CSC 240 Winter 2024 Homework Assignment 1

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1. Let U be the set of all people and let $\mathcal{P}(U)$ be the set of all subsets of U.
   Let $In : U \times \mathcal{P}(U) \to \{\text{T,F}\}$ be the predicate such that $In(x, G) = \text{T}$ when person x is in the set of people G.
   Let $size : \mathcal{P}(U) \to \mathbb{N}$ be the function such that $size(G)$ is the number of people in the set G.
   Let $K : U \times U \to \{\text{T,F}\}$ be the predicate such that $K(a, b) = \text{T}$ when person a and person b know each other.
   You may assume that K is symmetric, i.e. $\forall a \in U.\forall b \in U.[K(a, b) \text{ IFF } K(b, a)]$.

   (a) Consider the following predicate about a set of people $G \in \mathcal{P}(U)$:

   $$\exists H \in \mathcal{P}(U).\forall x \in H.\forall y \in H.$$
   $$[(size(H) = 40) \text{ AND } (\text{NOT}(K(x, y)) \text{ OR } (x = y)) \text{ AND } In(x, G)].$$

   Express this predicate using at most 15 English words. The letter G counts as one word. Justify your answer.

   Solution

   G has a sub-group of 40 people where nobody knows each other.

   Justification: Firstly, the three quantifiers specified that H is a subset of U, and x, y are arbitrary elements of H. Secondly, the last atomic predicate $In(x, G)$ implies H is a subset of the group G. In addition, the first atomic predicate $(size(H) = 40)$ stated that this subgroup is of 40 people. Lastly, the middle part says unless x and y refers to the same person, person x does not know person y. This is equivalent to say nobody knows each other.

   (b) Consider the following sentence in predicate logic.

   $$\exists n \in \mathbb{N}.\forall G \in \mathcal{P}(U).\exists H \in \mathcal{P}(U).\forall x \in H.\forall y \in H.$$
   $$[((size(H) = 40) \text{ OR } (size(G) < n)) \text{ AND}$$
   $$(\text{NOT}(K(x, y)) \text{ OR } (x = y) \text{ OR } (size(G) < n)) \text{ AND}$$
   $$(In(x, G) \text{ OR } (size(G) < n))].$$

Express this sentence using at most 20 English words. Justify your answer.

Solution

Every non-finite group of people has a sub-group of 40 people where nobody knows each other.

Justification: First observe the quantifiers to conclude that n is a natural number, G, H are subsets of U, and x, y are arbitrary elements of H. We can then use the distributive law of conjunction to simplify the body of this predicate sentence. It is equivalent with saying either the size of group G is finite (size bounded by some natural number n), or it must satisfy the predicate in part (a). Putting it together, it means all non-finite group must satisfy the property in (a).

2. A graph consists of a set of vertices V and a set of edges between pairs of vertices. The set of edges can be represented by a symmetric, irreflexive predicate $e : V \times V \to \{T, F\}$, where $e(a, b) = T$ denotes that there is an edge between vertex a and b. Symmetric means $\forall a \in V. \forall b \in V.(e(a, b) \text{ IFF } e(b, a))$ and irreflexive means $\forall a \in V.\text{NOT}(e(a, a)))$.

   We say that two vertices a and b are *adjacent* if there is an edge between them. The *degree of a vertex* is the number of vertices that are adjacent to it. The *maximum degree of a graph* is the maximum of the degrees of its vertices.

   A *colouring* of the graph is a function from V to a set of colours. We will use the natural numbers as the set of colours. Then $\mathbb{N}^V$ denotes the set of all colourings of V.

   A *proper colouring* is a *colouring* such that no two adjacent vertices have the same colour.

   Using predicate logic, express the following statement about a graph with vertex set V and whose set of edges is represented by the predicate e:
   "If the graph has maximum degree two, then it can be properly coloured using at most three colours."
   You cannot use any constants. The only predicate (besides e) you can use is binary equality, $=$.

   To get full marks, you must use IMPLIES at most once and NOT at most four times.

   Use parentheses and brackets when necessary to avoid ambiguity.
   Justify your answer by briefly explaining what its various parts mean.

Solution

$(\forall v_1 \in V. \exists a \in V. \forall z \in V.[e(v_1, z) \text{ IMPLIES } z = a])$
OR

$(\exists v_2 \in V.\exists a \in V.\exists b \in V.\exists c \in V.[(\text{NOT}(a = b \text{ OR } a = c \text{ OR } b = c)) \text{ AND } e(v_2, a) \text{ AND } e(v_2, b)$
$\text{AND } e(v_2, c)]$
OR
$(\exists f \in \mathbb{N}^V.\exists i_1 \in \mathbb{N}.\exists i_2 \in \mathbb{N}.\exists i_3 \in \mathbb{N}.[(\forall v_3 \in V.[f(v_3) = i_1 \text{ OR } f(v_3) = i_2 \text{ OR } f(v_3) =$
$i_3]) \text{ AND NOT}(\exists v_4 \in V.\exists v_5 \in V.[e(v_4, v_5) \text{ AND } k(v_4) = k(v_5)]]))$

Explanation:
The first part of this predicate is before the first OR, started with $\forall v_1 \in V$. It says if z is an adjacent vertices for an arbitrary vertices V_1, then it most be the same one. This part means that every vertices has either no adjacent vertex or one. In other word, part 1 denotes the maximum degree is less than 2.

The second part of this predicate is after the first OR, started with $\exists v_2 \in V$. This part says there is a vertex with at least three distinct adjacent vertices a, b, c. In other word, part 2 says the maximum degree is at least 3.

Part 1 and 2 together says the maximum degree is not two.

Part 3 begins with $\exists f \in \mathbb{N}^V$. It first says every output falls into one of three colors. Since it requires at most three colours, the three possible output does not need to be distinct. In addition, there are no adjacent vertices with the same colors (output number under f). This is exactly what we meant when we say f is a proper colouring.

Since (if P then Q) is equivalent to (not P or Q), we correctly symbolized this if-then statement with only one IMPLIES and two NOTs.