Linear Programming (Simplex method)

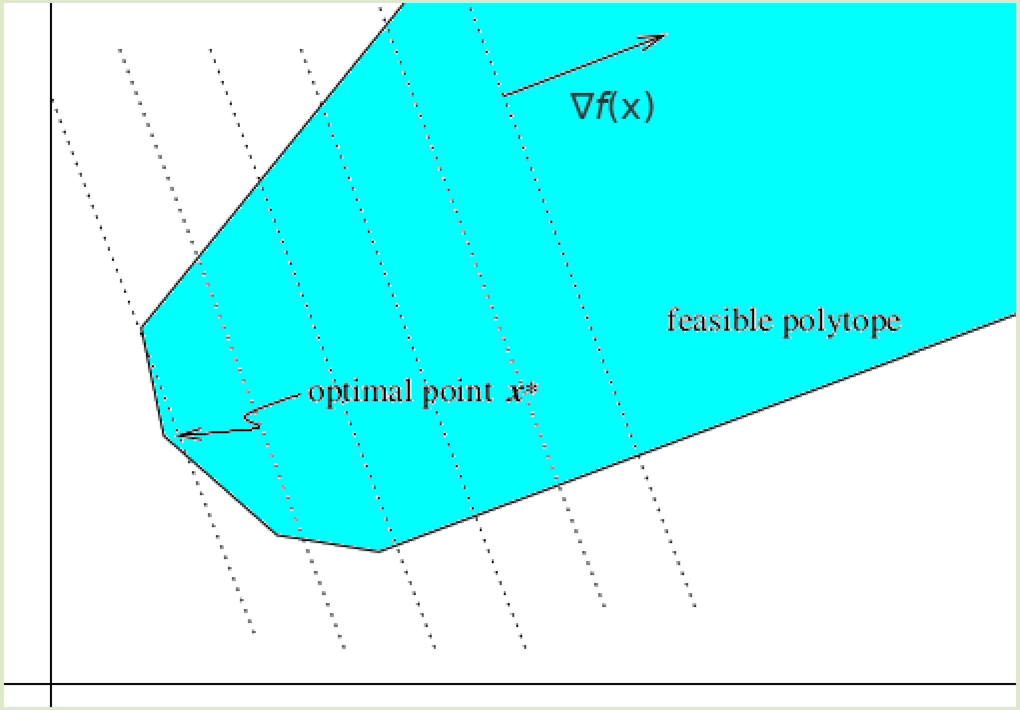
Numerical Optimization

Prepared by LimCK

Introduction

- Linear programming : linear objective function and linear constraints (Equalities + Inequalities).
- Example in figure
  - Dotted lines: contour of objective function.
  - Solution: vertex (point)
- In some other cases:
  - No solution
  - Single vertex
  - Infinite solutions

    (edge or face)



Introduction

Standard form:

$$\min \ c^T x \qquad \text{s.t } Ax = b, \ x \geq 0$$

where vectors *b, c, x* and matrix *A*:

$$c, x \in \mathbb{R}^n, \quad b \in \mathbb{R}^m, \quad A \in \mathbb{R}^{m \times n},$$

m – number of constraints, c_1, c_2, … c_m

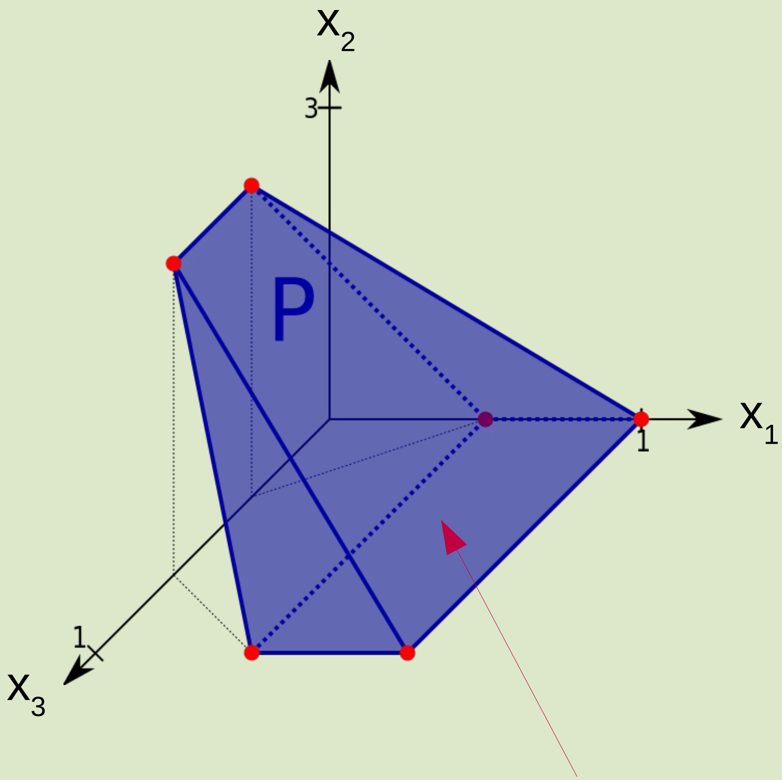
n – number of variables, x_1, x_2, … x_n

We assume:

- $m<n$, otherwise Ax=b has redundant row, or is infeasible, or defines a unique point
- *A* has full row rank (each of the rows of the matrix are linearly independent)

Graphical Explanation

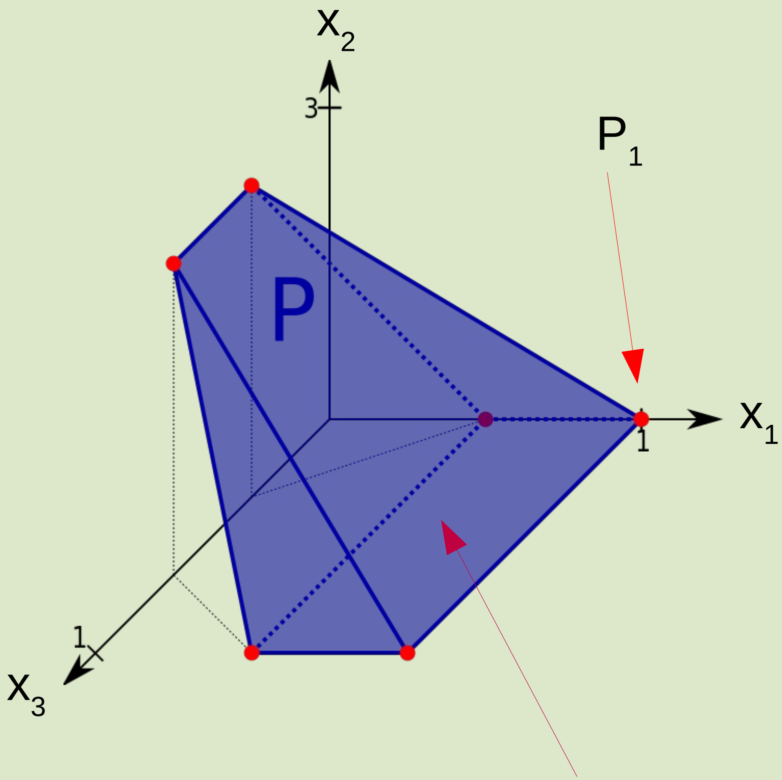
- The figure shows a polytope P that represent the feasible set of a particular linear programming problem.
- Simplex method seek for solutions at the vertices (red dots).
- Each of this vertex represents a possible solution (not necessarily optimal solution) to the problem – they are called Basic Feasible Solution.



3 dimensional polytope

Graphical Explanation

- At each basic feasible solution, a variable is called Basic Variable if it is not zero, otherwise, it is non basic variable.
- Example: P_1 is one of the basic feasible solution. At P_1, x_1=1, whereas x_2 and x_3 are 0. So x_1 is basic variable but x_2 and x_3 are non basic variables.



3 dimensional polytope

Simplex Method

Basic idea of how simple method works:

1 Restate the problem in standard form by adding slack, surplus or artificial variables.

2 Start at a vertex (basic feasible solution) and form a tableau.

3 Check whether that vertex represents optimal feasible solution

4 If not, move to another basic feasible solution by replacing exactly one basic variable. Form a new tableau so that the permutation of columns corresponding to basic variables forms identity matrix - “pivoting”

5 Repeat 3 and 4 until optimal solution is found.

1

Slack, Surplus and Artificial Variables

Some problems may not come in standard form. However we can transform:

- Maximize problem: max $c^T x$
  - we can work on: $-\min\ (-c)^T x$
- Inequality constraints like : subject to $Ax \leq b$
  - Convert to equality by introducing slack variable s

$$Ax \leq b \quad \Rightarrow \quad Ax + s = b,\ s \geq 0$$

Slack, Surplus and Artificial Variables

1

- Inequality constraints like : subject to $Ax \geq b$
  - Convert to equality by introducing surplus variable s

$$Ax \geq b \quad \Rightarrow \quad Ax - s = b,\ s \geq 0$$

Alternatively, introduce slack variable s:

$$Ax \geq b \quad \Rightarrow \quad -Ax \leq -b \quad \Rightarrow \quad -Ax + s = -b,\ s \geq 0$$

- x is not bounded to positive
  - splitting x into its nonnegative and nonpositive parts
    - $x = u - v$ where $u, v \geq 0$
- In some cases, we may need artificial variables. (discuss in example 2)

Example

$$\boxed{\min f(x) = -2x_1 + x_2 - 3x_3}$$

Subject to :

$$x_1 + x_2 + x_3 \leq 10$$

$$2x_2 - 3x_3 \geq -12$$

$$x_1, x_2, x_3 \geq 0$$

Subject to :

$$x_1 + x_2 + x_3 + s_1 = 10$$

$$-2x_2 + 3x_3 + s_2 = 12$$

$$x_1, x_2, x_3, s_1, s_2 \geq 0$$

Tableau form

- Collect the coefficients of the constraints, followed by the objective functions in an augmented matrix.
- The last column should be the values of the constraints, or 0 for the objective function.
- One should be able to see the basic feasible solution from the matrix.
- Should have *m* (number of constraints) basic variables – others are non basic variables.

Example

$$x_1 + x_2 + x_3 + s_1 \qquad = 10$$

$$-2x_2 + 3x_3 \qquad + s_2 = 12$$

$$\min f(x) = -2x_1 + x_2 - 3x_3$$

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & \\ \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 0 & 10 \\ 0 & -2 & 3 & 0 & 1 & 12 \\ \hline -2 & 1 & -3 & 0 & 0 & 0 \end{array}\right]$$

Basic variables

We reach the first basic feasible solution with s_1 and s_2 are basic variables. This is when $x_1=x_2=x_3= 0$, $s_1 = 10$ and $s_2 = 12$.

We write this point as (0, 0, 0, 10, 12) and f(x) = 0

3

Optimal condition

- Check whether there is a chance to improve the basic feasible point.
- For a min problem, the objective function can be further reduced if there is a variable with negative coefficient
- For a max problem, the objective function can be further reduced if there is a variable with positive coefficient
- Example:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 0 & 10 \\ 0 & -2 & 3 & 0 & 1 & 12 \\ \hline -2 & 1 & -3 & 0 & 0 & 0 \end{array}\right]$$

Can be further reduced

Pivoting

- Entering variable : the one that bringing most reduction (for min problem) or increment (for max problem) if it is included.

  In another words, it is the variable in objective function with most negative or positive coefficient, respectively.

- Exit variable: The one with stricter constraint if the entering variable is choose.

  Can be identify by dividing the constrain value to the coefficient of entering variable – choose the smallest non negative one.

- Use Gaussian elimination to get the permutation of identity matrix for basic variables.
- The new set of basic variables bring a new basic feasible solution.

3

Example

(2) Most negative, entering variable x_3

(1) Current basic Variables, s_1 and s_2

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 0 & 10 \\ 0 & -2 & 3 & 0 & 1 & 12 \\ \hline -2 & 1 & -3 & 0 & 0 & 0 \end{array}\right]$$

4

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & \\ \end{array}$$
$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 0 & 10 \\ 0 & -2 & 3 & 0 & 1 & 12 \\ \hline -2 & 1 & -3 & 0 & 0 & 0 \end{array}\right]$$

← 10/1 = 10

← 12/3 = 4

(3) Pivot this element: The second row, which corresponding to s_2 has smallest positive ratio. s_2 quit.

4

Example (continue)

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & \\ \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 0 & 10 \\ 0 & -2 & 3 & 0 & 1 & 12 \\ \hline -2 & 1 & -3 & 0 & 0 & 0 \end{array}\right]$$

Need to get this:

```
0 1
1 0
0 0
```

R1 ← R1 – R2 * 1/3

R2 ← R2 * 1/3

R3 ← R3 + R2

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & \\ \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & 1.67 & 0 & 1 & -0.33 & 6 \\ 0 & -0.67 & 1 & 0 & 0.33 & 4 \\ \hline -2 & -1 & 0 & 0 & 1 & 12 \end{array}\right]$$

Here, the basic variables are x_3 and s_1.

The new basic feasible solution is (0, 0, 4, 6, 0) and f(x) = -12

$$\begin{aligned} x_1 + x_2 + x_3 + s_1 \quad &= 10 \\ -2x_2 + 3x_3 \quad + s_2 &= 12 \\ \min f(x) = -2x_1 + x_2 - 3x_3 & \end{aligned}$$

3

Check whether it is optimal solution

- The optimal solution is not reached if we still can identify a exit variable from the matrix.
- Example:

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 1.67 & 0 & 1 & -0.33 & 6 \\ 0 & -0.67 & 1 & 0 & 0.33 & 4 \\ \hline -2 & -1 & 0 & 0 & 1 & 12 \end{array}\right] \end{array}$$

Still have negative coefficients, need more iterations

4

Continue pivoting:

$$\left[\begin{array}{ccccc|c} 1 & 1.67 & 0 & 1 & -0.33 & 6 \\ 0 & -0.67 & 1 & 0 & 0.33 & 4 \\ \hline -2 & -1 & 0 & 0 & 1 & 12 \end{array}\right] \quad \begin{array}{r} 6 \\ \text{#DIV/0!} \\ \end{array}$$

R1 ← R1

R2 ← R2

R3 ← R3 + 2*R1

$$\begin{array}{c} \left[\begin{array}{ccccc|c} 1 & 0.33 & 0 & 1 & -0.33 & 6 \\ 0 & -0.67 & 1 & 0 & 0.33 & 4 \\ \hline 0 & 3.67 & 0 & 2 & 0.33 & 24 \end{array}\right] \\ \begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & \end{array} \end{array}$$

5

Obtaining Optimal Solution

- When a matrix cannot be optimized any more, we reach the optimal solution.
- Obtain the values of variables, as well as function value from the matrix.
- Example:

(2) Values of variables for optimal solution: x_1 = 6 and x_3 = 4. x_2 = 0 since it is nonbasic.

$$
\begin{array}{cccccc}
x_1 & x_2 & x_3 & s_1 & s_2 & \\
\end{array}
$$

$$
\left[\begin{array}{ccccc|c}
1 & 0.33 & 0 & 1 & -0.33 & 6 \\
0 & -0.67 & 1 & 0 & 0.33 & 4 \\
\hline
0 & 3.67 & 0 & 2 & 0.33 & 24
\end{array}\right]
$$

(1) no more negative coefficient, optimization ends.

(3) negative of f. i.e., min f(x) = -24

$$\min f(x) = -2x_1 + x_2 - 3x_3 = -2(6) + 0 - 3(4) = -24$$

In some cases, we may need to add artificial variables

Artificial Variables

- In some cases, we may need to add artificial variables to constraints so that they help us in the process of pivoting.
- Sum of artificial variables must be minimum.
- Example: if $x_1, x_2, x_3 \geq 0$

  c_1: $2x_1 + x_3 \leq 10$ c_2: $2x_2 + 5x_2 \geq 6$ c_3: $2x_1 + 2x_2 + x_3 = 8$

- We can easily add slack variable for c_1, and use s_1 becomes basic variable since it only appears in c_1.
- For c_3, for the same purpose, we can add artificial variable a_1.
- For c_2, transform $2x_2 + 5x_2 \geq 6$ to $2x_2 + 5x_2 - s_2 = 6$ is good, but s_2 has negative coeficient. To solve this problem, we add a_2.

1

Example 2

Example:

$$\max \quad 3x_1 - 2x_2 + x_3$$

Subject to:

c_1: $2x_1 + x_3 \leq 10$

c_2: $2x_2 + 5x_3 \geq 6$

c_3: $2x_1 + 2x_2 + x_3 = 8$

$x_1 , x_2 , x_3 \geq 0$

Convert to:

$$-\min \quad -3x_1 + 2x_2 - x_3$$

Subject to:

c_1: $2x_1 + x_3 + s_1 = 10$

c_2: $2x_2 + 5x_3 - s_2 + a_1 = 6$

c_3: $2x_1 + 2x_2 + x_3 + a_2 = 8$

$x_1, x_2, x_3, s_1, s_2, a_1, a_2 \geq 0$

2

Example 2

Additional columns for artificial variables

Cost function

$2x_1 + x_3 + s_1 = 10$
$2x_2 + 5x_3 - s_2 + a_1 = 6$
$2x_1 + 2x_2 + x_3 + a_2 = 8$

$-3x_1 + 2x_2 - x_3$

x_1	x_2	x_3	s_1	s_2	a_1	a_2	
2	0	1	1	0	0	0	10
0	2	5	0	-1	1	0	6
2	2	1	0	0	0	1	8
-3	2	-1	0	0	0	0	0

2

Example 2

Additional columns for artificial variables

Cost function

$2x_1 + x_3 + s_1 = 10$

$2x_2 + 5x_3 - s_2 + a_1 = 6$

$2x_1 + 2x_2 + x_3 + a_2 = 8$

$-3x_1 + 2x_2 - x_3$

$-2x_1 - 4x_2 - 6x_3 + s_2$

x_1	x_2	x_3	s_1	s_2	a_1	a_2	
2	0	1	1	0	0	0	10
0	2	5	0	-1	1	0	6
2	2	1	0	0	0	1	8
-3	2	-1	0	0	0	0	0
-2	-4	-6	0	1	0	0	-14

- Negation of the sum of rows with artificial variables (i.e. $c_2 + c_3$).
- Instead of looking for negatives in cost function, we must look for most negatives one here, to drive the values of artificial variables to minimum

Example 2

<u>First iteration</u>

x_1	x_2	x_3	s_1	s_2	a_1	a_2		
2	0	**1**	1	0	0	0	10	10
0	2	**5**	0	-1	1	0	6	1.2
2	2	**1**	0	0	0	1	8	8
-3	2	**-1**	0	0	0	0	0	
-2	-4	**-6**	0	1	0	0	-14	

x_1	x_2	x_3	s_1	s_2	a_1	a_2	
2	-0.4	0	1	0.2	-0.2	0	8.8
0	0.4	1	0	-0.2	0.2	0	1.2
2	1.6	0	0	0.2	-0.2	1	6.8
-3	2.4	0	0	-0.2	0.2	0	1.2
-2	-1.6	0	0	-0.2	1.2	0	-6.8

3 4

Example 2

Second iteration

x_1	x_2	x_3	s_1	s_2	a_1	a_2		
2	-0.4	0	1	0.2	-0.2	0	8.8	4.4
0	0.4	1	0	-0.2	0.2	0	1.2	#DIV/0!
2	1.6	0	0	0.2	-0.2	1	6.8	3.4
-3	2.4	0	0	-0.2	0.2	0	1.2	
-2	-1.6	0	0	-0.2	1.2	0	-6.8	

x_1	x_2	x_3	s_1	s_2	a_1	a_2	
0	-2	0	1	0	0	-1	2
0	0.4	1	0	-0.2	0.2	0	1.2
1	0.8	0	0	0.1	-0.1	0.5	3.4
0	4.8	0	0	0.1	-0.1	1.5	11.4
0	0	0	0	0	1	1	0

5

Example 2

<u>Result</u>

x_1	x_2	x_3	s_1	s_2	a_1	a_2	
0	-2	0	1	0	0	-1	2
0	0.4	1	0	-0.2	0.2	0	1.2
1	0.8	0	0	0.1	-0.1	0.5	3.4
0	4.8	0	0	0.1	-0.1	1.5	11.4
0	0	0	0	0	1	1	0

1) No more negatives, so optimization complete.

2) Must be 0. If not, no feasible solution.

3) read solution from matrix:

$X_1 = 3.4$, $x_2 = 0$ and u, or $x_3 = 1.2$.

min f(x) = -11.4, so max f(x) = 11.4

<u>Note</u>: Further optimize is required if no more negative in the last row, but still have negatives in the row for cost function. Replace the column like what we did before.

Octave Programming

- Octave has built-in function `glpk` to solve linear programming problems in the form:

  min Cx subject to Ax = b, x≥0

  or similar.

- Syntax:

```
[XOPT, FMIN, ERRNUM, EXTRA]

= glpk (C, A, b, lb, ub, CTYPE, VARTYPE, SENSE, PARAM)
```

Octave Programming

- C : A **column array** containing the **objective function** coefficients.
- A : A **matrix** containing the **constraints coefficients**.
- B : A **column array** containing the right-hand side **value for each constraint** in the constraint matrix.
- LB : An **array** containing the **lower bound** on each of the variables. If lb is not supplied, the default lower bound for the variables is zero.
- UB : An **array** containing the **upper bound** on each of the variables. If ub is not supplied, the default upper bound is assumed to be infinite.

Octave Programming

- CTYPE : An array of characters containing the sense of each constraint in the constraint matrix. Each element of the array may be one of the following values.
  - U : Ax <= b
  - L : Ax >= b
  - S : Ax = b
- VARTYPE : A column array containing the types of the variables.
  - C : continuous variables
  - I : integer variables
- Sense : 1 for minimization (default) and -1 for maximization
- Refer to this link for more detail:
- https://octave.org/doc/v4.4.1/Linear-Programming.html

Example

- For example 1 in this slide:

$$\min f(x) = -2x_1 + x_2 - 3x_3$$

Subject to :

$$x_1 + x_2 + x_3 \leq 10$$

$$2x_2 - 3x_3 \geq -12$$

$$x_1, x_2, x_3 \geq 0$$

```
%% Example 1 in lecture notes
C = [-2; 1; -3];
A = [1 1 1 ; 0 2 -3];
b = [10; -12];
lb = [0; 0; 0];
ub=[];
cType = "UL";
varType = "CCC";
sense = 1;

%execute function
[xmin, fmin, status, extra] = glpk (C, ...
    A, b, lb, ub, cType, varType, sense);
```

```
xmin
fmin
```

output

```
xmin =

   6
   0
   4

fmin = -24
```

Any Question?

Exercise (Q1)

- Given that $f(x) = 4x_1 - 3x_2 - 2x_3 - x_4$. Find the max of the function subject to the following constraints:

$$x_1 + x_2 + x_3 + x_4 \leq 20$$

$$2x_2 - 2x_3 - x_4 \geq -10$$

$$x_1 , x_2 , x_3 , x_4 \geq 0$$

- Check your answer using Octave programming.

Answer:

$x_1 = 20$ $x_2 = x_3 = x_4 = 0$ max $f(x) = 80$

Exercise (Q2)

- Given that $f(x) = 2x_1 + 6x_2 + 10x_3$. Find the min of the function subject to the following constraints:

  $x_1 + x_2 - 4x_3 = 4$

  $4x_1 - 3x_2 + 2x_3 \geq -1$

  $3x_1 - x_2 + 6x_3 \leq 8$

  $x_1 , x_2 , x_3 \geq 0$

- Check your answer using Octave programming.

Answer:
$x_1 = 3$ $x_2 = 1$ $x_3 = 0$ min $f(x) = 12$

Exercise (Q3)

- Given that $f(x) = 4x_1 – 6x_2 – 5x_3 + 3x_4$. Find the min of the function subject to the following constraints:
- $-2x_1 + 5x_2 – 3x_3 + x_4 \leq 20$

  $5x_1 + 2x_3 + 3x_4 = 10$

  $x_1 , x_2 , x_3 \geq 0$
- Check your answer using Octave programming.

Answer:
$x_1 = x_4 = 0$ $x_2 = 7$ $x_3 = 5$ min $f(x) = -67$