Complex Numbers

1. **Definition:** A number of the form $a+ib$ where a and b are real numbers and $i=\sqrt{-1}$, is called a complex number.

If $z=a+ib$, then a is called the real part of z and b is called the imaginary part of z.

2. Modulus and Amplitude:

If the polar coordinates of the point (a,b) be (r,θ), then $a=r\cos\theta$ and $b=r\sin\theta$

$\therefore$ $r=\sqrt{a^2+b^2}$ and $\theta=\tan^{-1}\left(\frac{b}{a}\right)$

The number r is called the modulus or absolute value and θ is called the amplitude or argument of the complex number $z=a+ib$.

In symbols, we write

$$r=\mathrm{mod}\,z=|z|=|a+ib|=\sqrt{a^2+b^2}$$

$$\theta=\mathrm{amp}\,z=\arg z=\tan^{-1}\left(\frac{b}{a}\right)$$

Now, $z=a+ib=r(\cos\theta+i\sin\theta)=re^{i\theta}$

3. State and Prove De Moivre's theorem.

Statement: For all rational values of n,

$$(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta.$$

Proof: Case-1: When n is a positive integer.

We have, $(\cos\theta_1+i\sin\theta_1)(\cos\theta_2+i\sin\theta_2)$

$$=\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2+i(\sin\theta_1\cos\theta_2+\cos\theta_1\sin\theta_2)$$

$$=\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)$$

Similarly, $(\cos\theta_1+i\sin\theta_1)(\cos\theta_2+i\sin\theta_2)(\cos\theta_3+i\sin\theta_3)$

$$= \{\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)\}(\cos\theta_3+i\sin\theta_3)$$

$$= \cos(\theta_1+\theta_2+\theta_3)+i\sin(\theta_1+\theta_2+\theta_3)$$

Proceeding in this way, the product of the n factors

$$(\cos\theta_1+i\sin\theta_1)(\cos\theta_2+i\sin\theta_2)\cdots\cdots(\cos\theta_n+i\sin\theta_n)$$

$$= \cos(\theta_1+\theta_2+\cdots+\theta_n)+i\sin(\theta_1+\theta_2+\cdots+\theta_n)$$

If we put $\theta_1=\theta_2=\cdots=\theta_n=\theta$, then we have

$$(\cos\theta+i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

<u>Case-2</u>: When n is a negative integer.

Let $n=-m$, where m is a positive integer.

$$(\cos\theta+i\sin\theta)^n = (\cos\theta+i\sin\theta)^{-m}$$

$$= \frac{1}{(\cos\theta+i\sin\theta)^m}$$

$$= \frac{1}{\cos m\theta+i\sin m\theta} \quad [\text{by Case-I}]$$

$$= \frac{1\times(\cos m\theta-i\sin m\theta)}{(\cos m\theta+i\sin m\theta)(\cos m\theta-i\sin m\theta)}$$

$$= \frac{\cos m\theta-i\sin m\theta}{\cos^2 m\theta+\sin^2 m\theta}$$

$$= \cos m\theta - i\sin m\theta$$

$$= \cos(-n)\theta - i\sin(-n)\theta \quad [\because m=-n]$$

$$= \cos n\theta + i\sin n\theta$$

<u>Case-3</u>: When n is a fraction, +ve or -ve.

Let $n=\frac{p}{q}$, where q is a positive integer and p is any integer, +ve or -ve.

Now, $\left(\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta\right)^q = \cos\left(q\cdot\frac{p}{q}\theta\right) + i\sin\left(q\cdot\frac{p}{q}\theta\right)$ [by case 1]

$= \cos p\theta + i\sin p\theta$

$= (\cos\theta + i\sin\theta)^p$, since p is any integer.

Taking qth root, we get

$\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta$ = one of the values of $(\cos\theta + i\sin\theta)^{\frac{p}{q}}$

or, $\cos n\theta + i\sin n\theta$ = One of the values of $(\cos\theta + i\sin\theta)^n$

Thus, De Moivre's theorem is completely established for all rational values of n.

4. If $x_r = \cos\frac{\pi}{2^r} + i\sin\frac{\pi}{2^r}$, prove that

$$x_1 x_2 x_3 \cdots\cdots \infty = -1$$

Solution: LHS $= x_1 x_2 x_3 \cdots\cdots \infty$

$= \left(\cos\frac{\pi}{2^1} + i\sin\frac{\pi}{2^1}\right)\left(\cos\frac{\pi}{2^2} + i\sin\frac{\pi}{2^2}\right)\left(\cos\frac{\pi}{2^3} + i\sin\frac{\pi}{2^3}\right)\cdots\infty$

$= \cos\left(\frac{\pi}{2^1} + \frac{\pi}{2^2} + \frac{\pi}{2^3} + \cdots\infty\right) + i\sin\left(\frac{\pi}{2^1} + \frac{\pi}{2^2} + \frac{\pi}{2^3} + \cdots\infty\right)$

$= \cos\left(\frac{\pi/2}{1-1/2}\right) + i\sin\left(\frac{\pi/2}{1-1/2}\right)$ $\left[\because S_\infty = \frac{a}{1-r}\right]$

$= \cos\pi + i\sin\pi$

$= -1 + i\cdot 0$

$= -1$

$=$ R.H.S

5. If $x_r = \cos\frac{\pi}{3^r} + i\sin\frac{\pi}{3^r}$, show that

$$x_1 x_2 x_3 \cdots\cdots \infty = i$$

Solution: LHS $= x_1 x_2 x_3 \cdots\cdots \infty$

$= \left(\cos\frac{\pi}{3^1} + i\sin\frac{\pi}{3^1}\right)\left(\cos\frac{\pi}{3^2} + i\sin\frac{\pi}{3^2}\right)\cdots\cdots\infty$

$= \cos\left(\frac{\pi}{3^1} + \frac{\pi}{3^2} + \frac{\pi}{3^3} + \cdots\infty\right) + i\sin\left(\frac{\pi}{3^1} + \frac{\pi}{3^2} + \frac{\pi}{3^3} + \cdots\infty\right)$

$\therefore$ LHS $= \cos\left(\frac{\pi/3}{1-1/3}\right) + i\sin\left(\frac{\pi/3}{1-1/3}\right)$

$= \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

$= 0 + i\cdot 1$

$= i$

$=$ R.H.S

6. If n be a positive integer, prove that

(i) $(1+i)^n + (1-i)^n = 2^{\frac{n}{2}+1}\cos\frac{n\pi}{4}$

(ii) $(\sqrt{3}+i)^n + (\sqrt{3}-i)^n = 2^{n+1}\cos\frac{n\pi}{6}$

Solution: (i) Let $1 = r\cos\theta$ ... (1)

$1 = r\sin\theta$ ... (2)

Squaring (1) and (2), then adding

$2 = r^2(\cos^2\theta + \sin^2\theta)$

$= r^2\cdot 1$

$\therefore r = \sqrt{2}$

Dividing (2) by (1), we get $\tan\theta = 1$

or, $\tan\theta = \tan\frac{\pi}{4}$

$\therefore \theta = \frac{\pi}{4}$

Now, $1+i = 1 + i\cdot 1$

$= r\cos\theta + ir\sin\theta$

$= r(\cos\theta + i\sin\theta)$

$= \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$\therefore (1+i)^n = \left\{\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right\}^n$

$= 2^{n/2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^n$

$= 2^{n/2}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right)$ .... (3)

Similarly, $(1-i)^n = 2^{n/2}\left(\cos\frac{n\pi}{4} - i\sin\frac{n\pi}{4}\right)$ .... (4)

Adding (3) and (4), we get

$(1+i)^n + (1-i)^n = 2\cdot 2^{n/2}\cos\frac{n\pi}{4}$

$= 2^{n/2+1}\cos\frac{n\pi}{4}$

(ii) Let $\sqrt{3} = r\cos\theta$ --- (1)

$1 = r\sin\theta$ ---- (2)

Squaring (1) and (2), then adding

$$3+1 = r^2(\cos^2\theta + \sin^2\theta)$$

$$\text{or, } 4 = r^2 \cdot 1$$

$$\therefore r = 2$$

Dividing (2) by (1), $\tan\theta = \frac{1}{\sqrt{3}}$

$$\text{or, } \tan\theta = \tan\frac{\pi}{6}$$

$$\therefore \theta = \frac{\pi}{6}$$

Now, $\sqrt{3}+i = \sqrt{3}+i\cdot 1$

$$= r\cos\theta + i\, r\sin\theta$$

$$= r(\cos\theta + i\sin\theta)$$

$$= 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

$$\therefore (\sqrt{3}+i)^n = \left\{2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right\}^n$$

$$= 2^n\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^n$$

$$= 2^n\left(\cos\frac{n\pi}{6} + i\sin\frac{n\pi}{6}\right) \quad \text{--- (3)}$$

Similarly, $(\sqrt{3}-i)^n = 2^n\left(\cos\frac{n\pi}{6} - i\sin\frac{n\pi}{6}\right)$ -- (4)

Adding (3) and (4), we get

$$(\sqrt{3}+i)^n + (\sqrt{3}-i)^n = 2\cdot 2^n \cos\frac{n\pi}{6}$$

$$= 2^{n+1}\cos\frac{n\pi}{6}$$

7. Determine the locus represented by

(i) $|z-2| = 3$ (ii) $|z-2| = |z+4|$ (iii) $|z-3|+|z+3| = 10$

Solution: (i) $|z-2| = 3$

or, $|x+iy-2| = 3$

or, $|(x-2)+iy| = 3$

or $\sqrt{(x-2)^2+y^2} = 3$

or, $(x-2)^2 + y^2 = 3^2$ which represents a circle of radius 3 and centre $(2,0)$.

(ii) $|z-2| = |z+4|$

or, $|x+iy-2| = |x+iy+4|$

or, $(x-2)^2 + y^2 = (x+4)^2 + y^2$

or, $x = -1$ which represents a straight line.

(iii) $|z-3| + |z+3| = 10$

or, $\sqrt{(x-3)^2 + y^2} + \sqrt{(x+3)^2 + y^2} = 10$

or, $\sqrt{(x-3)^2 + y^2} = 10 - \sqrt{(x+3)^2 + y^2}$

or, $(x-3)^2 + y^2 = 100 - 20\sqrt{(x+3)^2 + y^2} + (x+3)^2 + y^2$

or, $12x + 100 = 20\sqrt{(x+3)^2 + y^2}$

or, $3x + 25 = 5\sqrt{(x+3)^2 + y^2}$

or, $9x^2 + 150x + 625 = 25(x^2 + 6x + 9 + y^2)$

or, $16x^2 + 25y^2 = 400$

or, $\frac{x^2}{25} + \frac{y^2}{16} = 1$

or, $\frac{x^2}{5^2} + \frac{y^2}{4^2} = 1$ which represents an ellipse.

8. If $x + \frac{1}{x} = 2\cos\theta$, show that $x^n + \frac{1}{x^n} = 2\cos n\theta$

Solution: We have, $x + \frac{1}{x} = 2\cos\theta$

or, $x^2 - 2x\cos\theta + 1 = 0$

$\therefore x = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$

$= \cos\theta \pm i\sin\theta$

Take +ve sign only, $x = \cos\theta + i\sin\theta$

Now LHS $= x^n + \frac{1}{x^n}$

$= (\cos\theta + i\sin\theta)^n + (\cos\theta + i\sin\theta)^{-n}$

$= \cos n\theta + i\sin n\theta + \cos(-n)\theta + i\sin(-n)\theta$ [By De Moivre's theorem

$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$= 2\cos n\theta$

$=$ RHS

9. If $x = \cos\theta + i\sin\theta$ and $1+\sqrt{1-a^2} = na$, prove that
$$1 + a\cos\theta = \frac{a}{2n}(1+nx)\left(1+\frac{n}{x}\right)$$

Solution: Given, $x = \cos\theta + i\sin\theta$

$$\therefore \frac{1}{x} = \frac{1}{\cos\theta + i\sin\theta}$$
$$= (\cos\theta + i\sin\theta)^{-1}$$
$$= \cos\theta - i\sin\theta$$
$$\therefore x + \frac{1}{x} = \cos\theta + i\sin\theta + \cos\theta - i\sin\theta$$
$$= 2\cos\theta$$

Also, $1+\sqrt{1-a^2} = na$

or, $\sqrt{1-a^2} = na - 1$

or, $1 - a^2 = n^2a^2 - 2na + 1$

or, $n^2a^2 + a^2 = 2na$

or, $a^2(1+n^2) = 2na$

or, $\frac{a^2(1+n^2)}{2na} = 1$

$\therefore \frac{a(1+n^2)}{2n} = 1$

Now, LHS $= 1 + a\cos\theta$
$$= \frac{a(1+n^2)}{2n} + a\cdot\frac{x+\frac{1}{x}}{2}$$
$$= \frac{a}{2n}\left\{(1+n^2) + n\left(x+\frac{1}{x}\right)\right\}$$
$$= \frac{a}{2n}\left(1+n^2 + nx + \frac{n}{x}\right)$$
$$= \frac{a}{2n}\left\{1\cdot(1+nx) + \frac{n}{x}(1+nx)\right\}$$
$$= \frac{a}{2n}(1+nx)\left(1+\frac{n}{x}\right)$$
$$= \text{R.H.S}$$

Important results: (i) $1 = e^{i2n\pi} = \cos 2n\pi + i\sin 2n\pi$

(ii) $-1 = \cos\pi + i\sin\pi = e^{i\pi}$

(iii) $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\pi/2}$

(iv) $-i = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2} = e^{-i\pi/2}$

10. If α, β be the roots of $x^2-2x+4=0$, prove that $\alpha^n+\beta^n = 2^{n+1}\cos\frac{n\pi}{3}$

Solution: Given, $x^2-2x+4=0$

$$\therefore x = \frac{-(-2)\pm\sqrt{(-2)^2-4\cdot 1\cdot 4}}{2\cdot 1}$$

$$= \frac{2\pm\sqrt{-12}}{2}$$

$$= 1\pm\sqrt{-3}$$

$$= 1\pm i\sqrt{3}$$

$$\therefore \alpha = 1+i\sqrt{3}$$

$$= 2\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$$

$$= 2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$$

and $\beta = 1-i\sqrt{3}$

$$= 2\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)$$

$$= 2\left(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}\right)$$

$$\therefore \alpha^n = 2^n\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^n$$

$$= 2^n\left(\cos\frac{n\pi}{3}+i\sin\frac{n\pi}{3}\right) \cdots (1)$$

and $\beta^n = 2^n\left(\cos\frac{n\pi}{3}-i\sin\frac{n\pi}{3}\right) \cdots (2)$

Adding (1) and (2) we get

$$\alpha^n+\beta^n = 2^n\cdot 2\cos\frac{n\pi}{3}$$

$$= 2^{n+1}\cos\frac{n\pi}{3}$$

11. Show that $\sin(\ln i^i) = -1$

Solution: $i = e^{\frac{i\pi}{2}}\cdot e^{i2n\pi}$ $\quad [\because 1 = e^{i2n\pi}]$

$$= e^{\frac{i\pi}{2}(4n+1)}$$

$$\therefore i^i = e^{-(4n+1)\frac{\pi}{2}}$$

$$\therefore \ln i^i = -(4n+1)\frac{\pi}{2}$$

Now, $\sin(\ln i^i) = \sin\left\{-(4n+1)\frac{\pi}{2}\right\}$

$$= -\sin\left(2n\pi+\frac{\pi}{2}\right)$$

$$= -\sin\frac{\pi}{2}$$

$$= -1$$

12. If $x = \cos\alpha + i\sin\alpha$, $y = \cos\beta + i\sin\beta$, $z = \cos\gamma + i\sin\gamma$ and $x+y+z=0$, then prove that

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=0.$$

Solution: We have, $x+y+z=0$

or, $\cos\alpha + i\sin\alpha + \cos\beta + i\sin\beta + \cos\gamma + i\sin\gamma = 0$

or, $(\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma) = 0$

Equating real and imaginary parts, we get

$\cos\alpha + \cos\beta + \cos\gamma = 0 \quad \cdots (1)$

and $\sin\alpha + \sin\beta + \sin\gamma = 0 \quad \cdots (2)$

Now LHS $= \frac{1}{x}+\frac{1}{y}+\frac{1}{z}$

$= (\cos\alpha + i\sin\alpha)^{-1} + (\cos\beta + i\sin\beta)^{-1} + (\cos\gamma + i\sin\gamma)^{-1}$

$= \cos\alpha - i\sin\alpha + \cos\beta - i\sin\beta + \cos\gamma - i\sin\gamma$

$= (\cos\alpha + \cos\beta + \cos\gamma) - i(\sin\alpha + \sin\beta + \sin\gamma)$

$= 0 - i.0$

$= 0$

13. Using De Moivre's theorem, solve the equations:

(i) $x^9 = 1$ (ii) $(x+1)^5 + (x-1)^5 = 0$ (iii) $x^4 + x^2 + 1 = 0$

Solution: (i) $x^9 = 1$

or, $x^9 = \cos 2n\pi + i\sin 2n\pi$ $\quad [\because 1 = e^{i2n\pi} = \cos 2n\pi + i\sin 2n\pi]$

or, $x = (\cos 2n\pi + i\sin 2n\pi)^{\frac{1}{9}}$

or, $x = \cos\frac{2n\pi}{9} + i\sin\frac{2n\pi}{9}$

Putting $n = 0, 1, 2, 3, 4, 5, 6, 7, 8$, the required solutions are

$\cos 0 + i\sin 0$, $\cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9}$, $\cos\frac{4\pi}{9} + i\sin\frac{4\pi}{9}$,
$\cos\frac{6\pi}{9} + i\sin\frac{6\pi}{9}$, $\cos\frac{8\pi}{9} + i\sin\frac{8\pi}{9}$, $\cos\frac{10\pi}{9} + i\sin\frac{10\pi}{9}$,
$\cos\frac{12\pi}{9} + i\sin\frac{12\pi}{9}$, $\cos\frac{14\pi}{9} + i\sin\frac{14\pi}{9}$, $\cos\frac{16\pi}{9} + i\sin\frac{16\pi}{9}$

(ii) $(x+1)^5+(x-1)^5=0$

or, $(x+1)^5=-(x-1)^5$

or, $\left(\frac{x+1}{x-1}\right)^5=-1$

or, $\frac{x+1}{x-1}=(-1)^{\frac{1}{5}}$

or, $\frac{x+1}{x-1}=\left(e^{i\pi}.e^{i2n\pi}\right)^{\frac{1}{5}}$

or, $\frac{x+1}{x-1}=\left\{e^{i(2n\pi+\pi)}\right\}^{\frac{1}{5}}$

or, $\frac{x+1}{x-1}=\left\{\cos(2n\pi+\pi)+i\sin(2n\pi+\pi)\right\}^{\frac{1}{5}}$

or, $\frac{x+1}{x-1}=\cos\left(\frac{2n\pi+\pi}{5}\right)+i\sin\left(\frac{2n\pi+\pi}{5}\right)$

or, $\frac{x+1}{x-1}=\frac{\cos\theta+i\sin\theta}{1}$ where $\theta=\frac{2n\pi+\pi}{5}$

Using Componendo and dividendo, we get

$$\frac{x+1+x-1}{x+1-x+1}=\frac{\cos\theta+i\sin\theta+1}{\cos\theta+i\sin\theta-1}$$

or, $\frac{2x}{2}=-\frac{2\cos^2\theta/_2+i.2\sin\theta/_2\cos\theta/_2}{2\sin^2\theta/_2-i.2\sin\theta/_2\cos\theta/_2}$

or, $x=-\frac{2\cos\theta/_2(\cos\theta/_2+i\sin\theta/_2)}{2\sin\theta/_2(\sin\theta/_2-i\cos\theta/_2)}$

or, $x=-i\cot\theta/_2\left[\frac{\cos\theta/_2+i\sin\theta/_2}{i\sin\theta/_2+\cos\theta/_2}\right]$

or, $x=-i\cot\theta/_2$

$\therefore x=-i\cot(2n+1)\frac{\pi}{10}$.

Putting $n=0,1,2,3,4$, the required solutions are

$-i\cot\frac{\pi}{10}$, $-i\cot\frac{3\pi}{10}$, $-i\cot\frac{\pi}{2}$, $-i\cot\frac{7\pi}{10}$, $-i\cot\frac{9\pi}{10}$.

(iii) Given, $x^4+x^2+1=0$

or, $(x^2-1)(x^4+x^2+1)=0$ [Multiplying bothsides by (x^2-1)]

or, $(x^2)^3-(1)^3=0$

or, $x^6-1=0$ or, $x^6=1$

or, $x^6=\cos 2n\pi + i\sin 2n\pi$ [$\because 1=\cos 2n\pi + i\sin 2n\pi$]

or, $x=(\cos 2n\pi + i\sin 2n\pi)^{\frac{1}{6}}$

or, $x=\cos\frac{2n\pi}{6}+i\sin\frac{2n\pi}{6}$

$\therefore x=\cos\frac{n\pi}{3}+i\sin\frac{n\pi}{3}$ .. (1)

Putting $n=0,1,2,3,4,5$ in (1), we get

$x=1$, $\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$, $\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$, -1, $\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}$, $\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}$

Among these values $x=\pm 1$ will be omitted as we have multiplied the equation by x^2-1. Hence the four roots of the given equation are

$x=\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}$, $\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}$, $\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}$, $\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}$.

14. Find all the values of (i) $(1+i)^{\frac{1}{5}}$ (ii) $(-i)^{\frac{1}{6}}$

Solution: Let us put $1=r\cos\theta$ and $1=r\sin\theta$

$\therefore r=\sqrt{2}$ and $\theta=\frac{\pi}{4}$

Now, $(1+i)^{\frac{1}{5}}=(1+i.1)^{\frac{1}{5}}$

$=(r\cos\theta+ir\sin\theta)^{\frac{1}{5}}$

$=\{r(\cos\theta+i\sin\theta)\}^{\frac{1}{5}}$

$=\{r.e^{i\theta}.e^{i2n\pi}\}^{\frac{1}{5}}$

$\therefore \quad (1+i)^{\frac{1}{5}} = \{r.\, e^{i(2n\pi+\theta)}\}^{\frac{1}{5}}$

$= \{\sqrt{2}\, e^{i(2n\pi+\frac{\pi}{4})}\}^{\frac{1}{5}}$

$= 2^{\frac{1}{10}}\{\cos(2n\pi+\frac{\pi}{4}) + i\sin(2n\pi+\frac{\pi}{4})\}^{\frac{1}{5}}$

$= 2^{\frac{1}{10}}\{\cos(8n+1)\frac{\pi}{20} + i\sin(8n+1)\frac{\pi}{20}\}$

Putting $n=0,1,2,3,4$, the required values are

$2^{\frac{1}{10}}(\cos\frac{\pi}{20}+i\sin\frac{\pi}{20})$, $2^{\frac{1}{10}}(\cos\frac{9\pi}{20}+i\sin\frac{9\pi}{20})$,

$2^{\frac{1}{10}}(\cos\frac{17\pi}{20}+i\sin\frac{17\pi}{20})$, $2^{\frac{1}{10}}(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4})$,

$2^{\frac{1}{10}}(\cos\frac{33\pi}{20}+i\sin\frac{33\pi}{20})$

(ii) $(-i)^{\frac{1}{6}} = (\cos\frac{\pi}{2} - i\sin\frac{\pi}{2})^{\frac{1}{6}}$

$= (e^{-i\frac{\pi}{2}})^{\frac{1}{6}}$

$= (e^{-\frac{i\pi}{2}} . e^{i2n\pi})^{\frac{1}{6}}$ $[\because 1 = e^{i2n\pi}]$

$= \{e^{i(4n-1)\frac{\pi}{2}}\}^{\frac{1}{6}}$

$= \{\cos(4n-1)\frac{\pi}{2} + i\sin(4n-1)\frac{\pi}{2}\}^{\frac{1}{6}}$

$= \cos(4n-1)\frac{\pi}{12} + i\sin(4n-1)\frac{\pi}{12}$

Putting $n=0,1,2,3,4,5$, the required values are

$\cos\frac{\pi}{12} - i\sin\frac{\pi}{12}$, $\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}$, $\cos\frac{7\pi}{12}+i\sin\frac{7\pi}{12}$,

$\cos\frac{11\pi}{12}+i\sin\frac{11\pi}{12}$, $\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}$, $\cos\frac{19\pi}{12}+i\sin\frac{19\pi}{12}$

Analytic functions-1

Complex variable: A symbol, such as z, which can stand for any one of a set of complex numbers is called a complex variable.

If x and y are real variables, then $z = x+iy$ is called a complex variable.

Function: If to each value which a complex variable z may assume there corresponds one or more values of a complex variable w, then w is called a function of z, written $w = f(z)$.

Single-valued function: If for each value of z there corresponds only one value of w, then w is called a single-valued function of z or that $f(z)$ is single-valued.

Multiple-valued function: If for each value of z there corresponds more than one value of w, then w is called a multiple-valued or many-valued function of z or that $f(z)$ is multiple-valued.

Derivative: If $f(z)$ is a single-valued function defined in a region R of the z-plane, the derivative of $f(z)$ is defined as

$$f'(z) = \frac{df(z)}{dz} = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

provided the limit exists.

1. Define an analytic function. Find the necessary conditions for a function $f(z) = u + iv$ to be analytic in a region R.

Solution:

Analytic function: A single-valued function $f(z)$ which is differentiable at every point of a region R, is called as an analytic function of z in R.

An analytic function can also be called as regular function or holomorphic function.

Necessary conditions for a function $f(z) = u + iv$ to be analytic:

Let $f(z) = u(x,y) + iv(x,y)$ be an analytic function in a region R.

$\therefore f(z)$ is differentiable in R.

$$\Rightarrow f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \text{ exists in R.}$$

Let $z = x + iy$

$\therefore \Delta z = \Delta x + i\Delta y$

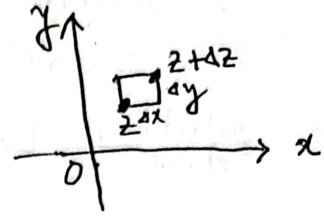
$$\therefore z + \Delta z = (x+iy) + (\Delta x + i\Delta y)$$
$$= (x + \Delta x) + i(y + \Delta y)$$

$$\therefore f(z+\Delta z) = u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)$$

If $\Delta z \to 0$ or, $\Delta x + i\Delta y \to 0$, then $\Delta x \to 0$, $\Delta y \to 0$.

$$\therefore f'(z) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{[u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y)] - [u(x,y) + iv(x,y)]}{\Delta x + i\Delta y}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{[u(x+\Delta x, y+\Delta y) - u(x,y)] + i[v(x+\Delta x, y+\Delta y) - v(x,y)]}{\Delta x + i\Delta y}$$



Along x-axis, $\Delta y = 0$, then the limit is

$$f'(z) = \lim_{\Delta x \to 0} \frac{[u(x+\Delta x, y) - u(x,y)] + i[v(x+\Delta x, y) - v(x,y)]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \quad \cdots\cdots (1)$$

(By definition)

Along y-axis, $\Delta x = 0$, then the limit is

$$f'(z) = \lim_{\Delta y \to 0} \frac{[u(x, y+\Delta y) - u(x,y)] + i[v(x, y+\Delta y) - v(x,y)]}{i\Delta y}$$

$$= \frac{1}{i} \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{\Delta y} + \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{\Delta y}$$

$$= \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad \cdots\cdots (2)$$

(By definition)

Since $f'(z)$ exists, so (1) and (2) must be equal.

i.e. $\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$

Comparing real and imaginary parts, we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \text{ or } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

i.e. $u_x = v_y$ and $u_y = -v_x$

These two equations are called Cauchy-Riemann (C-R) equations.

2. Derive the polar form of Cauchy-Riemann (C-R) equations.

<u>Solution:</u> Let $z = re^{i\theta}$ and $f(z) = u(r,\theta) + iv(r,\theta)$

[By using polar form]

Then $u(r,\theta) + iv(r,\theta) = f(re^{i\theta}) \quad \cdots (1)$

Differenting (1) partially wrt r, we get

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta}) \cdot e^{i\theta} \quad \cdots (2)$$

Differenting (1) partially wrt θ, we get

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta}) \cdot ire^{i\theta} \quad \cdots (3)$$

Now from (2) and (3), we get

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = \frac{1}{ir}\left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right)$$

or, $$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = \frac{1}{ir}\frac{\partial u}{\partial \theta} + \frac{1}{r}\frac{\partial v}{\partial \theta}$$

or, $$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = -\frac{i}{r}\frac{\partial u}{\partial \theta} + \frac{1}{r}\frac{\partial v}{\partial \theta}$$

Equating the real and imaginary parts, we get

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

The above equations are called Cauchy-Riemann (C-R) equations in Polar form.

Laplace's equation: An equation of the form $\frac{\partial^2 \varphi}{\partial x^2}+\frac{\partial^2 \varphi}{\partial y^2}=0$ or $\nabla^2\varphi=0$, is called a Laplace's equation (in two diemension).

Harmonic function: Any function having continuous second order partial derivatives which satisfies the Laplace's equation is called harmonic function.

Conjugate harmonic functions: Any two harmonic functions u and v such that $f(z)=u+iv$ is analytic, are called Conjugate harmonic functions.

3. Examine whethere the following functions are analytic or not:

(i) $e^{-x}(\cos y - i\sin y)$ (ii) $\frac{1}{z}$ $(z\neq 0)$ (iii) $\bar{z}$

(iv) $2xy+i(x^2-y^2)$

Solution: (i) let $f(z)=e^{-x}(\cos y - i\sin y)$

$$\text{or, } u+iv = e^{-x}\cos y - ie^{-x}\sin y$$

Equating real and imaginary parts, we get—

$$u=e^{-x}\cos y, \quad v=-e^{-x}\sin y$$

$$\therefore \frac{\partial u}{\partial x}=-e^{-x}\cos y, \quad \frac{\partial u}{\partial y}=-e^{-x}\sin y$$

$$\frac{\partial v}{\partial x}=e^{-x}\sin y, \quad \frac{\partial v}{\partial y}=-e^{-x}\cos y$$

$$\because \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

i.e. C-R equations are satisfied

$$\therefore f(z)=e^{-x}(\cos y - i\sin y) \text{ is analytic.}$$

(ii) Let. $f(z) = \frac{1}{z} \quad (z \neq 0)$

$$= \frac{1}{x+iy}$$

$$= \frac{x-iy}{x^2+y^2}$$

or, $u+iv = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$

Equating real and imaginary parts, we get—

$$u = \frac{x}{x^2+y^2}, \quad v = -\frac{y}{x^2+y^2}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{(x^2+y^2)\cdot 1 - x\cdot 2x}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(x^2+y^2)\cdot 0 - x\cdot 2y}{(x^2+y^2)^2}$$

$$= -\frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial x} = -\frac{(x^2+y^2)\cdot 0 - y\cdot 2x}{(x^2+y^2)^2}$$

$$= \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial y} = -\frac{(x^2+y^2)\cdot 1 - y\cdot 2y}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2}$$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

i.e., C-R equations are satisfied

$\therefore f(z) = \frac{1}{z}$ is analytic except at $z=0$.

(iii) Let $f(z) = \bar{z}$

or, $u+iv = x-iy \quad [\because z = x+iy \therefore \bar{z} = x-iy]$

$\therefore u = x, \; v = -y$

$\therefore \frac{\partial u}{\partial x} = 1, \; \frac{\partial u}{\partial y} = 0 \;\Big|\; \frac{\partial v}{\partial x} = 0, \; \frac{\partial v}{\partial y} = -1$

since $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$

$\Rightarrow$ C-R equations are not satisfied.

$\therefore f(z) = \bar{z}$ is not analytic.

(iv) Let $f(z) = 2xy + i(x^2-y^2)$

or, $u+iv = 2xy + i(x^2-y^2)$

$\therefore u = 2xy, \; v = x^2-y^2$

$\therefore \frac{\partial u}{\partial x} = 2y, \; \frac{\partial u}{\partial y} = 2x$

$\frac{\partial v}{\partial x} = 2x, \; \frac{\partial v}{\partial y} = -2y$

since $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$

$\Rightarrow$ C-R equations are not satified.

$\therefore f(z) = 2xy + i(x^2-y^2)$ is not analytic.

Construction of conjugate harmonic functions/ an analytic function whose real or imaginary part is given.

Method-1: suppose u is given.

Then $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ are known.

By total differentiation,

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$= -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \quad \left[\because \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\right]$$

integrating it, $v = \int \left(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy\right) + c_1$

since v is known we can construct $f(z) = u+iv$.

similarly, if v is given we can find $\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$

Then $du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$

$= \frac{\partial v}{\partial y} dx - \frac{\partial v}{\partial x} dy$

$\Rightarrow u = \int \left(\frac{\partial v}{\partial y} dx - \frac{\partial v}{\partial x} dy\right) + c_2$

$\therefore f(z) = u+iv$

Method-2: Milne's method / Milne Thomson method

Let $f(z) = u + iv$ is to be constructed

(i) Suppose the real part 'u' is given.

Then $\frac{\partial u}{\partial x} (= u_1(x,y))$ and $\frac{\partial u}{\partial y} (= u_2(x,y))$ are known.

Then by Milne's method we have,

$$f'(z) = u_1(z,0) - iu_2(z,0)$$

integrating it wrt z, we get

$$f(z) = \int [u_1(z,0) - iu_2(z,0)]\,dz + C_1$$

(ii) Similarly suppose the imaginary part 'v' is given.

Then $\frac{\partial v}{\partial y} (= v_1(x,y))$ and $\frac{\partial v}{\partial x} (= v_2(x,y))$ are known.

Then by Milne's method we have,

$$f'(z) = v_1(z,0) + iv_2(z,0)$$

integrating it wrt z, we get

$$f(z) = \int [v_1(z,0) + iv_2(z,0)]\,dz + C_2$$

Which is the analytic function.

4. Show that $u = 3x^2y + 2x^2 - y^3 - 2y^2$ is harmonic function. Also find its conjugate harmonic.

Solution: Given, $u = 3x^2y + 2x^2 - y^3 - 2y^2$

$$\therefore \frac{\partial u}{\partial x} = 6xy + 4x, \quad \frac{\partial^2 u}{\partial x^2} = 6y + 4 \quad \cdots (1)$$

$$\frac{\partial u}{\partial y} = 3x^2 - 3y^2 - 4y, \quad \frac{\partial^2 u}{\partial y^2} = -6y - 4 \cdots (2)$$

Adding (1) and (2), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

or, $\nabla^2 u = 0$

i.e. u satisfies Laplace's equation.

So, u is a harmonic function.

Let v be the conjugate harmonic function.

Then we have, $dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$

$$= -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$

$$= -(3x^2 - 3y^2 - 4y)dx + (6xy + 4x)dy$$

$$= d(3xy^2 + 4xy - x^3)$$

Integrating it, we get

$$v = 3xy^2 + 4xy - x^3 + c$$

5. In a two dimensional flow, the stream function is $\psi = \tan^{-1}(y/x)$. Find the velocity potential ϕ.

Solution: Given that $\psi = \tan^{-1}(y/x)$

$$\therefore \frac{\partial \psi}{\partial x} = \frac{1}{1+(y/x)^2} \cdot y \cdot \left(-\frac{1}{x^2}\right)$$

$$= \frac{-y}{x^2+y^2}$$

$$\frac{\partial \psi}{\partial y} = \frac{1}{1+(y/x)^2} \cdot \frac{1}{x}$$
$$= \frac{x}{x^2+y^2}$$

Then we have,

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy \quad \left[\because W = \phi + i\psi \quad \therefore \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}\right]$$
$$= \frac{\partial \psi}{\partial y}dx - \frac{\partial \psi}{\partial x}dy$$
$$= \frac{x}{x^2+y^2}dx + \frac{y}{x^2+y^2}dy$$

Integrating it, we get

$$\phi = \frac{1}{2}\ln(x^2+y^2) + \frac{1}{2}\ln(x^2+y^2) + C$$
$$= \ln(x^2+y^2) + C$$ is the required velocity potential.

6. (a) Prove that the function $u = 2x(1-y)$ is harmonic.
(b) Find a function v such that $f(z) = u + iv$ is analytic [i.e. find the conjugate function of u] (c) Express $f(z)$ in terms of z.

Solution: (a) Given that $u = 2x(1-y)$ ---(1)

$$\therefore \frac{\partial u}{\partial x} = 2 - 2y$$
$$= u_1(x,y), \text{ say}$$
$$\frac{\partial^2 u}{\partial x^2} = 0 \quad \cdots\cdots (2)$$
$$\frac{\partial u}{\partial y} = -2x$$
$$= u_2(x,y), \text{ say}$$
$$\frac{\partial^2 u}{\partial y^2} = 0 \quad \cdots\cdots (3)$$

Adding (2) and (3) we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Since u satisfies Laplace's equation, so u is a harmonic function.

(c) By Milne's method we have

$$f'(z) = u_1(z,0) - iu_2(z,0)$$
$$= 2 - i.(-2z)$$
$$= 2 + i2z$$

Integrating it, we get

$$f(z) = 2z + i2.\frac{z^2}{2} + c, \text{ where } c$$

is the complex const.

$\therefore f(z) = 2z + iz^2 + c$ is the required function.

(b) From (c) we have,

$$f(z) = 2z + iz^2 + c$$
$$= 2(x+iy) + i(x+iy)^2 + c_1 + ic_2$$

or, $u + iv = 2(x+iy) + i(x^2 + i2xy - y^2) + c_1 + ic_2$
$$= 2x - 2xy + i(2y + x^2 - y^2) + c_1 + ic_2$$

Equating the imaginary parts, we get

$v = 2y + x^2 - y^2 + c_2$ is the required conjugate harmonic.

7. Show that the function $f(z) = u + iv$, where

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, \quad z \neq 0$$
$$= 0, \quad z = 0$$

satisfies the Cauchy-Riemann conditions at $z=0$. Is the function analytic at $z=0$? Justify your answer.

Solution: Given, $f(z) = \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$

or, $u + iv = \dfrac{x^3 - y^3 + i(x^3+y^3)}{x^2+y^2}$

$\therefore u(x,y) = \dfrac{x^3-y^3}{x^2+y^2}, \; v(x,y) = \dfrac{x^3+y^3}{x^2+y^2}$

Also, $u(0,0)=0$, $v(0,0)=0$ [$\because$ At $z=0$, $f(0)=0$]

At the origin (i.e. at $z=0$)

$$\frac{\partial u}{\partial x}=\lim_{h\to 0}\frac{u(0+h,0)-u(0,0)}{h}$$
$$=\lim_{h\to 0}\frac{h^3/h^2}{h}$$
$$=1$$

$$\frac{\partial u}{\partial y}=\lim_{K\to 0}\frac{u(0,0+K)-u(0,0)}{K}$$
$$=\lim_{K\to 0}\frac{-K^3/K^2}{K}$$
$$=-1$$

$$\frac{\partial v}{\partial x}=\lim_{h\to 0}\frac{v(0+h,0)-v(0,0)}{h}$$
$$=\lim_{h\to 0}\frac{h^3/h^2}{h}$$
$$=1$$

$$\frac{\partial v}{\partial y}=\lim_{K\to 0}\frac{v(0,0+K)-v(0,0)}{K}$$
$$=\lim_{K\to 0}\frac{K^3/K^2}{K}$$
$$=1$$

$\therefore$ $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$

i.e., Cauchy - Riemann Conditions are satisfied at $z=0$.

Now $$f'(0)=\lim_{z\to 0}\frac{f(z)-f(0)}{z}$$
$$=\lim_{z\to 0}\left[\frac{x^3-y^3+i(x^3+y^3)}{x^2+y^2}\cdot\frac{1}{x+iy}\right]$$

Let $z\to 0$ along $y=x$, then

$$f'(0)=\lim_{x\to 0}\left[\frac{x^3-x^3+i(x^3+x^3)}{x^2+x^2}\cdot\frac{1}{x+ix}\right].$$
$$=\frac{2i}{2(1+i)}=\frac{i(1-i)}{1-i^2}=\frac{1}{2}(1+i)$$

Again let $z\to 0$ along $y=0$, then

$$f'(0)=\lim_{x\to 0}\frac{x^3+ix^3}{x^2}\cdot\frac{1}{x}=1+i$$

So we see that $f'(0)$ is not unique. Hence the function $f(z)$ is not analytic at $z=0$.

Analytic function-2

The complex potential function

The analytic function $W = \varphi(x,y) + i\psi(x,y)$ is referred to as the complex potential function. Its real part $\varphi(x,y)$ represents the velocity potential function and the imaginary part $\psi(x,y)$ represents the stream function.

Problem-1(a). If $W = \varphi + i\psi$ represents the complex potential for an electric field and $\psi = 3x^2y - y^3$, find the potential function φ.

Solution: Given $\psi = 3x^2y - y^3$

$$\frac{\partial \psi}{\partial y} = 3x^2 - 3y^2$$
$$= \psi_1(x,y), \text{ say}$$

and $$\frac{\partial \psi}{\partial x} = 6xy$$
$$= \psi_2(x,y), \text{ say}$$

By Milne's method we have,

$$W'(z) = \psi_1(z,0) + i\psi_2(z,0)$$
$$= 3z^2 + i\cdot 0$$
$$= 3z^2$$

Integrating w.r.t. z, we get

$$W(z) = z^3 + c$$

$$\text{or } \varphi + i\psi = (x+iy)^3 + c_1 + ic_2$$

$$\text{or, } \varphi + i\psi = x^3 + i3x^2y - 3xy^2 - iy^3 + c_1 + ic_2$$

$\therefore \varphi = x^3 - 3xy^2 + c_1$ is the required potential function.

Problem-1(b). An incompressible fluid flowing over the xy-plane has the velocity potential

$$\varphi = x^2 - y^2 + \frac{x}{x^2+y^2}$$

Examine if this is possible and find a stream function ψ.

Solution: Given, $\varphi = x^2 - y^2 + \dfrac{x}{x^2+y^2}$

$$\therefore \frac{\partial \varphi}{\partial x} = 2x + \frac{(x^2+y^2)\cdot 1 - x\cdot 2x}{(x^2+y^2)^2} = 2x + \frac{y^2-x^2}{(x^2+y^2)^2} = \varphi_1(x,y), \text{ say}$$

$$\frac{\partial^2 \varphi}{\partial x^2} = 2 + \frac{(x^2+y^2)^2\cdot(-2x) - (y^2-x^2)\cdot 2(x^2+y^2)\cdot 2x}{(x^2+y^2)^4}$$

$$= 2 + \frac{2x^3 - 6xy^2}{(x^2+y^2)^3}$$

$$\frac{\partial \varphi}{\partial y} = -2y + \frac{(x^2+y^2)(0) - x(2y)}{(x^2+y^2)^2} = -2y - \frac{2xy}{(x^2+y^2)^2} = \varphi_2(x,y), \text{ say}$$

$$\frac{\partial^2 \varphi}{\partial y^2} = -2 + \frac{(x^2+y^2)^2(-2x) - (-2xy)\cdot 2(x^2+y^2)\cdot 2y}{(x^2+y^2)^4}$$

$$= -2 + \frac{-2x^3 + 6xy^2}{(x^2+y^2)^3}$$

$$\therefore \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 \quad \text{i.e., } \varphi \text{ is harmonic.}$$

Hence it can be a possible form of the velocity potential function.

By Milne's method, we have

$$f'(z) = \varphi_1(z,0) - i\varphi_2(z,0)$$

$$= 2z - \frac{1}{z^2} - i\cdot 0$$

integrating it, we get $f(z) = z^2 + \dfrac{1}{z} + c$

$$\text{or, } \varphi + i\psi = (x+iy)^2 + \frac{1}{x+iy} + c_1 + ic_2$$

$$= x^2 - y^2 + i2xy + \frac{x-iy}{x^2+y^2} + c_1 + ic_2$$

$\therefore \psi = 2xy - \dfrac{y}{x^2+y^2} + c_2$ is the required.

Problem-1(c): If $W = \varphi + i\psi$ represents the complex potential for an electric field and $\psi = x^2 - y^2 + \dfrac{x}{x^2+y^2}$, find φ. Ans. $\varphi = -2xy + \dfrac{y}{x^2+y^2} + c_1$

Problem-1(d). Prove that the real and imaginary parts of an analytic function $f(z) = u(x,y) + iv(x,y)$ satisfies the Laplace's equation.

<u>Solution</u>: Since $f(z) = u(x,y) + iv(x,y)$ is an analytic function, so we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{---(1) and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{---(2)}$$

Differentiate (1) partially wrt x, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad \text{---(3)}$$

Differentiate (2) partially wrt y, we get

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \quad \text{---(4)}$$

Adding (3) and (4) we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{or,} \quad \nabla^2 u = 0$$

similarly $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ or $\nabla^2 v = 0$

i.e. u and v satisfy their Laplace's equations.
[Both u and v are harmonic functions]

Problem-2. Show that an analytic function with constant real part is constant.

<u>Solution</u>: Let $f(z) = u + iv$ be an analytic function.

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{---(1) and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{---(2)}$$

Given that $u = \text{constant} = c_1$, say

$$\therefore \frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial y} = 0$$

so from (1) and (2) we get, $\frac{\partial v}{\partial x} = 0$ and $\frac{\partial v}{\partial y} = 0$

i.e. v is independent of x and y

$\Rightarrow v = \text{Constant} = c_2$, say

$\therefore f(z) = u + iv = c_1 + ic_2$ is a constant

Problem-3(a). Show than an analytic function with Constant imaginary part is constant.

<u>Solution</u>: Let $f(z) = u+iv$ be an analytic function.

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{--- (1)}$$

Given that $v =$ Constant $= c_1$, say

$$\therefore \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = 0$$

$$\Rightarrow \frac{\partial u}{\partial y} = 0, \quad \frac{\partial u}{\partial x} = 0 \quad [\text{by (1)}]$$

$\Rightarrow u$ is independent of x and y

$\Rightarrow u =$ Constant $= c_2$, say

$$\therefore f(z) = u+iv = c_2 + ic_1 = \text{Constant}$$

Problem-3(b). Determine the analytic function whose real part is $x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$.

<u>Solution</u>: Given that $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

$$\therefore \frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x$$

$$= u_1(x,y), \text{ say}$$

$$\frac{\partial u}{\partial y} = -6xy - 6y$$

$$= u_2(x,y), \text{ say}$$

By Milne's method, we have

$$f'(z) = u_1(z,0) - iu_2(z,0)$$

$$= 3z^2 + 6z - i \cdot 0$$

Integrating it, we get $f(z) = 3 \cdot \frac{z^3}{3} + 6 \cdot \frac{z^2}{2} + c$

$= z^3 + 3z^2 + c$, where c is the complex constant.

Problem-4: Show that an analytic function with constant absolute value / modulus is constant.

Solution: Let an analytic function be $f(z)=u+iv$

$$\therefore \; |f(z)| = \sqrt{u^2+v^2}$$

But we are given, $|f(z)| = \text{Constant} = K$, say

$$\therefore \; u^2+v^2 = K^2$$

By differentiation, $uu_x + vv_x = 0$, $uu_y + vv_y = 0$

Now we use $v_x = -u_y$ in the first equation and $v_y = u_x$ in the second, we get

$$uu_x - vu_y = 0 \quad \text{--} \; (1)$$

$$uu_y + vu_x = 0 \quad \text{--} \; (2)$$

Multiplying (1) by u and (2) by v, then adding and also multiplying (1) by $-v$ and (2) by u, then adding we get

$$(u^2+v^2)u_x = 0, \quad (u^2+v^2)u_y = 0$$

If $K^2 = u^2+v^2 = 0$, then $u=0=v$, hence $f=0$.

If $K \neq 0$, then $u_x = u_y = 0$, hence by Cauchy-Riemann equations, also $v_x = v_y = 0$. Together, u = Constant and v = Constant, hence f = Constant.

Problem-5: Test whether the function $f(z) = z^3 + z$ is analytic or not.

Solution: We have, $f(z) = z^3 + z$

$$= (x+iy)^3 + (x+iy)$$

$$= x^3 + i3x^2y - 3xy^2 - iy^3 + x + iy$$

$$\text{or, } u + iv = (x^3 - 3xy^2 + x) + i(3x^2y - y^3 + y)$$

Equating the real and imaginary parts, we get

$$u = x^3 - 3xy^2 + x$$

$$v = 3x^2y - y^3 + y$$

$$\therefore \frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 1, \quad \frac{\partial u}{\partial y} = -6xy$$

$$\frac{\partial v}{\partial x} = 6xy, \quad \frac{\partial v}{\partial y} = 3x^2 - 3y^2 + 1$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\Rightarrow$ C-R equations are satisfied.

$\therefore f(z) = z^3 + z$ is analytic.

Problem-6: Find the constants a, b and c if $f(z) = x + ay + i(bx + cy)$ is analytic.

Solution: Given, $f(z) = x + ay + i(bx + cy)$

$$\text{or, } u + iv = x + ay + i(bx + cy)$$

$$\therefore u = x + ay, \quad v = bx + cy$$

$$\therefore \frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = a$$

$$\frac{\partial v}{\partial x} = b, \quad \frac{\partial v}{\partial y} = c$$

Since $f(z)$ is analytic, so Cauchy-Riemann (C-R) equations are satisfied.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\therefore 1 = c, \quad a = -b$$

$\Rightarrow c = 1,\ a = -b$, b may be any value.

Problem-7: Determine b such that $u = e^{bx}\cos 5y$ is harmonic.

Solution: Given, $u = e^{bx}\cos 5y$

$$\therefore \frac{\partial u}{\partial x} = b e^{bx}\cos 5y$$

$$\frac{\partial^2 u}{\partial x^2} = b^2 e^{bx}\cos 5y$$

$$\frac{\partial u}{\partial y} = e^{bx}(-5\sin 5y)$$

$$\frac{\partial^2 u}{\partial y^2} = -5e^{bx}\cdot 5\cos 5y$$

$$= -25e^{bx}\cos 5y$$

$\because u$ is harmonic function, so

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

or, $b^2 e^{bx}\cos 5y - 25e^{bx}\cos 5y = 0$

or, $e^{bx}\cos 5y\,(b^2 - 25) = 0$

or, $b^2 - 25 = 0 \quad [\because e^{bx}\cos 5y \neq 0]$

$$\therefore b = \pm 5$$

Problem-8: (a) Prove that $u = e^{-x}(x\sin y - y\cos y)$ is harmonic. (b) Find v such that $f(z) = u + iv$ is analytic.

Solution: Given, $u = e^{-x}(x\sin y - y\cos y)$

$$\therefore \frac{\partial u}{\partial x} = e^{-x} \cdot \sin y + (-e^{-x})(x\sin y - y\cos y)$$

$$= e^{-x}(\sin y - x\sin y + y\cos y)$$

$$= u_1(x,y), \text{ say}$$

$$\frac{\partial^2 u}{\partial x^2} = -e^{-x}(\sin y - x\sin y + y\cos y) + e^{-x}(-\sin y)$$

$$= -e^{-x}(2\sin y - x\sin y + y\cos y) \quad \text{--- (1)}$$

$$\frac{\partial u}{\partial y} = e^{-x}(x\cos y - 1\cdot\cos y + y\sin y)$$

$$= u_2(x,y), \text{ say}$$

$$\frac{\partial^2 u}{\partial y^2} = e^{-x}(-x\sin y + \sin y + 1\cdot\sin y + y\cos y)$$

$$= e^{-x}(-x\sin y + 2\sin y + y\cos y) \quad \text{--- (2)}$$

Adding (1) and (2), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

since u satisfies Laplace's equation, so u is harmonic function.

<u>2nd part</u>: (b) By Milne's method we have,

$$f'(z) = u_1(z,0) - iu_2(z,0)$$

$$= 0 - i(ze^{-z} - e^{-z})$$

integrating, $f(z) = -iz\cdot(-e^{-z}) + i\int 1\cdot(-e^{-z})\,dz + i\int e^{-z}\,dz + c$

$$= ize^{-z} + c$$

or, $u+iv = i(x+iy)\,e^{-(x+iy)} + C$

$$= i(x+iy)\,e^{-x}.\,e^{-iy} + C$$

$$= i(x+iy).\,e^{-x}(\cos y - i\sin y) + C$$

$$= ixe^{-x}\cos y + xe^{-x}\sin y - ye^{-x}\cos y + iye^{-x}\sin y + C_1 + iC_2$$

$$= (xe^{-x}\sin y - ye^{-x}\cos y + C_1) + i(xe^{-x}\cos y + ye^{-x}\sin y + C_2)$$

Equating imaginary parts, we get

$$v = e^{-x}(x\cos y + y\sin y) + C_2$$

Problem-9. In a two dimensional flow of a fluid, the velocity potential $\varphi = x^2 - y^2$. Find the stream function ψ.

Solution: Given that $\varphi = x^2 - y^2$

$$\therefore \frac{\partial \varphi}{\partial x} = 2x$$
$$= \varphi_1(x,y), \text{ say}$$

and $\frac{\partial \varphi}{\partial y} = -2y$
$$= \varphi_2(x,y), \text{ say}$$

By Milne's method we have,

$$W'(z) = \varphi_1(z,0) - i\varphi_2(z,0)$$
$$= 2z - i.0$$
$$= 2z$$

Integrating it, we get $W(z) = \cancel{2}\frac{z^2}{\cancel{2}} + C$

or, $W(z) = z^2 + C$

or, $\phi + i\psi = (x+iy)^2 + c_1 + ic_2$

$= x^2 - y^2 + i2xy + c_1 + ic_2$

Equating imaginary parts, we get

$\psi = 2xy + c_2$ is the required stream function.

Problem-10. Show that xy^2 cannot be real part of an analytic function.

<u>Solution</u>: Given $u = xy^2$

$\therefore \frac{\partial u}{\partial x} = y^2, \quad \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{---(1)}$

$\frac{\partial u}{\partial y} = 2xy, \quad \frac{\partial^2 u}{\partial y^2} = 2x \quad \text{---(2)}$

Adding (1) and (2) we get

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2x$

$\neq 0$

$\therefore$ u is not harmonic function.

i.e. u cannot be a real part of an analytic function.

Complex Integration

1. State and prove Cauchy's theorem / Cauchy's integral theorem.

Statement: If $f(z)$ is analytic inside and on a simple closed curve C, then $\oint_C f(z)\,dz = 0$.

Proof: Let $f(z) = u(x,y) + iv(x,y)$ be analytic.

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{---(1)}$$

Since $z = x+iy$, so $dz = dx + idy$

Now
$$\oint_C f(z)\,dz = \oint_C (u+iv)(dx+idy)$$
$$= \oint_C (udx + iudy + ivdx - vdy)$$
$$= \oint_C (udx - vdy) + i\oint_C (vdx + udy) \quad \text{..(1)}$$

By Green's theorem we have,

$$\oint_C (udx - vdy) = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dx\,dy$$

and
$$\oint_C (vdx + udy) = \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dx\,dy$$

where R is the region bounded by C.

Hence (1) becomes,

$$\oint_C f(z)\,dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dx\,dy$$

$$= \iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right)dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\right)dx\,dy$$

$$= 0 + i\cdot 0$$

$$= 0$$

2. State and prove Cauchy's integral formula.

Statement: If $f(z)$ is analytic inside and on a simple closed curve C, and 'a' is any point within C, then $\oint_C \frac{f(z)dz}{z-a} = 2\pi i f(a)$

Proof: Since $f(z)$ is analytic inside and on C, $\frac{f(z)}{z-a}$ is also analytic inside and on C, except at the point $z=a$. Hence, we draw a small circle with centre at $z=a$ and radius r lying entirely inside C.

Now, $\frac{f(z)}{z-a}$ is analytic in the region enclosed between C and C_1.

Hence, by Cauchy's extended theorem,

$$\oint_C \frac{f(z)dz}{z-a} = \oint_{C_1} \frac{f(z)\,dz}{z-a} \quad \text{---} (1)$$

On C_1, any point z is given by $z = a + re^{i\theta}$

$\therefore dz = ire^{i\theta}\,d\theta$

where θ varies from 0 to 2π.

$$\therefore \oint_{C_1} \frac{f(z)dz}{z-a} = \int_{\theta=0}^{2\pi} \frac{f(a+re^{i\theta}).ire^{i\theta}d\theta}{re^{i\theta}}$$

$$= i\int_{\theta=0}^{2\pi} f(a+re^{i\theta})\,d\theta$$

As $r \to 0$, the circle tends to a point.

Taking limit $r \to 0$, we get

$$\oint_{C_1} \frac{f(z)dz}{z-a} = i\int_{\theta=0}^{2\pi} f(a)\,d\theta$$

$$= if(a)\,[\theta]_{\theta=0}^{2\pi}$$

$$= 2\pi i\, f(a)$$

So from (1), we get $\oint_C \frac{f(z)dz}{z-a} = 2\pi i f(a)$

* In general, $\oint_C \frac{f(z)\,dz}{(z-a)^{n+1}} = \frac{2\pi i}{\lfloor n} f^{(n)}(a)$

where $n = 0, 1, 2, 3, \dots$

and $f^{(0)}(a) = f(a)$

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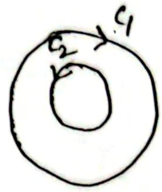
* Cauchy's extended theorem: If $f(z)$ is analytic within and on the boundary of a region bounded by two closed curves C_1 and C_2, then

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$$



3. Evaluate $\frac{1}{2\pi i}\oint_C \frac{e^z\,dz}{z-2}$ if C is (a) the circle $|z| = 3$, (b) the circle $|z| = 1$.

Solution: (a) Here $f(z) = e^z$ is analytic inside and on the circle $|z| = 3$ and $z = a = 2$ is a point inside the given circle.

Then by using the Cauchy's integral formula, we have

$$\oint_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

$$\therefore \oint_C \frac{e^z}{z-2}\,dz = 2\pi i \cdot e^2$$

$$\text{or, } \frac{1}{2\pi i}\oint_C \frac{e^z}{z-2}\,dz = e^2$$

(b) Here $f(z) = \frac{e^z}{z-2}$ is analytic inside and on the circle $|z| = 1$ and $z = 2$ is a point outside the given circle.

Then by using the Cauchy's integral theorem, $\oint_C f(z)\,dz = 0$ we get—

$$\oint_C \frac{e^z}{z-2}\,dz = 0 \quad \text{or, } \frac{1}{2\pi i}\oint_C \frac{e^z}{z-2}\,dz = 0$$

4. Evaluate $\oint_C \frac{\sin 3z}{z+\pi/2}\,dz$ if C is the circle $|z|=5$.

Solution: Here $f(z)=\sin 3z$ is analytic inside and on the circle $|z|=5$ and $z=a=-\frac{\pi}{2}$ lies inside the given circle.

Then by using Cauchy's integral formula,

$$\oint_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a) \text{ we get}$$

$$\oint_C \frac{\sin 3z}{z-(-\pi/2)}\,dz = 2\pi i f(-\pi/2)$$

$$\text{or, } \oint_C \frac{\sin 3z}{z+\pi/2} = 2\pi i$$

$\therefore f(-\frac{\pi}{2}) = \sin 3(-\frac{\pi}{2})$
$= -(-1)$
$= 1$

5. Evaluate $\oint_C \frac{e^{3z}}{z-\pi i}\,dz$ if C is (a) the circle $|z-1|=4$, (b) the ellipse $|z-2|+|z+2|=6$.

Solution: (a) Here $f(z)=e^{3z}$ is analytic inside and on the given circle $|z-1|=4$, and $z=\pi i$ is a point inside the given circle.

Then by using Cauchy's integral formula,

$$\oint_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a) \text{ we get}$$

$$\begin{aligned}\oint_C \frac{e^{3z}}{z-\pi i}\,dz &= 2\pi i f(\pi i)\\ &= 2\pi i \cdot e^{3\pi i}\\ &= 2\pi i(\cos 3\pi + i\sin 3\pi)\\ &= 2\pi i(-1+i\cdot 0)\\ &= -2\pi i\end{aligned}$$

(b) Here $f(z)=\frac{e^{3z}}{z-\pi i}$ is analytic inside and on the ellipse C, and $z=\pi i$ lies outside the given ellipse C.

Then by using Cauchy's integral theorem,

$$\oint_C f(z)\,dz = 0 \text{ we get}$$

$$\oint_C \frac{e^{3z}}{z-\pi i}\,dz = 0$$

** Locus of $|z-2|+|z+2|=6$ is $\frac{x^2}{3^2}+\frac{y^2}{(\sqrt{5})^2}=1$.

Its foci $(\pm ae, 0) = (\pm 3\cdot\frac{2}{3}, 0) = (\pm 2, 0)$, $e=\sqrt{1-\frac{5}{9}} = \frac{2}{3}$

and length of major axis $= 2\cdot 3 = 6$

6. Evaluate $\frac{1}{2\pi i}\oint_C \frac{\cos\pi z}{z^2-1}dz$ around a rectangle with vertices at: (a) $2\pm i, -2\pm i$ (b) $-i,\ 2-i,\ 2+i,\ i$.

Solution: We have $\frac{1}{2\pi i}\oint_C \frac{\cos\pi z}{z^2-1}dz = \frac{1}{4\pi i}\oint_C \left[\frac{\cos\pi z}{z-1} - \frac{\cos\pi z}{z+1}\right]dz$

$$= \frac{1}{4\pi i}\left[\oint_C \frac{\cos\pi z}{z-1}dz - \oint_C \frac{\cos\pi z}{z+1}dz\right] \quad \cdots (1)$$

(a) Here $f(z) = \cos\pi z$ is analytic inside and on C, and also both points $z = \pm 1$ lie inside the rectangle $2\pm i$, $-2\pm i$.

Then by using Cauchy's integral formula, $\oint_C \frac{f(z)}{z-a}dz = 2\pi i f(a)$, we get from (1)

$$\frac{1}{2\pi i}\oint_C \frac{\cos\pi z}{z^2-1}dz = \frac{1}{4\pi i}\left[2\pi i\cos\pi - 2\pi i\cos(-\pi)\right]$$

$$= \frac{1}{4\pi i}\left[-2\pi i + 2\pi i\right]$$

$$= 0$$

(b) Here only the point $z=1$ lies inside the rectangle $\pm i$, $2\pm i$.

Then by using the Cauchy's integral formula and also the Cauchy's integral theorem, we get from (1),

$$\frac{1}{2\pi i}\oint_C \frac{\cos\pi z}{z^2-1}dz = \frac{1}{4\pi i}\left[2\pi i\cos\pi - 0\right]$$

$$= -\frac{1}{2}$$

7. Show that $\frac{1}{2\pi i}\oint_C \frac{e^{zt}dz}{z^2+1} = \sin t$ if $t>0$ and C is the circle $|z|=3$

Solution: We have $\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2+1}dz = \frac{1}{2\pi i}\oint_C \frac{e^{zt}dz}{(z+i)(z-i)}$

$$= \frac{1}{2\pi i}\cdot\frac{1}{2i}\left[\oint_C \frac{e^{zt}}{z-i}dz - \oint_C \frac{e^{zt}dz}{z+i}\right] \quad \cdots (1)$$

Here $f(z) = e^{zt}$ is analytic inside and on the given circle $|z|=3$ and $z = \pm i$ are inside C.

Then by using Cauchy's integral formula, we get from (1)

$$\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{z^2+1}dz = \frac{1}{2\pi i}\cdot\frac{1}{2i}\left[2\pi i f(i) - 2\pi i f(-i)\right]$$

$$= \frac{1}{2i}\left[e^{it} - e^{-it}\right]$$

$$= \frac{1}{2i}\cdot 2i\sin t \quad \left[\because e^{it} - e^{-it} = 2i\sin t\right]$$

$$= \sin t$$

8. Evaluate $\oint_C \frac{e^{iz}}{z^3}dz$ where C is the circle $|z|=2$.

Solution: Here $f(z) = e^{iz}$ is analytic inside and on the circle $|z|=2$ and $z=a=0$ is a point inside the given circle.

Then by using Cauchy's integral formula,
$\oint_C \frac{f(z)}{(z-a)^{n+1}}dz = \frac{2\pi i}{\lfloor n} f^{(n)}(a)$, we get

$$\oint_C \frac{f(z)}{(z-0)^3}dz = \frac{2\pi i}{\lfloor 2} f^{(2)}(0)$$

$$\therefore \oint_C \frac{e^{iz}}{z^3}dz = \frac{2\pi i}{2}\cdot(-1)$$

$$= -\pi i$$

$f(z) = e^{iz}$
$\therefore f'(z) = ie^{iz}$
$f''(z) = -e^{iz}$
$f''(0) = -1 = f^{(2)}(0)$

9. Find the value of (a) $\oint_C \frac{\sin^6 z}{z-\frac{\pi}{6}}dz$, (b) $\oint_C \frac{\sin^6 z\, dz}{(z-\frac{\pi}{6})^3}$ if C is the circle $|z|=1$.

Solution: Here $f(z) = \sin^6 z$ is analytic inside and on the circle $|z|=1$ and $z=a=\frac{\pi}{6}$ is a point inside the given circle.

(a) Then by using Cauchy's integral formula,
$\oint_C \frac{f(z)}{z-a}dz = 2\pi i f(a)$ we get

$$\oint_C \frac{\sin^6 z}{z-\frac{\pi}{6}}dz = 2\pi i\left(\sin\frac{\pi}{6}\right)^6$$

$$= 2\pi i\cdot\left(\frac{1}{2}\right)^6$$

$$= 2\pi i\cdot\frac{1}{64}$$

$$= \frac{\pi i}{32}$$

(b) Then by using Cauchy's integral formula,

$$\oint_C \frac{f(z)}{(z-a)^{n+1}}dz = \frac{2\pi i}{\lfloor n} f^{(n)}(a) \text{ we get}$$

$$\oint_C \frac{\sin^6 z}{(z-\frac{\pi}{6})^3}dz = \frac{2\pi i}{\lfloor 2} f^{(2)}\left(\frac{\pi}{6}\right) \quad \cdots (1)$$

We have $f(z) = \sin^6 z$

$\therefore f'(z) = 6\sin^5 z \cdot \cos z$

$f''(z) = 30\sin^4 z \cos^2 z + 6\sin^5 z(-\sin z)$

$$\therefore f''\left(\frac{\pi}{6}\right) = f^{(2)}\left(\frac{\pi}{6}\right) = 30\cdot\left(\frac{1}{2}\right)^4\cdot\left(\frac{\sqrt{3}}{2}\right)^2 - 6\cdot\left(\frac{1}{2}\right)^6$$

$$= 30\cdot\frac{1}{16}\cdot\frac{3}{4} - 6\cdot\frac{1}{64}$$

$$= \frac{90-6}{64}$$

$$= \frac{84}{64}$$

$$= \frac{21}{16}$$

So from (1) we get

$$\oint_C \frac{\sin^6 z\, dz}{(z-\frac{\pi}{6})^3} = \frac{2\pi i}{2}\cdot\frac{21}{16}$$

$$= \frac{21\pi i}{16}$$

10. Evaluate $\frac{1}{2\pi i}\oint_C \frac{e^{zt}}{(z^2+1)^2}dz$ if $t>0$ and C is the circle $|z|=3$.

Solution: We have, $\frac{1}{(z^2+1)^2} = \frac{1}{(z+i)^2(z-i)^2}$

$$= \frac{1}{4i}\left[\frac{1}{(z-i)^2} - \frac{1}{(z+i)^2}\right]$$

$$\therefore \oint_C \frac{e^{zt}dz}{(z^2+1)^2} = \frac{1}{4i}\left[\oint_C \frac{\frac{e^{zt}}{z}dz}{(z-i)^2} - \oint_C \frac{\frac{e^{zt}}{z}dz}{(z+i)^2}\right]$$

Here $f(z) = \frac{e^{zt}}{z}$ is analytic inside on the given circle $|z|=3$ and $z=\pm i$ are inside C.

Then by using Cauchy's integral theorem,

$$\oint_C \frac{f(z)}{(z-a)^{n+1}}dz = \frac{2\pi i}{\lfloor n} f^{(n)}(a) \text{ we get}$$

$$\oint_C \frac{e^{zt}dz}{(z^2+1)^2} = \frac{1}{4i}\left[\frac{2\pi i}{\lfloor 1} f'(i) - \frac{2\pi i}{\lfloor 1} f'(-i)\right] \cdots (1)$$

We have, $f(z) = \frac{e^{zt}}{z}$

$$\therefore f'(z) = \frac{z.e^{zt}.t - 1.e^{zt}}{z^2}$$

$$\therefore f'(i) = \frac{ite^{it} - e^{it}}{i^2}$$

$$= e^{it} - ite^{it} \quad [\because i^2 = -1]$$

Also, $f'(-i) = \frac{-ite^{-it} - e^{-it}}{(-i)^2}$

$$= ite^{-it} + e^{-it}$$

So from (1) we get

$$\frac{1}{2\pi i}\oint_C \frac{e^{zt}dz}{(z^2+1)^2} = \frac{1}{4i}\left[e^{it} - ite^{it} - ite^{-it} - e^{-it}\right]$$

$$= \frac{1}{4i}\left[(e^{it} - e^{-it}) - it(e^{it} + e^{-it})\right]$$

$$= \frac{1}{4i}\left[2i\sin t - it.2\cos t\right]$$

$$= \frac{1}{4i}.2i\left[\sin t - t\cos t\right]$$

$$= \frac{1}{2}(\sin t - t\cos t) \quad \left[\begin{array}{l}\because e^{i\theta} = \cos\theta + i\sin\theta \\ e^{-i\theta} = \cos\theta - i\sin\theta \\ \Rightarrow 2\cos\theta = e^{i\theta} + e^{-i\theta} \\ 2i\sin\theta = e^{i\theta} - e^{-i\theta}\end{array}\right]$$

11. Evaluate $\oint_C \frac{e^z dz}{z(1-z)^3}$ if (i) 0 lies inside C and 1 lies outside C, (ii) 1 lies inside C and 0 lies outside C, (iii) 0 and 1 lie inside C.

Solution: (i) Since 0 lies inside C and 1 lies outside C.

$$\therefore \oint_C \frac{e^z dz}{z(1-z)^3} = \oint_C \frac{f(z)}{z-0}dz \text{ where } f(z) = \frac{e^z}{(1-z)^3}$$

$$= 2\pi i f(0)$$

$$= 2\pi i \cdot 1$$

$$= 2\pi i$$

(ii) Since 1 lies inside C and 0 lies outside C.

$$\therefore \oint_C \frac{e^z\,dz}{z(1-z)^3} = \oint_C \frac{f(z)\,dz}{(z-1)^3}$$

$$= \frac{2\pi i}{\lfloor 2}\, f''(1)$$

Where, $f(z) = -\dfrac{e^z}{z}$

$$\therefore f'(z) = \frac{e^z}{z^2} - \frac{e^z}{z}$$

$$f''(z) = -\frac{2e^z}{z^3} + \frac{e^z}{z^2} + \frac{e^z}{z^2} - \frac{e^z}{z}$$

$$\therefore f''(1) = -2e^1 + 2e^1 - e^1$$

$$= -e$$

$$\therefore \oint_C \frac{e^z\,dz}{z(1-z)^3} = \frac{2\pi i}{2}\cdot(-e)$$

$$= -\pi i e$$

(iii) Since 0 and 1 lie inside C, so we express $\dfrac{1}{z(1-z)^3}$ in partial fractions.

Let
$$\frac{1}{z(1-z)^3} = \frac{1}{z} + \frac{1}{1-z} + \frac{A}{(1-z)^2} + \frac{B}{(1-z)^3} \quad \cdots (1)$$

$$\Rightarrow 1 = (1-z)^3 + z(1-z)^2 + Az(1-z) + Bz \quad \cdots (2)$$

Putting $z=1$ in (2), we get $1 = B$

Equating the coefficients of z^2 from the bothsides of (2), we get

$$0 = 3 - 2 - A \quad \text{or,}\ A = 1$$

So from (1) we get

$$\frac{1}{z(1-z)^3} = \frac{1}{z} + \frac{1}{1-z} + \frac{1}{(1-z)^2} + \frac{1}{(1-z)^3}$$

$$\therefore \oint_C \frac{e^z\,dz}{z(1-z)^3} = \oint_C \frac{e^z\,dz}{z} + \oint_C \frac{e^z\,dz}{1-z} + \oint_C \frac{e^z\,dz}{(1-z)^2} + \oint_C \frac{e^z\,dz}{(1-z)^3}$$

$$= 2\pi i\cdot(e^0) - 2\pi i\cdot(e^1) + \frac{2\pi i}{\lfloor 1} f'(1) - \frac{2\pi i}{\lfloor 2} f''(1)$$

$$= 2\pi i - 2\pi i e + 2\pi i e - \frac{2\pi i}{2}\cdot e$$

$$= \pi i(2 - e)$$

12. What is the value of $\oint_C \frac{z^2+1}{z^2-1}\,dz$ if C is a circle of unit radius with centre at (i) $z=1$ and (ii) $z=-1$.

<u>Solution</u>: (i) If C is a circle of unit radius with centre at $z=1$, then

$$\oint_C \frac{(z^2+1)\,dz}{z^2-1} = \oint_C \frac{\frac{z^2+1}{z+1}}{z-1}\,dz$$

$$= 2\pi i f(1) \quad \text{where } f(z) = \frac{z^2+1}{z+1}$$

$$= 2\pi i \cdot 1$$

$$= 2\pi i$$

(ii) If C is a circle of unit radius with centre $z=-1$, then

$$\oint_C \frac{z^2+1}{z^2-1}\,dz = \oint_C \frac{\frac{z^2+1}{z-1}}{z+1}\,dz$$

$$= 2\pi i f(-1) \quad \text{where } f(z) = \frac{z^2+1}{z^2-1}$$

$$= 2\pi i\,(-1)$$

$$= -2\pi i$$

13. Using Cauchy's integral formula, evaluate $\oint_C \frac{z\,dz}{(z-1)(z-2)}$ where C is the circle $|z-2| = \frac{1}{2}$

<u>Solution</u>: Since $z=2$ is the only point lies inside the circle $|z-2| = \frac{1}{2}$,

$$\therefore \oint_C \frac{z\,dz}{(z-1)(z-2)} = \oint_C \frac{\left(\frac{z}{z-1}\right)dz}{z-2}$$

$$= 2\pi i f(2) \quad \text{where } f(z) = \frac{z}{z-1}$$

$$= 2\pi i \cdot 2$$

$$= 4\pi i$$

14. Evaluate $\oint_C \frac{dz}{(z^2+4)^2}$, where C is the circle $|z-i| = 2$

<u>Solution</u>: Let $F(z) = \frac{1}{(z^2+4)^2}$

$\therefore$ Singular points of $f(z)$ are $z = \pm 2i$. Among this only $z = 2i$ lies inside the circle $|z-i| = 2$.

$$\therefore \oint_C \frac{dz}{(z^2+4)^2} = \oint_C \frac{\frac{1}{(z+2i)^2}\,dz}{(z-2i)^2}$$

$$= \frac{2\pi i}{\underline{1}} f'(2i)$$

$$= 2\pi i \cdot \frac{1}{32 i}$$

$$= \frac{\pi}{16}$$

where $f(z) = \frac{1}{(z+2i)^2}$

$\therefore f'(z) = -\frac{2}{(z+2i)^3}$

$\therefore f(2i) = -\frac{2}{(4i)^3}$

$= \frac{1}{32 i}$

Singular Points, poles, Residues

Singular points: All the points of the z-plane at which an analytic function does not have a unique derivative are said to be singular points.

If $f(z) = \dfrac{1}{(z-3)^2}$, then $z=3$ is a singularity of $f(z)$.

Poles: If $f(z) = \dfrac{\varphi(z)}{(z-a)^n}$, $\varphi(a) \neq 0$, where $\varphi(z)$ is analytic everywhere in a region including $z=a$, and if n is a positive integer, then $f(z)$ has a singularity at $z=a$ which is called a pole of order n. If $n=1$, the pole is often called a simple pole; if $n=2$ it is called a double pole, etc.

If $f(z) = \dfrac{z}{(z-3)^2(z+1)}$ has two singularities; a pole of order 2 or double pole at $z=3$ and a pole of order 1 or simple pole at $z=-1$.

Residues: If $f(z)$ has a pole of order n at $z=a$ but is analytic at every other point inside and on a circle c with centre at a, then the Laurent's series about $z=a$ is given by

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$$

$$= \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} a_n (z-a)^{-n}$$

$$\therefore f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots$$

The part $a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots$ is called the analytic part, while the remainder consisting of inverse powers of $z-a$ is called the principal part.

The coefficient a_{-1}, called the residue of $f(z)$ at the pole $z=a$.

Method of finding residues:

(i) Residue of $f(z)$ at simple pole $z=a$ is

$$\lim_{x \to a} (z-a) f(z)$$

(ii) Residue of $f(z)$ at $z=a$ with pole of order n is

$$\lim_{x \to a} \frac{1}{\lfloor n-1} \frac{d^{n-1}}{dz^{n-1}} \left\{ (z-a)^n f(z) \right\}$$

Problem-1: Determine the residues of each function at its poles:

(i) $\dfrac{z^2}{(z-2)(z^2+1)}$

Let $f(z) = \dfrac{z^2}{(z-2)(z^2+1)}$

poles of $f(z)$ are given by

$$(z-2)(z^2+1) = 0$$

$\therefore\ z = 2, \pm i$

Residue at simple pole $z = 2$ is

$$\lim_{z \to 2} (z-2) f(z)$$

$$= \lim_{z \to 2} (z-2) \cdot \frac{z^2}{(z-2)(z^2+1)}$$

simple pole $= \frac{4}{5}$

Residue at $z = i$ is

$$\lim_{z \to i} (z-i) \cdot \frac{z^2}{(z-2)(z+i)(z-i)}$$

$$= \frac{i^2}{(i-2) \cdot 2i}$$

$$= \frac{-1}{2i^2 - 4i}$$

$$= \frac{-1}{-2(1+2i)}$$

$$= \frac{1-2i}{10}$$

Residue at simple pole $z = -i$ is

$$\lim_{z \to -i} (z+i) \frac{z^2}{(z-2)(z+i)(z-i)}$$

$$= \frac{1+2i}{10}$$

(ii) $f(z) = \dfrac{1}{z(z+2)^3}$

poles of $f(z)$ are given by $z(z+2)^3 = 0$

$\therefore z = 0, -2$

$\therefore z = 0$ is a simple pole and $z = -2$ is a pole of order 3.

Residue at simple pole $z=0$ is

$$\lim_{z\to 0} z \cdot \frac{1}{z(z+2)^3}$$

$$= \frac{1}{8}$$

Residue at $z=-2$ (pole of order 3) is

$$\lim_{z\to -2} \frac{1}{\lfloor 2} \frac{d^2}{dz^2}\left\{(z+2)^3 \cdot \frac{1}{z(z+2)^3}\right\}$$

$$= \lim_{z\to -2} \frac{1}{2} \frac{d^2}{dz^2}\left(\frac{1}{z}\right)$$

$$= \lim_{z\to -2} \frac{1}{2}\left(\frac{2}{z^3}\right)$$

$$= -\frac{1}{8}$$

Problem-2: Determine the residues of each function at its poles:

(i) $\frac{2z+3}{z^2-4}$ (ii) $\frac{z-3}{z^3+5z^2}$ (iii) $\frac{e^{zt}}{(z-2)^3}$

(iv) $\frac{z}{(z^2+1)^2}$

Cauchy's residue theorem: If $f(z)$ is analytic within and on a simple closed curve C except at a number of poles $a, b, c, \cdots$ interior to C at which the residues $a_{-1}, b_{-1}, c_{-1}, \cdots$ respectively, then

$$\oint_C f(z)\,dz = 2\pi i\,(a_{-1}+b_{-1}+c_{-1}+\cdots)$$

$$= 2\pi i \text{ (sum of residues)}$$

Problem-3: Evaluate $\oint_C \frac{e^z dz}{(z-1)(z+3)^2}$ where C is given by (i) $|z| = \frac{3}{2}$, (ii) $|z| = 10$.

Solution: Here $f(z) = \frac{e^z}{(z-1)(z+3)^2}$

poles of $f(z)$ are given by $(z-1)(z+3)^2 = 0$

$\therefore z = 1, -3$

Residue at simple pole $z = 1$ is

$$\lim_{z \to 1} \cancel{(z-1)} \cdot \frac{e^z}{\cancel{(z-1)}(z+3)^2}$$

$$= \frac{e}{16}$$

Residue at double pole $z = -3$ is

$$\lim_{z \to -3} \frac{1}{\underline{|1}} \frac{d^1}{dz^1} \left\{ (z+3)^2 \cdot \frac{e^z}{(z-1)(z+3)^2} \right\}$$

$$= \lim_{z \to -3} \frac{d}{dz} \left(\frac{e^z}{z-1} \right)$$

$$= \lim_{z \to -3} \frac{(z-1)e^z - e^z}{(z-1)^2}$$

$$= \frac{-5e^{-3}}{16}$$

(i) since $|z| = \frac{3}{2}$ encloses only the pole $z = 1$,

the required integral $= 2\pi i \left(\frac{e}{16} \right)$

$$= \frac{\pi i e}{8}$$

(ii) since $|z| = 10$ encloses both poles $z = 1$ and $z = -3$, the required integral

$$= 2\pi i \left(\frac{e}{16} - \frac{5e^{-3}}{16} \right)$$

$$= \frac{\pi i (e - 5e^{-3})}{8}$$

Problem-4: Evaluate $\oint_C \frac{z^2\,dz}{(z+1)(z+3)}$, where C is a simple closed curve enclosing all the poles.

Solution: Here $f(z) = \frac{z^2}{(z+1)(z+3)}$

poles of $f(z)$ are given by $(z+1)(z+3)=0$

$\therefore z = -1, -3$

Both poles are simple.

Residue at simple pole $z=-1$ is

$$\lim_{z\to -1} \cancel{(z+1)} \cdot \frac{z^2}{\cancel{(z+1)}(z+3)}$$

$$= \frac{1}{2}$$

Residue at simple pole $z=-3$ is

$$\lim_{z\to -3} \cancel{(z+3)} \cdot \frac{z^2}{(z+1)\cancel{(z+3)}}$$

$$= -\frac{9}{2}$$

Hence by Cauchy's residue theorem, we get

$$\oint_C f(z)\,dz = 2\pi i\left(\frac{1}{2} - \frac{9}{2}\right)$$

$$\therefore \oint_C \frac{z^2\,dz}{(z+1)(z+2)} = -8\pi i$$

OR (Use by Cauchy's integral formula)

$$\oint_C \frac{z^2\,dz}{(z+1)(z+3)} = \oint_C \left\{1 + \frac{-4z-3}{(z+1)(z+3)}\right\} dz$$

$$= \oint_C \left\{1 + \frac{\frac{1}{2}}{z+1} + \frac{-9/2}{z+3}\right) dz$$

$$= \oint_C dz + \frac{1}{2}\oint_C \frac{1}{z+1}\,dz - \frac{9}{2}\oint_C \frac{1}{z+3}\,dz$$

$$= 0 + \frac{1}{2}\cdot 2\pi i\cdot 1 - \frac{9}{2}\cdot 2\pi i \cdot 1$$

$$= -8\pi i$$

Laurent's series

SOS-91 / P-166 Expand $f(z) = \dfrac{1}{z-3}$ in a Laurent series valid for (a) $|z| < 3$, (b) $|z| > 3$.

Solution: We have, $f(z) = \dfrac{1}{z-3}$ ... (1)

(a) since $|z| < 3$, so from (1) we get

$$f(z) = -\frac{1}{3}\left(1-\frac{z}{3}\right)^{-1}$$
$$= -\frac{1}{3}\left(1+\frac{z}{3}+\frac{z^2}{9}+\frac{z^3}{27}+\cdots\cdots\right)$$
$$= -\frac{1}{3}-\frac{z}{9}-\frac{z^2}{27}-\frac{z^3}{81}-\cdots\cdots$$

(b) since $|z| > 3$, so from (1) we get

$$f(z) = \frac{1}{z}\left(1-\frac{3}{z}\right)^{-1}$$
$$= \frac{1}{z}\left(1+\frac{3}{z}+\frac{9}{z^2}+\frac{27}{z^3}+\cdots\cdots\right)$$
$$= \frac{1}{z}+\frac{3}{z^2}+\frac{9}{z^3}+\frac{27}{z^4}+\cdots\cdots$$

SOS-92 / P-166 Expand $f(z) = \dfrac{z}{(z-1)(2-z)}$ in a Laurent series valid for:

(a) $|z| < 1$, (b) $1 < |z| < 2$, (c) $|z| > 2$, (d) $|z-1| > 1$

(e) $0 < |z-2| < 1$

Solution: We have, $f(z) = \dfrac{z}{(z-1)(2-z)}$

$$= \frac{1}{z-1}+\frac{2}{2-z} \quad \cdots (1)$$

(a) since $|z| < 1$, so from (1) we get

$$f(z) = -\frac{1}{1-z}+\frac{1}{1-z/2}$$
$$= -(1-z)^{-1}+(1-z/2)^{-1}$$
$$= -(1+z+z^2+z^3+\cdots\cdots)+\left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots\right)$$
$$= -\frac{z}{2}-\frac{3z^2}{4}-\frac{7z^3}{8}-\cdots\cdots$$

(b) since $1 < |z| < 2$, so from (1) we get

$$f(z) = \frac{1}{z\left(1-\frac{1}{z}\right)}+\frac{1}{1-z/2}$$
$$= \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}+\left(1-z/2\right)^{-1}$$
$$= \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right)+\left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots\right)$$

$$\therefore f(z) = \cdots\cdots + \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z/2 + \frac{z^2}{4} + \frac{z^3}{8} + \cdots\cdots$$

(c) Since $|z|>2$, so from (1) we get—

$$f(z) = \frac{1}{z(1-\frac{1}{z})} - \frac{2}{z(1-2/z)}$$

$$= \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} - \frac{2}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$$= \frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right) - \frac{2}{z}\left(1+\frac{2}{z}+\frac{4}{z^2}+\cdots\right)$$

$$= \frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\cdots - \frac{2}{z} - \frac{4}{z^2} - \frac{8}{z^3} - \cdots$$

$$= -\frac{1}{z} - \frac{3}{z^2} - \frac{7}{z^3} - \cdots\cdots$$

(d) since $|z-1|>1$, so from (1) we get—

$$f(z) = \frac{1}{z-1} - \frac{2}{z-2}$$

$$= \frac{1}{z-1} - \frac{2}{(z-1)-1}$$

$$= \frac{1}{z-1} - \frac{2}{(z-1)(1-\frac{1}{z-1})}$$

$$= \frac{1}{z-1} - \frac{2}{z-1}\left(1-\frac{1}{z-1}\right)^{-1}$$

$$= \frac{1}{z-1} - \frac{2}{z-1}\left\{1+\frac{1}{z-1}+\frac{1}{(z-1)^2}+\cdots\cdots\right\}$$

$$= \frac{1}{z-1} - \frac{2}{z-1} - \frac{2}{(z-1)^2} - \frac{2}{(z-1)^3} - \cdots\cdots$$

$$= -\frac{1}{z-1} - \frac{2}{(z-1)^2} - \frac{2}{(z-1)^3} - \cdots\cdots$$

(e) since $0<|z-2|<1$, so from (1) we get

$$f(z) = \frac{1}{(z-2)+1} - \frac{2}{z-2}$$

$$= \{1+(z-2)\}^{-1} - \frac{2}{z-2}$$

$$= 1-(z-2)+(z-2)^2-(z-2)^3+\cdots\cdots - \frac{2}{z-2}$$

$$= 1 - \frac{2}{z-2} - (z-2) + (z-2)^2 - (z-2)^3 + \cdots\cdots$$

SOS-93
P-166

Expand $f(z) = \frac{1}{z(z-2)}$ in a Laurent series valid for (a) $0<|z|<2$, (b) $|z|>2$.

Solution: We have, $f(z) = \frac{1}{z(z-2)}$

$$= \frac{-\frac{1}{2}}{z} + \frac{\frac{1}{2}}{z-2}$$

$$= \frac{1}{2}\left[\frac{1}{z-2} - \frac{1}{z}\right] \cdots (1)$$

(a) Since $0<|z|<2$, so from (1) we get

$$f(z) = \frac{1}{2}\left[\frac{1}{-2(1-\frac{z}{2})} - \frac{1}{z}\right]$$

$$= \frac{1}{2}\left[-\frac{1}{2}(1-\tfrac{z}{2})^{-1} - \frac{1}{z}\right]$$

$$= \frac{1}{2}\left[-\frac{1}{2}\left(1+\tfrac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots\right) - \frac{1}{z}\right]$$

$$= -\frac{1}{2z} - \frac{1}{4} - \frac{z}{8} - \frac{z^2}{16} - \frac{z^3}{32} - \cdots$$

(b) Since $|z|>2$, so from (1) we get

$$f(z) = \frac{1}{2}\left[\frac{1}{z}\left(1-\frac{2}{z}\right)^{-1} - \frac{1}{z}\right]$$

$$= \frac{1}{2}\left[\frac{1}{z}\left(1+\frac{2}{z}+\frac{4}{z^2}+\frac{8}{z^3}+\cdots\right) - \frac{1}{z}\right]$$

$$= \frac{1}{z^2} + \frac{2}{z^3} + \frac{4}{z^4} + \cdots$$

SOS-94
P-166

Find an expression of $f(z) = \frac{z}{z^2+1}$ valid for $|z-3|>2$.

Solution: Since $|z-3|>2$, so from $f(z) = \frac{z}{z^2+1}$, we get

$$f(z) = \frac{z}{z^2(1+\frac{1}{z^2})}$$

$$= \frac{1}{z}\left(1+\frac{1}{z^2}\right)^{-1}$$

$$= \frac{1}{z}\left(1-\frac{1}{z^2}+\frac{1}{z^4}-\frac{1}{z^6}+\frac{1}{z^8}-\cdots\right)$$

$$= \frac{1}{z} - \frac{1}{z^3} + \frac{1}{z^5} - \frac{1}{z^7} + \frac{1}{z^9} - \cdots$$

SOS-95
P-166

Expand $f(z) = \dfrac{1}{(z-2)^2}$ in a Laurent series valid for (a) $|z| < 2$, (b) $|z| > 2$.

Solution: we have, $f(z) = \dfrac{1}{(z-2)^2}$ --- (1)

(a) Since $|z| < 2$, so from (1) we get

$$f(z) = \frac{1}{\{-2(1-z/2)\}^2}$$
$$= \frac{1}{4(1-z/2)^2}$$
$$= \frac{1}{4}(1-z/2)^{-2}$$
$$= \frac{1}{4}\{1+2\cdot(\tfrac{z}{2})+3\cdot(z/2)^2+4(z/2)^3+\cdots\infty\}$$
$$= \frac{1}{4}+\frac{z}{4}+\frac{3z^2}{16}+\frac{z^3}{8}+\cdots\cdots$$

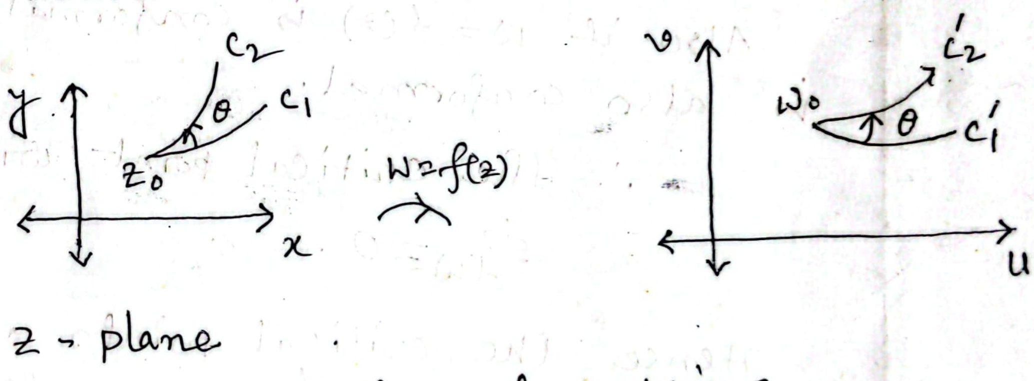
(b) Since $|z|>2$, so from (1) we get

$$f(z) = \frac{1}{\{z(1-\frac{2}{z})\}^2}$$
$$= \frac{1}{z}(1-\tfrac{2}{z})^{-2}$$
$$= \frac{1}{z}\{1+2(\tfrac{2}{z})+3(\tfrac{2}{z})^2+4(\tfrac{2}{z})^3+\cdots\infty\}$$
$$= \frac{1}{z}+\frac{4}{z^2}+\frac{12}{z^3}+\frac{32}{z^4}+\cdots\cdots$$

Conformal Mapping

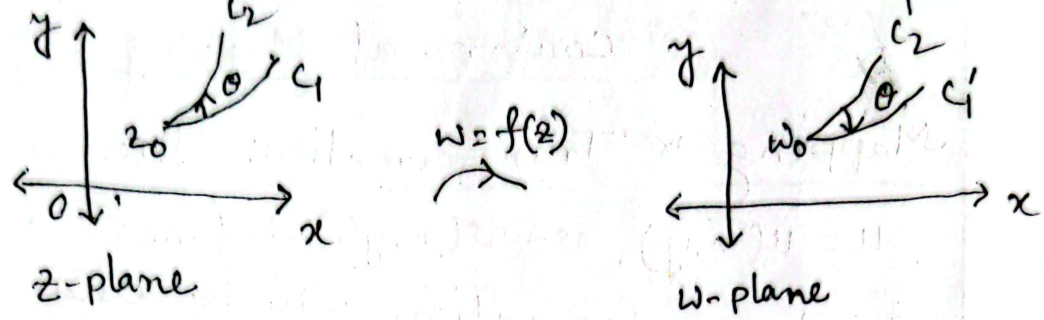
Mapping or Transformation: The set of equations $u = u(x,y)$, $v = v(x,y)$ defines a mapping or transformation which establishes correspondence between the points in the uv and xy planes.

Conformal Mapping: A mapping $w = f(z)$ is said to be conformal at $z = z_0$ if it preserves the angle between any two curves through z_0 in z plane both in magnitude and direction.



Conformal mapping

Isogonal Mapping: A mapping $w = f(z)$ is said to be isogonal if it preserves the angle between any two curves through z_0 in z plane only in magnitude but not necessarily in direction.



Isogonal mapping

Critical point: * A point z_0 is said to be a critical point of a transformation $w=f(z)$ if is not conformal at z_0 where $f'(z_0)=0$.

(i) The critical point will occur at $\frac{dw}{dz}=0$.

Also if $w=f(z)$ is conformal then $z=f^{-1}(w)$ is also conformal.

$\therefore$ The critical point will occur at $\frac{dz}{dw}=0$.

Hence the critical points of $w=f(z)$ will occur at $\frac{dw}{dz}=0$, $\frac{dz}{dw}=0$.

(ii) If $f(z)$ is not analytic, then $f(z)$ is not conformal.

(iii) An analytic function $f(z)$ is conformal everywhere except at its critical points where $f'(z)\neq 0$.

* A point at which $f'(z)=0$ is called a critical point of the transformation.

Problem-1. Find the critical points of the transformation $w = z + \frac{1}{z}$.

Solution: Given $w = z + \frac{1}{z}$

$$\therefore \frac{dw}{dz} = 1 - \frac{1}{z^2} = \frac{z^2-1}{z^2}$$

Now the critical points occur at $\frac{dw}{dz} = 0, \frac{dz}{dw} = 0$.

$$\therefore \frac{z^2-1}{z^2} = 0, \quad \frac{z^2}{z^2-1} = 0$$

$$\text{or, } z^2 - 1 = 0, \text{ and } z^2 = 0$$

$$\therefore z = \pm 1, 0$$

$\therefore$ The critical points are $z = \pm 1, 0$

Problem-2. Find the points such that $w = f(z) = \sin z$ is not conformal.

Solution: The points at which $f(z)$ is not conformal are called critical points.

Hence we require the critical points of $w = \sin z$.

$$\therefore \frac{dw}{dz} = \cos z, \quad \frac{dz}{dw} = \frac{1}{\cos z}$$

The critical points occur at $\frac{dw}{dz} = 0, \frac{dz}{dw} = 0$

$$\Rightarrow \cos z = 0, \quad \frac{1}{\cos z} = 0$$

$$\Rightarrow z = (2n+1)\frac{\pi}{2}, \; n = 0, \pm 1, \pm 2, \ldots$$

and $1 = 0$ which is impossible.

$\therefore$ The critical points are

$$z = (2n+1)\frac{\pi}{2}, \quad n = 0, \pm 1, \pm 2, \dots$$

Problem-3. Find the critical points of the transformation $w^2 = (z-\alpha)(z-\beta)$

Solution: Given $w^2 = (z-\alpha)(z-\beta) \quad \dots (1)$

$$\therefore 2w\frac{dw}{dz} = (z-\alpha)\cdot 1 + (z-\beta)\cdot 1$$

The critical points of $w = f(z)$ is given by $\frac{dw}{dz} = 0$ or, $\frac{(z-\alpha)+(z-\beta)}{2w} = 0$.

$$\text{or, } (z-\alpha)+(z-\beta) = 0$$

$$\text{or, } z = \frac{\alpha+\beta}{2}$$

Also, $\frac{dz}{dw} = 0$ or, $\frac{2w}{(z-\alpha)+(z-\beta)} = 0$

$$\text{or, } w = 0$$

$$\text{or, } w^2 = 0$$

$$\text{or, } (z-\alpha)(z-\beta) = 0 \quad [\text{from (1)}]$$

$$\therefore z = \alpha, \beta$$

$\therefore$ The critical points are $z = \alpha, \beta, \frac{\alpha+\beta}{2}$.

15 <u>Linear transformation</u>: A transformation of the form $w = az + b$ where a and b are complex constants, is called a linear transformation.

15 <u>Bilinear transformation</u>: A transformation of the form $w = \frac{az+b}{cz+d}$ where a, b, c, d are complex constants such that $ad - bc \neq 0$, is called a bilinear transformation.

** (i) The bilinear transformation is also called Mobius transformation or linear fractional transformation.

(ii) The transformation $w = \frac{az+b}{cz+d}$ can be expressed as

$$cwz + dw - az - b = 0.$$

It is linear both in w and z. That is why, it is called bilinear.

(iii) The inverse of the transformation $w = \frac{az+b}{cz+d}$ is $z = \frac{-dw+b}{cw-a}$ which is also a bilinear transformation except $w = \frac{a}{c}$.

(iv) The expression $ad - bc$ is called the determinant of the transformation $w = \frac{az+b}{cz+d}$ or $z = \frac{-dw+b}{cw-a}$.

(v) The transformation is conformal only when $\frac{dw}{dz} \neq 0$ or $\frac{dz}{dw} \neq 0$ i.e. $ad - bc \neq 0$.

(vi) If $ad-bc=0$, every point in the z or w-plane is a critical point.

15 Invariant or fixed points:

If z maps into itself (i.e., $w=z$), then $w=\frac{az+b}{cz+d}$ gives

$$z=\frac{az+b}{cz+d}$$

$$\text{or, } cz^2+(d-a)z-b=0$$

The roots of this equation are defined as the invariant or fixed points of the bilinear transformation.

15 Cross ratio: The cross ratio of four points z_1, z_2, z_3, z_4 is defined by

$$\frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$$

15 Problem-1. Prove that cross ratio of four points is invariant (unchanged) in a bilinear transformation.

Proof: Let the points z_1, z_2, z_3 and z_4 of the z-plane map onto the points w_1, w_2, w_3 and w_4 of w-plane respectively under the bilinear transformation $w=\frac{az+b}{cz+d}$, then to prove that

$$\frac{(w_1-w_2)(w_3-w_4)}{(w_2-w_3)(w_4-w_1)}=\frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$$

Since w_i is image of z_i, $i=1,2,3,4$ 7

$$\therefore w_1 - w_2 = \frac{az_1+b}{cz_1+d} - \frac{az_2+b}{cz_2+d}$$

$$= \frac{(ad-bc)(z_1-z_2)}{(cz_1+d)(cz_2+d)}$$

Similarly $w_3 - w_4 = \dfrac{(ad-bc)(z_3-z_4)}{(cz_3+d)(cz_4+d)}$

$$\therefore (w_1-w_2)(w_3-w_4) = \frac{(ad-bc)^2(z_1-z_2)(z_3-z_4)}{(cz_1+d)(cz_2+d)(cz_3+d)(cz_4+d)} \quad \text{---(1)}$$

Similarly $(w_2-w_3)(w_4-w_1) = \dfrac{(ad-bc)^2(z_2-z_3)(z_4-z_1)}{(cz_1+d)(cz_2+d)(cz_3+d)(cz_4+d)}$ --(2)

Now divide (1) by (2) we get

$$\frac{(w_1-w_2)(w_3-w_4)}{(w_2-w_3)(w_4-w_1)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$$

Thus the cross ratio of four points is invariant under bilinear transformation.

Problem-2: Find a bilinear transformation which maps points z_1, z_2, z_3 of the z-plane into points w_1, w_2, w_3 of the w-plane respectively.

Solution: We know that the bilinear transformation is given by $w = \dfrac{az+b}{cz+d}$

Since w_1, w_2, w_3 are the images of z_1, z_2, z_3 respectively.

$$\text{Now } w_1 - w_2 = \frac{(ad-bc)(z_1-z_2)}{(cz_1+d)(cz_2+d)}$$

$$w_2 - w_3 = \frac{(ad-bc)(z_2-z_3)}{(cz_2+d)(cz_3+d)}$$

$$w - w_1 = \frac{(ad-bc)(z-z_1)}{(cz+d)(cz_1+d)}$$

and $$w_3 - w = \frac{(ad-bc)(z_3-z)}{(cz_3+d)(cz+d)}$$

$$\therefore \frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{\dfrac{(z-z_1)(z_2-z_3)(ad-bc)^2}{(cz+d)(cz_1+d)(cz_2+d)(cz_3+d)}}{\dfrac{(z_1-z_2)(z_3-z)(ad-bc)^2}{(cz+d)(cz_1+d)(cz_2+d)(cz_3+d)}}$$

or, $$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

Solving for w in terms of z we get the required transformation.

15 Problem-3. Find a bilinear transformation which maps points $z = 0, -i, -1$ into $w = i, 1, 0$ respectively.

Solution: Let $z_1 = 0$, $z_2 = -i$, $z_3 = -1$ and $w_1 = i$, $w_2 = 1$, $w_3 = 0$

The required transformation is given by

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

or, $$\frac{(w-i)(1-0)}{(i-1)(0-w)} = \frac{(z-0)(-i+1)}{(0+i)(-1-z)}$$

or, $$\frac{(w-i)}{(1-i)w} = \frac{z(1-i)}{-i(1+z)}$$

or, $$-i(w + wz - i - iz) = wz(1-2i-1)$$

or, $$-iw - iwz - 1 - z = -2iwz$$

or, $iw(z-1) = z+1$

or, $w = \dfrac{z+1}{i(z-1)}$

$\therefore w = -\dfrac{i(z+1)}{z-1}$ which is the required transformation.

15 Problem-4. Find a bilinear transformation which maps the points $i, -i, 1$ of the z-plane into $0, 1, \infty$ of the w-plane respectively.

Solution: Since $w = \dfrac{az+b}{cz+d}$ --- (1)

So we have, $\dfrac{a \cdot i+b}{c \cdot i+d} = 0$ --- (2)

$\dfrac{a(-i)+b}{c(-i)+d} = 1$ --- (3)

$\dfrac{a \cdot 1+b}{c \cdot 1+d} = \infty$ --- (4)

from (4) we get $c+d=0$ or, $c=-d$

(2) gives, $ai+b=0$ or $b=-ai$

Thus from (3), $\dfrac{-ai-ai}{di+d} = 1$

or, $\dfrac{-2ai}{d(1+i)} = 1$

or, $d = \dfrac{-2ai}{1+i} = \dfrac{-2ai(1-i)}{2}$

$\therefore d = -a(i+1)$

$\therefore c = a(i+1)$

So from (1) we get $w = \dfrac{az-ai}{a(i+1)z-a(i+1)}$

$= \dfrac{a(z-i)}{a(i+1)(z-1)}$

$= \dfrac{(z-i)(1-i)}{2(z-1)}$ is the required transformation.

15 Problem-5. Find the bilinear transformation which maps the points $z = \infty, i, 0$ into $w = 0, i, \infty$ respectively. What are the invariant points of this transformation?

Solution: Let $z_1 = \infty$, $z_2 = i$, $z_3 = 0$ and

$w_1 = 0$, $w_2 = i$, $w_3 = \infty$

The required bilinear transformation is given by

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

or, $$\frac{(w-w_1)\,w_3\left(\frac{w_2}{w_3}-1\right)}{(w_1-w_2)\,w_3\left(1-\frac{w}{w_3}\right)} = \frac{z_1\left(\frac{z}{z_1}-1\right)(z_2-z_3)}{z_1\left(1-\frac{z_2}{z_1}\right)(z_3-z)}$$

or, $$\frac{(w-0)\left(\frac{i}{\infty}-1\right)}{(0-i)\left(1-\frac{w}{\infty}\right)} = \frac{\left(\frac{z}{\infty}-1\right)(i-0)}{\left(1-\frac{i}{\infty}\right)(0-z)}$$

or, $$\frac{-w}{-i} = \frac{-i}{-z}$$

or, $$w = \frac{i^2}{z}$$

$\therefore$ $w = \frac{-1}{z}$ which is the required transformation.

The invariant points of $w = \frac{-1}{z}$ are given by

$$z = \frac{-1}{z}$$

or, $z^2 = -1$

or, $z^2 = i^2$

$\therefore$ $z = \pm i$ are the required invariant points.

Problem-6. Find the fixed points of the transformation $W = \frac{3z-5}{1+z}$.

Solution: The fixed points of $W = \frac{3z-5}{1+z}$ are given by

$$z = \frac{3z-5}{1+z}$$

$$\text{or, } z + z^2 = 3z - 5$$

$$\text{or, } z^2 - 2z + 5 = 0$$

$$\therefore z = \frac{2 \pm \sqrt{4-20}}{2}$$

$$= \frac{2 \pm 4i}{2}$$

$$= 1 \pm 2i$$

Therefore, the fixed points are $1+2i$ and $1-2i$.

15 Problem-7. Find the bilinear transformation which maps the points $z = 0, 1, \infty$ onto $w = -5, -1, 3$ respectively. What are the invariant points of this transformation?

Solution: Given $z_1 = 0$, $z_2 = 1$, $z_3 = \infty$ and
$w_1 = -5$, $w_2 = -1$, $w_3 = 3$

The required bilinear transformation is given by

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

$$\text{or, } \frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)z_3\left(\frac{z_2}{z_3}-1\right)}{(z_1-z_2)z_3\left(1-\frac{z}{z_3}\right)}$$

$$\text{or, } \frac{(w+5)(-1-3)}{(-5+1)(3-w)} = \frac{(z-0)(0-1)}{(0-1)(1-0)}$$

$$\text{or, } \frac{(w+5)(-4)}{(-4)(3-w)} = z$$

or, $\frac{w+5}{3-w} = \frac{z}{1}$

or, $w+5 = 3z - wz$

or, $w(1+z) = 3z-5$

or, $w = \frac{3z-5}{1+z}$

To find the invariant points, put $w=z$.

$\therefore z = \frac{3z-5}{1+z}$

or, $z^2+z = 3z-5$

or, $z^2-2z+1 = -4$

or, $(z-1)^2 = i^2 4$

or, $z-1 = \pm i2$

$\therefore z = 1 \pm i2$ are the invariant points.

Problem-8. Define fixed points. Prove that a bilinear transformation has atmost two fixed points.

Solution: A point $z=a$ is said to be a fixed point of a mapping $w=f(z)$ if it's image under $f(z)$ is itself. Thus it is given by the equation $z=f(z)$.

The fixed points of the general bilinear transformation $w = \frac{az+b}{cz+d}$ are given by $z = \frac{az+b}{cz+d}$

or, $cz^2+dz = az+b$

or, $cz^2+(d-a)z-b=0$

This is a quadratic equation in z. Hence a bilinear transformation has atmost two fixed points.

Contoure Integration-1

Evaluate by Contour integration:

(i) $\displaystyle\int_0^{2\pi} \frac{\cos 3\theta \, d\theta}{5-4\cos\theta}$ (ii) $\displaystyle\int_0^{\pi} \frac{\cos 2\theta \, d\theta}{1-2a\cos\theta+a^2}$, $a^2<1$

(iii) $\displaystyle\int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta+a^2}$, $|a|<1$

(iv) $\displaystyle\int_0^{\pi} \frac{\sin^2\theta \, d\theta}{5-4\cos\theta}$ (v) $\displaystyle\int_0^{2\pi} \frac{d\theta}{5+3\sin\theta}$

Solution: (i) let $\displaystyle I = \int_0^{2\pi} \frac{\cos 3\theta \, d\theta}{5-4\cos\theta}$

$$= \text{Real part of } \int_0^{2\pi} \frac{e^{i3\theta} d\theta}{5-4\cos\theta}$$

$$= \text{R P of } \int_0^{2\pi} \frac{e^{i3\theta} d\theta}{5-4\cdot\frac{1}{2}(e^{i\theta}+e^{-i\theta})}$$

Put $e^{i\theta} = z$

$\therefore e^{i\theta}\cdot i d\theta = dz$

or, $d\theta = \dfrac{dz}{iz}$

$\therefore \displaystyle I = \text{R.P. of } \oint_C \frac{z^3 \cdot \frac{dz}{iz}}{5-2(z+\frac{1}{z})}$, where C is the unit circle, $|z|=1$.

$$= \text{RP of } \frac{1}{i}\oint_C \frac{z^3 dz}{5z-2z^2-2}$$

$$= \text{RP of } \frac{-1}{i}\oint_C \frac{z^3 dz}{2z^2-5z+2}$$

$$= \text{RP of } \frac{-1}{i}\oint_C f(z)dz \quad \ldots (1)$$

where $f(z) = \dfrac{z^3}{2z^2-5z+2}$

poles of $f(z)$ are given by

$2z^2-5z+2=0$

or, $(z-2)(2z-1)=0$

$\therefore z=2, \frac{1}{2}$

Only simple pole $z=\frac{1}{2}$ within C.

$1-a^2$

$= 0$

Residue at $z=\frac{1}{2}$ is

$$\lim_{z\to\frac{1}{2}} (z-\tfrac{1}{2})\cdot\frac{z^3}{2z^2-5z+2}$$

$$=\lim_{z\to\frac{1}{2}} (z-\tfrac{1}{2})\cdot\frac{z^3}{(z-2)(2z-1)}$$

$$=\lim_{z\to\frac{1}{2}} \cancel{(z-\tfrac{1}{2})}\cdot\frac{z^3}{2(z-2)\cancel{(z-\tfrac{1}{2})}}$$

$$=\frac{\frac{1}{8}}{2(\frac{1}{2}-2)}$$

$$=-\frac{1}{24}$$

By Cauchy's residue theorem we have,

$$\oint_C f(z)\,dz = 2\pi i\left(-\frac{1}{24}\right)$$

$$=-\frac{\pi i}{12}$$

$$\therefore\ I = \text{R.P. of } \frac{-1}{i}\oint_C f(z)\,dz$$

$$=\text{R.P. of } \frac{-1}{i}\cdot\left(-\frac{\pi i}{12}\right)$$

$$=\frac{\pi}{12}$$

(ii) let $I=\displaystyle\int_0^{\pi}\frac{\cos 2\theta\,d\theta}{1-2a\cos\theta+a^2}$

$$=\frac{1}{2}\int_0^{2\pi}\frac{\cos 2\theta\,d\theta}{1-2a\cos\theta+a^2}\quad\left[\because \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx \text{ if } f(2a-x)=f(x)\right]$$

$$=\tfrac{1}{2}\text{R.P of } \frac{1}{2}\int_0^{2\pi}\frac{e^{i2\theta}\,d\theta}{1-2a\cos\theta+a^2}$$

$$=\text{RP of } \frac{1}{2}\int_0^{2\pi}\frac{e^{i2\theta}\,d\theta}{1-2a\cdot\frac{1}{2}(e^{i\theta}+e^{-i\theta})+a^2}$$

Put $e^{i\theta}=z\ \therefore\ d\theta=\dfrac{dz}{iz}$

$$\therefore I = \text{RP of } \frac{1}{2}\oint_C \frac{z^2 \cdot \frac{dz}{iz}}{1-a\left(z+\frac{1}{z}\right)+a^2}$$

$$= \text{RP of } \frac{1}{2i}\oint_C \frac{z^2\,dz}{z-az^2-a+a^2z}$$

$$= \text{R.P of } \frac{-1}{2i}\oint_C \frac{z^2\,dz}{az^2-a^2z-z+a}$$

$$= \text{R.P. of } \frac{-1}{2i}\oint_C f(z)\,dz$$

where $f(z) = \dfrac{z^2}{az^2-a^2z-z+a}$

poles of $f(z)$ are given by

$$az^2-a^2z-z+a=0$$

$$\text{or, } (z-a)(az-1)=0$$

$$\therefore z = a, \frac{1}{a}$$

Only simple pole $z=a$ lies inside C.

Residue at $z=a$ is,

$$\lim_{z\to a} \cancel{(z-a)}\frac{z^2}{\cancel{(z-a)}(az-1)}$$

$$= \frac{a^2}{a^2-1}$$

Hence by Cauchy's residue theorem we get

$$\oint_C f(z)\,dz = 2\pi i\left(\frac{a^2}{a^2-1}\right)$$

$$\therefore I = \text{R.P. of } \frac{-1}{2i}\oint_C f(z)\,dz$$

$$= \text{R.P. of } \frac{-1}{2i}\cdot 2\pi i\left(\frac{a^2}{a^2-1}\right)$$

$$= \frac{\pi a^2}{1-a^2}$$

(iii) let $I = \int_0^{2\pi} \frac{d\theta}{1-2a\cos\theta+a^2}$

$$= \int_0^{2\pi} \frac{d\theta}{1-a(e^{i\theta}+e^{-i\theta})+a^2}$$

$$= \oint_C \frac{dz/iz}{1-a(z+1/z)+a^2}$$

Put $e^{i\theta} = z$ $\therefore d\theta = \frac{dz}{iz}$

Where C is the unit circle, $|z| = 1$.

$$= \frac{1}{i} \oint_C \frac{dz}{z-az^2-a+a^2z}$$

$$= \frac{-1}{i} \oint_C \frac{dz}{az^2-a^2z-z+a}$$

$$= \frac{-1}{i} \oint_C f(z)\,dz$$

where $f(z) = \frac{1}{az^2-a^2z-z+a}$

Poles of $f(z)$ are given by

$$az^2 - a^2z - z + a = 0$$

or, $(z-a)(az-1) = 0$

$\therefore z = a, 1/a$

Only simple pole $z=a$ lies inside C.

Residue at $z=a$ is $\lim_{z\to a} (z-a) \cdot \frac{1}{(z-a)(az-1)}$

$$= \frac{1}{a^2-1}$$

Hence by Cauchy's residue theorem we get

$$\oint_C f(z)\,dz = 2\pi i \left(\frac{1}{a^2-1}\right)$$

$\therefore$ $I = -\frac{1}{i} \oint_C f(z)\,dz$

$$= -\frac{1}{i} \cdot 2\pi i \left(\frac{1}{a^2-1}\right)$$

$$= \frac{2\pi}{1-a^2}$$

(iv) Let $I = \int_0^{\pi} \frac{\sin^2\theta \, d\theta}{5-4\cos\theta}$

$= \frac{1}{2}\int_0^{2\pi} \frac{\sin^2\theta \, d\theta}{5-4\cos\theta}$ $\left[\because \int_0^{2a} f(x)dx = 2\int_0^a f(x)\,dx \text{ if } f(2a-x) = f(x)\right]$

$= \frac{1}{2}\cdot\frac{1}{2}\int_0^{2\pi} \frac{(1-\cos 2\theta)\, d\theta}{5-4\cos\theta}$

$= \text{R.P of } \frac{1}{4}\int_0^{2\pi} \frac{(1-e^{i2\theta})\, d\theta}{5-4\cos\theta}$

$= \text{R.P of } \frac{1}{4}\int_0^{2\pi} \frac{(1-e^{i2\theta})\, d\theta}{5-2(e^{i\theta}+e^{-i\theta})}$

$= \text{RP of } \frac{1}{4}\oint_C \frac{(1-z^2)\cdot\frac{dz}{iz}}{5-2(z+\frac{1}{z})}$, Put $e^{i\theta} = z$ $\therefore d\theta = \frac{dz}{iz}$

$= \text{R.P of } \frac{1}{4i}\oint_C \frac{(1-z^2)dz}{5z-2z^2-2}$ where C is the unit circle, $|z|=1$.

$= \text{R.P. of } \frac{-1}{4i}\oint_C \frac{(1-z^2)\,dz}{2z^2-5z+2}$

$= \text{R.P. of } \frac{-1}{4i}\oint_C f(z)\,dz$

where $f(z) = \frac{1-z^2}{2z^2-5z+2}$

Poles of $f(z)$ are given by $2z^2-5z+2=0$
or, $(z-2)(2z-1)=0$
$\therefore z = 2, \frac{1}{2}$

Only simple pole $z=\frac{1}{2}$ lies inside C.

Residue at simple pole $z=\frac{1}{2}$ is

$\lim_{z\to\frac{1}{2}} (z-\frac{1}{2})\frac{1-z^2}{(z-2)(2z-1)}$

$= \lim_{z\to\frac{1}{2}} \frac{1-z^2}{2(z-2)}$

$= \lim_{z\to\frac{1}{2}} \frac{1-\frac{1}{4}}{1-4}$

$= -\frac{1}{4}$

By Cauchy's residue theorem we get
$\oint_C f(z)\,dz = 2\pi i\left(-\frac{1}{4}\right)$

$\therefore\ I = R.P.\ \text{of}\ \frac{-1}{4i}\cdot 2\pi i\left(-\frac{1}{4}\right)$

$= \frac{\pi}{8}$

(v) Let $I = \int_0^{2\pi} \frac{d\theta}{5+3\sin\theta}$

$= \int_0^{2\pi} \frac{d\theta}{5+3\cdot\frac{1}{2i}(e^{i\theta}-e^{-i\theta})}$

$= \oint_C \frac{dz/iz}{5+\frac{3}{2i}\left(z-\frac{1}{z}\right)}$ Put $e^{i\theta}=z$ $\therefore d\theta = \frac{dz}{iz}$

$= 2\oint_C \frac{dz}{10iz+3z^2-3}$

$= 2\oint_C f(z)\,dz$

where $f(z) = \frac{1}{3z^2+10iz-3}$

Poles of $f(z)$ are given by $3z^2+10iz-3=0$

or, $3z^2+10iz+i^2 3=0$

or, $3z^2+9iz+iz+i^2 3=0$

or, $3z(z+3i)+i(z+3i)=0$

or, $(z+3i)(3z+i)=0$

$\therefore z=-3i, -i/3$

since $|-i/3| = \frac{1}{3}$, so only simple pole $z=-i/3$ lies inside C.

Residue at simple pole $z=-\frac{i}{3}$ is

$\lim_{z\to -i/3} (z+i/3)\cdot\frac{1}{(z+3i)(3z+i)}$

$= \lim_{z\to -i/3} \frac{1}{3(z+3i)}$

$= \frac{1}{3(-i/3+3i)}$

$= \frac{1}{8i}$

Hence by Cauchy's residue theorem, we get

$\oint_C f(z)dz = 2\pi i\left(\frac{1}{8i}\right)$

$\therefore I = 2\cdot 2\pi i\left(\frac{1}{8i}\right)$

$= \frac{\pi}{2}$

Evaluate by contour integration: $\int_0^{2\pi} \frac{\sin 2\theta \, d\theta}{1-2a\cos\theta + a^2}, a^2<1$

Solution: Let $I = \int_0^{2\pi} \frac{\sin 2\theta \, d\theta}{1-2a\cos\theta+a^2}$

$= \text{imaginary part of } \int_0^{2\pi} \frac{e^{i2\theta} d\theta}{1-2a\cos\theta + a^2}$

$= \text{I.P. of } \int_0^{2\pi} \frac{e^{i2\theta} d\theta}{1-2a \cdot \frac{1}{2}(e^{i\theta}+e^{-i\theta})+a^2}$

Let us put $e^{i\theta} = z$ $\therefore e^{i\theta} i d\theta = dz$

or, $d\theta = \frac{dz}{iz}$

$\therefore I = \text{I.P. of } \oint_C \frac{z^2 \cdot \frac{dz}{iz}}{1-a(z+\frac{1}{z})+a^2}$, where C is the unit circle $|z|=1$.

$= \text{I.P. of } \frac{1}{i} \oint_C \frac{z^2 dz}{z - az^2 - a + a^2 z}$

$= \text{I.P. of } \frac{-1}{i} \oint_C \frac{z^2 dz}{az^2 - z - a^2 z + a}$

$= \text{I.P. of } \frac{-1}{i} \oint_C f(z)\, dz \quad \text{--- (1)}$

where $f(z) = \frac{z^2}{az^2 - z - a^2 z + a}$

poles of $f(z)$ are given by

$az^2 - z - a^2 z + a = 0$

or, $z(az-1) - a(az-1) = 0$

or, $(az-1)(z-a) = 0$

$\therefore z = \frac{1}{a}, a$

since $a^2 < 1$, so $z=a$ lies inside C.

Residue at simple pole $z=a$ is

$\lim_{z\to a} \cancel{(z-a)} \cdot \frac{z^2}{(az-1)\cancel{(z-a)}}$

$= \frac{a^2}{a^2-1}$

Hence by Cauchy's residue theorem, we get

$\oint_C f(z)\, dz = 2\pi i \left(\frac{a^2}{a^2-1}\right)$

So from (1) we get

$I = \text{I.P. of } \frac{-1}{i} \cdot 2\pi i \left(\frac{a^2}{a^2-1}\right)$

$= \text{I.P. of } \left(\frac{2\pi a^2}{1-a^2} + i \cdot 0\right)$

$= 0$