CSC240 Winter 2024 Homework Assignment 10

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For any language $S \subseteq \Sigma^*$, define $C(S) = \{x \in \Sigma^* \mid \exists w \in \Sigma^*.\exists y \in \Sigma^*.(xwxy \in S)\}$.
For example, if $S = \{ababc, aabaab\}$, then $C(S) = \{\lambda, a, aa, ab, aab\}$.

1. Describe the language $S = L(((01)^* + 1^*)^*) = \{z \in \{0,1\}^* \mid \dots\}$ by replacing the ... with at most 10 words. (z counts as one word.) Briefly justify your answer.

   Solution. $S = \{z \in \{0,1\}^* \mid \text{every 0 in } z \text{ must be followed by some 1(s)}\}$
   First observe that $(01)^*$ is the set of all strings that's either empty or concatenation of multiple 01. $((01)^* + 1^*)^*$ means either some multiple of 01 can be chosen or some multiple of 1 can be chosen, at each time. Thus, we can pick 01 from $(01)^*$ and any number of 1 from 1^* to obtain any non-zero number of 1s after a any zero.

2. Describe the language $T = L(\overline{\bar{\phi} \cdot 00 \cdot \bar{\phi}}) = \{x \in \{0,1\}^* \mid \dots\}$ by replacing the ... with at most 10 words. (x counts as one word.) Briefly justify your answer.

   Solution. $T = \{z \in \{0,1\}^* \mid z \text{ has no consecutive zeros}\}$
   $L(\bar{\phi})$ is the set of all strings. Thus, $L(\bar{\phi} \cdot 00 \cdot \bar{\phi})$ is the set of all strings having (at least) a pair of consecutive zeros. Hence T (the complement) is those without consecutive zeros.

3. Explain why $C(S) = T$.

   Solution. Let $t \in T$ be arbitrary. By def of T, it has no consecutive zeros ($00 \notin T$). Take $w = y = 1$, the string $twty \in S$ since [1] $00 \in T$ guarantees every two zeros have 1s in between [2] the last letter $y = 1$ guarantees the last zero is as well followed by 1. $T \subseteq C(S)$.

   Let $x \in C(S)$ be arbitrary. By definition, there are some $w, y \in \Sigma^*$ such that $xwxy \in S$. If $00 \in x$, then $00 \in xwxy$ is immediate and $xwxy$ cannot be in S. Hence, by contradiction, we have $00 \notin x$. This means $x \in T$ so $C(S) \subseteq T$.

   As both set are contained in each other, $C(S) = T$.

4. Given any deterministic finite automaton $M = (Q, \Sigma, \delta, q_0, F)$, construct a finite automaton $M' = (Q', \Sigma, \delta', q_0', F')$ such that $L(M') = C(L(M))$.

   Solution.

   For each $q_i \in Q$, define $SLeft(q_i) = (Q_i', \Sigma, \delta_i', z_i', F_i'), SRight(q_i) = (Q_i'', \Sigma, \delta_i'', z_i'', F_i'')$ be two deterministic finite automation where
   $Q_i' = \{q \in Q | \exists w \in \Sigma^*.\ q_i = \delta(q, w)\},\ F_i' = Q_i',\ \delta_i' = \delta\big|_{Q_i'}\ ,\ z_i' = q_0$ and

   $Q_i'' = \{q \in Q | \exists y \in \Sigma^*.\ q_i = \delta(q, y)\}\ ,\ F_i'' = Q_i'' \cap F,\ \delta_i'' = \delta\big|_{Q_i''}\ ,\ z_i'' = q_i$

   Further define $M_{q_i} = (Q_i, \Sigma, \delta_i, z_i, F_i) = (Q_i' \times Q_i'', \Sigma, \delta_i, (z_i', z_i''), F_i' \times F_i'')$ where δ_i is defined as $\delta_i((p_1, p_2), a) = (\delta_i'(p_1, a), \delta_i''(p_2, a))$ for all $p_1 \in Q_i', p_2 \in Q_i'', a \in \Sigma^*$

Finally, take

$$Q' = \left(\bigcup_{q_i \in Q} Q_i \right) \cup \{p_0\}, \quad F' = \bigcup_{q_i \in Q} F_i, \quad q'_0 = p_0$$

where $p_0 \notin \bigcup\limits_{q_i \in Q} Q_i$ and

$\delta'(p_0, \lambda) = \bigcup\limits_{q_i \in Q} \{z_i\}, \delta'(p_0, a) = \phi$ for $a \in \Sigma, \delta'(q_i, a) = \{\delta_i(q_i, a)\}, \delta'(q_i, \lambda) = \phi$ for $q_i \in Q_i, a \in \Sigma$. We claim that such $M' = (Q', \Sigma, \delta', q'_0, F')$ satisfy $L(M') = C(L(M))$.

5. Briefly describe how M' works.

   Solution.

   The goal of the construction is, for all strings that are in a language, we consider all half partitions of them, and see what prefixes are in common for the first half and the second half. For example, we can see $abccabc$ as the partitions $P_1 = abcc, P_2 = abc$. The common prefixes are $\{abc, ab, a\}$. To achieve this, for every state $q_i \in Q$, the construction of Q'_i, Q''_i represents all possible prefixes of strings up to, or starting at, q_i, respectively. In addition, for the second half, we only want those strings which can be accepted. Thus, the $SLeft, SRight$ are machines that filters the w, y part, respectively. The intersection of the language of these two machines represent $C(L(M))$.
   In addition, we would like to consider all possible states q_i. Thus, we take the union of the constructions for each q_i to obtain a machine that accepts strings that are in at least one of the construction.

6. Prove that $L(M') = C(L(M))$.

   Proof.

   Let $x \in \Sigma^*$ be arbitrary and assume $x \in L(M')$.
   Since M' is an nondeterministic finite automata, we have $\delta'(p_0, x) \cap F' \neq \phi$. Thus, there must be some $f \in \delta'(p_0, x) \cap F'$ .Consider such f. From $\delta'(p_0, \lambda) = \bigcup\limits_{q_i \in Q} \{z_i\}, \delta'(p_0, a) = \phi$, we can see such path x from p_0 to f must go from some lambda transition to a z_i first, where $z_i = (z'_i, z''_i) = (q_0, q_i) \in Q_i$ and $f \in \delta'(z_i, x)$. Since $\delta'(s_i, a) = \{\delta_i(s_i, a)\}$ for all $s_i \in Q_i, a \in \Sigma$, by instantiation, we have $\delta'(z_i, x) = \{\delta_i(z_i, x)\} = \{(\delta'_i(z'_i, x), \delta''_i(z''_i, x))\}$.

   Since $f \in F'$ and $f \in \delta'(z_i, x)$, we have $f \in F' \cap Q_i = F_i = F'_i \times F''_i$. Hence, $f = (f_1, f_2)$ for some $f_1 \in F'_i, f_2 \in F''_i$. Further by the definition of F'_i, F''_i, $f_1 \in Q'_i$ and $f_2 \in Q''_i \cap F$.

   From the result in the end of second paragraph, $f = (f_1, f_2) \in \delta'(z_i, x) = \{(\delta'_i(z'_i, x), \delta''_i(z''_i, x))\}$. Immediately, we have $f_1 = \delta'_i(z'_i, x), f_2 = \delta''_i(z''_i, x)$. Substitute $z_i = (z'_i, z''_i) = (q_0, q_i)$, there is a path labeled by x from q_0 to f_1, and a path labeled by x from q_i to f_2.

   Further more, by $f_1 \in Q'_i$, we have $\exists w \in \Sigma^*.\ q_i = \delta(f_1, w)$. By $f_2 \in Q''_i, \exists y \in \Sigma^*.\ q_i = \delta(f_2, y)$. By definition of transition function of deterministic finite automata, $\delta(q_0, xwxy) = \delta(\delta(q_0, x), wxy) = \delta(f_1, wxy) = \delta(\delta(f_1, w), xy) = \delta(q_i, xy) = \delta(f_2, y) = q_i$. Since $q_i \in Q''_i \cap F, \delta(q_0, xwxy) = q_i \in F$. In other word, $xwxy \in L(M)$.

Hence by construction, $\exists w \in \Sigma^*.\exists y \in \Sigma^*.(xwxy \in L(M))$. By definition of C, we have $x \in C(L(M))$

Since x is arbitrary, $\forall x \in \Sigma^*.[x \in L(M')$ IMPLIES $x \in C(L(M))]$. $L(M') \subseteq C(L(M))$

Let $x \in \Sigma^*$ be arbitrary and assume $x \in C(L(M))$.
By definition of C, $\exists w \in \Sigma^*.\exists y \in \Sigma^*.(xwxy \in L(M))$. Consider such w, y.

Since there if a path labeled by $xwxy$ from q_0 to a $f = \delta(p_0, xwxy) \in F$, we can consider the intermediate $q_i \in Q$ where xw leads q_0 to q_i and xw leads q_i to f (note: q_i can be q_0 or f, but we will discuss them together and there is no need to separate cases).

Formally, $q_i = \delta(q_0, xw)$ and $f = \delta(q_i, xy)$. Further, consider $s_1 = \delta(q_0, x), s_2 = \delta(q_i, x)$ be two intermediate states too. By our definition in question 4, we have $s_1 \in Q_i', s_2 \in Q_i''$.

$\delta'((z_i', z_i''), x) = \delta'((q_0, q_i), x) = (\delta_i'(q_0, x), \delta_i''(q_i, x)) = (\delta(q_0, x), \delta(q_i, x)) = (s_1, s_2) \in F_i' \times F_i''$. Since $F_i = F_i' \times F_i''$ and $F_i \in F'$, we have $\delta'((z_i', z_i''), x) \cap F' \neq \phi$. Recall that $q_0' = p_0$ and $z_i = (z_i', z_i'') \in \delta'(p_0, \lambda)$, we also have $\delta'((z_i', z_i''), x) \subseteq \delta'(p_0, \lambda \cdot x)$. Thus, if there is an $f \in F' and \delta'((z_i', z_i''), x)$ (by $\delta'((z_i', z_i''), x) \cap F' \neq \phi$), such f is also in $\delta'(p_0, x)$.

Hence, $\delta'(p_0, x) \cap F' \neq \phi$ In other words, $x \in L(M')$.

Since x is arbitrary, $\forall x \in \Sigma^*.[x \in C(L(M))$ IMPLIES $x \in L(M')]$. $C(L(M)) \subseteq L(M')$ As both $C(L(M)) \subseteq L(M')$ and $L(M') \subseteq C(L(M))$ are true, we conclude $L(M') = C(L(M))$