Solution of Cauchy's differential Equation

Parvin Akter

Assistant Professor

Department of Mathematics, CUET.

6.1 Homogeneous linear equations (or Cauchy-Euler-Equations)

A linear differential equation of the form

$$a_0 x^n (d^n y/dx^n) + a_1 x^{n-1} (d^{n-1} y/dx^{n-1}) + \ldots + a_{n-1}\, x\, (dy/dx) + a_n\, y = X \qquad \ldots (1)$$

i.e.,
$$(a_0 x^n D^n + a_1 x^{n-1} D^{n-1} + \ldots + a_{n-1}\, x\, D + a_n)\, y = X, \quad D \equiv d/dx \qquad \ldots (2)$$

where a_0, a_1, a_2, ... a_n are constants, and X is either a constant or a function of x only is called a *homogeneous linear differential equation*. Note that the index of x and the order of derivative is same in each term of such equations. These are also known as *Cauchy-Euler equations*.

6.2 Method of solution of homogeneous linear differential equation [Mumbai 2010]

$$(a_0 x^n D^n + a_1 x^{n-1} D^{n-1} + \ldots + a_{n-1}\, x\, D + a_n)\, y = X. \qquad \ldots (1)$$

In order to solve (1) introduce a new independent variable z such that

$$x = e^z \quad \text{or} \quad \log x = z \quad \text{so that} \quad 1/x = dz/dx. \qquad \ldots (2)$$

Now,
$$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}, \text{ using (2)} \qquad \ldots (3)$$

or
$$x\frac{dy}{dx} = \frac{dy}{dz} \quad \text{or} \quad xD = x\frac{d}{dx} \equiv \frac{d}{dz} = D_1, \text{ say} \qquad \ldots (4)$$

Again,
$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\cdot\frac{dz}{dx} = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2 y}{dz^2}, \text{ by (2)}$$

or
$$x^2 D^2 = x^2\frac{d^2 y}{dx^2} = \frac{d^2 y}{dz^2} - \frac{dy}{dz} = (D_1^2 - D_1)\, y = D_1 (D_1 - 1)\, y. \qquad \ldots (5)$$

Ex. 1. *(a) Solve* $x^2y_2 + xy_1 - 4y = 0$ **[Delhi Maths (G) 1993]**

Sol. Given $(x^2D^2 + xD - 4)\,y = 0,$ where $D \equiv d/dx.$ ... (1)

Let $x = e^z$ (or $z = \log x$) and $D_1 \equiv d/dz$ so that $xD = D_1$ and $x^2D^2 = D_1(D_1 - 1)$.

Then (1) reduces to $[D_1(D_1 - 1) + D_1 - 4]\,y = 0$ or $(D_1^2 - 4)\,y = 0.$ ... (2)

Its auxiliary equation is $D_1^2 - 4 = 0$ so that $D_1 = 2, -2$. Hence the general solution of (2) is

$$y = c_1\, e^{2z} + c_2\, e^{-2z} = c_1\, e^{2\log x} + c_2\, e^{-2\log x} = c_1x^2 + c_2x^{-2}, \text{ as } z = \log x,$$

where c_1 and c_2 are arbitrary constants.

Ex. 1. *(b) Solve* $x^2(d^2y/dx^2) - 3x\,(dy/dx) + 4y = 0$ **[I.A.S. Prel. 1994]**

Sol. Let $d/dx \equiv D$. Then the given equation reduces to $(x^2D^2 - 3xD + 4)\,y = 0.$ ... (1)

Let $x = e^z,$ *i.e.,* $z = \log x$ and $D_1 = d/dz$ ... (2)

Then, $xD = D_1$ and $x^2D^2 = D_1\,(D_1 - 1)$. Hence (1) reduces to

$\{D_1(D_1 - 1) - 3D_1 + 4\}\,y = 0$ or $(D_1 - 2)^2\,y = 0$

Its auxiliary equation is $(D_1 - 2)^2 = 0$ giving $D_1 = 2, 2.$

The general solution is $y = (c_1 + c_2z)\,e^{2z} = (c_1 + c_2z)\,(e^z)^2 = (c_1 + c_2\log x)\,x^2$, using (2).

where c_1 and c_2 are arbitrary constants.

Ex. 1. *(c) Solve* $x^3(d^3y/dx^3) + 2x^2\,(d^2y/d^2x) + 3x\,(dy/dx) - 3y = 0$ **[Meerut 2007]**

Sol. Rewriting the given equation, $(x^3D^3 + 2x^2D^2 + 3xD - 3)y = 0,\ D \equiv d/dx$ ....(1)

Let $x = e^z,$ *i.e.,* $z = \log x$ and $D_1 \equiv d/dz$ ...(2)

Then, $xD = D_1,$ $x^2D^2 = D_1(D_1 - 1)$ and $x^3D^3 = D_1\,(D_1 - 1)\,(D_1 - 2)$

Using (2) and (3), (1) becomes $\{D_1\,(D_1 - 1)\,(D_1 - 2) + 2D_1\,(D_1 - 1) + 3\,D_1 - 3\}\,y = 0$ ...(4)

Auxiliary equation of (4) is $D_1\,(D_1 - 1)\,(D_1 - 2) + 2D_1\,(D_1 - 1) + 3D_1 - 3 = 0$

or $(D_1 - 1)\,(D_1^2 + 3) = 0$ giving $D_1 = 1, \pm i\sqrt{3}$

$\therefore$ Solution of (4) is $y = c_1\,e^z + c_2\sin\,(z\sqrt{3}) + c_3\cos(z\sqrt{3})$

or $y = c_1x + c_2\sin\,(\sqrt{3}\log x) + c_3\cos(\sqrt{3}\log x)$, using (2); c_1, c_2, c_3 being arbitrary constants.

Ex. 3. *Solve the following differential equations :*

(*i*) $x^2y_2 + y = 3x^2$ **[Delhi Maths (G) 1993]**

(*ii*) $xy_3 + y_2 = 1/x$. **[Delhi Maths (G) 1995, 96]**

(*iii*) $(x^2D^2 - 3xD + 4)\, y = 2x^2$. **[Agra 2005, Lucknow 1992]**

(*iv*) $x^2D^2 - 2y = x^2 + (1/x)$ **[Rohilkhand 1993]**

Sol. (*i*) Given $x^2y_2 + y = 3x^2$ or $(x^2D^2 + 1)\, y = 3x^2$, where $D \equiv d/dx$. ... (1)

Let $x = e^z$ (or $z = \log x$) and $D_1 \equiv d/dz$ so that $x^2D^2 = D_1(D_1 - 1)$.

$\therefore$ (1) $\Rightarrow [D_1(D_1 - 1) + 1]\, y = 3\, e^{2z}$ or $(D_1^{\,2} - D_1 + 1) = 3e^{2z}$.

Its auxiliary equation is $D^2{}_1 - D_1 + 1 = 0$ so that $D_1 = (1 \pm i\sqrt{3})/2$.

$\therefore$ C.F. $= e^{z/2}\, [c_1 \cos\,(z\sqrt{3}/2) + c_2 \sin\,(z\sqrt{3}/2)] = (e^z)^{1/2}\, [c_1 \cos\,(z\sqrt{3}/2) + c_2 \sin\,(z\sqrt{3}/2)]$

$= x^{1/2}\, [c_1 \cos\,\{(\sqrt{3}/2)\, \log x\} + c_2 \sin\,\{(\sqrt{3}/2))\, \log x\}]$, as $x = e^z$;

c_1 and c_2 being arbitrary constants.

and
$$\text{P.I.} = \frac{1}{D_1^2 - D_1 + 1} 3e^{2z} = 3\frac{1}{2^2 - 2 + 1} e^{2z} = (e^z)^2 = x^2.$$

Hence the required general solution is $y =$ C.F. + P.I., *i.e.*,

$$y = x^{1/2}[c_1 \cos\{(\sqrt{3}/2)\log x\} + c_2 \sin\{(\sqrt{3}/2)\log x\} + x^2.$$

(*ii*) Given $x^3\, (d^3y/dx^3) + x^2\, (d^2y/dx^2) = x$ or $(x^3D^3 + x^2D^2)\, y = x$, $D \equiv d/dx$ ... (1)

Let $x = e^z$ (or $z = \log x$) and $D_1 \equiv d/dz$ ... (2)

so that $x^2D^2 = D_1(D_1 - 1)$, $x^3D^3 = D_1(D_1 - 1)\,(D_1 - 2)$. Then (1) transforms to

$[D_1(D_1 - 1)\,(D_1 - 2) + D_1\,(D_1 - 1)]\, y = e^z$ or $(D_1^{\,3} - 2D_1^{\,2} + D_1)\, y = e^z$.

Here the auxiliary equation is $D_1^{\,3} - 2D_1^{\,2} + D_1 = 0$ so that $D_1 = 0, 1, 1$.

$\therefore$ C.F. $= c_1e^{0.z} + (c_2 + c_3z)\, e^z = c_1 + (c_2 + c_3 \log x)\, x$, as $e^z = x$ and $z = \log x$.

$$\text{P.I.} = \frac{1}{D_1^3 - 2D_1^2 + D_1} e^z = \frac{1}{(D_1 - 1)^2} \frac{1}{D_1} e^z = \frac{1}{(D_1 - 1)^2} e^z, \quad \text{as} \quad \frac{1}{D_1} e^z = \int e^z dz = e^z$$

$$= \frac{z^2}{2!} e^z, \quad \text{since} \quad \frac{1}{(D_1 - \alpha)^m}\, e^{\alpha z} = \frac{z^m}{m!}\, e^{\alpha z}$$

$$= (x/2) \times (\log x)^2, \quad \text{since} \quad x = e^z \quad \text{and} \quad z = \log x$$

$\therefore$ The required solution is $y = c_1 + (c_2 + c_3 \log x)\, x + (x/2) \times (\log x)^2$,

c_1, c_2 and c_3 being arbitrary constants.

Ex. 4. *Solve the differential equations*

(*i*) $x^2(d^2y/dx^2)+2x(dy/dx)=\log x.$ **[Agra 1994]**

(*ii*) $(x^2D^2+7xD+13)\,y=\log x.$ **[Meerut 1997, 99]**

Sol. (*i*) given $(x^2D^2+2xD)y=\log x$, where $D=d/dx$ ... (1)

Let $x=e^z$ (or $z=\log x$ and $D_1\equiv d/dz$. Then (1) becomes

$[D_1(D_1-1)+2D_1]\,y=z$ or $(D_1{}^2+D_1)\,y=z.$

Its auxiliary equation is $D_1{}^2+D_1=0$ so that $D_1=0,-1.$

$\therefore$ C.F. $=c_1e^{0z}+c_2e^{-z}=c_1+c_2(e^z)^{-1}=c_1+c_2x^{-1}.$, c_1 and c_2 being arbitrary constants.

$$\text{P.I.}=\frac{1}{D_1^2+D_1}=\frac{1}{D_1(1+D_1)}z=\frac{1}{D_1}(1+D_1)^{-1}z=\frac{1}{D_1}(1-D_1+...)\,z=\frac{1}{D_1}(z-1)$$

$$=(1/2)\times z^2-z=(1/2)\times(\log x)^2-\log x,\ \text{ as }\ x=e^z\ \text{ and }\ z=\log x.$$

$\therefore$ The required solution is $y=c_1+c_2x^{-1}+(1/2)\times(\log x)^2-\log x,$

(*ii*) Given that $(x^2D^2+7xD+13)\,y=\log x,$ $D\equiv d/dx$ ...(1)

Let $x=e^z$ (or $z=\log x$) and $D_1\equiv d/dz$. Then, (1) becomes

$[D_1(D_1-1)+7D_1+13]\,y=z$ or $(D_1{}^2+6D_1+13)\,y=z.$

Its auxiliary equation is $D_1{}^2+6D_1+13=0$ so that $D_1=-3\pm 2i.$

$\therefore$ C.F. $=e^{-3z}(c_1\cos 2z+c_2\sin 2z)=x^{-3}[c_1\cos(2\log x)+c_2\sin(2\log x)],$

where c_1 and c_2 being arbitrary constants.

$$\text{P.I.}=\frac{1}{D_1^2+6D_1+13}z=\frac{1}{13[1+(6/13)D_1+(1/13)D_1^2]}z=\frac{1}{13}\left[1+\left(\frac{6}{13}D_1+\frac{1}{13}D_1^2\right)\right]^{-1}z$$

$$=\frac{1}{13}\left[1-\left(\frac{6}{13}D_1+\frac{1}{13}D_1^2\right)+...\right]z=\frac{1}{13}\left(z-\frac{6}{13}\right)=\frac{1}{13}\left(\log x-\frac{6}{13}\right)=\frac{1}{169}(13\log x-6)$$

$\therefore$ Required solution is $y=x^{-3}[c_1\cos(2\log x)+c_2\sin(2\log x)]+(1/169)\times(13\log x-6)$

Ex. 5. *Solve* $x^3 (d^3y/dx^3) + 3x^2 (d^2y/dx^2) + x (dy/dx) + y = \log x + x.$

[Agra 1995, Lucknow 1996, Meerut 1995, Rohilkhand 1997]

Sol. Given $(x^3D^3 + 3x^2D^2 + xD + 1)\, y = \log x + x$, where $D \equiv d/dx$. ... (1)

Let $x = e^z$ (or $z = \log x$) and $D_1 \equiv d/dz$. Then (1) becomes

$[D_1(D_1 - 1)(D_1 - 2) + 3D_1(D_1 - 1) + D_1 + 1]\, y = z + e^z$ or $(D_1^{\,3} + 1)\, y = e^z + z.$

Its auxiliary equation is $D_1^{\,3} + 1 = 0$ or $(D_1 + 1)(D_1^{\,2} - D_1 + 1) = 0$

so that $D_1 = -1, \quad (1 \pm i\sqrt{3})/2$ *i.e.*, $D_1 = -1, (1/2) \pm i(\sqrt{3}/2).$

$\therefore \text{C.F.} = c_1 e^{-z} + e^{z/2}\, [c_2 \cos\{(\sqrt{3}/2)z\} + c_3 \sin\{(\sqrt{3}/2)z\}]$

$= c_1 x^{-1} + x^{1/2}\, [c_2 \cos\{(\sqrt{3}/2) \log x\} + c_3 \sin\{(\sqrt{3}/2) \log x\}]$, as $x = e^z$

where c_1 and c_2 being arbitrary constatns

$$\text{P.I.} = \frac{1}{D_1^3+1}(e^z + z) = \frac{1}{D_1^3+1} e^z + \frac{1}{D_1^3+1} z = \frac{1}{1^3+1} e^z + (1 + D_1^3)^{-1} z$$

$= (1/2) \times e^z + (1 - D_1^{\,3} + \ldots)\, z = (1/2) \times e^z + z = x/2 + \log x$

Hence the required general solution is $y = \text{C.F.} + \text{P.I.}$ *i.e.*,

$$y = c_1 x^{-1} + x^{1/2} [c_2 \cos\{(\sqrt{3}/2) \log x\} + c_3 \sin\{(\sqrt{3}/2) \log x\}] + x/2 + \log x$$

Ex. 7. *Solve the following differential equations :*

(i) $x^2(d^2y/dx^2) + 5x\,(dy/dx) + 4y = x \log x.$ **[Allahabad 1994]**

(ii) $\{x^2D^2 - (2m-1)\,xD + (m^2+n^2)\}\,y = n^2 x^m \log x$, where $D \equiv d/dx$

Sol. (*i*) Given $(x^2D^2 + 5xD + 4)\,y = x \log x,$ where $D \equiv d/dx$ ... (1)

Let $x = e^z$ (or $z = \log x$) and $D_1 \equiv d/dz$. Then (1) becomes

$[D_1(D_1 - 1) + 5D_1 + 4]\,y = ze^z$ or $(D_1 + 2)^2\,y = ze^z.$

Its auxiliary equation is $(D_1 + 2)^2 = 0$ so that $D_1 = -2, -2.$

$\therefore$ C.F. $= (c_1 + c_2z)\,e^{-2z} = (c_1 + c_2z)\,(e^z)^{-2} = (c_1 + c_2 \log x)\,x^{-2},$

where c_1 and c_2 are arbitrary constants.

$$\text{P.I.} = \frac{1}{(D_1+2)^2} ze^z = e^z \frac{1}{[(D_1+1)+2]^2} z = e^z \frac{1}{(3+D_1)^2} z = \frac{e^z}{9}\frac{1}{(1+D_1/3)^2} z = \frac{e^z}{9}\left(1+\frac{D_1}{3}\right)^{-2} z$$

$$= \frac{e^z}{9}\left(1-\frac{2\mathrm{D}_1}{3}+\ldots\right)z = \frac{e^z}{9}\left(z-\frac{2}{3}D_1z\right) = \frac{e^z}{9}\left(z-\frac{2}{3}\right) = \frac{e^z}{27}(3z-2) = \frac{x}{27}(3\log x - 2).$$

Hence the solution is $y = (c_1 + c_2 \log x)\,x^{-2} + (x/27) \times (3 \log x - 2)$

(*ii*) Let $x = e^z$ or $z = \log x$ and $D_1 \equiv d/dz$. So the given equation becomes

$[D_1(D_1 - 1) - (2m-1)\,D_1 + (m^2 + n^2)]\,y = n^2 e^{mz} z$ or $[D_1^{\,2} - 2\,mD_1 + (m^2 + n^2)]\,y = n^2 e^{mz} z.$

Its auxilary equations is $D_1^{\,2} - 2\,mD_1 + (m^2 + n^2) = 0$ so that $D_1 = m \pm in.$

$\therefore$ C.F. $= e^{mz}\,[c_1 \cos nz + c_2 \sin nz] = x^m\,[c_1 \cos (n \log x) + c_2 \sin (n \log x)]$, as $x = e^z$

where c_1 and c_2 are arbitrary constants

$$\text{P.I.} = \frac{1}{D_1^2 - 2mD_1 + (m^2+n^2)} n^2 e^{mz} z = n^2 e^{mz} \frac{1}{(D_1+m)^2 - 2m(D_1+m) + m^2 + n^2} z$$

$$= n^2 e^{mz} \frac{1}{D_1^2 + n^2} z = n^2 e^{mz} \frac{1}{n^2(1 + D_1^2/n^2)} z = e^{mz}\{1 + (D_1^2/n^2)\}^{-1} z$$

$$= e^{mz}\{1 - (D_1^2/n^2) + \ldots\} z = e^{mz} z = (e^z)^m z = x^m \log x, \text{ as } x = e^z$$

$\therefore$ Solution is $y = C.F. + P.I. = x^m\,[c_1 \cos (n \log x) + c_2 \sin (n \log x)] + x^m \log x.$

Ex. 9. *Solve* $x^2(d^2y/dx^2) - 2x(dy/dx) + 2y = x + x^2 \log x + x^3$.

Sol. Given $(x^2D^2 - 2xD + 2)\,y = x + x^2 \log x + x^3$, where $D \equiv d/dx$ ... (1)

Let $x = e^z$ or $z = \log x$ and $D_1 \equiv d/dz$. Then (1) becomes

$[D_1(D_1 - 1) - 2D_1 + 2]\,y = e^z + ze^{2z} + e^{3z}$ or $(D_1^2 - 3D_1 + 2)\,y = e^z + ze^{2z} + e^{3z}$.

Here auxiliary equation is $D_1^2 - 3D_1 + 2 = 0$ so that $D_1 = 1, 2$.

$\therefore$ C.F. $= c_1 e^z + c_2 e^{2z} = c_1 e^z + c_2 (e^z)^2 = c_1x + c_2x^2$, c_1, c_2 being arbitrary constants

P.I. corresponding to $(e^z + e^{3z})$

$$= \frac{1}{D_1^2 - 3D_1 + 2}(e^z + e^{3z}) = \frac{1}{(D_1 - 1)(D_1 - 2)}e^z + \frac{1}{(D_1 - 1)(D_1 - 2)}e^{3z}$$

$$= \frac{1}{D_1 - 1}\frac{1}{1 - 2}e^z + \frac{1}{(3 - 1)(3 - 2)}e^{3z} = -\frac{1}{(D_1 - 1)}e^z + \frac{1}{2}e^{3z} = -\frac{z}{1!}e^z + \frac{1}{2}(e^z)^3$$

$= -z\,e^z - (1/2) \times (e^z)^3 = -x \log x + (x^3/2)$, as $x = e^z$ and $z = \log x$

P.I. corresponding to ze^{2z}

$$= \frac{1}{D_1^2 - 3D_1 + 2}ze^{2z} = e^{2z}\frac{1}{(D_1 + 2)^2 - 3((D_1 + 2) + 2}z = e^{2z}\frac{1}{D_1^2 + D_1}z$$

$$= e^{2z}\frac{1}{D_1}(1 + D_1)^{-1}z = e^{2z}\frac{1}{D_1}(1 - D_1 + \ldots\ldots)z = e^{2z}\frac{1}{D_1}(z - 1) = (e^z)^2\{(z^2/2) - z\}$$

$= x^2\,[(1/2) \times (\log x)^2 - \log x] = (x^2/2) \times [(\log x)^2 - 2 \log x]$

$\therefore$ Solution is $y = c_1x + c_2x^2 - x \log x + x^3/2 + (x^2/2) \times [(\log x)^2 - 2 \log x]$.

H.W. 2,6,8,10,11

Ex. 1(*a*). *Solve* $(1+x)^2 (d^2y/dx^2) + (1+x)(dy/dx) + y = 4 \cos \log (1+x)$.

[Andhra 1997, Delhi Maths (H) 1993, Delhi Maths (G) 2005, Meerut 1997, Purvanchal 1999]

Sol. Given $[(1+x)^2 D^2 + (1+x) D + 1]\, y = 4 \cos \log (1+x)$, $\quad D \equiv d/dx$. ... (1)

Let $1 + x = e^z$ or $\log (1+x) = z$. Also, let $D_1 \equiv d/dz$. ... (2)

Then, we have $(1+x) D = D_1$, $(1+x)^2 D^2 = D_1(D_1 - 1)$ and hence (1) gives

$[D_1(D_1 - 1) + D_1 + 1] = 4 \cos z$ or $(D_1^2 + 1)\, y = 4 \cos z$. ... (3)

Its auxiliary equation is $D_1^2 + 1 = 0$ so that $D_1 = 0 \pm i$.

$\therefore$ C.F. $= e^{0 \cdot z}(c_1 \cos z + c_2 \sin z) = c_1 \cos \log(1+x) + c_2 \sin \log(1+x)$, using (2)

where, c_1 and c_2 are arbitrary constants.

$$\text{P.I.} = \frac{1}{D_1^2+1} 4 \cos z = \text{R.P. of } \frac{1}{D_1^2+1} 4e^{iz}, \text{ where R.P. stands for real part}$$

$$= \text{R.P. of } \frac{1}{D_1^2+1} e^{iz} \cdot 4 = \text{R.P. of } e^{iz} \frac{1}{(D_1+i)^2+1} \cdot 4$$

$$= \text{R.P. of } e^{iz} \frac{1}{D_1^2+2Di} 4 = \text{R.P. of } e^{iz} \frac{1}{2D_1 i(1+D_1/2i)} \cdot 4$$

$$= \text{R.P. of } \frac{e^{iz}}{2i} \frac{1}{D_1}\left(1+\frac{D_1}{2i}\right)^{-1} 4 = \text{R.P. of } \frac{e^{iz}}{2i} \frac{1}{D_1}\left(1-\frac{D_1}{2i}+\ldots\right) 4$$

$= \text{R.P. of } e^{iz}(1/2i) \times (4z) = \text{R.P. of } (-2iz) \times (\cos z + i \sin z)$, as $1/i = -i$

$= 2z \sin z = 2 \log (1+x) \sin \log (1+x)$ as $z = \log (1+x)$

$\therefore$ Solution is $y = c_1 \cos \log (1+x) + c_2 \sin \log (1+x) + 2 \log (1+x) \sin \log (1+x)$.

Ex. 1(*b*) Solve $\{(x+1)^4 D^3 + 2(x+1)^3 D^2 - (x+1)^2 D + (x+1)\}y = 1/(x+1)$, $D \equiv d/dx$.

[I.A.S. 2005]

Sol. Dividing both sides by $(x+1)$, the given equation reduces to

$$\{(x+1)^3 D^3 + 2(x+1)^2 D^2 - (x+1)\, D + 1\}y = (1+x)^{-2} \qquad ...(1)$$

Let $1 + x = e^z$ or $\log(1+x) = z$, Also, let $D_1 \equiv d/dz$ ...(2)

Then, we have $xD = D_1$, $x^2 D^2 = D_1(D_1 - 1)$ and $x^3D^3 = D_1(D_1-1)(D_1-2)$ ...(3)

Using (2) and (3), (1) reduces to

$$\{D_1(D_1-1)(D_1-2) + 2D_1(D_1-1) - D_1 + 1\}\, y = e^{-2z}$$

or $(D_1^3 - D_1^2 - D_1 + 1)y = e^{-2z}$ or $(D_1-1)^2 (D_1+1)y = e^{-2z}$ ...(4)

Here auxiliary equation for (4) is $(D_1-1)^2(D_1+1) = 0$ giving $D_1 = 1, 1, -1$

$\therefore$ C.F. $= (c_1 + c_2 z)\, e^z + c_3 e^{-z}$, c_1, c_2 and c_3 being arbitrary constants

and $$\text{P.I.} = \frac{1}{(D_1-1)^2(D_1+1)} e^{-2z} = \frac{1}{(-2-1)^2(-2+1)} e^{-2z} = -\frac{1}{9} e^{-2z}$$

$\therefore$ The required solution is $y = (c_1 + c_2 z)\, e^z + c_3 (e^z)^{-1} - (1/9) \times (e^z)^{-2}$

or $y = \{c_1 + c_2 \log(1+x)\}(1+x) + c_3(1+x)^{-1} - (1/9) \times (1+x)^{-2}$, using (2)

Ex. 2(*a*). *Solve* $(x+a)^2 (d^2y/dx^2) - 4(x+a)(dy/dx) + 6y = x$. **[Kanpur 2011]**

Sol. Given $[(x+a)^2 D^2 - 4(x+a)\, D + 6]\, y = x$. ... (1)

Let $x + a = e^z$ or $\log(x+a) = z$. Also, let $D_1 \equiv d/dz$. ... (2)

Then, $(a+x)\, D = D_1$, $(a+x)^2 D^2 = D_1(D_1-1)$ and (1) hence gives

$[D_1(D_1-1) - 4D_1 + 6]\, y = e^z - a$ or $(D_1^{\,2} - 5D_1 + 6)\, y = e^z - a$. ... (3)

Its auxiliary equation is $D_1^{\,2} - 5D_1 + 6 = 0$ so that $D_1 = 2, 3$.

$\therefore$ C.F. $= c_1 e^{2z} + c_2 e^{3z} = c_1 (e^z)^2 + c_2 (e^z)^3 = c_1 (x+a)^2 + c_2 (x+a)^3$.

$$\text{P.I.} = \frac{1}{D_1^2 - 5D_1 + 6}(e^z - ae^{0.z}) = \frac{1}{D_1^2 - 5D_1 + 6} e^z - a\frac{1}{D_1^2 - 5D_1 + 6} e^{0.z}$$

$$= \frac{1}{1^2 - (5 \cdot 1) + 6} e^z - a\frac{1}{0^2 - (5 \cdot 0) + 6} e^{0.z} = \frac{x+a}{2} - \frac{a}{6} = \frac{3x + 2a}{6}$$

$\therefore$ Solution is $y = c_1 (x+a)^2 + c_2 (x+a)^3 + (3x + 2a)/6$.

Ex. 2(*a*). *Solve* $(x+a)^2 (d^2y/dx^2) - 4(x+a)(dy/dx) + 6y = x.$ **[Kanpur 2011]**

Sol. Given $[(x+a)^2 D^2 - 4(x+a) D + 6]\, y = x.$ ... (1)

Let $x + a = e^z$ or $\log(x+a) = z.$ Also, let $D_1 \equiv d/dz.$ ... (2)

Then, $(a+x) D = D_1$, $(a+x)^2 D^2 = D_1(D_1 - 1)$ and (1) hence gives

$[D_1(D_1 - 1) - 4D_1 + 6]\, y = e^z - a$ or $(D_1^2 - 5D_1 + 6)\, y = e^z - a.$ ... (3)

Its auxiliary equation is $D_1^2 - 5D_1 + 6 = 0$ so that $D_1 = 2, 3.$

$\therefore$ C.F. $= c_1 e^{2z} + c_2 e^{3z} = c_1 (e^z)^2 + c_2 (e^z)^3 = c_1 (x+a)^2 + c_2 (x+a)^3.$

$$\text{P.I.} = \frac{1}{D_1^2 - 5D_1 + 6}(e^z - ae^{0.z}) = \frac{1}{D_1^2 - 5D_1 + 6}e^z - a\frac{1}{D_1^2 - 5D_1 + 6}e^{0.z}$$

$$= \frac{1}{1^2 - (5\cdot 1) + 6}e^z - a\frac{1}{0^2 - (5\cdot 0) + 6}e^{0.z} = \frac{x+a}{2} - \frac{a}{6} = \frac{3x + 2a}{6}$$

$\therefore$ Solution is $y = c_1 (x+a)^2 + c_2 (x+a)^3 + (3x + 2a)/6.$

Ex. 2(*b*). *Solve* $(x+3)^2 y^2 - 4(x+3) y_1 + 6y = x.$ **[Delhi Maths (G) 1998]**

Sol. Given $[(x+3)^2 D^2 - 4(x+3) D + 6]\, y = x$, $D \equiv d/dx$ ...(1)

which is the same as equation (1) of Ex. 2(*a*). Here $a = 3$. Proceeding as before, the solution is

H.W. 3,4,6