Fourier Analysis

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Contents

1 Introduction

1.1 Periodic Functions

Definition 1.1.1: Periodic Functions

A function $f(x)$ is said to be have a *period* P or to be *periodic* with period P if for all x, $f(x+P) = f(x)$ where P is a positive constant. The least value of $P > 0$ is called the *least period* or simply the *period* of $f(x)$.

Example 1.1: Some examples of periodic functions

1. $\sin x$ has periods $2\pi, 4\pi, 6\pi, \cdots$ and $-\pi, -3\pi, -5\pi, \cdots$ and hence the least period is 2π.
2. $\cos x$ has the least period 2π.
3. $\tan x$ has the least period π.

Some other examples:

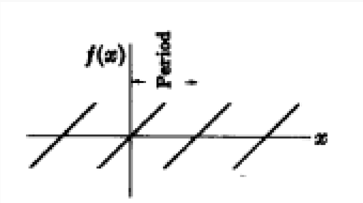

Figure 1.1.1

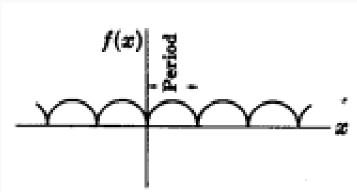

Figure 1.1.2

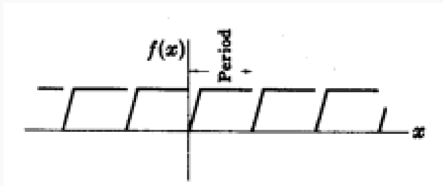

Figure 1.1.3

1.2 Piecewise Continuous Functions

Definition 1.2.1: Piecewise Continuous Functions

A function $f(x)$ is said to be *piecewise continuous* in the interval $[a,b]$ if $f(x)$ is continuous in the interval (a,b) and has a finite number of finite discontinuities in the interval $[a,b]$.

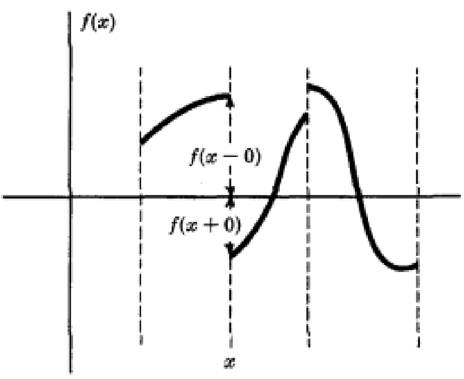

Figure 1.2.1

The right-hand limit of $f(x)$ is often denoted by $\lim_{\epsilon\to 0} f(x+\epsilon) = f(x+0)$, where $\epsilon > 0$.

Similarly, the left-hand limit of $f(x)$ is denoted by $\lim_{\epsilon\to 0} f(x-\epsilon) = f(x-0)$, where $\epsilon > 0$. The values of $f(x+0)$ and $f(x-0)$ at the point x in (1.2.1) are as indicated.

2 Fourier Expansion

2.1 Definition

Definition 2.1.1: Fourier Expansion

Let $f(x)$ be defined in the interval $(-L, L)$ and determined outside of this interval by $f(x+2L) = f(x)$, i.e. assume that $f(x)$ has the period $2L$. The *Fourier series* or *Fourier expansion* corresponding to $f(x)$ is defined to be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right] \tag{1}$$

where the *Fourier coefficients* a_n and b_n are given by

$$\begin{cases} a_0 = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)dx \\ a_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\cos\left(\dfrac{n\pi x}{L}\right)dx \qquad n = 0, 1, 2, \cdots \\ b_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\sin\left(\dfrac{n\pi x}{L}\right)dx \end{cases} \tag{2}$$

2.2 Some pre-derivations

$$\begin{aligned} I &= \int_{-L}^{L} \sin^2\frac{n\pi x}{L}\,dx \\ &= \sin\frac{n\pi x}{L}\cdot\frac{L}{n\pi}(\cos n\pi - \cos n\pi) + \frac{n\pi}{L}\cdot\frac{L}{n\pi}\int_{-L}^{L}\cos\frac{n\pi x}{L}\cos\frac{n\pi x}{L}\,dx \\ &= \int_{-L}^{L}\cos^2\frac{n\pi x}{L}\,dx \\ &= \frac{1}{2}\int_{-L}^{L}\left(\cos\frac{2n\pi x}{L} + 1\right)dx \\ &= \frac{1}{2}\int_{-L}^{L}\cos\frac{2n\pi x}{L}\,dx + \frac{1}{2}\int_{-L}^{L}dx \\ &= 0 + \frac{1}{2}\cdot 2L \\ &= L \end{aligned}$$

$$
\begin{aligned}
I_1 &= \int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx \qquad [m \neq 0] \\
&= \cos\frac{m\pi x}{L}\int_{-L}^{L}\cos\frac{n\pi x}{L}\,dx + \frac{m}{n}\int_{-L}^{L}\sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx \\
&= \frac{L}{n\pi}\cos\frac{m\pi x}{L}(\sin n\pi + \sin n\pi) + \frac{m}{n}I_2 \\
&= 0 + \frac{m}{n}\left[\int_{-L}^{L}\sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx\right] \\
&= \frac{m}{n}\left[\sin\frac{m\pi x}{L}\int_{-L}^{L}\sin\frac{n\pi x}{L}\,dx + \frac{m}{n}\int_{-L}^{L}\cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx\right] \\
&= \frac{m}{n}\left[\frac{L}{n\pi}\sin\frac{m\pi x}{L}(-\cos n\pi + \cos n\pi) + \frac{m}{n}\right] \\
&= 0 + \frac{m^2}{n^2}I_1 \\
I_1 &= 0 = I_2
\end{aligned}
$$

To summarize, we have

$$
\int_{-L}^{L}\sin^2\frac{n\pi x}{L}\,dx = \int_{-L}^{L}\cos^2\frac{n\pi x}{L}\,dx = L \tag{3}
$$

$$
\int_{-L}^{L}\cos mx\,dx = \int_{-L}^{L}\sin mx\,dx = 0 \tag{4}
$$

$$
\int_{-L}^{L}\sin\frac{n\pi x}{L}\cos\frac{n\pi x}{L}\,dx = 0 \tag{5}
$$

$$
\int_{-L}^{L}\sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \int_{-L}^{L}\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = 0 \qquad [m \neq n] \tag{6}
$$

2.3 Derivation of a_0

Taking integral on both sides of (1) from $-L$ to L, we get

$$
\begin{aligned}
\int_{-L}^{L} f(x)\,dx &= \frac{a_0}{2}\int_{-L}^{L}dx + \sum_{n=1}^{\infty}\int_{-L}^{L}\left[a_n\cos\frac{m\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right]dx \\
&= \frac{a_0}{2}\cdot 2L \qquad \text{[All the other terms are 0 according to equation (4)]}
\end{aligned}
$$

$$
\boxed{a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx}
$$

2.4 Derivation of a_n

Multiplying both sides of (1) by $\cos\dfrac{m\pi x}{L}$ and integrating from $-L$ to L, we get

$$
\begin{aligned}
\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx &= \frac{a_0}{2}\int_{-L}^{L}\cos\frac{m\pi x}{L}\,dx \\
&\quad + \sum_{n=1}^{\infty}\int_{-L}^{L}\left[a_n\cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\right]dx \\
&= a_n\int_{-L}^{L}\cos^2\frac{m\pi x}{L}\,dx \\
&= a_n\cdot L \qquad \text{[All the other terms are 0 according to equation (2)]}
\end{aligned}
$$

$$
\boxed{a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx}
$$

2.5 Derivation of b_n

Multiplying both sides of (1) by $\sin\dfrac{m\pi x}{L}$ and integrating from $-L$ to L, we get

$$
\begin{aligned}
\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\,dx &= \frac{a_0}{2}\int_{-L}^{L}\sin\frac{m\pi x}{L}\,dx \\
&\quad + \sum_{n=1}^{\infty}\int_{-L}^{L}\left[a_n\sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\right]dx \\
&= a_n\int_{-L}^{L}\sin^2\frac{m\pi x}{L}\,dx \\
&= a_n\cdot L \qquad \text{[All the other terms are 0 according to equation (2)]}
\end{aligned}
$$

$$
\boxed{b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\,dx}
$$

2.6 Examples

Example 2.1: Obtain the fourier series for $f(x) = x - x^2$ in the interval $(-\pi, \pi)$ and hence evaluate

$$
\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots
$$

$$
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)
$$

where

$$
a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)dx = \frac{1}{\pi}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-\pi}^{\pi} = -\frac{2\pi^2}{3}
$$

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\cos nx dx \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\cos nx\, dx - \int_{-\pi}^{\pi} x^2\cos nx\, dx\right] \\
&= -\frac{2}{\pi}\int_{0}^{\pi} x^2\cos nx\, dx \\
&= -\frac{2}{\pi}\left[\frac{x^2}{n}\sin nx - \frac{2}{n}\int x\sin nx\, dx\right]_0^{\pi} \\
&= \frac{4}{n\pi}\left[-\frac{x}{n}\cos nx + \frac{1}{n^2}\sin nx\right]_0^{\pi} \\
&= -\frac{4}{n^2}(-1)^n
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} (x-x^2)\cos nx\, dx \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\sin nx\, dx - \int_{-\pi}^{\pi} x^2\sin nx\, dx\right] \\
&= \frac{2}{\pi}\int_{0}^{\pi} x\sin nx\, dx \\
&= \frac{2}{\pi}\left[-\frac{x}{n}\cos nx + \frac{1}{n^2}\sin nx\right]_0^{\pi} \\
&= -\frac{2}{n}(-1)^n
\end{aligned}
$$

$$\therefore f(x) = -\frac{\pi^2}{3} - \sum_{n=1}^{\infty}\left(\frac{4}{n^2}(-1)^n\cos nx + \frac{2}{n}(-1)^n\sin nx\right)$$

For $x = 0$, we get

$$
\begin{aligned}
0 &= -\frac{\pi^2}{3} - \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n \\
\frac{\pi^2}{12} &= -\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^2}
\end{aligned}
$$

$$\boxed{\therefore 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}}$$

Example 2.2: Find a fourier series to represent the function $f(x) = e^x$ **for** $-\pi < x < \pi$ **and hence derive a series for** $\dfrac{\pi}{\sinh\pi}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x dx = \frac{1}{\pi}\, e^x\Big|_{-\pi}^{\pi} = \frac{e^{\pi} - e^{-\pi}}{\pi} = \frac{2\sinh x}{\pi}$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos nx\, dx \\
&= \frac{1}{\pi}\left[e^x \frac{\sin nx}{n} - \frac{1}{n}\int e^x \sin nx\, dx\right]_{-\pi}^{\pi} \\
&= -\frac{1}{n\pi}\left[-e^x \frac{\cos nx}{n} + \frac{1}{n}\int e^x \cos nx\, dx\right]_{-\pi}^{\pi} \\
a_n\left(1 + \frac{1}{n^2}\right) &= \frac{(-1)^n}{n^2\pi}\left(e^{\pi} - e^{-\pi}\right) \\
a_n &= \frac{(-1)^n}{n^2\pi}\left(e^{\pi} - e^{-\pi}\right)\left(1 + \frac{1}{n^2}\right)^{-1} \\
a_n &= 2\,\frac{(-1)^n}{n^2\pi}\sinh x\left(1 + \frac{1}{n^2}\right)^{-1}
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin nx\, dx \\
&= \frac{1}{\pi}\left[-e^x \frac{\cos nx}{n} + \frac{1}{n}\int e^x \cos nx\, dx\right]_{-\pi}^{\pi} \\
&= \frac{1}{\pi}\left[-e^x \frac{\cos nx}{n} + \frac{1}{n}\left\{e^x \frac{\sin nx}{n} - \frac{1}{n}\int e^x \sin nx\, dx\right\}\right]_{-\pi}^{\pi} \\
&= \frac{1}{\pi}\left[-e^x \frac{\cos nx}{n} - \frac{1}{n^2}\int e^x \sin nx\, dx\right] \\
b_n &= -\frac{(-1)^n}{n\pi}\left(e^{\pi} - e^{-\pi}\right)\left(1 + \frac{1}{n^2}\right)^{-1} \\
b_n &= -2\,\frac{(-1)^n}{n\pi}\sinh x\left(1 + \frac{1}{n^2}\right)^{-1}
\end{aligned}$$

$$\boxed{f(x) = e^x = 2\,\frac{\sinh x}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\left(1 + \frac{1}{n^2}\right)^{-1}\frac{(-1)^n}{n}\left(\frac{1}{n}\cos nx - \sin nx\right)\right]}$$

For $x = 0$, we get

$$\begin{aligned}
\frac{\pi}{\sinh x} &= 1 + 2\sum_{n=1}^{\infty}\left(1 + \frac{1}{n^2}\right)^{-1}\frac{(-1)^n}{n^2} \\
&= 1 + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2 + 1}
\end{aligned}$$

$$\boxed{\frac{\pi}{\sinh x} = 1 + 2\left(-\frac{1}{2} + \frac{1}{5} - \frac{1}{10} + \cdots\right)}$$

3 Fourier Integral

3.1 Definition

Definition 3.1.1: Fourier Integral

The Fourier integral of a function f defined on the interval $(-\infty, \infty)$ is given by

$$f(x) = \frac{1}{\pi} \int_0^\infty \left[A(\alpha) \cos \alpha x + B(\alpha) \sin \alpha x \right] d\alpha \tag{7}$$

where the coefficients $A(\alpha)$ and $B(\alpha)$ are given by

$$\begin{cases} A(\alpha) = \displaystyle\int_{-\infty}^{\infty} f(x) \cos \alpha x \, dx \\ \\ B(\alpha) = \displaystyle\int_{-\infty}^{\infty} f(x) \sin \alpha x \, dx \end{cases} \tag{8}$$

The Fourier integral can also be written in the form

$$f(x) = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^{\infty} f(t) \cos \lambda(t - x) \, dt \, d\lambda \tag{9}$$

where $\lambda = \dfrac{n\pi}{L}$

Fourier series were used to represent a function f defined on the finite interval $(-L, L)$ or $(0, L)$. It converged to f and to its periodic extension. In this sense, Fourier series is assosiated with periodic functions.

Fourier integral represents a certain type of non-periodic functions that are defined on $(-\infty, \infty)$ or $(0, \infty)$.

3.2 Derivation

Let a function f be defined on $(-L, L)$. The fourier series of the function is then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \tag{10}$$

where the coefficients are given by

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(t) \, dt$$
$$a_n = \frac{1}{L} \int_{-L}^{L} f(t) \cos \frac{n\pi t}{L} \, dt$$
$$b_n = \frac{1}{L} \int_{-L}^{L} f(t) \sin \frac{n\pi t}{L} \, dt$$

Now, let $a_n = \dfrac{n\pi}{L}$,

then $\Delta\alpha = \alpha_{n+1} - \alpha_n = \dfrac{\pi}{L}$

So, we get

$$\begin{aligned}\Delta f(x) = \frac{1}{2\pi}\left(\int_{-L}^{L} f(t)\,dt\right)\Delta\alpha + \frac{1}{\pi}\sum_{n=1}^{\infty}\Bigg[&\left(\int_{-L}^{L} f(t)\cos\alpha_n t\,dt\right)\cos\alpha_n x + \\ &\left(\int_{-L}^{L} f(t)\sin\alpha_n t\,dt\right)\sin\alpha_n x\Bigg]\Delta\alpha\end{aligned} \tag{11}$$

We now expand the interval $(-L, L)$ by taking $L \to \infty$, which implies that $\Delta\alpha \to 0$. Consequently, we get

$$\lim_{\Delta\alpha\to 0}\sum_{n=1}^{\infty} F(\alpha_n)\Delta\alpha \to \int_0^{\infty} F(\alpha)\,d\alpha \tag{12}$$

Thus, the limit of the first term in the Fourier series $\displaystyle\int_{-L}^{L} f(t)\,dt$ vanishes, and the limit of the sum becomes

$$\boxed{f(x) = \frac{1}{\pi}\int_0^{\infty}\left[\left(\int_{-\infty}^{\infty} f(t)\cos\alpha t\,dt\right)\cos\alpha x + \left(\int_{-\infty}^{\infty} f(t)\sin\alpha t\,dt\right)\sin\alpha x\right]d\alpha} \tag{13}$$

This is the Fourier integral of f on the interval $(-\infty, \infty)$.

3.3 Alternative Derivation

Substituting the values of a_0, a_n and b_n in (7), we get

$$\begin{aligned}f(x) &= \frac{1}{2L}\int_{-L}^{L} f(t)\,dt + \frac{1}{L}\sum_{n=1}^{\infty}\left[\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}\cos\frac{n\pi x}{L}\,dt + \int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}\sin\frac{n\pi x}{L}\,dt\right] \\ &= \frac{1}{2L}\int_{-L}^{L} f(t)\,dt + \frac{1}{L}\sum_{n=1}^{\infty}\left[\int_{-L}^{L} f(t)\left\{\cos\frac{n\pi t}{L}\cos\frac{n\pi x}{L} + \sin\frac{n\pi t}{L}\sin\frac{n\pi x}{L}\right\}dt\right]\end{aligned}$$

$$f(x) = \frac{1}{2L}\int_{-L}^{L} f(t)\,dt + \frac{1}{L}\sum_{n=1}^{\infty}\int_{-L}^{L} f(t)\cos\frac{n\pi}{L}(t-x)\,dt \tag{14}$$

Now, if we assume that $\displaystyle\int_{-\infty}^{\infty}|f(x)|\,dx$ converges, the first term on the right side of (11) approaches 0 as $L \to \infty$, since

$$\left|\frac{1}{2L}\int_{-L}^{L} f(t)\,dt\right| \le \frac{1}{2L}\int_{-\infty}^{\infty}|f(t)|\,dt$$

The second term on the right side of (11) approaches

$$\begin{aligned}&\lim_{L\to\infty}\frac{1}{L}\sum_{n=1}^{\infty}\int_{-\infty}^{\infty} f(t)\cos\frac{n\pi}{L}(t-x)\,dt \\ &= \lim_{\delta\lambda\to 0}\frac{1}{\pi}\sum_{n=1}^{\infty}\delta\lambda\int_{-\infty}^{\infty} f(t)\cos n\delta\lambda(t-x)\,dt\end{aligned}$$

where $\lambda = \dfrac{n\pi}{L}$ which implies that $\delta\lambda = \dfrac{\pi}{L}$.

We know,

$$\lim_{\delta\lambda \to 0} \sum_{n=1}^{\infty} \delta\lambda F(\lambda_n) = \int_0^{\infty} F(\lambda)\, d\lambda$$

Thus we get

$$\boxed{f(x) = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} f(t) \cos\lambda(t - x)\, dt\, d\lambda} \tag{15}$$

This is another form of the Fourier integral.