Methods of Variation of Parameters

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7.1 Method of variation of parameters for solving $dy/dx + P(x)\, y = Q(x)$

Consider a first order linear differential equation

$$dy/dx + Py = Q, \qquad i.e., \qquad y_1 + Py = Q, \qquad \text{where} \qquad y_1 = dy/dx \qquad \dots (1)$$

and P and Q are functions of x or constants. Suppose that the general solution of

$$y_1 + Py = 0, \qquad \dots (2)$$

be given by

$$y = au, \qquad \dots (3)$$

where a is an arbitrary constant and u is a function of x. Since u must be a solution of (2), we have

$$u_1 + Pu = 0 \qquad \text{where} \qquad u_1 = du/dx. \qquad \dots (4)$$

When $Q \neq 0$, (3) cannot be the general solution of (1).

Now assume that

$$y = Au \qquad \dots (5)$$

is the general solution of (1), where A is no longer constant but function of x to be so chosen that (1) is satisfied.

From (3) and (5), we note that the form of y is the same for two equations (1) and (2), but the constant which occurs in the former case is changed in the latter into a function of the independent variable x. For this reason, the present method is known as *variation of parameters*.

Differentiating (5) w.r.t. 'x', we have

$$y_1 = A_1 u + A u_1, \qquad \text{where} \qquad A_1 = dA/dx \qquad \dots (6)$$

Putting the values of y and y_1 given by (5) and (6) in (1), we get

$$A_1 u + A u_1 + PAu = Q \qquad \text{or} \qquad A_1 u + A(u_1 + Pu) = Q$$

or

$$A_1 u = Q, \text{ using (4)} \qquad \dots (7)$$

From (7), $\quad A_1 = Q/u \quad$ or $\quad dA/dx = Q/u \quad$ or $\quad dA = (Q/u)\, dx$

Integrating, $\quad A = \int (Q/u)\, dx + c,\;$ where c is an arbitrary constant $\qquad \dots (8)$

Using (8) in (5), the general solution of (1) is given by

$$y = u(x)\{c + \int (Q/u) dx\} \qquad \text{or} \qquad y = c\, u(x) + u(x)\int (Q/u) dx \qquad \dots (9)$$

Ex. 1. *Apply the method of variation of parameters to solve*

(*i*) $y_2 + n^2 y = sec\ nx$ **[Agra 2006, Madras 2005; Gulbarga 2005; Delhi Maths 99, 2004; Bundelhand 2001; Himachal 2004; Kanpur 2007; Meerut 2008, 09; Madurai 2001; Rohilkhand**

2004; Ravishanka 2002, 2004;Purvanchal 2007, I.A.S. 99 Venkenkateshwar 2003]

(*ii*) $y_2 + y = sec\ x$ **[Mysore 2004;Delhi Maths (P) 2001; 02, Delhi Maths (G) 2002]**

(*iii*) $y_2 + 4y = sec\ 2x$

(*iv*) $y_2 + 9y = sec\ 3x$ **[Meerut 2007; Delhi Maths (H) 1999]**

Sol. (*i*) Given $y_2 + n^2 y = \sec nx$ ... (1)

Comparing (1) with $y_2 + Py_1 + Q\,y = R$, we have $R = \sec nx$

Consider $y_2 + n^2 y = 0$ or $(D^2 + n^2)\,y = 0$, where $D \equiv d/dx$ ... (2)

Auxiliary equation of (2) is $D^2 + n^2 = 0$ so that $D = \pm\, in$.

C.F. of (1) = $C_1 \cos nx + C_2 \sin nx$, C_1 and C_2 being arbitrary constants ... (3)

Let $u = \cos nx$, $v = \sin nx$ Also, here $R = \sec nx$ ... (4)

Here $$W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} \cos nx & \sin nx \\ -n \sin nx & n \cos nx \end{vmatrix} = n \neq 0$$ ... (5)

Then, P.I. of (1) = $u\,f(x) + v\,g(x)$ ... (6)

where $$f(x) = -\int \frac{vR}{W}dx = -\int \frac{\sin nx \sec nx}{n}\,dx = \frac{1}{n^2}\log\cos nx, \text{ by (4) and (5)}$$

and $$g(x) = \int \frac{uR}{W}\,dx = \int \frac{\cos nx \ \ \sec nx}{n}\,dx = \frac{x}{n}, \textit{ by } \text{(4) and (5)}$$

$\therefore$ P.I. of (1) = $(\cos nx) \times (1/n^2) \log \cos nx + (\sin nx) \times (x/n)$, by (6)

Hence the general solution of (1) is $y = \text{C.F.} + \text{P.I.}$

i.e., $y = C_1 \cos nx + C_2 \sin nx + (1/n^2) \times \cos nx \log \cos nx + (x/n) \times \sin nx$

Ex. 2. *Apply the method of variation of parameters to solve*

(*i*) $y_2 + a^2 y = cosec\ ax$ **[Meerut 2004, 10; Kakatiay 2003; S.V. University A.P. 199, Rajsthan 2003, 01]**

(*ii*) $y_2 + y = cosec\ x$ **[Meerut 2007, 11; Bangalore 1996, Delhi Maths (G) 1998, 2003] Nagpur 2002, Delhi Maths (H) 1997; Guwahati 1996; Bilaspur 2000, 04 Indore 2001, 07]**

(*iii*) $y_2 + 9y = cosec\ 3x$ **[Delhi Maths (Pass) 2004]**

Sol. (*i*) Given $$y_2 + a^2 y = \operatorname{cosec} ax \qquad \text{... (1)}$$

Comparing (1) with $y_2 + Py_1 + Qy = R$, we have $R = \operatorname{cosec} ax$

Consider $y_2 + a^2 y = 0$ or $(D^2 + a^2)\,y = 0$, $D \equiv d/dx$ ... (2)

Auxiliary equation of (2) is $D^2 + a^2 = 0$ so that $D = \pm\, ai$

$\therefore$ C.F. of (1) $= C_1 \cos ax + C_2 \sin ax$, C_1 and C_2 being arbitrary constants ... (3)

Let $u = \cos ax$, $v = \sin ax$. Also, here $R = \operatorname{cosec} ax$ ... (4)

Here $$W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a \sin ax & a \cos ax \end{vmatrix} = a \neq 0 \qquad \text{... (5)}$$

Then, $$\text{P.I. of (1)} = u\, f(x) + v\, g(x), \qquad \text{... (6)}$$

where $$f(x) = -\int \frac{vR}{W}\, dx = -\int \frac{\sin ax \operatorname{cosec} ax}{a}\, dx = -\frac{x}{a}, \text{ by (4) and (5)}$$

and $$g(x) = \int \frac{uR}{W}\, dx = \int \frac{\cos ax \operatorname{cosec} ax}{a}\, dx = (1/a^2) \times \log \sin ax, \text{ by (4) and (5)}$$

$\therefore$ P.I. of (1) $= (\cos ax) \times (-x/a) + (\sin ax) \times (1/a^2) \times \log \sin ax$, by (6)

Hence the general solution of (1) is $y = \text{C.F.} + \text{P.I.}$

i.e., $$y = C_1 \cos ax + C_2 \sin ax - (x/a) \times \cos ax + (1/a^2) \times \sin ax \log \sin ax$$

(*ii*) Proceed as in part (*i*). Note that here $a = 1$.

Ans. $y = C_1 \cos x + C_2 \sin x - x \cos x + \sin x \log \sin x$

(*iii*) Proceed as in part (*i*). Note that have $a = 3$

Ans. $y = C_1 \cos 3x + C_2 \sin 3x - (x/3) \times \cos 3x + (1/9) \times \sin 3x \log \sin 3x$

Ex. 3. *Apply the method of variation of parameters to solve*

(*i*) $y_2 + a^2 y = \tan ax$ **[Osmania 2004]**

(*ii*) $y_2 + 4y = 4 \tan 2x$ **[Himiachal 2002, 03;Garhwal 2005, Delhi Maths (G) 1997, 2001; Rohilkhanad 2001; Delhi B.A. (Prog) II 2010; Kanpur 2002, 08; Nagpur 1996]**

(*iii*) $y_2 + y = \tan x$ **[Delhi B.A (Prog.) H 2007, 08, 11; Delhi B.A (G) 2000; Bangalore 2005; Delhi B.Sc. (Prog.) II 2008; Delhi Maths (H.) 1996, 2002]**

(*iv*) $y_2 + a^2 y = \cot ax$ **[Delhi Maths (G) 2005]**

(*v*) $y_2 + 4y = \cot 2x$

Sol. (*i*) Given $$y_2 + a^2 y = \tan ax \quad \text{... (1)}$$

Comparing (1) with $y_2 + Py_1 + Q\,y = Q$, we have $R = \tan ax$

Consider $y_2 + a^2 y = 0$ or $(D^2 + a^2)\,y = 0$, where $D \equiv d/dx$ ... (2)

Auxiliary equation of (2) is $D^2 + a^2 = 0$ so that $D = \pm\, ia$

$\therefore$ C.F. of (1) $= c_1 \cos ax + c_2 \sin ax$, c_1 and c_2 being arbitrary constants ... (3)

Let $u = \cos ax$, $v = \sin ax$. Also, here $R = \tan ax$ ... (4)

Here $$W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a \sin ax & a \cos ax \end{vmatrix} = a \neq 0 \quad \text{... (5)}$$

Then $$\text{P.I. of (1)} = u\,f(x) + v\,g(x), \quad \text{... (6)}$$

where $$f(x) = -\int \frac{vR}{W}dx = -\int \frac{\sin ax \tan ax}{a}dx = -\frac{1}{a}\int \frac{1-\cos^2 ax}{\cos ax}dx \text{, using (4) and (5)}$$

$$= -\frac{1}{a}\int (\sec ax - \cos ax)dx = -\frac{1}{a}\left[\frac{1}{a}\log(\sec ax + \tan ax) - \frac{\sin ax}{a}\right]$$

$$= (1/a^2) \times \{\sin ax - \log(\sec ax + \tan ax)\}$$

and $$g(x) = \int \frac{uR}{W}dx = \int \frac{\cos ax \tan ax}{a}dx = \frac{1}{a}\int \sin ax\, dx = -\frac{1}{a^2}\cos ax \text{, using (4) and (5)}$$

Using (6), P.I. of (1) $= \cos ax \times (1/a^2)\{\sin ax - \log(\sec ax + \tan ax)\} + \sin ax \times (-1/a^2)\cos ax$

$$= -(1/a^2) \times \cos ax \log(\sec ax + \tan ax)$$

Hence the general solution of (1) is $y = \text{C.F.} + \text{P.I.}$

i.e., $$y = c_1 \cos ax + c_2 \sin ax - (1/a^2) \times \cos ax \log(\sec ax + \tan ax)$$

(*ii*) Given $y_2 + 4y = 4\tan 2x$ ... (1)

Comparing (1) with $y_2 + Py_1 + Qy = R$, here $R = 4\tan 2x$

Consider $y_2 + 4y = 0$ or $(D^2 + 4)y = 0$, $D \equiv d/dx$ ... (2)

Auxiliary equation of (2) is $D^2 + 4 = 0$ so that $D = \pm 2i$.

C.F. of (1) $= C_1 \cos 2x + C_2 \sin 2x$, C_1 and C_2 being arbitrary constants ... (3)

Let $u = \cos 2x$, $v = \sin 2x$. Also, here $R = 4\tan 2x$ ... (4)

Here $$W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2 \neq 0 \qquad \dots (5)$$

Then, $$\text{P.I. of (1)} = uf(x) + vg(x), \qquad \dots (6)$$

where $$f(x) = -\int \frac{vR}{W} dx = -4\int \frac{\sin 2x \tan 2x}{2} dx = -2\int \frac{1-\cos^2 2x}{\cos 2x} dx, \text{ using (4) and (5)}$$

$$= 2\int (\cos 2x - \sec 2x)\, dx = \sin 2x - \log(\sec 2x + \tan 2x)$$

and $$g(x) = \int \frac{uR}{W} dx = 4\int \frac{\cos 2x \tan 2x}{2} dx = -\cos 2x, \text{ by (4) and (5)}$$

$\therefore$ P.I of (1) = $(\cos 2x)\{\sin 2x - \log(\sec 2x + \tan 2x)\} + (\sin 2x)(-\cos 2x)$, by (6)

or $$\text{P.I. of (1)} = -\cos 2x \log(\sec 2x + \tan 2x)$$

Hence the general solution of (1) is $y = \text{C.F.} + \text{P.I.}$

i.e., $$y = C_1 \cos 2x + C_2 \sin 2x - \cos 2x \log(\sec 2x + \tan 2x).$$

(*iii*) Proceed as in part (*i*) by taking $a = 1$. The general solution is

$$y = C_1 \cos x + C_2 \sin x - \cos x \log(\sec x + \tan x)$$

(*iv*) Given $$y_2 + a^2 y = \cot ax \qquad \dots (1)$$

Comparing (1) with $y_2 + Py_1 + Qy = R$, we have $R = \cot ax$

Consider $y_2 + a^2 y = 0$ or $(D^2 + a^2)y = 0$, $D \equiv d/dx$ ...(2)

Auxiliary equation of (2) is $D^2 + a^2 = 0$ so that $D = \pm ia$.

$\therefore$ C.F. of (1) = $c_1 \cos x + c_2 \sin ax$, c_1 and c_2 being arbitrary constants ... (3)

Let $u = \cos ax$, $v = \sin ax$. Also, here $R = \cot ax$ ...(4)

Here $$W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a \neq 0 \qquad \dots (5)$$

Then $$\text{P.I. of (1)} = u f(x) + v g(x), \qquad \dots (6)$$

where $$f(x) = -\int \frac{vR}{W} dx = -\int \frac{\sin ax \cot ax}{a} dx = -\frac{1}{a}\int \cos ax\, dx = -\frac{\sin ax}{a^2}, \text{ using (4) and (5)}$$

and $$g(x) = \int \frac{uR}{W} dx = \int \frac{\cos ax \cot ax}{a} dx = \frac{1}{a}\int \frac{1-\sin^2 ax}{\sin ax} dx, \text{ by (4) and (5)}$$

Using (6), P.I. of (1) $= \cos ax \times (-1/a^2) \times \sin ax + \sin ax \times (1/a^2) \times \{\log \tan (ax/2) + \cos ax\}$

$$= (1/a^2) \times \log \tan (ax/2)$$

Hence, the general solution of (1) is $y = \text{C.F.} + \text{P.I.}$

i.e., $$y = c_1 \cos ax + c_2 \sin ax + (1/a^2) \times \log \tan (ax/2)$$

(v) Proceed like part (iv) with $a = 2$. The solution is

$$y = c_1 \cos 2x + c_2 \sin 2x + (1/4) \times \log \tan x$$

Ex. 4. *Apply the method of variation of parameters to solve*

(i) $y_2 - y = 2/(1 + e^x)$ **[Delhi Maths (H) 2001; Delhi Maths (G) 1999; Rohilkhand 2002; Allahabad 2000, 05; Kanpur 2007; Nagpur 2001, 06; Bangalore 2004**

(ii) $y_2 - 3y_1 + 2y = e^x/(1 + e^x)$ **Delhi B.Sc. (Prog) 2009]**

(iii) $y_2 - 4y_1 + 3y = e^x/(1 + e^x)$.

H.W. 4,5,8