Theory of
Constrained Optimization

Numerical Optimization

Intro: Constrained Optimizations

- Generally:

$$\min_{x \in \mathbf{R}^n} f(x) \text{ subject to } \begin{cases} c_i(x) = 0 & i \in \mathbf{E} \\ c_i(x) \geq 0 & i \in \mathbf{I} \end{cases} \tag{1}$$

Where $\mathbf{E}$, $\mathbf{I}$ are index sets for equality and inequality constraints.

Feasible set / region:

$$\Omega = \{x \mid c_i(x) = 0,\ i \in \mathbf{E};\ c_i(x) \geq 0,\ i \in \mathbf{I}\} \tag{2}$$

Example

Given : $f(x) = -2x_1 + x_2 - 3x_3$

$\min f(x)$ Subject to :

$$2x_2 - 3x_3 = -12$$ Equality constraint

$$x_1 + x_2 + x_3 \leq 10$$ Inequality constraint

$$x_1, x_2, x_3 \geq 0$$

$x_1 = 0, x_2 = 0, x_3 = 0$ Not in feasible set

$x_1 = 0, x_2 = 3, x_3 = 6$ In feasible set, but not the optimal solution

$x_1 = 1, x_2 = 3, x_3 = 6$ In feasible set, but not the optimal solution

$x_1 = 6, x_2 = 0, x_3 = 4$ In feasible set, the optimal solution

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- At a feasible point x, the inequality constraint c_i ($i \in \mathbf{I}$) is:
  - Active iff $c_i(\mathrm{x}) = 0$ (on the boundary)
  - Inactive iff $c_i(\mathrm{x}) > 0$ (interior point)
- Active set : $\mathbf{A}(x) = \mathbf{E} \bigcup \{\, i \in \mathbf{I} \;:\; c_i(x) = 0 \,\}$
- In the last example, x = (0,3,6) is inactive, whereas x = (1,3,6) is active

Intro: Constrained Optimizations

- Smoothness of objective functions and constraints is an important issue in characterizing solutions.
- The non-smooth boundaries can often be described by a collection of smooth constraint functions.
- This applies to objective functions as well.

Intro: Constrained Optimizations

Example :

$$f(x) = \max(x^2, x)$$

A non-smooth cost function
can be reformulated as:

$$\max t \qquad \text{s.t.} \qquad t \geq x\ , \qquad t \geq x^2$$