Simple Harmonic Motion

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1 Oscillation and Vibration

1.1 Oscillation

- Oscillation is the repetitive variation, typically in time, of some measure about a central value (often a point of equilibrium) or between two or more different states.
- The term vibration is precisely used to describe mechanical oscillation.
- Familiar examples of oscillation include a swinging pendulum and alternating current.

1.2 Vibration

- Vibration is a mechanical phenomenon whereby oscillations occur about an equilibrium point.
- The word comes from Latin vibrationem ("shaking, brandishing").
- The oscillations may be periodic, such as the motion of a pendulum—or random, such as the movement of a tire on a gravel road.

1.3 Differences between Oscillation and Vibration

- Oscillation is the definite displacement of a body in terms of distance or time, whereas vibration is the movement brought about in a body due to oscillation.
- Oscillation takes place in physical, biological systems, and often in our society, but vibrations is associated with mechanical systems only.
- Oscillation is about a single body, whereas vibration is the result of collective oscillation of atoms in the body.
- All vibrations are oscillations, but not all oscillations are vibrations.

2 Simple Harmonic Motion

2.1 Definition

Definition 2.1.1: Simple Harmonic Motion

Simple harmonic motion is a type of periodic motion where the restoring force is directly proportional to the displacement and acts in the direction opposite to that of displacement.

A particle is said to execute SHM when it will

(a) Trace and retrace the same path over and over again.

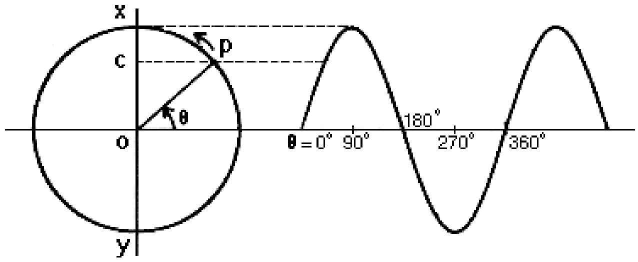
(b) Change direction at a regular interval of time.

(c) Move along a straight line.

(d) Have acceleration proportional to its displacement from the mean position.

A particle which satisfies the condition (a) only is said to execute **periodic motion**. A particle which satisfies condition (a) and (b) is said to execute **vibratory motion**.

Let P be a particle moving on the circumference of a circle of radius r with a uniform velocity v. Let angular velocity be $\omega = v/r$.



Displacement of the particle from the mean position is given by $y = r\sin\omega t$

So, velocity of the particle is given by $v = \dfrac{dy}{dt} = \omega r\cos\omega t$

And acceleration of the particle is given by $a = \dfrac{dv}{dt} = -\omega^2 r\sin\omega t = -\omega^2 y$

Angle	Position of vibrating particle	Displacement $y = r\sin\omega t$	Velocity $\frac{\mathrm{d}y}{\mathrm{d}t} = \omega r\cos\omega t$	Acceleration $-\omega^2 r\sin\omega t = -\omega^2 y$
0	O	0	ωr	0
$\pi/2$	X	r	0	$-\omega^2 a$
π	O	0	$-\omega r$	0
$3\pi/2$	Y	$-r$	0	$\omega^2 r$
2π	O	0	ωr	0

2.2 Differential Equation of SHM

Let y be the displacement of the particle from the mean position at time t, r be the amplitude, and α be the epoch of the vibrating particle

$$y = r\sin(\omega t + \alpha) \tag{2.2.1}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = r\omega\cos(\omega t + \alpha) \tag{2.2.2}$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = -r\omega^2\sin(\omega t + \alpha) \tag{2.2.3}$$

Hence the differential equation of SHM is

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \omega^2 y = 0 \tag{2.2.4}$$

2.3 Solution of the Differential Equation of SHM

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \omega^2 y = 0 \tag{2.3.1}$$

Here,

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}y}$$

$$v\,dv + \omega^2 y\,dy = 0$$
$$\int v\,dv + \omega^2 \int y\,dy = 0$$
$$\frac{v^2}{2} + \frac{\omega^2 y^2}{2} = C'$$
$$\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \omega^2 y^2 = C^2$$

At maximum displacement, $y = r$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = 0$
So, $C^2 = \omega^2 r^2$

$$\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = \omega^2 (r^2 - y^2)$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = \omega\sqrt{r^2 - y^2}$$
$$\int \frac{1}{\sqrt{r^2 - y^2}}\,dy = \int \omega\,dt$$
$$\sin^{-1}\frac{y}{r} = \omega t + \alpha$$
$$\boxed{y = r\sin(\omega t + \alpha)} \tag{2.3.2}$$

By expanding equation (2.3.2), we get

$$y = r\sin\omega t\cos\alpha + r\cos\omega t\sin\alpha \tag{2.3.3}$$

If $y = 0$ at $t = 0$, then $\alpha = 0$

$$y = r\sin\omega t \tag{2.3.4}$$

If $y = r$ at $t = 0$, then $\alpha = \pi/2$

$$y = r\cos\omega t \tag{2.3.5}$$

Hence, the general solution of the differential equation of SHM is

$$\boxed{y = A\sin\omega t + B\cos\omega t} \tag{2.3.6}$$

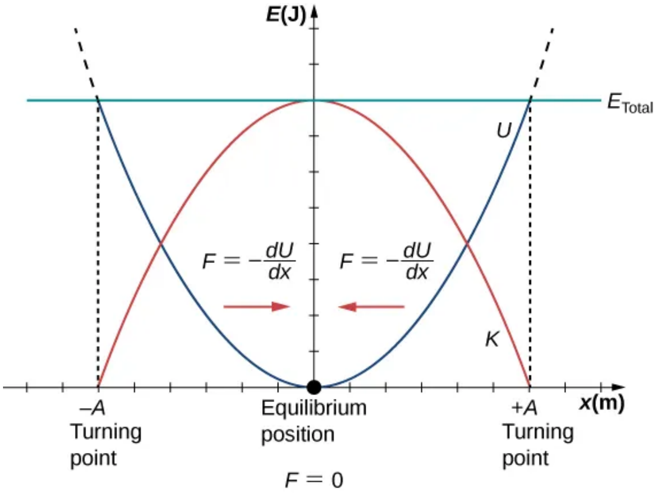
3 Energy in SHM

3.1 Total Energy of a Vibrating Particle

$$\begin{aligned}
\text{Kinetic Energy} &= \frac{1}{2}m\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 \\
&= \frac{1}{2}m\omega^2 r^2\cos^2(\omega t + \alpha) \\
\text{Potential Energy} &= \frac{1}{2}ky^2 \\
&= \frac{1}{2}m\omega^2 r^2 \\
&= \frac{1}{2}m\omega^2 r^2\sin^2(\omega t + \alpha)
\end{aligned}$$

Thus, the total energy of the vibrating particle is

$$\boxed{E = \frac{1}{2}kr^2 = \frac{1}{2}m\omega^2 r^2} \tag{3.1.1}$$



3.2 Average Kinetic Energy

Kinetic energy of the particle is given by

$$K = \frac{1}{2}m\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = \frac{1}{2}m\omega^2r^2\cos^2(\omega t+\alpha) \tag{3.2.1}$$

Hence, average kinetic energy is

$$\begin{aligned}
\overline{K} &= \frac{1}{T}\int_0^T K\,dt \\
&= \frac{1}{T}\int_0^T \frac{1}{2}m\omega^2r^2\cos^2(\omega t+\alpha)\,dt \\
&= \frac{1}{2}m\omega^2r^2\frac{1}{T}\int_0^T \cos^2(\omega t+\alpha)\,dt \\
&= \frac{1}{2}m\omega^2r^2\frac{1}{T}\int_0^T \frac{1+\cos 2(\omega t+\alpha)}{2}\,dt \\
&= \frac{1}{2}m\omega^2r^2\frac{1}{T}\left[\frac{t}{2}+\frac{\sin 2(\omega t+\alpha)}{4\omega}\right]_0^T \\
&= \frac{1}{2}m\omega^2r^2\frac{1}{T}\left[\frac{T}{2}+\frac{\sin 2(\omega T+\alpha)}{4\omega}-\frac{\sin 2\alpha}{4\omega}\right] \\
&= \frac{1}{2}m\omega^2r^2\frac{1}{T}\left[\frac{T}{2}+\frac{\sin 2\alpha}{4\omega}-\frac{\sin 2\alpha}{4\omega}\right] \\
&= \frac{1}{4}m\omega^2r^2
\end{aligned}$$

$$\boxed{\overline{K} = \frac{1}{4}m\omega^2r^2 = \frac{1}{4}kr^2 = \frac{1}{2}E} \tag{3.2.2}$$

3.3 Average Potential Energy

Potential energy of the particle is given by

$$U = \frac{1}{2}ky^2 = \frac{1}{2}m\omega^2r^2\sin^2(\omega t+\alpha) \tag{3.3.1}$$

Hence, average potential energy is

$$\overline{U} = \frac{1}{T}\int_0^T U\,dt$$

$$
\begin{aligned}
&= \frac{1}{T}\int_0^T \frac{1}{2}m\omega^2 r^2 \sin^2(\omega t+\alpha)\,dt \\
&= \frac{1}{2}m\omega^2 r^2 \frac{1}{T}\int_0^T \sin^2(\omega t+\alpha)\,dt \\
&= \frac{1}{2}m\omega^2 r^2 \frac{1}{T}\int_0^T \frac{1-\cos 2(\omega t+\alpha)}{2}\,dt \\
&= \frac{1}{2}m\omega^2 r^2 \frac{1}{T}\left[\frac{t}{2} - \frac{\sin 2(\omega t+\alpha)}{4\omega}\right]_0^T \\
&= \frac{1}{2}m\omega^2 r^2 \frac{1}{T}\left[\frac{T}{2} - \frac{\sin 2(\omega T+\alpha)}{4\omega} + \frac{\sin 2\alpha}{4\omega}\right] \\
&= \frac{1}{2}m\omega^2 r^2 \frac{1}{T}\left[\frac{T}{2} + \frac{\sin 2\alpha}{4\omega} - \frac{\sin 2\alpha}{4\omega}\right] \\
&= \frac{1}{4}m\omega^2 r^2
\end{aligned}
$$

$$
\boxed{\overline{U} = \frac{1}{4}m\omega^2 r^2 = \frac{1}{4}kr^2 = \frac{1}{2}E} \tag{3.3.2}
$$

4 Composition of Two SHMs of Same Frequency

4.1 Same Direction

Let y_1 and y_2 be the displacements of two SHM of same frequency ω, amplitude r_1 and r_2, and phases α_1 and α_2 respectively.

$$
y_1 = r_1 \sin(\omega t + \alpha_1) \tag{4.1.1}
$$
$$
y_2 = r_2 \sin(\omega t + \alpha_2) \tag{4.1.2}
$$

If the two SHMs are in the same direction, then the resultant displacement is

$$
\begin{aligned}
y &= y_1 + y_2 \\
&= r_1 \sin(\omega t + \alpha_1) + r_2 \sin(\omega t + \alpha_2) \\
&= r_1 \sin\omega t \cos\alpha_1 + r_1 \cos\omega t \sin\alpha_1 + r_2 \sin\omega t \cos\alpha_2 + r_2 \cos\omega t \sin\alpha_2
\end{aligned}
$$

$$
\therefore y = (r_1\cos\alpha_1 + r_2\cos\alpha_2)\sin\omega t + (r_1\sin\alpha_1 + r_2\sin\alpha_2)\cos\omega t \tag{4.1.3}
$$

In equation (4.1.3), let

$$
r_1\cos\alpha_1 + r_2\cos\alpha_2 = A\cos\varphi \tag{4.1.4}
$$
$$
r_1\sin\alpha_1 + r_2\sin\alpha_2 = A\sin\varphi \tag{4.1.5}
$$

Then, equation (4.1.3) becomes

$$
y = A\cos\varphi\sin\omega t + A\sin\varphi\cos\omega t \tag{4.1.6}
$$
$$
y = A\sin(\omega t + \varphi) \tag{4.1.7}
$$

Here, A is the amplitude of the resultant SHM and φ is the phase of the resultant SHM.
From equations (4.1.4) and (4.1.5), we get

$$
A^2 = A^2\sin^2\varphi + A^2\cos^2\varphi
$$

$$
\begin{aligned}
A^2 &= r_1^2 \sin^2 \alpha_1 + r_2^2 \sin^2 \alpha_2 + 2r_1 r_2 \sin \alpha_1 \sin \alpha_2 \\
&\quad + r_1^2 \cos^2 \alpha_1 + r_2^2 \cos^2 \alpha_2 + 2r_1 r_2 \cos \alpha_1 \cos \alpha_2 \\
A^2 &= r_1^2 + r_2^2 + 2r_1 r_2 (\sin \alpha_1 \sin \alpha_2 + \cos \alpha_1 \cos \alpha_2) \\
A^2 &= r_1^2 + r_2^2 + 2r_1 r_2 \cos (\alpha_1 - \alpha_2)
\end{aligned}
$$

$$\boxed{A = \sqrt{r_1^2 + r_2^2 + 2r_1 r_2 \cos (\alpha_1 - \alpha_2)}} \tag{4.1.8}$$

And,

$$\boxed{\varphi = \tan^{-1} \frac{r_1 \sin \alpha_1 + r_2 \sin \alpha_2}{r_1 \cos \alpha_1 + r_2 \cos \alpha_2}} \tag{4.1.9}$$

4.1.1 Special Cases

(I) Same phase : $\alpha_1 = \alpha_2$

In this case, equation (4.1.8) becomes

$$A = \sqrt{r_1^2 + r_2^2 + 2r_1 r_2 \cos 0} = \sqrt{(r_1 + r_2)^2} = r_1 + r_2$$

$$\boxed{A = r_1 + r_2} \tag{4.1.10}$$

And,

$$\varphi = \tan^{-1} \frac{r_1 \sin \alpha + r_2 \sin \alpha}{r_1 \cos \alpha + r_2 \cos \alpha} = \tan^{-1} \left(\frac{r_1 + r_2}{r_1 + r_2} \tan \alpha \right) = \tan^{-1} (\tan \alpha)$$

$$\boxed{\varphi = \alpha} \tag{4.1.11}$$

(II) Opposite phase : $\alpha_1 - \alpha_2 = (2n+1)\pi$**, where** $n = 0, 1, 2, \cdots$
In this case, equation (4.1.8) becomes

$$A = \sqrt{r_1^2 + r_2^2 + 2r_1 r_2 \cos (\alpha_2 + \pi - \alpha_2)} = \sqrt{(r_1 - r_2)^2} = r_1 - r_2$$

$$\boxed{A = r_1 - r_2} \tag{4.1.12}$$

And,

$$\varphi = \tan^{-1} \frac{r_1 \sin (\alpha + \pi) + r_2 \sin \alpha}{r_1 \cos (\alpha + \pi) + r_2 \cos \alpha} = \tan^{-1} \left(\frac{r_1 \sin \alpha - r_2 \sin \alpha}{r_1 \cos \alpha - r_2 \cos \alpha} \right) = \tan^{-1} (-\tan \alpha)$$

$$\boxed{\varphi = \alpha + \pi} \tag{4.1.13}$$

4.2 Right Angle

Let x and y be the displacements of two SHM of same frequency ω, amplitude a and b respectively, and phase difference α, acting at right angle to each other.

$$x = a \sin (\omega t + \alpha) \tag{4.2.1}$$

$$y = b \sin (\omega t) \tag{4.2.2}$$

Or,

$$\frac{x}{a} = \sin (\omega t + \alpha) \tag{4.2.3}$$

$$\frac{y}{b} = \sin(\omega t) \tag{4.2.4}$$

Thus, we get

$$\begin{aligned}
\frac{x}{a} &= \sin\omega t\cos\alpha + \cos\omega t\sin\alpha \\
\frac{x}{a} &= \frac{y}{b}\sqrt{1-\sin^2\alpha} + \sqrt{1-\frac{y^2}{b^2}}\sin\alpha \\
\frac{x}{a} - \frac{y}{b}\sqrt{1-\sin^2\alpha} &= \sqrt{1-\frac{y^2}{b^2}}\sin\alpha \\
\frac{x^2}{a^2} + \frac{y^2}{b^2}(1-\sin^2\alpha) - \frac{2xy}{ab}\sqrt{1-\sin^2\alpha} &= \left(1-\frac{y^2}{b^2}\right)\sin^2\alpha
\end{aligned}$$

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\sqrt{1-\sin^2\alpha} = \sin^2\alpha} \tag{4.2.5}$$

Equation (4.2.5) represents the general equation of the resultant SHM of the two perpendicular SHMs. The resulting curves are also known as Lissajous firgures.

4.2.1 Special Cases

(I) If $\alpha = 0$ or 2π

$$\cos\alpha = 1, \qquad \sin\alpha = 0$$

Then equation (4.2.5) becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab} = 0$$

Or,

$$\frac{x}{a} = \frac{y}{b}$$

This represents a straight line passing through the origin.

(II) If $\alpha = \pi$

$$\cos\alpha = -1, \qquad \sin\alpha = 0$$

Then equation (4.2.5) becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{2xy}{ab} = 0$$

Or,

$$\frac{x}{a} = -\frac{y}{b}$$

This represents a straight line with negative slope passing through the origin.

(III) If $\alpha = \pi/2$ or, $3\pi/2$

$$\cos\alpha = 0, \qquad \sin\alpha = 1$$

Then equation (4.2.5) becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

This represents an ellipse.

(IV) If $\alpha = \pi/2$ or, $3\pi/2$, and $a = b$

$$\cos\alpha = 0, \qquad \sin\alpha = 1$$

Then equation (4.2.5) becomes

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1$$

Or,

$$x^2 + y^2 = a^2$$

This represents a circle of radius a.

(V) If $\alpha = \pi/4$ or, $7\pi/4$

$$\cos\alpha = \frac{1}{\sqrt{2}}, \qquad \sin\alpha = \frac{1}{\sqrt{2}}$$

Then equation (4.2.5) becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}\frac{1}{\sqrt{2}} = \frac{1}{2}$$

This represents an oblique ellipse.

(VI) If $\alpha = 3\pi/4$ or, $5\pi/4$

$$\cos\alpha = -\frac{1}{\sqrt{2}}, \qquad \sin\alpha = \frac{1}{\sqrt{2}}$$

Then equation (4.2.5) becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{2xy}{ab}\frac{1}{\sqrt{2}} = \frac{1}{2}$$

This represents an oblique ellipse (negative slope).

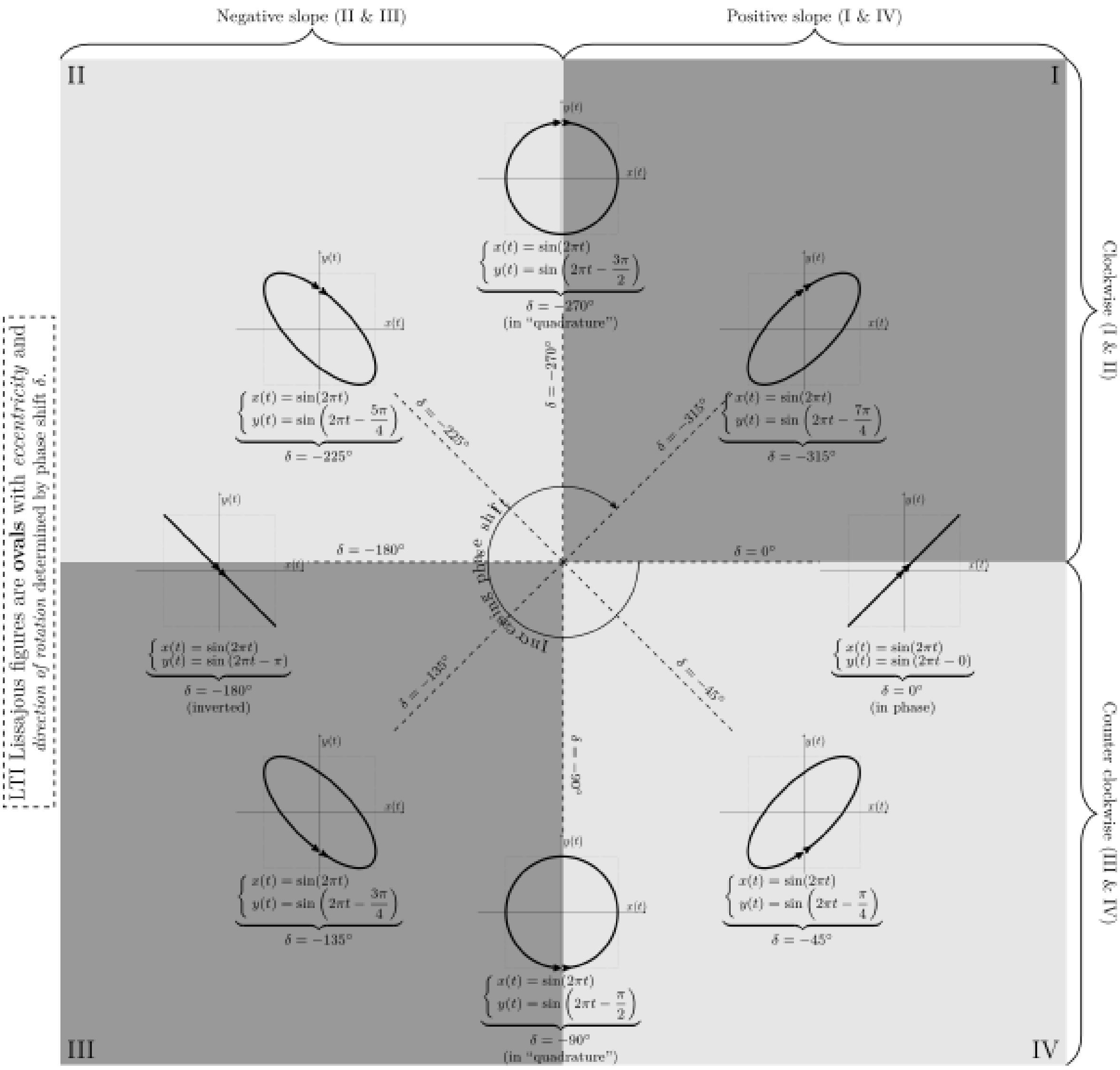


Figure 1: Lissajous figures

5 Damped Harmonic Motion

5.1 Definition

Definition 5.1.1: Damped Harmonic Motion

Damped harmonic motion is a type of periodic motion where the amplitude of the motion decreases over time.

Definition 5.1.2: Damping and Damped Oscillation

Damping is an influence within or upon an oscillatory system that has the effect of reducing, restricting or preventing its oscillations. In physical systems, damping is produced by processes that dissipate the energy stored in the oscillation.
In other words, the reduction in amplitude (or energy) of an oscillator is called damping, and the oscillation is said to be damped

Definition 5.1.3: Damping Force

Damping force is a force, that opposes the motion of the vibrating particle and is directly proportional to the velocity of the particle.

- The damping force is always directed opposite to the direction of motion of the particle.
- The magnitude of the damping force is directly proportional to the velocity of the particle.
- The direction of the damping force is opposite to the direction of the velocity of the particle.
- Damping force is defined as $F_d = -bv$, where b is the damping constant, and v is the velocity of the object.

5.2 Differential Equation of Damped Harmonic Motion

The damping force can be represented as

$$F_d = -bv = -b\frac{\mathrm{d}y}{\mathrm{d}t}$$

Thus, the differential equation can be written as

$$\begin{aligned} m\frac{\mathrm{d}^2y}{\mathrm{d}t^2} &= -ky - b\frac{\mathrm{d}y}{\mathrm{d}t} \\ m\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + b\frac{\mathrm{d}y}{\mathrm{d}t} + ky &= 0 \\ \frac{\mathrm{d}^2y}{\mathrm{d}t^2} + \frac{b}{m}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{k}{m}y &= 0 \end{aligned}$$

$$\boxed{\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 2\lambda\frac{\mathrm{d}y}{\mathrm{d}t} + \omega^2 y = 0} \tag{5.2.1}$$

Here, $\lambda = \dfrac{b}{2m}$ and $\omega = \sqrt{\dfrac{k}{m}}$

5.3 Solution of the Differential Equation

Equation (5.1.1) is a second order linear differential equation with constant coefficients. So, it must have at least one solution of the form

$$y = Ae^{kt}$$

, where A and k are arbitrary constants.
Hence, the auxiliary equation would be

$$k^2 + 2\lambda k + \omega^2 = 0 \tag{5.3.1}$$

Equation (5.1.2) is a quadratic equation in k. So, it has two roots

$$k = -\lambda \pm \sqrt{\lambda^2 - \omega^2}$$

Thus, the general solution of the differential equation is

$$\boxed{y = A_1 e^{(-\lambda + \sqrt{\lambda^2-\omega^2})t} + A_2 e^{(-\lambda - \sqrt{\lambda^2-\omega^2})t}} \tag{5.3.2}$$

Here, A_1 and A_2 are arbitrary constants.
Differentiating equation (5.1.3) with respect to t, we get

$$\frac{\mathrm{d}y}{\mathrm{d}t} = (-\lambda + \sqrt{\lambda^2 - \omega^2})A_1 e^{(-\lambda + \sqrt{\lambda^2-\omega^2})t} + (-\lambda - \sqrt{\lambda^2 - \omega^2})A_2 e^{(-\lambda - \sqrt{\lambda^2-\omega^2})t} \tag{5.3.3}$$

Let $y_{max} = a_0$ at $t = 0$. Then, equation (5.3.3) becomes

$$y_{max} = a_0 = A_1 + A_2 \tag{5.3.4}$$

Again, the velocity is zero at maximum displacement. So, equation (5.3.3) becomes

$$\begin{aligned}
\left(-\lambda + \sqrt{\lambda^2 - \omega^2}\right) A_1 + \left(-\lambda - \sqrt{\lambda^2 - \omega^2}\right) &= 0 \\
-\lambda(A_1 + A_2) + \sqrt{\lambda^2 - \omega^2}(A_1 - A_2) &= 0 \\
-\lambda a_0 + \sqrt{\lambda^2 - \omega^2}(A_1 - A_2) &= 0 \\
\sqrt{\lambda^2 - \omega^2}(A_1 - A_2) &= \lambda a_0
\end{aligned}$$

$$A_1 - A_2 = \frac{\lambda a_0}{\sqrt{\lambda^2 - \omega^2}} \tag{5.3.5}$$

From equations (5.3.4) and (5.3.5), we get

$$A_1 = \frac{a_0}{2}\left(1 + \frac{\lambda}{\sqrt{\lambda^2 - \omega^2}}\right) \tag{5.3.6}$$

$$\text{And,} \quad A_2 = \frac{a_0}{2}\left(1 - \frac{\lambda}{\sqrt{\lambda^2 - \omega^2}}\right) \tag{5.3.7}$$

Substituting the values of A_1 and A_2 in equation (5.1.3), we get

$$\boxed{y = \frac{a_0}{2} e^{-\lambda t}\left[\left(1 + \frac{\lambda}{\sqrt{\lambda^2 - \omega^2}}\right) e^{\sqrt{\lambda^2-\omega^2}t} + \left(1 - \frac{\lambda}{\sqrt{\lambda^2 - \omega^2}}\right) e^{-\sqrt{\lambda^2-\omega^2}t}\right]} \tag{5.3.8}$$

5.4 Over Damping, Critical Damping, and Under Damping

5.4.1 Over Damping

When $\lambda > \omega$, then $\sqrt{\lambda^2 - \omega^2}$ is a positive value, less than λ. Thus, each of the two terms on the RHS of equation (5.3.8) has exponential terms with negative power. In this case, the equation has no oscillating terms. So, the displacement falls asymptotically to zero without oscillating.

5.4.2 Critical Damping

When $\lambda = \omega$, then $\sqrt{\lambda^2 - \omega^2} = 0$. Thus, the two terms on the RHS of equation (5.3.8) become

$$\frac{a_0}{2} e^{-\lambda t} \left[\left(1 + \frac{\lambda}{\sqrt{\lambda^2 - \omega^2}}\right) + \left(1 - \frac{\lambda}{\sqrt{\lambda^2 - \omega^2}}\right) \right] = a_0 e^{-\lambda t}$$

In this case, the displacement falls asymptotically to zero without oscillating in the shortest possible time.

Let's consider, however, that $\lambda^2 \to \omega^2$. Let $\sqrt{\lambda^2 - \omega^2} = h$, a very small quantity. Then we have

$$\begin{aligned}
y &= A_1 e^{(-\lambda+h)t} + A_2 e^{(-\lambda-h)t} \\
&= e^{-\lambda t} \left[A_1 e^{ht} + A_2 e^{-ht} \right] \\
&= e^{-\lambda t} \left[A_1 (1 + ht + \frac{h^2 t^2}{2!} + \cdots) + A_2 (1 - ht + \frac{h^2 t^2}{2!} - \cdots) \right] \\
&= e^{-\lambda t} \left[(A_1 + A_2) + (A_1 - A_2) ht + \frac{h^2 t^2}{2!} (A_1 + A_2) + \cdots \right]
\end{aligned}$$

We have $A_1 + A_2 = a_0$ and $A_1 - A_2 = \dfrac{\lambda a_0}{h}$ [From equation (5.3.5)]
Neglecting the higher powers of h, we get

$$\boxed{y = a_0 e^{-\lambda t} (1 + \lambda t)} \tag{5.4.1}$$

This is the equation of motion of a critically damped oscillator.

5.4.3 Under Damping

When $\lambda < \omega$, then $\sqrt{\lambda^2 - \omega^2}$ is an imaginary value. Let $i = \sqrt{-1}$ and $g = \sqrt{\omega^2 - \lambda^2}$. So, equation (5.3.8) becomes

$$\begin{aligned}
y &= A_1 e^{(-\lambda+ig)t} + A_2 e^{(-\lambda-ig)t} \\
&= e^{-\lambda t} (A_1 e^{igt} + A_2 e^{-igt}) \\
&= e^{-\lambda t} \left[A_1 (\cos gt + i \sin gt) + A_2 (\cos gt - i \sin gt) \right] \\
&= e^{-\lambda t} \left[(A_1 + A_2) \cos gt + i (A_1 - A_2) \sin gt \right] \\
y &= e^{-\lambda t} \left[A \cos gt + B \sin gt \right]
\end{aligned}$$

If A and B can be related to a_0 as $A = a_0 \sin\theta$ and $B = a_0 \cos\theta$ then we get

$$y = e^{-\lambda t} \left[a_0 \cos gt \frac{A}{a_0} + a_0 \sin gt \frac{B}{a_0} \right]$$

$$= a_0 e^{-\lambda t} \left[\cos gt \sin\theta + \sin gt \cos\theta \right]$$

$$\boxed{y = a_0 e^{-\lambda t} \sin\left(gt + \theta\right)} \tag{5.4.2}$$

This is the equation of a damped harmonic oscillator with amplitude $a_0 e^{-\lambda t}$ and frequency $f = \dfrac{g}{2\pi} = \dfrac{\sqrt{\omega^2 - \lambda^2}}{2\pi}$.

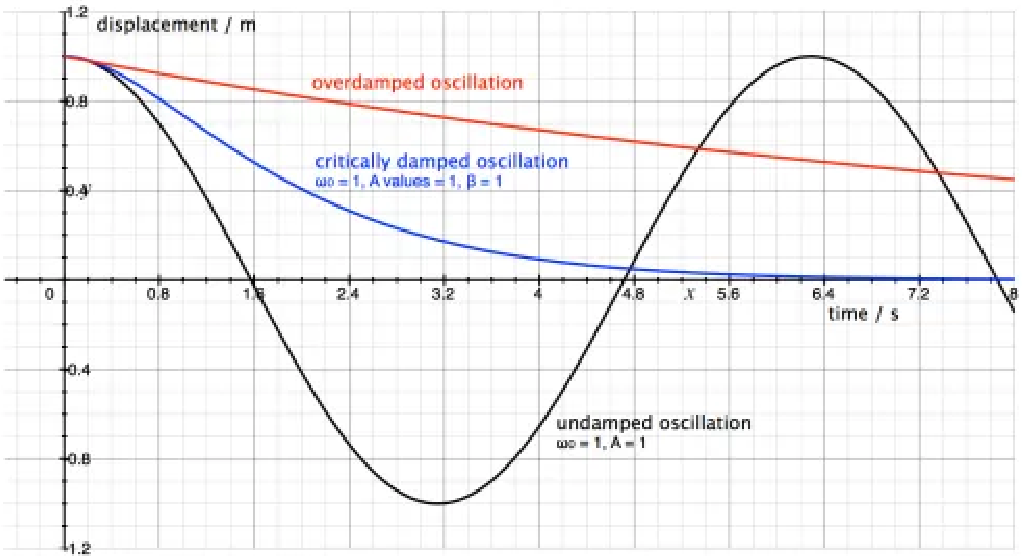


Figure 2: Over Damping, Critical Damping, and Under Damping

6 Forced Oscillation and Resonance

6.1 Forced Oscillation

Definition 6.1.1: Forced Oscillation

When a vibrating system is subjected to an external periodic force, the system is said to be under forced oscillation.