CSC240 Winter 2024 Homework Assignment 5

due February 15, 2024

My name and student number: Haoyun (Bill) Xi 1009992019
The list of people with whom I discussed this homework assignment: Anna Li

1. Give a well-structured informal proof by **strong induction** that for any propositional formula f built from propositional variables using only conjunction, disjunction, and negation, there exists a propositional formula g in negation normal form that is logically equivalent to f and has the same number of occurrences of propositional variables as in f.

   Define $M \subseteq F_{PV}$ (PV is the set of propositional variables) to be the set of propositional formulas built from propositional variable using only conjunction, disjunction, and negation using recursion:
   Base Case.
   If $P \in PV$, then $P \in M$.
   Constructor Case.
   If $f_1, f_2 \in M, \star \in \{$ AND , OR $\}$, then $(f_1 \star f_2) \in M$ AND $\text{NOT}(f_1) \in M$.

   Further define for all $n \in \mathbb{Z}^+, P \in M$:
   $N_v(P) : M \to \mathbb{Z}^+$ denote the number of occurrences of variables in P.
   $S_n \subseteq M = \{P \in M : N_v(P) = n\}$
   $Q(n) =$ "$\forall P \in S_n.\exists f \in NNF.[f \text{ IFF } P \text{ AND } N_v(f) = N_v(p)]$."

   $Q(n)$ says all propositional formulas in M of n-occurrences has a corresponding equivalent formula in negation normal form of the same occurrences.
   We will prove $\forall n \in \mathbb{Z}^+.Q(n)$. by strong induction.
   Proof.

   Assume $\forall i \in \mathbb{Z}^+.\forall j \in \mathbb{Z}^+.[(j < i) \text{ IMPLIES } Q(j)]$.

   Assume $i = 1$. (Base Case).

   Let $P \in S_1$ be arbitrary.
   Since $N_v(P) = 1$. It must be the case that either $P = f_1$ or $P = \text{NOT}(f_1)$. Both case P is a literal. Pick $f = P \in NNF$.

   Hence, $Q(1)$ is true.
   Assume $i > 1$ and $i \in \mathbb{Z}^+$.

   Let $P \in S_i$ be arbitrary.
   Since $N_v(P) = i > 1$, directly by the definition of M, its either the case where there are some $g_1, g_2 \in M$ and $\star \in \{$ AND , OR $\}$ such that P can be simplified down to $(g_1 \star g_2)$ or $\text{NOT}(g_1 \star g_2)$, and that $N_v(g_1) + N_v(g_2) = N_v(P)$. [This is because if P has even or 0 number of NOTs outside of some $(p_1 \star p_2)$, they would cancel out and become case 1. If there are odd number of NOTs, it is case 2.]
   Furthermore, we know the number of occurrences of g_1, g_2 are non-zero. Thus if $N_v(g_1) = j_1, N_v(g_2) = j_2$, we have $j_1 < N_v(P) = i$ and $j_2 < N_v(P) = i$.

By specialization of the induction hypothesis, we have $Q(j_1)$ and $Q(j_2)$.
By specialization of Q and instantiation, we know there are some $f_1, f_2 \in NNF$ such that f_1 IFF g_1, and f_2 IFF g_2, and $N_v(f_1) = N_v(g_1)$, and $N_v(f_2) = N_v(g_2)$.

Assume $P = g_1 \star g_2$. (case 1).
Pick $f = f_1 \star f_2$. $N_v(f) = N_v(f_1) + N_v(f_2) = N_v(g_1) + N_v(g_2) = N_v(P)$. It is also clear that f IFF P by law of substitution.
Hence by proof of construction, we have $\exists f \in NNF.[f$ IFF P AND $N_v(f) = N_v(p)]$.

$P = g_1 \star g_2$ IMPLIES $\exists f \in NNF.[f$ IFF P AND $N_v(f) = N_v(p)]$

Assume $P = \text{NOT}(g_1 \star g_2)$. (case 2).
This case, P IFF $\text{NOT}(f_1 \star f_2)$ by law of substitution.
$f_1, f_2 \in NNF$ implies $\text{NOT}(f_1), \text{NOT}(f_2) \in M$ because $NNF \subseteq M$.
Since adding a NOT does not affect the number of occurrences, we can use specialization of $Q(j_1), Q(j_2)$ on $\text{NOT}(f_1), \text{NOT}(f_2)$.
By instantiation, there are some $h_1, h_2 \in NNF$ such that h_1 IFF $\text{NOT}(f_1)$ and h_2 IFF $\text{NOT}(f_2)$ and $N_v(h_1) = N_v(\text{NOT}(f_1))$ and $N_v(h_2) = N_v(\text{NOT}(f_2))$.

Let $\cdot$ be the only element in $\{$ AND , OR $\}/\{\star\}$ (the connective other than $\star$).
Pick $f = h_1 \cdot h_2 \in NNF$.
Since $\text{NOT}(g_1 \star g_2)$

IFF $\text{NOT}(g_1) \cdot \text{NOT}(g_2)$ by de morgan's law
IFF $\text{NOT}(f_1) \cdot \text{NOT}(f_2)$ by substitution
IFF $h_1 \cdot h_2$ IFF f by substitution and double negation.

Together with $N_v(f) = N_v(h_1) + N_v(h_2) = N_v(f_1) + N_v(f_2) = N_v(g_1) + N_v(g_2) = N_v(P)$, we have $\exists f \in NNF.[f$ IFF P AND $N_v(f) = N_v(p)]$.

Hence, $P = \text{NOT}(g_1 \star g_2)$ IMPLIES $\exists f \in NNF.[f$ IFF P AND $N_v(f) = N_v(p)]$.
We have exhausted all cases, hence $\exists f \in NNF.[f$ IFF P AND $N_v(f) = N_v(p)]$.

Since $P \in S_i$ is arbitrary, we have $Q(i)$

For $i = 1$ and arbitrary $(i \in \mathbb{Z}^+, i > 1)$, assuming our induction hypothesis, $Q(i)$ is true.

Therefore, by the principle of strong induction, $\forall n \in \mathbb{Z}^+.Q(n)$

2. Consider the following game between Alice and Bob. A *position* in the game is defined by a triple (X, Y, t), where X and Y are disjoint sets of truth assignments to a set of n propositional variables $\{P_i \mid 1 \leq i \leq n\}$ and $t \in \mathbb{Z}^+$. The initial position of the game is decided in advance by a referee.

   The game proceed by rounds. Each round begins with a move by Alice followed by a response from Bob. Starting from the current position (X, Y, t) of a round, Alice first chooses one of X or Y (say she chooses the set X). She then picks subsets X' and X'' such that $X = X' \cup X''$ and integers $t', t'' \in \mathbb{Z}^+$ such that $t = t' + t''$. She hands the triples (X', Y, t') and (X'', Y, t'') to Bob, who respond to this move by choosing either (X', Y, t') or (X'', Y, t'') as the new position for the next round.

The game ends when it reaches a position $(X^*, Y^*, 1)$, where $X^* \subseteq X$ and $Y^* \subseteq Y$. Alice wins the game if there exists some $1 \leq i \leq n$ such that $\tau(P_i) \neq \tau'(P_i)$ for all $\tau \in X^*$ and $\tau' \in Y^*$. Otherwise, Bob wins the game.

We say that a position (X, Y, t) is *promising* if Alice has a winning strategy starting from position (X, Y, t). A winning strategy means that Alice can choose her moves (which may depends on the responses from Bob) in a way such that she is guaranteed to win the game.

(a) Give a recursive definition for the set of all promising positions.
Let PP be the set of all promising positions defined below using recursion
Base Case:
Let $\mathcal{T}$ be sets of all sets of truth assignments to $\{P_i \mid 1 \leq i \leq n\}$.
If $X, Y \in \mathcal{T}$ are disjoints and $\exists i \in \mathbb{Z}^+.[1 \leq i \leq n$ AND $(\forall \tau \in X.\forall \tau' \in Y.[\tau(P_i) \neq \tau'(P_i)])]$, then $(X, Y, 1) \in PP$.
Explanation:
The most "basic" situation is when $t = 1$. At this point, game ends and there is a predicate to determine whatever Alice wins or loses.

Constructor Case:
Let $X', X'', Y', Y'' \in \mathcal{T}$, $t', t'' \in \mathbb{Z}^+$.
(1) If $(X', Y', t') \in PP$ AND $(X'', Y', t'') \in PP$, then $(X' \cup X'', Y', t' + t'') \in PP$
(2) If $(X', Y', t') \in PP$ AND $(X', Y'', t'') \in PP$, then $(X', Y' \cup Y'', t' + t'') \in PP$
Explanation:
As we see in the above description, Alice can distribute one of the two set (X, Y) and t into smaller components. She then has no control over how Bob will pick it. Therefore, if at a position she is guaranteed to win, there must be a way for her to break the position down to two other promising positions. The converse is true too, because Bob's choice does not affect Alice winning. (1) and (2) are the reverses of the only two legal distribution Alice can perform. Therefore by the above reasoning, the set defined is the smallest set that spans all possible promising positions.
Also since if A, C are disjoints and B, C are disjoints, then $A \cup B, C$ are disjoints too.

(b) Give a well-structured informal proof by **structural induction** that for any propositional formula f in negation normal form, for any set X of truth assignments that make f true, for any set Y of truth assignments that make f false, and for any integer $t \geq N_v(f)$, (X, Y, t) is a promising position.

We will use the notation from our textbook "Course Notes for CSCB36/236/240" (p.g. 115): If $\tau : PV \to \{0, 1\}$ is a truth value assignment, $\tau^* : \mathcal{F}_{PV} \to \{0, 1\}$ extends τ to take in a formula of propositional variables and returns its truth value.
For example, a truth assignment τ that make f true will have $\tau^*(f) = 1$.

For all $f \in NNF$, define the functions
$M_T(f) = \{\tau \mid \tau^*(f) = 1\}$ be the set of all truth assignments that make f true, and
$M_F(f) = \{\tau \mid \tau^*(f) = 0\}$ be the set of all truth assignments that make f false.

Define the predicate
$R(f) : NNF \to \{T, F\}$ = "$\forall X \in \mathcal{T}.\forall Y \in \mathcal{T}.\forall t \in \mathbb{Z}.[((X \subseteq M_T(f))$ AND $(Y \subseteq M_F(f))$ AND $(t \geq N_v(f)))$ IMPLIES $(X, Y, t) \in PP]$".
This is equivalent to say given a formula f in negation normal form, for any set X of truth assignments that make f true, for any set Y of truth assignments that make f false, and for any integer $t \geq N_v(f)$, (X, Y, t) is a promising position.
We will show $\forall f \in NNF.R(f)$. by structural induction on f.

Proof.

Let $f \in NNF$ be arbitrary.
Base Case: f is a single literal
Let P_1 be an arbitrary propositional variable.
Assume $f = P_1$ or $f = \text{NOT}(P_1)$.
Let $X, Y \in \mathcal{T}$, and $t \in \mathbb{Z}$ be arbitrary.
Assume $(X \subseteq M_T(f))$ AND $(Y \subseteq M_F(f))$ AND $(t \geq N_v(f))$.
By the definition of $M_T(f)$ and $M_F(f)$ and subset relation, if $\tau \in X$, and $\tau' \in Y$, then
When $f = P_1$, $\tau(P_1) = 1$ and $\tau'(P_1) = 0$.
When $f = \text{NOT}(P_1)$, $\tau(P_1) = 0$ and $\tau'(P_1) = 1$.
Hence, no matter what case, $\tau(P_1) \neq \tau'(P_1)$
Since this is true for all $\tau \in X$, and $\tau' \in Y$, if we pick $i = 1$, we have $\exists i \in \mathbb{Z}^+.[1 \leq i \leq n$ AND $(\forall \tau \in X.\forall \tau' \in Y.[\tau(P_i) \neq \tau'(P_i)])]$.
Thus, $(X, Y, 1) \in PP$, by the definition of promising positions.
In other words, when $t = 1, (X, Y, t) \in PP$.
Assume $t \in \mathbb{Z}$ AND $t > 1$.
Pick $X' = \emptyset \in \mathcal{T}$ and $t' = t - 1 > 0 \in \mathbb{Z}^+$.
It is easy to note that $\forall \tau \in X'.\forall \tau' \in Y.[\tau(P_i) \neq \tau'(P_i)]$ (say if we pick $i = 1 \in \mathbb{Z}^+$), is vacuously true.
Hence, $(X', Y, 1)$ is also a promising position.
By the definition of promising position, we can take $(X' \cup X', Y, 1 + 1) = (\emptyset, Y, 2) \in PP$. Repeat this for $t - 1$ times will give us $(X', Y, t - 1) \in PP$.
Apply the definition of PP again, we have $(X \cup X', Y, 1 + (t - 1))$ $= (X, Y, t) \in PP$ because $(X, Y, 1) \in PP$.
Thus we have shown when $t > 1, (X, Y, t) \in PP$ is also true.
Hence by direct proof, we have $((X \subseteq M_T(f))$ AND $(Y \subseteq M_F(f))$ AND $(t \geq N_v(f)))$ IMPLIES $(X, Y, t) \in PP$.
As $X, Y \in \mathcal{T}$, and $t \in \mathbb{Z}$ are arbitrary, generalization gives us $R(f)$.
To conclude, when f is a literal, $R(f)$ is true. This completes the base case.
Constructor cases: $f = f_1 \star f_2$ for some $f_1, f_2 \in NNF$ and $\star \in \{$ AND , OR $\}$
Assume $R(f_1), R(f_2)$ are true.
Let $X, Y \in \mathcal{T}$, and $t \in \mathbb{Z}$ be arbitrary.
Assume $(X \subseteq M_T(f))$ AND $(Y \subseteq M_F(f))$ AND $(t \geq N_v(f))$.
Case 1: $f = f_1$ AND f_2
Let $\tau \in X$ be an arbitrary truth assignment that makes f true ($\tau^*(f) = 1$).

By the definition of AND, we know that $\tau^*(f) = 1$ IFF $\tau^*(f_1) = \tau^*(f_2) = 1$.
In plain English, τ must make both f_1 and f_2 true to make f true.
This suggests that $\tau \in M_T(f_1)$ and $\tau \in M_T(f_2)$, by definition of M_T.
Hence, $X \subseteq M_T(f_1)$ and $X \subseteq M_T(f_2)$.
Consider an arbitrary $\tau' \in Y$. Since τ' makes f false, at least one of the $\tau'^*(f_1)$, $\tau'^*(f_2)$ must be 0 to make $\tau'^*(f_1 \text{ AND } f_2) = 0$.
Thus, $\tau' \in M_F(f_1)$ or $\tau' \in M_F(f_2)$.
Let $Y_1 = \{\tau' \in Y \mid \tau'^*(f_1) = 0\}$ be the set of truth assignments make f_1 false.
Let $Y_2 = \{\tau' \in Y \mid \tau'^*(f_2) = 0\}$ be the set of truth assignments make f_2 false.
By our above reasoning, $Y = Y_1 \cup Y_2$, as $\forall \tau' \in Y.[\tau'^*(f_1) = 0 \text{ OR } \tau'^*(f_2) = 0]$.
In addition, by definition of M_F, we have $Y_1 \subseteq M_F(f_1), Y_2 \subseteq M_F(f_2)$.
Finally, $t \geq N_v(f) = N_v(f_1) + N_v(f_2)$. Thus, $t - N_v(f_1) \geq N_v(f_2)$.
Recall that $X \subseteq M_T(f_1)$ and $X \subseteq M_T(f_2)$.
By specialization and modus ponus of $R(f_1)$, $(X, Y_1, N_v(f_1)) \in PP$.
By specialization and modus ponus of $R(f_2)$, $(X, Y_2, t - N_v(f_1)) \in PP$.
By constructor 2 of PP, $(X, Y_1 \cup Y_2, N_v(f_1) + (t - N_v(f_1))) \in PP$
Since $Y = Y_1 \cup Y_2$, we have $(X, Y, t) \in PP$.

Case 2: $f = f_1$ OR f_2

Let $\tau \in X$ be arbitrary. Since τ makes f true, at least one of the $\tau^*(f_1)$, $\tau^*(f_2)$ must be 1 to make $\tau^*(f_1 \text{ OR } f_2) = 1$.
Thus, $\tau \in M_T(f_1)$ or $\tau \in M_T(f_2)$.
Let $X_1 = \{\tau \in X \mid \tau^*(f_1) = 1\}$ be the set of truth assignments make f_1 true.
Let $X_2 = \{\tau \in X \mid \tau^*(f_2) = 1\}$ be the set of truth assignments make f_2 true.
By our above reasoning, $X = X_1 \cup X_2$, as $\forall \tau \in X.[\tau^*(f_1) = 1 \text{ OR } \tau^*(f_2) = 1]$.
By definition of M_T, we have $X_1 \subseteq M_T(f_1), X_2 \subseteq M_T(f_2)$.
Let $\tau' \in Y$ be an arbitrary truth assignment that makes f false ($\tau'^*(f) = 0$).
By the definition of OR, we know that $\tau'^*(f) = 0$ IFF $\tau'^*(f_1) = \tau'^*(f_2) = 0$.
In plain English, τ' must make both f_1 and f_2 false to make f false.
This suggests that $\tau' \in M_F(f_1)$ and $\tau' \in M_F(f_2)$, by definition of M_F.
Hence, $Y \subseteq M_F(f_1)$ and $Y \subseteq M_F(f_2)$.
Finally, $t \geq N_v(f) = N_v(f_1) + N_v(f_2)$. Thus, $t - N_v(f_1) \geq N_v(f_2)$.
Recall that $X_1 \subseteq M_T(f_1)$ and $X_2 \subseteq M_T(f_2)$.
By specialization and modus ponus of $R(f_1)$, $(X_1, Y, N_v(f_1)) \in PP$.
By specialization and modus ponus of $R(f_2)$, $(X_2, Y, t - N_v(f_1)) \in PP$.
By constructor 1 of PP, $(X_1 \cup X_2, Y, N_v(f_1) + (t - N_v(f_1))) \in PP$
Since $X = X_1 \cup X_2$, we have $(X, Y, t) \in PP$.

In both cases, $(X, Y, t) \in PP$ is true.

Hence by direct proof, we have $((X \subseteq M_T(f))$ AND $(Y \subseteq M_F(f))$ AND $(t \geq N_v(f)))$ IMPLIES $(X, Y, t) \in PP$.

As $X, Y \in \mathcal{T}$, and $t \in \mathbb{Z}$ are arbitrary, generalization gives us $R(f)$.

By principle of structural induction, we have shown $\forall f \in NNF.R(f)$.