Gaussian Elemination and Gauss-Jordan Elemination (a revision)

What is Gaussian Elimination?

- An algorithm for solving systems of linear equations
- Named after Carl Friedrich Gauss (1777–1855), who invented this method to solve a system of 17 linear equations.
- This method was introduced earlier in a Chinese book “The Nine Chapters on the Mathematical Art” around 250BC.
- Method: Use elementary row operations to bring an augmented matrix to row echelon form.

Augmented Matrix

- Given a system of linear equations as follow:

$$a_1x_1 + b_1x_2 + c_1x_3 = y_1$$
$$a_2x_1 + b_2x_2 + c_2x_3 = y_2$$
$$a_3x_1 + b_3x_2 + c_3x_3 = y_3$$

- It can be written in a matrix form ***Ax*** = ***y***, and we get an Augmented matrix

$$\left[\begin{array}{ccc|c} a_1 & b_1 & c_1 & y_1 \\ a_2 & b_2 & c_2 & y_2 \\ a_3 & b_3 & c_3 & y_3 \end{array}\right]$$

Row echelon form

- The following matrices are in row echelon form:

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

- The leading entries (■) may have any nonzero value;
- the starred entries (*) may have any value (including zero)

Reduced echelon form

- Wilhelm Jordan suggest that we can further reduce the matrices into a reduced echelon form : Gauss–Jordan elimination (1888):

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * \end{bmatrix}$$

- The leading entries (■) have value 1;
- the starred entries (*) may have any value (including zero)

Elementary row operations

- 3 operations:
  - Interchange 2 rows
  - Multiply a row with a non zero constant
  - Add a multiple of one row to another row
- The aim of these operations is to bring an augmented matrix to its row echelon form.

Example

- Given :

$$4x_1 - 2x_2 + 5x_3 = 41$$
$$-2x_1 + 3x_2 + 2x_3 = -2$$
$$3x_1 + 3x_2 - 6x_3 = -27$$

- Augmented matrix :

$$\left[\begin{array}{ccc|c} 4 & -2 & 5 & 41 \\ -2 & 3 & 2 & -2 \\ 3 & 3 & -6 & -27 \end{array}\right]$$

- Solution : perform elementary row operations as shown in following pages

$$\left[\begin{array}{ccc|c} 4 & -2 & 5 & 41 \\ -2 & 3 & 2 & -2 \\ 3 & 3 & -6 & -27 \end{array}\right]$$

Divide row 1 by 4: $R_1 \to \dfrac{R_1}{4}$.

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{5}{4} & \frac{41}{4} \\ -2 & 3 & 2 & -2 \\ 3 & 3 & -6 & -27 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{5}{4} & \frac{41}{4} \\ -2 & 3 & 2 & -2 \\ 3 & 3 & -6 & -27 \end{array}\right]$$

Add row 1 multiplied by 2 to row 2: $R_2 \rightarrow R_2 + 2R_1$.

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{5}{4} & \frac{41}{4} \\ 0 & 2 & \frac{9}{2} & \frac{37}{2} \\ 3 & 3 & -6 & -27 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{5}{4} & \frac{41}{4} \\ 0 & 2 & \frac{9}{2} & \frac{37}{2} \\ 3 & 3 & -6 & -27 \end{array}\right]$$

Subtract row 1 multiplied by 3 from row 3: $R_3 \to R_3 - 3R_1$.

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{5}{4} & \frac{41}{4} \\ 0 & 2 & \frac{9}{2} & \frac{37}{2} \\ 0 & \frac{9}{2} & -\frac{39}{4} & -\frac{231}{4} \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{5}{4} & \frac{41}{4} \\ 0 & 2 & \frac{9}{2} & \frac{37}{2} \\ 0 & \frac{9}{2} & -\frac{39}{4} & -\frac{231}{4} \end{array}\right]$$

Divide row 2 by 2: $R_2 \blacktriangleright \dfrac{R_2}{2}$.

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{5}{4} & \frac{41}{4} \\ 0 & 1 & \frac{9}{4} & \frac{37}{4} \\ 0 & \frac{9}{2} & -\frac{39}{4} & -\frac{231}{4} \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{5}{4} & \frac{41}{4} \\ 0 & 1 & \frac{9}{4} & \frac{37}{4} \\ 0 & \frac{9}{2} & -\frac{39}{4} & -\frac{231}{4} \end{array}\right]$$

Add row 2 multiplied by $\frac{1}{2}$ to row 1: $R_1 \rightarrow R_1 + \frac{R_2}{2}$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{19}{8} & \frac{119}{8} \\ 0 & 1 & \frac{9}{4} & \frac{37}{4} \\ 0 & \frac{9}{2} & -\frac{39}{4} & -\frac{231}{4} \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{19}{8} & \frac{119}{8} \\ 0 & 1 & \frac{9}{4} & \frac{37}{4} \\ 0 & \frac{9}{2} & -\frac{39}{4} & -\frac{231}{4} \end{array}\right]$$

Subtract row 2 multiplied by $\frac{9}{2}$ from row 3: $R_3 \rightarrow R_3 - \frac{9R_2}{2}$.

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{19}{8} & \frac{119}{8} \\ 0 & 1 & \frac{9}{4} & \frac{37}{4} \\ 0 & 0 & -\frac{159}{8} & -\frac{795}{8} \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{19}{8} & \frac{119}{8} \\ 0 & 1 & \frac{9}{4} & \frac{37}{4} \\ 0 & 0 & -\frac{159}{8} & -\frac{795}{8} \end{array}\right]$$

Multiply row 3 by $-\frac{8}{159}$: $R_3 \rightarrow -\frac{8R_3}{159}$.

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{19}{8} & \frac{119}{8} \\ 0 & 1 & \frac{9}{4} & \frac{37}{4} \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{19}{8} & \frac{119}{8} \\ 0 & 1 & \frac{9}{4} & \frac{37}{4} \\ 0 & 0 & 1 & 5 \end{array}\right]$$

Subtract row 3 multiplied by $\frac{19}{8}$ from row 1: $R_1 \rightarrow R_1 - \frac{19R_3}{8}$.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & \frac{9}{4} & \frac{37}{4} \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & \frac{9}{4} & \frac{37}{4} \\ 0 & 0 & 1 & 5 \end{array}\right]$$

Subtract row 3 multiplied by $\frac{9}{4}$ from row 2: $R_2 \to R_2 - \frac{9R_3}{4}$.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$x_1 = 3$ $x_2 = -2$ $x_3 = 5$

Exercise

$$-3x_1 - 2x_2 - 2x_3 = 6$$
$$3x_1 + 5x_2 + 7x_3 = -12$$
$$2x_1 + 2x_2 + 6x_3 = -16$$

Answer:

$x_1 = -2$ $x_2 = 3$ $x_3 = -3$

Exercise

Ali invests a total of RM10,000 in three accounts with interests 5%, 8% and 9% respectively. The annual interest earned on the three investments last year was RM770. The amount invested at the one with 9% interest is twice the amount invested at the one with 5% interest. How much was Ali invested at each account?

$$x+y+z=10{,}000$$
$$0.05x+0.08y+0.09z=770$$
$$2x-z=0$$

<u>Answer:</u>

$x = 3000$ $y = 1000$ $z = 6000$

Modified from libretexts resource

Exercise

nd the third paying 9 interest. The annual interest earned on the three investments
nvested at 5. How much was invested at each rate?

$$x+y+z=10,000$$
$$0.05x+0.08y+0.09z=770$$
$$2x-z=0$$

Answer:

$x = 3000$ $y = 1000$ $z = 6000$