta$$

Then, $\log r^2=\log(a^2\cos 2\theta)=\log a^2+\log\cos 2\theta$

or, $2\log r=2\log a+\log\cos 2\theta$

$$\therefore\ \frac{2}{r}\frac{dr}{d\theta}=-\frac{2\sin 2\theta}{\cos 2\theta}=-2\tan 2\theta$$

or, $\cot\phi=-\tan 2\theta=\cot(\pi)/2+2\theta)$. $\left[\because\ r\dfrac{d\theta}{dr}=\tan\phi\right]$

$\therefore\ \phi=\pi/2+2\theta$

Lx. 2. Show that the curves $r^2\sin 2\theta=a^2$ and $r^2\cos 2\theta=b^2$ intersect orthogonally.

Consider $r^2\sin 2\theta=a^2$

Taking logarithm of both sides,

$$2\log r+\log\sin 2\theta=2\log a$$

$$\therefore\ \frac{2}{r}\frac{dr}{d\theta}+\frac{2\cos 2\theta}{\sin 2\theta}=0$$

or. $\cot\phi=-\cot 2\theta=\cot(\pi-2\theta)$

$\therefore\ \phi=\pi-2\theta$ (1)

Similarly for the 2nd curve, $r^2\cos 2\theta=b^2$

we have $\dfrac{2}{r}\dfrac{dr}{d\theta}-\dfrac{2\sin 2\theta}{\cos 2\theta}=0$. **Differentiate** $w.\ r.$ **to** θ

or, $\cot \phi_1 = \tan 2\theta$. $\therefore$ $\phi_1 = \frac{1}{2}\pi - 2\theta$.

Hence the angle of intersection of the curves is

$\angle = \phi - \phi_1 = \pi - 2\theta - \frac{1}{2}\pi + 2\theta = \frac{1}{2}\pi$.

Thus the curves intersect orthogonally (at right angles).

Ex. 3. Prove that locus of the extremity of the polar subtangent for the curve.

$$\frac{1}{r} + f(\theta) = 0 \text{ is } \frac{1}{r} = f\left(\frac{\pi}{2} + \theta\right)$$

Let the entrimity of the polar subtangent be (r_1, θ_1). then

$r_1 = OT =$ polar subtangent (see fig. 15)

$$= r^2 \frac{d\theta}{dr} \qquad (1)$$

From the figure

$$-\theta_1 = \angle XOT = \angle POT - \angle POX = \tfrac{1}{2}\pi - \theta$$

or, $\theta = \frac{1}{2}\pi + \theta_1$ (2)

From the given equation $\frac{1}{r} = -f(\theta)$

$$\therefore \quad -\frac{1}{r^2} \cdot \frac{dr}{d\theta} = -f'(\theta) \quad \text{or,} \quad r^2 \frac{d\theta}{dr} = \frac{1}{f'(\theta)}$$

or, $r_1 = \dfrac{1}{f'(\frac{1}{2}\pi + \theta_1)}$ [by (1) and (2)]

or, $\dfrac{1}{r_1} = f'(\frac{1}{2}\pi + \theta_1)$

Drop the suffixes from r and θ getting equation is

$$\frac{1}{r} = f'(\tfrac{1}{2}\pi + \theta)$$

which is the required locus.

Ex. 4. Show that the tangents to the cardioide $r = a(1 + \cos\theta)$ at the points whose vectorial angles are $\pi/3$ and $2\pi/3$ are respectively parallel and perpendicular to the initial line.

Let the angle between the radius vector and the tangent at (r, θ) be ϕ, then

$$\tan\phi = r\frac{d\theta}{dr} = \frac{r}{(dr/d\theta)}$$

$$= a(1 + \cos\theta) \times \frac{1}{-a\sin\theta}$$

$$= -\cot\tfrac{1}{2}\theta = \tan(\tfrac{1}{2}\pi + \tfrac{1}{2}\theta)$$

$\therefore$ $\phi = \frac{1}{2}\pi + \frac{1}{2}\theta$.

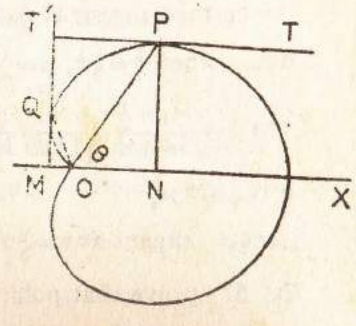


Fig. 16

If the tangent QT is parallel to the initial line OX then

$\phi = \pi - \theta \Rightarrow \frac{1}{2}\pi + \frac{1}{2}\theta = \pi - \theta$

or, $3\theta/2 = \frac{1}{2}\pi$ or, $\theta = \frac{1}{3}\pi$

Hence the tangent is parallel to the initial line at $\theta = \pi/3$

Let the tangent be perp. to the initial line at θ, when

$\phi = \angle OQT' = \angle QMO + \angle QOM = \frac{1}{2}\pi + \pi - \theta = 3\pi/2 - \theta$

$\Rightarrow \frac{1}{2}\pi + \frac{1}{2}\theta = 3\pi/2 - \theta$ or, $3\theta/2 = \pi$ or, $\theta = \frac{2}{3}\pi$

Hence the tangent at $\theta = \frac{2}{3}\pi$ is perp to the initial line

Alternative method

Let ψ be the angle made by the tangent with the x-axis (here initial line). Then $\psi = \theta + \phi$; $r = a(1 + \cos\theta)$.

$$\therefore \quad \frac{1}{r} \frac{dr}{d\theta} = \frac{-a\sin\theta}{a(1+\cos\theta)} = -\tan\tfrac{1}{2}\theta$$

or, $\cot\phi = \cot(\frac{1}{2}\pi + \frac{1}{2}\theta)$ $\therefore$ $\phi = \frac{1}{2}\pi + \frac{1}{2}\theta$

Now $\tan\psi = \tan(\theta + \phi) = \dfrac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$

At $\theta=\frac{1}{3}\pi$. then $\phi=\frac{1}{2}\pi+\frac{1}{2}\cdot\frac{1}{3}\pi=\frac{2}{3}\pi$

$$\tan\psi=\frac{\tan\frac{1}{3}\pi+\tan\frac{2}{3}\pi}{1-\tan\frac{1}{3}\pi\tan\frac{2}{3}\pi}=\frac{\sqrt{3}-\sqrt{3}}{1+3}=0 \quad \therefore \quad \psi=0$$

Hence the tangent is parallel to the initial line at $\theta=\pi/3$.

Again when $\theta=\frac{2}{3}\pi$, $\phi=\frac{1}{2}\pi+\frac{1}{2}\cdot\frac{2}{3}\pi=5\pi/6$

$$\tan\psi=\frac{\tan\frac{2}{3}\pi+\tan 5\pi/6}{3-\tan\frac{2}{3}\pi\tan 5\pi/6}=\frac{-\sqrt{3}-1/\sqrt{3}}{1-\sqrt{3}(1/\sqrt{3})}=\infty$$

$\therefore \quad \psi=90^\circ$,

Hence tangent at $\theta=\frac{2}{3}\pi$ is perp. to the initial line.

Ex. 5. prove that polar subtangent and the polar subnormal of the equiangular spiral $r=e^{a\theta}$ varies as the radius vector r.

Here $r=e^{a\theta}$ $\quad\therefore\quad \dfrac{dr}{d\theta}=ae^{a\theta}=ar$

Polar subtangent $r^2\dfrac{d\theta}{dr}=r^2\dfrac{1}{ar}=\dfrac{r}{a}$

$\therefore$ Polar subtangent varies as r as a is constant.

Polar subnormal$=\dfrac{dr}{d\theta}=ar$

$\therefore$ Polar subnormal varies as r as a is constant.

Exercise X (B)

1. Express ϕ in terms of θ for the following curves

(i) $r=a\theta$ (ii) $2a=r(1-\cos\theta)$ (iii) $r=a(1-\cos\theta)$

(iv) $r^2=\sin 2\theta$ (v) $r=ae^{\theta\cot\alpha}$ (vi) $r^n=a^n\cos n\theta$ C. U. 1983

2. Find the angle of intersection of the following curves

(i) $r=a(1+\cos\theta)$, $r=a(1-\cos\theta)$ (ii) $r=2\cos\theta$, $r\cos\theta=-\frac{1}{2}$

(iii) $r=4(1+\sin\theta)$, $r=3(1-\sin\theta)$ (iv) $r=1+\sin\theta$, $r=1-\sin\theta$

(v) $r^n=a^n\cos n\theta$, $r^n=a^n\sin n\theta$

3. Show that the following curves intersect orthogonally (at right angles).

(i) $r=a\sec^2\theta/2$, $r=b\operatorname{cosec}^2\theta/2$

(ii) $r^m=a^m\cos m\theta$, $r^m=b^m\sin m\theta$

(iii) $r=a(1+\cos\theta)$, $r=b(1-\cos\theta)$

(iv) $r^2=a^2\cos 2\theta$; $r^2=a^2\sin 2\theta$.

4. Show that polar subtangent is of constant length for the curve. $r\theta=a$

5. Show that in the equiangular spiral $r=ae^{\theta\cot a}$ the tangent is inclined at a constant angle to the radius vector.

6. Show that for the curve $r=a\theta$ the polar subnormal is constant and for the curve $r\theta=a$ the polar subtangent is constant a being a constant.

7. Prove that locus of the extremity of the polar subnormal of the curve $r=f(\theta)$ is $r=f'(\theta-\frac{1}{2}\pi)$

8. Show that the curves

$r^n = a^n \sec(n\theta + \alpha)$ and $r^n = b^n \sec(n\theta + \beta)$

intersect at an angle $(\alpha - \beta)$

9. Show that the locus of the extremity of the polar subnormal of the equiangular spiral $r = ae^{m\theta}$ is another equiangular spiral.

10. Show that Polar subtangent for the curve,

$r = a\ (1+\cos\theta)$ is $-\dfrac{2a \cos^3 \theta/2}{\sin \theta/2}$

11. Prove that

$$\frac{ds}{d\theta} = a(\sec n\theta)^{(n-1)/n}$$

for the curve $r^n = a^n \cos n\theta$

12. For the curve $r^n = a^n \cos n\theta$, prove that

$$\alpha^{2n}.\ \frac{d^2r}{ds^2} + nr^{2n-1} = 0$$

13. Show that tangent drawn at the extremities of any chord of the cardioide $r = a(1+\cos\theta)$ which passes through the pole are perpendicular to each other.

14. Show that in the curve $r^5 = a(1-\cos\theta)$ the radius vector of the pt. whose vectorial angle is $2\cos^{-1}\dfrac{5}{\sqrt{26}}$ makes an angle of 45° with the tangent at the point. [D. U. 1954]

15. Find the polar subtangents, polar subnormals, length of the tangent, length of normal of the following curves.

(i) $r = -6 \sin\theta$ at $\pi/3$

(ii) $r = 2 \sec\theta$ at $\pi/4$

(iii) $r = 5 + 2\sin\theta$ at $\pi/6$

(iv) $r^2 = 4\cos\theta$ at $\pi/6$

(v) $r = \dfrac{4}{1+\sin\theta}$ at $\pi/4$.

16. Show that the locus in polar co-ordinates of the intersection of two perpendicular tangents to the curve

$x = a\cos^3\phi,\ y = a\sin^3\phi$ is

$r^2 = \frac{1}{2}a^2\cos^2 2\theta.$

প্রশ্নমালা X (B)

1. নিম্নলিখিত বক্ররেখাগুলির জন্য ϕ কে θ-এর মাধ্যমে প্রকাশ কর।

(i) $r = a\theta$ (ii) $2a = r(1-\cos\theta)$ (iii) $r = a(1-\cos\theta)$

(iv) $r^2 = \sin 2\theta$ (v) $r = ae^{\theta \cot x}$ (vi) $r^n = a^n \cos n\theta$

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2. নিম্নলিখিত বক্ররেখাগুলির ছেদক কোণ নির্ণয় কর।

(i) $r = a(1+\cos\theta)$, $r = a(1-\cos\theta)$ (ii) $r = 2\cos\theta$, $r\cos\theta = -\frac{1}{2}$. (iii) $r = 4(1+\sin\theta)$, $r = 3(1-\sin\theta)$

(iv) $r = 1+\sin\theta$, $r = 1-\sin\theta$

(v) $r^n = a^n \cos n\theta$, $r^n = a^n \sin n\theta$

3. দেখাও যে নিম্নলিখিত বক্ররেখাগুলি পরস্পরকে লম্বভাবে ছেদ করে।

(i) $r=a\sec^2\theta/2,\quad r=b\,\mathrm{cosec}^2\theta/2$

(ii) $r^m=c^m\cos m\theta,\quad r^m=b^m\sin m\theta$

(iii) $r=a(1+\cos\theta),\quad r=b(1-\cos\theta)$

(iv) $r^2=a^2\cos 2\theta\,;\quad r^2=a^2\sin 2\theta$.

4. দেখাও যে বক্ররেখার $r\theta=a$ এরজন্য পোলার উপস্পর্শকের দৈর্ঘ্য ধ্রুব থাকে।

5. দেখাও যে সমান-কৌণিক কুণ্ডলী $r=ae^{\theta\cot\alpha}$ (eqiangular spiral) এর যে কোন বিন্দুতে স্পর্শক ঘূর্ণায়মান রেখার সহিত সর্বদা সমান কোণ উৎপন্ন করে।

6. যদি 'a' একটি ধ্রুব সংখ্যা হয় তবে দেখাও যে $r=a\theta$ বক্ররেখার পোলার উপলম্ব এবং $r\theta=a$ রেখায় পোলার উপ-স্পর্শকের দৈর্ঘ্য ধ্রুব থাকে।

7. প্রমাণ কর যে $r=f(\theta)$ বক্ররেখার যে কোন বিন্দুতে পোলার উপলম্বের প্রান্তবিন্দুর সঞ্চার পথ $r=f'(\theta+\pi/2)$.

8. দেখাও যে $r^n=a^n\sec(n\theta+\alpha)$ এবং $r^n=b^n\sec(n\theta+\beta)$ বক্ররেখাদ্বয়ের মধ্যে কোন হইল $(\alpha-\beta)$.

9. দেখাও যে সমানকৌণিক কুণ্ডলী $r=ae^{m\theta}$ এর যে কোন বিন্দুতে পোলার উপলম্বের প্রান্তবিন্দুর সঞ্চার পথ অপর একটি সমান-কৌণিক কুণ্ডলী (Equiangular spiral).

10. দেখাও যে $r=a(1+\cos\theta)$ বক্ররেখার পোলার উপস্পর্শক হইবে

$$-\frac{2a\cos^3(\theta/2)}{\sin(\theta/2)}$$

11. প্রমাণ কর যে $r^n=a^n\cos n\theta$ এর জন্য

$$\frac{ds}{d\theta}=a(\sec n\theta)^{\frac{n-1}{n}}$$

12. বক্ররেখা $r^n=a^n\cos n\theta$, এর জন্য প্রমাণ কর যে

$$a^{2n}\cdot\frac{d^2r}{ds^2}+nr^{2n-1}=0$$

13. দেখাও যে মেরুবিন্দু দিয়া অতিক্রান্ত কার্ডিওইড $r=a(1+\cos\theta)$ এর কোন জ্যা-এর প্রান্তবিন্দুদ্বয়ে অংকিত স্পর্শক পরস্পর লম্ব হইবে।

14. দেখাও যে বক্ররেখা $r^3=a(1-\cos\theta)$ এর যে বিন্দুর ভেক্টর কোন $\cos^{-1}\dfrac{5}{\sqrt{26}}$, সে বিন্দুতে ঘূর্ণায়মান রেখা এবং ঐ বিন্দুতে স্পর্শক-এর মধ্যে কোণ হইবে 45°. D. U. 1954

15. নিম্নলিখিত বক্ররেখাগুলির পোলার উপস্পর্শক, পোলার উপলম্ব, স্পর্শকের দৈর্ঘ্য, অভিলম্বের দৈর্ঘ্য নির্ণয় কর।

(i) $r=-6\sin\theta$ এর $\pi/3$ বিন্দুতে।

(ii) $r=2\sec\theta$ এর $\pi/4$ বিন্দুতে।

(iii) $r=5+2\sin\theta$ এর $\pi/6$ বিন্দুতে।

(iv) $r^2=4\cos\theta$ এর $\pi/6$ বিন্দুতে।

(v) $r=\dfrac{4}{1+\sin\theta}$ এর $\pi/4$ বিন্দুতে।

16. দেখাও যে বক্ররেখা $x=a\cos^3\phi,\ y=a\sin^3\phi$ এর উপর অংকিত দুইটি স্পর্শক পরস্পরকে যদি লম্বভাবে ছেদ করে তবে ছেদবিন্দুর পোলার স্থানাংকে সঞ্চারপথ হইবে $r^2=\frac{1}{2}a^2\cos^2 2\theta$.

ANSWERS

1. (i) $\tan\phi=\theta$, (ii) $-\theta/2,\ \pi-\theta/2$, (iii) $\theta/2$, (iv) 2θ, (v) α.

2. (i) $\pi/2$ (ii) 2θ (iii) $\pi/2$ (iv) $\pi/2$ (v) $\pi/2$.

15. (i) $-9, -3, \sqrt{108}, 6$, (ii) $2\sqrt{2}, 2\sqrt{2}, 4, 4$.

(iii) $12\sqrt{3},\ \sqrt{3},\ 6\sqrt{13},\ \sqrt{39}$

(iv) $\sqrt[4]{3},\ \sqrt[4]{3},\ 1/\sqrt{(2/\sqrt[4]{3})}\ 26\sqrt{3},\ \left(\dfrac{13}{\sqrt{3}}\right)^{1/2}$

(v) $-4\sqrt{2},\quad \dfrac{-4\sqrt{2}}{3+2\sqrt{2}}$, 6.15, 3 2 প্রায়।

PEDAL EQUATIONS X (C)

10. 11. Pedal equation :—The relation between p and r for a given curve is called the pedal equation of the curve, p is the length of the perpendicular from the pole or origin to the tangent and r is the distance of any point on the curve from the pole. It is denoted by $f(p, r)=0$.

10. 12. The Pedal equation of a curve given by $r=f(\theta)$.

Let O be the pole and OA the initial line. Let PT be the tangent at P(r, θ) to the curve $r=f(\theta)$. The radius vector $OP=r$.

Draw OT perpendicular to the tangent PT

Let $OT=p$, $\angle OPT=\phi$

From the right angled $\triangle OPT$,

$OT=OP \sin OPT=r \sin\phi$

or, $P=r \sin\phi$...... (45)

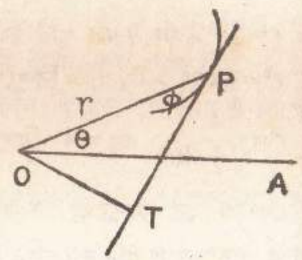


Fig. 17

or, $\dfrac{1}{p}=\dfrac{1}{r}\operatorname{cosec}\phi$ or, $\dfrac{1}{p^2}=\dfrac{1}{r^2}\operatorname{cosec}^2\phi=\dfrac{1}{r^2}(1+\cot^2\phi)$

$$=\frac{1}{r^2}\left\{1+\left(\frac{dr}{rd\theta}\right)^2\right\}\left[\because\ \tan\phi=\frac{rd\theta}{dr}\right]$$

$$\text{or,}\quad \frac{1}{p^2}=\frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \qquad (46)$$

If we put $u=\dfrac{1}{r}$, then $\dfrac{du}{d\theta}=-\dfrac{1}{r^2}\dfrac{dr}{d\theta}$

and hence

$$\frac{1}{p^2}=u^2+\left(\frac{du}{d\theta}\right)^2 \qquad (47)$$

By eliminating θ from (46) or (47) with the help of $r=f(\theta)$, the required pedal equation is obtained.

10. 13. To determine the pedal equation of a curve whose cartesian equation is given by $f(x, y)=0$.

The equation of the tangent at (x, y) on the curve $f(x, y)=0$ is

$$Y-y=\frac{dy}{dx}(X-x),\ \text{or,}\ Y-y-(X-x)\frac{dy}{dx}=0$$

The perpendicular distance from the origin (0, 0) to the tangent is

$$p=\frac{-y+x(dy/dx)}{\sqrt{\{1+(dy/dx)^2\}}}=\frac{xy_1-y}{\sqrt{(1+y_1^2)}}$$

Also $r^2=x^2+y^2$ ... ... (ii)

and $f(x, y)=0$ ... ... (iii)

Now eliminating x, y from (i), (ii) and (iii) we get a relation between p and r. This relation is the required pedal equation. See example 3,

Pedal Curves

10. 14. First Positive Pedal. If a perpendicular is drawn from a fixed point on a movable tangent to a given, the locus of the foot of the perpendicular is called the First Positive pedal of the orginal curve $w.\ r.$ to the given point.

The pedal of the first positive pedal is called the 2nd positive pedal and so on.

P is a point on the curve APQ. A tangent PT is drawn at P on the curve APQ and OT is perpendicular from O to the tangent PT.

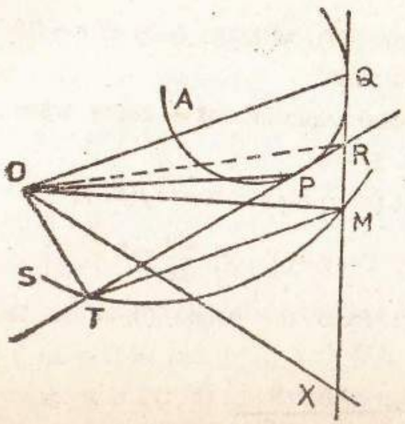


Fig. 18

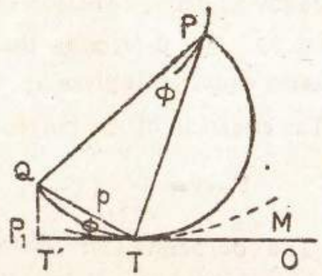


Fig. 19

T is the foot of the perpendicular from the origin to tangent at P of curve APQ. Similarly we get feet of the perpendiculars from the origin to the tangents drawn at the points on the curves APQ. All feet of the perpendiculars will lie on a curve STM

The curve STM is called the pedal curve of the curve APQ.

Consider two adjacent points P and Q on the curve APQ. Draw tangents PT and QM at P and Q respectively of the curve. Draw OT and OM perpendiculars from O to the tangents PT and QM respectively. The two tangents PT and QM intersect at R. Join OR and TM is a chord of the circle described on OR as diameter. P, Q, R coincide, when the chord TM becomes a tangent to the circle described on OP as diameter at a point where the circle cuts the tangent PT, i e, TM is tangent to the circle as well as the pedal curve at T.

Let OT_1 be perpendicular drawn from O the tangent T_1TM i, e, on first Pedal.

$$\angle OPT=\phi, \text{ then } \angle OTT_1=\phi$$

Let $OP=r$, $OT_1=p_1$, $OT=p$

From $\triangle OTT_1$, $OT_1=OT\sin\phi$ or, $p_1=p\sin\phi$

From $\triangle OTP$, $OT=OP\sin\phi$ or $p=r\sin\phi$

Therefore, from the above two equations. we get a relation between p and p_1.

$$\therefore\ p^2=p_1 r \text{ or, } r=p^2/p_1$$

(If $f(p, r)=0$ be the equation of the original curve, we will get a relation between p and p_1 by putting the value fo r in $f(p, r)=0$ i, e, $f(p, p^2/p_1)=0$.

$f(p, p^2/p_1)=0$ is the pedal equation of the pedal curve

put $p=r$, $p_1=r^2p$ in the pedal equation,

then $f(r, r^2/p)=0$

is the pedal curve of the pedal equation $f(p, p^2/p_1)=0$

Working Rule for determining pedal curve.

Find the pedal equation of the given curve say $f(p, r)=0$ Now put $p=r$, and $r=r^2/p$ in the pedal equation $f(p, r)=0$, to get $f(r, r^2/p)=0$ which is the pedal curve of the original pedal equation.

$f(r, r^2/p)=0$ is called positive pedal of the original equation. Similarly by applying the above rule we may get 1st positive pedal, 2nd positive pedal, 3rd positive pedal and so on,

Ex. Given the pedal equation of the circle $ap=r^2$. Determine the 4th positive pedal.

The pedal equation of the circle is

$$f(p, r)=ap-r^2=0 \quad \cdots \quad \cdots \quad \cdots \quad (1)$$

For the 1st positive pedal

put $p=r$, and $r=r^2/p$ in (1)

Then $f(r, r^2/p)=ar-(r^2/p)^2=0$ or, $p^2a=r^3$ ... (2)

The equation of the Ist positive pedal of (1)

$r^2a=(r^2/p)^3$ or, $p^3a=r^4$ ... ... (3)

Similarly, 3rd positive pedal is $p^4a=r^5$

Therefore the 4th positive pedal is $p^5a=r^6$

10. 15. To determine the positive pedal w, r, to the origin of any curve where cartesian equation is given.

Let the equation of the curve be $f(x, y)=0$ ... ... (1)

Let $P=X\cos a+Y\sin a$ (2)

be only straight line which touches the curve.

But the equation of the tangent to the curve $f(x, y)=0$ at (X, Y)

is $X\frac{\delta f}{\delta x}+Y\frac{\delta f}{\delta y}+Z\frac{\delta f}{\delta z}=0$ (3) see Art. 10.2 (iii)

If (2) and (3) are identical then their co-efficient will be proportional,

Hence $\frac{\delta f}{\delta x}\Big| \cos \alpha=\frac{\delta f}{\delta y}\Big| \sin \alpha=\frac{\delta f}{\delta z}\Big| (-p)=\lambda$ (say) ... (4)

∴ From the above four equation

we can eliminate x, y and λ get the result in terms of p and α.

p and α are the polar co-ordinates of the foot of the perpendicular.

If we replace p by r and α by θ then we get the polar equation

which is the required locus of the foot of the perpendicular. The locus is called the first positive pedal of the given curve w, r. to the origin.

Ex. Show that the first positive pedal of the parabola $y^2=4ax$ w. r to the vertex is $x(x^2+y^2)+ay^2=0$

$$y^2=4ax \quad \cdots \quad \cdots \quad \cdots \quad (1)$$

The equation to the tangent at (X, Y) of the curve (1) is

$Yy=2a(X+x)$ or, $2aX-Yy+2ax=0$ ... ... (2)

Let $p=X\cos\alpha+Y\sin\alpha$ ... ... (3)

be any straight line which touches the curve (1)

Thus two straight lines are identical if their co-efficients are proportional. Compare the co-efficients of these equations. Then

$\frac{\cos\alpha}{2a}=\frac{\sin\alpha}{-y}=\frac{-p}{2ax}$ or, $y=-2a\tan a$, $x=-p\sec\alpha$

and $4a^2\tan\alpha=-4ap\sec\alpha$

Now from (1) we have

$4a^2\tan^2\alpha=-4ap\sec\alpha$ or, $a\sin^2\alpha+p\cos\alpha=0$ ... ... (4)

Now replace α by θ and p by r then the locus is

$a\sin^2\theta+r\cos\theta=0$ or, $a.\ y^2/r^2+x=0$ ∴ $r^2=x^2+y^2$

or, $ay^2+xr^2=0$ or, $ay^2+x(x^2+y^2)=0$ ∴ $x=r\cos\theta$, $y=r\sin\theta$

Thus the first positive pedal w. r. to the vertex is

$x(x^2+y^2)+ay^2=0.$

***10. 16. Determine the first positive pedal w, r, to the pole of any curve whose polar equation is given.**

Let the equation to the curve be $f(r, \theta)=0$ ... ... (1)

Let PT be the tangent to the curve at $P(r, \theta)$

OQ is drawn perpendicular to the tangent. Let $Q(r_1, \theta_1)$ be the foot of the perpendicular. Let OX be the initial line.

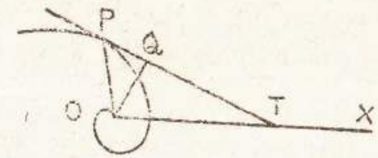


Fig. 20

$\triangle XOP=\theta$. $\triangle OPT=\phi$, $PTX=\psi$

$\therefore \quad \theta=\angle XOP=\angle XOQ+\angle POQ$

$\theta=\theta_1+\frac{1}{2}\pi-\angle OPQ=\theta_1+\frac{1}{2}\pi-\phi$ or, $\theta=\theta_1+\frac{1}{2}\pi-\phi$ ... ... (2)

আবার আমরা জানি, $\tan\phi=r\dfrac{d\theta}{dr}$ ... ... (3)

$\triangle OPQ$ হইতে $OQ=OP\sin\angle OPQ=r\sin\phi$

or, $r_1=r\sin\phi$ ... ... (4)

or, $\dfrac{1}{r_1^2}=\dfrac{1}{r^2}+\dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2$ ... ... (5)

From the equations (1), (2), (3), (4) or, (5) we can eliminate r, θ, ϕ and the result will be in terms of r_1 and θ_1.

The dashes may be dropped and the required locus will be obtained in r and θ only.

The required locus is called the first positive pedal (pedal curve) w. r. to the pole.

Ex. Show that the equation of first positive pedal of $r^n=a^n\cos n\theta$ w. r. to the pole is

$$r^{\frac{n}{n+1}}=a^{\frac{n}{n+1}}\cos\frac{n}{n+1}\theta.$$

$r^n=a^n\cos n\theta$ ... ... (1)

or, $n\log r=n\log a+\log\cos n\theta$

$n.\dfrac{1}{r}\dfrac{dr}{d\theta}=0+\dfrac{1}{\cos n\theta}.(-\sin n\theta).\ n=-n\tan n\theta$

or, $\cot\phi=-\tan n\theta=\cot(\pi/2+n\theta)$

$\therefore \quad \phi=\pi/2+n\theta$ ... ... (2)

But $\theta=\theta_1+\pi/2-\phi=\theta_1+\pi/2-\pi/2-n\theta=\theta_1-n\theta$

$\therefore \quad (n+1)\theta=\theta_1$ or, $\theta=\dfrac{\theta_1}{n+1}$ ... ... (3)

we know $r_1=r\sin\phi=\sin(\pi/2+n\theta)$

$=r\cos n\theta=a(\cos n\theta)^{1/n}.\cos n\theta$ by (1)

$\therefore \quad r_1=a(\cos n\theta)^{\frac{n+1}{n}}$

But $(r_1)^{\frac{n}{n+1}}=a^{\frac{n}{n+1}}\cos\dfrac{n\theta_1}{n+1}$

or, $(r_1)^{\frac{n}{n+1}}=a^{\frac{n}{n+1}}\cos\left(\dfrac{n\theta_1}{n+1}\right)$ by (3)

Now make r_1 and θ_1 as current, then the required first positive pedal is

$$r^{\frac{n}{n+1}}=a^{\frac{n}{n+1}}\cos\frac{n\theta}{n+1} \text{ Proved}$$

*10.17. Negative pedals

Let $C, C_1, C_2, C_3, C_4, C_5$ ... ...C_n be a series of curves. C_1 is called the first positive pedal of C, C_2 called the 2nd positive pedal of C. Similarly C_5 is called the 5th positive pedal of C.

Now if we consider C_4 as the original curve, then C_5 is called the first positive pedal, C_6 the 2nd positive pedal of C_4.

The curve C_3 is called the first negative pedal of C_4, C_2 the 2nd negative pedal of C_4, C_1, the 3rd negative pedal of C_4, C the 4th negative pedal of C_4. In this way we get negative pedal from a series of curves. see Ex. 4 and Ex. 5.

Working Rule for determining negative pedals

Put $r=p$ and $p=p^2/r$ in the given pedal equation $f(p, r)=0$ i. e. $f(r, p^2/r)=0$ is the first negative pedal. Repeat this process for the 2nd negative pedal, 3rd negative pedal and so on.

*Ex. Determine the pth negative pedal of the curve $p^4a=r^5$

Put $r=p$ and $p=p^2/r$ in $p^4a=r^5$

The first negative pedal is $(p^2/r)^4 a=p^5$ or, $p^3a=r^4$

2nd negative pedal is $(p^2/r)^3 a=p^4$ or, $p^2a=r^3$

3rd negative pedal is $(p^2/r)^2 a=p^3$ or, $pa=r^2$

The nth negative pedal is $p^{n-4}=ar^{n-5}$

***10. 18. Inverse curve**

Let P be a point on a curve and O be the origin. Another point Q is taken on OP such that OP. OQ$=a$ constant, say k^2.

The locus of Q is called the inverse of the curve along which P moves, with respect to a circle of radius k and centre O.

***(a) To find the inverse of a given curve whose cartesian equation is given.**

Let $P\,(x, y)$ be the co-ordinates of any point on the curve APC $f(x\ y)=0$. Let Q (x', y') be another point on OP such that OP. OQ$=k^2$ ... ... (1)

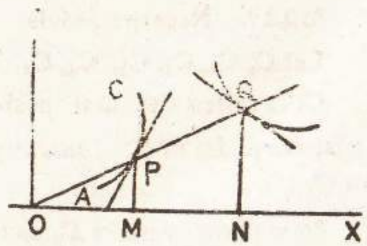


Fig. 21

PM and QN perpendicular are drawn from P and Q on OX respectively.

Ex. 1. Find the polar reciprocal of the hyperbola

Now from $\triangle$OPM and $\triangle$OQN

$$\frac{x}{x'}=\frac{\text{OM}}{\text{O}N}=\frac{\text{O}P}{\text{O}Q}=\frac{\text{O}P.\text{O}Q}{\text{O}Q^2}=\frac{k^2}{\text{OQ}^2}\ [\text{ by (1)}]$$

$$\text{or, }\ x^2=\frac{k^2x'}{\text{OQ}^2}=\frac{k^2x'}{x'^2+y'^2}\ \text{as}\ [\ \therefore\ \ \text{OQ}^2=\text{ON}^2+N\text{Q}^2\ ]$$

$$\text{Similarly } y=\frac{k^2y'}{x'^2+y'^2}$$

The equation of the given curve is $f(x.\ y)=0$.

Replace x and y by their new values.

$$\text{or, }\ f\left(\frac{k^2x'}{x'^2+y'^2},\ \frac{k^2y'}{x'^2+y'^2}\right)=0\ \cdots\ \ldots\ \ldots\ (2)$$

The locus of Q is another curve which is obtained by removing dashes from eq. (2)

$$i\ e. f\left(\frac{k^2x}{x^2+y^2},\ \frac{k^2y}{x^2+y^2}\right)=0.$$

Working Rule : The inverse of a curve is obtained by replacing x by $\frac{k^2x}{x^2+y^2}$ and y by $\frac{k^2y}{x^2+y^2}$ in the cartesian equation of the curve.

(b) Polar equation :—The inverse of a curve is obtained by replacing r by k^2/r in the polar equation $f(r,\ \theta)=0$ of the curve i. e.

$$f\left(\frac{k^2}{r},\ \theta\right)=0$$

(c) Inverse of a curve when its **pedal equation is given**

Let $p=f(r)$ be the pedal equation of a curve

The pedal equation of the inverse of a curve whose equation is $y=f(r)$ is given by

$$p=f\left(\frac{k^2}{r}\right)$$

(d) The polar reciprocal of a curve is the **inverse of its pedal.**

*** 10. 19. Polar Reciprocal**

(A) **Polar reciprocal of a curve w. r. to a given circle.**

Let OP be the perpendicular from the pole, O to a tangent to the curve, a point Q is taken on OP or OP produced such that

$OP.OQ$=constant$=k^2$ (say)

The locus of Q is called the polar reciprocal of the given curve *w. r.* to a circle of radius k and centre at O.

The polar reciprocal of a curve is the inverse of its First pedal. The equation of the polar reciprocal of a given curve is obtained by finding of its pedals.

Working Rule :--Find the condition that $p=x\cos\alpha+y\sin\alpha$ touches the given curve. Then replace p by k^2/r and α by θ from the condition.

The result will be in terms of r and θ is the required polar reciprocal *w. r.* to the circle of radius k and centre at the origin.

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with regard to circle of radius k and the centre the origin of the curve.

Let $p=x\cos\alpha+y\sin\alpha$ ... ... (1)

touch the conic $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ ... ... (2)

The equation of tangent at (x_1, y_1) to (2) is

$$\frac{xx_1}{a^2}-\frac{yy_1}{b^2}=1 \quad \dots \quad \dots \quad \dots \quad (3)$$

If (1) and (3) are identical, then their co-efficients will be proportional. Then compare their co-efficients. Thus

$$\frac{x_1}{a^2\cos\alpha}=-\frac{y_1}{b^2\sin\alpha}=\frac{1}{p}$$

$$\therefore \quad x_1=\frac{a^2}{p}\cos\alpha, \; y_1=-\frac{b^2\sin\alpha}{p}$$

The required condintion of tangency of line (1) is

$$p=\frac{a^2\cos^2\alpha}{p}-\frac{b^2\sin^2\alpha}{p} \text{ or, } p^2=a^2\cos^2\alpha-b^2\sin^2\alpha$$

Now remove p by $\frac{k^2}{r}$ and α by θ

$$\therefore \quad \frac{k^4}{r^2}=a^2\cos^2\theta-b^2\sin^2\theta$$

or, $k^4=a^2r^2\cos^2\theta-b^2r^2\sin^2\theta$ ... ... (4)

which is the required polar reciprocal of the curve, (2)

If $x=r\cos\theta$, $y=r\sin\theta$, then (4)

becomes $k^4=a^4x^2-b^4y^2$ or, $a^2x^2-b^2y^2=k^4$.

which is also the polar reciprocal of the hyperbola *w. r.* to the circle of radius k and centre at the origin.

***10.20. (B) Polar Reciprocal with respect to a given conic.**

Let $S=0$ be any curve and F=0 be the given conic. The locus of the poles *w. r.* to the conic F=0 of tangents to $S=0$ is called the polar reciprocal of the curve $S=0$ *w. r.* to the conic F=0.

Let $p=X\cos\alpha+Y\sin\alpha$ ... ... (1)

touch the curve $S=0$ and the condition of tangency is in terms of p and α i. e. $p=f(\alpha)$ ... ... (2)

Let (x, y) be the pole of the tangent (1) *w. r.* to the conic F=0. The polar of the conic F=0 *w. r.* to the pole (x, y) is

$XF_x+YF_y+ZF_z=0$ (3) $\therefore$ (Z=1)

This polar must coincide with the tangent (1) and their co-efficients are proportional. Now compare their co-efficients,

$$\frac{\cos\alpha}{F_x} = \frac{\sin\alpha}{F_y} = -\frac{p}{F_z} \text{ or, } \frac{\cos\alpha}{p} = -\frac{F_x}{F_z},\ \frac{\sin\alpha}{p} = -\frac{F_y}{F_z}$$

$$\therefore \frac{\cos^2\alpha}{p^2}+\frac{\sin^2\alpha}{p^2} = \frac{F_x^2+F_y^2}{F_z^2} \text{ or, } \frac{1}{p^2} = \frac{F_x^2+F_y^2}{F_z^2}$$

Also $\tan\alpha = \dfrac{F_y}{F_x}$

Put them in (2), then

$$\sqrt{\left\{\frac{F_x^2+F_y^2}{F_z^2}\right\}^{-1}} = f\left\{\tan^{-1}\frac{F_y}{F_x}\right\}^2$$

or, $F_x^2+F_y^2 = F_z^2\{f\tan^{-1} F_y/F_x\}^2$

which is required polar reciprocal w. r. to the conic $F=0$

see Ex. 7

Examples

Ex. 1. Show that the pedal equation of the ellipse

$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is C.H. 1992

$\dfrac{b^2}{p^2}=\dfrac{2a}{r}-1$ R. U. 1960, D. U. H. 1963, C. H. 1965.

we know the polar equation of the ellipse by taking one of the focus as pole is $l/r=1+e\cos\theta$ ... ... (1)

From (1), we have $\log l - \log r = \log(1+e\cos\theta)$

$$\therefore 0-\frac{1}{r}\cdot\frac{dr}{d\theta} = -\frac{e\sin\theta}{1+e\cos\theta} \text{ or, } \cot\phi = \frac{e\sin\theta}{1+e\cos\theta} \cdots (3)$$

$$\left[\therefore \frac{1}{r}\frac{dr}{d\theta} = \cot\phi\right]$$

But we know, $p = r\sin\phi$

$$\text{or, } \frac{1}{p^2} = \frac{1}{r^2\sin^2\phi} = \frac{\operatorname{cosec}^2\phi}{r^2} = \frac{1}{r^2}(1+\cot^2\phi)$$

$$= \frac{1}{r^2}\left[1+\frac{e^2\sin^2\theta}{(1+e\cos\theta)^2}\right] = \frac{1+2e\cos\theta+e^2(\sin^2\theta+\cos^2\theta)}{r^2(1+e\cos\theta)^2}$$

$$\text{or, } \frac{1}{p^2} = \frac{2+2e\cos\theta+e^2-1}{r^2(2+e\cos\theta)^2} = \frac{1}{r^2}\left[\frac{2(1+e\cos\theta}{(1+e\cos\theta)^2}+\frac{e^2-1}{(1+e\cos\theta)^2}\right]$$

$$= \frac{1}{r^2}\left[\frac{2}{1+e\cos\theta}+\frac{e^2-1}{1+e\cos\theta)^2}\right] = \frac{1}{r^2}\left[\frac{2}{l/r}+\frac{e^2-1}{l^2/r^2}\right]$$

$$= \frac{1}{r^2}\left[\frac{2r}{l}+\frac{r^2}{l^2}(e^2-1)\right] = \frac{1}{r^2}\left[\frac{2ra}{b^2}+\frac{r^2a^2}{b^4}(e^2-1)\right]$$

$$= \frac{2a}{b^2r}+\frac{a^2}{b^4}\left[-\frac{b^2}{a^2}-1\right] = \frac{2a}{b^2r}-\frac{b^2}{a^2}\times\frac{a^2}{b^4}$$

$$\therefore \frac{1}{p^2} = \frac{2a}{b^2r}-\frac{1}{b^2} \text{ or, } \frac{b^2}{p^2} = \frac{2a}{r}-1 \text{ Proved.}$$

Ex. 2. Obtain the pedal equation of the curve.

$r^m = a^m\sin m\theta$ R. U. 1967

Take logarithm of both sides $m\log r = m\log a + \log\sin m\theta$

$$\therefore m\cdot\frac{1}{r}\frac{dr}{d\theta} = 0+\frac{1}{\cos m\theta}(-\sin m\theta)m$$

or, $\cot\phi = \cot m\theta \ \therefore \ \phi = m\theta$, ... (1)

But $p = r\sin\phi = r\sin m\theta$ ... ... (2)

From the given equation $r^m = a^m\sin m\theta$

We have $\sin m\theta = r^m/a^m$ ... ... (3)

$\therefore p = r.\ r^m/a^m$ ... ... by (4)

or, $pa^m = r^{m+1}$

which is the required pedal equation.

Ex. 3 Find the pedal equation of the curve $x^2+y^2-2ax=0$

The equation of the tangent to the curve

$$x^2+y^2-2ax=0 \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

at (x_1, y_1) is $xx_1+yy_1-a(x+x_1)=0$

$$\text{or,} \quad x(x_1-a)+yy_1-ax_1=0 \quad \ldots \quad \ldots \quad (2)$$

The length of the perp. from the contre $(0, 0)$ to the tangent is

$$p=\frac{-ax_1}{\sqrt{\{(x_1-a)^2+y_1^2\}}}=\frac{-ax_1}{\sqrt{(x_1^2+y_1^2-2ax_1+a^2)}}=\frac{-ax_1}{-a}=x_1$$

Again from (1), $x_1^2+y_1^2-2ax_1=0$, or, $x_1=\dfrac{x_1^2+y_1^2}{2a}=\dfrac{r^2}{2a}$.

$\therefore \quad 2ap=r^2.$

* **Ex. 4.** Find the kth positive pedal of the cardioide $r=a(1+\cos\theta)$.

The equation can be written as $r=2a\cos^2\theta/2 \quad \ldots \quad \ldots \quad (1)$

$$\text{or,} \quad \log r=\log 2a+2\log\cos\tfrac{1}{2}\theta$$

$$\therefore \quad \frac{1}{r}\frac{dr}{d\theta}=2\cdot\frac{1}{2}\cdot\frac{-\sin\theta/2}{\cos\theta/2}=-\tan\tfrac{1}{2}\theta=\cot(\tfrac{1}{2}\pi+\tfrac{1}{2}\theta)$$

or, $\cot\phi_1=\cot(\frac{1}{2}\pi+\frac{1}{2}\theta)$

$\therefore \quad \phi_1=\frac{1}{2}\pi+\frac{1}{2}\theta \quad \ldots \quad \ldots \quad \ldots \quad (2)$

But from the Art 10.17 ... we have

$\theta=\theta'+\frac{1}{2}\pi-\phi_1=\theta'+\frac{1}{2}\pi-\pi/2-\frac{1}{2}\theta$ or, $\theta=\frac{2}{3}\theta' \quad \ldots \quad (3)$

Again we know

$r_1=r\sin\phi_1=r\sin(\frac{1}{2}\pi+\frac{1}{2}\theta)=r\cos\frac{1}{2}\theta$

$=2a\cos^2\theta\cdot 2\cos\frac{1}{2}\theta=2a\cos^3\theta/2=2a\cos^3\frac{1}{2}\cdot\frac{2}{3}\theta_1$

or, $r_1=2a\cos^3\frac{1}{3}\theta_1 \quad \ldots \quad - \quad (4)$

which is the first positive pedal Simplify as before, then

$$\frac{1}{r_1}\cdot\frac{dr_1}{d\theta_1}=-\tan\theta_1/3=\cot(\pi/2+\theta_1/3)$$

or, $\cot\phi_2=\cot(\pi/2+\theta_1/3)$

$\therefore \quad \phi_2=\theta/2+\theta_1/3, \quad \theta_1=\theta_2+\pi/2-\phi_2$ or, $\theta_1=3\theta_2/4$

Again $r_2=r_1\sin\phi_2=r_1\cos\theta_1/3=2a\cos^3\theta_1/3\cos\theta_1/3$

$$=2a\cos^4\theta_1/3=2a\cos^4(\theta_2/4)$$

$\therefore \quad r^2=2a\cos^4\theta_2/4 \quad \ldots \quad \ldots \quad \ldots \quad (5)$

which is the 2nd positive pedal of (1). In the same way we get the k th positive pedal of (1), and this is

$$r_K=2a\cos^{K+2}\left(\frac{\theta_k}{k+2}\right)$$

Alternative method

$r=2a\cos^2\theta/2 \quad \ldots \quad \ldots \quad \ldots \quad (1)$

Let us consider it as

$r=2a\cos^m(\theta/m) \quad \ldots \quad \ldots \quad (2)$

or, $\log r=\log 2a+m\log\cos\theta/m.$

$$\therefore \quad \frac{1}{r}\cdot\frac{dr}{d\theta}=m\frac{-\sin(\theta/m)}{\cos(\theta/m)}\cdot\frac{1}{m}=-\tan\theta/m$$

or, $\cot\phi=\cot(\pi/2+\theta/m) \quad \therefore \phi=\pi/2+\theta/m$

and $\theta=\theta_1+\pi/2-\phi$ or, $\theta=\theta_1+\pi/2-\pi/2-\theta/m$

or, $\theta+\dfrac{\theta}{m}=\theta_1 \quad \therefore \theta\dfrac{(1+m)}{m}=\theta_1 \quad \therefore \theta=\dfrac{m\,\theta_1}{m+1}$

But we know $r_1=r\sin\phi=r\sin(\pi/2+\theta/m)=r\cos\theta/m$

$$=2a\cos^m(\theta/m)\cos(\theta/m)=2a\cos^{m+1}(\theta/m)$$

or, $r_1=2a\cos^{m+1}\left(\dfrac{\theta}{m+1}\right) \quad \ldots \quad \ldots \quad (3)$

Therefore first positive pedal equation of (2)

$$r=2a \cos^{m+1}\left(\frac{\theta}{m+1}\right) \quad \cdots \quad \cdots \quad (4)$$

If we put $m_1=m+1$, then

$r=2a \cos^{m_1} \frac{\theta}{m_1}$, $m_1=m+1$

which is the first positive pedal of the curve (2)

Similarly the 2nd positive pedal of (2) is

$r=2a \cos^{m_2} \frac{\theta}{m_2}$, $m_2=m_1+1=m+2$

and the kth positive pedal is

$r_k=2a \cos^{m_k} \frac{\theta}{m_k}$, $m_k=m+k$

Put $m=2$, then $m_k=k+2$.

Thus $r=2a \cos^{k+2} \frac{\theta}{k+2}$ is the pedal equation of

$r=2a \cos^2 \theta/2=a(1+\cos\theta)$

Ex. 5. Find the pth negative pedal of the curve $r=a(1+\cos\theta)$

The equation can be written as

$r=2a \cos^2 \theta/2=2a \cos^m \theta/m$, $m=2$.

The kth positive pedal of $r=2a \cos^n (\theta/n)$ is

$r=2a \cos^m (\theta/m)$

where $m=n+k$ or, $n=m-k$

Hence kth negative pedal

of $r=2a \cos^m \theta/m$ is $r=2a \cos^n\theta/m$

where $n=m-k$

Similarly the pth negative pedal of $r=2a \cos^m\theta/m$ is

$r=2a \cos^{m-p} \frac{\theta}{m-1}$

Hence the pth negative pedal for $r=2a \cos^2\theta/2$

is $r=2a \cos^{2-p} \frac{\theta}{2-p}$ or, $r=2a \cos^n \frac{\theta}{n}$, $n=2-p$

Ex. 6. Show that the positive pedal of the curve

$r^{2/9} \cos 2\theta/9=a^{2/9}$ is $r^2 \cos 2\theta=a^2$

Let $r^{2/9} \cos (2/9)\theta=a^{2/9}$ become $r^m \cos m\theta=a^m$, $m=2/9$

or, $m \log r+\log \cos m\theta=m \log a$

$$\therefore \quad \frac{m}{r}\frac{dr}{d\theta}-m\frac{\sin (2/9)\theta}{\cos (2/9)\theta}=0$$

or, $\cot\phi=\tan m\theta=\cot(\frac{1}{2}\pi-m\theta)$, $\phi=\frac{1}{2}\pi-m\theta$.

But $\theta=\theta_1+\frac{1}{2}\pi-\phi=\theta_1+\frac{1}{2}\pi-\frac{1}{2}\pi+m\theta$ or, $\theta=\theta_1+m\theta$

or, $\theta(1-m)=\theta_1$ or, $\theta=\frac{\theta_1}{1-m}$

Again $r_1=r \sin\phi=r \sin(\frac{1}{2}\pi-m\theta)=r \cos m\theta$

or, $r_1^m=r^m \cos^m m\theta=a^m \cos^{m-1} m\theta$

or, $r_1^{m/(m-1)}=a^{m/(m-1)} \cos \frac{m}{1-m}\theta_1$

or, $r^{m/(1-m)} \cos \frac{m}{1-m}\theta=a^{m/(1-m)}$

Put $m_1=\frac{m}{1-m}$

The first positive pedal of (1) is $r_1^{m_1} \cos m_1 \theta = a^{m_1}$

Similarly the 2nd positive pedal is

$$a^{m_2} = r^{m_2} \cos m_2\theta ; \quad m_2 = \frac{m}{1-m_1} = \frac{m}{1-2m}$$

Similarly 4th positive pedal is

$$a^{m_4} = r^{m_4} \cos m_4 \theta, \quad m_4 = \frac{m}{1-4m}$$

But $m = \frac{2}{9}$, then $m_4 = \frac{2/9}{1-8/9} = 2$

Therefore the 4th positive pedal of $r^{2/9} \cos \frac{2}{9}\theta = a^{2/9}$.

is $a^2 = r^2 \cos 2\theta$.

Ex. 7. Show that inverse of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

with regard to the origin is $(x^2+y^2)^2 = k^4 \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)$

Put $\frac{k^2 X}{X^2+Y^2}$ for x, $\frac{k^2 Y}{X^2+Y^2}$ for y in the equation of ellipse.

Then $$\frac{k^4 X^2}{a^2 (X^2+Y^2)^2} + \frac{k^4 Y^2}{b^2 (X^2+Y^2)^2} = 1$$

or, $$k^4 \left(\frac{X^2}{a^2} + \frac{Y^2}{b^2}\right) = (X^2+Y^2)^2 \quad \dots \dots (1)$$

for the equation of inverse of the ellipse.

Now replace X by x, Y by y in (1), then

$$k^4 \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = (x^2+y^2)^2.$$

Ex. 8. Show that the inverse of the equiangular spiral $r = ae^{\theta \cot\alpha}$ is $r = \frac{k^2}{a} e^{-\theta \cot\alpha}$ another equiangular spiral.

Replace r by $\frac{k^2}{r_1}$ in $r = ae^{\theta \cot\alpha}$ then $\frac{k^2}{r_1} = ae^{\theta \cot\alpha}$

for the equation of the inverse of $r = ae^{\theta \cot\alpha}$

Now replace r_1 by r. then $\frac{k^2}{r} = ae^{\theta \cot\alpha}$ or, $r = \frac{k^2}{a} e^{-\theta \cot\alpha}$

which is the inverse of the given conic.

Ex. 9. Find the polar reciprocal of the parabola $y^2 = 4ax$ with regard to its focus.

The eq. of the tangent to curve $y^2 = 4ax$ at (x_1, y_1) is

$$yy_1 = 2a(x+x_1) \text{ or, } 2ax - yy_1 - 2ax_1 = 0 \quad (1)$$

If $p = x \cos\alpha + y \sin\alpha$ touches the curve $y^2 = 4ax$ then it will be identical to (1) if the co-efficients are proportional.

Thus $2ax - yy_1 + 2ax_1 = 0$; $x \cos\alpha + y \sin\alpha - p = 0$

Comparing co-efficients, we have

$$\frac{2a}{\cos\alpha} = \frac{y_1}{\sin\alpha} = -\frac{2ax_1}{p}$$

or, $$x_1 = -\frac{2ap}{2a \cos\alpha}, y_1 = -\frac{2a \sin\alpha}{\cos\alpha}$$

Thus the condition of tangency of the line $p = x_1 \cos\alpha + y_1 \sin\alpha$

is $$p = \frac{-p \cos\alpha}{\cos\alpha} - \frac{2a \sin^2 a}{\cos\alpha}$$

or, $2p = -2a \sin\alpha \tan\alpha$ or, $p = -a \sin\alpha \tan\alpha \quad \dots \dots (1)$

The polar equation of the pedal with respect to the vertex is

$r = -a \sin\theta \tan\theta \quad \dots \quad \dots \quad \dots \quad (2)$

The inverse of this curve is

$$k^2/r = -a \sin\theta \tan\theta \qquad (rr_1 = k^2)$$

which is the polar reciprocal of $y^2 = 4ax$.

Put $x = r\cos\theta$, $y = r\sin\theta$, $k^2 = -r\sin\theta\tan\theta$

or, $k^2 = -y\dfrac{y}{x}$ or, $y^2 + k^2x = 0$

Thus $r\sin\theta\tan\theta = -k^2$ or, $y^2 + k^2x = 0$

is the polar reciprocal of the parabola w. r. to a circle with centre at the vertex and radius k.

*Ex. 10. Show that the polar reciprocal of the curve $r^m = a^m \cos m\theta$ with regard to the hyperbola $r^2\cos 2\theta = a^2$ is

$$r^{\frac{m}{m+1}} \cos\frac{m\theta}{m+1} = a^{\frac{m}{m+1}}$$

Let the pedal equation of $r^m = a^m \cos m\theta$ ... (1)

be $p = r\sin\phi$

Take logarithmic differentiation of (1)

then $r\dfrac{dr}{d\theta} = -\tan m\theta$

or, $\cot\phi = \cot(\frac{1}{2}\pi + m\theta)$ $\therefore$ $\phi = (\frac{1}{2}\pi + m\theta)$

Hence $p = r\sin(\frac{1}{2}\pi + m\theta) = r\cos m\theta$

$$p = a\cos^{(m+1).m}\frac{m\theta_1}{m+1}$$

$\therefore$ $\theta = \theta'_1 + \frac{1}{2}\pi - \phi = \theta'_1 + \frac{1}{2}\pi - \frac{1}{2}\pi - m\theta$ or, $\theta = \dfrac{m}{m+1}\theta_1$

$\therefore$ $p^{m/(m+1)} = a^{m/(m+1)} \cos\dfrac{m\theta_1}{m+1}$ (2)

Let $p = X\cos\alpha + Y\sin\alpha$ ... ... (3)

touch the curve (1)

From the 2nd equation we have

$r^2\cos 2\theta = a^2$ or, $r^2\cos^2\theta - r^2\sin^2\theta = a^2$

or, $x^2 - y^2 = a^2$ ... ... (4)

If (x, y) be the pole of tangent (3) w. r. to $x^2 - y^2 = a^2$ then the tangent must coincide with the polar of (x, y).

Therefore $Xx - Yy = a^2$. ... ... (5)

Compare (3) and (5)

$$\frac{\cos\alpha}{x} = \frac{-\sin\alpha}{y} = \frac{p}{a^2} \quad\text{or,}\quad \frac{\cos\alpha}{p} = \frac{x}{a^2}, \quad \frac{\sin\alpha}{p} = -\frac{y}{a^2}$$

$$\therefore \frac{1}{p^2}(\cos^2\alpha + \sin^2\alpha) = \frac{x^2 + y^2}{a^4} = \frac{r^2}{a^4}$$

or, $p = \dfrac{a^2}{r}$ ... ... (6)

From (2) and (6), we have

$$\left(\frac{a^2}{r}\right)^{m/(m+1)} = a^{m/(m+1)} \cos\left(\frac{m\theta}{m+1}\right)$$

or, $a^{m/(m+1)} = r^{m/(m+1)} \cos\left(\dfrac{m\theta}{m+1}\right)$

$r^{m/(m+1)} \cos\dfrac{m\theta}{m+1} = a^{m\theta/(m+1)}$ Proved.

Exercise X (C)

1. Find the pedal equation of a parabola either from its polar equation or from cartesian with the focus as origin of co-ordinates. D. U. 1952, '56

2. Find the pedal equation of the curve

$r^4 = a^4 \cos 4\theta$ D. U. 1965.

4. Find the pedal equation of an ellipse w. r to one extremity of the major axis. D U. 1953

5. Find the pedal equation of the parabola

$y^2 = 4a(x+a)$ D. U. 1961

6. Show that the pedal equation of the parabola $y^2 = 4ax$ with its vertex is $a^2(r^2 - p^2) = p^2(r^2 + 4a^2)(p^2 + 4a^2)$ R.U.H.1964

7. Show that the pedal equation of the astroid

$x^{2/3} + y^{2/3} = a^{2/3}$ is $r^2 + 3p^2 = a^2$ R. H. 67, C.U. 1982

8. Show that the pedal equation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with regard to the origin is $a^2b^2/p^2 = a + b^2 - r^2$. N. U. 1995

*9. Show that the first positive pedal with regard to the vertx of the parabola $y^2 + 4bx = 0$ can be written as R. U. 1959

$y^2(2a - x) = x^2$ where $b = 2a$

$$\left(\frac{\cos a}{p}\right)^{\frac{m}{m-1}} \frac{\frac{1}{1-m}}{a} + \left(\frac{\sin \alpha}{p}\right)^{\frac{m}{m-1}} \frac{\frac{1}{m-1}}{b} = 1$$

*11. Show that the first positive pedal of the curve

$x^3 + y^3 = a^3$ is $(x^2 + y^2)^{3/2} = (x^{3/2} + a^{3/2})$

*12. Prove that the kth positive pedal of $r^m = a^m \cos m\theta$ is

$r^{m_k} = a^{m_k} \cos m_k\theta$. where $m_k = \frac{m}{1+km}$ D. U. 1965

*13. Show that the 5th negative pedal of the cardioide

$r = a(1 - \cos\theta)$ is $8a/r = \cos\theta + 3\cos\frac{1}{3}\theta$

*14. Show that nth positive pedal of the spiral $r = ae^{\theta \cot \alpha}$

is $r = a \sin^n\alpha \, e^{n(\frac{1}{2}\pi - \alpha)\cot\alpha} \, e^{\theta\cot\alpha}$

*15. Show that kth negative pedal of the curve

$r^m = a^m \cos m\theta$ is $r^n = a^n \cos n\theta$ when $n = \frac{m}{1-km}$

*16. Show that pedal equation of the curve

$x = a(3\cos\theta - \cos^3\theta)$, $y = a(3\sin\theta - \sin^3\theta)$

is $3p^2(7a^2 - r^2) = (10a^2 - r^2)^2$

*17. Show that the pedal equation of the curve

$e^2(x^2+y^2) = x^2y^2$ is $\frac{1}{p^2} + \frac{3}{r^2} = \frac{1}{c^2}$

*18. Find the pedal equations of the following curves

(ii $r = a + b\cos\theta$ (ii) $r^m = a^m \sin m\theta + b^m \cos m\theta$

(iii) $l/r = a(1 + \cos\theta)$ (iv) $r^2 \cos 2\theta = a^2$

(v) $r^2 = a^2 \cos 2\theta$, (iv) r = a(1 - cos θ) N.U. 1995

*19. Show that the pedal equation of the curve

$y^2(3a - x) = (x - a)^3$ is $p^2 = 9a^2(r^2 - a^2)/r^2 + (5a^2)$

*20. Show that 5th negative pedal of $r^2 = a^2 \cos 2\theta$

is $r^{2/9} \cos 2/9\theta = a^{2/9}$

21. Write down the Ist, 2nd and nth positive and negative pedal of the curves

(i) $p + r = a$. (ii) $a^2r = p^3$ (iii) $r/p = a$

22. Show that the pedal equation of the hyperbolic spiral

$r\theta = a$ is $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{a^2}$

23. Show that pedal equation of the spiral of Achimedes

$r = a\theta$ is $r^4 = p^2(a^2 + r^2)$

24. Show that the inverse of a straight line is a circle and that of circle is another circle.

25. Show that the inverse of the conic $ax^2 + 2hxy + by^2 = 5y$ is the cubic $k^2(ax^2 + 2hxy + by^2) = 5y(x^2 + y^2)$

26. Show that the inverse of the parabola $\frac{l}{r}=1+\cos\theta$ is cardoide $r=a(1+\cos\theta)$ where $a=k^2/l$

27. Show that polar reciprocal $r=a\cos\theta$ with regard to a circle of radius k and centre at the origin of the curve is

$$\frac{l}{r}=1+\cos\theta$$

28. Show that polar reciprocal of $x^m y^m = a^{m+n}$ with regard to the circle of radius k and centre at the origin of the curve is

$$x^m y^m \left\{\frac{a(m+n)}{k^2}\right\}^{m+n} = m^m n^n$$

*29. Show that the polar reciprocal of ellipse with regard its centre is $a^2x^2+b^2y^2=k^4$

where k is the radius of the circle.

30. Show that first positive pedal of the curve

$p=\frac{r^{m+1}}{a^m}$ is $p^{m+1}a^m = r^{m+1}$

and that its polar reciprocal with regard to a circle of radius a whose centre is at the origin $p^{m+1}=a^m r$

*31. Show that the polar reciprocal of the curve $x^n+y^n=a^n$ with regard to the circle of radius k and centre at the origin of the given curve is $x^{\frac{n}{n-1}}+y^{\frac{n}{n-1}}=\left(\frac{k^2}{a}\right)^{\frac{n}{n-1}}$

প্রশ্নমালা X (C)

1. মূল বিন্দুকে উপকেন্দ্র (focus) ধরিয়া পোলার বা কার্টেসীয় স্থানাংকে অধিবৃত্তের সমীকরণ হইতে পেডেল সমীকরণ নির্ণয় কর। D. U. 1952, '56

2. কার্ডিয়য়েড (Cardioide) $r=a(1-\cos\theta)$ এর পেডেল (Pedal) সমীকরণ নির্ণয় কর। C. U. 1983, D. U. 1958.

3. বক্ররেখা $r^4=a^4\cos 4\theta$ এর (p, r) সমীকরণ নির্ণয় কর। D. U. 196

4. কোন উপবৃত্তের বৃহদক্ষের একপ্রান্তের সাপেক্ষে উহার পেডেল সমীকরণ নির্ণয় কর। D. U. 1953

5. পরাবৃত্ত (Parabola) $y^2=4a(x+a)$ এর পেডেল সমীকরণ নির্ণয় কর। D. U. 1961

6. দেখাও যে পরাবৃত্ত $y^2=4ax$ এর শীর্ষবিন্দুর সাপেক্ষে ইহার পেডেল সমীকরণ হইবে

$a^2(r^2-p^2)^2=p^2(r^2+4a^2)(p^2+4a^2)$ R. U. H. 1964

7. দেখাও যে এস্ট্রয়েড (Astroid) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ এর পেডেল সমীকরণ $r^2+3p^2=a^2$ R. U. H. 67, C. U. 1982

8. দেখাও যে মূলবিন্দুর সাপেক্ষে উপবৃত্ত

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ এর পেডেল সমীকরণ $a^2b^2/p^2=a^2+b^2-r^2$.

*9. দেখাও যে শীর্ষবিন্দুর সাপেক্ষে পরাবৃত্ত $y^2+4bx=0$ এর প্রথম ধনাত্মক পেডেল সমীকরণকে লিখা যায় $y^2(2a-x)=x^3$ যেখানে $b=2a$ R. U. '59

*10. দেখাও যে বক্ররেখা $ax^m+by^m=1$ এর প্রথম ধনাত্মক পেডেল সমীকরণ হইবে $\left(\frac{\cos\alpha}{p}\right)^{m/(m-1)} a^{1/(1-m)}+\left(\frac{\sin\alpha}{p}\right)^{m/(m-1)} b^{1/(1-m)}=1.$

*11. দেখাও যে বক্ররেখা $x^3+y^3=a^3$ এর প্রথম ধনাত্মক পেডেল সমীকরণ হইবে $(x^2+y^2)^{3/2}=a^{3/2}(x^{3/2}+y^{3/2})$

*12. দেখাও যে $r^m=a^m\cos m\theta$ সমীকরণের k-তম

ধনাত্মক পেডেল সমীকরণ হইবে $r^{m_k}=a^{m_k}\cos m_k\theta$

যেখানে $m_k=\dfrac{m}{1+km}$

D. U. 1965

*13. দেখাও যে কার্ডিয়য়েড (cardioide) $r=a\,(1-\cos\theta)$ এর প্রথম ঋণাত্মক পেডেল সমীকরণ হইবে

$\dfrac{8a}{r}=\cos\theta_1+3\cos\frac{1}{3}\theta_1,\ \theta_1=\frac{1}{2}\pi-\frac{1}{2}\theta$

*14. কুণ্ডলী $r=ae^{\theta\cot\alpha}$ এর জন্য দেখাও যে n-তম ধনাত্মক পেডেলের সমীকরণ হইবে

$r=a\sin^n\alpha\, e^{n(\pi/2-\alpha)\cot\alpha}\, e^{\theta\cot\alpha}$

*15. দেখাও যে বক্ররেখা $r^m=a^m\cos m\theta$-এর k তম ঋণাত্মক পেডেল সমীকরণ হইবে $r^n=a^n\cos n\,\theta$ যেখানে $n=\dfrac{m}{1-km}$

*16. দেখাও যে বক্ররেখা $x=a(3\cos\theta-\cos^3\theta)$, $y=a\,(3\sin\theta-\sin^3\theta)$ এর পেডেল সমীকরণ হইবে $3p^2(7a^2-r^2)=(10\,a^2-r^2)^2$

*17. দেখাও যে বক্ররেখা $c^2(x^2+y^2)=x^2y^2$ এর পেডেল সমীকরণ হইবে $1/p^2+3/r^2=1/c^2$

*18. নিম্নলিখিত বক্ররেখাগুলির পেডেল সমীকরণ নির্ণয় কর :-

(i) $r=a+b\cos\theta$ (ii) $r^m=a^m\sin m\theta+b^m\cos m\theta$

(ii) $l/r=a(1+\cos\theta)$ (iv) $r^2\cos 2\theta=a^2$ (v) $r^2=a^2\cos 2\theta$

*19. দেখাও যে বক্ররেখা $y^2(3a-x)=(x-a)^3$ এর পেডেল সমীকরণ হইবে

$p^2=9a^2(r^2-a^2)/(r^2+15a^2)$

*20. দেখাও যে বক্ররেখা $r^2=a^2\cos 2\theta$ এর পঞ্চম ঋনাত্মক পেডেল সমীকরণ হইবে $r^{2/9}\cos 2/9\,\theta=a^{2/9}n$

*21. নিম্নলিখিত বক্ররেখাগুলির ১ম, ২য় এবং n-তম ধনাত্মক এবং ঋণাত্মক পেডেলস নির্ণয় কর।

(i) $p+r=a$ (ii) $a^2r=p^3$ (iii) $r/p=a$

22. দেখাও যে পরাবৃত্তিয় কুণ্ডলী $r\theta=a$-এর পেডেল সমীকরণ

$1/p^2=1/r^2+1/a^2$.

23. দেখাও যে আর্কিমিডিসের কুণ্ডলী $r=a\theta$ এর পেডেল সমীকরণ

$r^4=p^2(a^2+r^2)$.

24. দেখাও যে একটি সরলরেখার বিপরীত সমীকরণ একটি বৃত্ত এবং একটি বৃত্তের বিপরীত সমীকরণ অপর একটি বৃত্ত।

25. দেখাও যে কণিক $ax^2+2hxy+by^2=5y$ এর বিপরীত সমীকরণ হইল ত্রিঘাত সমীকরণ

$k^2(ax^2+2hxy+by^2)=5y\,(x^2+y^2)$

26. দেখাও যে অধিবৃত্ত $l/r=1+\cos\theta$ এর বিপরীত সমীকরণ হইল কার্ডিয়য়েড $r=a(1+\cos\theta)$, এখানে $a=k^2/l$.

27. দেখাও যে বক্ররেখার মূলবিন্দুকে কেন্দ্র এবং k ব্যাসার্ধ বিশিষ্ট বৃত্তের সাপেক্ষে বক্ররেখা $r=a\cos\theta$ এর পোলার উল্টা সমীকরণ হইবে $l/r=1+\cos\theta$.

28. দেখাও যে বক্ররেখার মূল বিন্দুকে কেন্দ্র এবং k ব্যাসার্ধ বিশিষ্ট বৃত্তের সাপেক্ষে বক্ররেখা

$x^m y^m=a^{m+n}$ এর পোলার উল্টা (Reciprocal) সমীকরণ হইবে

$x^m y^n\left\{\dfrac{a\,(m+n)}{k^2}\right\}^{m+n}=m^m n^n$.

*29. দেখাও যে উপবৃত্তের কেন্দ্রের সাপেক্ষে (with regard to its centre) উপবৃত্তের পোলার উল্টো সমীকরণ হইবে $a^2x^2+b^2y^2=k^4$, এখানে k হইল বৃত্তের ব্যাসার্ধ।

*30. দেখাও যে বক্ররেখা $p=\dfrac{r^{m+1}}{a^m}$ এর প্রথম ধনাত্মক পেডেল সমীকরণ হইবে $p^{m+1}a^m=r^{2m+1}$

এবং মূল বিন্দুতে কেন্দ্র ও 'a' ব্যাসার্ধ বিশিষ্ট বৃত্তের সাপেক্ষে ইহার পোলারের উল্টো সমীকরণ হইবে $p^{m+1}=a^m r$.

*31. দেখাও যে বক্ররেখা $x^n+y^n=a^n$ এর মূলবিন্দুতে কেন্দ্র এবং k ব্যাসার্ধ বিশিষ্ট বৃত্তের সাপেক্ষে বক্ররেখাটির পোলারের উল্টা সমীকরণকে লিখা যায়

$$x^{\frac{n}{n-1}}+y^{\frac{n}{n-1}}=\left(\frac{k^2}{a}\right)^{\frac{n}{n-1}}$$

Answers

1. $p^2=ar$ 2. $r^3=2ap^2$ 3. $r^5=a^4p$ 5. $p^2=ar$

18. (i) $r^4=(b^2-a^2+2ar)p^3$ (ii) $r^{m+1}=p\sqrt{(a^{2m}+b^{2m})}$

(iii) $rl=2ap^2$ (iv) $pr=a^2$ (v) $r^3=a^2p$.

21. nth $+ve$ pedal negative pedal

(i) nth $\frac{r^n}{p^n}(p+r)=a$ $\frac{p^n}{r^n}(p+r)=a$

(ii) $p^{2n+3}\ a^2=r^{2n-1}$, $r^{2n+1}\ a^2=p^{2n-3}$

(iii) $\frac{r}{p}=a$. $r/p=a$

Chapter xi

MAXIMA AND MINIMA

11. 1. Definition : A function $f(x)$ is said to have a maximum at $x=a$ if $f(a)\geqslant f(x)$ for every x in the neighbourhood of a.

Thus if h is any small positive number, then

$f(a)\geqslant f(x)$ for $a-h\leqslant x\leqslant a+h$.

This implies that

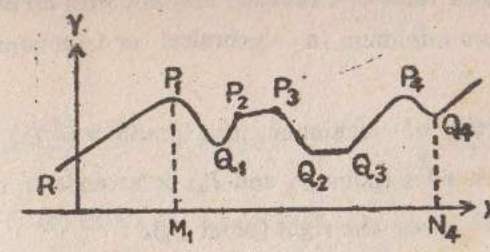
$f(a)\geqslant f(a-h)$ and $f(a)\geqslant f(a+h)$.

Similary, function $f(x)$ is minimum at $x=a$, if

$f(a)\leqslant f(x)$ for $a-h\leqslant x\leqslant a+h$

when $h>o$ and $h\to 0$. In this case

$f(a)\leqslant f(a-h)$ and $f(a)\leqslant f(a+h)$.



Feg.—1

Let Fig 1. represent the graph of some function $y=f(x)$. The function is maximum at P_1 P_2, P_3, P_4 and minimum at Q_1, Q_2 Q_3, Q_4.

(i) We note that a function may have several maxima and minima in an interval where the function is defined.

(ii) It is not necessary that a maximum value of a function is always greater than minimum value of the function. Maximum value of a function may be less than minimum value of a functions. For example the minimum value $Q_4 N_4$ at Q_4 is greater than the maximum value $P_1 M_1$ at P_1.

(iii) In between two maxima, there should be at least one minimum value of the function. Similary at least one maximum value of the function must lie between two minimum values of the function. There is a minimum value of the function between two consecutive maximum values and vice versa.

Thus we observe that maxmimum and minimum values of a function (continuous) occur alternately.

(iv) In Calculus we are concerned with a relative maximum or a relative minimum value of a function and not with an absolute maximum or absolute minimum in algebraical or trigonometrical examples.

(v) From a point of maximum, the graph $y=f(x)$ either descends on both sides (point P_1 and P_4) or ascends on the left if (point P_2) or descends on the right (point P_3).

From a point of minimum, the curve $y=f(x)$ either ascends on both sides (as at Q_1 and Q_4) or descends on the left (as at Q_2) or ascends on the right (as at Q_3)

A point of maximum or minimum of a function is called a ***turning point*** or a ***stationary point*** when the function is differentiable at the point.

11.2. Necessary Condition for Maxima or minima

If a function $f(x)$ is maximum or minimum at $x=a$ and if $f'(a)$ *exist*, then $f'(a)=0$.

Let $f(x)$ be a finite and continuous function of x in the neighbourghood of $x=a$

From the definition of maxima or minima, $f(x)$ is maximum or minimum at $x=a$ according as $f(a+h)-f(a)$ and $f(a-h)-f(a)$ are $+ve$ or both negative, h being indefinitely small $(h\to 0)$ and positive.

Proof : For the maximum value of $f(x)$ at $x=a$

$f(a+h)-f(a)\leqslant 0,\ f(a-h)-f(a)\leqslant 0$

$$\therefore \quad \frac{f(a+h)-f(a)}{h}\leqslant 0 \qquad \frac{f(a-h)-f(a)}{-h}\geqslant 0 \ (\because \ h>0)$$

Hence

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\leqslant 0 \text{ and} \lim_{h\to 0}\frac{f(a-h)-f(a)}{-h}\geqslant 0$$

Now if $f'(a)$ exists at $x=a$, then the above two limits must be equal. Hence $f'(a)=0$

Similary we can establish that $f'(x)=0$ is the necessary condition for a minimum value of $f(x)$ at $x=a$

Hence the necessary condition for maximum or minimum of $f(x)$ at $x=a$ is $f'(a)=0$

11.3.(A) Determination of Maxima and minima

If $f(x)$ is finite and continuous function of x in the vicinity of $x=a$ and if $f'(a)=0$ and $f''(a)\neq 0$, then

(i) **$f(a)$ is maximum if $f''(a)$ is negative**

(ii) **$f(a)$ is maximum if $f''(a)$ is positive.**

For $f(x)$ has maximum or minimum value at $x=a$, we have $f(a+h)-f(a)$ and $f(a-h)-f(a)$ are both negative or both positive h being indefinitely small ($h\to 0$).

By Taylor's Theorem

$$\begin{cases} f(a+h)=f(a)+hf'(a)+\dfrac{h^2}{\lfloor 2}f''(a)+\dfrac{h^3}{\lfloor 3}f'''(a+\theta h), 0<\theta<1 \\ f(a-h)=f(a)-hf'(a)+\dfrac{h^2}{\lfloor 2}f''(a)-\dfrac{h^3}{\lfloor 3}f'''(a-\theta h), 0<\theta<1 \end{cases}$$

$$\begin{cases} f(a+h)-f(a)=\dfrac{h^2}{\lfloor 2}f''(a)+\dfrac{h^3}{\lfloor 3}f'''(a+\theta h) \\ f(a-h)-f(a)=\dfrac{h^2}{\lfloor 2}f''(a)-\dfrac{h^3}{\lfloor 3}f'''(a-\theta h) \end{cases} \quad [\because f'(a)=0]$$

Since h is very small, we can neglect the 2nd term. Then

$$f(a+h)-f(a)=\frac{h^2}{\lfloor 2}f''(a),\; f(a-h)-f(a)=\frac{h^2}{\lfloor 2}f''(a)$$

approximately

As h^2 is always positive, so the sign of $f(a\pm h)-f(a)$ depends upon $f''(a)$

Since $f(a\pm h)-f(a)<0$ for a maximum of $f(x)$ at $x=a$, $f''(a)$ must be negative and similarly for a minimum value of $f(x)$ at $x=a$, $f''(a)$ must be positive.

(B) When $f''(a)=0$ at $x=a$ with $f'(a)=0$

we have from Taylor's Theorem ;

$$\begin{cases} f(a+h)-f(a)=hf'(a)+\dfrac{h^2}{\lfloor 2}f''(a)+\dfrac{h^3}{\lfloor 3}f'''(a)+\dfrac{h^4}{\lfloor 4}f^{iv}(a+\theta h) \\ f(a-h)-f(a)=-hf'(a)+\dfrac{h^2}{\lfloor 2}f''(a)-\dfrac{h^3}{\lfloor 3}f'''(a)+\dfrac{h^4}{\lfloor 4}f^{iv}(a+\theta h) \end{cases}$$

where $0<\theta<h$

$$\begin{cases} f(a+h)-f(a)=\dfrac{h^3}{\lfloor 3}f'''(a)+\dfrac{h^4}{\lfloor 4}f^{iv}(a+\theta h) \\ f(a-h)-f(a)=-\dfrac{h^3}{\lfloor 3}f'''(a)+\dfrac{h^4}{\lfloor 4}f^{iv}(a-\theta h) \end{cases} \quad (\because f''(a)=0)$$

In general,

(C) If $f'(a)=f''(a)=\ldots=f^{(n-1)}(a)=0$ and $f^{n}(a)\neq 0$, then

(I) $f(x)$ is maximum or minimum if n is even ;

$f(x)$ is maximum if $f^{(n)}(a)$ is negative and
$f(x)$ is minimum if $f^{(n)}(a)$ is positive.

(II) $f(x)$ is neither a maximum nor a minimum if n is odd.

Proof :—By Taylor's Theorem, we have

$$f(a+h)-f(a)=hf'(a)+\frac{h^2}{\lfloor 2}f''(a)+\ldots+\frac{h^{n-1}}{\lfloor (n-1)}f^{n-1}(a)+\frac{h^n}{\lfloor n}f^n(a+\theta h).\; 0<\theta<h$$

$$f(a-h)-f(a)=-hf'(a)+\frac{h^2}{\lfloor 2}f''(a)\ldots(-1)^{n-1}\frac{h^{n-1}}{\lfloor (n-1)}f^{(n-1)}(a)+(-1)^n\frac{h^n}{\lfloor n}f^{(n)}(a-\theta h),\; 0<\theta<h$$

or, $$\left.\begin{aligned} f(a+h)-f(a)&=\frac{h^n}{\lfloor n}f^{(n)}(a+\theta h) \\ f(a-h)-f(a)&=(-1)^n\frac{h^n}{\lfloor n}f^{(n)}(a-\theta h) \end{aligned}\right\}\text{by condition }(c)$$

Now we can write $f^n(a+\theta h)=f^n(a)+\epsilon$, where ϵ is a very small quantity ($\epsilon \to 0$). So the sign of $f^n(a+\theta h)$ is the same as the sign of $f^n(a)$

Similarly for $f^n(a-\theta h)$ has the same sign of $f^n(a)$

$$\therefore \quad f(a+h)-f(a)=\frac{h^n}{\lfloor n}f^n(a)$$

$$f(a-h)-f(a)=(-1)^n\frac{h^n}{\lfloor n}f^n(a) \text{ approximately}$$

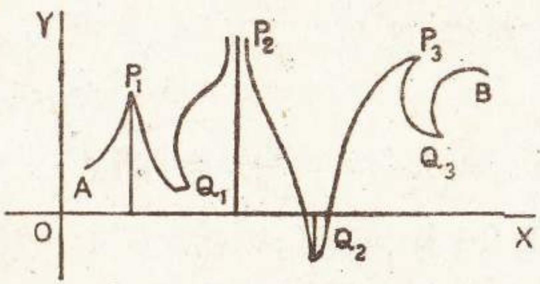
If n is even, then $f(a+h)-f(a)$ and $f(a-h)-f(a)$ have the same sign. So $f(a+h)-f(a)$ and $f(a-h)-f(a)$ are both negative if $f^{(n)}(a)$ is negative.

Similarly $f(x)$ has a minimum value at $x=a$ if $f^{(n)}(a)$ is positive.

If n is odd then $f(a+h)-f(a)$ and $f(a-h)-f(a)$ have different signs and $f(a)$ is neither a maximum nor a minimum if n is odd.

11.5. Maximum and minimum values when dy/dx or, $f'(x)$ is discontinuous.

Let $y=f(x)$ be a function of x The function is not continuous at points shown in the figure and $f'(x)$ is not continuous at some points, P_1, Q_1 and Q_2. (dy/dx) is undefined but y is finite



At P_2, both dy/dx and y are infinite At P_3 and Q_3, dy/dx is discontinuous.

How can we determine the maxima or minima values of $f(x)$ at these points?

Immediately before P_1, the curve rises and so $\frac{dy}{dx}>0$; immediately after P_1, $\frac{dy}{dx}<0$ since the curve falls. At P_1, the tangent to the curve is vertical and so $\frac{dy}{dx}$ is undefined. Thus at P_1, $y=f(x)$ is *maximum* and $\frac{dy}{dx}$ *changes sign from* $+ve$ to $-ve$.

Immediately before Q_1, the curve $y=f(x)$ falls and immediately after Q the curve rises. Hence Q is a point of *minimum* of $y=f(x)$ where $\frac{dy}{dx}$ *changes sign from* $-ve$ *to* $+ve$, although $\frac{dy}{dx}$ does not exist at Q.

Similar conclusions hold for at maxima and minima at points where $\frac{dy}{dx}$ is either dicontinuous or undefined.

11. 6. *In a rational integral algebraical function of the nth degree the greatest number of critical values is $n-1$ and these are alternately maxima and minima.*

Let $y=f(x)=a_0x^n+ax^{n-1}+\ldots+a_{n-1}x+a_n$ ... (1)

be an integral algebraical function (integral means there is no x in the denominator)

$\therefore \quad dy/dx=f'(x)=a_0nx^{n-1}+a_1(n-1)x^{n-2}+\ldots+a_{n-1}$ ... (2)

The necessary condition for maxima and minima is $f'(x)=0$

or, $a_0nx^{n-1}+a_1(n-1)x^{n-2}+\ldots+a^{n-1}=0$ ... (3)

The eq (3) is of $(n-1)$th degree, so it has $(n-1)$ roots, real or complex.

So, $f(x)$ has $(n-1)$ critical values if all the roots of $f(x)$ are real. From (2)

$$f'(x)=dy/dx=a_0n(x-\alpha_1)^2(x-\alpha_3)\cdots(x-\alpha_{n-1})\cdots(4)$$

where $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n$ are the roots of $f'(x)=0$

Let the roots be different and

$$\alpha_1<\alpha_2<\alpha_3<\ldots\ldots<\alpha_{n-1}$$

$$\text{Sign of}\left(\frac{dy}{dx}\right) \underset{\alpha_1}{---0} \underset{\alpha_2}{++++0} \underset{\alpha_3}{----0} \underset{\alpha_n}{---0} \underset{\alpha_{n-1}}{---0} ++++ \rightarrow x$$

Supposing $\frac{dy}{dx}<0$ for $x\rightarrow\alpha_1^-$ (which is true n is even), we see $\frac{dy}{dx}$ changes sign from $-ve$ to $+ve$ at α_1, from $+$ve to $-$ve at α^2, from $-ve$ to $+ve$ or α_3 and so on. Hence $y=f(x)$ is minimum at α_1, maximum at α_2, minimum at α_3 and so on. Hence maxima and minima occur alternately.

11.7. Inflexions

The necessary condition for maxima and minima if a function $y=f(x)$ is $f'(x)=0$

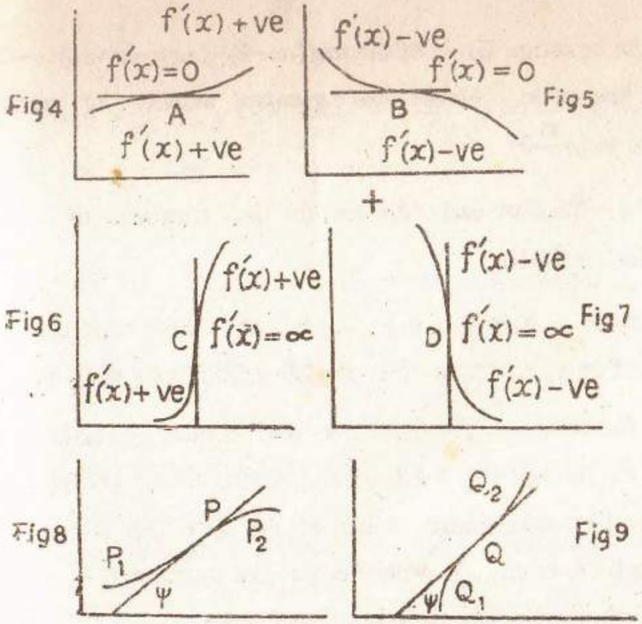
but this condition is not sufficient.

At a point P, $f'(x)$ may be zero, but it is *either* positive *or* negative on both sides of it, that is $f'(x)$ *does not change its sign* although it becomes zero at P. Such a point is called a point of inflexion.

In general, a point of inflexion is one at which a curve changes the sign of curvature or the direction of bending. Let $x=\alpha$ be point of inflexion. Then, if $f''(\alpha-h)>0$, we must have $f''(\alpha+h)<0$ or vice versa, where h is a small number, If the curve is smooth at $x=\alpha$, then $f''(\alpha)=0$. Hence $f''(\alpha)=0$ *is the necessary condition for the point $x=\alpha$ to be a point of inflexion.*

If d^2y/dx^2 is first $+ve$ for a given value of x say $x=\alpha$ at P then d^2y/dx^2 passes through zero and then becomes $-ve$, dy/dx first increases, then becomes stationary and then decreases, then the point P where $d^2y/dx^2=0$ is called a point of inflexion (Fig 8.)

Similarly if d^2y/dx^2 is first $-ve$, then passes through zero at Q and becomes $+ve$, then dy/dx first decreases then becomes stationary at Q and then increases. The necessary condition that dy/dx is maximum or minimum is $d^2y/dx^2=0$



Cor. The greatest number of points of inflexion is $(n-2)$ in an intergral algebraical function.

Let $y = a_0x^n + a_1x^{n-1} + \text{.........} + a_{n-1}\,x + a_n$

Then

$$\frac{dy}{dx} = a_0nx^{n-1} + a_1(n-1)x^{n-2} + \text{......} + a_{n-1}2x + a_{n-1}$$

$$\frac{d^2y}{dx^2} = a_0n(n-1)x^{n-2} + \text{......} + 2a_{n-1} \text{............} (1)$$

The condition for the existence of a point of inflexion is

$$\frac{d^2y}{dx^2} = 0$$

As the equation (i) is of degree $(n-2)$ if may have $(n-2)$ real roots at the most. **Hence the greatest number of points of inflexions is** $(n-2)$

11. 18. Maxima and Minima for the function of several independent variables.

Definition : Let $\phi(x, y, z\text{......})$ be any finite and continuous function of x, y, z...... at the neighbourhood of $(a, b, c,\text{.........})$.

The function $\phi(x, y, z,\text{........})$ is said to have a maximum value at $(a, b. c...)$if $\phi\,(a+h, b+k, c+l...) < \phi(a, b, c...)$ and $\phi(x, y, z...)$ is said to have a minimum value at $(a, b, c...)$ if $\phi\,(a+h, b+k, c+l...) > \phi\,(a, b, c\text{.......})$, whatever be the increments $h, k, l,$...etc. provided they are sufficiently small and finite.

11. 19. Necessary conditions for the existence of Maxima and Minima.

A function $\phi\,(x, y, z...)$ has maximum or minimum value at $(a, b, c,\text{.......})$ if the partial dervatives exist and

$$\frac{\delta\phi}{\delta x} = \frac{\delta\phi}{\delta y} = \frac{\delta\phi}{\delta z\phi} = \text{.........} = 0$$

As the function $\phi\,(x, y, z,\text{.....})$ is continuous at the neighbourhood of $(a, b, c\text{......})$ then we have by extended Taylors's Theorem

$$\phi\,(x+h, y+k, z+l...) = e^{\left(h\frac{\delta}{\delta x} + k\frac{\delta}{\delta y} + l\frac{\delta}{\delta z} + ...\right)}\phi(x, y, z...)$$

$$= [1 + (h\frac{\delta}{\delta x} + k\frac{\delta}{\delta y} + l\frac{\delta}{\delta z} + ...) + \frac{1}{\underline{|2}}(h\frac{\delta}{\delta x} + k\frac{\delta}{\delta y}$$

$$+ l\frac{\delta}{\delta z} + ...)^2 + \text{......})\phi\,(x, y, z\,\text{.....})$$

$$= \phi(x, y, z,\text{.......}) + (h\frac{\delta\phi}{\delta x} + k\frac{\delta\phi}{\delta y} + l\frac{\delta\phi}{\delta z} + \text{... ...}) + \text{terms}$$

of the 2nd and higher orders.

or, $(x+h, y+k, z+l...) - \phi\,(x, y, z\,..) = h\frac{\delta\phi}{\delta x} + k\frac{\delta\phi}{\delta y}$

$+ l\frac{\delta\phi}{\delta z} + ... +$ terms of the 2nd higher order h, k, l......(1)

As h, k, l...are very small, then $h\frac{\delta\phi}{\delta z} + k\frac{\delta\phi}{\delta y} + l\frac{\delta\phi}{\delta y} + \text{......}$ dominates the sign of Right handside of (1),

If we change the sign of $h, k, l\text{......}$, the sign of the Right side of (1) will change.

Hence the necessary condition for a maximum or minimum value is $h\frac{\delta\phi}{\delta x} + k\frac{\delta\phi}{\delta y} + l\frac{\delta\phi}{\delta z} + \text{......} + = 0$ .....(2)

Since (2) is always true whatever be the values of h, k, l.. (......as as h, k, l......etc, are all independent of one other,) we have

$$\frac{\delta \phi}{\delta x}=0, \frac{\delta \phi}{\delta y}=0, \frac{\delta \phi}{\delta z}=0 \quad \ldots \quad \ldots \qquad (3)$$

As there are n independent variables, so we will get n equations. Solving these equations we get the value of $a, b, c,$...... ..etc. Now put the values of x, y, z......in $\phi(x, y, z, \ldots)$ which is either a maximum or a minimum.

The conditions

$$\frac{\delta \phi}{\delta x}=0, \frac{\delta \phi}{\delta y}=0, \frac{\delta \phi}{\delta z}=0, \text{ etc.}$$

are necessary but not sufficient for the existence of maxima or minima.

11.19. Determination of the sign of quadratic expressions.

(i) $ax^2+2hxy+by^2$

(ii) $ax^2+2hxy+by^2+cz^2=2fyz+2gzx$

Let $I_2=ax^2+2hxy+by^2$

$=(1/a)(a^2x^2+2haxy+aby^2)=(1/a)\{(ax+hy)^2+(ab-h^2)y^2\}$

$(ax+hy)^2+(ab-h^2)y^2$ is positive if $ab-h^2>0$. Thus the sign of I_2 depends on the sign of a the co efficient of x^2

If $ab-h^2$ is negative, we cannot say anything about the sign of the expression. Let

$I_3=ax^3+by^2+cz^2+{}^{\cdot}hxy+2fyz+?gzx$

$=(1/a)\{a^2x^2+aby^2+acz^2+2ahxy+2afyz+2agzx\}$

$=(1/a\{a^2x^2+2ax(gz+hy)+aby^2+acz^2+2fazy\}$

$=(1/a)\{(ax+hy+gz)^2+(ab-h^2)y^2+2(af-gh)yz+(ac-g^2)z^2\}$

I_3 will have the same sign as a for real values of x, y, z if $(ab-h^2)y^2+2(af-gh)yz+(ac-g^2)z^2$ is positive

i. e., $(ab-h^2)$ and $\{(ab-h^2)(ab-g^2)-(af-gh)^2\}$ are positive i e., $(ab-h^2)$ and $a(abc+2fgh-af^2-bg^2-ch^2)$ are all $+ve$

Thus I_3 will be positive

if a, $\begin{vmatrix} a & h \\ h & b \end{vmatrix}$, $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$ are all positive

I_3 will be negative if these determinants are alternately negative and positive.

These findings will be used in determining maximum or minimum value of a function of two and three variables.

11.21. Two Independent variables. Lagrange's Conditions.

Let $\phi(x, y)$ be function of two variables in x and y

Let $r=\frac{\delta^2\phi}{\delta x^2}$, $s=\frac{\delta^2\phi}{\delta x\delta y}$, $t=\frac{\delta^2\phi}{\delta y^2}$, For $x=a$ and $y=b$,

By Taylors theorem, we have

$$\phi(a+h, b+k)=\phi(a, b)+h\left(\frac{\delta\phi}{\delta x}+\frac{\delta\phi}{\delta y}\right)+\frac{1}{\lfloor 2}\left(h^2\frac{\delta^2\phi}{\delta x^2}+\right.$$

$$\left.k^2\frac{\delta^2\phi}{\delta y^2}+2hk\frac{\delta^2\phi}{\delta x dy}\right)+\ \ldots \ \ldots \ \ldots$$

or, $\phi(a+h\ b+k)-\phi\ (a, b)=\frac{1}{2}(h^2r+2hks+tk^2)+R_3 \quad \ldots (1)$

where $\frac{\delta\phi}{\delta x}=\frac{\delta\phi}{\delta y}=0$

and R_3 contains terms with higher pawers of h and k and h k are very small. So the sign of $\phi(a+h, b+k)-\phi\ (a, b)$ depends upon the sign of

$I_2=(h^2r+2hks+tk^2)$

If the expression I_2 is positive, then $\phi(a, b)$ is minimum If I_2 is negative then $\phi(a, b)$ is maximum. Now,

I_2 will be positive if $h^2r+2hks+tk^2$ is positive *i. e,*

$rt-s^2$ is positive and r is positive. (See Art. 11 20)
Not that $rt-s^2>0 \Rightarrow rt>s^2>0$.

Thus if $rt-s^2$ is positive, $\phi(a, b)$ is maximum or minimum according as r and t are both negative or are both positive.

These are called Lagrange's condittons as these conditions were pointed out by him first.

If $rt-s^2$ is negative, $\phi(a,b)$ is neither a maximum nor a minimum.

If $rt=s^2>0$ then $I_2=rh^2+2shk+tk^2=(1/r)(hr+ks)^2$

$\therefore$ Thus the sign of I_2 is the same as the sign of r or t. So $\phi(a\ b)$ will be maximum or minimum if r is positive or negative.

If $rt=s^2$ and $(h/k)=-(s/r)=\beta$ (say) the 2nd degree terms in (1) will vanish. We are to consider the cubic terms of (1) which will again vanish as $h/k=\beta$ and then consider the 4th degree terms of (1) where we will get the sign of the entire 4th degree term depending on the signs of r and t when $(h/k)=\beta$.

11 22. Maximum and Minimum values of a function of two variables x and y, when x and y are connected by a relation.

Let $u=\phi(x, y)$ (1)

be a finite and continuous function of x and y. Let variables x and y be connected by a relation $f(x, y)=0$ ... ... (2)

Now eliminate y from (1) and (2), then a relations between u and x will be obtained. Now u is the function of x only.

From (2), we have $\dfrac{\delta f}{\delta x}dx+\dfrac{\delta f}{\delta y}dy=0$ or, $f_x dx+f_y dy=0$

or, $\dfrac{dy}{dx}=-\dfrac{\delta f}{\delta x}\Big/\dfrac{\delta f}{\delta y}=-f_x/f_y$... ... (3)

But from (1), the total differential is $du=\dfrac{\delta\phi}{\delta x}dx+\dfrac{\delta\phi}{\delta y}dy$

$$\frac{du}{dx}=\frac{\delta\phi}{\delta x}+\frac{\delta\phi}{\delta y}\frac{dy}{\delta x}=\frac{\delta\phi}{\delta x}+\frac{\delta\phi}{\delta y}(-f_x/f_y) \qquad [by\ (3)\ (4)]$$

As u is the function of x, so u will be maximum or minimum if

$$\frac{du}{dx}=0 \text{ i. e., } \frac{\delta\phi}{\delta x}+\frac{\delta\phi}{\delta y}(-f_x/f_y)=0$$

$$\text{or, } \frac{\delta\phi}{\delta x}f_y+\frac{\delta\phi}{\delta y}f_x=0 \quad \ldots \quad \ldots \quad (5)$$

Now solve (2) and (5) for x and y. These values are required for determining the maximum and minimum of u.

Now find d^2u/dx^2 put the value of x and y in d^2u/dx^2.

If d^2u/dx^2 is positive, then u is minimum.

If d^2u/dx^2 is negative, then u is maximum.

11.23. To determine the maximum or minimum values of a function of three independent variables at a point.

Let $\phi(x, y, z)$ be a function of x, y, z. We are to investigate at (a, b, c) whether $\phi(x, y, z)$ is maximum or minimum.

$$\text{Let } A=\frac{\delta^2\phi}{\delta x^2},\ B=\frac{\delta^2\phi}{\delta x^2},\ C=\frac{\delta^2\phi}{\delta z^2},\ F=\frac{\delta^2\phi}{\delta y\delta z},\ G=\frac{\delta^2\phi}{\delta z\delta x},\ H=\frac{\delta^2\phi}{\delta x\delta y}$$

For the existence of a maximum of a minimum at $(a\ b,\ c)$

$$\frac{\delta\phi}{\delta x}=\frac{\delta\phi}{\delta y}=\frac{\delta\phi}{\delta z}=0$$

By Taylor's Theorem

$$\phi(a+h,\ b+k,\ c+l)-\phi(a,\ b,\ c)=\tfrac{1}{2}(Ah^2+Bk^2+Cl^2+2Fkl+2Glh+2Hk)+R_3.$$

R_3 consists of terms of 3rd and higher powers of h, k, l which are very small. So the sign of (1) depends on the sign of

$I_3=Ah^2+Bk^2+Cl^2+2Fkl+2Glh+2Hhk$

Therefore ϕ (a b, b) is minimum

if A, $\begin{vmatrix} A & H \\ H & B \end{vmatrix}$, $\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$ are all positive.

and ϕ (a, b, c) is maximum if

A, $\begin{vmatrix} A & H \\ H & B \end{vmatrix}$, $\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$

are alternately negative and positive.

If these conditions are not satisfied, then ϕ (a, b, c) is neither a maximum nor a minimum.

11.24 Lagrange's method of Finding Undetermined Multipliers.
For Several Independent variables

Let $u=\phi\,(x_1, x_2, \ldots\ldots x_n)$ be a function of n variables $x_1, x_2, \ldots x_n$

Let these variables be connected by m equations

$f_1\,(x_1, x_2\ldots\ldots x_n)=0,\ f_2(x_1, x_2\ldots\ldots\ldots x_n)=0\ldots\ldots\ldots$

$f_m\,(x_1, x_2, \ldots\ldots\ldots x_n)=0$

So, only $(n-m)$ of the variables are independent. We are to find equations for $(n-m)$ remaining variables which will determine a point where u is minimum or maximum.

When u is a maximum or minimum.

$$\left.\begin{aligned}
du &= \frac{\delta u}{\delta x_1}dx_1 + \frac{\delta u}{\delta x_2}dx_2 + \ldots\ldots\ldots + \frac{\delta u}{\delta x_n}dx_n = 0 \\
df_1 &= \frac{\delta f_1}{\delta x_1}dx_1 + \frac{\delta f_1}{\delta x_2}\delta x_2 + \ldots + \frac{\delta f_1}{\delta x_n}dx_n = 0 \\
df_2 &= \frac{\delta f_2}{\delta x_1}dx_1 + \frac{\delta f_2}{\delta x_2}dx_2 + \ldots\ldots + \frac{\delta f_2}{\delta x_n}dx_n = 0 \\
&\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
&\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
df_m &= \frac{\delta f_m}{\delta x_1}dx_1 + \frac{\delta f_m}{\delta x_2}dx_2 + \ldots + \frac{\delta f_m}{\delta x_n}dx_n = 0
\end{aligned}\right\} \ldots(1)$$

In order to avoid total elimination we may make use of undetermined multipliers. Now multiply equations of (1) by $1, \lambda_1, \lambda_2 \ldots\ldots, \lambda_m$ respectively and add then $P_1dx_1+P_2dx_2+\ldots P_ndx_n=0\ldots(2)$

Where $P_r = \dfrac{\delta u}{\delta x_r} + \lambda_1\dfrac{\delta f_1}{\delta x_r} + \dfrac{\delta f_2}{\delta x_r}\lambda_2 \ldots\ldots + \lambda_m\dfrac{\delta f_m}{\delta x_r}$ (3)

Let us select $\lambda_1, \lambda_2 \ \ldots \ \ldots \ \ldots \ \lambda_m$ in such a way that they satisfy m linear equations.

If $P_1=P_2=P_3= \quad \ldots \quad \ldots \quad \ldots = P_m = 0 \quad \ldots \quad \ldots(4)$

then the equation ( ) reduces to

$P_{m+1}\,dx_{m+1} + P_{m+2}dx_{m+2} + \ldots \ldots + P_n\,dx_n = 0 \quad \ldots \quad \ldots \quad (5)$

Let $n-m$ quantities $x_{m+1}, x_{m+2} \ \ldots \ \ldots \ x_n$ be independent variables, As $n-m$ quantities.

$dx_{m+1}, dx_{m+2}, \ \ldots \ \ldots \ \ldots dx_n$ are all independent, so their co-efficients must be separately zero. Thus from (5)

$P_{m+1} = P_{m+2} = \quad \ldots \quad \ldots \quad = P_n = 0 \quad \ldots \quad \ldots \quad (6)$

Therefore we get

$$\left.\begin{aligned}
&P_1=P_2=P_3 \quad \ldots \quad \ldots \quad = P_m = 0 \\
&P_{m+1}=P_{m+2}= \quad \ldots \quad \ldots \quad = P_n = 0 \\
&\text{along with } f_1=f_2=f_3 \quad \ldots \quad = f_n = 0
\end{aligned}\right\}$$

These constitute a set of $m+n$ equations.

These will determime m multipliers $\lambda_1, \lambda_2 \ \ldots \ \ldots \ \lambda_m$, the value of u and a point where u is maximum or minimum.

Art 24(a) The significance of the Lagranges multiplier

Let m be the maximum (or minimum) value of $f(x,y)$ subject to the contraint $g(x,y)=k$ As the value of k, of the constraint function values, so does the corresponding optimal value m of the function f. The Lagrange multiplier λ is the rate of change of m with respect to k. That is $\lambda = dm/dk$

Art.11.25 If $F(x, y, z)$ be subject to constraint $G(x, y, z)=0$, prove that a necessary condition that $F(x, y, z)$ have an extreme value is $F_xG_y - F_yG_x = 0$

As $G(x, y, z)=0$, so we may consider z as function of x and y,

$z=f(x, y)$;

Now $F(x, y, z)$ is equivalent to $F[x, y, f(x, y)]$

A necessary condition that $F[x, y, f(x, y)]$ have an extreme value is that partial derivatives with respect to x and y be zero. Thus

$F_x + F_zZ_x = 0$ ... (1), $F_y + F_zZ_y = 0$ ... (2)

Again $G(x, y, z)=0$, so

$G_x + G_zZ_x = 0$ ... (3) ; $G_y + G_zZ_y = 0$ ... (4)

Now eliminate Z_x from (1) and (3),

$F_xG_z - F_zG_x = 0$ ... ... (5)

Eliminate Z_y from (2) and (4), $F_yG_z - F_zG_y = 0$ ... (6)

Eliminate G_z and F_z from (5) then

$\begin{vmatrix} F_x & -G_x \\ F_y & -G_y \end{vmatrix} = 0$ or, $F_xG_y - F_y\, G_x = 0$ D. H. 1987

Alternative Method

Let $\phi = F + \lambda G$ λ is a constant

The necessary conditions for extreme value of ϕ are

$\phi_x = 0$, $\phi_y = 0$ *i. e.*

$F_x + \lambda G_x = 0$, $F_y + \lambda G_y = 0$. Eliminate λ, then the condition is

$F_xG_y - F_y\, G_x = 0$

Ex. 1. Find the maximum and minimum values of

$2x^3 - 9x^2 + 12x - 3$ C. U. 1989

Let $f(x) = 2x^3 - 9x^2 + 12x - 3$ ... ... (1)

$\therefore$ $f'(x) = 6x^2 - 18x + 12$ ... ... (2)

As $f(x)$ is maximum or minimum $f'(x)=0$ *i. e.*,

$6x^2 - 18x + 12 = 0$

or, $x^2 - 3x + 2 = 0$ or, $(x-1)(x-2) = 0 \Rightarrow x = 1, 2$

Again $f''(x) = 12x - 18$

when $x=1$, $f''(x) = 12 - 18 = -ve$

when $x=2$, $f''(x) = 24 - 18 = 6 = +ve$.

Hence $f(x)$ will be maximum when $x=1$ and the maximum value is $f(1) = 2.1 - 9.1 + 12.1 - 3 = 2$

$f(x)$ will be minimum when $x=2$ and the minimum value is $f(2) = 2.8 - 9.4 + 12.2 - 3 = 1$

Ex. 2. Find the maximum and minimum values of

$x^3 - 3x^2 + 3x + 1$ R. U. 1966

Let $f(x) = x^3 - 3x^2 + 3x + 1$ ... ... (1)

$\therefore$ $f'(x) = 3x^2 - 6x + 3$ ... ... (2)

The necessary condition for the existence of maxima or minima is $f'(x)=0$ or, $3x^2 - 6x + 3 = 0$ or, $x^2 - 2x + 1 = 0$

or, $x=1$ Again $f''(x) = 6x - 6$

when $x=1$, $f''(x) = 0$ Again $f'''(x) = 6$

Thus $f(x)$ has no maximum or minimum.

Alternative method ;—

Let $y=x^3-3x^2+2x+1$

$\therefore \quad \frac{dy}{dx}=3x^2-6x+3=3(x^2-2x+1)=3(x-1)^2$

which is always positive can never change sign.

Hence y has neither a maximum nor a minimum.

Ex. 3. Discuss the maximum and minimum values of the function $f(x)=x^5-5x^4+5x-1$ R. U. 1987, D.U. 1983

$f(x)=x^5-5x^4+5x^3-1 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$

$\therefore \quad f'(x)=5x^4-20x^3+15x^2 \quad \ldots \quad \ldots \quad \ldots \quad (2)$

For the existence of maximum or minimum ; $f'(x)=0$

or, $5x^4-20x^3+15x^2=0$

or, $x^2(x^2-4x+3)=0$ or, $x^2(x-1)(x-3)=0 \therefore x=0, 1, 3$

Therefore $f(x)$ may have exterme values for $x=0, 1, 3$ only.

Now we are to investigate for the maximum or minimum at $x=0$.

$f'(x)=5x^4-20x^3+15x^2=5x^2(x-1)(x-3) \quad \ldots \quad \ldots \quad (3)$

sign of $f'(x)$

$$\underset{}{+ + + + + + \underset{0}{0} + + + + \underset{1}{0} - - - - - \underset{3}{0} + + +} \rightarrow 0$$

Now for $x<0, f'(x)=(+)(-)(-)=+ve$ i. e. $x<0, f'(x)>0$.

for $0<x<1$ i. e : $f'(x)=(+)(-)(-)=+ve$

Thus $f'(x)$ does not change sign when passes through $x=0$.

Therefore $f(x)$ has no maximum or minimum at $x=0$

At $x=1$, from (3), if $x<1, f'(x)=(+)(-)(-)=+ve$

if $1<x<3$ i. e. for $x>1$ but $x<3$, $f'(x)=(+)(+)(-1)=-ve$

Thus $f'(x)$ changes sign from $+ve$ to $-ve$ when passes through $x=1$

Hence $f(x)$ is maximum at $x=1$ and the maximum value of $f(x)$ is $f(1)=1-5+5-1=0$ from (1) At $x=3, f'(x)=0$;

if $1<x<3$, $f'(x)=(+)(+)(-1)=-ve$,

if $x>3$, $f'(x)=(+)(+1)(+)=+ve$.

Thus $f'(x)$ changes sign from $-ve$ to $+ve$ when passes through $x=3$

Hence $f'(x)$ is minimum for $x=3$ and the minimum value is $f(3)=-28$ [from (1)]

Ex. 4. Find the maximum or the minimum value of y when $y^7=(x-3)^4 \quad \ldots \quad \ldots \quad (1)$

Let $f(x)=y=(x-3)^{4/7} \quad \therefore \quad f'(x)=4/7 \dfrac{1}{(x-3)^{3/7}} \ldots (2)$

Here $f'(x)=0$ means $4/7=0$ which is absurd

If $dy/dx=f'(x)=\infty$, then $x-3=0$ or, $x=3$

We are to investigate maximum or minimum for $x=3$

If $x<3$ i. e., $x=3-h$ where $h\to 0$, then from (2)

$$f'(x)=\frac{4}{7}\frac{1}{(3-h-3)^{3/7}}=\frac{4}{7}\left(-\frac{1}{h}\right)^{3/7}=-ve$$

If $x>3$ i. e $x=3+h \quad h\to 0$

$$f'(x)=\frac{4}{7}\frac{1}{(3+h-3)^{3/7}}=\left(\frac{1}{h}\right)^{3/7}=+ve$$

Thus dy/dx or, $f'(x)$ changes sign from $-ve$ to $+ve$ when passes through $x=3$. Hence $f(x)$ is minimum for $x=3$.

Ex.5. Find the maximum or the minimum value of $y=f(x)$, if

$$f'(x)=\frac{2^{1/x}+1}{1-2^{1/x}}$$

If $x<0$ i, e , $x=0-h, h\to 0$, then

$$f'(x)=\lim_{h\to 0}\frac{2^{-1/h}+1}{1-2^{-1/h}}=\frac{2^{-\infty}+1}{1-2^{-\infty}}=\frac{0+1}{1-0}=1 \text{ i. e., } f'(x)$$

is positive $x=0+h, h\to 0^+$then

$$f'(x)=\lim_{h\to 0}\frac{2^{1/h}+1}{1-2^{1/h}}=\lim_{h\to 0}\frac{2^{1/h}(1+2^{-1/h})}{2^{1/h}(2^{-1/h}-1)}=\frac{1+0}{0-1}=-1$$

i. e., $f'(x)$ is negative.

L. H. Limit $\neq$ *R. H.* Limit. Here $f'(x)$ or, dy/dx is discontinuous at $x=0$ but dy/dx changes from $+ve$ to $-ve$ when passes through $x=0$. Hence $f'(x)$ or, y is maximum at $x=0$.

Ex. 6. Show that $e^x+e^{-x}-x^2$ has a minimum value for $x=0$.

Let $f(x)=e^x+e^{-x}-x^2 \quad \cdots \quad \cdots \quad (1)$

$\therefore \; f'(x)=e^x-e^{-x}-2x \quad \cdots \quad \cdots \quad (2)$

Putting $f'(x)=0$ we have $e^x-e^{-x}-2x=0 \quad \cdots \quad (3)$

or. $\left(1+x+\dfrac{x^2}{\lfloor 2}+\cdots\cdots\right)-\left(1-x+\dfrac{x^2}{\lfloor 2}-\cdots\right)-2x=0$

Thus we see that $x=0$ is a solution of (3)

Again $f''(x)=e^x+e^{-x}-2$, $f'''(x)=e^x-e^{-x}$, $f^{iv}(x)=e^x+e^{-x}$

When $x=0$. $f''(x)=1+1-2=0$, $f'''(x)=0$, $f^{iv}(x)=2=$ positive. Thus we se that $f^{iv}(x)$ is the first derivative which does not vanish and $f^{iv}(x)$ is positive *i. e.*, $f(x)$ is minimum at $x=0$

Ex. 7. Determine the maximum and minimum value of the function $\sin x+\cos 2x$; $0<x<2\pi$

Let $f(x)=\sin x+\cos 2x \quad \cdots \quad \cdots \quad (1)$

$\therefore \; f'(x)=\cos x-2\sin 2x=\cos x(1-4\sin x)$

Put $f'(x)=0$. then $\cos x\,(1-4\sin x)+0\cos x=0$ or, $\sin x=\frac{1}{4}$

Now $\cos x=0=\cos \pi/_2=\cos 3\pi/_2 \qquad x=\pi/2, 3\pi/2$

When $\sin x=\frac{1}{4}=\sin\alpha$ (say) then $x=\alpha$, $\pi-\alpha$, $\alpha=\sin^{-1}\frac{1}{4}$

Again $f''(x)=-\sin x-4\cos 2x$

when $x=\pi/_2$, $f''(x)=-\sin \pi/_2-4\cos 2\,\pi/_2=3$

When $x=\pi/_2$! $f''(x)=-\sin 3\pi/_2-4\cos 2.\,3\pi/_2=5$

Thus $f(x)$ is minimum for $x=\pi/2$ and $x=3\pi/2$.

From (1), values are $f(\frac{1}{2}\pi)=0$, $f(3\pi/_2)=-2$

Again for $x=\alpha$, or, $\sin\alpha=\frac{1}{4}$.

$f''(x)=-\sin x-4\cos x 2x=-\frac{1}{4}-4\,(1+2.\,1/16)=-15/4$

for $x=\pi-\alpha=\pi-\sin^{-1}\frac{1}{4}$.

$f''(x)=-\sin(\pi-\sin^{-1}\frac{1}{4})-4\cos(2\pi-2\sin^{-1}\frac{1}{4})$

$=-\sin^{-1}\frac{1}{4}-4\cos(2\sin^{-1}\frac{1}{4})=-\frac{1}{4}-4\,\{1-2\sin^2(\sin^{-1}\frac{1}{4})\}$

$=-\frac{1}{4}-4\,(1-2.\,1/16)=-\frac{1}{4}-7/2=-15/4$

Thus $f(x)$ is maximum for $x=\sin^{-1}\frac{1}{4}$ and $x=\pi-\sin^{-1}\frac{1}{4}$

And from (1), the values are

$f(\sin^{-1}\frac{1}{4})=\sin(\sin^{-1}\frac{1}{4})+1-2\sin^2(\sin^{-1}\frac{1}{4})=9/8.$

$f(\pi-\sin^{-1}\frac{1}{4})=\sin(\pi-\sin^{-1}\frac{1}{4})+1-2\sin^2(\pi-\sin^{-1}\frac{1}{4})=\frac{1}{4}+1-\frac{1}{8}$

$=9/8.$

Ex. 8. Show that the maximum value of $(1/x)^x$ is $e^{1/e}$

Let $y=(1/x)^x \quad \cdots \quad \cdots \quad \cdots \quad (1)$

or, $\log y=x\log(1/x)=-x\log x$

$\therefore \; \dfrac{1}{y}\dfrac{dy}{dx}=-x\times 1/x-\log x.=-1-\log x$

$\therefore \; \dfrac{dy}{dx}=y(-1-\log x) \quad \cdots \quad - \quad (2)$

For the maximum or minimum value of y, $dy/dx=0$

or, $y(-1-\log x)=0$. or, $\log x=-1$ as $y\neq 0$, $x=e^{-1}$

Again $\dfrac{d^2y}{dx^2}=\dfrac{dy}{dx}(-1-\log x)-(y/x)$

$=(1/x^x)(-1-\log x)^2-(1/x)^x\,1/x$ by (1) and (2) when

$x=e^{-1}$, then $d^2y/dx^2=e^{1/e}\,\{(-1-\log e^{-1})^2-e\}=-e^{1/e}e$,

which is negative.

Hence y is maximum for $x=e^{-1}$, the maximum value is

$y=\left(\dfrac{1}{e^{-1}}\right)^{e^{-1}}=e^{e^{-1}}=e^{1/e}$ Proved

Ex. 9. Find the maximum and minimum values of

$x^2+2y^2-4x+4y-3$ R. U, 1965

Let $\phi(x,y)=x^2+2y^2-4x+4y-3$

$\phi_x=2x-4$; $\phi_y=4y+4$; $r=\phi_{xx}=2$

$t=\phi_{yy}=4$; $s=\phi_{xy}=0$

$\phi(x,y)$ may have critical values if $\phi_x=0$ and $\phi_y=0$

or, $2x-4=0$ and $4y+4=0$ $\therefore$ $x=2$ and $y=-1$

Thus $\phi(x,y)$ may have a critical value at $(2,-1)$

Now $rt-s^2=2.4-0=8=+ve$ and r and t are both positive

Hence $\phi(x,y)$ has a minimum value at $(2,-1)$

Ex. 10. Show that the maximum values of $xy(a-x-y)$ is $1/27a^3$

Let $\phi(x,y)=xy(a-x-y)=axy-x^2y-xy^2$ ... ... (1)

$\therefore$ $\phi_x=ay-3xy-y^2$, $\phi_y=ax-x^2-2xy$

$\phi(x,y)$ is maximum. or, minimum if

$\phi_x=0$ and $\phi_y=0$ i.e. ; $ay-2xy-y^2=0$, $ax-x^2-2xy=0$

Solve for x and y and the points are $(0,0)$, $(0,a)$, $(a,0)$, $(\frac{1}{3}a, \frac{1}{3}a)$

Now we are to investigate the nature of the critical points at these points

Again $r=\phi_{xx}=-2y$, $s=\phi_{xy}=a-2x-2y$, $t=-2x$

For $(0,0)$

$rt-s^2=(-2y)(-2x)-(a-2x-2y)^2=0-a^2=-a^2=-ve$

There is no maximum or, minimum value of $\phi(x,y)$ at $(0,0)$.

For $(0,a)$ $rt-s^2=0-(a-2a)^2=-ve$

No critical value of $\phi(x,y)$ at $(a,0)$

For $(\frac{1}{3}a, \frac{1}{3}a)$ $rt-s^2=(-\frac{2}{3}a)(-\frac{2}{3}a)-(a-\frac{2}{3}a-\frac{2}{3}a)^2=\frac{1}{3}a^2$

Thus $rt-s^2$ is positive and r and t both negative

Hence $\phi(x,y)$ is maximum at $(\frac{1}{3}a, \frac{1}{3}a)$ and the maximum value is $\phi(\frac{1}{3}a, \frac{1}{3}a)=1/27a^3$ from (1)

Ex. 11. Show that the function ϕ, where

$\phi(x,y,z)=x^2+y^2+z^2+x-2z-xy$ has a minimum value at $(-2/3, -1/3, 1)$

Let $\phi=x^2+y^2+z^2+x-2z-xy$

$\therefore$ $\phi x=2x+1-y$, $\phi y=2y-x$, $\phi z=2z-2$

$\phi(x,y,z)$ will be maximum or minimum if $\phi_x=0$, $\phi_y=0$, $\phi_z=0$

or $2x-y+1=0$, $-x+2y=0$, $2z-2=0$

Solve for x, y and z, then $x=-2/3, y=-\frac{1}{3}, z=1$

We are to investigate the nature of critical point at $(-2/3, -1/3, 1)$ if there is any.

Again $A=\phi_{xx}=2$, $B=\phi_{yy}=2$, $C=\phi_{zz}=2$,

$F=\phi_{yz}=0$, $G=\phi_{zx}=0$, $H=\phi_{xy}=-1$

Hence from Art, 11.23

$$A=2. \begin{vmatrix} A & H \\ H & B \end{vmatrix}=\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}=3 \text{ and}$$

$$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}=\begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix}=6$$

As all them are positive, $\phi(x,y,z)$ is a minimum at $(-2/3,-1/3,1)$

The minimum value is $\phi(-\frac{2}{3},-\frac{1}{3},1)=-4/3$

Ex. 12. Show that the minimum value of C. H, 1977

$x^2+y^2+z^2$ when $ax+by+cz=p$ is $p^2/(a^2+b^2+c^2)$

Let $\phi=x^2+y^2+z^2$ (1)

and $ax+by+cz=p$ or, $z=(p-ax-by)/c$ (2)

Now $\phi=x^2+y^2+(p-ax-by)^2/c^2$ [by (2)]

$\therefore$ $\phi_x=2x-(2a/c^2)(p-ax-by)$, $\phi_y=2y-(2b/c^2)(p-ax-by)$

ϕ may be maximum or minimum if $\phi_x=0$ and $\phi_y=0$

or, $2x-(2a/c^2)(p-ax-by)=0$ and $2y-(2b/c^2)(p-ax-by)=0$

or, $x(c^2+a^2)-ap+aby=0$ ... ... (3)

and $y(c^2+b^2)-bp+abx=0$ ... ... (4)

solving these

$x = ap/(a^2+b^2+c^2),\ y = bp/(c^2+b^2+a^2)$ (5)

Again $r = \phi_{xx} = 2+2a^3/c^2,\ s = \phi_{xy} = 2ab/c^2,\ t = \phi_{yy} = 2+2b^2/c^2$

and $rt - s^2 = 4(1+a^2/c^2+b^2/c^2) = +ve$

Also r and t are both positive. Hence ϕ is minimum.

For values of x and y given by (5).

$z = \{p - a^2p/(a^2+b^2+c^2) - b^2p/(a^2+b^2+c^2)\}/c = cp/(a^2+b^2+c^2)$

Now from (1),

$$\phi = \frac{a^2p^2+b^2p^2+c^2p^2}{(a^2+b^2+c^2)^2} = \frac{p^2(a^2+b^2+c^2)}{(a^2+b^2+c^2)^2} = p^2/(a^2+b^2+c^2)$$

The minimum value of ϕ is $p^2/(a^2+b^2+c^2)$.

Ex. 13. Find the minimum value of $x^2+y^2+z^2$ with conditions $ax+by+cz=1,\ a_1x+b_1y+c_1z=1$ and interpret the result geometrically. R, H, 1966, D, H. 1961.

Let $u = x^2+y^2+z^2$ ... ... (1)

$ax+by+cz=1$ ... ... (2)

$a_1x+b_1y+c_1z=1$ ... ... (3)

$\therefore\ du = 2xdx+2ydy+2zdz = 0$ ... (4)

$adx+bdy+cdz=0$ ... ... (5)

$a_1dx+b_1dy+c_1dz=0$ ... ... (6)

Taking $(4)\times\frac{1}{2}+(5)\times\lambda+(6)\times\lambda_1$.

$(x+a\lambda+a_1\lambda_1)\,dx+(b+b\lambda+b_1\lambda_1)dy+(z+c\lambda+c_1\lambda_1)dz=0$

$$\text{or,}\quad \left.\begin{aligned} x+a\lambda+a_1\lambda_1&=0\\ y+b\lambda+b_1\lambda_1&=0\\ z+c\lambda+c_1\lambda_1&=0\end{aligned}\right\}\quad \ldots\quad \ldots\quad (7)$$

Now multiply equations of (7) by x, y, z respectively and then add

$x^2+y^2+z^2+\lambda(ax+by+cz)+\lambda_1(a_1x+b_1+c_1z)=0$

or, $u+\lambda+\lambda_1=0$ ... ... (8)

Again multiply the equations of (7) by a, b, c, respectively and then add

$(ax+by+cz)+(a^2+b^2+c^2)\lambda+(aa_1+bb_1+cc_1)\lambda_1=0$

or, $1+\lambda\Sigma a^2+\lambda\ \Sigma aa_1=0$ ... ... (9)

Again multiply the equations of (7) by $a, b_1\ c_1$ respectively and add :

$a_1x+b_1y+c_1z+(aa_1+bb_1+cc_1)\lambda+(a^2+b^2-c^2)\lambda_1=0$

or, $1+\lambda\Sigma aa_1+\lambda_1\Sigma a^2=0$ ... ... (10)

Thus $u+\lambda+\lambda_1=0,\ 1+\lambda\Sigma a^2+\lambda_1\Sigma aa_1=0,\ 1+\lambda\Sigma aa_1+\lambda_1\Sigma a^2=0$

Eliminate λ and λ_1 from the above equations.

$$\begin{vmatrix} u & 1 & 1\\ 1 & \Sigma a^2 & \Sigma aa_1\\ 1 & \Sigma aa_1 & \Sigma a^2\end{vmatrix}=0$$

which gives the maximum or minimum value of u.

λ, λ_1, can be obtained by solving (9) and (10) and then x, y, z can be found from (7)

Ex. 14, Divide a number a into three parts such that their product shall be maximum.

Let a be the number. Let x = first part, y = 2nd part and $z = a-x-y$ = third part. Their product is

$f(x,y) = xy(a-x-y) = axy-x^2y-xy^2$

$\delta f/\delta x = ay-2xy-y^2$ and $\delta f/\delta y = ax-x^2-2xy$

$f(x, y)$ will be maximum or minimum if $\delta f/\delta x=0$ and $\delta f/\delta y=0$

or, $ay-2xy-y^2=0$ xnd $ax-x^2-2yx=0$

or, $a-2x-y=0$ and $a-x-2y=0$

Solve for x and y. They are $x=\frac{1}{3}a$ ard $y=\frac{1}{3}a$

Again $\delta^2 f/\delta x^2=r=2y$, $\delta^2 f/\delta y^2=t=-2x$,

and $\delta^2 f/\delta x\delta y=s=a-2x-2y$ at $x=\frac{1}{3}a$, $y=\frac{1}{3}a$

$r=-2a/3$, $t=-2a/3$, $s=-a/3$

$\therefore$ $rt-s^2=(-2a/3)(-2a/3)-a^2/9=a^2/3=+ve$ and r and t are negative. Hence maximum if $x=y=z=\frac{1}{3}a$.

This is the same problem as Ex10

Ex. 15. Find the shape of a quart-can open at the top which requires for its construction the least amount of tin.

Let $r=$radius of the base $h=$depth of the cone.

Area of the base and curved surface$=\pi r^2+2\pi rh$ ... ... (1)

Let v be the volume of the cone

$\therefore$ $v=\pi r^2h$, v is constant or, $h=v/\pi r^2$ ... ... (2)

$\therefore$ $A=\pi r^2+2\pi rv/\pi r^2=\pi r^2+2v/r$... ... ... (x)

$\delta A/\delta r=2\pi r-2v/r^2$, area is to be minimum.

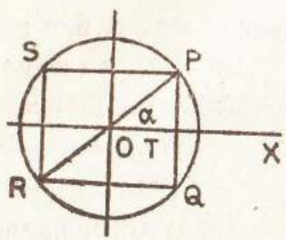
$\delta A/\delta r=0$ i. e., $2\pi r=(2v/r^2)=0$ or. $r=(v/\pi)^1/^3$

$\therefore$ $\delta^2A/\delta r^2=2\pi+4v/r^2$

$\therefore$ $\delta^2A/\delta r^2=2\pi+4\pi=+ve$ for $r(v/\pi)^1/^3$

Hence Area is minimum when $r=(v/\pi)^1/^3$ and $h=(v/\pi)^1/^3$.

Ex. 16. Prove that of all rec angles that can be inscribed in a given circle the square has the greatest area. Also show that the square will have the maximum perimeter as well. D. U. 1967, 83



Let $P(a\cos\theta, a\sin\theta)$ be any point on the circle $x^2+y^2=a^2$ ... (1)

It is evidened that

$PQ=2PT=2OP\sin\alpha=2a\sin\alpha$

$QR=2a\cos\alpha$

Now area $PQRS=2a\sin\alpha\ 2a\cos\alpha$

$=2a^2\sin 2\alpha$

Area will be maximum if $\sin 2\alpha$ is maximum i, e.,

$\sin 2\alpha=1=\sin\frac{1}{2}\pi$ or. $\alpha=\frac{1}{4}\pi$

In this case

$PQ=2a\sin\alpha=2a\sin\frac{1}{4}\pi=2a/\sqrt{2}=\sqrt{2}a$

also $QR=2a\cos\alpha=2a/\sqrt{2}=\sqrt{2}a$. Hence $PQ=QR$.

Thus the rectangle PQRS is a square

again perimeter$=C=2(PQ+QR)=2\{2a\sin\alpha+2a\cos\alpha\}$

$=4a(\sin\alpha+\cos\alpha)=4a\{\sin\alpha+\sin(\pi/2-\alpha)\}$

$=4a\ .\ 2\sin\left\{\frac{\pi/2-\alpha+\alpha}{2}\right\}\cos\left\{\frac{\pi/2-\alpha-\alpha}{2}\right\}$

$\therefore$ $C=8a\sin\pi/4\cos(\pi/4-\alpha)$ $\therefore$ Value of C (Perimeter) would be greatest if $\cos(\pi/4-\alpha)$ be greatest. i, e $\cos(\pi/4-\alpha)=1=\cos 0$

$\therefore$ $\pi/4-\alpha=0$ or ; $\alpha=\frac{1}{4}\pi$

$\therefore$ $PQ=2a\sin\alpha=2a\sin\pi/4=2a\ .\ 1/\sqrt{2}=\sqrt{2}a$

$PR=2a\cos\alpha=2a\cos\pi/4=2a\ .\ 1/\sqrt{2}=\sqrt{2}a$ $\therefore$ PQ=PR i.e. when a rectangle becomes a square, it will have the maximum perimeter.

Ex. 17. If a triangle has a given base and if the sum of the other two sides be given, prove that the area is greatest when these two sides ar. equal. R. U. 1954

Let ABC be the triangle, $BC=$base of the triangle$=a$ (given)

Let $AB=x$, $CA=y$ where $x+y=$constant$=k$ (say). We know

$2S=$Perimeter of the triangle$=AB+BC+CA$

$=a+x+y=a+k=$constant, i. e. S constant.

Area of the ABC.

$\Delta=\sqrt{\{s(s-a)(s-x)(s-y)\}}=\sqrt{\{m(s-x)(s-y)\}}$

where $m=s(s-a)$=constant.

Thus Δ will be greatest if the product $(s-x)(s-y)$ is greatest and $(s-x)(s-y)$ will be greatest if factors are equal

i. e; $s-x=s-y$ or, $x=y$. *i. e*; $AB=CA$.

Hence area of $\Delta\ ABC$ will be greatest if $AB=CA$.

Ex. 18. Show that the volume of greatest cylinder which can be inscribed in a cone of height h and semivertical angle α is $4/27\pi h^3 \tan^2\alpha$.

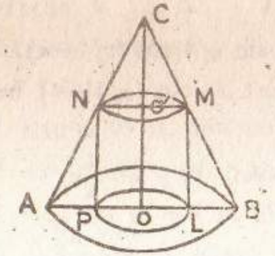
Let $LMNP$ be the inscribed cylinder of the cone ABC

Let

height of the cone$=h$

radius of the cone$=a$

radius of the cylinder$=x$



height of the cylinder$=ML=LB\cot\alpha=(OB-OL)\cot\alpha$ $=(a-x)\cot\alpha$ where a and α are constants.

Let V=volume of the cylinder.

$V=\pi.\ OL^2.\ LM=\pi^2x(a-x)\cot\alpha=(\pi\cot\alpha)(ax^2-x^3)$.

$dV/dx=(\pi\cot\alpha)(2ax-3x^2)$. If V is to be maximum or minimum

$dV/dx=0$ *i. e.* $2ax-3x^2=0$ or, $x=0,\ 2a/3$

Again $(d^2V/dx^2)=(x\cot\alpha)(2a-6x)$: When $x=2a/3$

(rejecting $x=0$,) $d^2V/dx^2=(\pi\cot\alpha)(2a-6.2\,a.3)=-ve$

Hence V is maximum when $x=2a/3$

Then cylinder of greatest volume.

$V=\pi/x^2(a-x)\cot\alpha=\pi(4a^2/9)(a-2a/3)\cot\alpha=4/27\,\pi a^2\cot\alpha$

is inscrived in the cone. But $a=h\tan\alpha$.

$$\therefore\ V=\frac{4}{27}.\ \pi(h\tan\alpha)^3\frac{1}{\tan\alpha}=\frac{4}{27}\pi h^3\tan^2\alpha \text{(Proved)}$$

Ex. 19. Prove that of all rectangalar parallelopipeds of the same volume the cube has the least surface.

Let x, y, z be the length, breadth and height of the cuboid.

Let V be the volume and S, the surface of the cuboid.

Then

$V=xyz$ ... ... (1) ; $S=2xy+2yz+2zx$ ... ... (2)

$\therefore$ $dS=2(v+z)dx+2(z+x)dy+2(x+y)dz$ ... ... (3)

$dV=0=yzdx+zxdy+xydz$ ... ... (4)

as the volume V is constant

For the surface S is to be maximum or minimum if $dS=0$,

or, $(y+z)dx+(z+x)dy+(x+y)dz=0$ ... (5)

Also $yzdx+zxdy+xydz=0$ ... ... (6) from (4).

Now multiply (5) $\times$ 1. and (6) $\times\lambda$ and add.

$(y+z+\lambda yz)dx+(z+x+\lambda xz)dy+(x+y+\lambda xy)dz=0$

Equate the co-efficients of dx, dy, dz to zero.

$\therefore$ $y+z+\lambda yx=0$ ... (7), $z+x+\lambda zx=0$ ... ... (8),

$x+y+\lambda xy=0$ ... (2)

or, $1/z+1/y=1/x+1/z=1/y+1/x=-\lambda$

From two equations we have $\therefore$ $1/y=1/x$ or. $x=y$

Similarly $y=z$;

Now from (1) $x = y = z = V^{1/3}$ ... ... (10)

Let us assume that x and y are independent of each other.

Then $S = 2xy + 2yz + 2zx$

$\therefore\ (\delta S/\delta x) = 2y + 2y(\delta z/\delta x) + 2z + 2x(\delta z/\delta x)\cdots$ ... (11)

Now from (1), differentiating w. r. to x

$yz + xy(\delta z/\delta x) = 0$ or. $(\delta z/\delta x) = -z/x$

$\therefore\ (\delta S/\delta x) = 2y - 2y(z/x) + 2z - 2zx/x = 2y - 2yz/x$ ... (12)

Again $r = (\delta^2 s/\delta x^2) = 2yz/x^2 - 2y/x\ (\delta z/\delta x) = 2yz/x^2 - 2y/x(-z/x)$

$= (2yz/x^2) + (2yz/x^2) = 4yz/x^2 = 4$ at $x = y = z$

Similarly $t = \delta^2 S/\delta x^2 = 4$ at $x = y = z$

$s = (\delta^2 S/\delta y \delta x) = 2 - 2z/x - (2y/x)(\delta z/\delta x) = 2 - 2z/x - 2y/x(-z/x)$

$= 2 - 2 + 2 = 2$ at $x = y = z$

Hence S is least when $x = y = z$.

Ex. 20. The cost of fuel for running a train is proportional to the square of the speed generated in kms. per hour and costs Tk. 48 per/hr. at 16 kms/hr. What is the most economical speed if the fixed charges are Tk. 300/hr.

[একটি রেলগাড়ীতে তেলের খরচ পড়ে ঘণ্টায় গতির বর্গের সমানুপাতিক ; ঘণ্টায় গতি ১৬ কিমি হইলে খরচ পড়ে ৪৮ টাকা ঘণ্টায়। কত কম বেগে গাড়িটি চলিলে ঘণ্টায় খরচ ৩০০ টাকা পড়িবে]

Sol : Let the speed of the train be v km/hr. and the distance travelled d kms. The time $= d/v$ hrs. The given cost of fuel according to the law $= kv^2$, where k is a constant. When cost $= 48$; $v = 16$, then

$48 = k(16)^2$ or. $k = 3/16$.

Hence the cost of fuel $= \dfrac{3}{16} v$ rupees per hour

For d kms, the cost $= \dfrac{3}{16} v^2 . d = \dfrac{3}{16} v^2 . \dfrac{d}{v} = \dfrac{3}{16} vd$.

Also the cost of fixed charges $= 300/$ per hour $= 300 \times \dfrac{d}{v}$

If p be the total cost of running per km in Taka, then

$p = (3/16)dv + 300\ (d/v)$

$\therefore\ \dfrac{dp}{dv} = \dfrac{3}{16} d - \dfrac{300d}{v^2},\quad \dfrac{d^2p}{dv^2} = \dfrac{600d}{v^3}$

For extremum value, $dp/dv = 0$

or, $\dfrac{3}{16} d - \dfrac{300d}{v^2} = 0$ or, $v^2 = 16 \times 100$ or, $v = 40$ km/hr

$\dfrac{d^2p}{dv^2} = \dfrac{600}{40} d = +ve.$

Hence p is minimum for $v = 40$km/hr, for most economical speed.

Ex. 21. Show that $f(x, y) = 4 + x^2 - y^2$ has a saddle point at (0, 0,4)

Let $z = f(x, y) = 4 + x^2 - y^2$

$f_x = 2x.\ f_{xx} = 2,\ f_{xy} = 0,\ f_y = -2y,\ f_{yy} = -2$

Critical points are $f_x = 0\ f_y = 0\ \therefore\ x = 0\ y = 0$

$D(x,y) = f_{xx}\ f_{yy} - f_{xy}^2 = rt - s^2 = 2(-2) - 0 = -4 < 0$

So, no critical point at (0,0) we know if

$D(x\ y) < 0$ i. e., $rt < s^2$, then the point $\{a, b, f(a,b)\}$ is a saddle point.

$z = f(x,0) = 4 + x^2,\ z = f(0,y) = 4 - y^2,\ z = (0, 0) = 4$

The saddle point is at $\{0,0,\ f(0,0)\} = (0,0,4)$

Ex. 22 Find the maximum and minimum values of the function $f(x,y) = xy$ subject to the condition $x^2 + y^2 = 8$

Ans. [$f(x,y) = xy$ এর গুরুমান ও লঘুমান নির্ণয় কর যদি $x^2 + y^2 = 8$ হয়]

শর্ত $g\ (x,y) = 8 = x^2 + y^2,\ f(x,y) = xy$

$\therefore\ f_x = y,\ f_y = x,\ g_x = 2x,\ g_y = 2y$

Lagrang's এর সমীকরণগুলি

$\frac{\delta f}{\delta x}=y=2\lambda x,\ \frac{\delta f}{\delta y}=x=2\lambda y$ এবং $x^2+y^2=8$

$\therefore\ 2\lambda=y/x,\ x\neq 0\ ;\ 2\lambda=x/y;\ y\neq 0$

$\therefore\ y/x=x/y \qquad x^2=y^2$

Lagrange এর সমীকরণগুলির অস্তিত্ব থাকতে হলে, $x\neq 0$, অতএব $2x^2=8$ or, $x^2=4$ or. $x=\pm 2$

যদি $x=2$, তখন $y=2,-2$

যদি $x=-2,\ y=2,-2$

$\therefore\ f(2,2)=4,\ f(2,-2)=4,\ f(2,-2)=-4,\ f(-2,2)=-4$

$f(x,y)$ এর মান সর্বোচ্চ হবে এবং ইহা 4 ; (2, 2) এবং (—2, —2) বিন্দুতে এবং —4 হবে (2, —2), (—2, 2) বিন্দুতে. লঘুমান হইবে।

Ex. 23. A publisher has been alloted Tk. 60,000/- to spend on the improvement and development of a new Calculus Text. He estimates that if he spends x thousand taka on improvement and y thousand taka for development,, approximately $20\,x^{3/2}y$ copies of the books will be sold. How much money should the Publisher allocate to improvement and how much to development in order to maximize sales ?

If the allotment is Tk. 61000/-, what is the effect of the sale of the maximum number of books for the addition of Tk. 1000 /-

(একজন প্রকাশককে একটি নূতন Calculus বই এর মানোন্নয়ন এবং উন্নতির জন্য Tk. 60,000/- দেওয়া হল। তিনি মানোন্নয়নের জন্য x হাজার টাকা এবং উন্নয়নের জন্য y হাজার টাকা বরাদ্দ করিলে, $20\,x^{3/2}y$ কপি বই বিক্রয় হয়। মানোন্নয়ন ও উন্নতির জন্য বরাদ্দের টাকার পরিমাণ কত ?

যদি বরাদ্দ Tk. 61,000/- হয়, এই বর্ধিত Tk. 1000/- জন্য সর্বোচ্চ কত কপি বিক্রয়কে প্রভাবান্বিত করিবে)

Sol : $g(x,y)=60,\ g(x,y)=x+y,\ M=20x^{3/2}y$. The Lagranges equations are

$\frac{\delta M}{\delta x}=30\sqrt{x}\,y=\lambda\ ;\ \frac{\delta M}{\delta y}=20\,x^{3/2}=\lambda$

and $x+y=60$

Solving, $30\sqrt{x}\,y=20\,x^{3/2}$ or, $x=\frac{3}{2}y$.

Put it in $x+y=60$ or, $3/2y+y=60$ or, $y=24$ and the $x=36$

Hence to maximize the sales, the publisher should spend Tk. 36,000/- for inprovement and Tk. 24000/- for development.

$M=f(36,24)=20x^{3/2}y=20\ (36)\ \ 24=103680.$

The value of λ is from

$\lambda=\frac{\delta M}{\delta y}=20\,x^{3/2}=20\,(36)^{3/2}$, when $x=36$

$=20\times 216=$ Tk. 4320 copies

$\lambda=dM/dk$, it follows that the unit increase in k from $k=x+y=60$ to $k=61$ will increase the maximal sales M of the book by approximately 4320 copies.

Exercise XI

1. Find the maximum aad minimum values of the following experessions.

(i) $x^2-8x^3+22x^2-24x+1$ C. U. 1980

(ii) $2x^3-21x^2+36x-20$ (a) $5x^6-18x^5+15x^4-10$. N. U. 1994

(iii) $x^6-5x^4+5x^2-1$ R. U. 1987

(iv) x^3-3x (a). x^4-4x^3+10 D. U. 1991

(v) x^3-3x^2-93 C. U. 1983

(vi) $2x^3-6x^2-18x+7$ C. U. 1984

(vii) $x^9-12x^6-36x^4+4=4$ C. U. 1986

2. Show that $y=x+1/x$ has a maximum and a minimum values and that the latter is greater than the former. D. U. 1961

3. Show that the curve $y=xe^x$ has a minimum ordinate where $x=-1$ R. U. 1962 '82

3. (a) If $f(x)=|x|$ Show that $f(o)$ is minimum although $f'(o)$ does not exist.

Sol : $f(x)=x, \quad x>0$

$\quad = 0 \quad x=0$

$\quad = -x \quad x<0$

$\lim\limits_{x\to 0+} f(x) = \lim\limits_{h\to 0} h = 0, \ \lim\limits_{x\to 0-} f(x) = \lim\limits_{h\to 0} -h = 0, \ f(o)=0$

Then $f(a+h)=f(a-h)=f(0)=0$

Hence $f(0)$ is continuous.

$$f'(x)=\lim_{h\to 0^+}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0^+}\frac{(0+h)-0}{h}=1$$

$$f'(x)=\lim_{h\to 0^-}\frac{f(x-h)-f(x)}{h}=\lim_{h\to 0^-}\frac{-h}{h}=-1$$

L.H Lim $\neq$ R.H Lim

Hince $f'(0)$ does not exist.

4. (a) Find the maximum and minimum value of $\sin x\cos^2 x$

(b) $\dfrac{x^4}{(x-1)(x-3)^3}$ R. U.. 1978

5. Find the maximum and minimum value of the function.

$t^3=3m^2t+2n$ when m and n are real constants. D.U. 1966

6. Find the maximum and minimum values of $\cos^4 x-\sin^4 x$.

7. Find the minimum value of $\dfrac{a^2}{\sin^2 x}+\dfrac{b^2}{\cos^2 x}$

8. Find the minimum value of $y=\frac{1}{2}a(e^{x/a}+e^{-x/a})$

R. U. 1958

9. Determine the values of x which will make $\dfrac{a^2}{x}+\dfrac{b^2}{a-x}$ a maximum and minimum

10. Show that $4\cos x+\cos 2x$ is maximum or minimum whenever $\cos x$ is maximum or minimum.

11. Prove that $x^{1/x}$ is a maximum when $x=e$

D. U. 1984, C. U. 1988

12. Show that $\dfrac{x^2-7x+6}{x-1}$ has a maximum value when $x=4$ and a minimum value $x=16$.

13. Prove that $\sin x\,(1+\cos x)$ has a maximum for $x=\frac{1}{3}\pi$

R. U. 1983.

(i) In what intervals is the function $f(x)=17-15x+9x^2-x^3$ increasing and in what intervals decreasing ?

Also find the relative maximum and maximum values of the function. Sketch the graph of f.

(উল্লেখিত ফাংশানের কোন ব্যবধিতে হ্রাস এবং কোন ব্যবধিতে বৃদ্ধি পায় নির্ণয় কর। ফাংশানটির আপেক্ষিক সর্বোচ্চ মান ও আপেক্ষিক সর্বনিম্ন মানও নির্ণয় কর। লেখচিত্র অংকন কর) D. U. 1987

Ans. $(-\infty, 1)\cup(5, \infty)$; $x=1$, the greatest value is 42, least is 10 at $x=5$

13 (ii) Find in what intervals the function $f(x)=x^2-2x+1$, $-1\leqslant x\leqslant 4$ is decreasing and in what intervals the function is increasing. Find the maximum and minimum values $f(x)$

13 (iii) Find the intervals where the function $f(x) = x^4 + 2x^3-4x + 4$ is increasing and decreasing. Also find the relative maximum and minimum values of the function and hence sktch its graph

$(fx) = x^4 + 2x^3-3x^2-4x + 4$ ফাংশনটির কোন্ ব্যবধিতে বর্ধমান এবং কোন্ ব্যবধিতে হ্রাসমান নির্ণয় কর। ইহার আপেক্ষিক গরিষ্ঠ ও লঘিষ্ঠমান নির্ণয় কর এবং চিত্র অঙ্কন কর)

C. U. 1992

14. Show that $(\log x/x)$ is a maximum for $x=e$ and the value is: $1/e$.

15. Show that x^x is a minimum for $x=1/e$ and the value is $(1/e)^{1/e}$ D. U. 1984

16. Find the maximum and minimum values of

(i) $\sin 2x - x$ (ii) $\sin^n x \sin nx$, n being a+ve integer.

17. Show that x^3-3x^2+3x+3 has neither a maximum nor a minimum value.

(i) Examine maxima and minima $f(x)=\frac{1}{5}x^5-\frac{1}{4}x^4$ R. U. 1988

18. Show that $(3-x)e^{2x}-4xe^x-x$ has no maximum or minimum value for $x=0$.

19. Find the extreme values of

(i) $a^{x+1}-a^x-x$, when $a>1$. (ii) $4x-8x\log 2$,

20. Show that the following function has neither a maximum nor a minimum value $x-\sin x$,

21. Test the following function for maxima and minima.

(i) $x^2-3xy+y^2+13x-12y+13$ (ii) $x^2-y^2+2x-4y-2$

(iii) $x^3+3xy^2-3x^2-3y^2+4$ (iv) $xy+(4/x)+(2/y)$

(v) $x^3y^2(1-x-y)$ (vi) $\sin x+\sin y+\sin(x+y)$

(vii) x^2+xy+y^2-6x+2 (viii) $9y^2+6xy+4x^2-24y-8x+4=0$

(ix) $x^2y^2-5x^2-8xy-5y^2=u$ C. H. 1985

(x) $f(x, y)=xy+8/x+8/y$, Also determine there nature.

D. H. 1986 Ans. $f(2,2)=12$

(xi) Show that $f(x,y)=4+x^2-y^2$ has a saddle point at $(0,0,4)$

22. Find the maximum and minimum values of the following functions.

(i) $\sin 2x-\sin 2x\cos 2x$ (ii) $4\cos x+4\sin^2 x-2$

(iii) $(3-x)\{\surd(1+x^2)-x\}$

23. Show that y is a minimum at $x=2$ of the function:

$$y^3=(x-2)^2$$

24. Find the maximum or minimum value of y, when $y^3=(x-3)^4$

*26. Find the maximum or minimum value of y if

$$\frac{dy}{dx}=\frac{3^{1/x}-2}{1+3^{1/x}}$$

*26. Show that y is max at $x=0$, point of inflexion for $x=1$ and minimum at $x=2$ of the function $dy/dx=x(x-1)^2(x-2)^3$

27. Show that the point $x=a$ is a maximum point of the $f(x)=b-\sqrt[3]{(x-a)^2}$ D. H. 1960

28. Show that points of inflexion of the curve $y^2=(x-a)^2(x-b)$ lie on the line $3x+a=4b$

29. A cubic function of x has a maximum value equal to 15 when $x=-3$, and a minimum value -17 when $x=1$ Find the function. D. U. 1963

(a) Show that the function $y=x+1/x$ has precisely one local maximum and a local minimum and that the latter is greater than the former. Give a rough graph of the function.

*30. Show that the curve $y=2+(1+\sin x)\cos x$ has a maximum, a minimum and four points of inflexions.

*31. Show that $e^x + e^{-x} + 2\cos x$ has a minimum value at $x=0$

*32. Show that $\tan^m x \tan^n(a-x)$ is a maximum when

$$\tan(a-2x) = \frac{n-m}{n+m}\tan\alpha$$

*33 Show that $4x^2 + \cos 2x - \frac{1}{2}(e^{2x} + e^{-2x})$ has a maximum value for $x=3$

34 Find the maximum, minimum points and the points of inflexion of $y = 2x^3 - 6x^2 + 18x + 7$ and show that point of inflexion lies between maximum and minimum points.

* 5 Show that the function $f(x) = (2x+3)(x+4)(x-2)(x-1)^3$ change signs from $+ve$ to $-ve$ as x passes through -4 and 1. and from $-ve$ to $+ve$ as x passes through $-3/2$ and 2,

36. Find the critical points of the following curves

(i) $f(x) = x - x^2 - x^3$

(ii) $y^3 = (x-a)^2$ (iii) $f'(x) = 2e^{\frac{-1}{x-a}} - 1$

37. Show that if $f(x) = (x^2+3x+2)^{2/5} + x^{2/5}$, $f'(x) = \infty$ gives a minimum for $x=-2$ $x=-1$ and $x=0$; and $f(x)=0$ gives two intermediate maxima

*38. Examine the function $z = 3axy - x^3 - y^3$ for maxima and minima

*39 Show that the maximum value of $u = \sin A \sin B \sin C$ is when $A = B = C = \pi/3$

*40. Find the maximum and minimum values of $ax + by$ when $xy = c^2$.

*41. If $z = a^2\{x + b^2/y\}$ where $x+y$ show that z has a minimum value when $x = a^2/(a+b)$ and a maximum value when $x = a^2/(a-b)$.

*42 Find the maximum and minimum values of u of the following curves,

(i) $u = x^3 + 2y^2 + 3x^2 - 2xy - 2yz - 2 = 0$

(ii) $u = 2a^2xy - 3ax^2y - ay^3 + x^3y + xy^3$.

(iii) $u = \dfrac{xyz}{(a+x)(x+y)(y+z)(z+b)}$

(iv) $u = 2xyz - 4zx - 2yz + x^2 + y^2 + z^2 - 2x - 4y + 4z$

(v) $u = x^2y^2 - 5x^2 - 8xy - 5y^2$

(vi) $u = axy^2z^3 - z^3x^2y^2 - xy^3z^3 - xy^2z^4$

(vii) Show that $u = 2xyz + x^3 + y^2 + z^2$ min. at (0,0,0)

(viii) $u = 2x^2 + 3y^2 + 4z^2 - 3xy + 8z$ mini. at (0, 0, -1)

(ix) $u = x^2 - 3xy + y^2 + 13x - 12y + 13$. D.U. 1989

42. (a) Prove that a necessary condition that $F(x, y, z)$ have an extreme value is that $F_xG_y - F_yG_x = 0$ subject to the constraint condition $G(x, y, z) = 0$ D. H. 1987

42(b) Use the method of Lagrange multiplyer to find the maximum and minimum values of the function.

(i) $f(x, y) = x^2 + 2y^2 + 2x + 3$

subject to the condition $x^2 + y^2 = 4$

Ans. Max. $f(1, \sqrt{3}) = f(1, -\sqrt{3}) = 12$ min. $f(-2,0) = 3$

(ii) $f(x, y) = 8x^2 - 24xy + y^2$

subject to the condition $x^2 + y^2 = 1$

Ans, max. $f(4/5, -3/5) = f(-4/5, 3/5) = 17$, min. $f(3/5, 4/5) = f(-3/5, -4/5) = 8$

(iii) $f(x, y) = x^2 + 2y^2 - xy$ subject to the condition $2x + y = 2$

Ans. min. $f(9,4) = 77$

(iv) $f(x,y) = xy$ subject to the condition $x + y = 1$

Ans. max. $f(\frac{1}{2}, \frac{1}{2}) = \frac{1}{4}$

#43. If $x^2/a^2+y^2/b^2+z^2/c^2=1$ Show that maximum value of xyz is abc. Interpret the result geometrically. R. H. 1967

43 (a) Show that $(x+y+z)^3-3(x+y+z)-24xyz+a^3$ is min. at (1, 1, 1) and max. at (—1, —1, —1)

#44. Show that maximum and minimum values of

$$u=4x+y+y^2 \text{ where } x^2+y^2+2x+y-1=0$$

are at $(\frac{1}{2}, -\frac{1}{2})$ and at $(-5/2, -1/2)$ respectively Is it always possible to express u as a function of x to have the result ?

#45. Show that the minimum value of $u=x+y+z$ when $a/x+b/y+c/z=1$ is $x/\sqrt{a}=y/\sqrt{b}=z/\sqrt{c}=\sqrt{a}+\sqrt{b}+\sqrt{c}$

#46. Show that maximum and minimum values of $u=x^2+y^2+z^2$, when $ax^2+by^2+cz^2=1$. are given y the roots of $(1/a-u)(1/b-u)(1/c-u)=0$.

47. Show that the minimum value of $u=x^4+y^4+z^4$,

is $xyx=c^3$ when $x=y=z=c$.

47. (a) Show that $F(x, y, z)=xy^2z^3$ is max and its value is 108.

#48. Find the maximum and minimum value of $x^2+y^2+z^2$ subject to the following conditions $ax^2+by^2+cz^2=1$ and $lx+my+nz=0$ C. H. 1972, D. U. H. 1958.

#49. Show that the maximum and minimum values of

$x^2+y^2+z^2$ with the condition C. H. 1988

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=1$$

are given by the roots of

$$\begin{vmatrix} a-1/u & h & g \\ h & b-1/u & f \\ g & f & c-1/u \end{vmatrix}$$

#50. Find the minimum value of $u=x^2+y^2+z^2$ when $xy+yz+zx=3a^2$.

*51. Find the maximum and minimum velues of xy when $x^2+xy+y^2=a^2$. (ii) $\frac{x}{3}+\frac{y}{3}=1$ N.H.1995

*52. Show that the maximum and minimum value of x^2+y^2 where $ax^2+2hxy+by^2=1$ are given by the roots of the quadratic $(a-1/r^2)(b-1/r^2)=h^2$

*53. Show that critical values of $u=x^py^qz^r$ when $a/x+b/y+c/z=1$ is maximum and the value is $(p+q+r)^{p+q+r}\,a/p)^p(b/q)^q(c/r)^r$

*54. Find the maximum value of $x^2+y^2+z^2$ subject to $6x^2+3y^2+2z=12$, $3x+2y+z=0$

*55. Find the critical value of 55(a) See Bengali versio

$u=x^2+y^2+z^2$ when $x+y+z=3a$

*56. Show that the critical values of u^2, when $u^2=a^2x^2+b^2y^2+c^2z^2$, $x^2+y^2+z^2=1$ and $lx+my$ $nz=0$. are the roots of

$$\frac{l^2}{u^2-a^2}+\frac{m^2}{u^2-b^2}+\frac{n^2}{u^2-b^2}=0. \quad \text{C.U.1993}$$

57. Show that maximum and minimum values of $u=x^2+y^2+z^2$. if $px+qy+rz=0$ and $x^2/a^2+y^2/b^2+z^2/c^2=1$ are given by the roots of

$$\frac{a^2p^2}{u-a^2}+\frac{b^2q^2}{u-b^2}+\frac{c^2r^2}{u-c^2}=0$$

For 57(a) See Bengali version

*58. If (x_1, y_1, z_1) : (x_2, y_2, z_2) are two points on the curve of intersection of $lx+my+nz=0$ and $ax^2+by^2+cz^2=1$ the distance r between these points is stationary when

$$\frac{l^2}{1-ar^2}+\frac{m^2}{1-br^2}+\frac{n^2}{1-cr^2}=0$$

*59. Show that the maximum and minimum values of $u=x^2/a^4+y^2/b^4+z^2/b^4$, when $lx+my+nz=0$ and $x^2/a^2+y^2/b^2+z^2/c^2=1$ are given by the roots of

$$\frac{l^2a^4}{1-a\ u}+\frac{m^2b^4}{1-b^2u}+\frac{n^2c^4}{1-c^2u}=0$$

59 (a) Find the minimum value of

$yz+zx+xy$ if $xyz=a^2(x+y+z)$ Ans. $9a^2$ D. H. '87

(b) Find the points where the value of $F=x^2y^2z^2$ will be maximum $x^2+y^2+z^2=c^2$ and find the greatest value of F.

60. Show that the semivertical angle of the right cone of a given curved surface and maximum volume is $\sin^{-1} 1/\sqrt{3}$.

61. A farmer can afford for buying 8800 ft. of wire fencing. He wishes to enclose a rectangular field of largest possible area. What should be the dimensions of the field ?

*62. Find the fraction which exceeds its second power by the greatest number possible.

63. Show that of all rectangles of a given area, the square has the smallest perimeter.

64. Show that semivecrtical angle of a cone of maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$.

65. Given the total surface area $2\pi a^2$ of a right circular cylinder, show that the cylinder of maximum volume is $2\pi a^3/3\sqrt{3}$

66. Find the surface area of the right circular cylinder of great st surface which can be inserted into a sphere of radius r.

67. A ladder is to be taken in a horizontal position round a right angled corner from a passage of width a to a passage of width b Show that the greatest possible length of it is $(a^{2/3}+b^{2/3})^{3/2}$. D. U. 1265

68. Prove that the least perimeter of isosceles triangle in which a circle of radius r can be inscribed is $6r\sqrt{3}$.

69. Show that the maximum right cone inserted into a given sphere is $(32/81)\pi a^3$, a being the radius of the sphere.

70. Find the maximum cylinder that can be inserted into a sphere of radius a. D. U. 1986

71. Show that the radius of the right circular cylinder of greatest curved surface which can be inserted within a given cone is half that of the cone.

72. Determine the cone of minimum volume described about of a given sphere of radius r.

*73. Divide a number into two parts such that square of one part multiplied by the cube of the other should give the greatest possible product.

74. What is the height of a right cone of greatest volume which can be kept within a sphere of radius 'a' ?

75. The sum of the surfaces of a cube and sphere is given, show that when the sum of their volume is least the diameter of the sphere is equal to the edge of the cube.

75 A thin closed rectangular box has one edge n times the length of another edge and the volume of the box is V. Prove that the least surface S is given by $ns^2 = 54(n+1)^2V^2$.

77. P is a point on the ellipse whose centre is C and N the foot of the perpendicular from C upon the tangent to the ellipse at P, find the maximum value of PN.

*78. A variable sphere is described with its centre on the surface of a fixed sphere. Find for what value of the radius the area of its surface intercepted by the fixed sphere is greatest.

*79. Through a given point $P(h, k)$, a straight line is drawn meeting the co-ordinate axes OX and OY in A and B respectively, determine the position of the line in order that $OA+OB$ may be a minimum.

80. If r_1, r_2 be the local distances of a point on an ellipse whose major axis is $2a$, find the maximum and minimum values of $r_1 r_2 (r_1 - r_2)$; $(r_1 > r_2)$ distinguishing between the cases where the eccentricity is greater than or less than $1/\sqrt{3}$ D. H. 1962

*81. What is the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid,

*82. Find the dimensions of the rectangular box open at the top, which has a maximum volume if its surface is 12.

*83. Find a point such that the sum of the squares of the perpendiculars drawn on the sides of a given triangle shall be a minimum.

*84. Show that the points such that the sum of the squares of its distance from n given points shall be minimum is the centre of mean position of the given points.

*85. P is a point in the $\triangle ABC$, PA; PB, PC are joined. If $\angle APB = \angle BPC = \angle CPA = 120^0$, show that the angular distance is maximum

*86. Prove that, of all polygons of a given number of sides circumscribed to a circle the regular polygon is of minimum area, and of all polygones inscribed in a circle, the regular polygon has a maximum area.

87. During the course of on epidemic, the number of people inficted at the time t is given by

$$N = f(t) = kt^{5/2}e^{-t}$$

where t is measured in weeks from start of the epidemic, and k is a constant. When is the maximum number of people infected? Find also the maximum value of N. R. H. 1988

Ans. $t = 5/2$ weeks, $N = k(5/2)^{5/2} e^{-5/2}$

88. Show that the surface area of all ellipsoid of constant volume is minimum when it a sphere. R. H. 1987

89. A firm sells all the product it makes at Tk. 12.00 per unit. The cost of making x units $c = 20 + 0.6x + 0.01x^2$. Find the maximum profit. D H. 1989

(একটি ফার্ম উৎপাদিত দ্রব্যের প্রত্যেকটির দাম ১২.০০ টাকা দরে বিক্রয় করে। x একক উৎপাদিত ব্যয় $c = 20 + 0.6x + 0.01x^2$ হইলে সর্বোচ্চ মুনাফা নির্ণয় কর।) Ans. Tk, 3229/-

90. A farmer with a field adjacent to a straight river wishes to fence a rectangular area for grazing. If no fence is needed along the river, and has 1600m of fencing, what should be the dimensions of the field in order that it have a maximum area?

95. The percel post regulation restrict parcels to be such that the length plus the girth ($=2\pi r$, r is the radius of the cylinder) must not exceed 180 cms. Determine the percel of greatest volume that can be sent by post at the form of the percel be a right circular cylinder,

[পোষ্ট অফিসে পার্শেল করিবার নিয়ম পার্শেলের দৈর্ঘ্য এবং পার্শেলের তলের পরিসীমার যোগফল ১৮০ সেমি বেশী হইবে না এবং দৈর্ঘ্য ১০০ সেমি এর বেশী হইবে না। সুষম চোঙা আকৃতি একটি পার্শেল পোষ্টে পাঠাতে ইহার বৃহত্তম ঘন কত হইবে।]

উঃ চোঙের দৈর্ঘ্য 60 cms, $2\pi r = 120$ cm,

96. A farmer can afford for 800 metre of wire fencing. He wishes to enclose a rectangular field to largest possible area. What should the dimensions of the field be? (একজন কৃষক ৮০০ মিটার দীর্ঘ তারের বেড়া দেওয়ার সামর্থ রাখে, সে সম্ভাব্য বৃহত্তম আয়তাকার মাঠকে ঘেরার ইচ্ছা রাখে, মাঠের দৈর্ঘ্য-প্রস্থ কিরূপ হবে) C.H. 1992

97. Show that th ratio of the height of a right circular cone of greatest curved surface which can be inscribed in a given sphere to the radius of the sphere is 4 : 3 (দেখাও যে, একটি গোলকের মধ্যে সুষম বৃত্তাকার কোণকের বৃহত্তম বক্রতলের উচ্চতা এবং গোলকের ব্যাসার্ধের অনুপাত 4 : 3 হইবে)

C.U. 1993

98. Find all the local maximum or minimum values (whichever exists) of the function $f(x) = x^4 + 2x^3 + 3$. Determine the inflexion points and the concavity of the graph of this fuinction and draw a rough sktch of the graph. N.U. 1994

($f(x) = x^4 + 2x^3 + 3$ ফাংশনের স্থানীয় আপেক্ষিক গুরু ও লঘুমান নির্ণয় কর, যাহা বিদ্যমান। ইনফ্লেকশান বা বাক বিন্দুগুলি নির্ণয় কর এবং ফাংশনটির চিত্রের অবতল নির্ণয় করিয়া লেখচিত্রটি অঙ্কন কর।)

অনুশীলনী– XI

1. নিম্নলিখিত রাশিমালাগুলির বৃহত্তম ও ক্ষুদ্রতম মান নির্ণয় কর।

(i) $x^4 - 8x^3 + 22x^2 - 24x + 1$ C. U. 1980

(ii) $2x^3 - 21x^2 + 36x - 20$ C. U. 1979

(iii) $x^5 - 5x^4 + 5x^2 - 1$ ... ... R. U. 1987

(iv) $x^3 - 3x$ (a) $x^4 - 4x^3 + 10$ D. U. 1991

(v) $x^3 - 3x^2 - 93$ C. U. 1983

(vi) $2x^3 - 6x^2 - 18x + 7$ C. U. 1984

(vii) $x^6 - 12x^5 - 36x^4 + 4 = 0$ C. U. 1986

(viii) $5x^6 - 18x^5 + 15x^4 - 10$ C. U. 1991

2. দেখাও যে $y = x + {}^1/_x$ -এর বৃহত্তম চরমমান ও ক্ষুদ্রতম চরমমান আছে। আরো দেখাও যে ক্ষুদ্রতম মান বৃহত্তম মানের চেয়ে বৃহত্তম। D. U. 1961

3. দেখাও যে বক্ররেখা $y = xe^x$ -এর একটি সর্বনিম্ন মানের কোটি (ordinate) আছে যেখানে $x = -1$. R. U. 1961

4. (a) $4 \sin x \cos^2 x$ (b) $\dfrac{x^4}{(x-1)(x-3)^3}$ এর বৃহত্তম ও ক্ষুদ্রতম চরমমান নির্ণয় কর। R. U. 1987

5. ফাংশন $t^3 = 3m^2t + 2n$, এখানে m ও n বাস্তব এবং ধ্রুব। এই ফাংশনের বৃহত্তম এবং ক্ষুদ্রতম চরমমান নির্ণয় কর। D. U. 196[illegible]

6. $\cos^4 x - \sin^4 x$ এর বৃহত্তম ও ক্ষুদ্রতম চরমমান নির্ণয় কর।

7. $y = \frac{1}{2}a(e^{x/a} - e^{-x/a})$ এর ক্ষুদ্রতম চরমমান নির্ণয় কর। R. U. 19[illegible]

8. $\dfrac{a^2}{\sin^2 x} + \dfrac{b^2}{\cos^2 x}$ এর ক্ষুদ্রতম চরমমান নির্ণয় কর।

9. x-এর কোন মানের জন্য $\dfrac{a^2}{x} + \dfrac{b^2}{a-x}$ এর বৃহত্তম ও ক্ষুদ্রতম মান পাওয়া যাইবে।

10. দেখাও যে $4 \cos x + \cos 2x$ এর মান বৃহত্তম অথবা ক্ষুদ্রতম হয় যখন $\cos x$ এর মান বৃহত্তম বা ক্ষুদ্রতম হইবে।

11. প্রমাণ কর যে যখন $x = e$ হয় তখন $x^{1/x}$ মান বৃহত্তম।

D. U. 1984

12. দেখাও যে $\frac{x^2-7x+6}{x-10}$ এর মান বৃহত্তম হইবে যখন $x=4$ এবং ক্ষুদ্রতম হইবে যখন $x=16$ হয়।

13. প্রমাণ কর যে $x=\pi/3$ এর জন্য $\sin x(1+\cos x)$ এর মান বৃহত্তম হইবে। R. U. 1983

(i) In what intervals is the function $f(x)=17-15x+9x^2-x^3$ increasing and in what intervals decreasing ?

Also find the relative maximum and maximum values of the function. Sketch the graph of f.

(উল্লেখিত ফাংশানের কোন ব্যবধিতে হ্রাস এবং কোন ব্যবধিতে বৃদ্ধি পায় নির্ণয় কর। ফাংশানটির আপেক্ষিক সর্বোচ্চ মান ও আপেক্ষিক সর্বনিম্ন মানও নির্ণয় কর। লেখচিত্র অংকন কর) D. U. 1987

Ans. $(-\infty,1)U\ (5, \infty)$; $x=1$, the greatest value is 42, least is 10 at $x=5$

13. (iii) Determine the maximum and minimum values of $f(x) = x^3 - 3x + 2$ on the intervals [-3, 3/2] Hence sketch the graph. ($f(x)$ এর জন্য ব্যবধি [- 3, 2/2] এর মধ্যে লঘিষ্ঠ ও গরিষ্ঠ মান নির্ণয় কর এবং চিত্রটি অংকন কর। D. U. 1990

(ii) Find in what intervals the function $f(x) = x^2 = x^2 - 2x + 1$, $-1 \le x \le 4$ is decreasing and in what intervals the function is increasing. Find the maximum and minimum values $f(x)$

(ফাংশনটি কোন ব্যবধিতে হ্রাস পায়, কোন ব্যবধিতে বৃদ্ধি পায় নির্ণয় কর। ফাংশনটির গরিষ্ঠমান এবং লঘিষ্ঠমান নির্ণয় কর) D. U. 1986

Ans. [- 1,1), $x = 1$, greatest value 4 and 9 and at $x = 4$, least value is zero

14. দেখাও যে $x=e$ এর জন্য $\left(\frac{\log x}{x}\right)$ এর মান ক্ষুদ্রতম এবং ক্ষুদ্রতম মান হইবে $1/e$.

15. দেখাও যে, $x=1/e$ এর জন্য x^x এর মান ক্ষুদ্রতম এবং এই ক্ষুদ্রতম মান হইল $(1/e)^{1/e}$. D. U. 1984

16. (i) $\sin 2x - x$ (ii) $\sin^n x \sin nx$ (এখানে n একটি $+ve$ পূর্ণ সংখ্যা) এর বৃহত্তম এবং ক্ষুদ্রতম মান নির্ণয় কর।

17. দেখাও যে $x^3 - 3x^2 + 3x + 3$ এর বৃহত্তম বা ক্ষুদ্রতম কোন মানই নাই।

(i) Examine maxima and minima $f(x) = \frac{1}{5}x^5 - \frac{1}{4}x^4$ R. U. 1988

18. দেখাও যে, $x = 0$ এর জন্য $(3-x)e^{2x} - 4xe^x - x$ এর কোন বৃহত্তম বা ক্ষুদ্রতম মান নাই।

19. (i) $a^{x+1} - a^x - x$ যখন $a > 1$ (ii) $4^x - 8x \log 2$. ইহাদের চরম মান সমূহ নির্ণয় কর।

20. দেখাও যে; $x - \sin x$ এর বৃহত্তম বা ক্ষুদ্রতম মান নাই।

21. বৃহত্তম এবং ক্ষুদ্রতম মানের জন্য নিম্নলিখিত ফাংশনগুলি পরীক্ষা কর।

(i) $x^2 - 3xy + y^2 + 13x - 12y + 13$ (ii) $x^2 - y^2 + 2x - 4y - 2$

(iii) $x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$ (iv) $xy + (4/x) + (2/y)$

(v) $x^3y^2(1 - x - y)$ (vi) $\sin x + \sin y + \sin(x + y)$

(vii) $x^2 + xy + y^2 - 6x + 2$ (viii) $9y^2 + 6xy + 4x^2 - 24y - 8x + 4 = 0$

(ix) $x^2y^2 - 5x^2 - 8xy - 5y^2 = u$ C. U. 1985

(x) $f(x, y)=xy+8/x+8/y$, Also determine theie nature.

D. H. 1986 Ans. $f(2, 2)=12$

(xi) Show that $f(x, y)=4+x^2-y^2$ has a saddle point at (0, 0, 4)

22. নিম্নলিখিত ফাংশনগুলির বৃহত্তম ও ক্ষুদ্রতম চরমমান নির্ণয় কর।

(i) $\sin 2x - \sin 2x \cos 2x$ (ii) $4\cos x + 4\sin^2 x - 2$

(iii) $(3-x)\{\sqrt{(1+x^2)}-x\}$

23. দেখাও যে $y^3=(x-2)^2$ ফাংশানটির $x=2$ বিন্দুতে y এর মান ক্ষুদ্রতম।

24. y এর বৃহত্তম বা ক্ষুদ্রতম মান বাহির কর যখন $y^3=(x-3)^2$

*25. y-এর বৃহত্তম বা ক্ষুদ্রতম মান নির্ণয় কর যখন

$$\frac{dy}{dx}=\frac{3^{1/x}-2}{1+3^{1/x}}$$

*26. দেখাও যে $dy/dx = x(x-1)^2(x-2)^3$ ফাংশনের $x=0$ বিন্দুতে বৃহত্তম, $x=1$ বিন্দুতে বক্রতার বিন্দু (point of inflexion) এবং $x=2$ বিন্দুতে ক্ষুদ্রতম মান আছে।

*27. দেখাও যে $f(x) = b - \sqrt[3]{(x-a)^2}$ ফাংশানে $x=a$ বিন্দুটি একটি বৃহত্তম চরমমান বিন্দুতে অবস্থিত। D. H. 196

28. দেখাও যে বক্ররেখা $y^2 = (x-a)^2(x-b)$ এর বক্রতার বিন্দুগুলি (Point of inflexion) $3x+a=4b$ এই সরলরেখাটির উপর অবস্থিত।

29. x এর একটি ত্রিঘাত সমীকরণের বৃহত্তম মান 15 যখন $x=-3$, এবং ক্ষুদ্রতম মান -17 যখন $x=1$. ফাংশনটি নির্ণয় কর। D. U. 1963

29. (a) $u = x\phi\left(\frac{y}{x}\right) + \phi\left(\frac{y}{x}\right)$ হয়, তবে দেখাও যে

$$x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} = x\phi\left(\frac{y}{x}\right)$$

(b) দেখাও যে ফাংশন $y = x + 1/x$ এর নিদ্দিষ্ট একটি স্থানীয় বৃহত্তম ও স্থানীয় ক্ষুদ্রতম মান আছে। আরো দেখাও যে ক্ষুদ্রতম মানটি বৃহত্তম মান অপেক্ষা বৃহত্তর। ফাংশনটির মোটামুটি একটি লেখচিত্র অংকন কর।

[Show that the function $y = x + 1/x$ has precisely one local maximum and a local minimum and that the letter is greater than the former. Give a rough sketch of the graph of the function]

*30. দেখাও যে বক্ররেখা $y = 2 + (1+\sin x)\cos x$-এর একটি বৃহত্তম ও ক্ষুদ্রতম চরমমান এবং চারটি বক্রতার বিন্দু (four points of inflexions) আছে।

*31. দেখাও যে $x=0$ বিন্দুতে $e^x + e^{-x} + 2\cos x$-এর একটি ক্ষুদ্রতম মান আছে।

*32. দেখাও যে যখন $\tan(a-2x) = \frac{n-m}{n+m}\tan \alpha$ হয় তখন

$\tan^m x \tan^n (a-x)$ এর মান বৃহত্তম হইবে।

*33. দেখাও যে $x=3$ হইলে রাশিমালা $4x^2 + \cos 2x - \frac{1}{2}(e^{2x} + e^{-2x})$ এর মান বৃহত্তম হইবে।

34. $y = 2x^3 - 6x^2 - 18x + 7$ এর বৃহত্তম, বা ক্ষুদ্রতম এবং বক্রতার বিন্দুসমূহ (Points of inflexion) নির্ণয় কর। দেখাও যে বক্রতার বিন্দু সমূহ বৃহত্তম ও ক্ষুদ্রতম বিন্দুর মাঝখানে অবস্থিত।

*35. দেখাও যে x, -4 এবং 1 বিন্দু দিয়া অতিক্রম করার সময় ফাংশন $f(x) = (2x+3)(x+4)(x-2)(x-1)^3$ এর চিহ্ন $+ve$ হইতে $-ve$ এর পরিবর্তিত হয়। এবং x-এর মান $-3/2$ হইতে 2-এ পরিবর্তিত হওয়ার সময় চিহ্ন $-ve$ হইতে $+ve$ এ পরিবর্তিত হয়।

36. নিম্নলিখিত বক্ররেখাগুলির জন্য সন্ধি বিন্দু (Critical points) নির্ণয় কর।

(i) $f(x) = x - x^2 - x^3$ (ii) $y^3 = (x-a)^2$, (iii) $f'(x) = 2e^{-\frac{1}{x-2}} - 1$.

37. দেখাও যে যদি $f(x) = (x^2+3x+2)^{2/5} + x^{2/5}$ এবং $f'(x) = \infty$ হয়, তবে $x=-2$, $x=-1$ এবং $x=0$ মানের জন্য $f(x)$ ক্ষুদ্রতম হইবে। আবার $f(x)=0$ ক্ষুদ্রতম মানগুলির মধ্যবর্তী বৃহত্তম মানগুলি প্রদান করিবে।

*38. বৃহত্তম ও ক্ষুদ্রতম চরমমানের জন্য ফাংশন $z = 3axy - x^3 - y^3$ পরীক্ষা কর।

39. দেখাও যে ফাংশন $u = \sin A \sin B \sin C$-এর মান বৃহত্তম চরম হইবে যখন $A = B = C = \pi/3$ হইবে।

40. $ax + by$-এর বৃহত্তম এবং ক্ষুদ্রতম চরমমান নির্ণয় কর যখন $xy = c^2$

41. যদি $z = \frac{a^2}{x} + b^2/y$ যেখানে $x + y = a$ হয় তবে দেখাও যে,

$x = a^2/(a+b)$ এর জন্য z এর মান ক্ষুদ্রতম হইবে এবং $x' = a^2/(a-b)$. এর জন্য z বৃহত্তম হইবে।

42. নিম্নলিখিত বক্ররেখাগুলির জন্য u-এর বৃহত্তম এবং ক্ষুদ্রতম চরমমান নির্ণয়কর।

(i) $u = x^3 + 2y^2 + 3z^2 - 2xy - 2yz - 2 = 0$

(ii) $u = 2a^2xy - 3ax^2y - ay^3 + x^3y + xy^3$.

(iii) $u = \frac{xyz}{(a+x)(x+y)(y+z)(z+b)}$

(iv) $u = 2xyz - 4zx - 2yz + x^2 + y^2 + z^2 - 2x - 4y + 4z.$

(v) $u = x^2y^2 - 5x^2 - 8xy - 5y^2$

(vi) $u = axy^2z^3 - z^3x^2y^2 - xy^3z^3 - xy^3z^4$

(vii) দেখাও যে $(0, 0, 0)$ বিন্দুতে $u = 2xyz + x^2 + y^2 + z^2$ ক্ষুদ্রতম।

(viii) দেখাও যে $(0, 0, -1)$ বিন্দুতে $u = 2x^2 + 3y^2 + 4z^2 - 3xy + 8z$ এর মান ক্ষুদ্রতম

(ix) $x^2 - 3xy + y^2 + 13x - 12y + 13$ D. U. 1989

42. (a) প্রমাণ কর যে $F(x, y, z)$ ফাংশনের সর্বোচ্চমান থাকিতে হইলে $F_xG_y - F_y G_x = 0$ একটি প্রয়োজনীয় শর্ত হইবে এবং বাধ্যতামূলক (constraint) শর্ত হইবে $G(x, y, z) = 0.$

43. যদি $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$ হয় তবে দেখাও যে xyz এর বৃহত্তম মান হইবে abc. এই ফলের জ্যামিতিক রূপ দাও। R. U. 1967

43. (a) দেখাও যে $(x + y + z)^3 - 3(x + y + z) - 24xyz + a^3$ এর মান $(1, 1, 1)$ বিন্দুতে ক্ষুদ্রতম এবং $(-1, -1, -1)$ বিন্দুতে বৃহত্তম।

*44 দেখাও যে $u = 4x + y + y^2$ সর্বোচ্চ ও সর্বনিম্ন মান যথাক্রমে $(\frac{1}{2}, -\frac{1}{2})$ এবং $(-5/2, -\frac{1}{2})$ বিন্দুতে যখন $x^2+y^2+2x+y-1=0.$ এই ফল পাইতে হইলে u কে x-এর ফাংশনরূপে সর্বদা প্রকাশ করা সম্ভব কি ?

*45. দেখাও যে $u=x+y+z$ এর ক্ষুদ্রতম মান $\frac{x}{\sqrt{a}}=\frac{y}{\sqrt{b}}=\frac{z}{\sqrt{c}}$

$=\sqrt{a}+\sqrt{b}+\sqrt{c}$ যখন $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=1$ হয়।

*46. দেখাও যে $u=x^2+y^2+z^2$ এর বৃহত্তম ও ক্ষুদ্রতম মান

$$\left(\frac{1}{a}-u\right)\left(\frac{1}{b}-u\right)\left(\frac{1}{c}-u\right)=0$$

সমীকরণের মূল হইতে পাওয়া যাইবে যখন $ax^2+by^2+cz^2=1$ হইবে।

47. দেখাও যে যখন $x=y=z=c$ যখন তখন $u=x^4+y^4+z^4$ এর ক্ষুদ্রতম মান হইবে $xyz=c^3$.

47. (a) দেখাও যে $F(x, y, z)=xy^2z^3$ এর কেবল বৃহত্তম মান আছে এবং এই মান হইবে 108.

*48. $ax^2+by^2+cz^2=1$ এবং $lx+my+nz=0$ এই শর্তে $x^2+y^2+z^2$ এর বৃহত্তম ও ক্ষুদ্রতম মান নির্ণয় কর।

C. H. 1972, D. U. H. 1958.

49. $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=1$।

এই শর্তে $\begin{vmatrix} a-\frac{1}{u} & h & g \\ h & b-\frac{1}{u} & f \\ g & f & c-\frac{1}{u} \end{vmatrix}=0$ এর মূলগুলি হইতে

C. H. 1988

$x^2+y^2+z^2$ এর বৃহত্তম ও ক্ষুদ্রতম মান পাওয়া যাইবে, ইহা দেখাও।

*50. $u=x^2+y^2+z^2$ এর ক্ষুদ্রতম মান নির্ণয় কর যখন $xy+yz+zx=3a^2$ হয়।

51. xy এর বৃহত্তম ও ক্ষুদ্রতম মান নির্ণয় কর যখন $x^2+xy+y^2=a^2$ হয়।

52. দেখাও যে $(a-1/r^2)(b-1/r^2)=h^2$ এর মূলগুলিদ্বারা x^2+y^2 এর বৃহত্তম ও ক্ষুদ্রতম মান সমূহ দেওয়া যায়।

যখন $ax^2+2hxy+by^2=1$ হয়।

53. দেখাও যে যখন $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=1$ হয় তখন $u=x^py^qz^r$ এর সন্ধি মানই বৃহত্তম মান হইবে এবং এই বৃহত্তম মান

$$(p+q=r)^{p+q+r}\left(\frac{a}{p}\right)^p\left(\frac{b}{q}\right)^q\left(\frac{c}{r}\right)^r$$

54. $6x^2+3y^2+2z^2=12,\ 3x+2y+z=0$ শর্ত সাপেক্ষে $x^2+y^2+z^2$ এর বৃহত্তম মান নির্ণয় কর।

55. যখন $x+y+z=3a$ হয় তখন $u=x^2+y^2+z^2$ এর সন্ধিমান নির্ণয় কর। (critical values)

(a) $2x^2+2y^2+9a^3-6ax-6ay+2xy$

56. দেখাও যে যখন $u=a^2x^2+b^2y^2+c^2z^2,\ x^2+y^2+z^2=1$ এবং $lx+my+nz=0$ হয়, তখন

$$\frac{l^2}{u^2-a^2}+\frac{m^2}{u^2-b^2}+\frac{n^2}{u^2-c^2}=0.$$

এর মূলগুলিই হইবে u^2 এর সন্ধিমান সমূহ। (Critical values)

57. দেখাও যে যদি $px+qy+rz=0$ এবং

$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ হয় তবে

$\frac{a^2p^2}{u-a^2}+\frac{b^2q^2}{u-b^2}+\frac{c^2r^2}{u-c^2}=0$ এর মূলগুলি হইবে $u=x^2+y^2+z^2$ বৃহত্তম ও ক্ষুদ্রতম মান সমূহ।

57. (a). find the maximum and minimum values of $x^2+y^2+z^2$ Subject to the conditions $x^2/4+y^2/5+z^2/z=1$ and $z=x+y$

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58. সরলরেখা $lx+my+nz=0$ এবং বক্ররেখা $ax^2+by^2+cz^2=1$ এর মধ্যে ছেদ বিন্দু (x_1, y_1, z_1) ও (x_2, y_2, z_2).
এই দুই বিন্দুর মধ্যে দূরত্ব স্থির হইবে যদি

$$\frac{l^2}{1-ar^2}+\frac{m^2}{1-br^2}+\frac{n^2}{1-cr^2}=0.$$

*59. দেখাও যে যখন $lx+my+nz=0$ এবং $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ হয় তখন $u=\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}$, এর বৃহত্তম ও ক্ষুদ্রতম মান পাওয়া যাইবে

$\frac{l^2a^4}{1-a^2u}+\frac{m^2b^4}{1-b^2u}+\frac{n^2c^4}{1-c^2u}=0$ এর মূলগুলি হইতে।

(a) Find the minimum value of

$yz+zx+xy$ if $xyz=a^2(x+y+z)$ Ans. $9a^2$ D. H. 1987

(b) Find the points where the value of $F=x^2y^2z^2$ will be maximum $x^2+y^2+z^2=c^2$ and find the greatest value of F.

60. দেখাও যে নির্দিষ্ট ক্ষেত্রফল বিশিষ্ট বক্রতল ও বৃহত্তম মানের আয়তন বিশিষ্ট কোন খাড়া বৃত্তাকার কোণকের (right cone) অর্ধ-শির কোণ $=\sin^{-1}(1/\sqrt{3})$ হইবে।

61. একজন কৃষক ৮০০ ফুট তারের বেড়া কিনিতে পারে। সে ঐ বেড়া

দিয়া সর্বাধিক ক্ষেত্রফল যুক্ত একখণ্ড জমি ঘিরিতে মনস্থ করিল। ঐ জমির পরিমান কি হইবে নির্ণয় কর।

62. এমন একটি ভগ্নাংশ নির্ণয় কর যাহা হইতে ইহার বর্গ বিয়োগ করিলে বিয়োগফল বৃহত্তম হইবে।

[সংকেত $f(x, y)=\frac{y}{x}-\left(\frac{y}{x}\right)^2$]

63. দেখাও যে প্রদত্ত ক্ষেত্রফলের সকল আয়তক্ষেত্রের মধ্যে বর্গক্ষেত্রের পরিসীমাই ক্ষুদ্রতম।

64. দেখাও যে প্রদত্ত হেলানো উন্নতি এবং বৃহত্তম আয়তন বিশিষ্ট কোন কোণের (cone) অর্ধ-শির-কোণ $\tan^{-1}\sqrt{2}$ হইবে।

(Show that semivertical angle of a cone of maximum volume and given slant heigh is $\tan^{-1}\sqrt{2}$.)

65. একটি বেলন বা সমবৃত্তভূমিক সিলিণ্ডারের মোট ক্ষেত্রফল দেওয়া আছে $2\pi a^2$. ইহার বৃহত্তম আয়তন হইবে $2\pi a^3/3\sqrt{3}$. ইহা প্রমাণ কর।

[Given the total surface area $2\pi a^2$ of a right circular cylinder, show that the maximum volume of such a cylinder is $2\pi a^3/3\sqrt{3}$.]

66. 'r' ব্যাসার্ধ বিশিষ্ট একটি গোলকের মধ্যে রক্ষিত সর্বাপেক্ষা বৃহৎ ক্ষেত্রফল বিশিষ্ট সমবৃত্তভূমিক সিলিণ্ডারের l পৃষ্ঠের মোট ক্ষেত্রফলের পরিমাণ নির্ণয় কর।

67. 'a' প্রস্থ বিশিষ্ট একটি রাস্তা, 'b' প্রস্থ বিশিষ্ট অপর একটি রাস্তার সহিত সমকোণে সংযুক্ত। একটি মইকে আনুভূমিকভাবে ঐ সংযোগস্থল পার করাইতে হইবে। দেখাও যে ঐ মইয়ের সর্ববৃহৎ দৈর্ঘ্য $(a^{2/3}+b^{2/3})^{3/2}$ এর বেশী হইবে না।

68. r ব্যাসার্ধ বিশিষ্ট একটি বৃত্ত একটি সমদ্বিবাহু ত্রিভুজের মধ্যে অঙ্কিত হইলে, দেখাও যে ত্রিভুজের ন্যূনতম পরিসীমা হইবে $6r/\sqrt{3}$।

69. দেখাও যে a ব্যাসার্ধ বিশিষ্ট একটি গোলকের মধ্যে অঙ্কিত বৃহত্তম সমবৃত্তভূমিক কোণের আয়তন হইবে $\frac{32}{81}\pi a^3$.

70. a ব্যাসার্ধ বিশিষ্ট একটি গোলকের মধ্যে রক্ষিত সর্ববৃহৎ সিলিণ্ডারের আয়তন কত?

71. দেখাও যে প্রদত্ত কোণের (cone) মধ্যে রক্ষিত সর্বাপেক্ষা বৃহত্তম বক্রতল বিশিষ্ট সম বৃত্তভূমিক সিলিণ্ডারের ব্যাসার্ধ প্রদত্ত কোণের ব্যাসার্ধের অর্দ্ধেক হইবে। D. U. 1986

72. একটি কোণ (cone) দ্বারা, 'r' ব্যাসার্ধ বিশিষ্ট একটি গোলককে (sphere) পরিবেষ্টিত করিতে ন্যূনতম কোণের আয়তন নির্ণয় কর।

*73. একটি প্রদত্ত সংখ্যাকে এমন দুই অংশে বিভক্ত কর যেন প্রথম অংশের বর্গের সহিত ২য় অংশের ঘণফল গুণ করিলে গুণফল সর্ব্বৃহৎ হয়।

73. (a) 100 কে এমন দুই অংশে বিভক্ত কর যেন ইহাদের গুণফল বৃহত্তম হয়।

74. 'a' ব্যাসার্ধের একটি গোলকের মধ্যে রক্ষিত বৃহত্তম সমবৃত্ত ভূমির কোণের উচ্চতা কত হইবে?

75. একটি ঘনক এবং একটি গোলকের পৃষ্ঠদ্বয়ের ক্ষেত্রফলের যোগফল দেওয়া আছে। দেখাও যে যখন ইহাদের আয়তনের যোগফল ন্যূনতম হয় তখন গোলকের ব্যাসার্ধ ঘনকের ধারের সমান হইবে।

76. একটি পাতলা আয়তাকার বন্ধ বাক্সের একটি ধার অপর ধারের দৈর্ঘ্যের n গুণ এবং আয়তন V, দেখাও যে ঐ বাক্সের পৃষ্ঠের ন্যূনতন ক্ষেত্রফল s পাওয়া যাইবে $ns^3=54(n+1)^2V^2$ সমীকরণ হইতে।

77. C কেন্দ্র বিশিষ্ট কোন উপবৃত্তের উপর P একটি বিন্দু এবং P বিন্দুতে অংকিত স্পর্শকের উপর C হইতে অংকিত লম্বের পাদ বিন্দু N হইলে PN এর ক্ষুদ্রতম মান নির্ণয় কর।

*78. স্থির একটি গোলকের পৃষ্ঠের যে কোন বিন্দুকে কেন্দ্র করিয়া একটি সম্প্রসারণশীল গোলক নেওয়া হইল। ব্যাসার্ধ কত হইলে স্থির গোলক পৃষ্ঠদ্বারা কর্তিত পৃষ্ঠের ক্ষেত্রফল বৃহত্তম হইবে?

*79. একটি নির্দিষ্ট বিন্দু $P(h, k)$ এর মধ্য দিয়া একটি সরল রেখা এমন-

ভাবে আঁকা হইল যেন উহা অক্ষদ্বয় OX ও OY কে যথাক্রমে A ও B বিন্দুতে ছেদ করে। সরলরেখাটির এরূপ একটি অবস্থান নির্ণয় কর যেখানে $OA+OB$ এর মান ক্ষুদ্রতম হইবে।

*80. একটি উপবৃত্তের (Ellipse) বৃহদাক্ষ $2a$, ইহার উপর কোন বিন্দুর ফোকাস দূরত্ব r_1 ও r_2; রাশিমালা $r_1 r_2(r_1-r_2)$ $[r_1>r_2]$ এর বৃহত্তম ও ক্ষুদ্রতম চরমমান নির্ণয় কর। এই মান উৎকেন্দ্রতার (eccentricity) মান $1/\sqrt{3}$ অপেক্ষা বৃহত্তর বা ক্ষুদ্রতর হওয়ার সাথে সাথে কি রকম পরিবর্তিত হয় তাহা দেখাও। D. H. 1962

*81. একটি উপবৃত্তকের (Ellipsoid) মধ্যে অবস্থিত সর্ববৃহৎ আয়তাকার ঘনবস্তুর (rectangular parallelopiped) আয়তন নির্ণয় কর।

*82. একটি আয়তাকার বক্সের উপরিতল খোলা। যদি ইহার তল-সমূহের ক্ষেত্রফলের যোগফল 12 হয় এবং আয়তন বৃহত্তম হয় তবে ইহার দৈর্ঘ্য, প্রস্থ ও উচ্চতা নির্ণয় কর।

*83. এমন একটি বিন্দুর অবস্থান নির্ণয় কর যাহা হইতে একটি নির্দিষ্ট ত্রিভুজের বাহুগুলির উপর অংকিত লম্বগুলির বর্গের যোগফল ন্যূনতম হয়।

*84. দেখাও যে, যে বিন্দু হইতে nটি বিন্দুর দূরত্বের বর্গের যোগফল ক্ষুদ্রতম হয়, সে বিন্দুটি প্রদত্ত n বিন্দুগুলির গড় অবস্থানের কেন্দ্র হইবে।

*85. ΔABC ত্রিভুজের মধ্যে P যে কোন বিন্দু। PA, PB, PC যুক্ত করা হইল। দেখাও যে কৌণিক দূরত্ব সর্বাধিক হইবে যদি

$\angle APB=\angle BPC=\angle CPA=120°$ হয়।

*86. প্রমাণ কর যে কোন বৃত্তে পরিলিখিত কোন নির্দিষ্ট সংখ্যক বাহুদ্বারা গঠিত বহুভূজগুলির মধ্যে সুষম বহুভূজের ক্ষেত্রফল ক্ষুদ্রতম এবং কোন বৃত্তে অন্তর্লিখিত কোন নির্দিষ্ট সংখ্যক বাহুদ্বারা গঠিত বহুভূজগুলির মধ্যে সুষম বহুভূজের ক্ষেত্রফল বৃহত্তম হইবে।

Exercise XI

Answers

1. (i) max $x=2$ min $x=1$, and 3

   (i) max $x=1$, min $x=6$

   (iii) max $x=-1$ min $x=1$

   (iv) max $x=1$ min $x=3$, none. $x=0$

5. mini. for $=\pi/2$, max for $\tan x=1/\sqrt{2}$ in the Ist quadrant

   (i) no. max or mini.

5. max. value $2m^3+2n$ when $t=-m$

   min. value $-2m^2+2n$ when $t=m$

6. max. 1, mir. 1

7. minimum value is a when $x=0$ 8. $(a+b)^2$

9. max. $\frac{a^2}{a+b}=x$, min $x=\frac{a^2}{a-b}$

16. (i) max $x=n\pi+\pi/6$, mini $x=n\pi-\pi/6$,

    (ii) max. and mini. alternately for $x=\frac{k\pi}{n+1}$

starting from $k=1$, omiting even values of n for which $k=0$ or multiples $(n+1)$

19. (i) min value $=\{\log(ae-e)/\log a\}\log a$

    (ii) min. at $x=1$, value $=4-8\log 2$.

21. (i) nothing

    (ii) no maximum or minimum value at $(-1, 2)$

    (iii) max $(0, 0)$, min. $(2, 0)$

    (iv) min. at $(2, 1)$ (v) max at $(\frac{1}{2}, \frac{1}{2})$

    (iv) $x=y=\pi/3$ max, mini; $x=y=5\pi/3$

    (vii) min at $(4, -2)$ (viii) mini. at $(0, 4/3)$.

22. (ii) max $x=\pi/3$

    (ii) max at $x=\pi/3$, $5\pi/3$, min at $x=0, \pi, 2\pi$

    (iii) max. value $=3$, $x=4/3$ 24. min at $x=3$

26. min at $x=0$ 29. $x^3+3x^2-9x-12$

34. max at $(-1, 17)$, min at $(3, -47)$ inflexion at $(1, -15)$

36. (i) max at $(\frac{1}{3}, 5/27)$, mini at $(-1, -1)$ (i) min, at $x=a$

    (iii) $f'(a)$ is discontinuous but $f(a)$ is max at $x=a$

38. mini. at (a, a) nothing at $(0, 0)$

40. mini at $x=\sqrt{(b/a)}$ max. at $x=-c\sqrt{(b/a)}$

42. (i) min at $(2/5, 6/25, 2/25)$. $u=-254/125$,

    (ii) min. at $(a/2, a/2)$ max. at $(3a/2, -a/2)$

    max at $(a/2, a/2)$. min at $(a/2, -a/2)$

    (iii) min value is $(a^{1/4}+b^{1/4})^{-4}$ at (ar, ar^2, ar^3), $r=(b/a)^{1/4}$

    (iv) min. at $(1, 2, 0)$ (v) max at $(0, 0)$

    (vi) max value $=108a^7/7^7$.

44. This example illustrates how it is not sufficient to express u as a function of x say $2x-x^2+1$

48. $\frac{l^2}{au-1}+\frac{m^2}{bu-1}+\frac{n^2}{cu-1}=0$

50. $3a^2$, at (a, a, a) and at $(-a, -a, -a)$

51. $\frac{1}{3}a^2$, max. at $(a^2/\sqrt{3}, a/\sqrt{3})$, $(-a/\sqrt{3}, -a/\sqrt{3})$,

    $-a^2$, min, at $(a, -a)$; $(-a, a)$.

54. $28/5$, 3 55. (i) min at $(a\ a\ a)$ $3a^2$

61. 200, 200. 62. $\frac{1}{2}$. 66. max surface $=\pi r^2(5+\sqrt{5})/\sqrt{5}$.

67. $h=\frac{2}{3}a\sqrt{3}$ 71. $h=2r$.

72. v is mini. for $x=3r$, r is the radius of the sphere,

73. $x=2a/5$, a being a number y maxi, 74. $h=4a/3$

79. $a-b$. 78. $r=4a/3$

79. $\frac{x}{b+\sqrt{(hk)}}+\frac{y}{k+\sqrt{(hk)}}=1$

81. $8abc/3\sqrt{3}$

82. V max at $(2, 2)$ 83. $\frac{4\Delta^2}{a^2+b^2+c^2}$

CHAPTER XII
ASYMPTOTES

12. **Definition.** *If a straight line meets a curve into two coincident points at infinity, and yet is not itself wholly at infinity, it is called an asymptote to the curve.*

An asymptote can also be defined as "a straight line whose distance d from the moving point P on the curve to the straight line approaches zero as the point recedes to infinity.

When we say that (Px, y) moves along the curve to infinity, we mean that at least one of the Co-ordinates x and y tends to $+\infty$ or $-\infty$ and this is denoted by $P \to \infty$

12.1 Determination of asymptotes (*not parallel to the axes*)

Let $y = mx + c$ (1)

where m and c are finite, be an asymptote of a curve.

Then $m = \lim\limits_{x \to \infty} \frac{y}{x}$ and $c = \lim\limits_{x \to \infty} (y - mx)$

where x, y are the co-ordinates of a moving point P on the curve.

Let d be perpendicular distance between the point $P(x, y)$ of the curve and the straight line $y - mx - c = 0$. Then

$$PM = d = \frac{y - mx - c}{\sqrt{(1 + m^2)}} \qquad (2)$$

$\therefore \lim\limits_{x \to \infty} (y - mx - c) = \lim\limits_{d \to 0} d\sqrt{(1 + m^2)} = 0$; Since m is finite.

Again putting $d\sqrt{(1 + m^2)} = z$. from (2) ; we have

$y - mx - c = z$ or, $y - mx = z + c$ or ; $y/x - m = (z + c)/x$

$\therefore \lim\limits_{x \to \infty} (y/x - m) = \lim\limits_{x \to \infty} (z + c)/x = 0$

Since $z \to 0$. as $x \to \infty$ and c is finite with

or $m = \lim\limits_{x \to \infty} \frac{y}{x}$ (4)

From (3) and (4) we notice that we are to find out $m = \lim\limits_{x \to \infty} (y/x)$ first from the equation ; m may have more than one value. For each value of m, we get a corresponding value of $c = \lim\limits_{x \to \infty} (y - mx)$

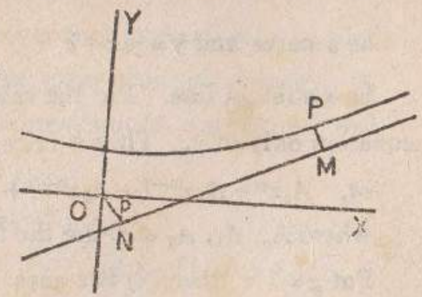
If c is finite, then $y = mx + c$ will be an asymptote of the curve.

This method will determine all asymptotes of the curve not parallel to the co-ordinate axes. Seperate methods will be discussed in Art. 124 for parallel asymptotes

Alternative Definition

An Asymptote is defined as a tangent whose points of contact are at infinity and again it is not itself wholly at infinity.

Let the equation of the tangent at (x, y) of the curve $y = f(x)$ be



Fig—12

$$Y - y = \frac{dy}{dx}(X - x) \text{ or. } Y = \frac{dy}{dx} X + \left(y - x\frac{dy}{dx} \right) \quad (1)$$

Exclude the asymptote parallel to the co-ordinate axes (that is $\frac{dy}{dx} \neq 0, \infty$). The tangent (1) will be asymptote if

$\lim_{x\to\infty} \frac{dy}{dx} = m$ and $\lim_{x\to\infty} \left(y - x\frac{dy}{dx}\right) = c$ are finite. If these conditions are satisfied, then the asymptote is given by **y=mx+c.**

NOTE : The converse is not always true.

12.2. To determine the Asymptotes of an algebraic curve.

In. Art. 12.1. We have discussed definition of asymptotes and also discussed how to determine asymptotes.

In this Article we are giving another method for determining asymptotes.

Let $\phi(x, y) = (a_0y^n + a_1y^{n-1}x + \dots \dots + a_nx^n) + (b_0y^{n-1}$
$b_1y^{n-2}x + \dots + b_n{}^{-1}) + (c_0y^{n-2} + c_0y^{n-3}x + \dots + c_{n-2}x^{n-2}) +$
$\dots \quad \dots \quad \dots \quad \dots \quad \dots \quad = 0 \quad \dots \quad \dots \quad (1)$

be a curve and $y = mx + c$ (2)

be a straight line. Put the value of y in (1), then we get an equation only in x. Thus $\varphi(x, mx+c) = 0 \quad \dots \quad \dots \quad (3)$

or, $A_0x^n + A_1x^{n-1} + A_2x^{n-2} + \dots + A_n = 0 \quad \dots \quad (4)$

where A_0, A_1, A_2 etc. are the functions of m and c.

Put $x = 1/z$, then (4) becomes

$A_0 + A_1z + A_2z^2 + \dots + A_{n-1}z^{n-1} + A_nz^n = 0$

If two roots of this equation are zero, then

$A_0 = 0$ and $A_1 = 0$

But $x = 1/z$, when $z \to 0$, x becomes infinite.

Thus we see that we are to select the values of m and c in such a way that two of the roots of x in (4) become infinite. In this case the straight line $y = mx + c$ cuts the curve at two coincident points on the curve whose distances from the origin are infinite.

From $A_0 = 0$ and $A_1 = 0$ we will get values of m and c. Putting these values in $y = mx + c$ we will get the equations of asymptote of the curve (1)

12.3. The asymptotes of the general algebraic curve.

(A) Let the equation to the curve be

$$(a_0y^n + a_1y^{n-1}x + a_2y^{n-2}x^2 + \dots + a_{n-1}yx^{n-1} + a_nx^n)$$
$$+ (b_1y^{n-1} + b_2y^{n-2}x + \dots + b^{n-1}yx^{n-2} + b_nx^{n-1})$$
$$+ (c_2y^{n-2} + \dots + c_nx^{n-2}) + = 0 \quad \dots \quad (1)$$

or ; $P_n + P_{n-1} + P_{n-2} + \quad \dots \quad \dots \quad + P_0 = 0 \quad \dots \quad (2)$

Now we see that P_n is homogeneous expression in x. y of degree n ; P_{n-1}, is a homogeneous expression of degree $n-1$, P_{n-2} is also an homogeneous expression of degree $n-2$ and so on.

$$P_n = a_0y^n + a_1y^{n-1}x + \dots + a_{n-1}yx^{n-1} + a_nx^n$$
$$= x^n \{a_0(y/x)^n + a_1(y/x)^{n-1} + \dots \dots \dots a_{n-1}(y/x) + a_n\}$$
$$= x^n\phi_n(y/x) \quad \dots \quad \dots \quad \dots \quad (3)$$
$$= x^n\phi(m), \text{ where } (y/x) = m \quad \dots \quad (4)$$

$\phi_n(m)$ is a homogenous expression of degree n, and so $\phi_n(m) = 0$ has n roots. Let $m_1, m_2 \dots m_n$ be the roots of $\phi_n(m) = 0$

Thus $\phi_n(m) = (m-m_1)(m-m_2)\ldots(m-m_n)$

$= (y/x-m_1)(y/x-m_2)\ldots(y/x-m_n)$

$\therefore \quad P_n = x^n\phi_n(m) = (y-m_1x)(y-m_2x)(y-m_3x)\ \ldots(y-m_nx)$

The possible asymptotes of the curve are parallel to $y-m_1x=0$

$y-m_2x=0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad y-m_nx=0.$

Case 1. If P_n has no repeated linear factor, then the equation (2) can be written as

$P_n+(P_{n-1}+P_{n-2}\ \ldots\ \ldots\ \ldots +P_0)=0$ or, $P_n+F_{n-1}=0$ ... ... (6)

Let $y-m_1x$ be a non-repeated factor of P_n

Then $P_n=(y-m_1x)\ Q_{n-1}$,

where $Q_{n-1}=(y-m_2x)(y-m_3x)\ldots(y-m_nx)$ and

F_{n-1} contains all the terms of $(n-1)$ the degree and lower degree.

$\therefore \quad (y-m_1x)\ Q_{n-1}+F_{n-1}=0 \quad \ldots \quad \ldots \quad \ldots \quad (7)$

It there is an asymptote parallel to $y-m_1x=0$ let it be

$$y-m_1x=c_1$$

From the definition of asymptotes

$$c_1=\lim_{x\to\infty}(y-m_1x)=\lim_{x\to\infty}\frac{-F_{n-1}}{Q_{n-1}}$$

Therefore, the asymptote parallel to $y-m_1x=0$ is

$$y-m_1x+\lim_{x\to\infty}\frac{F_{n-1}}{Q_{n-1}}=0 \quad \ldots \quad \ldots \quad (8)$$

In this way other asymptotes parallel to $y-m_2x=0$.

$y-m_3x=0$ etc can be determined.

Cese II. Let P_n consist of repeated factors, say $(y-m_1x)^2$

Then Eq (7) becomes

$(y-m_1x)^2\ Q_{n-2}+P_{n-1}+(P_{n-2}+\ldots+P_0)=0$

or, $(y-m_1x)^2\ Q_{n-2}+P_{n-1}+F_{n-2}=0 \quad \ldots \quad \ldots \quad (9)$

$Q_{n-2}=(y-m_3x)(y-m_4x)\ldots(y-m_nx)=$ terms of $(n-2)$th degree,

$P_{n-1}=$ terms of $(n-1)$th degree

$F_{n-2}=$ terms of $(n-2)$th degree and lower degree terms.

Asymptotes parallel to $y-m_1x=0$ is given by

$y-m_1x=c_2$, Where c_2 is obtained from (9)

$$c_2=\lim_{x\to\infty}(y-m_1x)^2=-\lim_{x\to\infty}\frac{P_{n-1}+F_{n-2}}{Q_{n-2}}$$

If P_{n-1} does not contain $y-m_1x$ as a factor, then c_2 does not tend to a finite limit, so there will be no asymptote parallel to $y-m_1x=0$,

If P_{n-1} has a factor $y-m_1x$. then eq. (9) becomes

$(y-m_1x)^2\ Q_{n-2}+(y-m_1x)\ R_{n-1}+F_{n-2}=0$

If there are asymptotes parallel to $y-m_1x=0$, they are given by

$$(y-m_1x)^2+(y-m_1x)\lim_{x\to\infty}\frac{R_{n-1}}{Q_{n-2}}+\lim_{x\to\infty}\frac{F_{n-2}}{Q_{n-2}}=0 \ \ldots \quad (10)$$

In the same way we can find out asymptotes parallel to the factors of $P_n=0$.

All the asymptotes of curve (1) can be determined by the method shown above.

Cor. If P_{n-1} in eq (2) is absent, then all the asymptotes parallel to the factors of P_n are given by $P_n=0$ if there are repeated factors in P_n;

i, e ; if the eq, (2) is written in the form.

$P_n+(P_{n-2}+P_{n-3}+\ldots P_0)=0$

or. $P_n+R_{n-2}=0$ ... ... (10a)

then all the asympototes are given by $P_n=0$ ... ... (1)

Ex. Find the asympotes of $x^4-xy^3+x^2+y^2-a^2=0$

The eq can be written as $(x^4-xy^3)+(x^2+y^2)-a^2=0$

or ; $P_4+(P_2+P_0)=0$ or ; $P_4+R_2=0$

Here P_2 is absent. So all the asymptotes are given by

$P_n=0$ or, $(x^4-xy^3)=0$ or, $x(x-y)(x^2+xy+y^2)=0$ are the asymptotes of given curve.

Working Rule :—

(a) Group the highest homogeneous degree terms in bracket It is P_n. Equate $P_n=0$. The asymptotes of the curve will be parallel to the factors of $P_n=0$.

Let $y=m_1x$ be of the factor of P_n. Then the asynptotes parallel to $y-m_1x=0$ is from (7)

$$y=m_1x+\lim_{\substack{x\to\infty\\ y=m_1x}}\frac{F_{n-1}}{Q_{n-1}}=0$$

Similarly for other asymptotes parallel to the factors of P_n.

(b) If P_n has repeated factors say $(y-m_1x)^2$ then the asymptotes parallel to $y-m_1x=0$ are given by

$$(y-m_1x)^2+(y-m_1x)\lim_{\substack{x\to\infty\\ y=m_1x}}\frac{R_{n-2}}{Q_{n-2}}+\lim_{\substack{x\to\infty\\ y=m_1x}}\frac{F_{n-2}}{Q_{n-2}}=0$$

12.3. (B) Let th eq. (1) ...... (A) be written as

$$P_n+P_{n-1}+P_{n-2}+\dots\dots+P_n=0$$

or, $x^n\phi_n\left(\frac{y}{x}\right)+x^{n-1}\phi_{n-1}\left(\frac{y}{x}\right)+\quad\dots\quad\dots=0$ ...(12)

where $\phi_n(y/x)$ represent an expession of nth degree in y/x

Let $y=mx+c$ ... ... (13)

be one of the asymptotes of the curve (12)

$\therefore\ y/x=m+c/x$ ... ... (14)

Put the value of y/x in (12) then

$$x^n\phi_n(m+c/x)+x^{n-1}\phi_{n-1}(m+c/x)+\dots\dots=0\quad\dots(15)$$

Expand it by Taylors Theorem.

$$x^n\{\phi_n(m)+\frac{c}{x}\phi'_n(m)+\frac{c^2}{\lfloor 2x^2}\phi''(m)+\dots\dots\}$$

$$+x^{n-1}\{\phi_{n-2}(m)+\frac{c}{x}\phi'_{n-1}(m)+\frac{c^2}{\lfloor 2x^2}\phi''_{n-1}(m)+\dots\}$$

$$+x^{n-2}\{\phi_{n-2}(m)+\frac{c}{x}\phi'_{n-2}(m)+\frac{c^2}{\lfloor 2x^2}\phi''_{n-2}(m)$$

$$+\dots\dots\}+\dots\dots=0$$

or, $x^n\phi_n(m)+x^{n-1}\{c\phi'_n(m)+\phi_{n-1}(m)\}+$

$$x^{n-2}\left\{\frac{c^2}{\lfloor 2}\phi''_n(m)+c\phi'_{n-1}(m)+\phi_{n-2}(m)\right\}+\dots=0\ \dots..(16)$$

Since $y=mx+c$ is an asymptote of the curve (12) therefore two roots of the equations become infinite for which we are to equate to zero the co-efficients of first two highest degree terms.

Now from (16) we have

$\phi_n(m)=0$ ... ... (17)

$c\phi'_n(m)+\phi_{n-1}(m)=0$ ... (18)

Equation $\phi_n(m)=0$ is of nth degree in m, so it gives us roots say, $m_1, m_2 \dots m_n$. From (18) we get the values c corresponding to the values of m say m_1, m_2 etc.

Thus the asymptotes are $y=m_1x+c_1$, $y=m_2x+c_2$ etc.

Cor, *If $\phi_n(m)=0$ has two equal roots.* From (16) we have

$$c_1=-\phi_{n-1}(m_1)/\phi'_n(m_1)$$

If $m_1=m_2$ in $\phi(m)=0$, then $\phi_n'(m)=0$

Hence $c_1=-\phi_{n-1}(m_1)/\phi_n'(m_1)$ is infinite if $\phi_{n-1}(m_1)\neq 0$,

The straight line $y=m_1x+c_1$ will intersect the y—axis at infinity, so is not an asymptote, through m_1 is a roots of $\phi_n'(m)=0$

In this case if $\phi_n(m)=0$ has two equal roots, then $\phi_n'(m_1)=0$ and $\phi_{n-1}(m_1)=0$. The eq. (18) vanish independently.

In such cases c will be obtained by equation to zero the co-efficient of x^{n-2} in (16).

Thus

$$\frac{c^2}{\underline{|2}}\phi''_n(m)+c\phi'_{n-1}(m)+\phi_{n-2}(m)=0 \quad \cdots \quad \cdots \quad (19)$$

The eq. (17) is a quadratic in c so it will give two values of c for the value of m_1.

Therefore, the asymptotes are $y=m_1x+c_1$, $y=m_1x+c_2$,

Cor. 2. If the equation $\phi_n(m)=0$ has three equal roots then we are to equate to zero the co-efficient of x^{n-2}, of eq. (16)

Thus $\phi_n(m)=0$

and $(c^3/\underline{|3}\phi'''_n(m)+c^2/\underline{|2}\phi''_{n-1}(m)+c\phi'_{n-2}(m)+\phi_{n-3}(m)=0$

There will be three values of c say c_1, c_2, c_3 for m_1. Then the asymptotes are $y=m_1x+c_1$, $y=m_1x+c_2$, $y=m_1x+c_3$.

Cor. 3. For imaginary roots, there will be no asymptotes of the curve.

Working Rule :—

(a) First group the highest homogeneous degree terms in a bracket : which is $\phi_n(y/x)$.

Put $x=1$, $y=m$ Then $\phi_n(m)$ is obtained.

Similarly find $\phi_{n-1}(m)$. Differentiate $\phi_n(m)$ w. r. to m.

Then $\phi'_n(m)$ is obtained

(b) Put $\phi_n(m)=0$. Find the roots of m. Consider only the real roots of m. Put one by one in $c\phi'_n(m)+\phi_{n-1}(m)=0$,

then for each value of m, there will be a value of c.

Put the values of m and c in $y=mx+c$. which will be an asymptote. In this way all the asymptotes can be determined for real values of m.

(c) If $\phi(m)=0$ has two eqal roots, then equate to zero the co-efficient of x^{n-1} i. e. ;

$$\frac{c^2}{\underline{|2}}\phi_n''(m)+c\phi'_{n-1}(m)+\phi_{n-2}(m)=0.$$

Two values of c will be obtained and thus two parallel asymptotes will be determined.

(d) If $\phi_n(m)=0$ has three equal roots, then

$$\frac{c^3}{\underline{|3}}\phi_n'''(m)+\frac{c^2}{\underline{|2}}\phi''_{n-1}(m)+c\phi'_{n-2}(m)+\phi_{n-3}(m)=0$$

Thus there will be three parallel asymptotes.

———

Asymptotes parallel to the co-ordinates axes

12.4. (A) Asymptotes parallel to the x—axis.

Let $\phi(x, y)=0$

be the equation as shown in Art. 12.2

The equation can be arranged in the descending powers of x *i.e.*

$$a_0x^n+(a_1y+b_1)x^{n-1}+(a_2y^2+b_2y+c_1)x^{n-2}+ \quad ... \quad =0 \quad ... (1)$$

If $a_0=0$ and y be selected in such a way that $a_1y+b_1=0$ ...(2)

Then there are two infinite roots of x.

Hence $a_1y+b_1=0$ is an asymptote of the curve (1)

$a_1y+b_1=0$ or. $y=-b_1/a$; $a_1\neq 0$, or, $y=k$, (say)

which is a straight line parallel to x-axis.

If $a_0=0$, $a_1=0$, $b_1=0$. then there are three infinite roots of x so $a_2y^2+b_2y+c_1=0$ giving two asymptotes parallel to x—axis if the roots of y are not imaginary.

Hence to determine the asymptotes parallel to the x-axis proceed as follows ;

Rule : In an algebraic equation of a degree if the highest power term of x (say x^n) is absent, all the asymptotes parallel to the x-axis are obtained by equating to zero the co-efficients to the next available highest power of x (say x^{n-1}) in the equation.

If the co-efficient is constant, then equate to zero the next available higher power.

If the co-efficients give imaginary factors or constant then there will be no asymptotes parallel to x—axis.

Ex. Determine the asymptote of the curve.

$$y^3-yx^2+y^2x+x^2-4=0$$

In this equation highest power of x. *i. e.* x^3 is absent. So there may be some asymptotes parallel to x—axis.

Arrange the equation in descending powers of x. Then

$$x^2(1-y)+xy^2+y^3-4=0 \quad ... \quad ... \quad (1)$$

Equate to zero the Co-efficiente of available highest power of x (i. e., x^2). Then the asymptot parallel to x—given by

$$1-y=0 \quad \text{or,} \quad y=1$$

12.4. (B) Asmptotes parallel to y—axis.

In an equation of nth degree in x and y, if the highest power of y (i. e., y^n) is absent, then equate the co-efficients of next available highest powers of y (i. e, y^{n-1}) to zero ; all asymptotes parallel to y—axis will be obtained.

Ex. Determine the asymptotes of the curve

$$x^4-x^2y^2+x^2+y^2-a^2=0$$

In this equation the highest powers of y (*i. e.*, y^4) is absent. There may be some asymptotes parallel to y—axis.

Arrange the equation in descending powers of y. Then

$$y^2(1-x^2)+x^4+x^2-a^2=0 \quad ... \quad ... \quad (1)$$

Equate to zero the available highest power of y in the equation (1), Thus $1-x^2=0$ or, $x=\pm 1$

Thus asymptotes parallel to y axis are given by

$$x-1=0 \text{ and } x+1=0$$

12.5. Intersection of a curve with its asymptotes

Let us consider an equation of nth degree. This equation has n asymptotes. If there are no two parallel asymptotes, we can write the equations of this curve and the asymptotes as in Cor, Art. 12.3

$$F_n=0 \quad \ldots \quad \cdots \quad (1)$$

$$F_n+F_{n-2}=0 \quad \cdots \quad (2)$$

If $S_1=0$ and $S_2=0$ be two curves then any other curve is represented by $S_1-\lambda S_2=0$.

If $\lambda=1$, then the curve is $S_1-S_2=0$

If $S_1=F_n+F_{n-2}$ and $S_2=F_n$

then $(F_n+F_{n-2})-F_n=0$ or, $F_{n-2}=0$

represents curve of intersection of the two curves $F_n+F_{n-2}=0$ and $F_{n-2}=0$. Thus we see that all points of intersection of the asymptotes and curve be on a curve $F_{n-2}=0$.

We know a straight line cuts generally a curve of nth degree in n points (real, imaginary). If this line is an asymptote it cuts a curve in two points at infinity. Hence it cuts the curve of nth degree in $(n-2)$ points. As there are n asymptotes for curve of nth degree so all asymptotes cut the curve in $n\,(n-2)$ point and all the points will lie on a curve. $F_{n-2}=0$

The equations of all asymptotes are given by $F_n=0$.

Thus the given curve is obtained by combining the equations $F_n+F_{n-2}=0$.

For example

(1) If the curve is a cubic equation the points of intersection of the asymptotes and the curve will lie on a curve.

$F_{n-2}=0$ or, $F_{3-2}=0$ or, $F_1=0$

which is a first degree equation in x and y. The asymptotes meet the curve at $n(n-2)$ i. e ; $3(3-2)=3$ points.

Hence three points will lie in a straight line.

(ii) If the curve is a 4th degree equation then $n=4$.

The points of intersection of the asymptotes and the curve will lie on a curve.

$F_{n-2}=0$ or, $F_{4-2}=0$ or $F_2=0$ which is an equation degree 2. This will represent a conic section.

Moreover the points of intersection of the asymptotes and the curve are

$n(n-2)=4(4-2)=8$ *i. e.*, 8 points will lie on a curve $F_2=0$ Similarly for curves of higher degree.

12.6. Asymptotes in Polar Co-ordinates.

If α be a root of the eqation $f(\theta))=0$, then

$r\sin(\theta-\alpha)=1/f'(\alpha)$ (1)

is a asymptote of the curve $1/r=f(\theta)$ $\cdot\cdot$ (2)

If $r\to\infty$, then from the eq. 2) we have $f(\theta)=0\ldots$ (4)

Let the roots of the equation $f(\theta)=0$ be α,β,γ ...

Let $P(r,\theta)$ be any point on a curve near the asymptote CM.

Draw

$OP=$ the radius vector

$PN=$ tangent at P

NO is perpendicular to $P.N$.

$ON=$ the polar subtangent $=r^2\dfrac{d\theta}{dr}$

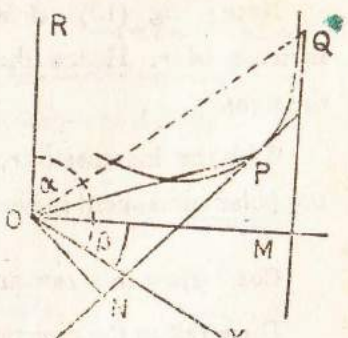


Fig. 13

If P moves along the curve up to infinity, the tangent PN becomes an asymptote QM and consquently ON becomes perpendicular to QM, ON equals OM, and becomes parallel to QM,

If $p = \underset{P \to Q}{Lim}\, ON = \underset{r \to \infty}{Lim}\, r^2 \frac{d\theta}{dr}$ and $\beta = \alpha - \frac{1}{2}\pi$ where $\underset{r \to \infty}{Lim}\, \theta = \alpha$,

The polar equation of the asymptote is $p = r \sin (\theta - \beta)$

or, $1/f'(\alpha) = r \sin (\theta - \alpha + \frac{1}{2}\pi) = r \sin (\theta - \alpha)$

or, $r \sin (\theta - \alpha) = 1/f'(\alpha)$

Working Rule : Arrange the given equation in the form $1/r = f(\theta)$

Equate $f(\theta) = 0$. Find the roots of θ, say α, β, γ etc. Differentiate $f(\theta)$, put $\theta = \alpha, \beta, \gamma$ etc in $f'(\theta)$ so that $f'(\alpha)$, $f'(\beta)$ etc, are obtained.

Put the values of α and $f'(\alpha)$ to get $r \sin (\theta - \alpha) = 1/f'(\alpha)$ which will give us the polar equation of an asymptote.

Note : fig (13). $dr/d\theta$ is positive as θ increase with the increase of r. Hence the polar subtangent is positive and is to the right.

With the increase of r, θ decreases, so $dr/d\theta$ is negative, so the polar subtangent is negative and is drawn to the left.

Cor. How to draw an asymptote in polar Co-ordinates.

Draw OR in the direction indicated by α

Draw OM parpendicular to OR where $OM = \underset{r \to \infty}{Lim}\, r^2 \frac{d\theta}{dr} = p$

If p positive, draw OM to the right side of OR, and p is negative draw to left side of OR.

Now draw MQ parallel to OR which dedermines the asymptote of the curve.

12.7. Circular Asymptotes.

In many polar equation when θ increases indefinitely and r remains finite, the equation involves only r. Such type of curve posseses circular asymptotes since r is ultimately constant.

In this case the curve represents one or more concentric circles.

Ex. 1. Find the asymptotes of

$$x^3 + 2x^2y - xy^2 - 2y^3 + xy - y^2 - 1 = 0 \quad \cdots \quad \cdots \quad (1)$$

Let $y = mx + c \quad \ldots \quad \cdots \quad (2)$

be one of the asymptotes.

Put the value of y in (1), then

$$x^2 + 2x^2(mx+c) - x(mx+c)^2 - 2(mx+c)^3 + x(mx+c) - (mx+c)^2 - 1 = 0$$

or, $x^3(1 + 2m - m^2 - 2m^3) + x^2(2c - 2mc - 6m^2c + m - m^2) + x(c - c^2 - 6mc^2 + 2mc) + (-2c^3 - c^2 - 1) = 0$,

or, $A_0x^3 + A_1x^2 + A_2x + A_3 = 0$.

Now equate to zero the co-efficients of x^3 and x^2 seperately.

$\therefore \quad A_0 = 0$ or, $1 + 2m - m^2 - 2m^3 = 0$

and $A_1 = 0$ or, $2c - 2mc - 6m^2c + m - m^2 = 0$

The first equation gives

$1 + 2m - m^2 - 2m^3 = 0$ or, $(1+2m)(1+m)(1-m) = 0$

$\therefore \quad m = 1, -1, -\frac{1}{2}$.

From the 2nd equation $2c-2mc-5m^2c+m-m^2=0$

or, $c=\dfrac{m^2-m}{2-2m-6m^2}$

For, $m=1,\ -1,\ -\frac{1}{2},\ c=0,\ -1,\ \frac{1}{2}$ respectively.

Hence the asymptotes are obtained from (2) by putting the values of m and c.

There are $y=x,\ y=-x-1,$ and $y=-\frac{1}{2}x+\frac{1}{2}$

or: $y=x,\ x+y+1=0$ and $x+2y-1=0$

Ex. 2, Find the asymptotes of D. H. 1987

$$4x^3-x^2y-4xy^2+y^3+3x^2+2xy-y^2-7x+5=0$$

The equation can be written as

$(4x^3-x^2y-4xy^2+y^3)+(3x^2+2xy-y^2)-7x+5=0$

or: $x^3\phi_3(y/x)+x^2\phi_2(y/x)-7x+5=0$ ... ... (1)

Let $y=mx+c$ ... ... (2)

be one of the asymptotes of (1)

Then Put $x=1$ and $y=m$ in $\phi_3(y/x)$ and $\phi_2(y/x)$ in (1) we have

$\phi_3(m)=4-m-4m^2+m^3$; $\phi_2(m)=3+2m-m^2$

$\phi'_3(m)=-1-8m+3m^2$

Now $\phi_3(m)=0$ or, $4-m-4m^2+m^3=0$

or, $(4-m)(1+m)(1-m)=0$ or, $m=1,-1,4$

Also $c=-\dfrac{\phi_{n-1}(m)}{\phi'_n(m)}=-\dfrac{\phi_2(m)}{\phi'_3(m)}=-\dfrac{3+2m-m^2}{-1-8m+3m^2}$

For $m=1,-1,4$, the values of $c=\frac{2}{3},\ 0,\frac{1}{3}$ respectively

Hence required asymptotes are, from (2).

$y=x+\frac{2}{3},\ y=-x+0,\ y=4x+\frac{1}{3}$

or, $3y=3x+2,\ y+x=0,\ 3y=12x+1$

Ex. 3. Find the asymptotes of

$x^3-x^2y-xy^2+y^3+2x^2-4y^2+2xy+x+y+1=0$ R. U. 1954

Write the equation as

$(x^3-x^2y-xy^2+y^3)+(2x^2-4y^2+2xy)+(x+y)+1=0$ ... (1)

or, $P_3+P_2+P_1+P_0=0$

Now $P_3=x^3-x^2y-xy+y^3=(y-x)^2(y+x)$

[Put $x=1,\ y=m,\ 1-m-m^2+m^3=(1-m)^2(1+m)$.

Then put $m=y/x$. The factors of P_3 will be obtained].

The asymptotes of the curve (1) are parallel to $y-x=0$.

$y+x=0$

Now the asymptote parallel to $y+x=0$

$$y+x+\lim_{\substack{x\to\infty\\ y=-x}}\frac{2x^2-4y^2+2xy}{(y-x)^2}+\lim_{\substack{x\to\infty\\ y=-x}}\frac{x+y+1}{(y-x)^2}=0$$

or, $y+x+\lim_{x\to\infty}\dfrac{2x^2-4x^2-2x^2}{(-x-x)^2}+\lim_{x\to\infty}\dfrac{x-x+1}{(-x-x)^2}=0$

or, $y+x-1+0=0$

or, $y+x=1$

Again the asymptote parallel to $y-x=0$ is

$$(y-x)^2-(y-x)\lim_{\substack{x\to\infty\\ y=x}}\frac{2(x+2y)}{y+x}+\lim_{\substack{x\to\infty\\ y=x}}\frac{x+y}{x+y}+\lim_{\substack{x\to\infty\\ y=x}}\frac{1}{x+y}=0$$

or, $(y-x)^2-(y-x).\ 3+1+0=0$

or, $(y-x)^2-3(y-x)+1=0$

Hence asymptotes are $x+y-1=0$ $y-x=\frac{1}{2}(3\pm\sqrt{5})$

Ex. 4. Find the asymptotes of the curve N.U. 1994

$(y-x)^2x-3y(y-x)+2x=0$ R. U. 1951, D.H. 1965.

The equation is of 3rd degree but y^3 is absent

So, there are asymptotes parallel to y—axis. The highest power of y available here is y^2

$$y^2x-2x^2y+x^3-3y^2+3xy+2x=0$$

or, $y^2(x-3)-2x^2y+x^2+3xy+2x=0$

Now equate the Co-efficients of y^2 to zero. Thus

$x-3=0$ is an asymptote.

For other asymptotes, from the original equation we notice that

$$P_3=(y-x)^2x$$

Hence the asymptotes are parallel to $y-x=0$, $x=0$.

The asymptotes parallel to $y-x=0$ are

$$(y-x)^2+(y-x)\lim_{\substack{x\to\infty\\ y=x}}\frac{-3y}{x}+\lim_{\substack{x\to\infty\\ y=x}}\frac{2x}{x}=0$$

or, $(y-x)^2+(y-x)(-3)+2=0$ or; $(y-x-2)(y-x-1)=0$

or, $y-x-2=0$, $y-x-1=0$

Hence the asymptotes are $x=3$, $y-x-2=0$, $y-x-1=0$

Ex. 5. Find the asymptotes of the curve $y\sqrt{(x^2-1)}=x^2$ C. U. '88

The equation is written as $y^2(x^2-1)=x^4$ ... ... (1)

The equation (1) is of 4th degree, so it may have four asymptotes. As the highest power of y, *i. e.*

y^4 is absent, so there are some asymptotes parallel to y—axis.

Now equate the co-efficient of y^2 (as y^2 is available higher power) to zero, then $x^2-1=0$ or, $x=\pm 1$.

For two more asymptotes, the equation (1) is written as

$$x^4-y^2x^2-y^2=0$$

or, $x^2(x^2-y^2)-y^2=0$ ... (2)

or, $x^2(x+y)(x-y)-y^2=0$ ... (3)

The other two asymptotes are parllel to $x+y=0$, $x-y=0$

Now asymptote parallel to $x+y=0$ is

$$x+y+\operatorname*{Lt}_{\substack{x\to\infty\\ y=-x}}\frac{-y^2}{x^2(y-x)}=0$$

or, $x+y+0=0$ or, $x+y=0$.

Similarly other asymptote is $x-y=0$.

Hence asymptotes are $x+1=0$, $x-1=0$, $x+y=0$, $x-y=0$

Ex. 6. Find the asymptotes of the curve

$$xy^2(x-y)-5x^2y-y^3+6x^2-5=0$$

The equation is of 4th degree, so there are four asymptotes.

From the equation, the asymptotes are parallel to $x=0, y=0$, $x-y=0$

The asymptotes parallel to $x-y=0$ is

$$x-y+\lim_{\substack{x\to\infty\\ y=x}}\frac{-y(5x^2+y^2)}{xy^2}+\lim_{\substack{x\to\infty\\ y=x}}\frac{6x^2-5}{xy^2}=0$$

or, $x-y-6=0$ or, $x-y=6$

The equation is written as

$$x^2y^2-xy^3-5x^2y-y^3+6x^2-5=0 \quad \cdots \quad \cdots \quad (1)$$

or, $x^2(y^2-5y+6)-xy^3-y^3-5=0$ ... ... (2)

or, $y^3(-x-1)+y^2x^2-5yx^2+6x^2-5=0$ ... (3)

The equation is of 4th degree. The highest powers of x and y. *i. e*, x^4 and y^4 are absent. So there are some asymptotes parallel to co-ordinate axes.

Now equate the co-efficient of x^2 (available highest power of x) to zero from (2) then the asymptotes are

$y^2-5y+6=0$ or, $(y-3)(y-2)=0$ $\therefore$ $y-2=0$ and $y-3=0$.

Again equate the co-efficient of y^2 (available highest power of y) to zero from (3). then the asymptote parallel to y axis is

$$-x-1=0 \quad \text{or,} \quad x+1=0$$

Thus the asymptotes are $y=2$, $y=3$, $x+1=0$, $x-y=6$.

Ex. 7. Find the asymptotes of the curve $y=\dfrac{\log x}{x}$

Let us define x in the interval $0<x<+\infty$

The function y is not discontinuous in the defined domain but when x tends to zero, then

$$\underset{x\to 0}{Lim}\ y=\underset{x\to 0}{Lim}\ \frac{\log x}{x}=-\infty$$

Hence the straight line $x=0$ *i. e.* ; y—axis is the asymptotes of the given curve.

For an asymptote of the from $y=mx+c$, where m and c finite we have,

$$m=\underset{x\to+\infty}{Lim}\ (y/x)=\underset{x\to\infty}{Lim}\ (\log x)/x^2=0$$

$$\text{and } c=\underset{x\to+\infty}{Lim}\ (y-mx)=\underset{x\to\infty}{Lim}\left(\frac{\log x}{x}-mx\right)$$

$$=\underset{x\to\infty}{Lim}\left(\frac{\log x}{x^2}x-0\right)=0$$

$\therefore$ There is another asymptote which is $y=0$, *i. e.* the x—axis is also an asymptote of the curve.

The asymptotes are $x=0$, $y=0$.

Note : Note the asymptotes of the following curves-

(i) $y=a^x$; x—axis is the asymptote.

(ii) $y=\log_a x$; y—axis is the asymptote See Art. 1. 15.

Ex. 8. Find the asymptotes of the curve $y=e^{-x}\sin x+x$

There are no asymptotes parallel to the co-ordinate axes.

Let us try for obeique asymptote of the form $y=mx+c$

$$m=\underset{x\to\infty}{\text{Lim}}\ \frac{y}{x}=\underset{x\to\infty}{\text{Lim}}\ \frac{e^{-x}\sin x+x}{x}=0+1=1$$

$$c=\underset{x\to\infty}{\text{Lim}}\ (y-mx)=\underset{x\to\infty}{\text{Lim}}\ (e^{-x}\sin x+x-x)$$

$$=\underset{x\to\infty}{\text{Lim}}\ e^{-x}\sin x=0$$

Hence $y=x$ is the asymptote of the curve as $x\to\infty$.

When $x\to-\infty$.

$$m=\underset{x\to-\infty}{\text{Lim}}\ \frac{y}{x}=\underset{x\to\infty}{\text{Lim}}\ \left(\frac{e^{-x}\sin x}{x}+1\right)=\text{does not exist as}$$

the first term increases to infinity as $x\to-\infty$

So, there is no asymptote when $x\to-\infty$

Ex. 9. Find the asymptote of the curve $r(3\theta-\pi)=a\sin\theta$

The equation is written as

$$\frac{1}{r}=\frac{3\theta-\pi}{a\sin\theta}=f(\theta) \quad \ldots \quad (1)$$

For the asymptotes $r\to\infty$ and so $f(\theta)\to 0$

$$\Rightarrow \frac{3\theta-\pi}{a\sin\theta}=0 \quad \text{or,} \quad \theta=\tfrac{1}{3}\pi$$

We are to investigate the possibility of an asymptote at $\theta=\pi/3$

Differentiate (1) $w.\,r$ to θ, $f'(\theta)=\dfrac{3\sin\theta-(3\theta-\pi)\cos\theta}{a\sin^2\theta}$

When $\theta=\pi/3$,

$$f'(\pi/3)=\frac{3\sin\pi/3-(3\pi/3-\pi)\cos\pi/3}{a\sin^2\pi/3}=\frac{3\sqrt{3}/4}{2a.3}=\frac{2\sqrt{3}}{a}$$

Therefore the asymptote is

$$r\sin(\theta-\tfrac{1}{3}\pi)=1/f'(\pi/3)=\frac{a}{2\sqrt{3}}$$

or ; $2\sqrt{3}\, r\sin(\theta-\pi/3)=a$

Ex. 10. Find the asymptotes of $r=\sec\theta+b\tan\theta$.

The equation is written as $r=a/\cos\theta+(b\sin\theta)/\cos\theta$

$$=(a+b\sin\theta)/\cos\theta$$

or ; $1/r=\cos\theta/(a+b\sin\theta)=f(\theta)$

If $f(\theta)=0$, then $\cos\theta=0$

$\therefore\ \theta=(2n+1)\pi/2$, n is any integer.

$\therefore\ \theta=\frac{1}{2}\pi, 3\pi/2, 5\pi/2, 7\pi/2$ and so on.

$$\text{Now } f'(\theta)=\frac{-\sin\theta\,(a+b\sin\theta)-\cos\theta\, b\cos\theta}{(a+b\sin\theta)^2}=\frac{-a\sin\theta-b}{(a+b\sin\theta)^2}$$

$f'(\pi/2)=-1/(a+b), f'(3\pi/2)=(a-b)/(a-b)^2=1/(a-b)$,

$f'(5\pi/2)=-1/(a+b)$, $f'(7\pi/2)=1/(a-b)$

Thus the equation of the asymptote at $\theta=\pi/2$ is

$r\sin(\theta-\pi/2)=1/f'(\pi/2)=-(a+b)$ or ; $r\cos\theta=a+b$

the asymptote at $\theta=3\pi/2$ is

$$r\sin(\theta-3\pi/2)=a-b \quad\text{or ; } r\cos\theta=a-b$$

Proceding this way we see that the asymptotes at $\theta=\pi/2, 5\pi/2, 9\pi/2$ each has equation

$$r\cos\theta=a+b$$

and asymptotes at $\theta=3\pi/2$, $7\pi/2$, ete. each has equation

$$r\cos\theta=a-b.$$

Therefore there are only two different asymptotes whose equations are

$$r\cos\theta=a+b \text{ and } r\cos\theta=a-b$$

Ex. 11. Find the circular asymptotes of $(r-2)\,\theta=\sin\theta$

The circular asymptote is

$$\therefore\ r-2=\lim_{\theta\to\infty}(3\sin\theta)/\theta=0 \qquad \text{as } |\sin\theta|\leqslant 1$$

or ; $r=2$

Ex. 12. Show that the asymptotes of the curve

$$(x+a)\,y^2-(y+b)x^2=0$$

cut the curve in three points which lie on the straight line

$$b^2(x+a)=a^2(y+b)$$

The given equation is of 3rd degree and is written as

$$(x+a)y^2-(y+b)\,x^2=0 \quad \cdots\ \cdots \quad (1)$$

$$\text{or ; } xy^2-yx^2+ay^2-bx^2=0 \quad \cdots\ \cdots \quad (2)$$

The asymptotes parallel to x axis is given by equating the co-efficient of x^2 to zero i. e $y+b=0$ ... ... (3)

Similarly asymptote parallel to y axis is given by

$x+a=0$ ... ... (4)

From eq. (2), $xy(y-x)+ay^2-bx^2=0$ ... (5)

Now asymptote parallel to $y-x=0$ is

$$y-x+\lim_{\substack{x\to\infty\\ y=x}}\frac{ay^2-bx^2}{xy}=0$$

or ; $y-x+a-b=0$ ... ... (6)

Hence the asymptotes are

$x+a=0$, $y+b=0$, $y-x+a-b=0$.

The joint equation of the asymptotes is

$$(x+a)(y+b)(y-x+a-b)=0$$

or ; $xy^2-x^2y-bx^2+ay^2-b^2x+a^2b-ab^2+a^2y=0$

or ; $(x+a)y^2-(y+b)x^2-b^2(x+a)+a^2(y+b)=0$

The equation of the given curve is written as

$(x+a)y^2-(y+b)x^2+\{a^2(y+b)-b^2(x+a)\}=0$

which is of the from $F_n+F_{n-2}=0$ i. e. $F_3+F_1=0$

Hence $3(3-2)$ *i. e.*; 3 points of their intersection lie on the line $F_{n-2}=0$ or, $F_{n-2}=0$ or, $F_1=0$ i. e.

$$a^2(y+b)-b^2(x+a)=0$$

$\therefore\ a^2(y+b)-b^2(x+a)=0$ Proved.

Ex. 13. Show that the asymptotes of the curve

$$x^2y^2-a^2(x^2+y^2)-a^3(x+y)+a^4=0$$

form a square, two of whose angular points lie on the curve.

The equation is of 4th degree and highest powers of x and y are absent. The equation is written as

$$x^2(y^2-a^2)-a^2y^2-a(x+y)+a^4=0 \quad \cdots \quad \cdots \quad (1)$$

or, $y^2(x^2-a^2)-a^2x^2-a(x+y)+a^4=0 \quad \cdots \quad \cdots \quad (2)$

The asymptotes parallel to x axis from (1). are $y=\pm a$

and asymptotes parallel to y-axis, from (2), are $x=\pm a$

The four asymptotes are $x=a$, $x=-a$, $y=a$, $y=-a$.

which form a square of sides $2a$.

Thus $A(a,a)$, $B(-a,a)$, $C(-a,-a)$, $D(a,-a)$ are vertices of the square $ABCD$

Put the co-ordinates of $B(-a,a)$ and $D(a,-a)$ satisfy the equation of the curve.

Hence $(a,-a)$ and $(-a,a)$ are on the curve.

Ex. 14. Show that the asymptotes of the curve

$$4(x^4+y^4)-17x^2y^2-4x(4y^2-x^2)+2(x^2-2)=0$$

pass through the point of intersection of the ellipse

$x^2+4y^2=4$ D. U. H. 1958, C. H. 77.

The equation is written as

$$(4x^4+4y^4-17x^2y^2)-4x(4y^2-x^2)+2(x^2-2)=0 \quad \cdots \quad (1)$$

Then put $x=1$, $y=m$ in the first and 2nd terms of (1)

$\phi_4(m)=4+4m^4-17m^2$; $\quad \phi_2(m)=4(4m^2-1)$

$\phi_4(m)=(2-m)(2+m)(2m+1)(1-2m)$

$\therefore\ \phi_4(m)=0 \Rightarrow$

or, $m=2,-2,-\frac{1}{2},\frac{1}{2}$

$\phi'_4(m)=16m^3-34m$

Now $c=-\dfrac{\phi_3(m)}{\phi'_4(m)}=-\dfrac{+16m^3-4}{16m^2-34m}$

When $m=2,-2,-\frac{1}{2},\frac{1}{2}$, then values of $c=1,+1,0,0$

Thus the asymptotes are $(y=mx+c)$

$y=2x-1$, $y=-2x+1$, $y=-\frac{1}{2}x$, $y=\frac{1}{2}x$

or ; $y-2x+1=0$. $y+2x-1=0$, $2y+x=0$, $2y-x=0$

The combined equation of all asymptotes is

$(y-2x+1)(y+2x-1)(2y+x)(2y-x)=0$

or ; $(y-2x)(y+2x)(2y+x)(2y-x)+4x(4y^2-x^2)-(4y^2-x^2)=0$

or , $4(x^4+y^4)-17x^2y^2+4x(4y^2-x^2)-(4y^2-x^2)=0 \quad \cdots \quad (2)$

The equation of the given curve is written as

$4(x^4+y^4)-17x^2y^2+4x(4y^2-x^2)-4y^2+x^2+(4y^2+x^2-4)=0$

or ; $F_4+F_2=0$

The asymptotes meet the curve at $n(n-2)=4(4\ \ 2=8$ points and these points lie on the curve $F_{n-2}=0$ or, $F_{4-2}=F_2=0$

or, $4y^2+x^2-4=0$ or, $4y^2+x^2=4$.

Ex. 15. Show that the equation of the cubic which has the same asymptotes as the curve $x^2y-xy^2+xy+y^2+x-y=0$ and which passes through the points (0,0), (0,1), (1,0) is

$$yx^2-xy^2+xy+y^2+3y=0$$

The given equation is of 3rd degree and x^3 and y^3 both the terms are absent. There are some asymptotes parallel to co-ordinate axes.

The equation is written as

$$yx^2-xy^2+xy+y^2+3y=0 \quad \cdots \quad \cdots \quad (1)$$

$$\text{or ; } xy(x-y)+xy+y^2+3y=0 \quad \cdots \quad \cdots \quad (2)$$

$$y^2(-x+1)+yx^2+xy+3y=0 \quad \cdots \quad \cdots \quad (3)$$

From (2), the asymptotes are parallel to $x=0$, $y=0$, $x-y=0$.

Now asymptotes parallel to x-axis from (1) is $y=0$

Aaymptotes parallel to $y=x$ from (2) is

$$x-y+\lim_{\substack{x\to\infty\\ y=x}}\frac{xy+y^2}{xy}+\lim_{\substack{x\to\infty\\ y=x}}\frac{3y}{yx}=0 \quad \text{or,} \quad x-y+2=0$$

And asymptotes parallel to $y-$axis from (3) is $x=1$

$\therefore$ The asymptotes are $y=0$, $x=1$, $x-y+2=0$

and the joint equation of asymptotes is

$$y(x-1)(x-y+2)=0 \quad \cdots \quad \cdots \quad (4)$$

Let the equation of the new curve be

$$y(x-1)(x-y+2)+ax+by+c=0 \quad \cdots \quad (5)$$

Since it passes through (0,0), (1,0) (0,1)

then, $c=0$, $a=0$, $b=1$

Hence the new curve which has the same asymptotes as (1) is

$$y(x-1)(x-y+2)+y=0.$$

Exercise X11

Find the asymptotes of following curves.

1. $2y^3-2x^2y-4xy^2+4x^3-14xy+6y^2+4x^2+6y+1=0$
2. $3x^3+2x^2y-7xy^2+2y^3-14xy+7y^2+4x+5y=0$
3. $y^3-6xy^2+11x^2y-6x^3+x+y=0$
4. $x^3+x^2y-xy^2-y^3+x^2-y^2-2=0$
5. $x^3-2y^3+xy(2x-y)+y(x-y)+1=0$

6. $y^3-xy^2-x^2y+x^3+x^2-y^2-1=0$

(i) $x^3+2x^2y-xy^2-2y^3+4y^2+2xy-5y+6=0$ D.U. 1991

7. $x^3-5x^2y+8xy^2-4y^3+x^2-3xy+2y^2-7=0$

(i) $x^3-y^3-x^2+2y^2=0$ N.U. 1994

8. $xy^2-x^2y=a^2(x+y)+b^2$ (i) $y^2(x^2-y^2)-2ay^3+2a^3x=0$ N.U. 1995

D.U. 1959

9. $x^3+2x^2y+xy^2-x^2-xy+2=0$

10. $x^2y^2=a^2y^2-b^2x^2$ (ii) $y(x^2-y^2)=y(x-y)+2$ N.H. 1994

(i) $x^3-2x^2y+xy^2+x^2-xy+2=0$ D.U. 1977

11. $xy^2+x^2y+xy+y^2+3x=0$
12. $x^3-xy^2+6y^2=0$
13. $y^2(x^2-a^2)=x^2(x^2-4a^2)$
14. $y^2(x^2-a^2)=x$ R. U. 1979
15. $y^3+x^2y+2xy^2-y+1=0$ C. U. 1969

(i) $x^4-x^2y^2+x^2+y^2-a^2=0$

16. $x^2y-xy^2+xy+y^2+x-y=0$
17. $x^2y^2=x^2+y^2$ 18. $x^2y=x^3+x+y$
19. $y^2=x^3+ax^2$ 20. $x^2y^2-xy^2+x+y+1=0$
21. $x^3+y^3=3axy$ 22. $x^3-y^3=a^2xy$
23. $\{y/(x+a)\}^3=(x-a)/(x-2a)$
24. $(x-y)^2(x-2y)(x-3y)-2a(x^3-y^3)$
   $-2a^2(x-2y)(x+y)=0$ D. H, 1961
25. $y^4-2xy^3+2x^3y-x^4-3x^3+3x^2y+3xy^3-3y^3-2x^2+2y^2-1=0$

26. $x^2(y^2-4)-4y^2+8xy+2y-3=0$

27. $x^2y^2-9(x^2+y^2)-5(x+y)+27=0$

28. $x^2(y+3)-y^2(x+2)=0.$

29. $x^2(x^2+y^2-2xy)-2x^3-2y^2=0$ parallel to $y=x$.

30. $x^2(x-y)^2-a^2(x^2+y^2)=0$

31. $(x+3y)^2(2x+y)+2x-3y+7=0$ D. U. 1984

32. $x(y-3)^3=4y(x-1)^3$ 33. $2x(y-3)^3-3y(x-1)^2=0$

34. $xy(x^2-y^2)(x^2-4y^2)+3xy(x^2-y^2)+x^2+y^2-7=0$

35. $x^3+3x^2y-xy^2-3y^3+x^2-2xy+3y^2+4x+7=0$

36. $y=e^{1/x}$ 37. $y=x+\log x$

38. $y=e^{-x^2}$ 39. $y=e^{ax}$

40. $y=e^{1/x}-1$ 41. $y=\log x$

42. $y=e^{-2x}\sin x$ 43. $y=xe^{1/x^2}$

44. $y=e^{-x}\sin 2x+x.$

Find the rectilinear asymptotes of the following curves.

45 $r\theta=a$ 46. $r\cos\theta=a\theta$

47. $r\sin\theta/2=a$ 48. $r\cos 3(\theta+\alpha)=a\sin(\theta+\alpha)$

49 $r\sin n\theta=a$ 50. $r=4(1-\sec 2\theta)$

51. $r=a(\sec\theta+\cos\theta)$ 52. $r=a\dfrac{\theta-\alpha}{\theta+\alpha}$

53. $r=a\dfrac{\sin^2\theta}{\cos\theta}$ 54. $r\tan 3\theta=\alpha$

55. $r^n\sin n\theta=a^n$ 56. $r\cos\theta=a\sin\theta$

57. $r(\frac{1}{2}-\cos\theta)=a$

Find the circular asymptotes of the following curves.

58. $r=\dfrac{\theta}{\theta+1}$ 59. $r=\dfrac{\theta^2-1}{\theta^3+1}$

60. $r=\dfrac{5\theta^2+3\theta+3}{3\theta^2+7\theta+5}$ 61. $(r-1)(\theta-1)=1$

62. $r(e^{\theta}-1)=a(e^{\theta}+1).$

63. (i) Show that the asymptotes of

$(x^2-y^2)^2=2(x^2+y^2)$ form a square

63. (ii) Show that the asymptotes of the curve

$(x^2-y^2)y-2ay^2+5x-7=0$

form a triangle of area equal to a^2 R.U.H. 1969. C. H. 1969

(iii) Show that the asymptotes of the curve $x^2y^2=a^2(x^2+y^2)$ form a square of the side $2a$ R. H. 1986 ; R. U. 1972 '87

(iv) Show that the asymptotes of $x^2y^2=9(x^2+y^2)$ form a square of area 36 square units. R. H. 1988

64. Show that the asymptotes of the curve

$x^3-2y^3+xy(2x-y)+y(x-y)+1=0$

cut the curve in three points whieh lie on the straight line, $x-y+1=0.$

65. Show that the asymptotes of the curve

$x^2y-xy^2+xy+y^2+x-y=0$

cut the curve in three points which lie on the straight lins $x+y=0$

66. Show that points of intersection of the curve

$2y^3-2x^2y-4xy^2+4x^3-14xy+6y^2+4x^2+6y+1=0$

and its asymptotes lie on the straight line, $8x+2y+1=0$

67. Show that the asymptotes of the curve

$x^4-5x^2y^2+4y^4+x^2-y^2+x+y+1=0$

cut the curve again in eight point lying upon a rectangular hyperbola. $x^2-y^2+x+y=0.$

68. Show that the asymptotes of the curve

$(x^2-4y^2)(x^2-9y^2)+5x^2y-5xy^2-30y^3+xy+7y^2-1=0$

cut the curve in the eight points lying on a circle, $x^2+y^2=1.$

69. Show that the points of intersection of the curve

$$4x^4-13x^2y^2+9y^4+32x^2y-42y^3-20x^2+74y^2-56y+4x+16=0$$

and its asymptotes lie on the curve $y^2+4x=0$.

70. Show that the equation of the curve which has the same asymptote as the curve

$$x^3-6x^2y+11xy^2-6y^3+4x+5y+7=0$$

and which passes through the points (0, 0), (2, 0), (0, 2) is

$$x^3-6x^2y+11xy^2-6y^3-4x+24y=0$$

71. Show that there is an infinite series of parallel asymptotes to the curve $r=\dfrac{a}{\theta\sin\theta}+b$, and show that their distances from the pole are in Harmonic progression.

72. Show that all the asymptote of the curve $r\tan n\theta=a$ touch a circle $r=a/n$.

73. Find the equation of the curve which has $x=0$, $y=0$, $y=x$, $y+x=0$ for asymptotes, which passes through the point (a, b) and which cuts its asymptotes again in lying upon the circle $x^2+y^2=a^2$.

Find the asymptote of the following curves.

74. (i) $xy(x^2-y^2)=x^2+y^2$ (a) $x^2y^2-9(x^2+y^2)-5(x+y)+27=0$ C.H. 1992

(ii) $y^2x^2-3yx^2-5xy^2+2x^2+6y^2-x-3y+2=0$ R. U. 1982

(iii) $x^2y^2-x^2y-xy^2+2x-3y+4=0$ R. U. 1983

(iv) $xy+(x^2-4y^2)=2x^2+7y^2$ R. U. 1984

(v) $x^3=8y^3+3x^2+y^2-7x+2=0$ C. U. 1984, 92

(vi) $y=a\log\sec x/a$ D. H. 1983

(vii) $2x(y-5)^2=3(y-2)(x-1)^2$ C. U. 1987

(viii) $x^3-x^2y-xy^2+y^3+2x^2-4y^2+2xy+x+y+1=0$ C.H. 1993

প্রশ্নমালা XII

নিম্নলিখিত বক্ররেখাগুলির অটরেখা সমূহ নির্ণয় কর।

1. $2y^3-2x^2y-4xy^2+4x^3-14xy+6y^2+4x^2+6y+1=0$

2. $3x^3+2x^2y-7xy^2+2y^3-14xy+7y^2+4x+5y=0$

3. $y^3-6xy^2+11x^2y-6x^3+x+y=0$

4. $x^3+x^2y-xy^2-y^3+x^2-y^2-2=0$

5. $x^3-2y^3+xy(2x-y)+y(x-y)+1=0$

6. $y^3-xy^2-x^2y+x^3+x^2-y^2-1=0$

6(i) $x^3+2x^2y-xy^2-2y^3+4y^2+2xy-5y+6=0$ D.U 1991

7. $x^3-5x^2y+8xy^2-4y^3+x^2-3xy+2y^2-7=0$ R. U. 1962

8. $xy^2-x^2y=a^2(x+y)+b^2$ D. U. 1959

9. $x^3+2x^2y+xy^2-x^2-xy+2=0$

10. (i) $x^2y^2=a^2y^2-b^2x^2$ D. U. 1977

(ii) $x^3-2x^2y+xy^2+x^2-xy+2=0$

11. $xy^2+x^2y+xy+y^2+3x=0$

12. $x^3-xy^2+6y^2=0$

13. $y^2(x^2-a^2)=x^2(x^2-4a^2)$ (i) $y^2(x^2-y^2)-2ay^3+2a^3x=0$ N.U. 1995

14. $y^2(x^2-a^2)=x$ (i) $y(x^2-y^2)=y(x-y)+2$. N.U. 1994. R. U. 1979

15. (i) $y^3+x^2y+2xy^2-y+1=0$ R. U. 1960

(ii) $x^4-x^2y^2+x^2+y^2-a^2=0$

16. $x^2y-xy^2+xy+y^2+x-y=0$

17. $x^2y^2=x^2+y^2$

18. $x^2y=x^3+x+y$

19. $y^3=x^3+ax^2$

20. $x^2y^2-xy^2+x+y+1-x^2y=0$

21. $x^3+y^3=3axy$

22. $x^5-y^5=a^2xy$

23. $\left\{\dfrac{y}{x+a}\right\}^3=\dfrac{x-a}{x+2a}$

24. $(x-y)^2(x-2y)(x-3y)-2a(x^3-y^3)$ D. H. 1961
$-2a^2(x-2y)(x+y)=0$

25. $y^4-2xy^3+2x^3y-x^4-3x^3+3x^2y+3xy^2-3y^3-2x^2+2y^2-1=0$

26. $x^2(y^2-4)-4y^2+8xy+2y-3=0$

27. $x^2y^2-9(x^2+y^2)-5(x+y)+27=0$

28. $x^2(y+3)-y^2(x+2)=0$.

29. $x^2(x^2+y^2-2xy)-2x^3-2y^2=0$, $y=x$ এর সমান্তরাল।

30. $x^2(x-y)^2-a^2(x^2+y^2)=0$

31. $(x+3y)^2(2x+y)+2x-3y+7=0$ D. H. 1954

32. $x(y-3)^3=4y(x-1)^3$, 33. $2x(y-3)^2-3y(x-1)^2=0$

34. $xy(x^2-y^2)(x^2-4y^2)+3xy(x^2-y^2)+x^2+y^2-7=0$

35. $x^3+3x^2y-xy^2-3y^3+x^2-2xy+3y^2+4x+7=0$

36. $y=e^{1/x}$ 37. $y=x+\log x$

38. $y=e^{-x^2}$ 39. $y=e^{ax}$

40. $y=e^{1/x}-1$ 41. $y=\log x$

42. $y=e^{-2x}\sin x$ 43. $y=xe^{1/x^2}$

44. $y=e^{-x}\sin 2x+x$.

নিম্নলিখিত বক্ররেখাগুলির সরলরৈখিক তটরেখাসমূহ নির্ণয় কর।

45. $r\theta=a$ 46 $r\cos\theta=a\theta$

47. $r\sin\theta/2=a$ 48. $r\cos 3(\theta+\alpha)=a\sin(\theta+\alpha)$

49. $r\sin n\theta=a$ 50. $r=4(1-\sec 2\theta)$

51. $r=a(\sec\theta+\cos\theta)$ 52. $r=a\dfrac{\theta-\alpha}{\theta+\alpha}$

53. $r=a\dfrac{\sin^3\theta}{\cos\theta}$ 54. $r\tan 3\theta=\alpha$

55. $r^n\sin n\theta=a^n$ 56. $r\cos\theta=a\sin\theta$

57. $r(\frac{1}{2}-\cos\theta)=a$

নিম্নলিখিত বক্ররেখাগুলির বৃত্তীয় তটরেখা সমূহ নির্ণয় কর।

58. $r=\dfrac{\theta}{\theta+1}$ 59. $r=\dfrac{\theta^3-1}{\theta^3+1}$

60. $r=\dfrac{5\theta^2+3\theta+3}{3\theta^2+7\theta+5}$ 61. $(r-1)(\theta-1)=1$

62. $r(e^{\theta}-1)=a(e^{\theta}+1)$.

63. (i) দেখাও যে বক্ররেখা $(x^2-y^2)^2=2(x^2+y^2)$ এর তটরেখা সমূহ একটি বর্গক্ষেত্র তৈরী করে। C. U. 1993

(ii) দেখাও যে বক্ররেখা $(x^2-y^2)y-2ay^2+6x-7=0$ এর তটরেখা সমূহ একটি ত্রিভুজ তৈরী করে যাহার ক্ষেত্রফল সমান a^2, N.H. 1993.

R. U. H. 1969, 92, C. H. '69

(iii) দেখাও যে, বক্ররেখা $x^2y^2=a^2(x^2+y^2)$ এর তটরেখাসমূহ একটি বর্গক্ষেত্র তৈরী করে যাহার বাহু $2a$ R. U. 1972

64 দেখাও যে বক্ররেখা $x^3-2y^3+xy(2x-y)+y(x-y)+1=0$ এর তটরেখা সমূহ বক্ররেখাটিকে এমন তিনটি বিন্দুতে ছেদ করে যাহারা সরলরেখা $x-y+1=0$ এর উপর থাকে।

65 দেখাও যে বক্ররেখা $x^2y-xy^2+xy+y^2+x-y=0$ এর তটরেখা সমূহ বক্ররেখাটিকে এমন তিনটি বিন্দুতে ছেদ করে যাহারা সরলরেখা $x+y=0$ এর উপর অবস্থিত।

66. দেখাও যে বক্ররেখা $2y^3-2x^2y-4xy^2+4x^3-14xy+6y^2+4x^2+6y+1=0$ এবং এর তটরেখা সমূহের ছেদবিন্দুগুলি $8x+2y+1=0$ সরলরেখার উপর অবস্থিত।

67. দেখাও যে বক্ররেখা $x^4-5x^2y^2+4y^4+x^2-y^2+x+y+1=0$ এর তটরেখা সমূহ বক্ররেখাটিকে আটটি বিন্দুতে ছেদ করে যাহারা $x^2-y^2+x+y+1=0$ এই আয়ত অধিবৃত্তটির (Rectangular hyperbola) উপর অবস্থিত।

68. দেখাও যে বক্ররেখা $(x^2-4y^2)(x^2-9y^2)+5x^2y-5xy^2-30y^3+xy+7y^2-1=0$ এবং এর তটরেখা সমূহের আটটি ছেদবিন্দু আছে যাহার $x^2+y^2=1$ এই বৃত্তটির উপর অবস্থিত।

69. দেখাও যে বক্ররেখা $4x^4-13x^2y^2+9y^4+32x^2y-42y^3-20x^2+74y^2-56y+4x+16=0$ এবং ইহার তটরেখা সমূহের ছেদবিন্দুগুলি বক্ররেখা $y^2+4x=0$ এর উপর অবস্থিত।

70. একটি বক্ররেখার তটরেখাগুলি এবং $x^3-6x^2y+11xy^2-6y^3+4x+5y+7=0$ বক্ররেখার তটরেখাগুলি একই। ঐ বক্ররেখা (0, 0), (2, 0), (0, 2) বিন্দুগুলি দিয়া অতিক্রম করিলে দেখাও যে ইহার সমীকরণ হইবে

$$x^3-6x^2y+11xy^2-6y^3-4x+24y=0.$$

71. দেখাও যে বক্ররেখা $r=\dfrac{a}{\theta\sin\theta}+b$ এর অসীম সংখ্যক সমান্তরাল তটরেখা আছে এবং আরো দেখাও যে মেরু বিন্দু বা মূল বিন্দু হইতে (Pole) ইহাদের দূরত্ব একটি ধারাবাহিক শ্রেণী গঠন করে।

72. দেখাও যে বক্ররেখা $r \tan n\theta = a$ এর সমস্ত তটরেখাগুলি বৃত্ত $r = a/n$ কে স্পর্শ করে।

73. একটি বক্ররেখার সমীকরণ নির্ণয় কর যাহার তটরেখাগুলি হইল $x=0, y=0, y=x, y+x=0$, যাহা (a, b) বিন্দু দিয়া যাইবে এবং ঐ বক্ররেখা ও তটরেখাগুলির ছেদ বিন্দু বৃত্ত $x^2+y^2=a^2$ এর উপর অবস্থিত হইবে।

74. নিম্নলিখিত বক্ররেখাগুলির তটরেখাগুলি নির্ণয় কর :—

(i) $xy(x^2-y^2)=x^3+y^2$

(ii) $y^2x^2-3yx^2-5xy^2+2x^2+6y^2-x-3y+2=0$ R. U. 1982

(iii) $x^2y^2-x^2y-xy^2+2x-3y+4=0$ R. U. 1983

(iv) $xy+(x^2-4y^2)=2x^2+7y^2$ R. U. 1984

(v) $x^3-8y^3+3x^2+y^2-7x+2=0$ C. U. 1984

(vi) $y=a \log \sec x/a$. D. U. 1983

(vii) $2x(y-5)^2=3(y-5)(x-1)^2$ C. U. 1987

(viii) $x^3-x^2y-xy^2+y^3+2x^2-4y^2+2xy+x+y+1=0$ C. H-1993

উত্তরমালা XII

1. $y=2x, y=-x-2, y=x-1$
2. $6y=6x-7, 2y=6x+15, 2y+x+1=0$
3. $y=x, y=2x, y=3x$, 4. $y=x, y+x=0, y+x+1=0$
5. $x-y=0, x+y+1=0, x+2y-1=0$
6. $y+x=0, y=x, y-x=1$
7. $y-x=0, 2y=x, 2y=x+1$
8. $x=0, y=0, y=x$.
9. $x=0, x+y=0, x+y=1$

10(i) $x=\pm a$. (ii) $x=0, x-y=0, x-y+1=0$

11. $y=0, x+1=0, x+y=0$
12. $x=6$; $y+x=3, y=x+3$
13. $x=\pm a, y=\pm x$. (i) $y=0, y=0, x+y+a=0, x-y-a=0$

14. $y=0, y=0, x=a, x=-a$

15. (i) $x+y=1, x+y+1=0$

15. (ii) $x=\pm 1, x\pm y=0$

16. $y=0, x-1=0, y=x+2$

17. $y=\pm 1, x=\pm 1$.

18. $x=\pm 1, y=x$ 19. $y=x+a/3$

20. $x=0, x=1, y=0, y=1$ 21. $x+y=a$

22. $y=x$ 23. $x+2a=0, y=x$

24. $x-y=a, x-y=2a, x-2y=13a, x-2y+14a=0$

25. $y=\pm x, y=x+1, y=x+2$ 26. $y=\pm 2, x=\pm 2$

27. $x=\pm 3, y=\pm 3$, 28. $x+2=0, y=x+1, y+3=0$

29. $x-y=\pm 2$ 30. $x=\pm a; x-y=\pm\sqrt{2}a$

31. $2x+y=0$ 32. $x=0, y=0. 2y=4x+3, 2y+4x=15$

33. $y=0, x=0, 2y=3x+9$ 34. $y=\pm x, 2y=\pm x, x=0, y=0$

35. $4x-4y+1=0, 2x+2y-3=0, 4x+12y+9=0$

36. $x=0$ 37. $x=0$ 38. $y=0$ 39. $y=0$

40. $x=0, y=0$ 41. $x=0$ 42. $y=0$ 43. $y=x, x=0$

44. $y=x$ 45. $r\sin\theta=a$ 46. $r\cos\theta=-a\pi/2, 3a\pi/2$

47. $r\sin\theta=2a$ 48. $a=6r\sin(\frac{1}{6}\pi-\alpha-\theta), -a=3r\sin(\frac{1}{2}\pi-\alpha-\theta)$; $a=6r\sin\left(\frac{5\pi}{6}-\alpha-\theta\right)$ 49. $rn\sin\left(\theta-\frac{m}{n}\pi\right)=a\sec m\pi$

50. $r\sin(\theta-\pi/4)=\pm 2$, $r\cos(\theta-\frac{1}{4}\pi)=\pm 2$. 51. $r\cos\theta=a$

52. $2ax=-r\sin(\theta+a)$ 53. $a=r\cos\theta$

54. $\theta=0, \pi/3, 2\pi/3$ ইত্যাদি।

এখানে ছয়টি তটরেখা আছে যারা একটি সুষম ষড়ভুজ তৈরী করে যাহার কেন্দ্র মূলবিন্দু O

55. তটরেখাগুলির সমীকরণ $\theta=\frac{k\pi}{n}$ যদি $n\geq 1$ হয়। কিন্তু যদি $n<1$ হয় তবে কোন তটরেখা হইবেনা।

56. $r\cos\theta=\pm a$. 57. $\pm 4a=\sqrt{r}(3\sin\theta\pm 3\cos\theta)$

58. $r=1$ 59. $r=1$ 60. $r=5/3$. 61. $r=1$ 62. $r=a$.

73. $bxy(y^2-x^2)+a(a^2-b^2)(x^2+y^2-a^2)=0$

74. (i) $x=0,\ y=0,\ x+y-2=0,\ x+y+2=0$

(ii) $x-2=0,\ x-3=0,\ y-1=0,\ y-2=0$

(iii) $x=0,\ y=0,\ x=1,\ y=1$.

(iv) $x=0,\ y=0,\ x\pm 2y=0$

(v) $24x-48y+13=0$

(vi) $x=\pm\pi a/2$. (vii) $x=0,\ y=2,\ 2y-3x-14=0$

— 0 —

CHAPTER XIII

CURVATURE

13.1. Definitions :—

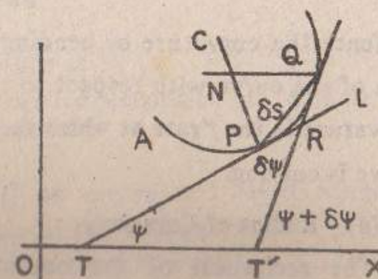
(a) Angle of Contignance.

Let P and Q be two neighbouring points on the curve APQ.

Let arc $AP=s$,

arc $AQ=s+\delta s$

then arc $PQ=\delta s$.



Let the tangents PT and QT' at P and Q of the curve make angles ψ and $\psi+\delta\psi$ respectively with the positive direction of X-axis

$$\angle QRL=\angle TRT'=\angle RT'X-\angle RTT'=\psi+\delta\psi-\psi=\delta\psi$$

Thus $\delta\psi=(\psi+\delta\psi-\psi)$ is the change in inclination of the tangent line as the point of contact of the tangent describes the arc δs. The angle $\delta\psi$ is called the **angle of contignance** of PQ provided the bending of the curve between P and Q is continuous in one direction only.

(b) Average Curvature or Average bending

The average curvature of arc PQ is the ratio of the corresponding angle of contignence $\delta\psi$ to the length of the arc δs, that is average curvature or average bending of the arc $PQ=\frac{\delta\psi}{\delta s}$

For one and the same curve the average curvature of its difference parts may be different.

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56. $r \cos\theta=\pm a$. 57. $\pm 4a=\sqrt{r}(3 \sin\theta \pm 3 \cos\theta)$

58. $r=1$ 59. $r=1$ 60. $r=5/3$. 61. $r=1$ 62. $r \mp a$.

73. $bxy(y^2-x^2)+a(a^2-b^2)(x^2+y^2-a^2)=0$

74. (i) $x=0,\ y=0,\ x+y-2=0,\ x+y+2=0$

(ii) $x-2=0,\ x-3=0,\ y-1=0,\ y-2=0$

(iii) $x=0,\ y=0,\ x=1,\ y=1$.

(iv) $x=0,\ y=0,\ x \pm 2y=0$

(v) $24x-48y+13=0$

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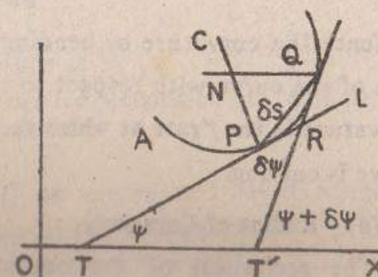

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For one and the same curve the average curvature of its difference parts may be different.

(c) Curvature

The curvature at a given point P is the limit (if it exists) of the average **curvature (bending)** of the arc PQ when the length of this arc δs approaches zero. The curvature at P is denoted by λ.

$\therefore$ **Curvature** at $P = \lambda = \lim\limits_{\delta s \to 0} \dfrac{\delta\psi}{\delta s} = \dfrac{d\psi}{ds}$.

Hence the curvature or bending is the rate of change of direction of the curve with respect to the arc or roughly speaking the curvature is the **"rate at which the curve curves"** or how much the curve is curving.

(d) Radius of Curvature.

The reciprocal of the curvature λ is called the radius of curvature of the curve at P.

The radius of curvature is usually denoted by ρ.

Thus $\rho = \dfrac{1}{\lambda} = \dfrac{ds}{d\psi}$ or, $\rho = \dfrac{ds}{d\psi}$.

(e) Radius of Curvature (Geometrically)

Let the normals at P and Q to the given curve intersect at N. The limiting position of N as $Q \to P$ is called the centre of curvature at P. In the fig. 14 ; C is the centre of curvature at P.

The radius of curvature at P.

$$\rho = \lim_{\delta s \to 0} (PN)$$

From the $\triangle NPQ$,

$$\frac{PN}{\text{chord } PQ} = \frac{\sin NQP}{\sin PNQ} = \frac{\sin NQP}{\sin TRT'} = \frac{\sin NQP}{\sin \delta\psi}$$

$$\rho = \lim_{\delta s \to 0} PN = \lim_{\delta s \to 0} \text{chord } PQ \frac{\sin NQP}{\sin \delta\psi}$$

$$= \lim_{\delta\psi \to 0} \frac{\text{chord } PQ}{\delta s} \cdot \frac{\delta s}{\delta\psi} \cdot \frac{\delta\psi}{\sin \delta\psi} \cdot \sin NQP.$$

Now, $\delta\psi \to 0$, $NQP \to \frac{1}{2}\pi$ as $\delta s \to 0$.

$$\therefore \quad \rho = \lim_{\delta\psi \to 0} \frac{\text{chord } PQ}{\text{Arc } PQ} \cdot \frac{\delta s}{\delta\psi} \frac{\delta\psi}{\sin \delta\psi} \sin NQP$$

$$= 1 . \frac{ds}{\delta\psi} 1 \cdot 1 = \frac{ds}{\delta\psi}$$

or, $\rho = \dfrac{ds}{d\psi}$

which is the radius of curvature at P,

Note : ρ is positive or negative according as C is on the positive or negative side of the normal.

(f) Circle of curvature

If a circle is drawn having C as centre and $\rho = PC$ as radius then the circle is called the circle of curvature at P.

(g) Curvature and radius of Curvature of a circle.

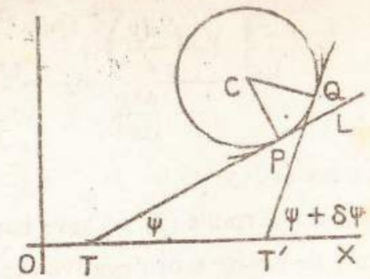
Let C be the centre of a circle of radius a, P and Q be two points very near to each other on the circle. Let PT and QT' be tangents drawn at P and Q respectively

Let

$\angle PTX = \psi$

$\angle QT'X = \psi + \delta\psi$

Join CP and CQ.



$\therefore$ $\angle PCQ = \angle TPT' = \delta\psi$

Fig 15

$$\therefore \quad \frac{\delta\psi}{\delta s} = \frac{\angle TPT'}{\delta s} = \frac{\angle PCQ}{\delta s} = \frac{\delta s / a}{\delta s} = \frac{1}{a} \text{ as } \angle PCQ = \frac{\delta s}{a}$$

$\lambda =$ Curvature at any point $P = \lim\limits_{\delta s \to 0} \dfrac{\delta\psi}{\delta s} = \dfrac{1}{a}$

$$= \text{constant} = \frac{1}{\text{radius of the circle}}$$

$\therefore\ \rho=\frac{1}{\lambda}$ = radius of curvature of a circle $=a=$ the radius of the curve.

13.2. Formula for the radius of Curvature.

(a) Explicit equation (i. e. when y is expressed directly in terms of x) or, Cartesian formula for radius of curvature for y=f(x)

We know that $\frac{dy}{dx}=\tan\psi$ ... ... (1)

Differentiate w. r. to x. Then

$$\frac{d^2y}{dx^2}=\sec^2\psi\,\frac{d\psi}{dx}=\sec^2\psi\,\frac{d\psi}{ds}\cdot\frac{ds}{dx}$$

$$=\sec^2\psi\,\frac{1}{\rho}\sec\psi \qquad \left(\because\ \frac{dx}{ds}=\cos\psi\right)$$

$$\therefore\ \rho=\sec^3\psi\Big/\frac{d^2y}{dx^2}=(1+\tan^2\psi)^{3/2}\Big/\frac{d^2y}{dx^2}$$

$$\therefore\ =\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}}=\frac{(1+y_1^2)^{3/2}}{y_2}\ \dots\ \dots\ (2)$$

provided $y_2\neq 0$.

In the formula (2) we have not mentioned about the sign of ρ, ρ may be positive or negative according as y_2 is positive or negative. It is customary to attach that sign to the radical which will give a positive sign to ρ. The radius of curvature is zero at point of inflexion.

The above formula fails when y_1 is infinite (*i. e*) when tangent is parallel to y—axis.

In the case, follow the process shown in the corollary.

Cor. The value of ρ does not depend on the axes but depends on the curve. Hence interchanging x and y the formula (2) can be written as

$$\rho=\frac{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}^{3/2}}{\frac{d^2x}{dy^2}}=\frac{(1+x_1^2)^{3/2}}{x_2}\quad \text{if } x_2\neq 0,$$

(b) Implicit equations f(x, y)=0

we know $\frac{dy}{dx}=-\frac{f_x}{f_y}\Rightarrow -\frac{\delta f}{\delta x}\Big|\frac{\delta f}{dy}$, where $f_y\neq 0$

or, $f_x+f_y\frac{dy}{dx}=0$

Differentiate w. r. to x, again

$$f_{xx}+f_{xy}\frac{dy}{dx}+\left(f_{yx}+f_{yy}\right)\frac{dy}{dx}+f_y\frac{d^2y}{dx^2}=0$$

or, $$f_{xx}+2f_{xy}\frac{dy}{dx}+f_{yy}\left(\frac{dy}{dx}\right)^2+f_y\frac{d^2y}{dx^2}=0\quad\therefore\ f_{xy}=f_{yx}$$

Put $\frac{dy}{dx}=-\frac{f_x}{f_y}$. Then

$$\frac{d^2y}{dx^2}=\frac{-f_{xx}f_y^2-2f_{xy}f_xf_y+f_{yy}f_x^2}{f_y^3}$$

Now put the Value of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ in (a)

$$\therefore\ \rho=\frac{(1+y_1^2)^{3/2}}{y_2}=\frac{(f_x^2+f_y^2)^{3/2}}{f_{xx}f_y^2-2f_{xy}f_yf_x+f_{yy}f_x^2}$$

where $f_{xx}f_y^2-2f_{xy}f_xf_y+f_{yy}f_x^2\neq 0$

(c) **Parametric equation : $x=\phi(t)$, $y=\psi(t)$**

$$\frac{dy}{dx}=\frac{dy}{dt}\Big/\frac{dx}{dt}=y'/x', \quad \text{where } x'\neq 0$$

$$\frac{d^2y}{dx^2}=\frac{d}{dt}\left(\frac{y'}{x'}\right)=\frac{d}{dt}\left(\frac{y'}{x'}\right)\frac{dt}{dx}=\frac{x'y''-x''y'}{(x')^2}\cdot\frac{1}{x'}=\frac{x'y''-x''y'}{(x')^2}$$

Put the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (a) and simplify

$$\rho=\frac{(1+y_1^2)^{3/2}}{y_2}=\frac{\{x')^2+(y')^2\}^{3/2}}{x'y''-y'x''} \quad \therefore \rho=\frac{\{(x')^2+(y')^2\}^{3/2}}{x'y''-y'x''}$$

where $x'y''-y'x''\neq 0$.

Dashes indicate the number of differentiation w. r. to t.

(d) **Polar equation : $r=f(\theta)$**

The radius of curvature

$$\rho=\frac{ds}{d\psi}=\frac{ds}{d\theta}\cdot\frac{d\theta}{d\psi}=\frac{ds}{d\theta}\Big/\frac{d\psi}{d\theta}$$

we know $\psi=\theta+\phi$ $\quad\therefore$ see Art, 10.8. Eq. 35

$$\tan\phi=r\frac{d\theta}{dr}=r/r_1 \quad \therefore \phi=\tan^{-1}r/r_1$$

$$\therefore \psi=\theta+\tan^{-1}r/r_1, \quad \text{where } r_1=\frac{dr}{d\theta}$$

Differentiate $w.$ $r.$ to θ

$$\frac{d\psi}{d\theta}=1+\frac{1}{1+r^2/r_1^2}\cdot\frac{r_1^2-rr_2}{r_1^2}=\frac{r^2+2r_1^2-rr_2}{r^2+r_1^2}$$

But $\frac{ds}{d\theta}=\sqrt{(r^2+r_1^2)}$ $\quad\therefore$ see Art. 10.8. Eq. .34 (i)

Now put the Values of $\frac{d\psi}{d\theta}$ and $\frac{ds}{d\theta}$ in ρ.

$$\text{Hence } \rho=\frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$$

Cor. If $r=1/u$. then

$$\rho=\frac{\{u^2+(u')^2\}^{3/2}}{u^3(u+u'')} \quad \text{if } u^3(u+u'')\neq 0.$$

(e) **Pedal Equation : $p=f(r)$**

The radius of curvature $\rho=\frac{ds}{d\psi}$

$$\text{or,} \quad \frac{1}{\rho}=\frac{d\psi}{ds}=\frac{d}{ds}(\theta+\phi) \quad \therefore \psi=\theta+\phi$$

$$=\frac{d\theta}{ds}+\frac{d\phi}{ds} \quad \cdots \quad \cdots \quad (1)$$

we know $p=r\sin\phi$

$$\therefore \frac{dp}{dr}=\sin\phi+r\cos\phi\frac{d\phi}{dr}\sin\phi+r.\frac{d\phi}{ds}\cdot\frac{dr}{ds}$$

$$=r\frac{d\theta}{ds}+r\frac{dr}{ds}\cdot\frac{d\phi}{ds}\cdot\frac{ds}{dr}=r\frac{d\theta}{ds}+r\frac{d\phi}{ds}$$

$$\therefore \sin\phi=r\frac{d\theta}{ds}, \cos\phi=\frac{dr}{ds}$$

$$=r\left(\frac{d\theta}{ds}+\frac{d\phi}{ds}\right)=\frac{r}{\rho} \quad \text{or,} \quad \rho=r\frac{dr}{dp} \text{ by (1)}$$

13. 3. Carvature at the origin.

Method of substitution : In the formula Art. 13. (a).

$$\rho=\frac{(1+y_2^2)^{3/2}}{y_2}$$

put $x=0$ and $y=0$ in the Value of ρ or, by substituting the Valuc of $(y_1)_0$ and $(y_2)_0$ in ρ

If y is expanded is powers of x by any method and

$$y=px+qx^2/\underline{|2}+\cdots \quad \cdots$$

which shows that the curve passes through the origin,

$$p=\left(\frac{dy}{ax}\right)_{x=0,\,y=0}, \quad q=\left(\frac{d^2y}{dx^2}\right)_{x=0,\,y=0}$$

The radius of curvature at the origin is

$$\rho=\frac{(1+p^2)^{3/2}}{q}$$

(b) Newton's Method :

(i) If the curve passes through the origin and axis of x is the tangent at the origin then

$$\rho=\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2}{2y}$$

At the origin, $x=0$. $y=0$ and $y=\frac{dy}{dx}=0$ (for the x-axis)

Expand $y=f(x)$ by Maclaurin's theorem

$y=0+0\,x+qx^2/\underline{|2}+ \quad \ldots \quad \ldots \quad \ldots;$

or, $\frac{2y}{x^2}=q+\ldots$term x containing x and higher point of x

$$\therefore \lim_{(x,y)\to(0,0)} 2y/x^2=q$$

So, $\rho=\frac{(1+p^2)^{3/2}}{q}=\frac{1}{q}$ $[\because\ p=y_1(0),\ q=y_2(0)]$

$$=\lim_{\substack{x\to 0\\ y\to 0}}\frac{x^2}{2y}$$

(iii) If the curve passes through the origin and the y-axis is the tangent at the origin the radius of curvature at the origin is

$$\rho=\lim_{\substack{x\to 0\\ y\to 0}} \frac{y^2}{2x}$$

(ii) **Generalised Newtonian Formula.**

Let $ax+by=0$ be the tangent OT at the origin O. Take point $P(x, y)$ on the curve. Draw PM perpendicular to the tangent. $OP^2=x^2+y^2$,

$$PM=\frac{ax+by}{\sqrt{(a^2+b^2)}}$$

Let OB be the diameter of the circle through O and PN be the perpendicular to OB.

We have

or, $ON(OB-ON)=PN^2$

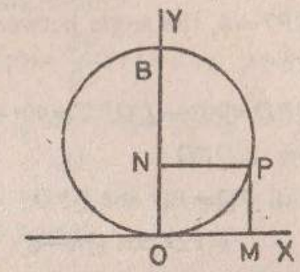


Fig 16

or, $OB=PN^2/ON+ON=(PN^2+ON^2)/ON=OP^2/ON=OP^2/PM$

or, $2r=OB=\frac{OP^2}{PM}=\frac{x^2+y^2}{(ax+by)/(a^2+b^2)}$

where r is the radius of the circle.

If $p\to 0$, $x\to 0$ $y\to 0$ then $r\to\rho$ where ρ is the radius of curvature at the origin. Hence

$$\rho=\tfrac{1}{2}\lim_{x\to 0,\ y\to 0}\frac{OP^2}{PM}=\tfrac{1}{2}\sqrt{(a^2+b^2)}\ \lim_{x\to 0,\ y\to 0}\frac{x^2+y^2}{ax+by}$$

$$\therefore\ \rho=\tfrac{1}{2}\sqrt{(a^2+b^2)}\ \lim_{\substack{x\to 0\\ y\to 0}}\frac{x^2+y^2}{ax+by}$$

13. Chord of the Curvature through the origin

Let APB be a curve and QPD be the circle of curvature at P with centre at C. Join OP meeting the circle at Q. Thus PQ is the chord passing through the origin O. Join PG and produce it to meet. D. Join DQ.

$\angle PQD=90°$, as PD is the diameter of the circle.

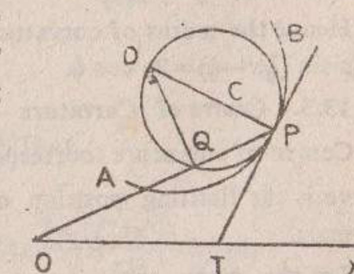


Fig 17

Let PT be the tangent to the curve at P.

$\angle QPT=\phi$, the angle between the radius vector and the tangent.

$\angle QPD=90^\circ-\angle OPT=90^\circ-\phi$, as PD is perp. to PT.

From $\triangle QPD$

Chord $PQ=PD\cos QPD$

$=PD\cos(\frac{1}{2}\pi-\phi)$

or, Chord $PQ=2p\sin\phi$

$$=2r\frac{dr}{dp}\cdot\frac{p}{r}\quad\therefore\quad p=r\sin\phi$$

$$\text{Chord } PQ=2p.\ \frac{dr}{dp}$$

Cor. If the chord does not pass through the origin, the angle $\angle QPT=\alpha$ (say)

Hence the chord of curvature,

$PQ=2\rho\sin\alpha$

Cor. 2. If the chord PQ is parallel to x—axis then $\angle PDQ=\psi$

Hence the chord of curvature parallel to x—axis$=2\rho\sin\psi$

Cor. 3. If the chord PQ is parallel to y—axis, then $\angle PDQ=\frac{1}{2}\pi-\psi$,

Hence the radius of curvature parallel to the y—axis $=2\rho\sin(\frac{1}{2}\pi-\psi)=2\rho\cos\psi$.

13.5. Centre of Curvature

Centre of curvature corresponding to any point $P(x, y)$ on a curve is the limiting position of intersection of two consecutive normals.

Let the given curve be $y=f(x)$ and the two neighbouring points be $P(x, y)$ and $Q(x+\delta x, y+\delta y)$

Let $G(\alpha, \beta)$ be the centre of curvature for P.

Now $f'(x)=\frac{dy}{dx}$ at $P(x, y)$, $f'(x+\delta x)=\frac{dy}{dx}$ at $(x+\delta x,\ y+\delta y)$

The equations of normals at P and Q are respectively

$$(Y-y)f'(x)+(X-x)=0 \qquad (1)$$

$$(Y-y-\delta y)f'(x+\delta x)+(X-x-\delta x)=0 \qquad (2)$$

Now find the intersection of two normals.

Substract (1) from (2)

$$(Y-y)\{f'(x+\delta x)-f'(x)\}-\delta y f'(x+\delta x)-\delta x=0$$

$$\text{or, } (Y-y)\frac{f(x+\delta x)-f'(x)}{\delta x}-f'(x+\delta x)\frac{\delta y}{\delta x}=1 \quad (3)$$

If Q tends to P, δx tends to zero and hence $X\to\alpha$ and $Y\to\beta$

$$\text{Also } \lim_{\delta x\to 0}\frac{f'(x+\delta x)-f'(x)}{\delta x}=f''(x)=\frac{d^2y}{dx^2}.$$

$$\text{and } \lim_{\delta x\to 0}\frac{\delta y}{\delta x}=f'(x)=\frac{dy}{dx}$$

$\therefore$ from (3)

$$\lim_{\delta x\to 0}(Y-y)\frac{f'(x+\delta x)-f'(x)}{\delta x}-\lim_{\delta x\to 0}f'(x+\delta x)\frac{\delta y}{\delta x}-1=0$$

or, $(\beta-y)f''(x)-f'(x)f'(x)-1=0$

$$\left.\begin{aligned}\text{or, } \beta&=y+\frac{1+\{f'(x)\}^2}{f''(x)}\\ \text{or, } \beta&=y+\frac{1+(dy/dx)^2}{d^2y/dx^2}\end{aligned}\right\}\quad (4)$$

Since the point $C(\alpha, \beta)$ is on (1), then

$$(\beta-y)f'(x)+\alpha-x=0$$

$$\text{or, } \alpha=x-(\beta-y)f'(x)=x-f'(x)\frac{1+\{f'(x)\}^2}{f''(x)} \text{ by (4)}$$

$$\therefore\quad \alpha=x-\frac{(dy/dx)\{1+(dy/dx)^2\}}{d^2y/dx^2}\quad \cdots \quad (5)$$

Thus $C(\alpha, \beta)$, the centre of curvature, can be determined from (4) and (5).

Note 1.

Let the tangent PT at (x, y) make an angle ϕ with x-axis. PC is the normal which makes an angle $90^\circ+\psi$ with the x—axis, Let (α, β) be the co-ordinates of the centre of curvature C at $P(x, y)$ of the curve.

Therefore $PC=\rho$, Then

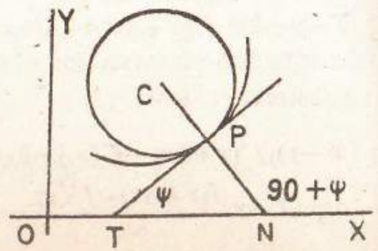


Fig 18

$$\frac{\alpha-x}{\cos\left(\frac{1}{2}\pi+\psi\right)}=\frac{\beta-y}{\sin\left(\frac{1}{2}\pi+\psi\right)}=\rho$$

or, $\alpha=x-\rho\sin\psi,\ \beta=y+\rho\cos\psi$ (1)

But $\tan\psi=\dfrac{dy}{dx}=y_1$. So ; $\sin\psi=\dfrac{y_1}{\sqrt{(1+y_1^2)}}$

$$\cos\psi=\frac{1}{\sqrt{(1+y_1^2)}} \text{ and } \rho=\frac{(1+y_1^2)^{3/2}}{y_2}$$

Putting the values of ρ, $\sin\psi$ and $\cos\psi$ in (1), the co-ordinates of centre of curvature α and β will be determined,

Note. 2 Let $C(\alpha, \beta)$ be the centre of curvature at $P(x, y)$ of curve $y=f(x)$.

The equation of the normal at $P(x, y)$

$$X-x+(Y-y)\,y_1=0$$

Since it passes through $C(\alpha, \beta)$,

$$\alpha-x+(\beta-y)\,y_1=0 \qquad (1)$$

But $PC=\rho$ (See) fig. (18)

$$\therefore\ (\alpha-x)^2+(\beta-y)^2=\rho^2 \qquad (2)$$

where $\rho=\dfrac{(1+y_1^2)^{3/2}}{y_2}$

From (1) and (2),

$$\beta=y+\frac{1+y_1^2}{y^2}\ \dots\ \dots\ (3)$$

From (1) and (3)

$$\alpha=x-\frac{y_1(1+y_1^2)}{y^2}\ \dots\ \dots\ (4)$$

Thus from (3) and (4), (α, β) the co-ordinates of centre of curvature can be determined.

Ex. 1. Find the radius of curvature at the point (s, ψ) on the curve $s=c\log\sec\psi$,

The radius of curvature.

$$\rho=\frac{ds}{d\psi}=\frac{d}{d\psi}(c\log\sec\psi)=c\,\frac{\sec\psi\tan\psi}{\sec\psi}$$

$\therefore\ \rho=c\tan\psi$.

Ex. 1. (a) What is the geometrical shape of the curve for which $s=5\psi$.

We have, $s=5\psi$

$$\therefore\ \frac{ds}{d\psi}=5=\text{constant or, } \rho=5$$

The curve is such that the radius of curvature at every point is 5. The curve is a circle of radius 5.

Ex. 2. Find the radius of curvature at the point (x, y) of the curve $ay^2=x^3$.

Here $ay^2=x^3$ $\therefore\ 2ay\dfrac{dy}{dx}=3x^2$

or, $y_1=\dfrac{dy}{dx}=\dfrac{3x^2}{2ay}=\dfrac{3x^2}{2ax^{3/2}/\sqrt{a}}=3/2(x/a)^{1/2}$ as $ay^2=x^3$

$$\therefore\ y_2=\frac{d^2y}{dx^2}=3/2\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{a}}\cdot\frac{1}{\sqrt{x}}=\frac{3}{4}\,\frac{1}{\sqrt{(ax)}}$$

Hence $\rho=\frac{(1+y_1^2)^{3/2}}{y_2}=\left(1+9/4.\frac{x}{a}\right)^{3/2}4/3\sqrt{(ax)}$

$=\frac{1}{6a}(4a+9x)^{3/2}\sqrt{x}.$

Ex 3. Show that for the curve $r^m=a^m\cos m\theta$. R. U. 1962

$$\rho=\frac{a^m}{(m+1)r^{m-1}}$$

Now $r^m=a^m\cos n\theta$

or, $m\log r=m\log a+\log\cos m\theta$.

$\therefore\quad \frac{m}{r}\frac{dr}{d\theta}=-\frac{m\sin m\theta}{\cos m\theta}$ or, $r_1=\frac{dr}{d\theta}=-r\tan m\theta$

$\therefore\quad \frac{d^2r}{d\theta^2}=-\frac{dr}{d\theta}\tan m\theta-rm\sec^2 m\theta$

or, $r_2=r\tan^2 m\theta-rm\sec^2 m\theta$

But $\rho=\frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$

$=\frac{(r^2+r^2\tan^2 m)^{3/2}}{(r^2+2r^2\tan^2 m\theta-r^2\tan^2 m\theta+mr^2\sec^2 m\theta)}$

$=\frac{r^3\sec^3 m\theta}{(m+1)a^2\sec^2 m\theta}=\frac{r}{(m+1)\cos m\theta}=\frac{ra^m}{(m+1)r^m}$

$\therefore\quad \rho=\frac{a^m}{(m+1)r^{m-1}}$ Proved.

Ex. 4. For the curve $x=a\cos^3\theta$, $y=a\sin^3\theta$

Show that the radius of curvature is

$\rho=3a\sin\theta\cos\theta$. D. U. 1966

Here $x=a\cos^3\theta$, $y=a\sin^3\theta$

$\therefore\quad dx/d\theta=-3a\cos^2\theta\sin\theta$, $dy/d\theta=3a\sin^2\theta\cos\theta$

$\therefore\quad \frac{dy}{dx}=\frac{dy}{d\theta}\Big/\frac{dx}{d\theta}=-\frac{3a\sin^2\theta\cos\theta}{3a\cos^2\theta\sin\theta}=-\tan\theta$

and $\frac{d^2y}{dx^2}=-\frac{d}{dx}(\tan\theta)=-\sec^2\theta\frac{d\theta}{dx}$

$=-\sec^2\theta\left(-\frac{1}{3a\cos^2\theta\sin\theta}\right)=\left(\frac{1}{3a}\sec^4\theta\operatorname{cosec}\theta\right)$

Hence $\rho=\frac{(1+y_1^2)^{3/2}}{y_2}=\frac{3a(1+\tan^2\theta)^{3/2}}{\sec^4\theta\operatorname{cosec}\theta}$

$=\frac{3a\sec^3\theta}{\sec^4\theta\operatorname{cosec}\theta}=3a\sin\theta\cos\theta$

$\therefore\quad \rho=3a\sin\theta\cos\theta$. Proved.

Ex. 5. Find the radius of curvature at the point (r,θ) on the curve $r=a(1-\cos\theta)$

$r=a(1-\cos\theta)=a\sin^2\frac{1}{2}\theta$ ... (1)

or, $\log r=\log 2a+\log\sin^2\frac{1}{2}\theta$ $\therefore\ \frac{1}{r}\frac{dr}{d\theta}=\frac{\cos\frac{1}{2}\theta}{\sin\frac{1}{2}\theta}=\cot\frac{1}{2}\theta$

or, $\cot\phi=\cot\frac{1}{2}\theta$ $\left[\text{As } r\frac{d\theta}{dr}=\tan\phi\right]$

$\therefore\quad \phi=\frac{1}{2}\theta$.

But $p=r\sin\phi=r\sin\frac{1}{2}\theta$ ... ... (2)

From (1) and (2), we have

$r=2ap^2/r^2$ or, $r^3=2ap^2$ ... ... (3)

Differentiate (3) w. r. to p, then

$3r^2\frac{dr}{dp}=4ap$ or, $\frac{dr}{ap}=4/3\frac{ap}{r^2}$

But $\rho=r\frac{dr}{dp}=r.\,4/3\frac{a}{r^2}p$

$=4/3a.\frac{1}{r}.\frac{r^{3/2}}{\sqrt{2}\sqrt{a}}=\frac{2\sqrt{2}}{2}\sqrt{r}\sqrt{a}=\frac{2}{3}\sqrt{(2ar)}$

$\therefore\quad \rho=\frac{2}{3}\sqrt{(2ar)}$

Ex. 6. Find the radius of curvature at the origin of the curve

$$y = x^4 - 4x^3 - 18x^2$$

The tangent at the origin is $y=0$, *i. e.* ; x—axis is the tangent to the given curve at the origin.

Hence by Newton's formula, we have

$$\rho = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2}{2y} \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Divide the given equation by y, then

$$\frac{x^4}{y} - \frac{4x^3}{y} - 18\frac{x^2}{y} - 1 = 0$$

or, $x^2 . \dfrac{x^2}{y} - 4x . \dfrac{x^2}{y} - 18 \dfrac{x^2}{y} - 1 = 0$

when $x \to 0$, $y \to 0$, then

or, $0. \, 2\rho - 4.0.2\rho - 18.2\rho - 1 = 0$ by (1)

or, $36\,\rho = -1$ or, $\rho = -1/36$

Hence the radius of curvature at the origin is $1/36$

Ex. 7 Find the radius of cu vature at the origin of the curve.

$$y^2 - 2xy - 3x^2 - 4x^3 - x^2y^2 = 0$$

The tangents at the origin are given by $y^2 - 2xy - 3x^2 = 0$

or, $(y+x)(y-3x) = 0$ *i. e.* $y+x=0$ and $y-3x=0$

Let $y = px + qx^2/\underline{|2} + \ldots \quad \ldots$

Put the value of y in the given equation.

Then

$$(px + qx^2/\underline{|2} + \ldots)^2 - 2x(px + qx^2/\underline{|2} + \cdots) - 3x^2 - 4x^3$$
$$- x^2(px + qx^2/\underline{|2} + \ldots)^2 = 0$$

or, $(p^2 - 2p - 3)x^2 + (2pq\,\underline{|2} - 2q/\underline{|2} - 4)x^3 - \cdots = 0$

Equate the co-efficient of x^2 to zero, then

$$p^2 - 2q - 3 = 0 \quad \text{or,} \quad (p-3)(p+1) = 0 \quad \therefore \quad p = 3, -1$$

Also equate the co-efficient of x^3 to zero, then

$pq - q - 4 = 0$ when $p=3$, $q=2$.

and when $p=-1$, then $q=-2$.

$\therefore$ $p=3, q=2$, $\quad \rho = \dfrac{(1+p^2)^{3/2}}{q} = \dfrac{(1+9)^{3/2}}{2} = 5\sqrt{10}$

when $p=-1$, $q=-2$, $\rho = \dfrac{(1+1)^{3/2}}{-2} = \sqrt{2}$

Hence radius of curvature are $5\sqrt{10}, -\sqrt{2}$

Ex. 8. Show that the chord of Curvature through the pole of the cardioide $r = a(1+\cos\theta)$ is $4/3r$.

Now $r = a(1+\cos\theta) = 2a\cos^2\theta/2 \quad \ldots \quad \ldots \quad \ldots \quad (1)$

or, $\log r = \log 2a + 2\log\cos\theta/2$

$\therefore$ $\dfrac{1}{r}\dfrac{dr}{d\theta} = -2\,\tfrac{1}{2}\dfrac{\sin\theta/2}{\cos\theta/2}$ or, $\cot\phi = -\tan\tfrac{1}{2}\theta$

or, $\cot\phi = \cot\ (\tfrac{1}{2}\pi + \tfrac{1}{2}\theta)$

$\therefore$ $\phi = \tfrac{1}{2}\pi + \tfrac{1}{2}\theta \quad \ldots \quad \ldots \quad (2)$

we know $p = r\sin\phi = r\sin(\tfrac{1}{2}\pi + \tfrac{1}{2}\theta)$ by (2)

or, $p = r\cos\theta/2 = \sqrt{(r/2a)}$ by (1)

or, $2ap^2 = r^3$ or, $r^3 = 2ap^2 \quad \ldots \quad \ldots \quad (3)$

$\therefore$ $3r^2\dfrac{dr}{dp} = 4ap$ or, $r\dfrac{dr}{dp} = \dfrac{4ap}{3r}$

$\therefore$ $\rho = r\dfrac{dr}{dp} = \dfrac{4a}{3r}p = \dfrac{4a}{3r}\,\dfrac{r^3/2}{\sqrt{(2a)}}$ by (3)

Hence the chord of curvature through the origin (pole)

$$2\rho\ \sin\ \phi = 2.\ \frac{4a}{3r}\,\frac{r^{3/2}}{\sqrt{(2a)}}\sin\cdot(\tfrac{1}{2}\pi + \tfrac{1}{2}\theta) = \frac{8a}{3.\ (2a)}\sqrt{r}\cos\tfrac{1}{2}\theta$$

$$= \frac{8a}{3\sqrt{(2a)}}\sqrt{r}\left(\frac{r}{2a}\right)^{\frac{1}{2}} = \frac{8a}{3.2a}.\ r = \frac{4}{3}r.$$

Hence the cnord of carvature through Pole is $4/3r$.

Ex. 9. Find centre of curvature of $xy=16$ corresponding to the point (4, 4) R. U. 1965 ; D. U. 1983

Here $xy=16$ or, $y=16/x$.

$$\therefore \frac{dy}{dx}=y_1=-16/x^2, \text{ when } x=4, \text{ then } y_1=-1$$

$$\frac{d^2y}{dx^2}=y_2=32/x^3, \text{ when } x=4, \text{ then } y_2=\tfrac{1}{2}$$

Let (α, β) be the co-ordinates of the centre of curvature at (4,4)

$$\alpha=x-\frac{y_1}{y_2}(1+y_1^2)=4-\frac{-1}{\frac{1}{2}}(1+1)=4+4=8$$

$$\beta=y+\frac{1+y_1^2}{y_2}=4+\frac{1+1}{\frac{1}{2}}=4+4=8$$

Hence the centre of curvature is at (8, 8)

Ex. 10. Show that the radius for curvature at the origin for the curve $r=a\sin(2\theta/m)$ is a/m.

By Newtons formula.

$$\rho=\lim \frac{x^2}{2y} \text{ when } x\to 0, \quad y\to 0$$

$$=\lim_{\theta\to 0}\frac{r^2\cos^2\theta}{2r\sin\theta}=\lim_{\theta\to 0}\frac{r}{2\theta} \quad \text{[to the first approximation)}$$

In this case,

$$\rho=\lim_{\theta\to 0}\frac{a\sin(2\theta/m)}{2\theta}=\lim_{\theta\to 0}\frac{a}{2\theta}\left[\frac{2\theta}{m}-\frac{2^3\theta^3}{\lfloor 3\, m^3}+\right]$$

$$=\lim_{\theta\to 0}\left(\frac{a}{m}-\frac{2\theta^2 a}{3m^3}+\cdots\right)=\frac{a}{m}$$

$\rho=a/m$. (Proved).

Ex. 11. If C_x and C_y be the chords of curvature parallel to the axes at any point of the curve $y=ae^{x/a}$, prove that

$$\frac{1}{C_x^2}+\frac{1}{C_y^2}=\frac{1}{2aC_x}$$

Here $y=ae^{x/a}$ ... ... (1)

$$y_1=a\frac{1}{a}e^{x/a}=e^{x/a}=y/a \quad \cdots \quad (2)$$

$$y_2=a\frac{1}{a^2}e^{x/a}=y/a^2 \cdots \quad \cdots \quad (3)$$

C_x=chord of curvature parallel to x-axis$=2\rho\sin\psi$

$$=2\frac{(1+y_1^2)^{3/2}}{y_2}\cdot\frac{y_1}{\sqrt{(1+y_1^2)}}$$

$$\text{or, } C_x=2.\frac{y_1}{y_2}(1+y_1^2)=2.\frac{y}{a}.\frac{a^2}{y}\left(1+\frac{y^2}{a^2}\right)$$

$$\left[\therefore \tan\psi=y_1, \sin\psi=\frac{y_1}{\sqrt{(1+y_1^2)}}\right] \text{ by (2) and (3)}$$

or, $C_x=2(a^2+y^2)/a$ ... ... (4)

Similarly,

$$C_y=2\rho\cos\psi=2\frac{(1+y_1^2)^{3/2}}{y_2}\cdot\frac{1}{\sqrt{(1+y_1^2)}}$$

$$\text{as } \cos\psi=\frac{1}{\sqrt{(1+y_1^2)}}$$

$$C_y=2\frac{(1+y_1^2)}{y_2}=2\frac{(1+y^2/a^2)}{y/a^2}=\frac{2(a^2+y^2)}{y}$$

$$\therefore \frac{1}{C_x^2}+\frac{1}{C_y^2}=\frac{a^2}{4(a^2+y^2)^2}+\frac{y^2}{4(a^2+y^2)^2}=\frac{(a^2+y^2)}{4(a^2+y^2)^2}$$

$$=\frac{1}{4(a^2+y^2)}=\frac{1}{2aC_x} \quad \text{by (4)}$$

Hence $\dfrac{1}{C_x^2}+\dfrac{1}{C_y^2}=\dfrac{1}{2aC_x}$ Proved.

Exercise—XIII

1 (a) Give an example of a curve of constant curvature.

1 (b) Find the radius of curvature at any point (s, ψ) on the following curves.

(i) $s=a\tan\psi$ (ii) $s=a\psi$

(ii) $s=a\log\tan(\frac{1}{4}\pi+\frac{1}{2}\psi)$ (iv) $s=a(e^{m\psi}-1)$

(v) $\rho=a(1+\sin\psi)$ at any point. R. U. 1987

2. Find the radius of curvature at the indicated points of the curve. $y=x^3-2x^2+7x$ at $(0,0)$ N.U. 1994

(a) (i) $y=\tan x$ at $x=\pi/4$ (ii) $y=x^4$ at $x=1$ R. U. 1965,

(iii) $y=\log\cos x$ at $x=\pi/4$.

(iv) $x=1+\cos 2\pi t$, $y=3+2\sin\pi t$, at $t=\frac{1}{6}$

(v) $x^3+y^3=3axy$ at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ D. H. 1965

(vi) $y^2x^2=a^2(x^2-a^2)$ at $(a, 0)$ R. H. 1962

(vii) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at $(0, b)$ C. H. 1988

(viii) $y=e^{-x^2}$ at $(0, 1)$ (ix) $x^4+y^4=2$ at $(1,1)$ 1984

(b) (i) $r=a\sin 3\theta$ at $(a, \pi/6)$

(ii) $r^2=a^2\cos 2\theta$ at (a, π)

(iii) $r=a\sec\theta$ at $(a, 0)$

(iv) $r=a/(1-\cos\theta)$ at $(a, \pi/2)$

(v) $x=a\cos\theta$, $y=a\sin\theta$ at $(a, 0)$ C. U. 1990, '84

(vi) $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$ at vertex $(a\pi, 2a)$ R. U. 1979

(c) (i) $r^3=2ap^2$ at (p, r)

(ii) $pa^n=r^{n+1}$ at (p, r) C. U. 1983

3. (a) Prove that $\rho=y^2/c$ for the curve

$$y^2=c^2+s^2$$

3, (b) Prove that for the curve C. U. 1993

$s=a\log\cot(\frac{1}{4}\pi-\frac{1}{2}\psi)+a(\sin\psi/\cos^2\psi)$,

$\rho=2a\sec^2\psi$.

3. (c) Show that the radius of curvature of the catenary $y=c\cosh x/c$ at $(0, c)$ is y^2/c. D. H. 1983

3. (d) Find the radius of curvature of the curve

$r^2=a^2\cos 2\theta$, at (r, θ) $a>0$ D. U. 1967, D. U 1966

4. If $x=a(\cos t+t\sin t)$, $y=a(\sin t-t\cos t)$ prove that $\rho=at$ D. U. 1955

5. In the curve $p=r^{n+1}a^n$, show that the radius of curvature varies inversely at the $(n+1)$th power of the radius vector.

6. Find the chord of curvature through the pole of the following curve.

(i) $r^2=a^2\cos 2\theta$; R. U. 1964

(ii) $r=a(1-\cos\theta)$

(iii $r^n=a^n\cos(n\theta)$

7. Find the radius of curvature at the origin of the following curve.

(i) $y^2=x^3+5x^2+6x$

(ii) $y^2=3xy-2x^3+x^3-y^4$

(iii) $x^3+y^3=2x^2-6y$

(iv) $y^2(a^2-x^2)=a^2x$,

(v) $y^2=x^2(a+x)/(a-x)$

(vi) $5x^3+7y^3+4x^2y+xy^2+2x^2+3xy+y^2+4x=0$

(viii) $x^3-2x^2y+3xy^2-4y^3+5x^2-6xy+7y^2-8y=0$

7. Find the radius of curvature of the parabols $y^2=16x$ at an end of the tatus rectum.

8. Show that the curvature of the point $(3a/2, 3a/2)$ of the curve $x^3+y^3=3axy$ is $-8\sqrt{2}/3a$. R. U. 1961, D. H. 1963

9. Prove that chord of carvature parallel to the axis of y for the curve. R. H. 1960

$y=a\log\sec x/a$ is of constant length.

10. Show that in the curve $y^2-3xy-4x^2+x^3+x^4y+y^5=0$. the radii of curvature at the origin are $\frac{1}{2}85\sqrt{17}$ and $5\sqrt{2}$. C. H. 1986, D. H. 1960

11. Show that the chord of curvature through the pole of the enquiangular spiral $r=ae^{m\theta}$ is $2r$.

12. Show that the chord of curvature through the pole of the curve $p=f(r)$ is $2f(r)/f'(r)$.

13. Prove that the points on the curve $r=f(\theta)$ the circle of curvature at which pass through the origin are given by the equation $f(\theta)+f''(\theta)=0$.

14. Find the radius of curvature of the curve $r=a\sin n\theta$ at the origin.

14. (a) For the curve $x=2\cos^3\theta$, $y=2\sin^3\theta$ show that the radius of curvature. is 3 at $\theta=\pi/4$ D. U. 1987

15. Find the centre of curvature of the following curves at the indicated points. D. U. 1989

(i) $xy=x^2+4$ at $(2, 4)$

(ii) $y=3x^3+2x^2-3$ at $(0,-3)$

(iii) $x^3+y^3=3axy$ at $(3a/2, 3a/3)$

16. Prove that the centre of curvature at the point to $(a\cos\theta, b\sin\theta)$ of the ellipse.

$x^2/a^2+y^2/b^2=1$ is $\left(\frac{a^2-b^2}{a}\cos^2\theta, \frac{b^2-a^2}{b}\sin^2\theta\right)$ D. U. 1969

17. Show that the centre of curvature of the cycloid $x=a(\theta-\sin\theta)$, $y=a(1-\cos\theta)$ lies on a similar cycloid. D H. 1960

18. For the equiangular spiral $r=ae^{\theta\cot\alpha}$ prove that the centre of curvature is at the point where the perpendicular to the radius vector intersect the normals. R. H. 1965

19. If (α, β) be the co-ordinates of centre of curvature of the parabola $\sqrt{x}+\sqrt{y}=\sqrt{a}$ at (x, y) then prove that $\alpha+\beta=3(x+y)$ R. U. 1983

20. Obtain the formula for radius of curvature $\rho=r\,dr/dp$. Use the formula to obtain the radius of curvature to the parabola at one end of the latus rectum. D. U. 1954

21. If ρ_1 and ρ_2 be the radii of curvature at the extremities of a focal chord of $y^2=4ax$, show that R. U. 1988

$$\frac{1}{(\rho_1)^{2/3}}+\frac{1}{(\rho_2)^{2/3}}=\frac{1}{(2a)^{2/3}}$$

22. The tangents at two points, P, Q, on the cycloid $x=a(\theta-\sin\theta)$; $y=a(1-\cos\theta)$ are at right angles, show that if ρ_1, ρ_2 be the radii of curvatures at these points.

then $\rho_1^2+\rho_2^2=16a^2$

23. Find the circle of curvature of the following curves.

(I) $y=x^2-6x+10$.

Hints :—

$\alpha=3$, $\beta=32$ Find $\rho=\frac{1}{2}$. The equation is

$(x-3)^2+(y-3/2)^2=(\frac{1}{2})^2$

(II) $y=x^3+2x^2+x+1$ at $(0, 1)$

24. Prove that the locus of the centre of curvature of the parabola $x^2=4ay$ is

$4(y-2a)^3=27ax^2$ C. H. 1983

25. Find the radius of curvature of the curve given by $x=a(1+\cos 2\theta)\sin 2\theta$ $y=a(1-\cos 2\theta)\cos 2\theta$.

প্রশ্নমালা XIII

1. (a) ধ্রুব বক্রতাবিশিষ্ট একটি বক্ররেখার উদাহরণ দাও।

(b) যে কোন বিন্দু (s, ψ) এ নিম্নলিখিত বক্ররেখাগুলির বক্রতার ব্যাসার্ধ নির্ণয় কর।

(i) $s=a\tan\psi$ (ii) $s=a\psi$

(iii) $s=a\log\tan(\pi/4+\frac{1}{2}\psi)$ (iv) $s=a\left(e^{m\psi}-1\right)$

2. নিম্নলিখিত বক্ররেখা সমূহের পার্শ্বে বর্ণিত বিন্দুগুলির বক্রতার ব্যাসার্ধ নির্ণয় কর,

(a) (i) $y=\tan x$ এর $x=\pi/4$ বিন্দুতে।

(ii) $y=x^4$ এর $x=1$ বিন্দুতে R. U. 1965

(iii) $y=\log \operatorname{cose} x$ এর $x=\pi/4$ বিন্দুতে

(iv) $x=1+\cos 2\pi t,\ \ y=3+2\sin\pi t$ এর $t=\frac{1}{2}$ বিন্দুতে

(v) $x^3+y^3=3axy$ এর $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ বিন্দুতে। D. H. 1965

(vi) $y^2x^2=a^2(x^2-a^2)$ এর (o, b) বিন্দুতে। R. H. 1962

(vii) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ এর $(0, b)$ বিন্দুতে (vii) $x^4+y^4=9(x+y)$ মূল বিন্দুতে C. U. 1990

(b) (i) $r=a\sin 3\theta$ এর $(a, \pi/6)$ বিন্দুতে।

(ii) $r^2=a^2\cos 2\theta$ এর (a, π) বিন্দুতে।

(iii) $r=a\sec\theta$ এর $(a, 0)$ বিন্দুতে।

(iv) $r=\frac{a}{(1-\cos\theta)}$ এর $(a, \pi/2)$ বিন্দুতে।

(v) $x=a\cos\theta.\ \ y=a\sin\theta$ এর $(a, 0)$ বিন্দুতে। C. U. 19:0, '84

(vi) $x=a(\theta+\sin\theta),\ y=a(1-\cos\theta)$ এর শীর্ষবিন্দু $(a\pi, 2a)$-এ।

(c) (i) $r^3=2ap^2$ এর (p, r) বিন্দুতে। C. U. 1983

(ii) $pa^n=r^{n+1}$ এর (p, r) বিন্দুতে।

3. (a) বক্ররেখা $y^2=c^2+s^2$ এর জন্য প্রমাণ কর যে $\rho=y^2/c$

(b) বক্ররেখা $s=a\log\cot\left(\frac{\pi}{4}-\frac{\psi}{2}\right)+a\left(\frac{\sin\psi}{\cos^2\psi}\right)$ এর জন্য প্রমাণ কর যে $\rho=2a\sec^3\psi$.

(c) দেখাও যে ক্যাটেনারী (catenary) $y=c\cosh\left(\frac{x}{c}\right)$ এর $(0, c)$ বিন্দুতে বক্রতার ব্যাসার্ধ হইবে y^2/c.

D. H. 1983

(d) বক্ররেখা $r^2=a^2\cos 2\theta$ এর (r, θ) বিন্দুতে $[a>0]$ বক্রতার ব্যাসার্ধ নির্ণয় কর। D. U. 1967; D. U. 1966

4. যদি $x=a(\cos t+t\sin t)$; $y=a(\sin t-t\cos t)$ হয় তবে প্রমাণ কর যে $\rho=at$. D.U. 1990 D. U. 1955

5. বক্ররেখা $p=r^{n+1}a^n$-এর জন্য দেখাও যে, ইহার যে কোন বিন্দুর বক্রতার ব্যাসার্ধ ঐ বিন্দুর ভেক্টর ব্যাসার্ধের (radius vector) $(n+1)$ তম শক্তির (Power) ব্যাস্তানুপাতিক।

6. নিম্নলিখিত বক্ররেখাগুলির মেরুবিন্দু (Pole) গামী বক্রতার জ্যাসমূহের সমীকরণ নির্ণয় কর। R. U. 1964

(i) $r^2=a^2\cos 2\theta$

(ii) $r=a(1-\cos\theta)$

(iii) $r^n=a^n\cos(n\theta)$

7. মূলবিন্দুতে নিম্নলিখিত বক্ররেখাগুলির বক্রতার ব্যাসার্ধ নির্ণয় কর।

(i) $y^2=x^3+5x^2+6x$ (ii) $y^2=3xy-2x^2+x^3-y^4$

(ii) $x^3+y^3=2x^2-6y$ (iv) $y^2(a^2-x^2)=a^2x$

(v) $y^2=x^2(a+x)/(a-x)$

(vi) $5x^3+7y^3+4x^2y+xy^2+2x^2+3xy+y^2+4x=0$

(vii) $x^3-2x^2y+3xy^2-4y^2+6x^3-6xy+7y^2-8y=0$

7. (a) অধিবৃত্ত (Parabola) $y^2=16x$ এর উপকেন্দ্রিক লম্বের (latus rectum) এক প্রান্তের বিন্দুর বক্রতার ব্যাসার্ধ নির্ণয় কর।

8. দেখাও যে বক্ররেখা $x^3+y^3=3axy$ এর $(3a/2, \ 3a/2)$ বিন্দুর বক্রতা $-8\sqrt{2}/3y$. R. U. 1961, D. H. 1963

9. প্রমাণ কর যে বক্ররেখা $y=a \log \sec \ x/a$ এর y-অক্ষের সমান্তরাল বক্রতার জ্যা-এর দৈর্ঘ্য শূন্য হইবে। R. U. 1960

10. দেখাও যে বক্ররেখা $y^2-3xy-4x^2+x^3+x^4y+y^5=0$ এর উপর মূল বিন্দুতে বক্রতার ব্যাসার্ধ সমূহ হইবে $\frac{85}{2}\sqrt{17}$ এবং $5\sqrt{2}$ D. H. 196

11. দেখাও যে সমানকৌনিক শঙ্খিল রেখা (Equiangular spiral) $r=ae^{m\theta}$-এর মেরুবিন্দুগামী বক্রতার জ্যা-এর দৈর্ঘ্য $2r$.

12. দেখাও যে $p=f(r)$ বক্ররেখার মেরুবিন্দুগামী (Pole) বক্রতা জ্যা এর দৈর্ঘ্য= $\frac{2f(r)}{f'(r)}$.

13. প্রমাণ কর যে $r=f(\theta)$ বক্ররেখার উপর যে সববিন্দুর বক্রতার বৃত্ত মূলবিন্দুগামী হইবে সেসব বিন্দুগামী রেখার সমীকরন হইবে $f(\theta)+f''(\theta)=0$.

14. $r=a \sin n\theta$ বক্ররেখার মূলবিন্দুতে-ব্যাসার্ধ নির্ণয় কর।

15. প্রদত্ত বিন্দুতে নিম্নলিখিত বক্ররেখাগুলি বক্রতার কেন্দ্র নির্ণয় কর।

(i) $xy=x^2+4$ বক্ররেখার উপর $(2, 4)$ বিন্দুর।

(ii) $y=3x^3+2x^2-3$ বক্ররেখার উপর $(0, -3)$ বিন্দুর।

(iii) $x^3+y^3=3axy$ বক্ররেখার উপর $(3a/2, 3a/2)$।

16. প্রমাণ কর যে ($a \cos \theta$, $b \sin \theta$) বিন্দুতে উপবৃত্ত

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ এর বক্রতার কেন্দ্রের স্থানাংক হইবে

$$\left(\frac{a^2-b^2}{a}\cos^3\theta, \frac{b^2-a^2}{b}\sin^3\theta\right).$$ D. U. 1969.

17. দেখাও যে বৃত্তাকার (cycloid)

$x=a(\theta-\sin\theta)$, $y=a(1-\cos\theta)$ এর বক্রতার কেন্দ্র সমূহ অনুরূপ একটি বৃত্তাকার ক্ষেত্রের (cycloid) উপর অবস্থিত হইবে। D. H. 1960,

18. প্রমাণ কর যে, সমানকৌনিক শঙ্খিল রেখা (equiangular spiral) $r=ae^{\theta \cot \alpha}$ এর বক্রতার কেন্দ্র সমূহ হইবে ব্যাসার্ধ ভেক্টরের উপর অঙ্কিত লম্ব এবং অভিলম্বের ছেদ বিন্দু সমূহ। R. H. 1965

19. অধিবৃত্ত $\sqrt{x}+\sqrt{y}=\sqrt{a}$ এর উপর (x, y) বিন্দুর বক্রতার কেন্দ্রের স্থানাংক (α, β) হইলে দেখাও যে

$\alpha+\beta=3(x+y)$ R. U. 1983

২০। বক্রতার ব্যাসার্ধ সূত্র অর্থাৎ $\rho=r\frac{dr}{dp}$ সূত্রটি প্রতিপাদন কর। এই সূত্র প্রয়োগে অধিবৃত্তের উপকেন্দ্রীক লম্বের (latus reetum) এক প্রান্তের বিন্দুর বক্রতার ব্যাসার্ধ নির্ণয় কর। D. U. 1954

২১। অধিবৃত্ত $y^2=4ax$-এর উপকেন্দ্রিক জ্যা এর প্রান্তবিন্দুদ্বয়ের বক্রতার ব্যাসার্ধ ρ_1 এবং ρ_2 হইলে দেখাও যে

$$\frac{1}{(\rho_1)^{2/3}}+\frac{1}{(\rho_2)^{2/3}}=\frac{1}{(2a)^{2/3}}$$

22. বৃত্তাকার ক্ষেত্র (Cycloid) $x=a(\theta-\sin\theta)$, $y=a(1-\cos\theta)$ এর উপর P ও Q বিন্দুদ্বয়ে অংকিত স্পর্শকদ্বয় পরস্পরকে লম্বভাবে ছেদ করে। যদি ঐ বিন্দুদ্বয়ের বক্রতার ব্যাসার্ধদ্বয় যথাক্রমে ρ_1 ও ρ_2 হয় তবে দেখাও যে $\rho_1^2+\rho_2^2=16a^2$.

২৩। নিম্নলিখিত বক্ররেখাগুলির বক্রতার বৃত্ত নির্ণয় কর।

(i) $y=x^2-6x+10$

ইঙ্গিত : এখন $\alpha=3$, $\beta=3/2$ এবং $\rho=\frac{1}{2}$ পাওয়া যাইবে।

$\therefore$ বৃত্তের সমীকরণ $(x-3)^2+(y-3/2)^2=(\frac{1}{2})^2$.

(ii) $y=x^3+2x^2+x+1$ বক্ররেখার $(0, 1)$ বিন্দুতে।

— 0 —

ANSWERS

Exercise XIII

1. (a) $s = a\psi$, the curve is a circle of radius a.

1. (b) (i) $a \sec^2\psi$ (ii) a (iii) $a \sec \psi$ (iv) $ame^{m\psi}$

2 (a) (i) $\frac{5\sqrt{5}}{4}$ (ii) $\frac{(17)^{3/2}}{12}$ (iii) -2 (iv) $7\sqrt{7}/16$

(v) $\frac{3\sqrt{2}a}{16}$ (vi) a (vii) a^2/b

(b) (i) $a/10$, (ii) $\frac{1}{8}a$ (iii) ∞ (iv) $2\sqrt{2}a$

(c) (i) $2\sqrt{(2ar)}/3$ (ii) $\frac{a^n}{(n+1)r^{n-1}}$ 3(d) $a^2/3r$

6. (i) $2r/3$ (ii) $4r/3$ (iii) $2r/(n+1)$

7. (i) 3. (ii) $2\sqrt{5}/2, -\sqrt{2}$ (iii) $3/2$ (iv) $1/2$

(v) $\pm a\sqrt{2}$ (vi) 2 (vii) $4/5$, 7. (a) $16\sqrt{2}$ 14. $an/2$

15. (i) $(2, 5)$ (ii) $(0, -2\frac{3}{4})$ (iii) $21a/16, 21a/(16)$

23. (ii) $x^2 + y^2 + x - 3y + 2 = 0$

CHAPTER XIV

SINGULARITIES

14. Concavity and Convexity :—

Let P be a point of the curve $y = f(x)$. Let AB be a straight line which does not pass through the point P. Draw a tangent at P.

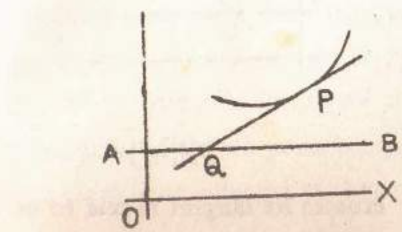


Fig. 19

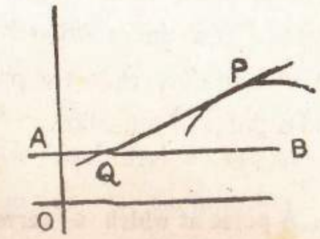


Fig. 20

Then the sufficiently small arc containing P lies entirely with in (fig. 20) or without (fig 19) the acute angle made by the tangent and the line AB.

The curve at P in the fig. 19 is called convex to AB and the curve at P in fig. 20 is called the concave to AB.

Mathematically a curve is convex or concave at P to the axis of x according as $y \frac{d^2y}{dx^2}$ is positive or negative at P. (For proof any Higher Calculus),

Similarly a curve is convex at P w. r. to y-axis if $x \frac{d^2x}{dy^2}$ is positive and the curve is concave at P. w. r. to y-axis if $x \frac{d^2x}{dy^2}$ is negative.

14. 2. Point of inflexion : As regards the point of inflexion we discused in chapter XI. Chapter on maxima and minima. A point P of inflexion is a point on the curve such that on one side of it the curve is concave and on the other side it is convex with respect to any line AB. The point P is called a point of inflexion.

In other way we can define the point of inflexion ; We know that ordinarily a curve does not cross its tangent. If a curve crosses its tangent at a point, then the point is called a point of inflexion.

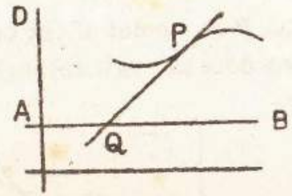


Fig. 21

i. e. **A point at which a curve crosses its tangent is said to be a point of inflexion.**

Test for point of Inflexion.

(a) For Cartesian curves :—

A point of inflexion at P exists if $\frac{dy}{dx}=0$, $\frac{d^2y}{dx^2}=0$ but $\frac{d^3y}{dx^3}\neq 0$

(b) For Polar curves :—

A point of inflexion at P exists if $u+\frac{d^2u}{d\theta^2}$ changes sign

Put $u+\frac{d^2u}{d\theta^2}=0$ and find for what values of θ changes of sign can occur.

(c) For pedal curve

A point of inflexion at P exists if

$$\frac{dp}{dr}=0 \text{ as } \rho=r/\frac{dp}{dr}$$

For a point of inflexion $\frac{dp}{dr}$ changes sign.

14. 3. Double point—If two branches of a curve pass through a point then the point is called double point.

A curve has two tangents at a double point, one for each branch.

If three branches of a curve pass through a point, then the point is called triple point.

Multiple Points :—If more than one branch of a curve passes through a point, then the point is called a multiple point. If r branches of a curve pass through a point, then the point is called **a multiple point of the rth order** on the curve. The curve has r tangents (real or imaginary) at that point one for each branch.

Singular point :—A multiple point is sometimes called a singular point.

Point of Undulation :—When a straight line meets a curve at four coincident point of contact is called **a point of undulation.** In this case the tangent does not cross the curve but is indistinguishable from an ordinary tangent.

For example, is $y-x=x^4+y^4$ there is point of undulation at the origin.

14. 4. Classification of double points.

At a double point of a curve, there are two tangents one for each branch.

Case (i) If the two tangents are real and not coincident, then the two real branches of the curve passing through the point is called **node or crunode.**

Case (ii) If the two tangents are coincident the point is called **a cusp, stationary point or spinode.**

Case (iii) If tangents are imaginary, there are no real points on the curve in the neighbourhood of the point considered ; such a point is called an **isolated point or conjugate point or, acnode.**

At a conjugate point the tangents are usually imaginary but sometimes tangents at such point may be real.

14. 5. Find the necessary condition for the existence of double points.

Let $f(x, y)=0$ be the equation of a curve and $P(x, y)$ be any point on it.

The slope of the tangent at $P(x, y)$ to the curve $f(x, y)=0$ is

$$\frac{dy}{dx}=-f_x/f_y \text{ whence } f_x+f_y\frac{dy}{dx}=0 \quad \cdots \quad (1)$$

At a multiple point of a curve the curve has a least two tangents at that point and $\frac{dy}{dx}$ must have two values at the multiple point. But the eq. (1) is of first degree in $\frac{dy}{dx}$ can be satisfied by two values of $\frac{dy}{dx}$ if and only if

$$f_x=0 \text{ and } f_y=0$$

Hence the necessary condition for any point (x, y) of the curve to be a multiple point is then

$$f_x=0 \text{ and } f_y=0$$

Solve the equation $f_x=0$ and $f_y=0$ for x and y. Put the values of x and y in eq. $f(x, y)=0$. The pairs of values of x and y which satisfy $f(x, y)=0$ constitute the required double points. The values which do not satisfy $f(x, y)=0$ should be rejected.

The Co-ordinates of the multiple points then satisfy the three equations.

$$f(x, y)=0. \quad f_x=0, \quad f_y=0,$$

Differentiate $f_x+f_y\frac{dy}{dx}=0$ with respect to x.

Then

$$f_{xx}+f_{xy}\frac{dy}{dx}+\left(f_{yx}+f_{yy}\frac{dy}{dx}\right)\frac{dy}{dx}+f_y\frac{d^2y}{dx^2}=0$$

For a double point $f_x=0, f_y=0$

Also $f_{xy}=f_{yx}$

$$\text{Thus } f_{yy}\left(\frac{dy}{dx}\right)^2+2f_{yx}\left(\frac{dy}{dx}\right)+f_{xx}=0 \quad \cdots \quad \cdots \quad (2)$$

The above equation is quadratic in $\frac{dy}{dx}$

$$\therefore \quad \frac{dy}{dx}=\frac{-2f_{yx}+\sqrt{\{4f^2_{yx}-4f_{yy}f_{xx}\}}}{2f_{yy}}$$

The double point be a node, **cusp or conjugate** according as

$4f^2_{yx}-4f_{yy}f_{xx}>,=$ or, <0 or, $f^2_{yx}>,=$ or, $<f_{yy}f_{xx}$

If f_{xx}, f_{xy} and f_{yy} are not also zero.

$\therefore$ In general a double point is a **node, cusp, or conjugate** point if $f^2_{yx}>=$ or $<f_{yy}f_{xx}$ ... ... (3)

If $f_{xx}=f_{xy}=f_{yy}=0$, then such a point is a triple point.

14. 6. (a) Classification of cusps.

There are two types of cusps.

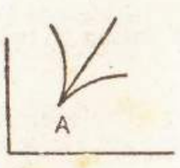


Fig. 22

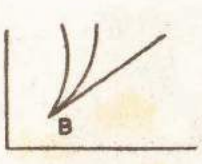


Fig. 23

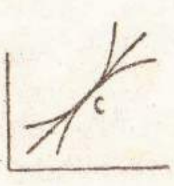


Fig. 24

(i) **Single cusps** (ii) **Double cusps**

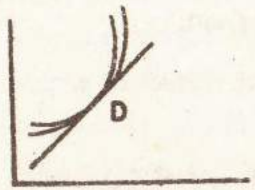


Fig. 25

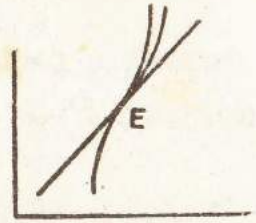


Fig. 26

(i) **Single cusps :** When branches of the curve do not extend on both sides of the point of contact. Fig. 22 & 23

(ii) **Double cusps :**—When branches of the curve extend on both sides of point of contact. Fig. 24, 25, 26.

(b) **Species of cusps.**

Cusps (single or double) are of two species.

First Species—A cusp of the first species or a keratoid cusp, (cusp like horns) is that in which the branches are on opposite sides of the common tangents. See fig. 22 and 24.

Second Species :—A cusp of the second species or a rhamphold cusp (cusp like beak) is that in which the branches of the curve lie on the same side of the common tangents, see fig. 23 and 25.

Oscul inflexion : – Double cusp with change of species is a point of oscul inflexion.

The point of oscul-inflexion is the combination of two species fig. 26.

The point of contact is a double cusp. The cusp is of 2nd species on the right side of the point of contact and the cusp is of first species on the left side of the point.

14. 7. Search for the nature of a cusp.

Let $P(x, y)$ be a cusp on a curve $f(x, y)=0$

To determine the nature of the cusp transfer the origin to the point $P(x, y)$. The equation of the curve is such that the lowest degree terms form a perfect square *i. e* ; there are two coincident tangent at $P(x, y)$. Draw perpendicular from a point near to $P(x, y)$ to the tangent $ax+by=0$, then $p=\dfrac{ax+by}{\sqrt{(a^2+b^2)}}$

As the point is very near to $P(x, y)$ then we can write $p=ax+by$ (roughly).

Now eliminate y between $f(x, y)=0$ and $p=ax+by$.

We get an equation in p and x only. As we want to consider only the small values of p and x, so we take only the terms involving p^2 and x^2. Thus we get a quadratic equation in x and p.

(a) If the roots of p are imaginary there is a conjugate point at the point (new origin).

(b) If the roots are real but of opposite signs (*i, e*, product of the roots is negative) then two perpendiculars lie on opposite sides of common tangent at that point. Hence the cusp of 1st species or 1st kind.

(c) If the roots of p are real and of same sign ; then two perpendiculars lie on same side of the common tangent. Hence the point is a cusp of the 2nd species.

(d) If the reality of the roots of the equation depends on the sign of x the cusp is single.

(e) If the reality of the roots of the equation is independent of the sign of x the cusp is double.

The above stated facts have been shown in the examples.

Ex 1. Find the points of the inflexions of the following curves.

(i) $y=2x+x^3+x^4$ (ii) $r(\theta^2-3)=2$

(i) $y=2x+x^3+x^4$ ... ... (1)

Differentiate with respect to x.

$$\frac{dy}{dx}=2+3x^2+4x^3 \quad \therefore \frac{d^2y}{dx^2}=6x+12x^2 \ldots(2)$$

For a points of inflexion $\frac{d^2y}{dx^2}=0$ and $\frac{d^3y}{dx^2}\neq 0$

$\therefore \frac{d^2y}{dx^2}=0$ or, $6x+12x^2=0$ or, $x=0,-2$

From (1) $y=0,\ 4$ when $x=0,-2$

The probable points of inflexion are $(0,0), (-2,4)$

Again $\frac{d^3y}{dx^3}=6+24x$ which is not zero for $x=0$ and -2.

Hence the points of inflexion are $(0,0)$ and $(-2,4)$

(ii) $r(\theta^2-3)=2$

or; $2u=\theta^2-3$, Put $u=1/r$

$2u_1=2\theta$; $2u_2=2$.

$u+u_2=\frac{1}{2}\theta^2-3/2+1=\frac{1}{2}\theta^2-\frac{1}{2}=\frac{1}{2}(\theta-1)(\theta+1)=0$

If $\theta=\pm 1$.

As $u+u_2$ changes sign as θ passes through these values.

From $r(\theta^2-3)=2$; $r=-1,-1$ for $\theta=\pm 1$. Hence the points of inflexion are at $(-1,1)$ and $(-1,1)$.

Ex. 1. (a) Determine the double points of the curve

$x^4+y^4-4a^2xy=0$ R. H. 1964

Let $f(x,y)=x^4+y^4-4a^2xy$ ... ... (1)

$\therefore f_x=4x^3-4a^2y=0$ ... ... (2)

and $f_y=4y^3-4a^2x=0$ ... ... (3)

From (2) and (3)

$4x^3-4a^2y=0$ and $4y^3-4a^2x=0$

Solve for x and y. Then

$x=0,\ a$, and $y=0,\ a$

The double points are $(0,0)$ and (a,a)

But $(0,0)$ only satisfies $f(x,y)=0$.

Hence only probable double point is at $(0,0)$

Again $f_{xx}=12x^2=0$ at $(0,0)$, $f_{yy}=12y^2=0$ at $(0,0)$

and $f_{yx}=-4a^2$ at $(0,0)$

Now $f^2_{yx}-f_{xx}f_{yy}=16a^4-0.\ 0=16a^4=+ve$

or, $f^2_{yx}>f_{xx}f_{yy}$

Which shows that the double point is a node at the origin.

Ex. 2. Find the position and nature of the double points on the curve.

$x(2x^2-5ax+4a^2)=ay(2a-y)$

Let $f(x,y)=x\ (2x^2-5x+4a^2)-ay(2a-y)$ ... (1)

$f_x=6x^2-10ax+4a^2=0$ ... ... (2)

$f_y=-2a+2y=0$ ... ... (3)

From (2) and (3), we have $x=a,\ \frac{2}{3}a$ and $y=a$,

of these only (a,a) staisfies eq. $f(x,y)=0$

Hence the only probable double point is at (a,a),

Again $f_{xx}=12x-10a$ at $(a,a)=2a$

$f_{yy}=2$ at (a,a)

$f_{yx}=0$ at (a,a)

Now $f^2_{yx}-f_{xx}f_{yy}=0-2a.\ 2=-4a$

or ; $f^2_{yx}<f_{xx}f_{yy}$

i, e ; the double point is a conjugate point.

Ex. 3. Examine the nature of the origin on the curve

$y^2=2x^2y+x^4y-2x^4$ ... ... (1) D. U. 1958

Tangents and the origin of the curve are given by

$y^2=0$, or, $y=0$ ... ... (2)

i. e. there are two coincident tangents at the origin. The probable double point is a cusps.

Let $y=p$ then (1) becomes,

$p^2=2x^2p+x^4p-2x^4$ or, $p^2-p\ (2x^2+x^4)+2x^4=0$

or, $p=\frac{1}{2}[(2x^2+x^4)\pm\sqrt{\{(2x^2+x^4)^2-8x^4\}}]$

$= \frac{1}{2}[(2x^2+x^4) \pm \sqrt{\{4x^4-4x^4+x^8\}}\}$

$= \frac{1}{2}\{(2x^2+x^4) \pm x^2\sqrt{(4x-4+x^4)}\}$

$= \frac{1}{2}\{2x^2+x^4+x^2\sqrt{(4x^2-4)}\}$ for small values of x near the origin $= x^2 \pm x^2\sqrt{(4x^2-4)}$.

For small values of x. $\sqrt{(4x^2-4)}$ is always imaginary. Hence p is imaginary near the origin, there is no real point near the origin. Hence the double point is a conjugate point.

Ex. 4. Find the nature of the curve

$$y^2 = x^3(1-x)$$

The tangent at the origin is

$$y^2 = 0 \quad \text{or,} \quad y = 0 \quad \cdots \quad \cdots \quad (1)$$

i. e. $x-$axis is the tangent at the origin

Put $y = p$ in $y^2 = x^3(1-x)$, then

$p^2 = x^3(1-x)$ or, $p^2 = x^3 - x^4$

$\therefore \quad p = \pm\sqrt{(x^3-x^4)} \quad \cdots \quad \cdots \quad (2)$

For small values of x. x^4 is neglected in comparison with x^3, then near the origin $p = \pm\sqrt{x^3}$.

For negative values of x, p is imaginary *i. e.*, there is no branch of the curve to the left of the origin.

For positive values of x, p has equal and opposite signs. Hence there is a cusp for first species of single cusp.

Ex. 5. Show that the curve $x+y=y^2(2+3\sqrt{y})$ has a single cusp of the 2nd species at the origin.

The tangent at the origin is $x+y=0$.

Let $p = x+y$ or, $y = p-x \quad \cdots \quad (1)$

Put the value of y in the equation, then

$p = 2(p-x)^2 + 3(p-x)^{5/2}$

or, $2p^2 - p(4x+1) + 2x^2 = 0 \quad \cdots \quad \cdots \quad (2)$

Near the origin, we neglect $p^{5/2}, x^{5/2}$ in comparison to x^2 and p^2.

or, $p = \frac{1}{4}[(4x+1) \pm \sqrt{(16x^2+8x+1-16x^2)}]$

$= \frac{1}{4}[4x+1) \pm \sqrt{(8x+1)}]$

The reality of roots of p depends on the sign of x. Hence the doub. point is a single cusp.

Again from (2), the product of roots of p is positive $(2x^2)$ *i. e*, the two roots of p have the same sign. Thus the cusp is of 2nd species.

Hence the double point is a single cusp of 2nd species.

Ex 6. Show that the curve $y^2 - 2x^2y - x^4y - x^4 = 0$ has a double cusps of first species at the origin.

Let $y^2 - 2x^2y - x^4y - x^4 = 0 \quad \cdots \quad (1)$

The tangents at the origin are $y^2 = 0$

There are two coincident tangents at the origin, so there is a cusp at the origin.

Let $y = p \quad \cdots \quad \cdots \quad (2)$

From (1) by (2), we have

$p^2 - 2x^2p - x^4p - x^4 = 0$ or, $p^2 - p(2x^2+x^4) - x^4 = 0 \quad \cdots \quad (3)$

$\therefore \quad p = \frac{1}{2}[2x^2+x^4) \pm \sqrt{\{(2x^2+x^4)^2+4x^4\}}]$

$= \frac{1}{2}\{(2x^2+x^4) \pm \sqrt{(8x^4+4x^6+x^8)}\}$

Near the origin we neglect x^3, x^4 in comparison with x^2, Then $p = \frac{1}{2}(2x^2 + 2\sqrt{2x^2}) = (1+\sqrt{2})x^2$

Therefore roots of p are real. Moreover the reality of roots does not depend on x (x may be positive or negative). Hence the origin is a cusp.

Again products of roots p^2 from (3), is equal to$-x^4$, *i, e.* the roots are opposite in signs.

Thus the two perpendiculars lie on the opposite sides of the common tangent at the cusp. The cusp is of 1st species.

Hence the origin is a double cusp of first species.

Ex. 7. Show that the curve $p^2 - x^2y + x^5 = 0$ has an osculinflexion at the origin.

The tangents at the origin are $y^2 = 0$

The two tangents are coincident at the origin ; so there is a cusp at the origin.

Let $y=p$ ... ... (2)

Then the equation (1) becomes by (2).

$p^2-x^2p+x^5=0$ ... ... (3)

or, $p=\frac{1}{2}(x^2\pm\sqrt{(x^4-4x^5)})$

As x is very small, so x^4-4x^5 is positive. Then the roots of p are real.

More over roots do not depend upon the sign of x.

Hence the double point is a double cusp.

The product of the roots of p in (3), is equal to x^5.

If x is positive, the product of the root is positive.

Hence the two perpendiculars lie on the same sides of the common tangent at the origin. The cusp is of 2nd species.

If x is negative the product of the roots is negative *i, e* ; the two perpendiculars lie on the opposite sides of the common tangent at the origin.

Hence the cusp is also of 1st species.

Thus cusp at the origin is of the double cusp of mixed species *i, e.* the point is a oscul-inflexion.

Ex. 8. Examine the nature of the double points on the curve $(x+y)^3-\sqrt{2}(y-x+2)^2=0$ D. U. H. 1959

Let $f(x,y)=(x+y)^3-\sqrt{2}(y-x+2)^2=0$ ... (1)

$f_x=3(x+y)^2+2\sqrt{2}(y-x+2)=0$ ... ... (2)

$f_y=3(x+y)^2-2\sqrt{2}(y-x+2)=0$ ... ... (3)

Add (2) and (3) ; $6(x+y)^2=0$ or $x+y=0$ ... (4)

Subtract (3) from (2) ;

$4\sqrt{2}(y-x+2)=0$ or ; $x-y=2$ ... ... (5)

Thus from (4) and (5) $x=1$, $y=-1$

The point $(1,-1)$ satisfies $f(x,y)=0$.

Hence $(1,-1)$ is a double point.

What type of double point is ?

Again $f_{xx}=6(x+y)-2\sqrt{2}=-2\sqrt{2}$ when $x=1$, $y=-1$

$f_{yx}=6(x+y)+2\sqrt{2}=2\sqrt{2}$ when $x=1$, $y=-1$

$f_{yy}=6(x+y)-2\sqrt{2}=-2\sqrt{2}$ when $x=1$, $y=-1$

Therefore at $(1,-1)$, we have $f^2_{yx}=f_{xx}f_{yy}$

Thus the curve $f(x,y)=0$ has a cusp at $(1,-1)$

What is the species of the cusp ?

Now transfer the origin at $(1,-1)$, the equation (1) is [Put $x=x+1$, $y=y-1$]

$(x+1+y-1)^3-\sqrt{2}(y-1-x-1+2)^2=0$

or, $(x+y)^3-\sqrt{2}(y-x)^2=0$ ... ... (6)

Tangents at the new origin of eq (6) are

$(y-x)^2=0$ or, $y-x=0$

Put $y-x=p$ ... ... (7)

Now the eq. (6) by (7) becomes

$(2x+p)^3-\sqrt{2}p^2=0$ or, $8x^3+12x^2p+6xp^2+p^3-\sqrt{2}p^2=0$

or, $p^2(6x-\sqrt{2})+12x^2p+8x^3=0$

neglecting p^3 in the neighbourhood of the origin ; x and p are small.

$$\text{or, } p=\frac{-12x^2\pm\sqrt{\{144x^4-32x^3(6x-\sqrt{2})\}}}{2(6x-\sqrt{2})}$$

$$=\frac{-12x^2+\sqrt{\{32\sqrt{2}x^3\}}}{2(6x-\sqrt{2})} \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

neglecting x^4 in comparison with x^3. For positive values of x, p is real.

For negative values of x, p is imaginary. Hence there is a single cusp at $(1,-1)$. From (8) for small positive values of x one value of p is positive and the other negative. Hence the cusp is of the first species.

Therefore there is a single cusp of first species at $(1,-1)$.

Exercise XIV

1. Find the points of inflexion of the following curves if any.

(i) $y(x-1)=x^3$ (ii) $xy^2=a^2(a-x)$

(iii) $x^2=y^2(a^2+y^2)$ (iv) $y^2=4x^4+x^3$

(v) $y=3x^4-4x^3+1$ (vi) $a=r\sqrt{\theta}$

(vii) $r^2\theta=a^2$ (viii) $r(\theta^2-1)=a\theta^2$

2. Show that the point of inflexion on the curve $r=a\theta^n$ are given by $r=a\{-n(n+1)\}^{n/2}$.

3. Show that the point of inflexion of the curve

$y^2=(x-a)^2(x-b)$ lie on the line $3x+a=-4b$.

4. Find the nature of the cusps, if any, in the following curves

(i) $y=x^{3/2}+x^2$ (ii) $x=y^2+y^{5/2}+y^3$

(iii) $(x-2)^2-y(y-1)^2=0$ (iv) $x^4-ax^2y+axy^2+a^2y^2=0$

(v) $x^3-y^2-7x^2+4y+15x-13=0$ (vi) $a^2y^2-2abx^2y-x^5=0$. R.U. 1991

(vii) $x^7+2x^4+2x^3y+x^2+2xy+y^2=0$

(viii) $y(y-6)=x^2(x-9)^3-9$ (ix) $y=x^2+x^{5/2}+x^3$

(x) $y=x^2+x^3\sqrt{(9-x)}$ (xi) $y^2=x^4(x^2-1)$ D. H. 1962

5. Show that the origin on the curve $y^2=bx\sin(x/a)$ there is a node or a conjugate point according as a and b have like or unlike signs.

6. Show that the curve $a^2y^2-2abx^2y+b^2(x^5+x^6)=0$ has a double point at the origin. Show that the double point is an oscul-inflexion.

7. Show that the curve $x^3+y^2+x^2-x-4y+3=0$ has a node at $(-1, 2)$ and a loop.

8. Show that the curve $x^4-2ax^2y-axy^2+a^2y^2=0$ has a cusp of the 2nd kind at the origin.

9. Show that the curve $x^5+ayx^4-a^3x^2y+a^4y^2=0$ has a double cusp of 2nd species at the origin.

10. Show that the curve $y^2=2x^2y+x^3y+x^3$ has a single cusp of the first species at the origin.

11. Show that the curve $y^2=2x^2y+x^4y+x^3$ has a double cusp of 1st. species at the origin.

12. Show that the curve $y-2=x(1+x+x^3y^2)$ has a single cusp of second species at a point where it cuts the y—axis.

13. Prove that the radius of curvature of the cartenary $y=\frac{1}{2}a(e^{x/a}+e^{-x/a})$ is y^2/a and that of the cartenary of uniform strength $y=c\log\sec x/c$ is $c\sec(x/c)$ R. U. 1958

প্রশ্নমালা XIV

1. নিম্নলিখিত বক্ররেখা গুলিতে যদি কোন আনতি (points of inflexion) বিন্দু থাকে তাহা নির্ণয় কর।

(i) $y(x-1)=x^3$ (ii) $xy^2=a^2(a-x)$

(iii) $x^2=y^2(a^2+y^2)$ (iv) $y^3=4x^2+x^3$

(v) $y=3x^4-4x^3+1$ (vi) $a=r\sqrt{\theta}$

(vii) $r^2\theta=a^2$ (viii) $2(\theta^2-1)=a\theta^2$

2. দেখাও যে $r=a\theta^n$ বক্ররেখার উপর আনতি বিন্দু সমূহকে $r=a\{-n(n+1)\}^{n/2}$ সমীকরণ দ্বারা দেওয়া যায়।

3. দেখাও যে $y^2=(x-a)^2(x-b)$ বক্ররেখার আনতি বিন্দু সমূহ $3x+a=-4b$ সরলরেখার উপর অবস্থিত।

4. নিম্নলিখিত বক্ররেখাগুলির মধ্যে সূক্ষাগ্র বিন্দু যদি থাকে তবে তাদের প্রকৃতি নির্ণয় কর।

(i) $y=x^{3/2}+x^2$ (ii) $x=y^2+y^{5/2}+y^3$

(iii) $(x-2)^2-y(y-1)^2=0$ (iv $x^4-ax^2y+axy^2+a^2y^2=0$

(v) $x^3-y^2-7x^2+4y+15x-13=0$ (vi) $a^3y^2-2abx^2y-x^5=0$

(vii) $x^7+2x^4+2x^3y+x^2+2xy+y^2=0$ R.U. 1991.

(viii) $y(y-6)=x^2(x-9)^3-9$ (ix) $y=x^2+x^{5/2}+x^3$

(x) $y=x^2+x^3\sqrt{(9-x)}$ (xi) $y^2=x^4(x^2-1)$ D. H. 1962

5. দেখাও যে $y^2=bx\sin(x/a)$ বক্ররেখার উপর মুলবিন্দু প্যাদবিন্দু অথবা যুগ্ম বিন্দু হবে যদি a এবং b এর চিহ্ন একই রকম বা ভিন্ন রকম হয়

6. দেখাও যে $a^2y^2-2abx^2y+b^2(x^5+x^6)=0$

বক্ররেখার উপর মূলবিন্দুতে একটি দ্বি-বিন্দু আছে। আরো দেখাও যে ঐ দ্বি-বিন্দুটি একটি আনত চুম্বন বিন্দু।

7. দেখাও যে $x^3+y^3+x^2-x-4y+3=0$ বক্ররেখার $(-1, 2)$ বিন্দুতে পাতবিন্দু (node) এবং একটি ফাঁস (a loop) আছে।

8. দেখাও যে $x^4-2a^2xy-axy^2+a^2y^2=0$ বক্ররেখার উপর মূলবিন্দুতে একটি ২য় প্রকারের সূক্ষ্মাগ্র বিন্দু আছে। (a cusp of the 2nd kind).

9. দেখাও যে. $x^5+ayx^4-a^3x^2y+a^4y^2=0$ বক্ররেখার উপর মূলবিন্দুতে একটি দ্বিতীয় প্রজাতির দ্বি সূক্ষ্মাগ্র বিন্দু আছে। (a double cusp of the 2nd species).

10. দেখাও যে $y^2=2x^2y+x^3y+x^3$ বক্ররেখার উপর মূলবিন্দুতে একটি প্রথম প্রজাতির একক সূক্ষ্মাগ্র বিন্দু আছে (a single cusp of the first species).

11. দেখাও যে $y^2=2x^2y+x^4y+x^3$ বক্ররেখার উপর মূলবিন্দুতে একটি প্রথম প্রজাতির দ্বি-সূক্ষ্মাগ্র বিন্দু আছে।

12. দেখাও যে $y-2=x(1+x+x^3y^2)$ বক্ররেখা যে বিন্দুতে y-অক্ষকে ছেদ করে সে বিন্দুতে একটি দ্বিতীয় প্রজাতির একক সূক্ষ্মাগ্র বিন্দু আছে।

(a single cusp of 2nd species).

13. প্রমাণ কর যে কাটেনারী $y=\frac{1}{2}a(e^{x/a}+e^{-x/a})$-এর বক্রতার ব্যাসার্ধ হবে y^2/a এবং সম-শক্তির কাটেনারী $y=c\log\sec(x/c)$-এর বক্রতার ব্যাসার্ধ হবে $c\sec(x/c)$.

R. U. 1958

উত্তরমালা XIV

1. (i) $(0,0)$ (ii) no (iii) no (iv) no (v) $\left(\frac{2}{3}, \frac{11}{27}\right)$ (vi) $(r=a\sqrt{2}, \theta=\frac{1}{2})$ (vii) $\theta=\pm\frac{1}{2}$ (viii) $\theta=\sqrt{3}$, $r=3a/2$

4. (i) একক কেরাটয়েড সূক্ষ্মাগ্র বিন্দু। (ii) দ্বিতীয় প্রজাতি (iii) পাত-বিন্দু (node) (iv) যুগ্মবিন্দু Coujugate point) (v) পাতবিন্দু (node) (vi) আনত-চুম্বনবিন্দু (vii) আনত-চুম্বনবিন্দু (oscul-inflexion) (viii) একক কেরাটয়েড সূক্ষ্মাগ্রবিন্দু (single keratoid cusp) যুগ্মবিন্দু (Conjugate point.) (ix) দ্বিতীয় প্রজাতির একক সূক্ষ্মাগ্র বিন্দু। (x) দ্বিতীয় প্রজাতির দ্বি-সূক্ষ্মাগ্র-বিন্দু।

EXERCISE XV

Envelopes and Evolutes

15. 1. Family of curves :—Let there be an equation of the form $f(x, y, c)=0 \cdots\cdots(1)$ where c is an arbitrary constant. For different values of c, the equation (1) will represent different curves. The quantity c which is constant for a particular curve but different for different curves is called a parameter of the family to which the curves belong. Since there is only one parameter c in $f(x, y, c)=0$. these curves are sometimes called the **one parameter family of curves.**

15. 2. Definition of Envelope :—A curve which touches each member of a family of curves and conversely if each point is touched by some members of the family, is called the envelope of that family of curves.

15.3. To find the equation of an envelope.

Let $f(x, y, c)=0 \quad \cdots \quad \cdots \quad (1)$

represent a family of curves.

$$\text{Let} \quad \left.\begin{array}{l} f(x, y, c)=0 \\ f(x, y, c+h)=0 \end{array}\right\} \quad \cdots \quad \cdots \quad (2)$$

be the two consecutive members of the family of curves (1)

Suppose the curve (1) touches the envelope at P.

The curves in (2) will intersect at a point P_1 which is near the points of contact of these curves with the envelope.

The equation of the curve through P_1 of (2) is

$$f(x, y, c+h)-f(x, y, c)=0$$

$$\text{or ;} \quad \frac{f(x, y, c+h)-f(x, y, c)}{h}=0$$

$$\therefore \lim_{h\to 0} \frac{f(x, y, c+h)-f(x, y, c)}{h}=0$$

or, $\frac{\delta}{\delta c} f(x, y, c)=0 \quad \cdots \quad \cdots \quad (3)$

i. e. the co-ordinates of P will satisfy the equation (3). *But* P is a point on the curve (1)

Hence P satisfies eq. (1) and (3). If we eliminate c from (1) and (3) we shall get the locus of that point of intersection for all values of parameter c.

Hence the locus of P is the envelope of $f(x, y, c)=0$

Let us explain the above definition by two illustrations.

Ex. 1. Consider the family of circles

$$(x-c)^2+y^2=r^2 \quad \cdots \quad \cdots \quad (1)$$

Where c is a parameter.

The centre of the circle is at $(c, 0)$. For different values of c, we will get different circles of radius 'r' The centers of the circles lie on the x-axis. This family of circles will lie between two straight lines $y=r$ and $y=-r$. All the circles touch the straight lines AB and CD. Straight lines AB and CD are the envelopes of the family of the circles (1), see fig. 27.

(**Ex. 2.** Consider the family of straight lines.

$x \cos \alpha+y \sin \alpha=a$, where x is a parameter.

We know that $x \cos \alpha+y \sin \alpha-a=0$ is the tangent to the circle at ($a \cos \alpha$, $a \sin \alpha$)

For each value of α, we get a fixed straight line which touches the fixed circle $C = x^2+y^2-a^2=0$. For different values of α we will get different straight lines which touch the circle C.

Hence the envelope of the straight lines $x \cos \alpha+y \sin \alpha-a=0$ is the circle $x^2+y^2=a^2$. See fig. 28.

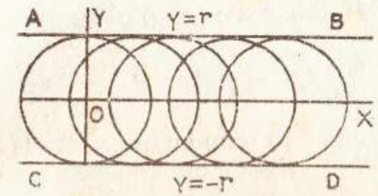


Fig. 27

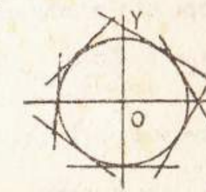


Fig. 28

The required envelope is the eliminant of c from

$$\left.\begin{array}{l} f(x, y, c)=0 \\ \frac{\delta}{\delta c} (x, y, c)=0 \end{array}\right\}$$

Cor. The envelope of $Aa^2+2Ba+C-0$ is $B^2=AG$

If the equation $f(x, y, \alpha)=0$ is a quadratic equation in α, let it be of the form

$$A\alpha^2+2B\alpha+C=0 \quad \cdots \quad \cdots \quad (1)$$

where A, B, C are functions of x and y.

Differentiate (1) *w. r* to α partially $2A\alpha+2B=0$

or, $\alpha=-B/A \quad \cdots \quad \cdots \quad (2)$

Put the value of α in (1), then

$A(-B/A)^2+2B(-B/A)+C=0$ or, $B^2/A-2B^2/A+C=0$

or, $B^2=AC$

Thus the envelope of $Aa^2+2Ba+C=0$ is $B^2=AC$.

Note :—The polar curves $f(r, \theta, c)=0$ may be treated in the same manner.

15. 4. The envelope touches each member of the family.

Let $f(x, y, c)=0$ ... ... (1)

represent a family of curves.

The slope of the tangent at any point (x, y) of $f(x, y, c)=0$ is

$$\frac{dy}{dx}=-\frac{\delta f/\delta x}{\delta f/\delta y} \quad \text{or,} \quad \frac{\delta f}{\delta y}\,dy+\frac{\delta f}{\delta x}\,dx=0 \quad \cdots \quad (2)$$

The envelope of (1) is obtained by eliminating c from (1) and

$$\frac{\delta f}{\delta c}(x, y, c)=0 \quad \cdots \quad \cdots \quad \cdots \quad (3)$$

Let x and y be the function of c. *i. e* ; $\left.\begin{array}{l} x=\phi(c) \\ y=\psi(c) \end{array}\right\} \cdots (4)$

Therefore values of x and y are obtained *by* solving the equation (1) and (3) in terms of c.

The slope of the tangent at $P(x, y)$ of the envelope is.

$$\frac{dy}{dx}=\frac{\psi'(c)}{\phi'(c)} \quad \cdots \quad \cdots \quad \cdots \quad (5)$$

where primes denote differentiation *w. r.* to c.

Now take the total differential of $f(x, y, c)=0$

$$df=\frac{\delta f}{\delta x}\,dx+\frac{\delta f}{\delta y}dy+\frac{\delta f}{\delta c}\,dc \quad \cdots \quad \cdots \quad \cdots \quad (6)$$

($\cdot$For convenience, $f(x, y, c)$ will be written as f)

If x and y satisfy equations (4). then $df=0$

$dx=\phi'(c)\,dc$, $dy=\psi'(c)\,dc$ and $\frac{\delta f}{\delta c}=0$ by (3)

Hence (6) becomes

$$\frac{\delta f}{\delta x}dx+\frac{\delta f}{\delta y}by=0 \text{ or, } \frac{\delta f}{\delta x}+\frac{\delta f}{\delta x}\frac{dy}{dx}=0$$

$$\frac{df}{dx}\;\frac{\delta f}{\delta y}\;\frac{\psi'(c)}{\phi'(c)}=0 \quad \cdots \quad - \quad (7)$$

From (2) and (7) we notice that both the gradients of the tangents are the same. Thus the curve and the envelope have the same tangent at the common points on the curve *i. e.* ; the envelope touches each member of his family.

Note :—If $\delta f/\delta x$ and $\delta f/\delta y$ are both zero then envelope may not touch a curve at points. These points are the singular points on the curve.

15. 5 Double Parameters

Let $f(x, y, \alpha, \beta)=0$ ... ... (1)

be an equation with two variables (parameters)

Let the parameters be related by

$\phi(\alpha, \beta)=0$ ... ... (2)

To remove the variables from (1) and (2) let us suppose α *an* independent parameter. Then from 1) and (2).

$$\frac{\delta f}{\delta \alpha}+\frac{\delta f}{\delta \beta}\frac{d\beta}{d\alpha}=0 \quad \cdots \quad \cdots \quad (3)$$

$$\text{where} \quad \frac{\delta f}{\delta \alpha}+\frac{\delta \phi}{\delta \beta}\quad\frac{d\beta}{d\alpha}=0 \quad \cdots \quad \cdots \quad (4)$$

There are four equation and three quantities such as α, β, $(d\beta/d\alpha)$. The eliminant of these quantities is the required envelope of (1)

15. 6. (i) Pedal curves as Envelopes :—

If circles are drawn on radius vector of a given curve as diameters ; they all touch the first positive pedal of the curve with respect to the origin. Thus the process of finding the first positive pedal of a curve is the same as the finding of envelopes of circles described on the radii vectors as diameters.

(ii) Envelope of a line as Negative pedals.

The first negative pedal of a curve is the envelope of a straight line drawn through any point of the curve and perpendicular to the radius vector to the point on the curve.

Let O be the pole OP the radius vector of a point P on the curve. PQ is perpendicular to OP at P. The locus of PQ is the envelope of such straight lines or the first negative pedal of the curve. See Examples 6 and 7.

EVOLUTES

15.7. Definition of evolutes :—The locus of centre of curvature for a curve is called its evolute.

or, the evolute of curve is the envelope of the normals of that curve.

15. 8. Properties of evolute

(*i*) The normal to a given curve is a tangent to its evolute.

(*ii*) The length of an arc of the evolute of a certain curve is the difference between the radii of curvature of the given curve, which are tangents to this arc of the evolute at its extremites.

(*iii*) **Radius of the curvature of the evolute** $\rho' = \dfrac{d^2s}{d\psi^2}$

where ρ' is the radius of curvature of the evolute.

15. 9. Involutes :—If one curve is the evolute of another then the later is called an involute of the former

If the curve $Q_1\,Q_2\,Q_3\,Q_4$ is the evolute of the curve $P_1\,P_2\,P_3\,P_4$ then $P_1P_2P_3P_4$ is called the involute of the curve $Q_1Q_2Q_3Q_4$.

Again $P'_1P'_2P'_3P'_4$ is also the involute of the curve $Q_1Q_2Q_3Q_4$. All curve parallel to the curve $P_1\,P_2\,P_3\,P_4$ are the involutes of $Q_1\,Q_2\,Q_3\,Q_4$

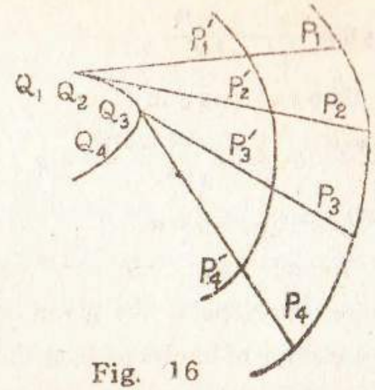


Fig. 16

Therefore every curve has an infinite number of involutes but there is only one evolute (in the fig $Q_1\,Q_2\,Q_3\,Q_4$)

Ex. 1. Find the envelope of the straight line

$y = mx + \sqrt{(a^2m^2+b^2)}$ where m is a parameter.

The equation is written as $(y-mx)^2 = a^2m^2+b^2$

or, $m^2(x^2-a^2) - 2mxy + y^2 - b^2 = 0$ ... ... (1)

This is quadratic in m.

Hence the envelope of (1) is given, by Art. *15. 3* Cor.

$4x^2y^2 = 4(x^2-a^2)(y^2-b^2)$ or, $x^2y^2 = x^2y^2 + a^2b^2 - b^2x^2 - a^2y^2$

or, $x^2/a^2 + y^2/b^2 = 1$ which is an ellipse

Ex. 2. Find the envelope of the straight line

$x\cos\theta + y\sin\theta = a\sin\theta\cos\theta$ where θ is the parameter

The given equation is written as

$$\frac{x}{\sin\theta} + \frac{y}{\cos\theta} = a \quad \cdots \quad \cdots \quad (1)$$

Differentiate (1) partially *w. r.* to θ, then

$$-\frac{x}{\sin^2\theta}\cos\theta - \frac{y}{\cos^2\theta}(-\sin\theta) = 0 \quad \text{or ; } \tan^3\theta = \frac{x}{y}$$

$$\text{or ; } \tan\theta = (x/y)^{1/3} \quad \therefore \quad \sin\theta = \frac{x^{1/3}}{\sqrt{(x^{2/3}+y^{2/3})}}$$

$$\cos\theta = \frac{y^{1/3}}{\sqrt{(x^{2/3}+y^{2/3})}}$$

put the value of $\sin\theta$ and $\cos\theta$ in (1)

then $\dfrac{x\sqrt{(x^{2/3}+y^{2/3})}}{x^{1/3}} + \dfrac{y\sqrt{(x^{2/3}+y^{2/3})}}{y^{1/3}} = a$

or ; $\sqrt{(x^{2/3}+y^{2/3})(x^{2/3}+y^{2/3})} = a$

or ; $(x^{2/3}+y^{2/3})^{3/2} = a$ or ; $x^{2/3}+y^{2/3} = a^{2/3}$

which is the required envelope of the given straight line.

Ex. 3. Find the envelope of circles passing though the origin and having their centres lie on the parabola $x^2=4ay$ D. H. 1966

In the parabola $x^2=4ay$: $P(2at,\ at^2)$ is any point. The distance OP from the origin $O\ (0,\ 0)$ is

$OP=\sqrt{(4a^2t^2+a^2t^4)}$ = radius of the circle.

The equetion of the circle whose centre is at $(2at,\ at^2)$ is

$$(x-2at)^2+(y-at^2)^2=OP^2=4a^2t^2+a^2t^4$$

or ; $x^2+y^2-4axt-2ayt^2=0$ ... ... (1)

Differentiate $w.\ r.$ to t. Then $-4ax-4ayt=0$ or, $t=-x/y$

putting the value of t in (1), we have

$x^2+y^2+4a.x.x/y-2ya.\ x^2/y^2=0$ or, $yx^2+y^3+4ay^2-2ax^2=0$

or, $y^3+yx^2+2ax^2=0$ which is the required envelope.

Ex. 4. Find the envelope of the family of ellipses $x^2/a^2+y^2/b^2=1$ where two parameters a, b are connected by the relation

$a+b=c$, c being a constant D. H. 1962

Here $x^2/a^2+y^2/b^2=1$ ... ... (i)

and $a+b=c$ ... ... (ii)

$\therefore\ -(2x^2/a^3)da-(2y^2/b^3)\ db=0$

or, $(x^2/a^3)da+(y^2/b^3)db=0$ ... ... (iii)

and From (ii) $da+db=0$ ... ... (iv)

from (iii) and (iv) Comparing we have

$x^2/a^3=y^2/b^3=\lambda$ (say) or, $x^2/a^2=a\lambda$ and $y^2/b^2=b\lambda$ ... (v)

Adding we have $x^2/a^2+y^2/b^2=(a+b)\lambda$

or, $1=c\lambda$ or, $\lambda=1/c$ by (i) and (ii)

From (v),

$x^2/a^3=a/c$ and $y^2/b^2=b/c$ or, $a=(x^2c)^{1/3}$, $b=(y^2c)^{1/3}$

Therefore putting the values of a and b in (ii)

$x^{2/3}+y^{2/3}=c^{2/3}$ which is the required envelope of (i)

Ex. 5. Find the envelope of the straight line $x/l+y/m=1$. where l and m are parameters connected by the relation

$l/a+m/b=1$, a and b being constants. R. U. 1964, '88

Here $x/l+y/m=1$ ... ... (i)

and $l/a+m/b=1$ ... ... (ii)

$\therefore\ -(x/l^2)\ dl-(y/m^2)\ dm=0$

or, $(x/l^2)dl+(y/m^2)\,dm=0$ ... (iii)

and $dl/a+dm/b=0$ ... ... (iv)

From (iii) and (iv), comparing we have

$\dfrac{x}{l^2}\Big/\dfrac{1}{a}=\dfrac{y}{m^2}\Big/\dfrac{1}{b}=\lambda$ (say) $\dfrac{x}{l}=\dfrac{\lambda l}{a}$, $\dfrac{y}{m}=\dfrac{\lambda m}{b}$ ... ... (v)

Adding : $x/l+y/m=\lambda(l/a+m/b)$ or, $1=\lambda.\ 1$ by (i) and (ii)

or, $y=1$

Therefore from (v)

$l^2=ax$, $m^2=by$ ... ... (vi)

or, $l=\sqrt{(ax)}$, $m=\sqrt{(by)}$

Putting the values of l and m in (ii)

we have $\sqrt{(ax)}/a+\sqrt{(by)}/b=1$ or, $\sqrt{(x/a)}+\sqrt{(y/b)}=1$

which is the required envelope of the above line (i)

Ex. 6. Show that envelope of straight lines at right angles to the radii vectors of the curve $r=a\ (1+\cos\theta)$ drawn through their extremities is $r=2a\cos\theta$.

The equation of the given curve is

$$r=a\ (1+\cos\theta) \qquad \ldots \quad \ldots \quad (i)$$

Let $P(l, \alpha)$ be any point on the curve (1)

then $\quad l=a\ (1+\cos\alpha) \quad \ldots \quad (2)$

The equation of a straight line right angles to OP {i. e PQ, $Q\ (r, \theta)$} is

$$\frac{OP}{OQ}=\cos(\theta-\alpha)$$

as $\angle OPQ=90°$, $\angle POQ=\theta\ \ \alpha$ and $OQ=r$, $OP=l$

$\therefore\ \ l=r\cos(\theta-\alpha)$ or, $a(1+\cos\alpha)=r\cos(\theta-\alpha)$ $\quad\ldots\quad\ldots\quad(3)$

or, $\ a=\cos\alpha\,(r\cos\theta-a)+r\sin\alpha\sin\theta \quad\ldots\quad\ldots\quad\ldots\quad(4)$

where α is a parameter.

Differentiate (3) $w.\ r.$ to α, then

$0=-\sin\alpha\,(r\cos\theta-a)+r\sin\theta\cos\alpha \quad\ldots\quad\ldots\quad(5)$

Square and add (4) and (5) to get $a^2=(r\cos\theta-a)^2+r^2\sin^2\theta$

or, $\ a^2=r^2+a^2-2ar\cos\theta$ or, $r=2a\cos\theta$. Proved

Ex. 7. Show that the envelope of the circles drawn on the radii vectors of the curve $r^n=a^n\cos n\theta$ as diameter is

$$r^{\frac{n}{n+1}}=a^{\frac{n}{n+1}}\cos\frac{n\theta}{n+1}$$

Let $P(l, a)$ be any point on the curve $r^n=a^n\cos n\theta \cdots \cdots$ (i). Then the equation (i) becomes

$l^n=a^n\ \cos n\alpha \cdots \cdots \cdots \cdots \cdots$ (ii)

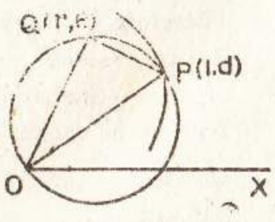


Fig 30

Draw the circle OPQ with OP as a diameter and take $Q(r, \theta)$ any point on the circle. Then

$\angle OQP=90°$, $\angle POX=\alpha$, $\angle QOP=\theta$, $OP=l$, $OQ=r$

$\therefore\ \ \angle POQ=\theta-\alpha \quad\ldots\quad\ldots\quad\ldots\quad(iii)$

The equation of the circle is

$OQ/OP=\cos(\theta-\alpha)$ or, $r=l\cos(\theta-\alpha)$

Put the value of l from (ii)

or, $\ r=a(\cos n\alpha)^{1/n}\cos(\theta-\alpha) \quad\ldots\quad\ldots\quad(iv)$

or, $\ \log r=\log a+\dfrac{1}{n}\log\cos n\alpha+\log\cos(\theta-\alpha)$

Differentiate $w.\ r.$ to α which is a parameter, then

$0=-(1/n)n\tan n\alpha+\tan(\theta-\alpha)\ \therefore\ n\alpha=\theta-\alpha$ or, $\alpha=\theta/(n+1)$

Put the value of α in (iv) then

$$r=a\left(\cos\frac{n\theta}{n+1}\right)^{1/n}\cos\left(\theta-\frac{\theta}{n+1}\right)=a\left(\cos\frac{n\theta}{n+1}\right)^{1/n}\cos\frac{n\theta}{n+1}$$

$$\text{or,}\quad r=a\left(\cos\frac{n\theta}{n+1}\right)^{\frac{1}{n}}=a\left(\cos\frac{n\theta}{n+1}\right)^{(n+1)/n}$$

$$\text{or,}\quad r^{\frac{n}{n+1}}=a^{\frac{n}{n+1}}\cos\left(\frac{n\theta}{n+1}\right)\ \text{Proved}$$

Ex. 8. Find the evolute of parabola $y^2=4ax$.

We know evolute is the envelope of normals, so the evolute of the parabola $y^2=4ax$ is the envelope of the normals.

$y=mx-2am-am^3 \quad\ldots\quad\ldots\quad(1)$

at $(am^2, 2am)$ of the parabola, where m is the parameter.

Differentiate (1) $w.\ r.$ to m then

$0=x-2a-3am^2$ or, $3am^2=x-2a \quad\ldots\quad\ldots\quad(2)$

From (1); $y=m(x-2a)-am^3=m.\ 3am^2-am^3$ by (2)

or, $\ y=2am^3$ or, $m^3=y/2a$ or, $m=(y/2a)^{1/3}$

Put the value of m in (2), then

$3a(y/2a)^{2/3}=(x-2a)$ or, $27a^3(y/2a)^2=(x-2a)^3$

or, $27ay^2=4(x-2a)^3$

which is the required evolute of $y^2=4ax$.

Note : Evolute is the locus of centres of curvature of a curve, The centre of curvature at any point (x,y) of $y^2=4ax$ is (α,β) such that $\alpha=3x+2a$ and $\beta=-\dfrac{2}{\sqrt{a}}x^{3/2}$ or, $x=\dfrac{\alpha}{3}$

$$\therefore\ \beta=-\frac{2}{\sqrt{a}}\left(\frac{\alpha-2a}{3}\right)^{3/2} \text{ or, } 27\beta^2a=4(\alpha-2a)^3$$

The required evolute is $27ay^2=4(x-2a)^3$

Ex. 9. Show that evolute of ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is $(ax)^{2/3}+(by)^{2/3}=(a^2-b^2)^{2/3}$

Evolute is the envelopes of normals to the ellipse

The equation of the normal at $(a\cos\theta, b\sin\theta)$ of

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \dots \quad \dots \quad (1)$$

is $$\frac{ax}{\cos\theta}+\frac{by}{\sin\theta}=a^2-b^2 \quad \dots \quad \dots \quad (2)$$

where θ is the parameter,

Differentiating (2) w. r. to θ

$$ax\times\frac{1}{\cos^2\theta}(-\sin\theta)+by\times\frac{1}{\sin^2\theta}(\cos\theta)=0 \quad \dots \quad \dots \quad (3)$$

or, $\tan^3\theta=\dfrac{by}{ax}$ or. $\tan\theta=\left(\dfrac{by}{ax}\right)^{1/3}$

$$\text{or, } \frac{\sin\theta}{(by)^{1/3}}=\frac{\cos\theta}{(ax)^{1/3}}=\sqrt{\left\{\frac{\sin^2\theta+\cos^2\theta}{(by)^{2/3}+(ax)^{2/3}}\right\}}=\frac{1}{\sqrt{\{(ax)^{2/3}+(by)^{2/3}\}}}$$

$$\text{or, } \sin\theta=\frac{(by)^{1/3}}{\sqrt{\{(ax)^{2/3}+(by)^{2/3}\}}} \text{ or, } \cos\theta=\frac{(ax)^{1/3}}{\sqrt{\{(ax)^{2/3}+(by)^{2/3}\}}}$$

Putting the value of $\sin\theta$ and $\cos\theta$ in (2)

$$ax\times\frac{\sqrt{\{(ax)^{2/3}+(by)^{2/3}\}}}{(ax)^{1/3}}+by\frac{\sqrt{\{(ax)^{2/3}+(by)^{2/3}\}}}{(by)^{1/3}}=a^2-b^2$$

or, $\{(ax)^{2/3}+(by)^{2/3}\}^{1/2}\times\{(ax)^{2/3}+(by)^{2/3}=a^2-b^2$

or, $\{(ax)^{2/3}+(by)^{2/3}\}^{3/2}=a^2-b^2$

or, $(ax)^{2/3}+(by)^{2/3}=(a^2-b^2)^{2/3}$ Proved.

Ex. 10. Find the evolute of the equiangular spiral $r=ae^{\theta\cot\alpha}$

we know $p=r\sin\phi \quad \dots \quad \dots \quad (1)$

$r=ae^{\theta\cot\alpha} \quad \dots \quad \dots \quad (2)$

$\therefore\ \dfrac{1}{r}\dfrac{dr}{d\theta}=\cot\alpha$ or, $\cot\phi=\cot\alpha$ $\therefore\ \phi=\alpha$

Hence $p=r\sin\alpha \quad \dots \quad \dots \quad (3)$

Let ρ be the radius of curvature at $\theta=\alpha$ $(\angle POX=\theta)$

$\therefore\ \rho \quad \dfrac{dr}{dp}=r/\sin\alpha=r\operatorname{cosec}\alpha$ by (4)

Let CP be the normal, then

$CP=\rho$

$\angle OPN=\phi=\alpha$; $\angle CPO=90°-\alpha$

Now $CP=OP\operatorname{cosec}\phi=OP\operatorname{cosec}(90°-\alpha)=OP\sec\alpha$ i. e, OC is perpendicular to OP

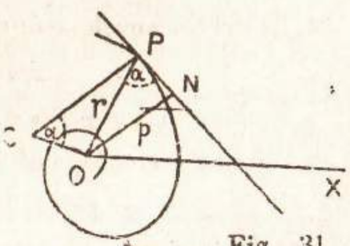


Fig. 31

[*See Author's Co-ordinate Geometry Art. 10]

Since C is the centre of curvature, the evolute is the locus of C. Let $C(r_1, \theta_1)$ be the co-ordinates of centre of curvature. Then $r_1=OC=OP\cot\alpha=r\cot\alpha$ or, $r=r_1\tan\alpha$

$\theta_1 = \angle COX = 90° + \angle POX = 90° + \theta°$ or, $\theta = -90° + \theta_1$

Put the values of r and θ in

$r = a e^{\theta \cot \alpha}$ or, $r_1 \tan \alpha = ae^{(-\frac{1}{2}\pi + \theta_1) \cot \alpha}$

The evolute of the equiangular spiral

is $r \tan \alpha = ae^{(-\frac{1}{2}\pi + \theta) \cot \alpha}$ or, $r = a \cot \alpha e^{(\theta - \frac{1}{2}\pi) \cot \alpha}$

Exercise XV

1. Find the envelopes of the following :—

(i) $y = mx + a/m$, m being a parameter

(ii) $x \cos 2\theta + y \sin 2\theta = a$, θ is variable

(iii) $c^2(y-a)^2 + (cx - a^2)^2 = (a^2 + c^2)$, a is a parameter.

(iv) $y = mx + am^p$, m is variable

(v) $x^2 \sec^2\theta + y^2 \operatorname{cosec}^2\theta = a^2$, θ is a variable.

2. Prove that the envelope of $x \cos^3\theta + y \sin^3\theta = a$

is $a^2(x^2 + y^2) = x^2y^2$

3 Show that the envelope of the straight lines joining the xtremities of a pair of conjugate diameters of an ellipse is imilar ellipse.

4. Find the envelope of the family of straight lines

$$x/a + y/b = 1$$

where (i) $a^2 + b^2 = c^2$ (ii) $ab = c^2$ R. U. 1987

5. Find the envelope of the curve

$$(x/a)^m + (y/b)^m = 1$$

where a and b are connected by

$$a^p + b^p = c^p$$

6. Show that the envelope of ellipses having the axes of co-ordinates as principal axes when

$ab = c$ is $4x^2y^2 = c^2$

7. Show that the envelope of the family of circles whose diameters are double ordinates of

$y^2 = 4ax$ is the parabola $y^2 = 4a(x + a)$

8. Show that envelope of the circles described on the radius vectors of $r^3 = a^3 \cos 3\theta$ as diameter is the curve $r^3 = a^3 \cos^4 \frac{3}{4}\theta$.

9. Find the envelope of the curve $(a/x)^{1/3} + (b/y)^{1/3} = 1$ where the paramaters a and b are connected by the equation $\sqrt{a} + \sqrt{b} = \sqrt{c}$,

10. Find the envelope of the straight lines drawn at right angles to the radius vectors of the spiral $r = ae^{\theta \cot \alpha}$ through their extremities.

11. Find the envelope of the straight lines drawn through the extremities of and at right angles to the radii vectors of the following curves.

(i) $r^n \cos n\theta = a^n$ (ii) $r = a + b \cos \theta$.

12. Show that the envelope of the circles described on the radii vectors of the curve $y^2 = 4ax$ as diameter is $ax^2 + x(x^2 + y^2) = 0$.

13. Show that envelope of all the cardioids described on radii vectors of cardioid $r = a(1 + \cos \theta)$ for axes and having their cusps at the pole is $r = 2a \cos^4 \frac{1}{4}\theta$.

14. Show that the envelope of the circles drawn on the radii vectors of the curve $r = 2a \cos \theta$ as diameter is the cordiid $r = a\ (1 + \cos \theta)$

15. Show that the envelope of a circle whose centre lies on the parabola $y^2 = 4ax$ and which passes through its vertex is

$$2ay^2 + x(x^2 + y^2) = 0.$$

16 If O be the pole and P be any point of the curve

$r=a\cos m\theta$ and if with O for pole and P for vertex a similar curve be described; the envelope of all such curves is $r=a\cos^2\frac{1}{2}m\theta$.

17. Show that the envelope of the family of curves

$a\cos^n\theta+b\sin^n\theta=c$, when θ is arbitrary parameter and a, b and c are functions of x and y is $a^{2/(2-n)}+b^{2/(2-n)}=c^{2/(2-n)}$.

18. Find the evolute of the following curves,

(i) $xy=c^2$ (ii) $x^{2/3}+y^{2/3}=a^{2/3}$

(ii) $x=a(\cos\theta+\log\tan\frac{1}{2}\theta)$, $y=a(1+\cos\theta)$

19. Prove that the evolute of the cardioid $r=a(1+\cos\theta)$ is the cardioid $r=\frac{1}{3}a(1-\cos\theta)$ the pole in the latter equation being at the point $(\frac{2}{3}a, 0)$. 1968

20. Show that the evolute of the cycloid

$$x=a(\theta+\sin\theta),\quad y=a(1-\cos\theta)$$

is given by $x=a(\theta-\sin\theta)$, $y=a(1+\cos\theta)$

Show that this is an equal cycloid by transfering the origin to $(a\pi, 2a)$ and by putting $\theta-\pi=\phi$.

21. Show that the whole length of the evolute of the astroid

$$x=a\cos^3\theta,\quad y=a\sin^3\theta \text{ is } 12a.$$

22. Show that the centre of curvature of cycloid $x=a(\theta-\sin\theta)$, $y=a(1-\cos\theta)$ lies on a similar cycloid.

D.H. 1960.

23. Show that the envelope of circles drawn on radii vectors of the cardioide $r=a(1+\cos\theta)$ is $r=2a\cos^2\frac{1}{2}\theta$.

24. Find the evolute of the parabola $y^2=4x$; find also the length of the evolute from the cusp to the point where it meets the parabola. R. U. 1987

25. Define singular points of an algebraic curve. Explain the classification of double points. R. H. 1987

উদাহরণমালা—(XV)

1. নিম্নলিখিত রেখাগুলির আচ্ছাদন নির্ণয় কর।

i) $y=mx+a/m$, এখানে m পরামিতিক রাশি।

ii) $x\cos 2\theta+y\sin 2\theta=a$, এখানে θ চলরাশি।

iii) $c^2(y-a)^2+(cx-a^2)^2=(a^2+c^2)$ এখানে a একটি পরামিতিক রাশি।

iv) $y=mx+am^p$, m একটি চলরাশি।

v) $x^2\sec^2\theta+y^2\operatorname{cosec}^2\theta=a^2$, এখানে θ একটি চলরাশি।

2. প্রমাণ কর যে $x\cos^3\theta+y\sin^3\theta=a$ এর আচ্ছাদন হবে
$$a^3(x^2+y^2)=x^2y^2.$$

3. দেখাও যে কোন উপবৃত্তের যুগল অনুবন্ধি ব্যাসের প্রান্ত বিন্দুর সংযোগ সরলরেখাসমূহের আচ্ছাদন অপর একটি উপবৃত্ত।

4. সরলরেখা-পরিবার $\frac{x}{a}+\frac{y}{b}=1$ এর আচ্ছাদন নির্ণয় কর

যখন (i) $a^2+b^2=c^2$ (ii) $ab=c^2$

5. বক্ররেখা $\left(\frac{x}{a}\right)^m+\left(\frac{y}{b}\right)^m=1$ এর আচ্ছাদন নির্ণয় কর

যখন a এবং b এর সম্পর্ক হবে $a^p+b^p=c^p$

6. দেখাও যে যদি অক্ষদ্বয় কোন উপবৃত্তের প্রধান অক্ষ হয় এবং $ab=c$ হয়, তবে $4x^2y^2=c^2$ হবে উপবৃত্তটির আচ্ছাদন।

7. দেখাও যে $y^2=4ax$ অধিবৃত্তের দ্বি-কোটিকে (double-ordinates) ব্যাস ধরে যে বৃত্তগুলি অঙ্কন করা যায় তাদের আচ্ছাদন হবে অধিবৃত্ত $y^2=4a(x+a)$ (show that the envelope of the family of circle whose diameters are double-ordinates of $y^2=4ax$ is the parabola $y^2=4a(x+a)$.

8. দেখাও যে $r^3=a^3\cos 3\theta$ বক্ররেখার কোন বিন্দুর ব্যাসার্ধ ভেক্টরকে ব্যাস ধরে যে বৃত্তগুলি অঙ্কন করা যায় তাদের আচ্ছাদন হবে $r^3=a^3\cos^4\frac{3}{4}\theta$.

9. বক্ররেখা $\left(\frac{a}{x}\right)^{1/3}+\left(\frac{b}{y}\right)^{1/3}=1$ -এর আচ্ছাদন নির্ণয় কর।

যেখানে পরামিতিক রাশি a এবং b এর মধ্যে সম্পর্ক হল $\sqrt{a}+\sqrt{b}=\sqrt{c}$.

10. কুণ্ডলী $r=ae^{\theta\cot\alpha}$ -এর ব্যাসার্ধ ভেক্টর সমূহের প্রান্ত বিন্দুগুলির মধ্য দিয়ে অংকিত লম্ব সরলরেখাগুলির আচ্ছাদন নির্ণয় কর।

11. নিম্নলিখিত বক্ররেখাদ্বয়ের ব্যাসার্ধ ভেক্টর সমূহের প্রান্ত বিন্দুগুলির মধ্য দিয়ে অংকিত লম্ব সরলরেখাগুলির আচ্ছাদন নির্ণয় কর,

(i) $r^n \cos n\theta=a^n$ (ii) $r=a+b\cos\theta$.

12. দেখাও যে $y^2=4ax$ বক্ররেখার ব্যাসার্ধ ভেক্টরকে ব্যাস ধরে যে বৃত্তগুলি অঙ্কন করা যায় তাদের আচ্ছাদন হবে $ax^2+x(x^2+y^2)=0$.

13. দেখাও যে $r=a(1+\cos\theta)$ কার্ডিয়েডের ব্যাসার্ধ ভেক্টরকে অক্ষরেখা এবং ইহাদের সূচ্যগ্র বিন্দুসহকে মেরু বিন্দু ধরে যে আচ্ছাদন পাওয়া যায় তা' হবে $r=2a\cos^4\frac{1}{2}\theta$. [Show that envelope of the cardioids described on radii vectors of cardioid $r=a(1+\cos\theta)$ for axes and having their cusps at the pole is $r=2a\cos^4\frac{1}{2}\theta$]

14. দেখাও যে $r=2a\cos\theta$ বক্ররেখার বিন্দুগুলির ব্যাসার্ধ ভেক্টরকে ব্যাস ধরে অঙ্কিত বৃত্ত সমূহের আচ্ছাদন হবে কার্ডিয়েড $r=c(1+\cos\theta)$.

15. কোন বৃত্তের কেন্দ্র $y^2=4ax$ এর উপর হলে এবং বৃত্তটি অধিবৃত্তের শীর্ষবিন্দুগামী হলে, দেখাও যে বৃত্তের আচ্ছাদন হবে $2ay^2+x(x^2+y^2)=0$.

16. যদি $r=a\cos m\theta$ বক্ররেখার মেরুবিন্দু O এবং P উহার উপর যে কোন বিন্দু হয় এবং O বিন্দুকে মেরুবিন্দু এবং P বিন্দুকে শীর্ষবিন্দু ধরে অনুরূপ কতকগুলি বক্ররেখা অংকন করা হয় তা হলে ঐ সব বক্ররেখার আচ্ছাদনে সমীকরণ হবে $r=a\cos^2\frac{1}{2}m\theta$.

17. দেখাও যে বক্ররেখ-পরিবার $a\cos^n\theta+b\sin^n\theta=c$ এর আচ্ছাদন হবে $a^{\frac{2}{2-n}}+b^{\frac{2}{2-n}}=c^{\frac{2}{2-n}}$, এখানে θ হল একটি অনিয়মিত পরামিতিক রাশি (arbitrary parameter) এবং a, b এবং c হল x এবং y এর ফাংশন।

18. নিম্নলিখিত বক্ররেখাগুলির লম্বাচ্ছাদন নির্ণয় কর। (Find the evolute of the following curves,

(i) $xy=c^2$ (ii) $x^{2/3}+y^{2/3}=a^{2/3}$ (iii) $x=a(\cos\theta+\log\tan\theta/2)$, $y=a\sin\theta$

19. দেখাও যে কার্ডিয়েড (cardioid) $r=a(1+\cos\theta)$ এর লম্বাচ্ছাদন হবে কার্ডিয়েড $r=\frac{1}{3}a(1-\cos\theta)$, যাহার মেরুবিন্দু হবে $(\frac{4}{3}a, 0)$, D. H. 1968

20. দেখাও যে বৃত্তাকার ক্ষেত্র (cycloid) $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$ এর লম্বাচ্ছাদন হবে অপর একটি বৃত্তাকার ক্ষেত্র (cycloid) $x=a(\theta-\sin\theta), y=a(1+\cos\theta)$, দেখাও যে ইহা প্রথমটির সমান বৃত্তাকার ক্ষেত্র হবে (cycloid যদি মূলবিন্দুটি $(a\pi, 2a)$ বিন্দুতে স্থানান্তর এবং $\theta-\pi=\phi$ বসানো হয়।

21. দেখাও যে এষ্ট্রয়েড (astroid) $x=a\cos^3\theta$, $y=a\sin^3\theta$ এর লম্বাচ্ছাদনের সম্পূর্ণ দৈর্ঘ্য হবে $12a$.

22. দেখাও যে বৃত্তাকার ক্ষেত্র $x=a(\theta-\sin\theta)$, $y=a(1-\cos\theta)$ এর বক্রতার ব্যাসার্ধ অনুরূপ একটি বৃত্তাকার ক্ষেত্রে (cycloid) উপর হবে। D.H 1960

23. দেখাও যে কার্ডিয়য়েড (cardioide) $r=a(1+\cos\theta)$ এর ব্যাসার্ধ ভেক্টরগুলির উপর অংকিত বৃত্ত সমূহের আচ্ছাদন হবে $r=2a\cos^3\theta/2$. [Show that the envelope of circles drawn on radii vectors of cardioide $r=a(1+\cos\theta)$ is $r=2a\cos^2\theta/2$,]

ANSWERS

Exercise XV

1. (i) $y^2=4ax$ (ii) $x^2+y^2=a^2$

(iii) $cy^2+(c+2x)(x^2+y^2-c^2)=0$

(iv) $x^p(p-1)^{p-1}+ap^p y^{p-1}=0$ (v) $x\pm y\pm a=0$

4. (i) $x^{2/3}+y^{2/3}=c^{2/3}$ (ii) $xy=c^2/4$

5. $x^{mp/(m+p)}+y^{mp/(m+p)}=c^{mp/(m+p)}$

9. $\frac{1}{x}\pm\frac{1}{y}=\frac{1}{c}$ 10. $r_1=be^{\theta_1\cot a}$

11. (i) $r^{n/(1+n)}\cos\{n\theta/(n+1)\}=a^{n/(1+n)}$

(ii) $(x-b)^2+y^2=a^2$

18. (i) $(x+y)^{2/3}-(x-y)^{2/3}=(4a)^{2/3}$.

(ii) $(x+y)^{2/3}+(x-y)^{2/3}=2a^{2/3}$ (iii) $y=a\cos k\frac{x}{a}$

CHAPTER XVI

TRACING OF CURVES

Art. 16.1. To determine roughly the shapes of the curves.

We shall in many cases have to form rough idea about the shape of the curve under discussion. Here we simply give some rules for tracing the curves which will throw some light about the shape of the curves.

Art. 16.2. (A) For Cartesian Curves :–

(a) Detect the symmetry in the curve.

(i) A curve is symmetrical about x-axis, if it contains only even powers of y.

(ii) It is symmetrical about y-axis, if it contains only even powers of x.

(iii) If on interchanging x and y, the equation does not change, the curve is symmetrical about the line $y=x$.

(iv) If by putting $-x$ for x and $-y$ for y, the equation does not change, the curve is symmetrical in opposite quadrants.

(b) If the curve passes throug the origin, find the tangent at the origin by equating to zero the lowest degree terms in the equation.

(c) Determine the points, where the curve cuts the axes of co-ordinates and find out the inclination of the tangent to the axis of x at these points, either by shifting the origin to the point and equating to zero the lowest degree term, or finding $\frac{dy}{dx}$ at that point.

(d) Find the asymptotes, if any.

(e) See if there are any limitations upon the values of x and y e. g. in the curve $x^2=\frac{y^3}{2a-y}$ if y is negative, x is imaginary and if $y>2a$, x is imaginary : so that the curve does not lie beyond $y=2a$.

(f) Find $\frac{dy}{dx}$ and find the point where it is zero or infinite.

(g) Determine the singularities of the curve, note that if there is a note, cusps or conjugate points at the origin or a multiple point of higher order than the second. If there is a cusp, determine its species.

(h) Find also the points of inflexions if any.

Art. 16.3. (B) For Polar Curves :–

(a) Examine the symmetry.

(i) If we put $-\theta$ for θ and the equation does not change, the curve is symmetrical about the initial line.

(ii) If we put $-r$ for r and the equation does not change, there is symmetry about the pole.

(iii) If we put $-\theta$ for θ and $-r$ for r and equation does not change, the curve is symmetrical about the line through the pole perpendicular to the initial line.

(b) Find if r and θ are confined between any limits, e. g if $r=a\sin n\theta$, r cannot be greater than a.

(c) Form a table of corresponding values of r for chosen values of θ

(d) Find for what value or values of θ, r is infinite. This will indicate the direction of the asymptotes if any.

Art. 16.4. (C) For Curves in which variables are connected by a parameter.

(a) Find dy/dx and form a table finding the values of x, y and dy/dx for corresponding value of parameter. Plot the points and mark the slopes of the tangent.

(b) If posible, eliminate the parameter and find a relation between x and y, and trace the curve by above methods.

16. (A) 1. কোন একটি বক্ররেখার সমীকরণ হইতে বক্ররেখার আকার সম্বন্ধে জানার জন্য স্বাভাবিক ভাবে আগ্রহ দেখা যায়। বিশ্লেষণ জ্যামিতিতে (Analytical geometry) 2nd Degree Equation হইতে standard বক্ররেখা সম্বন্ধে যেমন ($y^2=4ax$) পরাবৃত্ত, ($x^2/a^2+y^2/b^2=1$) উপবৃত্ত ($x^2/a^2-y^2/b^2=1$) অধিবৃত্ত ইত্যাদি সম্বন্ধে ধারণা আছে। এই আকার গুলি হইতে ছাত্রেরা

(Symmetry), প্রতিসাম্য অক্ষরেখাকে ছেদ করা ইত্যাদি সম্বন্ধে ধারণা আছে। জ্যামিতি ব্যতীত calculus এর সাহায্যে একটি সমীকরণের দ্বারা যে বক্ররেখা বুঝায় তাহা সম্বন্ধে অধিক তথ্য পাওয়া যায়। যেমন calculus প্রয়োগকালে (points of Inflexion), (Maxima and Minima) গুরুমান ও লঘুমান, (double points) দ্বিবিন্দু, (asympotes) অসীমতটরেখা ইত্যাদি তথ্য জানার ফলে একাধিক বিন্দু ছাড়াও একটি সমীকরণ হইতে বক্ররেখার ধারণা ও ইহার রূপরেখা তৈয়ারী করা যায়। তবে caulculus এর প্রয়োগ করার পূর্বে উল্লেখিত বিষয় সম্বন্ধে ছাত্রদের জানা থাকিতে হইবে। এই বিষয়গুলি পূর্ববর্তী অধ্যায়গুলিতে বিষদভাবে ব্যাখ্যা করা হইয়াছে।

সমীকরণ হইতে বক্ররেখার আকার জানার জন্য এবং ইহাকে অঙ্কন করার কয়েকটি নিয়ম নীচে দেওয়া হইল।

7.1. For Cartesian Equation

1. (i) If the equation of a curve remains unchanged when y is changred into $-y$, the curve is symmetrical about the x axis i. e. if the equation contains only even powers of y একটি সমীকরণে যদি y এর শক্তি যুগ্ম হয় তাহা হইলে বক্ররেখাটি x অক্ষরেখার সহিত প্রতিসাম্য হইবে। অথবা সমীকরণে y এর পরিবর্তে $-y$ বসাইলে যদি সমীকরণের আকারের কোন পরিবর্তন না হয় তাহা হইলেও বক্ররেখাটি x অক্ষরেখার প্রতিসাম্য হইবে।

$\therefore$ $y^2=4ax$, $y^2=x^3$ etc,

(ii) If the equation of the curve remains unchanged when x is changed into x or, if the equations only even powers of x, then the curve is symmetrical about the y—axis i e. if (x, y) lies on the curve, then $(-x_1, y)$ also lies on the curve. (সমীকরণে যদি x এর পরিবর্তে $-x$ বসান হয় এবং তাতে যদি সমীকরণের কোন পরিবর্তন না হয় অথবা সমীকরণে যদি শুধু x এর যুগ্ম শক্তি থাকে, তখন বক্ররেখাটি $-y$ অক্ষরেখায় প্রতিসাম্য হইবে। (x, y) যদি বক্ররেখার উপর থাকে তাহা হইলে $(-x, y)$ বিন্দুও বক্ররেখার উপর থাকিবে। $x^2=y$, $x^2=y^3$, $x^2y+x^2-y=0$. ইত্যাদি।

(iii) If the equation of the curve remain un changed when both x and y are changed into $-x$ and $-y$, the curve is symmetrical in opposite quadrant.

(কোন বক্ররেখার সমীকরণে (x, y) এর পরিবর্তে $(-x, -y)$ বসাইলে সমীকরণের যদি কোন পরিবর্তন না হয় তাহা হইলে বক্ররেখাটি তাহার বিপরীত কোয়াড্রেন্টে প্রতিসাম্য হইবে।

(iv) If the eq.. of the curve remains unchanged when x and y are interchanged the curve is symmetrical about the line $y=x$. If the point (x, y) lies on the curve, then (y, x) also lies on the curve. (কোন সমীকরণে x এর পরিবর্তে y এবং y এর পরিবর্তে x বসাইলে; সমীকরণের যদি পরিবর্তন না হয়; তাহা হইলে বক্ররেখাটি $y=x$ সরলরেখার প্রতিসাম্য হইবে। (x, y) বিন্দু বক্ররেখায় থাকিলে, (y, x) বিন্দুও ঐ বক্ররেখায় থাকিবে। যেমন $x^3+y^3=3x.\ xy$.

2. **Nature of the origin ; (মূল বিন্দুর প্রকৃতি) :—**

সমীকরণে যদি ধ্রুব সংখ্যা (constant) না থাকে, তাহা হইলে বক্ররেখাটি মূল বিন্দু দিয়া যাইবে। এইরূপ সমীকরণে মূলবিন্দুতে স্পর্শক বাহির করিতে হইবে (Taugents at the origin) যদি মূল বিন্দুতে দুইটি স্পর্শক, বা অধিক স্পর্শক থাকে তাহা হইলে মূল বিন্দুতে বিশেষ বিন্দুর (singularity) প্রকৃতি জানিতে হইবে এবং স্পর্শকের সম্পর্কে মূল বিন্দুতে বক্ররেখার অবস্থান বাহির করিতে হইবে। Tangents and Normals এবং singularities অধ্যায় দুইটি আলোচনা করিতে হইবে। $y=x^3-3ax^2$. Tangents at the origin is given by $y=0$. Near the origin, the eq is roughly $y=-3ax^2$ which shows that it will be a parabola along negative $y-$axis as axis of the parabola.

3. **Intersection with the axes : (অক্ষরেখার সহিত বক্ররেখার ছেদন) :**
সমীকরণে $x=0$ এবং $y=0$ বসাইয়া অক্ষরেখাকে বক্ররেখা যে সমস্ত বিন্দুতে ছেদ করে তাহাদের নির্ণয় করিতে হইবে। প্রয়োজনে $y=x$ বা $y=-x$ সমীকরণে বসাইয়া ইহাদের ছেদ বিন্দুর অবস্থান বাহির করিতে হইবে।

4. **Asymptotes : (অসীম তটরেখা) :** বক্ররেখার অসীমরেখা বাহির করিতে হইবে এবং বক্ররেখা ও অসীমতটরেখার সম্পর্কে জানিতে হইবে।

5. Regions containing the curves (**বক্ররেখার অধিনস্ত এলাকা সমূহ**) **:** প্রদত্ত সমীকরণকে পৃথকভাবে x অথবা y এর জন্য প্রকাশ করিয়া x এবং y এর ধনাত্মক ও ঋণাত্মক মানগুলি পর্যালোচনা করিতে হইবে। যদি কোন মানের জন্য রাশিমালাটি কাল্পনিক হয় তাহা হইলে বক্ররেখা ঐ মানের বাহিরে সীমাবদ্ধ হইবে।

ইহার বিশ্লেষণ উদাহরণে দেখান হইলে, $y^2(a-x)=x^2(a+x)$.

or, $y^2=x^2\dfrac{a+x}{a-x}$ যদি x ঋণাত্মক এবং $x<-a$ তখন y কাল্পনিক হইবে,

অর্থাৎ $x=-a$ পরে বক্ররেখার কোন অংশ থাকিবে না।

6. Stationary values (**নিশ্চল মানসমূহ**) **:**—প্রদত্ত সমীকরণের $\dfrac{dy}{dx}$ বা $\dfrac{dx}{dy}$ নির্ণয় করিয়া (Tangents) x অক্ষ বা y অক্ষরেখা বরাবর সমান্তরাল স্পর্শকগুলি নির্ণয় করিতে হইবে। ইহাতে y এর মান বাড়িতে বা কমিতে থাকিলে তাহা হইতে বক্ররেখার আকার সম্বন্ধে জানিতে সহজ হয়।

7. Inflexions and other Singularities.

Find the points of inflexions (**নিশ্চল বিন্দু**) and other Singularities. If the equation is conplicated, then examine only the nature of the curve.

8. Approximation (**ধারণাযোগ্য**) **:**—সমীকরণে x এবং y এর জন্য খুব ছোট মান বসাইয়া x এবং y এর বৃহত্তম মানগুলি বর্জন করা যায় এবং তাহাতে সমীকরণটি মূল বিন্দুর নিকটে কি আকার হয় তাহা অনুমান করা যায়। উদাহরণ স্বরূপ দেখান যায় $y^2=x+y^3$ | এই সমীকরণ হইতে x-এর জন্য ক্ষুদ্র মান ধরিলে, x^3 এর মান বাদ দিয়া $y^2=x$ মূল বিন্দুর নিকট প্রদত্ত বক্ররেখা সম্বন্ধে ধারণা করা যায়।

Art. 16. (B) For Polar Curves

1. To Examine the Symmetry (প্রতি সাম্যতা পরীক্ষা)

(a) যদি θ এর জন্য $-\theta$ সমীকরণ কোন পরিবর্তন না হইলে, বক্ররেখাটি আদিরেখা (Initial line) বরাবর প্রতিসাম্য (Symmetry) হইবে।

যেমন $r^2=a^2\cos^2\theta$, $r=a\cos\theta$, $a=r(1+\cos\theta)$

(b) যদি r এর পরিবর্তে $-r$ সমীকরণে বসান হয় এবং সমীকরণের কোন পরিবর্তন না হয় তাহা হইলে মেরুর (Pole)-এর বরাবর বক্ররেখাটি প্রতিসাম্য হইবে, θ এর পরিবর্তে $\pi+\theta$ বসাইলেও বক্ররেখাটি মেরুর বরাবর প্রতিসাম্য হইবে।

(c) যদি r এর জন্য $-r$ ও θ এর জন্য $-\theta$ সমীকরণে বসান হয় এবং সমীকরণের কোন পরিবর্তন না হয় তাহা হইলে বক্ররেখাটি মেরুগামী আদিরেখার (Initial line) উপর লম্বের বরাবর প্রতিসাম্য হইবে। (Symmetrical about a line through the pole perpendicular to the initial line).

2. **Regions** (**এলাকা**) **:**—বক্ররেখাটি যে সমস্ত এলাকায় (Regions) থাকেনা তাহা নির্ণয় করিতে হইবে। θ এর মানের জন্য r^2 ঋণাত্মক হইলে অথবা r এর কোন মানের জন্য কোন এলাকার বাহিরে বক্ররেখা যাইতে পারে না তাহা নির্ণয় করিতে হইবে।

যেমন $r^2=a^2\cos 2\theta$ এখানে বক্ররেখা $\theta=\pi/4$ এবং $\theta=3\pi/4$ এর মধ্যে থাকেনা, আবার $\theta=5\pi/4$ এবং $\theta=7\pi/4$ এর মধ্যে ও থাকে না। r এর মান a হইতে বড় হইতে পারেনা এবং বক্ররেখাটি $r=a$ এর বাহিরে যাইতে পারে না, ইহা একটি $(r=a)$ বৃত্তের মধ্যে থাকিবে।

3. **মূলবিন্দু Origion) :**— যদি : $r=0$, $\theta=a$-এর জন্য $\theta=a$ সরলরেখাটি মূলবিন্দুতে স্পর্শক হইবে।

4. **Asympotes** (**অসীম তটরেখা**) **:**—বক্ররেখার সমীকরণের যদি অসীম তটরেখা থাকে, তাহা নির্ণয় করিতে হইবে।

5. **কোণ ϕ (Angle)** ; স্পর্শক ও রেডিয়াস ভেক্টরের মধ্যে কোণ ϕ নির্ণয় করিতে হইবে।

6. **Variation of r and θ** (**r এবং θ এর পরিবর্তন**) **:**—

r এবং θ এর মানের জন্য একটা ছক্ অঙ্কন করিতে হইবে' এই ছক্ হইতে বক্ররেখাটি অঙ্কন করা যায়।

মন্তব্য :—প্রদত্ত সমীকরণ হইতে বক্ররেখা অঙ্কনের জন্য উল্লেখিত সকল বিষয়গুলির প্রয়োগ প্রয়োজন হয় না। বক্ররেখা অঙ্কনের একটা নিজস্ব পদ্ধতি আনা যায় এবং তিন চারটি নিয়মের প্রয়োগের মাধ্যমে বক্ররেখা অঙ্কন করা যায়। ছাত্রদের বক্ররেখা অঙ্কন সম্বন্ধে ধারণা করার জন্য কয়েকটি সমীকরণের বক্ররেখার অঙ্কন দেখান হইল।

Exercise XVI

Ex. 1. Trace the curve represented by the equation

$$ay^2 = x^2\,(a-x).$$

1. **Symmetry :**— As the equation involves only even powers of y, so the curve is symmetric about x-axis,

2 Put $x=0$ then $y=0$, Put $y=0$. then $x=0, a$; the curve meets the axis at (0, 0) and (a, 0).

3. **Tangent at the origin :**– Equate the higest degree terms of the equation to zero, then $ay^2 - ax^2 = 0$ or, $y = \pm x$.

Tangent at $(a, 0)$:—Put $x = a + x$, $y = 0 + y$ in the equation. Then $ay^2 = (a+x)^2\,(a-a-x)$ or, $ay^2 = (x+x)^2\,(-x)$, Equate the lowest degree terms to zero. Thus $-a^2 x = 0$ or, $x=0$ *i.e.* $x=a$ is tangent at $(a, 0)$.

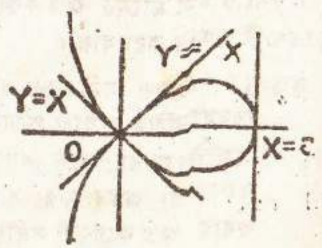
4. **Asymptotes :**— There is no asymptote of the curve.

5. **Singular points :**– The curve has a pair of real & distinct tangents at the origin. Hence the curve has n nodes at the origin.

6. **Regions :** $ay^2 = x^2(a-x)$ or, $y = \pm x\sqrt{\left(\dfrac{a-x}{a}\right)}$

If $x<a$, then y is real, y increases with the increasing values of x first and then gradually decreases and $y=0$ when $x=a$. If $x>a$, then y is imaginary. There lies no branch of the curve beyond $x=a$.

For negative values of x i. e. along the negative side of x-axis y increases with the increasing value of x. The shape of the curve is as shown in the figure 32, with the facts discussed above.



Ex. 2. Trace the curve $x^2y^2 = (1+y)^2\,(4-y^2)$ ... ... 1)

1. **Symmetry :**—The eq (1) involues only even powers of x So the curve is symmetric about y– axis.

2. **Where curve meets the axes :**—Put $x=0$ in (1)

$(1+y)^2(4-y^2) = 0$ or, $y^2 = \pm 2$, -1 *i. e* the curve meets the y—axis at (0, 2), (0, –2), (1,—1).

3. **Tangents at these point :**–

Differentiate (1) logarithmically.

$$\frac{2}{x} + \frac{2}{y}\frac{dy}{dx} = \left(\frac{2}{1+y} - \frac{2y}{4-y^2}\right)\frac{dy}{dx}$$

$$\text{or, } \frac{dy}{dx} = -\frac{y(1+y)(4-y^2)}{(y^2+4)x} \text{ or, } \left(\frac{dy}{dx}\right)^2 = \frac{y^2(1+y)^2(4-y^2)^2}{(y^3+4)^2x^4}$$

$$= \frac{y^2(1+y)^2(4-y^2)^2\,y^2}{(y^3+4)^2(1+y)^2(4-y^2)} = \frac{y^4(4-y)^2}{(y^3+4)^2}$$

$$\therefore \quad \frac{dy}{dx} = \pm\frac{y^2\,(4-y^2)}{(y^3+4)}$$ See fig. 33.

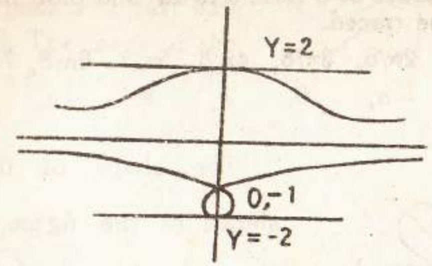


Fig. 33

The conchoid of Nicomedes,

Ex. 3. Trace the curve whose equation is

$$r = a\,(1+\cos\theta) \quad \cdots \quad \cdots \quad (1)$$

1. **Symmetry :** If θ is changed to $-\theta$ in (1), the equation will not change. Hence the curve is symmetrical about the initial line.

2. $\theta=0$ $\quad \pi/6 \quad \pi/4 \quad \pi/3 \quad \pi/2 \quad 2\pi/3 \quad \pi$
$r=2a \quad a(1+\sqrt{3}/2) \quad a(1+1/\sqrt{2}) \quad 3a/2 \quad a \quad \frac{1}{2}a \quad 0$

As θ increases from 0 to π r. gradually decreases from $2a$ to zero.

Since the curve is symmetrical about the initial line, so we get the same type of curve when θ varies from π to 2π.

The shap of curve is shown in the fig 34. The name of the curve is **Cardioide.**

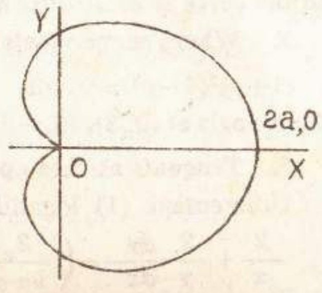


Fig. 34

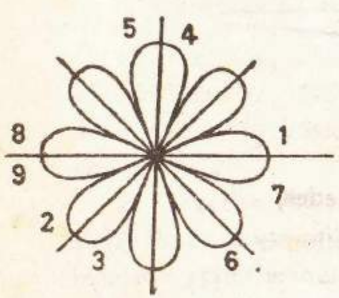
Note : $r=a(1-\cos\theta)$ Proceed as in Ex. 3. The shape of the curve will be the reverse of the figure shown in fig. 34.

Ex. 3. (*a*) Trace the curve $r=a\,(1+\sin\theta)$.
which is also a cardioide.

Ex. 4. Trace the curve $r=a\cos 4\theta$.

Let us take values of θ from 0 to 2π and plot these points. The curve will be traced.

$\theta=0:\ \pi/8;\ 2\pi/8,\ 3\pi/8,\ 4\pi/8,\ 5\pi/8,\ 6\pi/8,\ 7\pi/8,\ 8\pi/8 \ldots\ldots$
$r=a.\quad 0,\quad -a,\quad 0,\quad a,\quad 0,\quad -a,\quad 0,\quad a$



The shape of the curve is shown in the figure. There will be eight loops. *i. e,* twice as much loops if the θ is even multiple.

Fig. 35.

Notes :—If $r=a\cos 3\theta$ or, $r=a\sin 3\theta$ there will be only three loops.

If $r=a\cos n\theta$ or, $r=a\sin n\theta$, the number of loops will be $2n$ if n is even and number of loops will be only n if n is odd.

These curves are generally called Rose petales

Ex. 5. Trace the curve $r^2=a^2\cos 2\theta$

(The curves is called lemniscate of Bernouli)

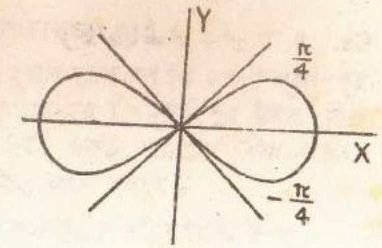
Symmetry :— The curve is symmetric about the initial line as by putting $\theta=-\theta$ the equation will not change.

The curve is also symmetric about the line perpendicular to the line and passing through the pole as by changing $r=-r$ and $\theta=-\theta$ the curve will not change.

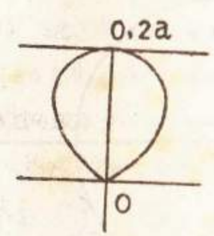
2. Form table of values of θ.

$\theta=0,\ \pi/6 \quad \pi/4 \quad 3\pi/4 \quad 5\pi/4 \quad \pi \cdots\cdots$
$r=a \quad a/\sqrt{2} \quad 0 \text{ imaginary } 0 \quad a/\sqrt{2}. \quad a \cdots\cdots$

Now plot these points. The portion of the curve above the initial line for $\theta=0$, $\theta=\pi$ will be obtained. As the curve is symmetric the remaining part will be easily traced. The shape of the curve is shown in the figure 36.



The cartesian eq. of the above curve is Fig. 36
$a^2y^2=x^2\,(a^2-x^2)$



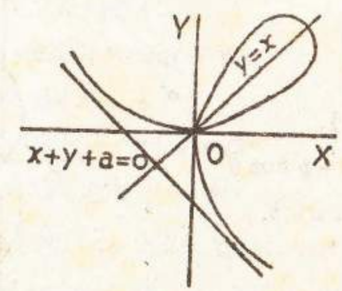


Fig. 37. Fig. 38

Ex. 6. $a^2x^2=y^3(2a-y)$ **Ex.** 7. $x^3+y^3=3axy$

Ex. 8. $x^6+y^6=a^6x^2y^2$

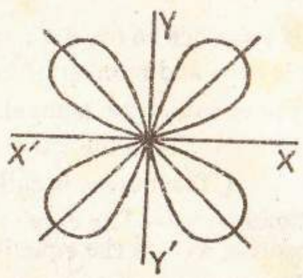


Fig. 39

Ex. 9. $x^5+y^5=5a^2x^2y$

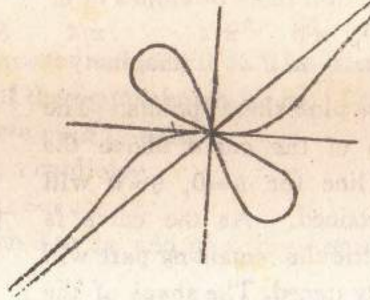


Fig. 40

Ex. 10. $x^2y^2=(a+y)^2(a^2-y^2)$ Ex. as in Ex. 2.

Ex. 11.

$r=a+b\cos\theta$

when $a<b$,

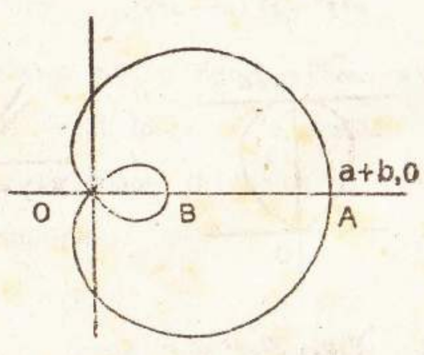


Fig. 41

Ex. 12. $r=a\cos 2\theta$. Fig. as in Ex. 8.

Ex. 13. $r=a\sin 3\theta$

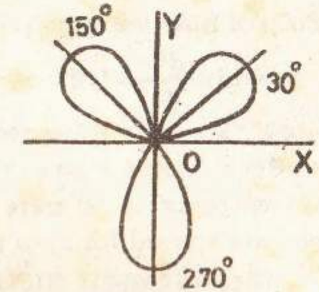


Fig. 42

Ex. 14. Trace the cruve $y=x^3-3ax^2$

($y=x^3-3ax^2$ এর দ্বারা নির্দ্দেশিত বক্ররেখাটি অঙ্কন কর।)

(i) প্রতিসাম্যতা :—এই সমীকরণে প্রতিসাম্যতা নাই।

(ii) বক্ররেখাটি মূলবিন্দু দিয়া যায়। মূলবিন্দুতে স্পর্শক $y=0$ মূলবিন্দুর খুবই নিকটে সমীকরণটি $y=-3ax^2$ এর ন্যায় হয়। মূলবিন্দুর উভয় দিকে স্পর্শকের নীচে বক্ররেখাটি থাকিবে কারণ ইহা একটি পরাবৃত্ত (Parabola বুঝায় যাহার অক্ষরেখা y—অক্ষরেখার নীচের অংশ হইবে।

(iii) বক্ররেখাটি x—অক্ষরেখাকে ছেদ করিলে, $y=0$ হইলে, তখন $x^3-3ax^2=0$ বা, $x^2(x-3a)=0$ বা, $x=0,\ 3a$ অর্থাৎ বক্ররেখাটি $(0, 0)$ ও $(3a, 0)$ দিয়া যাইবে।

(iv) বক্ররেখাটির কোন অসীমতটরেখা নাই।

(v) প্রত্যেক x এর মানের জন্য y এর মান পাওয়া যায়।

$y\to\infty$ যদি $x\to\infty$, এবং $y\to-\infty$ যদি $x\to-\infty$

সমীকরণকে $y=x^2(x-3a)$ ভাবে লিখিলে দেখা যায় যে $x<3a$ হইলে y ঋণাত্মক এবং $x>3a$ হইলে, y ধণাত্মক।

(vi) $\frac{dy}{dx}=3x(x-2a)=0$ হইলে; $x=0, 2a$. সুতরাং $x=0$ হইলে $y=0$ হয় ; $x=2a$ হইলে $y=-4a^2$ অর্থাৎ $(0, 0)$ এবং $(2a, -4a^2)$ বিন্দুতে স্পর্শক x—অক্ষরেখার সহিত সমান্তরাল হইবে।

(vii) $\frac{d^2y}{dx^2}=6(x-a)$ যাহা $x=a$ হইলে শূণ্য হয় এবং এই বিন্দুতে Point of Inflexion হইবে।

(viii) $\frac{d^2y}{dx^2}>0$ যদি $x>a$ এবং $\frac{d^2y}{dx^2}<0$ যদি $x<a$ অর্থাৎ x এর মান a হইতে বেশী হইলে স্পর্শক ধনাত্মক কোণ···অর্থাৎ বক্ররেখাটি বাড়িতে থাকিবে। আবার x এর মান a হইতে ছোট হইলে, স্পর্শকটি ঋনাত্মক কোণ (x—অক্ষরেখার সহিত) অর্থাৎ বক্ররেখাটি নীচে নামিবে। The cruve is concave upward for $x>a$ and convex upward for $x<a$.

বক্ররেখাটির আকার পার্শ্বের চিত্রে দেখান হইল।

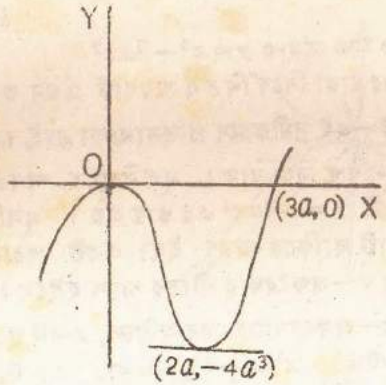


Fig. 43

Ex. 15. $y(1-x^2)=x^4$ সমীকরণের দ্বারা প্রকাশিত বক্ররেখার চিত্র অঙ্কন কর।

(i) বক্ররেখাটি y—অক্ষরেখা বরাবর প্রতিসাম্য, কারণ x এর শক্তি যুগ্ম।

(ii) বক্ররেখাটি মূল বিন্দু দিয়া যায়, ইহাকে $x^2y+x^4-y=0$ লিখিলে, $y=0$ মূলবিন্দুতে স্পর্শক হয়।

মূল বিন্দুর নিকটে x, y এর ক্ষুদ্রমানের জন্য x^2y পদ বাদ দিলে, $x^2-y=0$ বা, $y=x^2$ যাহা একটি পরাবৃত্ত (Parabola) বুঝায়। ইহার বক্রতা y অক্ষরেখার উপরের দিকে হইবে। অতএব বক্ররেখাটি মূলবিন্দুতে স্পর্শকের উপরের দিকে থাকিবে।

(iii) বক্ররেখাটি মূলবিন্দু ব্যতীত অন্য কোন বিন্দুতে x এবং y অক্ষরেখাকে ছেদ করিবেনা।

(iv) $x=\pm1$ এবং $y=1$ ইহাদের অসীমতটরেখা।

(v) সমীকরণকে $x^2=\frac{y}{1+y}$ লিখিলে দেখা যায় যে y এর মান 0 এবং 1 এর মধ্যে থাকেনা। অসীমতটরেখা $y=-1$ নীচের দিক হইতে অগ্রসর হইবে। y ধনাত্মক হইবে যদি x এর মান 1 হইতে ছোট হয় এবং ঋণাত্মক হইবে যদি x এর মান 1 হইতে বড় হয়।

সুতরাং $x=1$ অসীমতটরেখা উপর হইতে বাম দিকে আসিয়া নীচে ডানদিকে বক্ররেখার অসীমতটরেখা হইবে।

আবার অসীমতটরেখা $x=-1$ এর জন্য উপর হইতে ডানদিকে আসিয়া নীচে বক্ররেখাকে বাম দিকে রাখিবে। চিত্রে ইহা দেখান হইল।

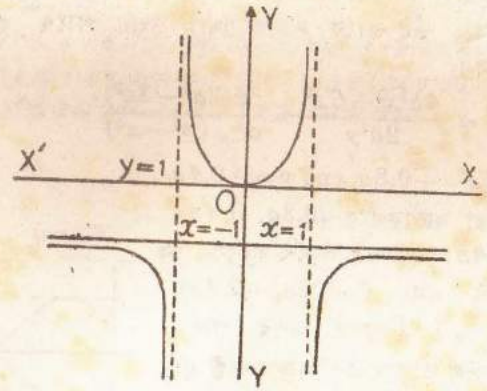


Fig. 44

Ex. 16. $a^2x^4-x^6=a^4y^2$ দ্বারা প্রকাশিত বক্ররেখাটি অঙ্কন কর।

(i) বক্ররেখাটি উভয় অক্ষরেখার প্রতিসাম্য যেহেতু x এবং y এর প্রত্যেকের শক্তি যুগ্ম।

(ii) ইহা মূলবিন্দু দিয়া যায় এবং মূলবিন্দুতে স্পর্শক $y^2=0$ মূলবিন্দুকে দ্বি-বিন্দু (Doubl- point) বলা হইবে এবং এই বিন্দুকে cusp বলা হইবে। মূলবিন্দুর খুব নিকটে x এবং y ক্ষুদ্র মানের জন্য সমীকরণটি $a^4y^2=a^2x^4$ or, $x^2=\pm ay$.

$x^2=ay$ এবং $x^2=-ay$ ধরিলে মূল বিন্দুর নিকটে y অক্ষরেখাকে অক্ষ ধরিয়া দুইটি (Parabola) পরাবৃত্ত পাওয়া যায়। পরাবৃত্ত দুইটি পরস্পরকে মূলবিন্দুতে স্পর্শ করিবে।

(iii) বক্ররেখাটি x-অক্ষরেখাকে $(0, 0)$ এবং $(\pm a, 0)$ বিন্দুতে ছেদ করিবে। ইহা y-অক্ষরেখাকে শুধু মূল বিন্দুতে ছেদ করিবে।

(iv) বক্ররেখাটির অসীমতটরেখা নাই।

(v) সমীকরণটিকে লিখা যায় $a^4y^2=x^4(a^2-x^2)$; $x \not> a$ (x-এর মান a হইতে

(vi) সমীকরণটিকে লিখা যায় $a^4y^2=x^4(a^2-x^2)$, $x \not> a$ (x-এর মান a হইতে বড় হইতে পারবেনা) $x=a$, $x=-a$ এর বাহিরে বক্ররেখার কোন অংশ থাকিবেনা কারণ $y=\pm\frac{x^2}{a^2}\sqrt{(a^2-x^2)}$, x-এর মান a-হইতে বড় হইলে y-কাল্পনিক হইবে। x-এর মান 0 হইতে বাড়িতে থাকিলে y-এর মান বাড়িতে থাকে এবং বাড়িয়া আবার কমিতে থাকিবে এবং $x=a$ হইলে y-এর মান শূন্য হইবে। এই ভাবে y অক্ষরেখার উভয় পার্শ্বে একরকম বক্ররেখা পাওয়া যাইবে।

(vi) $\frac{dy}{dx}=\frac{4a^2x^2-6x^6}{2a^4y}=\frac{x(2a^2-3x^2)}{a^2\sqrt{(a^2-x^2)}}=0$ হইলে,

$x=\pm\sqrt{\frac{2}{3}}a=\pm0.8a$ and $y=\pm0,4a$

ইহাদের দ্বারা বক্ররেখার $(0.8a, 0.4a)$, $(-0.8a, 0.4a)$ তে একটি স্পর্শক বক্ররেখাকে স্পর্শ করিবে এবং $(-0.8a, -0.4a)$, $(0.8a, -0.4a)$ বিন্দুতে অপর স্পর্শক বক্ররেখাকে স্পর্শ করিবে এবং বক্ররেখাটি এই দুই স্পর্শক এবং $x=a$, $x=-a$ এর ভিতরে থাকিবে। উল্লিখিত বিষয় হইতে বক্ররেখাটি অঙ্কন করা যায়।

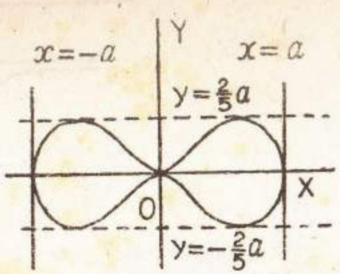


Fig. 45.

Ex. 17. $x^2y^2=x^2+1$ এর সমীকরণে বক্ররেখা অঙ্কন কর।

(i) x এবং y এর শক্তি যুগ্ম. সুতরাং বক্ররেখাটি উভয় অক্ষরেখা বরাবর প্রতিসাম্য।

(ii) ইহা মূল বিন্দু দিয়া যায় না।

(iii) ইহা অক্ষরেখাকে ছেদ করে না।

(iv) $x^2y^2=x^2+1$ বা, $x^2(y^2-1)=1$ লিখিলে, ইহার অসীমতটরেখাগুলি $y^2-1=0$ $y=1$বা, $y=-1$ হইবে, অপর অসীমতটরেখা, $x=0$ হইবে। এইগুলি চিত্রে অঙ্কন করিতে হইবে।

(v) $x^2y^2=x^2+1$

or, $y^2=\frac{x^2+1}{x^2}$

or, $y=\pm\frac{\sqrt{(x^2+1)}}{x^2}$ ...... (1)

আবার $x^2(y^2-1)=1$ ... ... (2)

এখানে y-এর মান 1 হইতে ছোট হইবে কারণ y-এর মান $+1$ অথবা -1হইতে বড় হইবে x-এর মান কাল্পনিক হইবে। অর্থাৎ $y=1$, $y=-1$ এই দুইরেখার মধ্যে বক্ররেখা থাকিবে না।

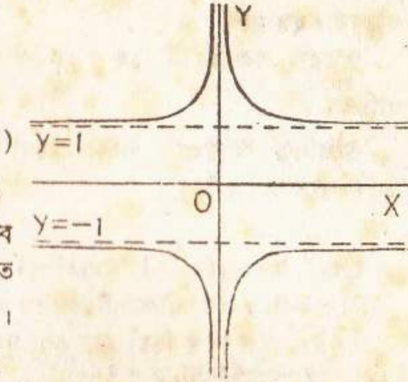


Fig. 46.

সমীকরণ (1) হইতে, x এর মান শূণ্য হইতে ∞ হইলে y এর মান $\pm\infty$ হইতে $y=\pm1$ হইবে, আবার x এর মান $-\infty$ হইতে শূণ্য হইলে y এর মান ±1 হইবে।

Ex. 18 $x^2y^2=x^2-1$ এর বক্ররেখাটি অঙ্কন কর।

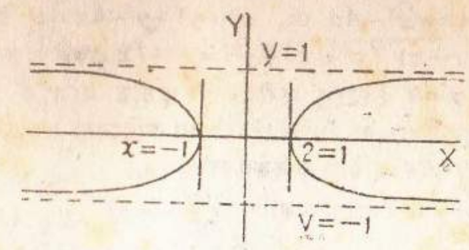


Fig. 47.

Ex. 19. $x^4+y^4=4a^2xy$ সমীকরণের বক্ররেখার চিত্র অঙ্কন কর।

(i) $x=0$, $y=0$ দিলে সমীকরণটি সিদ্ধ হয়। সুতরাং বক্ররেখাটি মূলবিন্দু দিয়া যাইবে।

(ii) x এবং y কে পরিবর্তন করিলে সমীকরণের পরিবর্তন হয় না। অতএব $y=x$ সরলরেখায় ইহা প্রতিসাম্য।

(iii) মূলবিন্দুতে স্পর্শগুলি, $xy=0$ বা, $x=0$, $y=0$ অর্থাৎ অক্ষরেখা দুইটি মূলবিন্দুতে বক্ররেখার স্পর্শক। যেহেতু মূলবিন্দুতে দুইটি পৃথক স্পর্শক পাওয়া যায়, অতএব মূলবিন্দুটি বক্ররেখার জন্য একটি গ্রন্থি (Node) হইবে।

(iv) ইহার অসমীতটরেখা নাই।

(v) বক্ররেখাটি $y=x$ সরলরেখায় ছেদ করে অতএব

$x^4+y^4=4a^2xy$ বা, $2y^4=4a^2y^2$ বা, $y^2=2a^2$ বা, $y=\pm\sqrt{2}a$; $y=x$ রেখাকে $(\pm\sqrt{2}a, \pm\sqrt{2}a)$ and $(0, 0)$ বিন্দুতে ছেদ করিবে।

(vi) x এবং y এর ধনাত্মক ও ঋণাত্মক উভয় মানের জন্য সমীকরণের মান পরিবর্তন হয় না।

সুতরাং বক্ররেখাটি ১ম ও ৩য় কোয়াড্রেণ্টে থাকিবে।

উল্লিখিত বিষয়গুলি পর্য্যালোচনা করিয়া চিত্রটি পার্শ্বরূপ হইবে।

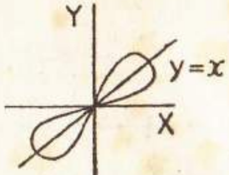


Fig. 48.

Ex. 20. $y(y^2-1)=x(x^2-4)$ সমীকরণের চিত্র অঙ্কন কর।

(i) $x=0, y=0$ সমীকরণটিকে সিদ্ধ করে। ইহা মূলবিন্দুর বরাবরে প্রতিসাম্য।

(ii) ইহা মূল বিন্দু দিয়া যায় এবং মূল বিন্দুতে স্পর্শক [$y^3-y=x^3-4x$] হইবে $-y=-4x$ বা, $y=4x$.

(iii) $y=0$ হইলে $x(x^2-4)=0$ বা, $x=0, \pm 2$
$x=0$ হইলে $y(y^2-1)=0$ বা, $y=0, \pm 1$

ইহা $(0, 0)$, $(2, 0)$ $(-2, 0)$ বিন্দুতে অক্ষরেখা এবং বিন্দুতে $(0, 0)$, $(0, 1)$. $(0, -1)$ বিন্দুতে y—অক্ষরেখাকে ছেদ করে।

(iv) $y^3-y=x^3-4a$ or, $x^3-y^3+y-4x=0$ ইহার অসীমতটরেখার মধ্যে $(y-x)(x^2+xy+y^2)=y-4x$ একটি প্রকৃত অসীমতটরেখা $y-x=0$ বা $y=x$ যেহেতু সমীকরণটি তৃতীয় মাত্রার (3rd degree) সেইহেতু ইহা বক্ররেখাকে দ্বিতীয়বার ছেদ করিবেনা। মূলবিন্দুতে বক্ররেখা অসীমতটরেখাকে ছেদ করিয়া অতিক্রম করিবে।

(v) $\frac{dy}{dx}(3y^2-1)=3x^2-4$ or, $\frac{dy}{dx}=\frac{3x^2-4}{3y^2-1}$; $\frac{dy}{dx}=0$ ধরিলে,

$$x=\pm 2/\sqrt{3}=\pm 1.16.$$

ইহাতে $x=1.16$, $x=-1.16$ এর জন্য দুইটি স্পর্শক যাহা x অক্ষরেখার সহিত সমান্তরাল হইবে।

উল্লিখিত বিষয়গুলি বিবেচনা করিলে বক্ররেখাটির রূপ চিত্রে দেওয়া হইল।

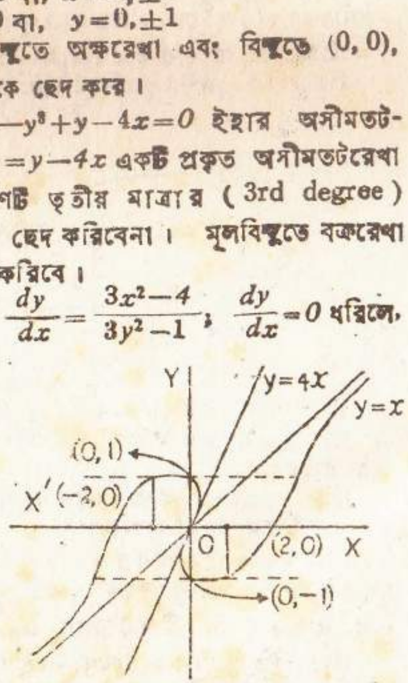


Fig. 49

Ex. 21. $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$

These equations are known as the equations of a **Cycolid.**

Cycolid : – When a circle rolls in a plane along a given straight line, the locus traced out by any point on the circumference of the rolling circk is called a cycloid.

Let P be a point on the circle which is called the generating circle. The path of the point P which rolls without sliding on a straight line is called a cycloid. The point P is originally at O Let O be the origin and OMX, the axis of x.

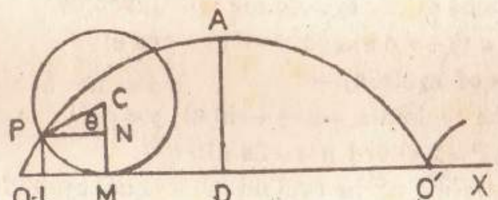


Fig. 50

Let a be the radius of circle the centre of the circle C.

$\angle PCN=\theta$ and the the co-ordinates of P be (x, y)

Therefore. $OM=\text{arc } PM=a\theta$

$x=OL=OM-LM=OM-PN=a\theta-a\sin\theta$

$y=PL=CM-CN=a-a\cos\theta=a(1-\cos\theta)$

Thus the parametric equations of the cycloid are

$$\mathbf{x}=\mathbf{a}(\theta-\sin\theta) \text{ and } \mathbf{y}=\mathbf{a}(1-\cos\theta)$$

The point A is the highest point on the curve from the base OMX. The point A is called the vertex

For the vertex, $y=a(1-\cos\theta)$, $\cos\theta$ should be minimum, *i e,* $\cos\theta=-1$ or, $\theta=\pi$.

$\therefore\ AD=y=a(1+1)=2a$

Co-ordinates of vertex $(a\pi, 2a)$

For O and O', $y=0$ or, $a(1-\cos\theta)=0$, or, $\theta=0$ or, π, If we consider the vertex $A(a\pi,\ 2a)$ as the origin and the tangent at A as the x—axis, the shape of the cycloid is shown below.

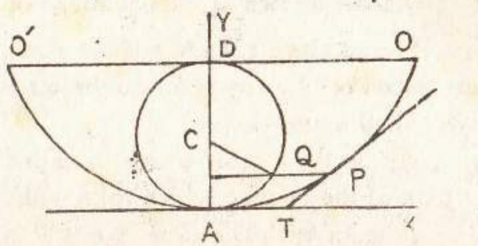


Fig. 51

The equations of the cycloid are now given by

$x=a\ (\theta+\sin\theta)$ and $y=a\ (1-\cos\theta)$

Properties of cycloid,

(i) For the cycloid $x=a\ (\theta+\sin\theta)$, $y=a\ (1-\cos\theta)$
Arc AP = 2 chord AQ = 2a sin θ/2.

(ii) The evolute of the cycloid is an equal cycloid.

(iii) For cycloid $x=a\ (\theta-\sin\theta)$, $y=a\ (1-\cos\theta)$
radius of curvature = twice the length of the normal.

22. $y^3(2a-x)=x^3$ or, $r=\dfrac{2a\ \sin^2\theta}{\cos\theta}$ is Cissoid of Diocles, Fig. 45

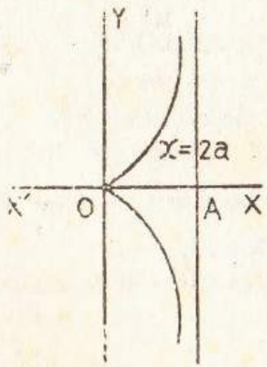


Fig. 52

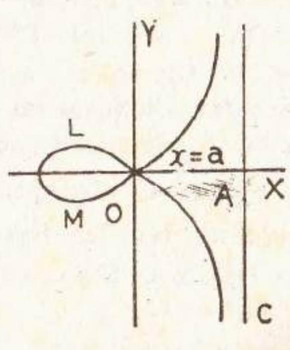


Fig. 53

23. $y^2(a-x)=x^2(a+x)$ Strophoid fig $\qquad r=ae^{\theta\cot\alpha}$ or, $r=ae^{m\theta}$

Logarithmic Spiral or, Equiangular spiral properties :—

(i) $\phi=\alpha$; the angle between any tangent at any point and the radius vector of that point is constant.

(ii) The pedal, inverse, polar reciprocal and evolute of this curve are all equiangular spirals.

(iii) The radius of curvature makes a right angle at the pole.

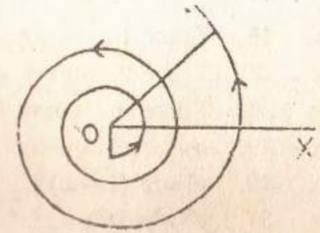


Fig 54

24. $r=a\theta$: Spiral of. Archimedes,

one of the main property of "Archimede's" spiral is that its polar subnormal is constant, fig. 18,

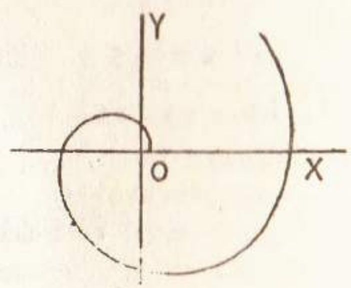


Fig. 55

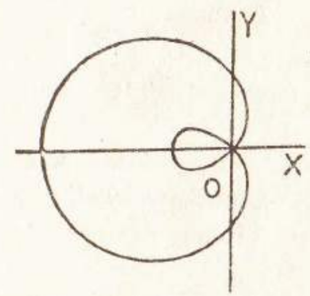


Fig 56

19. Limacon $r=a-b\cos\theta$: $a>b$

25. Probability curve

$y=e^{-x^2}$

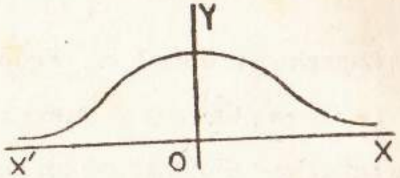


Fig. 57

Ex. 26. Trace the curve $x^4-y^4=x^2y+x^2-y^2$

Ex 27. Trace the curve $ay^2=x^2y+x^3$

Ex. 28. Trace the curve $r=a\ (2\cos\theta+\cos 3\theta)$

Ex. 29. $ay^2=x^2\ (x-a)$

Ex. 30. $y^2=x\ (1-x)^3$

Ex. 31. $y^2\ x^2+a^2)=a^2x^2$

Ex. 32. $y^2(x^2-a^2\ =a^2x^2$

Ex. 33. $x^2(x^2-4a^2)=y^2(x^2-a^2)$

Ex. 34. $r=2a\sin\theta$

Ex. 35. $r=a\cos 4\theta$

Ex. 36. $r=\cos 2\theta=a$

C. U. (S) 1989

1. (a) Define limit, continuity and differentiability of a function. Evaluate $\underset{x\to 0}{\text{Lt}}\dfrac{1-\cos x}{x^2}$.

(b) Consider the function $g\ (x)=\begin{cases} x\ ;\ 0\leq x\leqq \frac{1}{2} \\ 3-x\ ;\ \frac{1}{2}<x\leqq 3. \end{cases}$

(i) Find the domain and range of $g(x)$.

(ii) Determine whether $g(x)$ is continuous at $x=\frac{1}{2}$.

2. (a) Using definition, find the differential coefficient of $\log\cos x$.

(b) Find $\dfrac{dy}{dx}$ where—

(i) $y=(\sin x)^{\log x}\cot\{e^x\ (a+bx)\}$;

(ii) $x=a\ (\theta+\sin\theta)$, $y=a(1-\cos\theta)$.

3. (a) If $y=\sin\ (p\sin^{-1}x)$. Prove that

$(1-x^2)y_{n+2}=(2n+1)xy_{n+1}+(n^2-p^2)y_n$.

(b) Expand $\cos^{-1}x$ into an infinite series of ascending powers of x.

4. (a) State and prove Lagrange's Mean-Value theorem.

(b) If $p=x\cos\alpha+y\sin\alpha$ touches the curves

$$\left(\frac{x}{a}\right)^{n/n-1}+\left(\frac{y}{b}\right)^{n/n-1}=1,$$

prove that $p^n=(a\cos\alpha)^n+(b\sin\alpha)^n$.

১। (ক) একটি ফাংশনের সীমা, ছেদহীনতা এবং অন্তরীকরণ যোগ্যতার সংজ্ঞা দাও। $\underset{x\to 0}{\text{Lim}}\dfrac{1-\cos x}{x^2}$ এর মান নির্ণয় কর।

(খ) মনে কর, $g(x)=\begin{cases} x\ ;\ 0\leq x\leq \frac{1}{2} \\ 3-x\ ;\ \frac{1}{2}<x\leq 3 \end{cases}$ একটি ফাংশন।

(i) $g\ (x)$ ডোমেইন এবং রেইঞ্জ নির্ণয় কর।

(ii) $x=\frac{1}{2}$ বিন্দুতে $g(x)$ ছেদহীন কিনা পরীক্ষা কর।

২। (ক) সংজ্ঞা ব্যবহার করিয়া $\log\cos x$ এর অন্তরক সহগ নির্ণয় কর।

(খ) $\dfrac{dy}{dx}$ বাহির কর, যেখানে –

(i) $y=(\sin x)^{\log x}\cot\{e^x\ (a+bx)\}$;

(ii) $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$.

(৩) (ক) $y=\sin\ (p\sin^{-1}x)$ হইলে, প্রমাণ কর যে,

$(1-x^2)y_{n+2}=(2n+1)\ xy_{n+1}+(n^2-p^2)y_n$.

(খ) $\cos^{-1}x$ কে x এর ক্রমবর্ধমান সূচকের অসীম ধারায় প্রকাশ কর।

৪। (ক) ল্যাগরেইঞ্জের গড়মান উপপাদ্য বর্ণনা ও প্রমাণ কর।

(খ) $p=x\cos\alpha+y\sin\alpha$ যদি $\left(\dfrac{x}{a}\right)^{n/n-1}+\left(\dfrac{y}{b}\right)^{n/n-1}=1$ বক্ররেখাকে স্পর্শক করে, প্রমাণ কর যে, $p^n=(a\cos\alpha)^n+(b\sin\alpha)^n$.

UNIVERSITY QUESTIONS

জাতীয় বিশ্ববিদ্যালয়-১৯৯৫

অন্তরকলন

১। (ক) কোন বিন্দুতে একটি ফাংশনের বামপক্ষ সীমা, ডান পক্ষ সীমা এবং ফাংশনটির সীমার ব্যাখ্যা দাও।

$f(x)$ ফাংশনটি নিম্নরূপে বর্ণিত হইল :–

$$f(x) = \begin{cases} x, & \text{যখন } x > 0 \\ 0, & \text{যখন } x = 0 \\ -x, & \text{যখন } x < 0 \end{cases}$$

মান নির্ণয় কর :

$\lim\limits_{x\to 0-} f(x)$, $\lim\limits_{x\to 0+} f(x)$ এবং $\lim\limits_{x\to 0} f(x)$ (যদি আদও বিরাজ করে)।

(খ) দেখাও যে,

$$f(x) = \begin{cases} \dfrac{x}{|x|}, & \text{যখন } x \neq 0 \\ 0, & \text{যখন } x = 0 \end{cases}$$

ফাংশনটি $x=0$ বিন্দুতে বিচ্ছিন্ন। ফাংশনটির লেখচিত্র অংকন কর।

(গ) মান নির্ণয় কর :–

$$\lim_{x\to\infty} x(2^{1/x}-1).$$

২। (ক) নিম্নের যে-কোন দুইটির $\dfrac{dy}{dx}$ নির্ণয় কর :–

(i) $y = x + x^x$;

(ii) $\sin y = x\sin(x+y)$;

(iii) $\tan y = \dfrac{2t}{1-t^2}$; $\sin x = \dfrac{2t}{1+t^2}$.

(খ) $y=x^3\sin x^2 x$-এর n-তম অন্তরজ বাহির কর।

(গ) রোল-এর উপপাদ্যটির বর্ণনা দাও। ম্যাকলরিন শ্রেণীর সাহায্যে $\log_e(1+x)$ কে x-এর ঊর্দ্ধ ক্রমানুসারে বিস্তার কর।

৩। (ক) সমমাত্রিক ফাংশন-এর সংজ্ঞা দাও।

দেখাও যে,

$$x\frac{\delta u}{\delta x} + y\frac{\delta u}{\delta y} + z\frac{\delta u}{\delta z} = 0.$$

যখন $u = \dfrac{y}{z} + \dfrac{z}{x} + \dfrac{x}{y}$

প্রশ্ন :- ১

(খ) যেখানে $y=\frac{a}{4}$ সেই বিন্দুতে $y(x^2+a^2)=ax^2$ রেখাটির স্পর্শক সমীকরণ নির্ণয় কর।

(গ) xy-এর গরিষ্ঠ মান নির্ণয় কর যখন $\frac{x}{3}+\frac{y}{4}=1$.

৪। (ক) দেখাও যে, কেন্দ্রের সাপেক্ষে উপবৃত্তের $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ এর পাদ সমীকরণ

$$\frac{a^2b^2}{p^2}=a^2+b^2-r^2.$$

(খ) বক্রতা জ্যা-এর সংজ্ঞা দাও। $s=c\log\sec\psi$- এর (s,ψ) বিন্দুতে বক্রতা ব্যাসা বাহির কর।

(গ) অসীমতট নির্ণয় করঃ–

$$y^2(x^2-y^2)-2ay^3+2a^3x=0$$

যোগজকলন

৫। যোগজীকরণ নিষ্পন্ন কর (যে-কোন তিনটি) ঃ –

(i) $\int\frac{x+\sin}{1+\cos x}dx;$ (ii) $\int\sin^{-1}\frac{x}{\sqrt{1+x}}dx;$

(iii) $\int\frac{2x^2-1}{(x+1)^2(x-2)}dx;$ (iv) $\int\frac{dx}{\sqrt{4+3x-2x^2}}$

৬। মান নির্ণয় কর (যে-কোন তিনটি ঃ–

(i) $\int_0^1\frac{x}{1+\sqrt{x}}dx;$ (ii) $\int_0^\pi\frac{dx}{(1+x^2)\sqrt{1-x^2}};$

(iii) $\int_0^\pi\frac{dx}{1-2a\cos x+a^2}$, $0<a<1;$

(iv) $\int_0^\pi\frac{dx}{3+2\cos x}.$

৭। (ক) $\int_0^\pi f(x)\,dx$ কে সমষ্টির লিমিট হিসাবে সংজ্ঞায়িত কর এবং ইহা হইতে

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)^{1/2}\left(1+\frac{3}{n}\right)^{1/3}\cdots\left(1+\frac{n}{n}\right)^{1/n}$$

এর মান নির্ণয় কর।

(খ) $S_n=\int_0^{\pi/2}\frac{\sin(2n-1)x}{\sin x}dx$. ($n$ পূর্ণ সংখ্যা) হইলে দেখাও যে, $S_{n+1}=S_n=\frac{\pi}{2}$

৮। (ক) $x^{2/3}+y^{2/3}=a^{2/3}$রেখা দ্বারা বেষ্টিত ক্ষেত্রটির ক্ষেত্রফল নির্ণয় কর।

(খ) $r=a(1+\cos\theta)$ বক্র-রেখাটি আদি রেখার চতুর্দিকে আবর্ত্তনের ফলে যে ঘন উৎপন্ন হয় তাহার আয়তন নির্ণয় কর।

অন্তরক সমীকরণ

৯। (ক) যে-কোন দুইটির সমাধান কর ঃ–

(i) $(x^3y^2-y)\,dx-(x^2y^3+x)dy=0;$

(ii) $\frac{dy}{dx}=\frac{y}{x}+\tan\frac{y}{x};$

(iii) $\frac{dy}{dx}+\frac{y}{x}\log y=\frac{y}{x^2}(\log y)^2.$

(খ) দেখাও যে, $(x^2+y^2+x)dx+xydy=0$ যথাযথ নয়। সমীকরণটির জন্য একটি যোগজীকারক নির্ণয় কর ঃ–

$$(D^3-D^2+4D-4)y=e^x\sin 2x.$$

(খ) $\frac{d^2y}{dx^2}-\frac{3dy}{dx}+2y=e^{3x}$ সমাধান বাহির কর, যেখানে $y=0$, যখন $x=0$ এবং $x=\log_e 2$.

(গ) নিম্নের রেখাগুলির সমকোণচ্ছেদী রেখাগোত্রের সমীকরণ বাহির করঃ–

$$y^2=4a(x+a).$$

English version

Differential Calculus

1. (a) Explain what are meant by left hand limit, right hand limit and limit of a function at a point.

A function $f(x)$ is defined as follows :—

$$f(x)=\begin{cases}x, & \text{when } x>0\\ 0, & \text{when } x=0\\ -x, & \text{when } x<0\end{cases}$$

Evaluate :

$\lim_{x\to0-}f(x)$, $\lim_{x\to0+}f(x)$ and $\lim_{x\to0}f(x)$ (whichever exists).

(b) Show that the function

$$f(x)=\begin{cases}\frac{x}{|x|}, & \text{when } x\neq0\\ 0, & \text{when } x=0\end{cases}$$

is discontinuous at $x=0$

Draw the graph of the function.

(c) **Evaluate :**

$$\lim_{x\to\infty} x\,(2^{1/x}-1).$$

2. (a) Find the $\frac{dy}{dx}$ of any two of the following :—

(i) $y=x+x^x$; (ii) $\sin y\ (x+y)$; (iii) $\tan y = \frac{2t}{1-t^2}$; $\sin x = \frac{2t}{1+t^2}$

(b) Obtain the n-th derivative of $y = x^3\sin 2x$.

(c) State Rolle's Theorem. Expand $\log_e(1+x)$ is powers of x by Maclaurin's Theorem.

3. (a) Define homogeneous function.

Show that

$$x\frac{\delta u}{\delta x}+y\frac{\delta u}{\delta x}+z\frac{\delta u}{\delta x}=0,\ \text{when}\ u=\frac{y}{z}+\frac{z}{x}+\frac{x}{y}.$$

(b) Find the equations of the tangents to the curve $y(x^2+a^2) = ax^2$ at the points where $y = \frac{a}{4}$.

(c) Find the maximum value of xy when $\frac{x}{3}+\frac{y}{4}=1$.

4. (a) Show that the pedal equation of the curve $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with regard to the centre is $\frac{a^2b^2}{p^2}=a^2+b^2-r^2$.

(b) Define : Chord of curvature. Find the radius of curvature at the point (s,ψ) on the curves $s=c\log\sec\psi$.

(c) Find the asymptotes of the cure $y^2(x^2-y^2)-2ay^3+2a^3x=0$

Integral Calculus.

5. Perform any three of the following integrations:—

(i) $\int\frac{x+\sin x}{1+\cos x}dx$; (ii) $\int\sin^{-1}\frac{x}{\sqrt{1+x}}dx$;

(iii) $\int\frac{2x^2-1}{(x+1)^2(x-2)}dx$; (iv) $\int\frac{dx}{\sqrt{4+3x-2x^2}}$

6. Evaluate (any three):—

(i) $\int_0^1\frac{x}{1+\sqrt{x}}dx$; (ii) $\int_0^\pi\frac{dx}{(1+x^2)\sqrt{1-x^2}}$

(iii) $\int_0^\pi\frac{dx}{1-2a\cos x+a^2}$, $0<a<1$; (iv) $\int_0^\pi\frac{dx}{3+2\cos x}$.

7. (a) $\int_a^b f(x)\,dx$ as the limit of a sum and evaluate

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)^{1/2}\left(1+\frac{3}{n}\right)^{1/3}\cdots\left(1+\frac{n}{n}\right)^{1/n}$$

এর মান নির্ণয় কর।

(b) If $S_n=\int_0^{\pi/2}\frac{\sin(2n-1)x}{\sin x}dx$, ($n$ being an integer) then show that $S_{n+1}-S_n=\frac{\pi}{2}$.

8. (a) Find the whole area of the curve $x^{2/3}+y^{2/3}=a^{2/3}$.

(b) The curve $r=a(1+\cos\theta)$ revolves about the initial line. Find the volume of the figure formed.

Differential Equations.

9. (a) Solve any two of the foollowing :—

(i) $(x^3y^2-y)dx-(x^2y^3+x)dy=0$;

(ii) $\frac{dy}{dx}=\frac{y}{x}+\tan\frac{y}{x}$;

(iii) $\frac{dy}{dx}+\frac{y}{x}\cdot\log=\frac{y}{x^2}(\log y)^2$.

(b) Show that the equation $(x^2+y^2+x)dx+xydy=0$ is not exact. find an integating factor for this equation and hence solve the equation.

10.(a) Solve :—

$(D^3-D^2+4D-4)y=e^x\sin 2x$.

(b) Find a solution of

$$\frac{d^2y}{dx^2}\frac{3dy}{dx}+2y=e^{3x}$$

which shall vanish when $x=0$ and $x=\log_e 2$.

(c) Find the orthogonal trajectories of $y^2=4a(x+a)$.

জাতীয় বিশ্ববিদ্যালয়-১৯৯৫

অন্তরকলন

১। (ক) কোন বিন্দুতে একটি ফাংশনের অবিচ্ছিন্নতার সংজ্ঞা দাও।

$f(x)$ ফাংশনটি নিম্নরূপে বর্ণিতঃ –

$$f(x) = \begin{cases} 5x - 4 \; ; & 0<x\leq 1 \\ 4x^2-3x \; ; & 1<x<2 \end{cases}$$

$x = 1$ বিন্দুতে ফাংশনটি অবিচ্ছিন্ন কিনা তা আলোচনা কর।

(খ) মান নির্ণয় কর :–

$$\lim_{\theta\to\frac{\pi}{4}} \frac{\sqrt{2}-\cos\theta-\sin\theta}{(4\theta-\pi)^2}.$$

(গ) $\frac{dy}{dx}$ নির্ণয় কর (যে-কোন দুইটি):–

(i) $y = x^{\log x}+x^{\cos^{-1}x}$

(ii) $y = \frac{x\cos^{-1}x}{\sqrt{1-x^2}}$;

(iii) $y = \sin\sqrt[3]{1+x^2}\,\log\cos y$

২। (ক) $=a(t-\sin t)$,

$y = a\,(1-\cos t)$

রেখাটির যে-কোন বিন্দুতে স্পর্শকের ঢাল নির্ণয় কর, $(0\leq t\leq 2\pi)$.

(খ) $y = (x^3+2x^2+x+1)\,a^x$ হইলে y_u নির্ণয় কর।

(গ) $e^x\sin x$ কে x - এর উর্দ্ধ ক্রমানুসারে বিস্তার ক।

৩। (ক) $u= (x^2+y^2+z^2)^{-1/2}$ হইলে প্রমাণ কর যে

$$\frac{\delta^2u}{\delta x^2}+\frac{\delta^2u}{\delta y^2}+\frac{\delta^2u}{\delta z^2} = 0.$$

(খ) সমমাত্রিক ফাংশনের সংজ্ঞা দাও। অয়লারের সমমাত্রিক উপপাদ্যটি বর্ণনাসহ প্রমাণ কর।

(গ) $a_1x^2+b_1y^2=1$ এবং $a_2x^2+b_2y^2 =1$ রেখারদ্বয় পরস্পরকে লম্বভাবে ছেদ করার শর্ত নির্ণয় কর।

৪। (ক) কোন ফাংশনের আপেক্ষিক গরিষ্ঠ এবং লঘিষ্ঠ মান বলিতে কি বুঝায় তাহার সংজ্ঞা দাও। কোন ফাংশনের $f(x)$ এর আপেক্ষিক গরিষ্ঠ এবং লঘিষ্ঠ মান নির্ণয়ের ব্যবহারিক নিয়মগুলি সংক্ষেপে ব্যাখ্যা কর।

(খ) অসীমতট নির্ণয় কর :–

$x^3-8y^3+3x^2+y^2-7x+2= 0.$

(গ) $x^{2/3}+y^{2/3}=a^{2/3}$ রেখার (x,y) বিন্দুতে বক্রতা ব্যাসার্ধ নির্ণয় কর।

যোগজকলন

৫। যে-কোন তিনটির যোগজীকরণ কর :–

(i) $\int\frac{dx}{(x-1)\sqrt{x^2+1}}$; (ii) $\int\frac{2\sin x + 3\cos x}{7\sin x-2\cos x}\,dx$;

(iii) $\int\sqrt{1+\sec x}\,dx$; (iv) $\int\tan^2x\sqrt{\sec x}\,dx$

৬। মান নির্ণয় কর (যে-কোন তিনটি : –

(i) $\int_0^{\pi/2}\frac{x\sin x}{1+\cos^2x}\,dx$; (ii) $\int_0^{\pi/4}\log(1+\tan x)\,dx$;

(iii) $\int_0^{\pi/4}\sin^4\theta d\theta$;

(iv) $\lim_{n\to\infty}\left[\frac{1}{n}+\frac{1}{\sqrt{n^2-1}}+\ldots+\frac{1}{\sqrt{n^2-(n-1)^2}}\right].$

৭। (ক) মান নির্ণয় কর :–

$$\int_0^{\pi}\frac{x\,dx}{(a^2\cos^2x + b^2\sin^2x)^2}$$

(খ) $u_n =\int_0^{\pi/2} x^n(\sin x+\cos x)\,dx$ (যেখানে n যোগবোধক পূর্ণ সংখ্যা) হইলে দেখা যে

$$u_n+n(n-1)u_{n-2}=\left(n+\frac{\pi}{2}\right)\left(\frac{\pi}{2}\right)^{n-1}$$

৮। (ক) $r(1+ \cos\theta) =2$ রেখাটির $\theta=0$ হইতে $\theta=\frac{\pi}{2}$ পর্যন্ত অংশটির দৈর্ঘ্য নির্ণয় কর।

(খ) $y^2=x(2a-x)$ এবং $y^2=ax$ দ্বারা বেষ্টিত ক্ষেত্রটির ক্ষেত্রফল নির্ণয় কর।

অন্তরক সমীকরণ

৯। (ক) সমাধান কর :–

$(x^2+y^2)\,dx -2xy\,dy = 0.$

(খ) $\frac{dy}{dx}+P(x)y=Q(x)$ -এর সাধারণ সমাধানের জন্য একটি সূত্র বাহির কর, যেখানে $P(x)$ এবং $Q(x)$, x -এর অবিচ্ছিন্ন ফাংশন।

(গ) $\frac{dy}{dx}+\frac{x}{1-x^2}y=x\sqrt{y}$ এর সমাধান বাহির কর।

১০। (ক) সাধারণ নির্ণয় কর ঃ-

$\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2x=0$, যখন $x=0$, $y=2$ এবং $\frac{dy}{dx}=0$

(খ) নিম্নের রেখাগুলির সমকোণছেদী রেখা গোত্রের সমীকরণ বাহির কর ঃ-

$y^2=2x^2 9 1-cx)$.

(গ) সমাধান কর ঃ-

$\frac{d^2y}{dx^2}-y=xe^x\sin x$.

Differential Calculus

1. (a) Define the continuity of a function at a point.

A function $f(x)$ is defined as follows:—

$$f(x)=\begin{cases}5x-4 \,;\, 0<x\le 1\\ 4x^2-3x \,;\, 1<x<2\end{cases}$$

Discuss whether the function is continuous at $x=1$

(b) Evaluate :—

$$\lim_{\theta\to\frac{\pi}{4}}\frac{\sqrt{2}-\cos\theta-\sin\theta}{(4\theta-\pi)^2}$$

(c) Find $\frac{dy}{dx}$ of any two : —

(i) $y=x^{\log x}+x^{\cos^{-1}x}$

(ii) $y=\frac{x\cos^{-1}x}{\sqrt{1-x^2}}$;

(iii) $y=\sin\sqrt{1+x^2}\ \log\cos y$

2. (a) Find the slope of a tangent to the curve

$x=a(t-\sin t)$

$y=a(1-\cos t)$

at any point $(0\le t\le 2\pi)$.

(b) If $y=(x^3+2x^2+x+1)\,a^x$, find y_n.

(c) Expand $e^x\sin x$ in a series of ascending powers of x.

(a) If $u=(x^2+y^2+z^2)^{-1/2}$, then prove that

$$\frac{\delta^2u}{\delta x^2}+\frac{\delta^2u}{\delta y^2}+\frac{\delta^2u}{\delta z^2}=0.$$

(b) Define homogeneous function. State and prove Euler's Theorem on Homogeneous functions.

(c) Find the condition that the curves $a_1x^2+b_1y^2=1$ and $a_1x^2+b_1y^2=1$ should cut orthogonally.

4. (a) Define what is meant by relative maximum and minimum values of a function. Explain briefly the working rule of determining the relative maxima and minima of a function $f(x)$.

(b) Find the asymptotes of $x^3-8y^3+3x^2+y^2-7x+2=0$.

(c) Find the radius of curvature of the curve $x^{2/3}+y^{2/3}=a^{2/3}$ at any point (x,y)

Integral Calculus

5. Integrate any three of the following:—

(i) $\int\frac{dx}{(x-1)\sqrt{x^2+1}}$; (ii) $\int\frac{2\sin x+3\cos x}{7\sin x-2\cos x}dx$;

(iii) $\int\sqrt{1+\sec x}\,dx$; (iv) $\int\tan^3x\sqrt{\sec x}\,dx$

6. Evaluate any three of the following :—

(i) $\int_0^{\pi}\frac{x\sin x}{1+\cos^2x}dx$; (ii) $\int_0^{\pi/4}\log(1+\tan x)\,dx$;

(iii) $\int_0^{\pi/4}\sin^4\theta\,d\theta$;

(iv) $\lim\limits_{n\to\infty}\left[\frac{1}{n}+\frac{1}{\sqrt{n^2-1}}+\ldots+\frac{1}{\sqrt{n^2-(n-1)^2}}\right].$

7. (a) Evaluate : —

$$\int_0^{\pi}\frac{x\,dx}{(a^2\cos^2+b^2\sin^2x)^2}.$$

(b) If $u_n=\int_0^{\pi/2}x^n(\sin x+\cos x)\,dx$, then prove that

$$u_n+n(n-1)u_{n-2}=\left(n+\frac{\pi}{2}\right)\left(\frac{\pi}{2}\right)^{n-1}$$

where n is a positive integer.

8. (a) Find the length of the arc of the curve $r\,(1+\cos\theta)=2$ from $\theta=0$ to $\theta=\frac{\pi}{2}$.

(b) find the area of the region bounded by $y^2=x(2a-x)$ and $y^2=ax$.

Differential Equations

9. (a) Solve :—

$(x^2+y^2)\,dx-2xy\,dy=0.$

(b) Obtain a formula for the general solution of differential equation

$$\frac{dy}{dx}+P(x)\,y=Q(x).$$

where $P(x)$ and $Q(x)$ are continuous functions of x.

(c) Solve : —

$$\frac{dy}{dx}+\frac{x}{1-x^2}\,y=x\sqrt{y}.$$

10. (a) Find the particular solution of the differential equation

$$\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2x=0 \text{ when } x=0,\ y=2 \text{ and } \frac{dy}{dx}=0$$

(b) Find the orthogonal trajectories of $y^2=2x^2(1-cx)$.

(c) Solve the following :—

$$\frac{d^2y}{dx^2}-y=xe^x\sin x.$$

জাতীয় বিশ্ববিদ্যালয় -১৯৯৭

অন্তরকলন

১। (ক) একটি ফাংশন $f(x)$ নিম্নলিখিত উপায়ে সংজ্ঞায়িত ঃ –

$f(x)=\frac{1}{2}(b^2-a^2)$ যখন $0<x\le a$;

$=\frac{1}{2}b^2-\frac{x^2}{6}-\frac{a^3}{3x}$ যখন $a<x\le b.$

$=\frac{1}{3}\frac{b^3-a^3}{x},\ x>b$

দেখাও যে, $f(x)$ এবং $f'(x)$ অবিচ্ছিন্ন, কিন্তু $f''(x)$ বিচ্ছিন্ন।

(খ) মান নির্ণয় কর ঃ–

$$\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{\sin^2x}\right).$$

২। যে-কোন তিনটির উত্তর দাও ঃ–

(ক) যদি $y=(\cot x)^{\sin x}+(\tan x)^{\cos x}$ হয় তবে $\frac{dy}{dx}$ এর মান নির্ণয় কর।

(খ) মূল নিয়মে x-এর সাপেক্ষে $\tan\frac{1}{x}$ এর অন্তরক সহগ নির্ণয় কর।

(গ) $y=\log_e\left[\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}\right]^{1/2}$

হইলে, দেখাও যে, $\frac{dy}{dx}=\frac{-1}{x\sqrt{1-x^2}}$

(ঘ) যদি $\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$ হয়, তবে দেখাও যে, $\frac{dy}{dx}=\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}.$

৩। (ক) যদি $f(x)=x(x-1)(x-2)$ হয়, তবে $\left[0,\frac{1}{2}\right]$ ব্যবধিতে $f(x)$ -এর জন্য গড় মান উপপাদ্যটির সত্যতা যাচাই কর।

(খ) মূল বিন্দুতে $y^2-3xy-2x^2-x^3+y^4=0$ বক্ররেখার বক্রতার ব্যাসার্ধ নির্ণয় কর।

(গ) $\sin x\,(1+\cos x)$ - এর গরিষ্ঠ ও লঘিষ্ঠ মানের বিন্দুগুলি নির্ণয় কর। ইহার বৃহত্তম মানও নির্ণয় কর।

৪। (ক) যদি $x^{2/3}+y^{2/3}=a^{2/3}$ বক্ররেখার কোন বিন্দুতে অভিলম্ব অংকিত করিলে তাহ x-অক্ষের সহিত ϕ কোণ উৎপন্ন করে তবে দেখাও যে, উহার সমীকরণ হইবে $y\cos\phi-x\sin\phi=a\cos 2\phi$.

(খ) $(x+2)^2(x+2y+2) = x+9y+2$ বক্ররেখার অসীমতটসমূহ নির্ণয় কর।

যোগজকলন

৫। যে-কোন তিনটির যোগজীকরণ কর :-

(ক) $\int \frac{dx}{1+3\sin^2 x}$; (খ) $\int \frac{dx}{13+3\cos x+4\sin x}$

(গ) $\int \frac{2x^2+x+17}{(x-1)(x^2+2x+3)}\,dx$; (ঘ) $\int \frac{\log_e \sec^{-1}x}{x\sqrt{x^2-1}}\,dx$.

৬। যে-কোন তিনটি মান নির্ণয় কর : -

(ক) $\int_0^{\pi/2} \frac{(\sin\theta)^{3/2}}{(\sin\theta)^{3/2}+(\cos\theta)^{3/2}}\,d\theta$;

(খ) $\int_0^1 \log\sin\left(\frac{1}{2}\pi\theta\right)d\theta$;

(গ) $\int_0^1 \frac{dx}{(1+x^2)\sqrt{1-x^2}}$; (ঘ) $\int_0^{\pi/2} \frac{\cos x\,dx}{(1+\sin x)(2+\sin x)\,dx}$

৭। (ক) মান নির্ণয় কর :-

$$\lim_{n\to\infty}\left[\frac{1}{\sqrt{2n-1^2}}+\frac{1}{\sqrt{4n-2^2}}+\ldots\ldots+\frac{1}{n}\right].$$

(খ) দেখাও যে, $\int_0^1 \frac{dx}{(x^2-2x+2)^3} = \frac{3\pi+8}{32}$.

(গ) যদি $I_n = x^n \tan^{-1}x\,dx$ হয়, তবে দেখাও যে,

$(n+1)\,I_n + (n-1)\,I_{n-2} = \frac{\pi}{2} - \frac{1}{n}$.

৮। (ক) $r=a(1-\cos\theta)$ কারডিঅয়েডের পরিসীমা নির্ণয় কর।

(খ) দেখাও যে, $x^{2/3}+y^{2/3}=a^{2/3}$ এ্যাসট্রয়েড দ্বারা সীমাবদ্ধ ক্ষেত্রের ক্ষেত্রফল $\frac{3}{8}\pi a^2$.

অন্তরক সমীকরণ

৯। (ক) সমাধান কর (যে-কোন দুইটি) :-

(i) $\frac{dy}{dx}+1 = e^{x-y}$;

(ii) $x\frac{dy}{dx}+y = x^5y^4.\cos x$;

(iii) $(x^2+y^2)\,dx + 2xy + 2xy\,dy = 0$.

(খ) সমাধান কর (যে-কোন একটি) :-

(i) $(x^2+2xy-y^2)\,dx + (y^2+2xy-x^2)\,dy = 0$;

(ii) $y\,dx - x\,dy + \log x\,dx = 0$

১০। (ক) সমাধান কর (যে-কোন দুইটি) :-

(i) $(D^2+4)y = \sin 2x$;

(ii) $\frac{d^2y}{dx^2} - 9\frac{dy}{dx} + 20y = 20x$;

(iii) $(D^2-2D+1)y = e^x$.

(খ) সমাধান কর (যে-কোন একটি):-

(i) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 4x - 20\cos 2x$;

(ii) $(x+1)^2\frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} = (2x+3)(2x+4)$.

English version

Differential Calculus

1.(a) A function $f(x)$ is defined as follows :-

$f(x) = \frac{1}{2}(b^2-a^2)$ for $0<x\leq a$;

$= \frac{1}{2}b^2 - \frac{x^2}{6} - \frac{a^3}{3x}$ for $a<x\leq b$;

$= \frac{1}{3}\frac{(b^3-a^3)}{x}$ for $x>b$.

Prove that $f(x)$ and $f'(x)$ are continuous but $f''(x)$ is discontinuous.

(b) Evaluate $\lim_{x\to 0}\left(\frac{1}{x^2} - \frac{1}{\sin^2 x}\right)$

2. Answer (any three):—

(a) Find $\frac{dy}{dx}$ when $y = (\cot x)^{\sin x} + (\tan x)^{\cos x}$.

(b) Differntiate from 1st principle $\tan\frac{1}{x}$ w. r. t. x.

(c) If $y=\log_e\left[\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}\right]^{1/2}$, then prove that $\frac{dy}{dx}=\frac{-1}{x\sqrt{1-x^2}}$

(d) If $\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$, show that $\frac{dy}{dx}=\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$.

3. (a) If $f(x)=x(x-1)(x-2)$, then verify the mean value theorem for $f(x)$ on the interval $\left[0,\frac{1}{2}\right]$.

(b) Find the radii of curvature at the origin for the curve $y^2-3xy-2x^2-x^3+y^4=0$

(c) Find the points of maxima and minima for the function sinx (1+cosx). Also find the maxmum value of the function.

4. (a) If the normal to the curve $x^{2/3}+y^{2/3}=a^{2/3}$ makes an angle ϕ with the axis of x, show that its equation is $y\cos\phi-\sin\phi=a\cos2\phi$.

(b) Find the asymptotes of the curve $(x+2)^2(x+2y+2)=x+9y+2$.

Integral Calculus

5. Integrate any three of the following :-

(a) $\int\frac{dx}{1+3\sin^2x}$; (b) $\int\frac{dx}{13+3\cos x+4\sin x}$

(c) $\int\frac{2x^2+x+17}{(x-1)(x^2+2x+3)}dx$; (d) $\int\frac{\log_e\sec^{-1}x}{x\sqrt{x^2-1}}dx$.

6. Evaluate any three of the following:-

(a) $\int_0^{\pi/2}\frac{(\sin\theta)^{3/2}}{(\sin\theta)^{3/2}+(\cos\theta)^{3/2}}d\theta$;

(b) $\int_0^1\log\sin\left(\frac{1}{2}\pi\theta\right)d\theta$;

(c) $\int_0^1\frac{dx}{(1+x^2)\sqrt{1-x^2}}$; (d) $\int_0^{\pi/2}\frac{\cos x}{(1+\sin x)(2+\sin x)}dx$.

7. **(a) Evaluate:-**

$$\lim_{n\to\infty}\left[\frac{1}{\sqrt{2n-1^2}}+\frac{1}{\sqrt{4n-2^2}}+\dots\dots+\frac{1}{n}\right].$$

(b) Show that $\int_0^1\frac{dx}{(x^2-2x+2)^3}=\frac{3\pi+8}{32}$.

(c) If $I_n=\int_0^1x^n\tan^{-1}x\,dx$, than show that $(n+1)I_n+(n-1)I_{n-2}=\frac{\pi}{2}-\frac{1}{n}$.

8. (a) Find the perimeter of the cardioide $r=a(1-\cos\theta)$.

(b) Show that the area of the region bounded by the astriod $x^{2/3}+y^{2/3}=a^{2/3}$ is $\frac{3}{8}\pi a^2$.

Differential Equations

9. (a) Solve (*any two*):-

(i) $\frac{dy}{dx}+1=e^{x-y}$;

(ii) $x\frac{dy}{dx}+y=x^5y^4.\cos x$;

(iii) $(x^2+y^2)\,dx+2xy+dy=0$.

(b) Solve (*any one*):-

(i) $(x^2+2xy-y^2)\,dx+(y^2+2xy-x^2)\,dy=0$;

(ii) $y\,dx-x\,dy+\log x\,dx=0$

10. (a) Solve (*any two*):-

(i) $(D^2+4)y=\sin2x$;

(ii) $\frac{d^2y}{dx^2}-9\frac{dy}{dx}+20y=20x$;

(iii) $(D^2-2D+1)y=e^x$.

(U) Solve (any one):-

(i) $\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=4x-20\cos2x$;.

(ii) $(x+1)^2\frac{d^2y}{dx^2}+(x+1)\frac{dy}{dx}=(2x+3)(2x+4)$.

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1. (a) Draw the graph of the function defined by

$$f(x) = \begin{cases} 1+x, & \text{when } -1 \leq x < 0 \\ 1-x, & \text{when } 0 \leq x < 1 \\ 0, & \text{when } 1 < x \end{cases}$$

Determine the range and domain of the function.

(b) A function $f(x)$ is defined as follows :-

$$f(x) = \begin{cases} 1+\sin x, & 0 \leq x \leq \frac{\pi}{2} \\ 2+(x-\frac{\pi}{2})^2, & \frac{\pi}{2} \leq x < \infty \end{cases}$$

Discuss the continuity and differentiability of $f(x)$ at $x=\frac{\pi}{2}$.

2. (a) Find $\frac{dy}{dx}$ where (i) $\tan y = e^{\cos 2x}\sin x$.

(ii) $y = e^{\tan^{-1}y} \cdot \log \sec^2 x^2$

(b) Differentiate $\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}$ with respect to $\sqrt{1-x^4}$.

3. (a) If $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$ prove that

$(x^2-1)\, y_{n+2}+(2x+1)\, xy_{n+1}+(x^2-m^2)\, y_n=0$

(b) state and prove Rolle's theorem. Verify with an example.

4. (a) Find the asymptotes of $x^3-2x^2y+xy^2+n^2-xy+2=0$

(b) Find the maximum or minimum value of $u=\frac{4}{x}+\frac{36}{y}$, where $x+y=2$.

(c) show that the pedal equation of the parabola $y^2=4ax$ with regard to its vertex is $a^2(r^2-p^2) = p^2(r^2+4a^2)\,(p^2+4a^2)$

5. Integrate any three of the following :-

(i) $\int \frac{x^2-x+1}{x^2+x+1}\, dx$; (i) $\int \frac{dx}{(x-1)\sqrt{2x^2-8x-1}}\, dx$;

(i) $\int \sin^{-1}\sqrt{\frac{x}{a+x}}\, dx$; (i) $\int \frac{x+\tan^{-1}x}{(1+x^2)^2}\, dx$;

6. (a) If $P_n = \int \frac{\sin(2n-1)x}{\sin x}\, dx$; , $Q_n \int \frac{\sin^2 nx}{\sin^2 x}\, dx$;

then show that $n\,(p_{n+1}-p_{n+1}) = \sin 2nx$ and $Q_{n+1}-Q_n=P_{n+1}$

(b) Show that $\int_a^b \frac{dx}{x\sqrt{(b-x)(x-a)}} = \frac{\pi}{\sqrt{ab}}$;

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$$f(x) = \begin{cases} 1+x, & \text{যখন } -1 \leq x < 0 \\ 1-x, & \text{যখন } 0 \leq x < 1 \\ 0, & \text{যখন } 1 < x \end{cases}$$

ফাংশনটির ডোমেন ও রেঞ্জ নির্ণয় কর।

(খ) যদি $f(x) = 1+\sin x,\ 0 \leq x < \frac{\pi}{2}$

$= 2 + \left(x-\frac{\pi}{2}\right)^2,\ \frac{\pi}{2} \leq x < \infty$ হয়, তবে $x = \frac{\pi}{2}$ বিন্দুতে $f(x)$ এর অবিচ্ছিন্নতা

এবং অন্তরীকরণ যোগ্যতা পরীক্ষা কর।

২। (ক) $\frac{dy}{dx}$ বাহির কর যেখানে, (i) $\tan y = e^{\cos^2 x}\sin x$ এবং

(ii) $y = e^{\tan^{-1}y} \cdot \log \sec^2 x^2$

(খ) $\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}$ কে $\sqrt{1-x^4}$ সাপেক্ষে অন্তরকলন কর।

3. (ক) যদি $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$ হয় তাহলে প্রমাণ কর যে,

$(x^2-1)\, y_{n+2}+(2x+1)\, xy_{n+1}+(n^2-m^2)\, y_n=0$

(ক). রোল্‌স উপপাদ্যটির বর্ণনাসহ প্রমাণ কর। একটি উদাহরণ দ্বারা পরীক্ষাটি কর।

৪। (ক) $x^3-2x^2y+xy^2+x^2-xy+2=0$

রেখার অসীমতটগুলি বাহির কর।

প্রশ্ন:-২

(খ) $x+y=2$ হইলে $u=\frac{4}{x}+\frac{36}{y}$ এর গরিষ্ট ও লঘিষ্ঠ মান নির্ণয় কর।

(গ) দেখা যে, মূল বিন্দু সাপোয়া পরাবৃত্ত $y^2=4ax$ এর পেড়াল সমীকরণ হবে

$$a^2(r^2-p^2)^2 = p^2(r^2+4a^2)(p^2+4a^2)$$

৫। নিম্নের যেগুকোন তিনটির সমীকলন কর ঃ-

(i) $\int \frac{x^2-x+1}{x^2+x+1}dx$; (ii) $\int \frac{dx}{(x-1)\sqrt{2x^2-8x-1}}dx$;

(iii) $\int \sin^{-1}\sqrt{\frac{x}{a+x}}dx$; (iv) $\int \frac{m\tan^{-1}x}{(1+x^2)^2}dx$;

6. (a) $P_n = \int \frac{\sin(2n-1)x}{\sin x}dx$; এবং $Q_n \int \frac{\sin^2 nx}{\sin^2 x}dx$;

হইলে দেখাও যে, $n(p_{n+1}-p_n) = \sin 2nx$ এবং

$Q_{n+1}-Q_n=P_{n+1}$

(খ) দেখাও যে $\int_b^a \frac{dx}{x(b-x)(x-a)} = \frac{\pi}{\sqrt{ab}}$;

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1. (a) Find the domain and range of $f(x) = |x-1|$ and examine the continuity and differentiability of $f(x)$ at $x=1$.

(b) State and prove L'Hospital's theorem and henoe evaluate

$$\lim_{x\to 0}\left[\frac{1}{x^2}-\frac{1}{\sin^2 x}\right]$$

2. (a) Differontiate logtan $\sqrt{x^2-1}$ with respect to $\sqrt{x-1}$

(c) If $y = e^{ax}\{a^2x^2-2nax+n(n+1)\}$ find y_n.

3. (a) Find the maximum and minimum values of $\frac{a^n}{x}+\frac{b^2}{y}$ when $x+y=a$.

(b) define an asymptote. Find the asymptotes of the curve

$$y^3-5xy^2+8x^2y-4x^3-3y^2+9xy-6x^2+2y-2x+1=0$$

4. (a) If V is a function of x and y, where $x = r\cos\theta$ and $y = r\sin\theta$; then prove that

$$\frac{\delta^2 V}{\delta x^2}+\frac{\delta^2 V}{\delta y^2}=\frac{\delta^2 V}{\delta r^2}+\frac{1}{r}\frac{\delta V}{\delta r}+\frac{1}{r^2}\frac{\delta^2 V}{\delta\theta^2}$$

(b) Define the angle of intersection of two curves and show that the curves $y=x^3$ and $6y=7-x^2$ intersect orthogonally.

5. Integrate any three of the following:

(i) $\int \frac{dx}{(1+x)\sqrt{1+2x-x^2}}$; (ii) $\int \frac{dx}{x^2-b^2\cos^2 x}$, $a>b$

(iii) $\int \frac{dx}{\sin x(3+2\cos x)}$; (iv) $\int_\beta^\alpha \sqrt{(x-\alpha)(\beta-x)}\,dx$;

(v) $\int_0^{\pi/2} \frac{dx}{5+4\sin x}$

6. (a) If $I_n = \int e^{ax}\cos^n x\,dx$ where $n\geq 2$ is an integer, then prove that

$$I_n=\frac{1}{a^2+n^2}e^{ax}\cos^{n-1}x(a\cos x+n\sin x)+\frac{n(n-1)}{a^2+n^2}I_{n-2}.$$

(b) Using (a) or otherwise, $\int_0^{\pi/2} e^x\cos^4 x\,dx$.

(c) For a continuous function f on [0,1], express

$$\lim_{n\to\alpha}\frac{1}{n}\left[f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+\dots+f\left(\frac{n}{n}\right)\right]$$

as an integral and hence evaluate

$$\lim_{n\to\alpha}\frac{\sqrt{1}+\sqrt{2}+\dots\sqrt{n}}{n^{3/2}}$$

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(বিঃ দ্রঃ সকল প্রশ্নের মান সমান। যে কোন চারটি প্রশ্নের উত্তর দাও।)

১। (ক) $f(x) = |x-1|$ -এর ডোমেন ও রেঞ্জ নির্ণয় কর এবং $x=1$ বিন্দুতে $f(x)$ এর অবিচ্ছিন্নতা ও অন্তরকলনীয়তা পরীক্ষা কর।

(খ) ল'পিটালের উপপাদ্যটি বিবৃত কর ও প্রমান দাও এবং অতঃপর

$\underset{x \to 0}{Lt} \left[\frac{1}{x^2} - \frac{1}{\sin^2 x}\right]$ এর মান নির্ণয় কর।

২। (ক) $y = \frac{\tan x}{\log x + x^{1/2}}$ এর $\frac{dy}{dx}$ বাহির কর।

(খ) $\sqrt{x-1}$ এর প্রেক্ষিতে $\log \tan \sqrt{x^2-1}$ এর অন্তরক সহগ নির্ণয় কর।

(গ) $y = e^{ax}\{a^2x^2 - 2nax + n(n+1)\}$ হইলে y_n নির্ণয় কর।

৩। (ক) $x+y=a$ হইলে $\frac{a^2}{x} + \frac{b^2}{y}$ এর গরিষ্ঠ ও লঘিষ্ঠ মান নির্ণয় কর।

(খ) অসীম তটের সংজ্ঞা দাও

$$y^3 - 5xy^2 + 8x^2y - 4x^3 - 3y^2 + 9xy - 6x^2 + 2y - 2x + 1 = 0$$

এর অসমীমতটগুলি বাহির কর।

৪। (ক) যদি V, x, y এর ফাংশন হয়, যেখানে $x = r\cos\theta$

এবং $y = r\sin\theta$; তাহলে প্রমাণ কর যে,

$$\frac{\delta^2 V}{\delta x^2} + \frac{\delta^2 V}{\delta y^2} = \frac{\delta^2 V}{\delta r^2} + \frac{1}{r}\frac{\delta V}{\delta r} + \frac{1}{r^2}\frac{\delta^2 V}{\delta \theta^2}$$

(খ) দুইটি বক্ররেখার ছেদন-কোণের সংজ্ঞা দাও এবং দেখাও যে $y=x^3$

ও $6y=7-x^2$ বক্ররেখাদ্বয় লম্বীয়ভাবে ছেদ করে।

5. নিম্নের যে কোন তিনটির উত্তর দাও ঃ

(i) $\int \frac{dx}{(1+x)\sqrt{1+2x-r^2}}$; (ii) $\int \frac{dx}{a^2 - b^2\cos^2 x}$, $a>b$

(iii) $\int \frac{dx}{\sin x(3+2\cos x)}$; (iv) $\int_\alpha^\beta \sqrt{(x-\alpha)(\beta-x)}\,dx$;

(v) $\int_0^{\pi/2} \frac{dx}{5+4\sin x}$

৬। (ক) যদি $I_n = \int e^{ax}\cos^n x\, dx$ হয়, যেখানে $n \geq 2$ পুর্ণসংখ্যা, তবে প্রমান কর যে

$$I_n = \frac{1}{a^2+n^2} e^{ax}\cos^{n-1} x(a\cos x + n\sin x) + \frac{n(-1)}{a^2+n^2} I_{n-2}.$$

(খ) উপরের (ক) ব্যবহার করিয়া অথবা অন্যভাবে $\int_0^{\pi/2} e^x\cos^4 x\, dx.$

এর মান নির্ণয় কর।

(গ) $[0, 1]$, ব্যবধিতে অবিচ্ছিন্ন ফাংশন f- এর জন্য

$$\lim_{n \to \alpha} \frac{1}{n}\left[f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \dots + f\left(\frac{n}{n}\right)\right]$$

কে যোগজরূপে প্রকাশ কর এবং তাহার সাহায্যে

$$\lim_{n \to \alpha} \frac{\sqrt{1}+\sqrt{2}+\dots\sqrt{n}}{n^{3/2}}$$

এর মান নির্ণয় কর।

R.H. (201)-1995

1. (a) Regarding $f(x+ht, y+kt)$ as a functsion of t, expand the function in a series of ascending powers of t, and deduce a proof of taylor's theorem for two independent variables.

(b) Use Maclaurin's Theorem to obtain the expansion of e^{xy}.

2. (a) If $z = \left(x\frac{\delta}{\delta x} - 1\right)\{f(y+x) - \phi(y-x)\}$

prove that $x\left(\frac{\delta^2 z}{\delta x^2} - \frac{\delta^2 z}{\delta y^2}\right) = z\frac{\delta z}{zx}$

(b) If $xyz = a^2(x+y-z)$, prove that the minimum value of $yz+zx+xy$ is $9a^2$.

3. (a) Show that the circle of curvature at the origin for the curve

$x+y = ax^2+by^2+cx^2$ is $(a+b)(x^2+y^2) = 2x+2y$

(b) Trace the ourve $r^2 = a^2\cos 2\theta$.

4. (a) Evaluate

(i) $\int_0^1 \frac{\log x}{1-x^2}\, dx$ (ii) $\int_0^{\pi/2} \tan^n\theta, -1<n<1$

(b) Find the value of $\Gamma\frac{1}{n}\ \Gamma\frac{2}{n}\ \Gamma\frac{3}{n} \dots \Gamma\frac{n-1}{n}$

5. (a) Evaluate $\iint e^{\frac{y}{x+y}}\, dxdy$, taken over the triangle $x=0$, $y=0$, $x+y=1$.

(b) Evaluate $\int\int\int_R (x+y+z+1)^2\, dxdydz$ where R is the region defined by $x\geq 0$, $y\geq 0$ $z\geq 0$, $x+y+z\leq 1$.

(c) prove that $\int_0^1 \frac{x^a-x^{-a}}{1-x}\, dx = \pi \cot a\pi - \frac{1}{a}$

6. (a) $\int_0^\infty \frac{\tan^{-1}(ax)-\tan^{-1}(bx)}{x}\, dx = \frac{\pi}{2}\log\frac{a}{b}$

(b) Find the volume cut from the sphere $x^2+y^2+z^2=a^2$ by the cylinder $x^2+y^2=ax$

(c) Find the area of the surface of the cone $z^2=3(x^2+y^2)$ cut out by the paraboloid $z=x^2+y^2$.

বাংলা অনুবাদ

১। (ক) $f(x+ht,\ y+kt)$ কে t এর ফাংশন বিবেচনা করিয়া, t এর উর্দ্ধক্রম শক্তি আকারে সিরিজে বিস্তার কর এবং ইহা ইহতে স্বাধীন দুই চলকের টেলর উপপাদ্য প্রমাণ কর।

(খ) ম্যাক্লোরিন উপপাদ্য ব্যবহার করিয়া e^{xy} এ বিস্তার কর।

২। (ক) যদি $z=\left(x\frac{\delta}{\delta x}-1\right)\{f(y+x)-\theta(y-x)\}$

প্রমাণ কর যে $x\left(\frac{d^2z}{dx^2}-\frac{\delta^2 z}{\delta y^2}\right)=2\frac{\delta z}{\delta x}$

(খ) যদি $xyz = a^2(x+y+z)$ হয়, প্রমাণ কর যে $yz+zx+xy$ এর লঘিষ্ঠ মান $9a^2$ হবে।

৩। (ক) $x+y = ax^2+ by^2+ cx^2$ কার্ভ এর মূল বিন্দুতে বৃত্ত বক্রতা $(a+b)(x^2+y^2) = 2x+2y$ হইবে দেখাও।

(খ) $r^2= a^2\cos^2\theta$ এর লেখ চিত্র অংকন কর।

৪। (ক) মান নির্ণয় কর। (i) $\int_0^1 \frac{\log x}{1-x^2}\, dx$ (ii) $\int_0^{\pi/2} \tan^n\theta d\theta,\ -1<x<1$.

(খ) $\Gamma\frac{1}{n}\ \Gamma\frac{2}{n}\ \Gamma\frac{3}{n} \dots \Gamma\frac{n-1}{n}$ এর মান নির্ণয় কর।

৫। (ক) $x=0$, $y=0$, $x+y=1$ ত্রিভুজ এর উপর $\iint e^{\frac{y}{x+y}}\, dxdy$ এর মান নির্ণয় কর।

(খ) $x\geq 0$, $y\geq 0$, $x+y+z\leq 1$ অঞ্চলে

$\int\int\int (x+y+z+1)^2\, dxdy\, dz$ এর মান নির্ণয় কর।

(গ) প্রমাণ কর যে $\int_0^1 \frac{x^a-x^{-a}}{1-x}\, dx = \pi \cot a\pi - \frac{1}{a}$

৬। (ক) $\int_0^\infty \frac{\tan^{-1}(ax)-\tan^{-1}(bx)}{x}\, dx = \frac{\pi}{2}\log\frac{a}{b}$

(খ) $x^2+y^2= ax$ সিলিন্ডার দ্বারা $x^2+y^2+z^2=a^2$ গোলকের ছেদাংকের আয়তন নির্ণয় কর।

(গ) কোনক $z^2=3(x^2+y^2)$ এর যে অংশ অধিবৃত্তক $z=x^2+y^2$ এর দ্বারা কর্তিত পৃষ্ঠতলের ক্ষেত্রফল নির্ণয় কর।

1. (a) If $f(x,y,z)$ admits a continuous partial derivatives and satisfies the relation

$$x\frac{\delta f}{\delta x}+y\frac{\delta f}{\delta y}+z\frac{\delta f}{\delta z}=nf(x,y,z),$$

where n is a positive integer, prove that $f(x,y,z)$ is a homogeneous function of degree n.

(b) If $F(V^2-x^2,\ V^2-y^2,\ V^2-z^2)=0$

where V is a function of x, y, z, show that

$$\frac{1}{x}\frac{\delta V}{\delta x}+\frac{1}{y}\frac{\delta V}{\delta y}+\frac{1}{z}\frac{\delta V}{\delta z}=\frac{1}{V}$$

2. (a) Prove that the maximum and the minimum radii vectors of the surface

$$(x^2+y^2+z^2)^2=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}$$

by the plane $lx+my+nz=0$ are given by

$$\frac{a^2l^2}{1-a^2r^2}+\frac{b^2m^2}{1-b^2r^2}+\frac{c^2n^2}{1-c^2r^2}=0$$

(b) Prove that the centre of curvature at the point

$(a\cos\theta, b\sin\theta)$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$\left(\frac{a^2-b^2}{a}\cos^3\theta, \frac{b^2-a^2}{b}\sin^3\theta\right).$$

3. (a) Define the point of inflexion, node and cusp. Examine the nature of the origin of the curve $y^2=2x^2+x^4y-2x^4$.

(b) Trace the curve

$$y^2(x^2+y^2)+16x^2-4x(x^2+2y^2)=0$$

4. (a) Examine the convergence of the improper integral $\int_0^\pi \frac{dx}{\sin x}$

(b) Show that $\int_0^\infty e^{-x}x^n=n!$,
n being a positive integer.

5. (a) prove that $B(l,m)=\frac{\Gamma(l)\Gamma(m)}{\Gamma(l+m)}$

Hence show that $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$.

(b) Find the value of $\int_0^\infty \frac{\log(1+a^2x^2)}{1+b^2x^2}\,dx$

6. (a) Find the volume cut from the sphere $x^2+y^2+z^2=a^2$ by the cylinder $x^2+y^2=ax$.

(b) Show that area botween the parabola $y^2=4x$ and the atraight line $y=2x-4$ is 9.

(c) Change the order of integration in $\int_0^\infty\int_0^\infty \frac{e^{-y}}{y}\,dxdy$ and hence find its value.

জাতীয় বিশ্ববিদ্যালয়-১৯৯৮
গণিত
প্রথম পত্র

১। (ক) পূর্ণসংখ্যা সেটে "কনগ্রুয়েন্স মডুলো m" সংজ্ঞায়িত কর। দেখাও যে, যদি $a\equiv b$ (মড m) এবং $c\equiv d$ (মড n) হয়, তবে $na-mc\equiv nb-md$ (মড mn) হয়।

(খ) দেকার্তেজের চিহ্নের নিয়ম উদাহরণসহ বর্ণনা কর।

(গ) জটিল সংখ্যাকে জ্যামিতিক স্থানাঙ্কের নিয়মে প্রকাশ করে ইহার আরগুমেন্ট ও এম্প্লিসিউড বের কর।

(ঘ) দ্বিমাত্রিকে একটি দ্বিঘাত সাধারণ সমীকরণ দ্বারা একটি প্যারাবোলা প্রকাশ করার শর্তগুলো নির্ণয় কর।

(ঙ) ত্রিমাত্রিক জ্যামিতিতে একটি সরল রেখার সমীকরণ বের কর, যখন সরল রেখাটি নির্দিষ্ট একটি বিন্দু দিয়ে গমন করে এবং সরল রেখাটির নির্দিষ্ট দিক কোসাইন দেওয়া আছে।

ক বিভাগ

২।(ক) p একটি মৌলিক সংখ্যা হ'লে দেখাও যে, $\underline{|p-1}+1\equiv 0$ (মড p).

(খ) দেখাও যে $\underline{|28}+233$ সংখ্যাটি 899 দ্বারা বিভাজ্য।

(গ) সমাধান কর : $78x\equiv 26$ (মড 143.)

৩। (ক) যদি $p>1$ এবং a, b দু'টি ধনাত্মক পূর্ণ সংখ্যা $a>b$ হয়, তবে দেখাও যে, $\frac{p^a-1}{a}-\frac{p^b-1}{b}$
$>\frac{1}{2}(a-b)(p-1)^2$

(খ) যদি $x^3+5x^2+7x+9=0$ সমীকরণের মূলগুলো a, b, c হয়, তবে
$(b+c-3a)(c+a-3b)(a+b-3c)$-এর মান বের কর।

৪। (ক) যদি a, b, c যে- কোন বাস্তব সংখ্যা হয় তবে দেখাও যে,
$(b-c-a)^2+(c+a-b)^2+(a+b-c)^2>(bc+ca+ab)$.

(খ) দেখাও যে, $\left(\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^m=\cos\left(\frac{m\pi}{2}-m\theta\right)+i\sin\left(\frac{m\pi}{2}-m\theta\right)$.

৫। (ক) যদি $x^3+px^2+qx+r=0$ সমীকরণের মূলগুলো α, β, γ হয়, তবে $\alpha^2+\beta^2, \beta^2+\gamma^2, \gamma^2+\alpha^2$ মূল দ্বারা গঠিত সমীকরণটি বের কর।

(খ) $(3+4i)^{\alpha+i\beta}$ কে বাস্তব ও কাল্পনিক অংশে বিশ্লেষণ কর।

৬। যে-কোন দুইটি ধারার সমষ্টি নির্ণয় কর ঃ –

(i) $\frac{3}{2.4}+\frac{3.4}{2.4.6}+\frac{3.4.5}{2.4.6.8}+$ .... অসীম পদ পর্যন্ত;

(ii) $\tan\frac{1}{2}\theta\sec\theta+\tan\frac{1}{2^2}\theta\sec\frac{1}{2}\theta+\tan\frac{1}{2^3}\theta\sec\frac{1}{2^2}\theta+$ ...... অসীম পর্যন্ত;

(iii) $\cos^2\theta-\frac{1}{3}\cos^3\theta\cos3\theta+\frac{1}{5}\cos^5\theta\cos5\theta-$........n পদ পর্যন্ত।

খ বিভাগ

৭। (ক) $ax^2+2hxy+by^2+4gx+4fy+c=0$ সমীকরণটি এক জোড়া সরল রেখা নির্দেশ করলে তাদের অন্তর্ভুক্ত কোণগুলো বের কর। সরল রেখাদ্বয় পরস্পর লম্ব ও সমান্তরাল হওয়ার শর্তগুলো নির্ণয় কর।

(খ) প্রমাণ কর যে, মূল বিন্দুর সাথে সরল রেখা $ax+by=2ab$ ও বক্ররেখা $(x-b)+(y-a)^2=c^2$ এর ছেদ বিন্দুদ্বয়ের সংযোগকারী সরল রেখা দুইটি পরস্পরের সাথে লম্ব হবে যদি $a^2+b^2=c^2$ হয়।

৮।(ক) দেখাও যে, $9x^2+24xy+16y^2-2x+14y+1=0$ সমীকরণটি একটি প্যারাবোলা নির্দেশ করে। এই প্যারাবোলার শীর্ষ বিন্দু ও নাভির স্থানাঙ্ক এবং অক্ষরেখা ও শীর্ষ বিন্দুতে স্পর্শকের সমীকরণ নির্ণয় কর।

(খ) $(0, 2)$ ও $(0,-2)$ বিন্দুদ্বয়গামী দুইটি বৃত্ত $y=mx+c$ সরল রেখাকে স্পর্শ করে। প্রমাণ কর যে, যদি উহারা সমকোণে ছেদ করে, তবে $c^2=4(m^2+2)$.

৯।(ক) প্রমাণ কর যে, সরল রেখার দিক কোসাইন $pl+qm+rn=0$ এবং $alm+bmn+cnl=0$ সম্বন্ধযুক্ত হ'লে উহারা পরস্পর লম্ব হবে,

যদি $\frac{b}{p}+\frac{c}{q}+\frac{a}{r}=0$ হয়।

(খ) (a, b, c) বিন্দুগামী এবং $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ ও $\frac{x}{p}=\frac{y}{q}=\frac{z}{r}$ এর সমান্তরাল সমতলের সমীকরণ নির্ণয় কর।

১০।(ক) $\frac{x-3}{3}=\frac{y-5}{5}=\frac{z-7}{7}$ এবং $\frac{x-2}{2}=\frac{y-4}{4}=\frac{z-6}{6}$ দ্বারা নির্দেশিত সরল রেখা দুটোর ক্ষুদ্রতম দূরত্বের দৈর্ঘ্য এবং এর সমীকরণ বের কর।

(খ) গোলকের সমীকরণ নির্ণয় কর যাহা বৃত্ত $x^2+y^2+z^2+2x+3y+6=0$, $x-2y+4z-7=0$ দিয়ে এবং গোলক $x^2+y^2-z^2-2x+4y-6z+5=0$ এর কেন্দ্র দিয়ে যায়।

১১।(ক) যদি $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ উপবৃত্তটির এক জোড়া অনুবন্ধী অর্ধব্যাসের প্রান্ত বিন্দুদ্বয়ের উৎকেন্দ্রিক কোণ ϕ এবং ϕ' হয়, তবে প্রমাণ করে যে, $\phi'-\phi=\pm\frac{\pi}{2}$.

(খ) $ax+by+cz=1$ সমতলটি $x^2-2y^2=3z$ প্যারাবোলয়েডটিতে স্পর্শ সমতল হওয়ার শর্ত বের কর।

(English Version)

1. (a) Define "Congruence modulo m" on the set of integers. Show that if $a\equiv b \pmod m$ and $c\equiv d \pmod n$, then $na-mc\equiv nb-md \pmod{mn}$.

(b) State descarte's rule of signs with examples.

(c) Expressing a complex number in co-ordinate system of geometry, find its argument and amplitude.

(d) Find the conditions that a second degree general equation in two dimensions shall represent a parabola.

(e) Find the equation of a straight line in three dimensional geometry, when it passes through a fixed point and it has fixed direction cosines.

Group —A

2. (a) If p is a prime number, then show that $\left|\underline{p-1}\right.+1\equiv 0 \pmod p$.

(b) Show that the number $\left|\underline{28}\right.+233$ is divisible by 899.

(c) Solve : $78x\equiv 26 \pmod{143}$.

3. (a) If $p>1$ and a, b are positive integers with $a>b$, then show that

$$\frac{p^a-1}{a}-\frac{p^b-1}{b}>\frac{1}{2}(a-b)(p-1)^2$$

(b) If a, b, c are the roots of the equation $x^3+5x^2+7x+9=0$ then find the value of $(b+c-3a)(c+a-3b)(a+b-3c)$

4. (a) If a, b, c are any real numbers, show that $(b+c-a)^2+(c+a-b)^2+(a+b-c)^2>(bc+ca+ab)$.

(b) Show that—$\left(\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^m=\cos\left(\frac{m\pi}{2}-m\theta\right)+i\sin\left(\frac{m\pi}{2}-m\theta\right)$.

5. (a) If α, β, γ are the roots of the equation $x^3+px^2+qx+r=0$, form the equations whose roots are $\alpha^2+\beta^2, \beta^2+\gamma^2, \gamma^2+\alpha^2$

(b) Separate $(3+4i)^{\alpha+i\beta}$ into real and imaginary parts.

6. Find the sum of any two of the following series :—

(i) $\frac{3}{2.4}+\frac{3.4}{2.4.6}+\frac{3.4.5}{2.4.6.8}+$............up to infinity;

(ii) $\tan\frac{1}{2}\theta\sec\theta+\tan\frac{1}{2^2}\theta\sec\frac{1}{2}\theta+\tan\frac{1}{2^3}\theta\sec\frac{1}{2^2}\theta+$....... up to infinity;

(iii) $\cos^2\theta-\frac{1}{3}\cos^3\theta\cos3\theta+\frac{1}{5}\cos^5\theta\cos5\theta-$......up to n-terms.

Group-B

7. (a) If the equation $ax^2+2hxy+by^2+4gx+4fy+c=0$ respresents a pair of straight lines, then find the angles between them. Find the conditions for which these lines shall be perpendicular to each other and also parallel to each other.

(b) Prove that two straight lines joining the origin to the two points of intersection of the straight line $ax+by=2ab$ with the curve $(x-b)^2+(y-a)^2=c^2$ will be perpendicular to each other if $a^2+b^2=c^2$.

8. (a) Show that $9x^2+24xy+16y^2-2x+14y+1=0$ represents a parabola. Find the co-ordinates of the vertex, focus and the equations of the axis and the tangent at the vertex of this parabola.

(b) Prove that, if the two circles, which pass through the points $(0, 2)$ and $(0,-2)$ and touch the line $y=mx+c$, cut orthogonally, then $c^2=4(m^2+2)$.

9. (a) Prove that straight lines whose direction cosines are connected by the relation $pl+qm+rn=0$ and $alm+bmn+cnl=0$ are perpendicular if $\frac{b}{p}+\frac{c}{q}+\frac{a}{r}=0$.

(b) Find the equation of the plane through the point (a, b, c) and parallel to the lines $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ ও $\frac{x}{p}=\frac{y}{q}=\frac{z}{r}$.

10. (a) Find the shortest distance and the equation of the line of the shortest distance between the lines whose equations are $\frac{x-3}{3}=\frac{y-5}{5}=\frac{z-7}{7}$ and $\frac{x-2}{2}=\frac{y-4}{4}=\frac{z-6}{6}$.

(b) find the equation of the sphere through the circle $x^2+y^2+z^2+2x+3y+6=0$, $x-2y+4z-7=0$ and the centre of the sphere $x^2+y^2-z^2-2x+4y-6z+5=0$.

11.(a) If ϕ and ϕ' are eccentric angles of the end points of a pair of conjugate semi-diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

then prove that $\phi'-\phi=\pm\frac{\pi}{2}$.

(b) Find the condition that the plane $ax+by+cz=1$ may be tangent plane to the paraboloid $x^2-2y^2=3z$.

জাতীয় বিশ্ববিদ্যালয়-১৯৯৮
গণিত
দ্বিতীয় পত্র
ক বিভাগ- অন্তরকলন

১।(ক) কোন বিন্দুতে একটি ফাংশনের অবিচ্ছিন্নতার সংজ্ঞা দাও। f(x) ফাংশনটি নিম্নরূপে বর্ণিত :-

$$f(x)=\begin{cases}5x-4; & 0<x\le 1\\ 4x^2-3x; & 1<x<2.\end{cases}$$

x=1 বিন্দুতে ফাংশনটি অবিচ্ছিন্ন কিনা তা নির্ণয় কর।

(খ) আদি সূত্রের সাহায্যে x-এর সাপেক্ষে $\sin^2 x$-এর অন্তরক সহগ নির্ণয় কর।

২।(ক) $\frac{dy}{dx}$ নির্ণয় কর (যে-কোন দুইটি) :-

(i) $y=x^{\log x}+x^{\sin^{-1}x}$; (ii) $x^p y^q=(x+y)^{p+q}$;

(iii) $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$.

(খ) লিবনিজ-এর উপপাদ্যটি বর্ণনা ও প্রমাণ কর।

৩।(ক) যদি $u=\sin^{-1}\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$ হয়, তবে দেখাও যে, $x\frac{\delta u}{\delta x}+y\frac{\delta u}{\delta y}=0$.

(খ) $y(x^2-y^2)=y(x-y)+2$ বক্ররেখার অসীমতটগুলি নির্ণয় কর।

৪।(ক) প্রমাণ কর যে, প্রদত্ত বৃত্তের মধ্যে অঙ্কিত আয়তক্ষেত্রগুলির মধ্যে বর্গই বৃহত্তম।

(খ) $\left(\pm\frac{a}{\sqrt{3}}, \frac{a}{4}\right)$ বিন্দুতে $y(x^2+a^2)=ax^2$ রেখাটির স্পর্শকের সমীকরণ নির্ণয় কর।

খ বিভাগ-যোগজকলন

৫। যে-কোন তিনটির যোগজীকরণ কর :-

(ক) $\int\frac{\cos x\,dx}{5-3\cos x}$; (খ) $\int\frac{\sec x\,dx}{a+b\tan x}$;

(গ) $\int\frac{3x+2}{5x^2+2x+3}dx$; (ঘ) $\int e^x\left(\frac{1-x}{1+x^2}\right)^2dx$.

৬। যে-কোন তিনটির মান নির্ণয় কর :

(ক) $\int_0^{\pi/2}\frac{\cos x\,dx}{\sin x+\cos x}$; (খ) $\int_0^{\pi/2}\frac{(\sin\theta)^{3/2}d\theta}{(\sin\theta)^{3/2}+(\cos\theta)^{3/2}}$;

(গ) $\int_{-2}^{2}x^9(1-x^2)^7dx$; (ঘ) $\int_0^{\infty}\frac{x^2dx}{(x^2+a^2)(x^2+b^2)}$; [a, b>0].

৭।(ক) মান নির্ণয় কর :- $\lim_{n\to\infty}\left[\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\cdots\cdots+\frac{n}{2n^2}\right]$

(খ) দেখাও যে, $\int_0^{\pi/4}\log(1+\tan\theta)d\theta=\frac{\pi}{8}\log 2$.

(গ) যদি $I_n=\int_0^{\pi/4}\tan^n\theta\,d\theta$ হয়, তবে দেখাও যে, $I_n=\frac{1}{n-1}-I_{n-2}$ অতপর $\int_0^{\pi/4}\tan^6x\,dx$ এর মান নির্ণয় কর।

৮।(ক) দেখাও যে, $y^2=4ax$ এবং $x^2=4ay$ বক্ররেখাদ্বয় দ্বারা সীমাবদ্ধ ক্ষেত্রের ক্ষেত্রফল $16a^2/3$.

(খ) $r(1+\cos\theta)=2$ রেখাটির $\theta=0$ হইতে $\theta=\pi/2$ পর্যন্ত অংশটির দৈর্ঘ্য নির্ণয় কর।

গ বিভাগ -অন্তরক সমীকরণ

৯।(ক) সমাধান কর :

(i) $(x+y)(dx-dy)=dx+dy$; (ii) $(x^2+y^2)\frac{dy}{dx}=xy$;

(iii) $(2x-5y+3)dx-(2x+4y-6)dy=0$

(খ) সমাধান কর :

(i) $(D^2+4)y=\sin 3x$; (ii) $(D^2+5D+4)y=0$.

১০।(ক) নিম্নের রেখাগুলির সমকোণছেদী রেখা গোত্রের সমীকরণ বাহির কর :

$y^2=2x^2(1-cx)$.

(খ) সাধারণ সমাধান নির্ণয় কর :- $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2x=0$. যখন x=0, y=2 এবং $\frac{dy}{dx}=0$

(গ) সমাধান কর : $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=e^x\cos x$.

[English Version]
Group A—Differential Calculus

1. (a) Define the continuity of a function at a point. A function $f(x)$ is defined as follows :—

$$f(x)=\begin{cases}5x-4; & 0<x\le 1\\ 4x^2-3x; & 1<x<2.\end{cases}$$

Find whether the function is continuous at x=1 or not.

(b) Differntiate $\sin^2 x$ With respect to x from first principle.

2. (a) Find $\frac{dy}{dx}$ of any two of the following :—

(i) $y=x^{\log x}+x^{\sin^{-1}x}$; (ii) $x^p y^q=(x+y)^{p+q}$;

(iii) $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$.

(b) State and prove Leibnitz's theorem.

3. (a) If $u=\sin^{-1}\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$, then show that $x\frac{\delta u}{\delta x}+y\frac{\delta u}{\delta y}=0$.

(b) Find the asymptotes of the curve $y(x^2-y^2)=y(x-y)+2$.

4\. (a) Show that the maximum rectangle inscribed in a circle is a square.

(b) Find the equations of the tangents to the curve $y(x^2+a^2)=ax^2$ at the points $\left(\pm\frac{a}{\sqrt{3}}, \frac{a}{4}\right)$.

GROUP B— INTEGRAL CALCULUS

5\. Integrate any three of the following :—

(a) $\int \frac{\cos x\, dx}{5-3\cos x}$; (b) $\int \frac{\sec x\, dx}{a+b\tan x}$;

(c) $\int \frac{3x+2}{5x^2+2x+3}dx$; (d) $\int e^x\left(\frac{1-x}{1+x^2}\right)^2 dx$.

6\. Evaluate any three to following :—

(a) $\int_0^{\pi/2} \frac{\cos x\, dx}{\sin x+\cos x}$; (b) $\int_0^{\pi/2} \frac{(\sin\theta)^{3/2}d\theta}{(xin\theta)^{3/2}+(\cos\theta)^{3/2}}$;

(c) $\int_{-2}^{2} x^9(1-x^2)^7dx$; (d) $\int_0^\infty \frac{x^2dx}{(x^2+a^2)(x^2+b^2)}$; $[a, b>0]$.

7\. (a) Evaluate :— $\lim_{n\to\infty}\left[\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\dots\dots\dots+\frac{n}{2n^2}\right]$

(b) Show that $\int_0^{\pi/4} \log(1+\tan\theta)d\theta=\frac{\pi}{8}\log 2$.

(c) If $I_n=\int_0^{\pi/4}\tan^n\theta d\theta$, then show that $I_n=\frac{1}{n-1}-I_{n-2}$. Hence find the value of $\int_0^{\pi/2}\tan^6 x\, dx$.

8\. (a) Show that the area enclosed by the curves $y^2=4ax$ and $x^2=4ay$ is $16a^2/3$.

(b) Find the length of the arc of the curve $r(1+\cos\theta)=2$ from $\theta=0$ to $\theta=\pi/2$. 8

Group C—Differential Equations

9\. (a) Solve :—

(i) $(x+y)(dx-dy)=dx+dy$; (ii) $(x^2+y^2)\frac{dy}{dx}=xy$;

(iii) $(2x-5y+3)dx-(2x+4y-6)dy=0$

(b) Solve :—(i) $(D^2+4)y=\sin 3x$; (ii) $(D^2+5D+4)y=0$.

10\.(a) Find the orthogonal trajectories of $y^2=2x^2(1-cx)$.

(b) Find the general solution:—

$\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2x=0$. when $x=0$. $y=2$ and $\frac{dy}{dx}=0$

(c) Solve:— $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=e^x\cos x$.

জাতীয় বিশ্ববিদ্যালয়–১৯৯৮
তৃতীয় পত্র
গাণিতিক পদ্ধতি ও যোগাশ্রয়ী বীজগণিত

১। যে কোন চারটি প্রশ্নের উত্তর দাও ঃ–

(ক) পর্যায়ী, যুগ্ম ও অযুগ্ম ফাংশনের সংজ্ঞা দাও। দেখাও যে, $\sin x$ একটি পর্যায়ী ফাংশন। ইহার পর্যায়কাল কত?

(খ) কোন রৈখিক অন্তরক সমীকরণকে ধারায় সমাধানের ফ্রবিনিয়াস-এর পদ্ধতি বর্ণনা কর।

(গ) যদি একটি সার্বিক সেট U এর উপসেট A, B, C হয়, তবে ভেন-চিত্রের সাহায্যে নিম্নলিখিত সূত্রগুলোর সত্যতা যাচাই কর ঃ

(i) $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$;

(ii) $(A\cup B)'=A'\cap B'$. যেখানে "′" সেটের পূরক নির্দেশ করে।

(ঘ) $v_1, v_2, u_3, \dots\dots\dots\dots, vk$. ভেক্টরসমূহের যোগাশ্রয়ী নির্ভরশীলতা ও অনির্ভরশীলতার সংজ্ঞা দাও R^3 এ $u=(1, 1, -1)$. $v=(1, 0, 2)$ এবং $w=(1, 1, 1)$ ভেক্টরগুলোর যোগাশ্রয়ী নির্ভরশীলতা বা অনির্ভরশীলতা যাচাই কর।

(ঙ) দুইটি ভেক্টরের ডট ও ক্রশ গুণনের সংজ্ঞা দাও। $3i+2j-6k$ এবং $4i-3j+k$ ভেক্টরদ্বয়ের মধ্যবর্তী কোণ নির্ণয় কর।

ক বিভাগ–গাণিতিক পদ্ধতি

২।(ক) ফুরিয়ার ধারার সংজ্ঞা দাও এবং দেখাও যে, যুগ্ম ফাংশনের ফুরিয়ার ধারায় কোন সাইন পদ থাকে না।

(খ) $f(x)$ ফাংশনটিকে ফুরিয়ার ধারায় প্রকাশ কর যেখানে

$$f(x)=\begin{cases}-1, & -\pi\le x\le 0\\ 1, & 0\le x\le\pi.\end{cases}$$

অতঃপর দেখাও যে, $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\dots\dots\dots\infty=\frac{\pi}{4}$

৩।(ক) লেজেন্ডার বহুপদী $P_n(x)$ এর রড্রিগ্স এর সূত্র হতে $P_0(x)$, $P_1(x)$, $P_2(x)$, $P_3(x)$ বের কর এবং $2x^3+x^2-5x-4$ কে লেজেণ্ডার বহুপদীগুলোর মাধ্যমে প্রকাশ কর।

(খ) প্রমাণ কর যে, $\int_{-1}^{1}[P_n(t)]^2dt=\frac{2}{2n+1}$

৪।(ক) প্রথম প্রকারের বেসেল ফাংশন $J_n(x)$ এর সংজ্ঞা দাও এবং প্রমাণ কর যে,

$$\frac{d}{dx}\{x^nJ_n(ax)\}=ax^nJ_{n-1}(ax).$$

(খ) প্রমাণ কর ঃ $e^{\frac{x}{2}\left(t-\frac{1}{t}\right)}=\sum_{n=0}^{\infty}J_n(x)t^n$, $n\ge 0$. অতঃপর দেখাও যে,

$$x\{J_{n+1}(x)+J_{n-1}(x)\}=2nJ_n(x).$$

৫।(ক) বটা ফাংশন $\beta(m, n)$ এর সংজ্ঞা দাও এবং প্রমাণ কর যে,

$\int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}dt=\beta(m, n)$. আরও দেখাও যে, $\int_0^1 t^6\sqrt{(1-t^2)}\,dt=\frac{5\pi}{256}$

(খ) সমতলে গ্রীণ এর উপপাদ্যটি বর্ণনা কর এবং $\oint_c (2xy-x^2)dx+(x+y^2)dy$ এর জন্য ইহার সত্যতা যাচাই কর, যেখানে c হল $y=x^2$ এবং $y^2=x$ দ্বারা আবদ্ধ এলাকা।

English Version

1. (a) Define periodic, even and odd functions. Show that sin x is a periodic function. What is its period?

(b) Describe the method of Frobenious to find the solution in series of a linear differential equation.

(c) If A, B, C are subsets of a universal set U, then verify the following formulae by using Venn diagrams:—

(i) $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$;

(ii) $(A\cup B)' = A'\cap B'$, where "$'$" denotes the complement of a set.

(d) define linear dependence and independence of the vectors $v_1, v_2, v_3, \ldots\ldots\ldots vk$. Test the linear dependence or independence of the vectors $u=(1, 1, -1)$, $v = (1, 0, 2)$ and $w= (1, 1, 1)$ in R^3

(e) Define the dot and cross product of two vectors. Find the angle between the vetors $3i+2j-6k$ and $4i-3j+k$.

Group A—Mathematical Methods

2. (a) Define a Fourier series and show that an even function can have no sine terms in its Fourier expansion.

(b) Expand $f(x)$ Fourier series, where

$$f(x)=\begin{cases}-1, & -\pi\leq x\leq 0\\ 1, & 0\leq x\leq \pi.\end{cases}$$

Hence show that $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\ldots\ldots\ldots\infty=\frac{\pi}{4}$

3. (a) Find $P_0(x)$, $P_1(x)$, $P_2(x)$, $P_3(x)$ from Rodrigue's formula for Legendre polynomials $P_n(x)$ and express $2x^3+x^2-5x-4$ in terms of the Legendre polynomials.

(b) Prove that $\int_{-1}^{1} [P_n(t)]^2 dt= \frac{2}{2n+1}$

4. (a) Define Bessel function $J_n(x)$ of the first kind and prove that

$$\frac{d}{dx}\{x^n J_n(ax)\}=ax^n J_{n-1}(ax).$$

(b) Prove : $e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=0}^{\infty} J_n(x)t^n,\ n\geq 0.$

Hence show that $x\{J_{n+1}(x)+J_{n-1}(x)\} = 2nJ_n(x)$.

5. (a) Define Beta function $\beta(m, n)$ and prove that

$\int_0^{\infty} \frac{t^{m-1}}{(1+t)^{m+n}} dt=\beta(m, n)$. Also show that

$$\int_0^1 t^6\sqrt{(1-t^2)}\, dt = \frac{5\pi}{256}$$

(b) State Green's theorem in the plane and verify it for $\oint_c (2xy-x^2)dx+(x+y^2)dy$ where c is the closed curve of the region bounded by $y = x^2$ and $y^2 =x$