CSC240 Winter 2024 Homework Assignment 4

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1. Give a well-structured informal proof by induction that, for each positive integer n and each sequence $r = \{r_i\}_{i=1}^n$ of n positive real numbers,

$$\prod_{i=1}^{n} \frac{1-r_i}{1+r_i} \geq \frac{1-\sum\limits_{i=1}^{n} r_i}{1+\sum\limits_{i=1}^{n} r_i}.$$

Proof.
Define the predicate for all $n \in \mathbb{Z}^+$,

$$P(n) = \text{“}\forall r \in (\mathbb{R}^+)^n. \prod_{i=1}^{n} \frac{1-r_i}{1+r_i} \geq \frac{1-\sum\limits_{i=1}^{n} r_i}{1+\sum\limits_{i=1}^{n} r_i}\text{”}$$

We will show $\forall n \in \mathbb{Z}^+.P(n)$ by induction on n.

Base case: $n = 1$.
Let $r = \{r_1\} \in (\mathbb{R}^+)^1$ be arbitrary. Observe that LHS equals to RHS equals to $\dfrac{1-r_1}{1+r_1}$. Hence, $P(0)$ is true.

Induction Step:
Let $n \in \mathbb{Z}^+$ be arbitrary.
Assume $p(n)$.
Let $r = \{r_i\}_{i=1}^{n+1} \in (\mathbb{R}^+)^{n+1}$ be arbitrary.

$$\prod_{i=1}^{n+1} \frac{1-r_i}{1+r_i} = \frac{1-r_{n+1}}{1+r_{n+1}} \cdot \prod_{i=1}^{n} \frac{1-r_i}{1+r_i} \geq \frac{1-r_{n+1}}{1+r_{n+1}} \cdot \frac{1-\sum\limits_{i=1}^{n} r_i}{1+\sum\limits_{i=1}^{n} r_i} \text{ (by induction hypothesis)}$$

$$= \frac{1-r_{n+1}-\sum\limits_{i=1}^{n} r_i + r_{n+1}\sum\limits_{i=1}^{n} r_i}{1+r_{n+1}+\sum\limits_{i=1}^{n} r_i + r_{n+1}\sum\limits_{i=1}^{n} r_i} = \frac{1-\sum\limits_{i=1}^{n+1} r_i + r_{n+1}\sum\limits_{i=1}^{n} r_i}{1+\sum\limits_{i=1}^{n+1} r_i + r_{n+1}\sum\limits_{i=1}^{n} r_i} \text{ (we refer this as LHS)}$$

Define $a = 1 - \sum\limits_{i=1}^{n+1} r_i, b = 1 + \sum\limits_{i=1}^{n+1} r_i, x = r_{n+1}\sum\limits_{i=1}^{n} r_i$.
Since r is a sequence of positive reals, and positive reals are closed under addition and multiplication, we have $0 < b - a$, $x > 0$, and $b > 0$.
LHS$= \dfrac{a+x}{b+x}$. We will show that it is greater or equal to $\dfrac{a}{b}$.

$$\frac{a+x}{b+x} - \frac{a}{b} = \frac{b(a+x)-a(b+x)}{(b+x)b} = \frac{ab+bx-ab-ax}{(b+x)b} = \frac{x(b-a)}{(b+x)b}$$

Since $x, (b-a), (b+x), b$ are all greater than 0, $\dfrac{a+x}{b+x} - \dfrac{a}{b} \geq 0$.

Rearranging the second term to the left, we have $\dfrac{a+x}{b+x} \geq \dfrac{a}{b}$

Hence, $\displaystyle\prod_{i=1}^{n+1} \frac{1-r_i}{1+r_i} \geq \frac{1-\sum_{i=1}^{n+1} r_i}{1+\sum_{i=1}^{n+1} r_i}$.

By generalization $\big(r \in (\mathbb{R}^+)^{n+1}$ is arbitrary$\big)$, $p(n+1)$ is true.

Therefore, we showed $p(n)$ IMPLIES $p(n+1)$.

Since $n \in \mathbb{Z}^+$ is arbitrary, we proved $\forall n \in \mathbb{Z}^+.[p(n)$ IMPLIES $p(n+1)]$.
Combine with the fact that $p(1)$ is true, by the principle of induction, $\forall n \in \mathbb{Z}^+.p(n)$.

2. An n-bit *gradually changing sequence* consists of all 2^n length n bit strings such that

   - any two consecutive strings in the sequence differ in exactly one position and
   - the first string and the last string differ in exactly one position.

   For instance, the following is a 3-bit gradually changing sequence:

   $$000, 100, 101, 111, 110, 010, 011, 001.$$

   Note that this sequence is not the unique. The following is another example of a 3-bit gradually changing sequence:

   $$100, 101, 111, 110, 010, 011, 001, 000.$$

   Give a well-structured informal proof by induction that, for all $n \in \mathbb{Z}^+$, there exists an n-bit gradually changing sequence.

   Answer:
   Before we begin our proof, we first define some terms and notations:
   Let $\mathcal{B}$ denotes the set of all sequence of binary strings.
   Let $S_n(C) : \mathcal{B} \to \{T, F\}$ be a predicate returns True iff $C \in \mathcal{B}$ is a n-bit gradually changing sequence.
   If A is a sequence of some binary strings, we let A_i denote the i^{th} string, starting from 1 and up to its length. Furthermore, if $A_i, A_j \in A$, we use $A_i + A_j$ to denote string concatenation.
   Define for all $n \in \mathbb{Z}^+$, $P(n) =$ "$\exists C \in \mathcal{B}.S_n(C)$.". $P(n)$ is equivalent to say there exists an n-bit gradually changing sequence.
   We will prove our desired claim by induction.

   $$\forall n \in \mathbb{Z}^+.P(n).$$

   Proof.

   Base case:
   Let $C = 0, 1$. $C \in \mathcal{B}$ is a 1-bit gradually changing sequence, the only two consecutive strings

(as well as the first and last) differs in exactly one (the only one) position.
Hence, $P(1)$ is true.

Inductive step:

Let $n \in \mathbb{Z}^+$ be arbitrary.

Assume $P(n)$.

By instantiation of induction hypothesis, let $A \in \mathcal{B}$ be such that $S_n(A)$.

By the definition of n-bit gradually changing sequence, we know that (1) $|A| = 2^n$;
(2) for all $i \in \mathbb{N}$ AND $1 \leq i \leq 2^n - 1$, A_i and A_{i+1} differs in exactly one position; and
(3) A_0 and A_{2^n} differs in exactly one position.

We will construct another binary sequence $B \in \mathcal{B}$ such that $|B| = 2^{n+1}$ and for $1 \leq i \leq 2^{n+1}$:

if i rem $4 = 1, B_i = A_{\frac{i+1}{2}} +$ "0"
if i rem $4 = 2, B_i = A_{\frac{i}{2}} +$ "1"
if i rem $4 = 3, B_i = A_{\frac{i+1}{2}} +$ "1"
if i rem $4 = 0, B_i = A_{\frac{i}{2}} +$ "0"

We will then show that $S_{n+1}(B)$ is true. In plain words, we constructed B by duplicating each elements in A twice while maintaining the original order. Then add 0 to the first string, add 1 to the second one, (if $n \geq 1$) add 1 to the third one, and then add 0 to the forth. This sequence therefore contains all possible 2^{n+1} $(n+1)$-bit binary string.

We will then show every consecutive strings in the sequence differ in exactly one position.

Let $i \in \mathbb{N}$ AND $1 \leq i \leq 2^{n+1} - 1$ be arbitrary.

Case 1: i rem $4 = 1$. ($i + 1$ rem $4 = 2$)

$B_i = A_{\frac{i+1}{2}} +$ "0", $B_{i+1} = A_{\frac{i+1}{2}} +$ "1". If we count from left to right, B_i and B_{i+1} only differs in the last position (B_i, B_{i+1} have $0, 1$, respectively). Rest are identical.

Case 2: i rem $4 = 2$. ($i + 1$ rem $4 = 3$)

$B_i = A_{\frac{i}{2}} +$ "1", $B_{i+1} = A_{\frac{i}{2}+1} +$ "1". Since $A_{\frac{i}{2}}$ and $A_{\frac{i}{2}+1}$ differs in exactly one position (by (2)) and their (B) last positions are all 1, B_i and B_{i+1} would differs in exactly once too (in the same position where A differs, from left to right).

Case 3: i rem $4 = 3$. ($i + 1$ rem $4 = 0$)

This is very similar to case 1, only differs in the last position.

Case 4: i rem $4 = 0$. ($i + 1$ rem $4 = 1$)

This is very similar to case 2, B differs in the same position where A does.

Since we have exhausted all cases, we conclude that every consecutive strings in B differs in exactly one positions.

Finally, since the first string have index $i = 1$(i rem $4 = 1$) and the last string have index with $i = 2^{n+1}$. When $n = 0$, they are consecutive strings. When $n > 0$, i will be a multiple of 4. So i rem $4 = 0$. This implies B_1, B_{2^n} are the result of adding the same string "0" to the rightmost position of A_1 and A_{2^n}, which differs in exactly one position.

Hence, $S_{n+1}(B)$ is true. By proof of construction, we showed $\exists C \in \mathcal{B}.S_{n+1}(C)$.

Thus, $P(n+1)$ is true.

Therefore, $P(n)$ IMPLIES $P(n+1)$.

Since $n \in \mathbb{Z}^+$ is arbitrary, we proved $\forall n \in \mathbb{Z}^+.[P(n)$ IMPLIES $P(n+1)]$.
As we previously showed $P(1)$, by the principle of induction, $\forall n \in \mathbb{Z}^+.P(n)$.