MATH-281
Complex Variables

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Contents

1 Complex Numbers

1.1 Definition

Definition 1.1.1: Complex Numbers

A complex number is a number that can be expressed in the form $a + bi$, where a and b are real numbers, and i is a solution of the equation $x^2 = -1$, or simply, $i = \sqrt{-1}$. Because no real number satisfies this equation, i is called an imaginary number. For the complex number $a + bi$, a is called the real part, and b is called the imaginary part.

Note:-

- The set of all complex numbers is denoted by $\mathbb{C}$.
- The set of all real numbers is denoted by $\mathbb{R}$.

Definition 1.1.2: Modulus and Amplitude

Let $z = a + bi$ be a complex number. The modulus of z is the non-negative real number $|z| = \sqrt{a^2 + b^2}$. The amplitude of z is the angle θ such that $\cos(\theta) = \frac{a}{|z|}$ and $\sin(\theta) = \frac{b}{|z|}$.

If the polar form of the point (a, b) be (r, θ), then $a = r\cos\theta$ and $b = r\sin\theta$.

$$r = |z| = \sqrt{a^2 + b^2} \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{b}{a}\right) \tag{1.1.1}$$

Here, r is the modulus of z and θ is the amplitude of z.
In symbols, we write

$$r = \text{mod}(z) = |a + ib| \quad \text{and} \quad \theta = \arg(z) = \tan^{-1}\left(\frac{b}{a}\right) \tag{1.1.2}$$

1.2 De Moivre's Theorem

***Theorem 1.2.1* (De Moivre's Theorem):** *Let $z = r(\cos\theta + i\sin\theta)$ be a complex number. Then, for any positive integer n,*

$$z^n = r^n(\cos n\theta + i\sin n\theta) \tag{1.2.1}$$

Proof:
Case 1: $n \in \mathbb{Z}_+$
We have,

$$\begin{aligned} z_1 z_2 \ldots z_n &= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\ldots(\cos\theta_n + i\sin\theta_n) \\ &= \big\{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\big\}\ldots(\cos\theta_n + i\sin\theta_n) \\ &= \cos(\theta_1 + \theta_2 + \ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \ldots + \theta_n) \end{aligned}$$

Hence, we get

$$z^n = (\cos n\theta + i\sin n\theta) \quad \square$$

Case 2: $n \in \mathbb{Z}_-$
Let $n = -m$. We have,

$$\begin{aligned} z^n &= (\cos\theta + i\sin\theta)^{-m} \\ &= \frac{1}{(\cos\theta + i\sin\theta)^m} \\ &= \frac{1}{\cos m\theta + i\sin m\theta} \\ &= \frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta} \\ &= \cos m\theta - i\sin m\theta \\ &= \cos n\theta + i\sin n\theta \end{aligned}$$

Hence, we get

$$z^n = (\cos n\theta + i\sin n\theta) \quad \square$$

Case 3: $n \in \mathbb{Q}$, i.e. $n = \frac{p}{q}$, where $p, q \in \mathbb{Z}$ and $q \neq 0$.
Now,

$$\begin{aligned} \left(\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta\right)^q &= \cos\left(q \cdot \frac{p}{q}\theta\right) + i\sin\left(q \cdot \frac{p}{q}\theta\right) \\ &= \cos p\theta + i\sin p\theta \\ &= (\cos\theta + i\sin\theta)^p \end{aligned}$$

Taking the q^{th} root of both sides, we get

$$\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta = (\cos\theta + i\sin\theta)^{\frac{p}{q}} \quad \square$$

Note:-

Some Important Results:

(i) $1 = e^{i2n\pi} = \cos 2n\pi + i\sin 2n\pi$

(ii) $-1 = \cos\pi + i\sin\pi = e^{i\pi}$

(iii) $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\frac{\pi}{2}}$

(iv) $-i = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2} = e^{-i\frac{\pi}{2}}$

2 Analytic Functions

2.1 Definitions

Definition 2.1.1: Complex variable

A **complex variable** is a variable that can take on complex values. A complex variable is usually denoted by z.

If x and y are real variables, then $z = x + iy$ is a complex variable, where i is the imaginary unit.

Definition 2.1.2: Complex Function

A **complex function** is a function that takes complex variables as input and returns complex values. A complex function is usually denoted by $f(z)$.

If $z = x + iy$ and $w = u + iv$ are complex variables, then $f(z) = u(x, y) + iv(x, y)$ is a complex function, where $u(x, y)$ and $v(x, y)$ are real functions.

Definition 2.1.3: Single-valued Function

A **single-valued function** is a function that returns a unique value for each input.

A complex function $f(z)$ is single-valued if and only if $f(z_1) = f(z_2)$ implies $z_1 = z_2$. In other words, if $z_1 \neq z_2$, then $f(z_1) \neq f(z_2)$.

$$\forall z_1, z_2 \in \mathbb{C} \quad : \quad z_1 \neq z_2 \quad \implies \quad f(z_1) \neq f(z_2)$$

Definition 2.1.4: Multiple-valued Function

A **multiple-valued function** is a function that returns multiple values for each input.

A complex function $f(z)$ is multiple-valued if and only if $f(z_1) = f(z_2)$ for some $z_1 \neq z_2$.

$$\exists z_1, z_2 \in \mathbb{C} \quad : \quad z_1 \neq z_2 \quad \implies \quad f(z_1) = f(z_2)$$

Definition 2.1.5: Derivative

The **derivative** of a complex function $f(z)$ is defined as

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

where Δz is a complex number.

If the limit exists, then $f(z)$ is said to be **differentiable** at z. If $f(z)$ is differentiable at every point in a region R, then $f(z)$ is said to be **analytic** in R.

Definition 2.1.6: Analytic Function

A complex function $f(z)$ is **analytic** in a region R if it is differentiable at every point in R.

$$\forall z \in R \quad : \quad f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \quad \text{exists}$$

If $f(z)$ is analytic in a region R, then $f(z)$ is also said to be **regular** or **holomorphic** in R.

2.2 Necessary Conditions for Analyticity

Let $f(z) = u(x,y) + iv(x,y)$ be an analytic function in a region R.

That means, $f(z)$ is differentiable at every point in R.

or, $f'(z) = \lim\limits_{\Delta z \to 0} \dfrac{f(z+\Delta z) - f(z)}{\Delta z}$ exists at every point in R.

Now, let $z = x + iy$ and $\Delta z = \Delta x + i\Delta y$.

$$z + \Delta z = (x + \Delta x) + i(y + \Delta y)$$

Then,

$$\begin{aligned}
f'(z) &= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{f(x + \Delta x + i(y + \Delta y)) - f(x + iy)}{\Delta x + i\Delta y} \\
&= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{u(x+\Delta x, y+\Delta y) + iv(x+\Delta x, y+\Delta y) - u(x,y) - iv(x,y)}{\Delta x + i\Delta y} \\
&= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{u(x+\Delta x, y+\Delta y) - u(x,y)}{\Delta x + i\Delta y} + i \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{v(x+\Delta x, y+\Delta y) - v(x,y)}{\Delta x + i\Delta y}
\end{aligned}$$

Along the real axis, $\Delta y = 0$. Hence, the limit is

$$f'(z) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x}$$

$$\boxed{f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}} \tag{2.2.1}$$

Along the imaginary axis, $\Delta x = 0$. Hence, the limit is

$$f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x, y+\Delta y) - v(x,y)}{i\Delta y}$$

$$\boxed{f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}} \tag{2.2.2}$$

2.3 Cauchy-Riemann Equations

Since $f'(z)$ exists, (2.2.4) and (2.2.5) must be equal.

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \tag{2.3.1}$$

Comparing real and imaginary parts,

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \quad \text{and} \quad \boxed{\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}} \tag{2.3.2}$$

These are called the **Cauchy-Riemann equations**.

2.4 Cauchy-Riemann Equations in Polar Form

Let

$$z = re^{i\theta}$$

$$f(z) = u(r,\theta) + iv(r,\theta)$$

Then

$$f(re^{i\theta}) = u(r,\theta) + iv(r,\theta) \tag{2.4.1}$$

Differentiating (2.4.1) with respect to r, we get

$$e^{i\theta} f'(re^{i\theta}) = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} \tag{2.4.2}$$

Differentiating (2.4.1) with respect to θ, we get

$$ire^{i\theta} f'(re^{i\theta}) = \frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} \tag{2.4.3}$$

Now, from (2.4.2) and (2.4.3),

$$\begin{aligned}
\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} &= \frac{1}{ir}\left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right) \\
\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} &= \frac{1}{ir}\frac{\partial u}{\partial \theta} + \frac{1}{r}\frac{\partial v}{\partial \theta} \\
\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} &= -\frac{i}{r}\frac{\partial u}{\partial \theta} + \frac{1}{r}\frac{\partial v}{\partial \theta}
\end{aligned}$$

Equating the real and imaginary parts, we get

$$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}} \quad \text{and} \quad \boxed{\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}} \tag{2.4.4}$$

These are the **Cauchy-Riemann equations in polar form**.

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3 Harmonic Function

3.1 Laplace's Equation

Definition 3.1.1: Laplace's Equation

An equation of the form

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \quad \text{or} \quad \nabla^2\phi = 0 \tag{3.1.1}$$

is called **Laplace's equation** (in two dimentions).
Here, ∇^2 is the Laplacian operator.

3.2 Harmonic Function

Definition 3.2.1: Harmonic Function

A function $\phi(x, y)$ is called **harmonic** if it satisfies Laplace's equation

$$\nabla^2\phi = 0 \tag{3.2.1}$$

where ∇^2 is the Laplacian operator.

Theorem 3.2.2: *If $f(z) = u + iv$ is an analytic function, then u and v are both harmonic functions.*

Proof:
Since $f(z)$ is analytic, it satisfies the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{3.2.2}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{3.2.3}$$

Differentiating (3.2.2) w.r.t. x and (3.2.3) w.r.t. y, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial v}{\partial x\partial y} \tag{3.2.4}$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial v}{\partial y\partial x} \tag{3.2.5}$$

Adding (3.2.4) and (3.2.5), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{3.2.6}$$

Similarly,

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \tag{3.2.7}$$

Hence, both u and v are harmonic functions. $\square$

Definition 3.2.3: Conjugate Harmonic Function

Any two functions ϕ and ψ such that $f(z) = \phi + i\psi$ is analytic, are called **Conjugate Harmonic Functions**.

3.3 Velocity Potential

Consider a two-dimensional flow of an incompressible fluid. The velocity of the fluid at a point (x, y) is given by the vector

$$\mathbf{v} = v_x \hat{\mathbf{i}} + v_y \hat{\mathbf{j}} \tag{3.3.1}$$

Here, v is called the stream function.

The **velocity potential** $\phi(x, y)$ is defined as the scalar function such that

$$\boxed{\mathbf{V} = \nabla\phi = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y}\right)\phi = \hat{\mathbf{i}}\frac{\partial \phi}{\partial x} + \hat{\mathbf{j}}\frac{\partial \phi}{\partial y}} \tag{3.3.2}$$

Comparing (3.3.1) and (3.3.2), we get

$$v_x = \frac{\partial \phi}{\partial x} \quad \text{and} \quad v_y = \frac{\partial \phi}{\partial y} \tag{3.3.3}$$

The scalar function $\phi(x, y)$ gives the velocity components.
Since the fluid is incompressible,

$$\begin{aligned} \nabla v &= 0 \\ \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y}\right)\left(\hat{\mathbf{i}}v_x + \hat{\mathbf{j}}v_y\right) &= 0 \\ \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} &= 0 \end{aligned}$$

Putting the values of v_x and v_y from (3.3.3),

$$\begin{aligned} \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) &= 0 \\ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} &= 0 \end{aligned}$$

This is Laplace's equation. Hence, the velocity potential $\phi(x, y)$ is a harmonic function and is a real part of the analytic function

$$f(z) = \phi + i\psi$$

3.4 Method for Finding Conjugate Harmonic Function

A Method 1: Real or Imaginary Part of an Analytic Function is Given

If $f(z) = u + iv$ and u is known

We know that

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

Using C-R equations,

$$\begin{aligned} dv &= -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \\ v &= -\int \frac{\partial u}{\partial y}dx + \int \frac{\partial u}{\partial x}dy + C \end{aligned}$$

Since u is known, v can be found using the above method.

If v is known, then u can be found using the same method.

B Method 2: Milne's Method/ Milne Thomson Method

By this method, $f(z)$ is directly constructed without finding v.
Since

$$\begin{aligned} z &= x + iy \quad \text{and} \quad \bar{z} = x - iy \\ x &= \frac{z + \bar{z}}{2} \quad \text{and} \quad y = \frac{z - \bar{z}}{2i} \end{aligned}$$

Thus,

$$\boxed{f(z) = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)} \tag{3.4.1}$$

Case 1: u is given

Let $f(z) = u + iv$ be an analytic function and u is given.
Then,

$$\frac{\partial u}{\partial x} = u_1(x, y) \quad \text{and} \quad \frac{\partial u}{\partial y} = u_2(x, y)$$

By Milne's method, we get

$$f'(z) = u_1(z, 0) - iu_2(z, 0) \tag{3.4.2}$$

Integrating (3.4.1) w.r.t. z, we get

$$f(z) = \int \big[u_1(z, 0) - iu_2(z, 0)\big]\, dz + C_1 \tag{3.4.3}$$

Case 2: v is given

If v is given, then

$$\frac{\partial v}{\partial y} = v_1(x, y) \quad \text{and} \quad \frac{\partial v}{\partial x} = v_2(x, y)$$

By Milne's method, we get

$$f'(z) = v_1(z, 0) + iv_2(z, 0) \tag{3.4.4}$$

Integrating (3.4.3) w.r.t. z, we get

$$f(z) = \int \big[v_1(z, 0) + iv_2(z, 0)\big]\, dz + C_2 \tag{3.4.5}$$

3.5 Complex Potential Function

Definition 3.5.1: Complex Potential Function

The analytic function

$$W = \phi(x, y) + i\psi(x, y)$$

is called the **Complex Potential Function**.
The real part $\phi(x, y)$ represents the velocity potential function, and the imaginary part $\psi(x, y)$ represents the stream function.

Example 3.1: If $W = \phi + i\psi$ represents the complex potential for an electric field, and $\psi = 3x^2y - y^3$, then find ϕ.

Given,

$$\psi = 3x^2y - y^3$$

Hence,

$$\begin{aligned} \frac{\partial \psi}{\partial y} &= 3x^2 - 3y^2 \\ \frac{\partial \phi}{\partial x} &= 6xy \end{aligned}$$

By Milne's method, we have

$$\begin{aligned} W'(z) &= \psi_1(z, 0) + i\psi_2(z, 0) \\ &= 3z^2 + i \cdot 0 \\ &= 3z^2 \end{aligned}$$

Integrating $W'(z)$ w.r.t. z, we get

$$\begin{aligned} W(z) &= \int 3z^2 dz + C \\ \phi + i\psi &= z^3 + c_1 + ic_2 \\ \phi + i\psi &= x^3 - 3xy^2 + c_1 + i\left(3x^2y - y^3 + c_2\right) \end{aligned}$$

Comparing real and imaginary parts, we get the required potential function

$$\boxed{\phi = x^3 - 3xy^2 + c_1}$$

4 Complex Integration

4.1 Definitions

Definition 4.1.1: Simply Connected Region

A connected region is said to be a **Simply Connected** region if all the interior points of a closed curve C drawn in the region D are the points of the region D.

Definition 4.1.2: Multi-Connected Region

A **Multi-connected** region is bounded by more than one curve. A multi-connected region can be divided into simply connected regions.

4.2 Complex Line Integrals

Definition 4.2.1: Complex Line Integral

The **Complex Line Integral** of a function $f(z)$ along a curve C is defined as

$$\oint_C f(z)\,dz = \lim_{n\to\infty} \sum_{k=1}^{n} f(z_k^*)\Delta z_k \tag{4.2.1}$$

where z_k^* is a point on the curve C and Δz_k is the length of the curve C.

If $z = x + iy$ and $f(z) = u(x, y) + iv(x, y)$, then

$$dz = dx + i\,dy$$

and

$$f(z)\,dz = (u\,dx - v\,dy) + i(u\,dy + v\,dx)$$

Hence, the complex line integral can be written as

$$\boxed{\oint_C f(z)\,dz = \int_C (u\,dx - v\,dy) + i\int_C (u\,dy + v\,dx)} \tag{4.2.2}$$

4.3 Cauchy's Integral Theorem

***Theorem 4.3.1* (Cauchy's Integral Theorem):**
If $f(z)$ is analytic and its derivative $f'(z)$ is continuous at all points inside and on a simple closed curve C, then

$$\oint_C f(z)\,dz = 0 \tag{4.3.1}$$

Proof:

Let the region enclosed by the curve C be D, and let

$$\begin{aligned} f(z) &= u(x,y) + iv(x,y) \\ z &= x + iy \\ dz &= dx + i\,dy \end{aligned}$$

Now,

$$\begin{aligned} \oint_C f(z)\,dz &= \oint_C (u+iv)\,(dx+idy) \\ &= \oint_C (u\,dx - v\,dy) + i\oint_C (u\,dy + v\,dx) \\ &= \oint_C u\,dx - v\,dy + i\oint_C u\,dy + v\,dx \\ &= \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy \end{aligned}$$

[By Green's Theorem]

Since $f(z)$ is analytic, the Cauchy-Riemann equations hold, i.e.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Thus, we get

$$\oint_C f(z)\,dz = 0 \quad \square$$

4.4 Cauchy's Integral Formula

***Theorem 4.4.1* (Cauchy's Integral Formula):**
If $f(z)$ is analytic inside and on a simple closed curve C, and if a is a point inside the curve C, then

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz \tag{4.4.1}$$

Proof:

Let $f(z)$ be analytic inside and on a simple closed curve C, and let a be a point inside the curve C. Then, by Cauchy's Integral Theorem, we have

$$\oint_C f(z)\,dz = 0$$

Now, consider the function

$$g(z) = \frac{f(z)}{z-a}$$

This function is analytic inside and on the curve C, except at the point $z = a$. Thus, by Cauchy's Integral Theorem, we have

$$\oint_C g(z)\,dz = 0$$

Now, we have

$$\begin{aligned}
\oint_C g(z)\,dz &= \oint_C \frac{f(z)}{z-a}\,dz \\
&= \oint_C \frac{f(z)}{z-a}\,dz - \oint_C \frac{f(a)}{z-a}\,dz + \oint_C \frac{f(a)}{z-a}\,dz \\
&= \oint_C \frac{f(z)-f(a)}{z-a}\,dz + f(a)\oint_C \frac{1}{z-a}\,dz
\end{aligned}$$

For any point on the curve C_1, we have

$$z-a = re^{i\theta} \quad \text{and} \quad dz = ire^{i\theta}\,d\theta$$

$$\begin{aligned}
\oint_{C_1} \frac{f(z)-f(a)}{z-a}\,dz &= \int_{C_1} \frac{f(z)-f(a)}{z-a}\,dz \\
&= \int_0^{2\pi} \frac{f(re^{i\theta})-f(a)}{re^{i\theta}}\,ire^{i\theta}\,d\theta \\
&= \int_0^{2\pi} i\left[f(re^{i\theta})-f(a)\right]\,d\theta \\
&= 0 \\
\int_{C_1} \frac{1}{z-a}\,dz &= \int_0^{2\pi} \frac{ire^{i\theta}}{re^{i\theta}}\,d\theta \\
&= \int_0^{2\pi} i\,d\theta \\
&= 2\pi i
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\oint_C g(z)\,dz &= \oint_C \frac{f(z)-f(a)}{z-a}\,dz + f(a)\oint_C \frac{1}{z-a}\,dz \\
&= 0 + f(a)\cdot 2\pi i \\
&= 2\pi i f(a)
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\oint_C g(z)\,dz &= 2\pi i f(a) \\
\oint_C \frac{f(z)}{z-a}\,dz &= 2\pi i f(a) \\
f(a) &= \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz \quad \square
\end{aligned}$$

Theorem 4.4.2 **(Cauchy Integral Formula for the Derivative of an Analytic Function)***:*
If $f(z)$ is analytic inside and on a simple closed curve C, and if a is a point inside the curve C, then

$$f'(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^2}\, dz \tag{4.4.2}$$

$$f''(a) = \frac{2!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^3}\, dz \tag{4.4.3}$$

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz \tag{4.4.4}$$

Proof:
The proof of these formulas can be obtained by differentiating the Cauchy Integral Formula and using the Cauchy Integral Formula for $f(a)$.
We know

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\, dz \tag{4.4.5}$$

Differentiating both sides with respect to a, we get

$$f'(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^2}\, dz \tag{4.4.6}$$

Differentiating again, we get

$$f''(a) = \frac{2!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^3}\, dz \tag{4.4.7}$$

Continuing this process, we get

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz \quad \square \tag{4.4.8}$$

4.5 Cauchy's Extended Theorem

Theorem 4.5.1 **(Cauchy's Extended Theorem)***:*
If $f(x)$ is analytic within and on the boundary of a region bounded by two closed curves C_1 and C_2, then

$$\oint_{C_1} f(z)\, dz = \oint_{C_2} f(z)\, dz \tag{4.5.1}$$

5 Singularities and Residues

5.1 Definitions

Definition 5.1.1: Singular Points

All the points of the z-plane at which an analytic function does not have a unique derivative are called singular points.

For example, the function $f(z) = \frac{1}{z}$ has a singular point at $z = 0$ because the derivative of $f(z)$ at $z = 0$ is not unique.

Definition 5.1.2: Poles

A singular point z_0 of a function $f(z)$ is called a pole of order m if the function $f(z)$ can be written as

$$f(z) = \frac{g(z)}{(z - z_0)^m}$$

where $g(z)$ is analytic at z_0 and $g(z_0) \neq 0$.

The smallest positive integer m for which the above equation holds is called the order of the pole.
Poles of order 1 are called simple poles, poles of order 2 are called double poles, and so on.

Definition 5.1.3: Residues

If $f(z)$ has a pole of order n at $z = a$ but is analytic at every other point inside and on a circle C with center at a, then the **Laurent series** about $z = a$ is given by

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - a)^n \tag{5.1.1}$$

Or,

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n (z - a)^n + \sum_{n=1}^{\infty} a_{-n} (z - a)^{-n} \\ f(z) &= a_0 + a_1(z - a) + a_2(z - a)^2 + \cdots + \frac{a_{-1}}{z - a} + \frac{a_{-2}}{(z - a)^2} + \cdots \end{aligned}$$

The part of the Laurent series containing the positive powers of $(z - a)$ is called the **analytic part** of $f(z)$ at $z = a$ and is denoted by $P(f; a)$, and the part containing the negative powers of $(z - a)$ is called the **principal part** of $f(z)$ at $z = a$ and is denoted by $Q(f; a)$.

The coefficient a_{-1} is called the **residue** of $f(z)$ at $z = a$ and is denoted by $\mathrm{Res}(f; a)$.

5.2 Methods of Finding Residues

***Theorem 5.2.1* (Residue at a Simple Pole):**
If $f(z)$ has a simple pole at $z = a$, then the residue of $f(z)$ at $z = a$ is given by

$$\operatorname{Res}(f; a) = \lim_{z \to a} (z - a) f(z)$$

Proof:
Since $f(z)$ has a simple pole at $z = a$, we can write $f(z)$ as

$$f(z) = \frac{g(z)}{z - a}$$

where $g(z)$ is analytic at $z = a$ and $g(a) \neq 0$.
Multiplying both sides by $(z - a)$, we get

$$(z - a) f(z) = g(z)$$

Taking the limit as $z \to a$ on both sides, we get

$$\lim_{z \to a} (z - a) f(z) = \lim_{z \to a} g(z) = g(a)$$

Therefore, the residue of $f(z)$ at $z = a$ is given by

$$\operatorname{Res}(f; a) = \lim_{z \to a} (z - a) f(z) = g(a) \quad \square$$

***Theorem 5.2.2* (Residue at a Pole of Order m):**
If $f(z)$ has a pole of order m at $z = a$, then the residue of $f(z)$ at $z = a$ is given by

$$\operatorname{Res}(f; a) = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left[(z - a)^m f(z) \right]$$

Proof:
Since $f(z)$ has a pole of order m at $z = a$, we can write $f(z)$ as

$$f(z) = \frac{g(z)}{(z - a)^m}$$

where $g(z)$ is analytic at $z = a$ and $g(a) \neq 0$.
Multiplying both sides by $(z - a)^m$, we get

$$(z - a)^m f(z) = g(z)$$

Differentiating both sides $m - 1$ times, we get

$$\frac{d^{m-1}}{dz^{m-1}} \left[(z - a)^m f(z) \right] = \frac{d^{m-1}}{dz^{m-1}} g(z)$$

Taking the limit as $z \to a$ on both sides, we get

$$\frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left[(z - a)^m f(z) \right] = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} g(a)$$

Therefore, the residue of $f(z)$ at $z = a$ is given by

$$\operatorname{Res}(f; a) = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left[(z - a)^m f(z) \right] = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} g(a) \quad \square$$