Category Theory II

More notes towards a gentle introduction

Peter Smith

LOGIC MATTERS

I am slowly revising and expanding some earlier notes, and the unrevised chapters are clearly marked. I hope that despite its known inadequacies, you find this work-in-progress helpful and interesting. Two friendly requests:

1. Please do send any comments, suggestions and corrections – however small! – to this address. They are always welcome. It can help a lot if you give the date of the PDF you are commenting on: this version is dated December 22, 2023.
2. Please do spread the word about this gentle intro. But please don't directly share this pdf or place it on a website. Instead, share the url where the latest version can always be found, namely logicmatters.net/categories.

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Preface to Part II

There is no one best narrative path to take when telling the story of elementary category theory. For better or worse, I have structured these Notes by first concentrating in Part I on what happens inside single categories: then the central theme of this Part II concerns the way that categories can be mapped to each other by functors. I take up where the earlier discussions leave off; and chapters, definitions and theorems are numbered accordingly.

Cross-references are live links, with internal links in black, and external ones in blue. Put the PDFs of the two Parts in the same folder, and then the cross-references in blue in Part II will take you across to the mentioned definition or theorem in Part I. (In considerably revising the chapters both in Part I and Part II, I have broken a few cross-references yet to be repaired. There are also a few, I hope non-troubling, inconsistencies in notational conventions, which will eventually get ironed out.)

Everything here in Part II is very much work in progress: so caveat lector! At the moment, I am going through the old draft version from some years ago, trying to improve the presentation of the material there. I'm careful to advertise which old chapters still remain unrevised. When I have finished these first revisions, I'll need to think about what other material might be added. In the meantime, all corrections and suggestions are of course most welcome.

26 Matters of size

We have spent a good deal of time delving *inside* categories. In particular, we've been exploring various constructions that can recur across many different categories – such as forming products, quotients, exponentials. Indeed, a main point of packaging various familiar mathematical objects into categories is to "shift one's perspective from the particularities of each mathematical sub-discipline to potential commonalities between them" (as Riehl 2017, p. xi, nicely puts it).

Our next major task is to develop some apparatus for talking about relations *between* categories, tracking those commonalities. In particular we'll want to think about maps that can take a construction in one category to the same sort of construction in another category. After all, as I've said before, the spirit of category theory is to understand objects by looking at the morphisms between them. So, in that same spirit, we'll now seek to understand more about categories themselves by thinking about structure-preserving maps or morphisms between *them*. The standard term for such an inter-category map is 'functor'. The next chapter explains.

But first, it will be useful if this chapter makes some distinctions concerning matters of size.[1]

26.1 Categories, collections, classes, sets

One route into our topic is to return to our original preferred definition of the notion of a category and to compare it with another style of definition which you may very well encounter (this comparion is something we'll want to make anyway).

(a) We have been working with a definition – Defn. 13 – of the following type:

Type I. A category C comprises two kinds of things:

(1) Some C-objects,
(2) Some C-arrows.

These objects and arrows are governed by such-and-such axioms. △

[1]These distinctions are often made explicitly very near the beginning of presentations of category theory. However, we have only fleetingly needed to touch on issues of size before. So I've been able to leave more careful discussion until now.

Two comments. First, this leaves it quite unspecified what kinds of entities the objects and arrows of a category might be. And this seems entirely appropriate, since what category theory cares about are structural properties – the ways that the relevant objects and arrows inter-relate, not the intrinsic make-up of these entities.

Second, our definition says that what we need for a category is just some objects and some arrows. We don't require there to be *collections*, *classes* or *sets* of these objects or arrows, at least not in any sense which conceives of these as entities in their own right over and above their members.

(b) Here's a contrasting type of definition:[2]

Type II. A category C comprises:

(1) A class $Ob(\mathsf{C})$ whose elements are called the objects of the category,
(2) For each pair A, B of such objects, a set $Hom_{\mathsf{C}}(A, B)$ whose elements are called the arrows from A to B.

These objects and arrows are governed by such-and-such axioms. △

The old-school notation '*Hom*' reminds us that, at least in typical concrete categories, sets of arrows will be sets of homomorphisms of some kind. And the sets $Hom_{\mathsf{C}}(A, B)$ and $Hom_{\mathsf{C}}(A', B')$ for distinct pairs of objects (A, B) and (A', B') are usually required to be disjoint.

Two immediate things to explain. Why does a Type II definition talk, not of the arrows of a category all taken together as a class, but of lots of disjoint sets of arrows? And why *class* (or sometimes *collection*) in the definition's clause about objects, but *set* in the clause about arrows?

(c) Right back in §4.6, we noted that some important categories will comprise too many objects to form a set. Take Set for the obvious example. At least on the usual story, there is no universal set, no set containing all sets. So if we insisted that a category always has only a set's-worth of objects, Set wouldn't make the grade. That's why we'll want to avoid, one way or another, saying that a category must always comprise a *set* of objects. We can instead use Type I plural talk. Or, if you insist on Type II singular talk of a category as comprising a collection of objects, then this must be a collection-which-may-not-be-a-set: and 'class' is, of course, the common term for such a collection.

Now, the objects of a category are associated one-to-one with their identity arrows; hence if a category has too many objects to form a set, then it must also have too many arrows overall to form a set. However, in a category like Set (or Grp, etc.), when we focus in just on the arrows between two given objects, there can be a set of *them*. For example, take any two sets A and B in Set: then on

[2] As previously noted in §4.1, fn. 1, you will find definitions of categories of Type I in e.g. Awodey (2010, p. 4), Lawvere and Schanuel (2009, p. 21) and McLarty (1992, p. 13). For definitions of Type II in books old and new, you could see e.g. Borceux (1994, p. 4), Pareigis (1970, p. 1), Richter (2020, p. 7), Roman (2017, p. 2), Schubert (1972, p. 1), and Taylor (1999, p. 184). Sometimes, 'collection' is used in the first clause of a Type II definition instead of 'class', e.g. by Adámek et al. (2009, p. 21) and Spivak (2014, p. 93).

the standard construction, the set-functions from A to B live in $\mathcal{P}(\mathcal{P}(A \cup B)$, and there is therefore only a set's worth of such arrows. Hence clause (2) of our Type II definition does happily cover categories like Set.

(d) It remains the case, though, that Type I and Type II definitions do differ over the size of categories they allow.

Call a category *locally small* if there is only a set's worth of arrows between any pair of its objects (not necessarily distinct). Then a Type II definition builds into the very definition of a kosher category that it must be locally small (as even Set is). On the other hand, a Type I definition allows for categories where there can be too many arrows between certain objects to form a set.

Now, it is in fact rather easy to cook up examples which count as categories according to a Type I definition but not according to a Type II definition. Recall from §4.4 that any monoid $(\mathrm{M}, *, e)$ can be treated as a Type I category with a single object $\bullet$ and with each m from among M being counted as an arrow $m\colon \bullet \to \bullet$. So if there are too many *objects* M to form a set, then the resulting monoid-as-category will have too many *arrows* from $\bullet$ to $\bullet$ to form a set. Consider then the particular case of, say, the monoid of von Neumann ordinals under addition. There is no set of all ordinals; hence the corresponding Type I monoid-as-category will not be locally small and hence won't be a Type II category.

Still, does this sort of divergence between the two types of definition matter? After all, although our example is not *wildly* artificial, it doesn't seem to be of obvious interest either. So at least at the outset when exploring the world of categories, a Type II definition requiring categories to be locally small *is* perhaps quite generous enough to cover the initially significant cases. We will have to wait to see if it this style of definition in fact becomes too restrictive as our investigations proceed further.

(e) There remains a question of how, more precisely, we should interpret talk of classes (or collections) in Type II definitions. Category theorists who talk of classes typically treat these as entities in their own right, so are working with (at least) a two-level theory of collections – allowing ordinary sets and then, on top of these, a layer of collections-too-big-to-be-ordinary-sets. But do we really need more than virtual large collections here – can Type II class talk be cashed out as non-committal plural talk in thin disguise (see §3.1 again)?

Just to complicate matters, you can also often encounter a Type III style of definition which is like Type I in that it doesn't require categories to be locally small. But now, instead of talking of the ingredients of a category plurally, as some objects and some arrows, we define a category as comprising a collection of objects and a collection of arrows – where these collections are typically then treated as classes qua entities in their own right.[3] Again, we can wonder about the best way of thinking of these classes (or other potentially large collections) for category-theoretic purposes.

However, let's shelve that sort of question, for there's a much more immediate,

[3]For Type III definitions, see e.g. Pierce (1991, p. 1), Riehl (2017, p. 3) and Simmons (2011, p. 2).

lower-level, issue to discuss:[4] just how should we interpret the Type II requirement that for each pair of objects A, B of a category C there is a *set* $Hom_{\mathsf{C}}(A, B)$ of arrows from A to B? To put it snappily,

26.2 Where do hom-sets live?

(a) The standard assumption is that, whatever the category C, all its hom-sets $Hom_{\mathsf{C}}(A, B)$ live in the category Set.

Actually, this is surprisingly rarely announced up front. What usually happens is that authors at some point consider maps – functors, in fact – like the one which results when we choose a fixed object A, and then send each object X in the category C to the corresponding $Hom_{\mathsf{C}}(A, X)$, the set of arrows from the fixed source A to X. Such maps are then reported as operating between C and Set, so each hom-set $Hom_{\mathsf{C}}(A, X)$ is indeed taken to be an object of Set.[5]

Now, we left it open at the outset exactly which universe of sets equipped with functions between sets constitutes Set. But this universe is widely presumed to be a model of standard set theory, and standard set theory, recall, is a theory of *pure* sets where the members of sets are always more sets.[6] But if a set of C-arrows like $Hom_{\mathsf{C}}(A, X)$ is a pure set belonging to Set then its members, the C-arrows, must *themselves* be pure sets.

Assuming that the arrows of a category C are pure sets doesn't, strictly speaking, require that C's objects are pure sets too. But that would be the natural companion assumption. So the Type II definition according to which the C-arrows between two objects always assemble into a set pushes us towards the idea that the ingredients of any category C are sets, pure if not simple!

Now, some do explicitly take this line. As already mentioned in §3.4, fn. 7, the canonical example is Saunders Mac Lane, who initially gives in effect a Type I definition of what *he* calls metacategories, and then says that a category

[4]On the higher-level issue of the ways in which fully-caffeinated category theory can get entangled with questions about what sets/classes there are, see the rightly much-cited paper on 'Set Theory for Category Theory' by Michael A. Shulman (2008). But as you'll find if you follow up that reference, things rapidly become quite tangled, in a way that I think we can reasonably hope to side-step in a 'naive' presentation of entry-level category theory.

[5]Evidence? Just to take our Type II authors, you'll find this move in Borceux (1994, p. 9), Pareigis (1970, p. 11), Richter (2020, p. 20), Schubert (1972, p. 7), Taylor (1999, p. 237(f)).

[6]Evidence? Some authors on category theory do spell out the point. For example, Pareigis (1970, p. 248), explaining his background theory of sets and classes, tells us that "Only sets can occur as elements of classes or sets". Similarly Schubert (1972, §3.1) writes "One has to be aware that the set theory used here has no 'primitive (ur-)elements'; elements of sets ... are always themselves sets."

And many other authors too are equally committed to taking sets to be pure sets: thus Borceux (1994, §1.1) kicks off his triple-decker book by offering as a background theory of sets either Gödel-Bernays or standard set theory augmented with an axiom of universes, either way giving us a theory of pure sets. Roman (2017, p. 1) does just the same. Likewise Riehl (2017, p. 6) also notes that "common practice among category theorists is to work in an extension of the usual Zermelo–Fraenkel axioms of set theory, with new axioms allowing one to distinguish between 'small' and 'large' sets, or between sets and classes" and such an extension will still be a theory of pure sets.

proper "will mean any interpretation of the category axioms within set theory" (Mac Lane 1997, p. 10). And this is fine for many purposes. In fact, in Part I, I followed a conventional path in talking a lot about categories like Grp – where that category, for example, was defined not as the category of *all* groups whatever (for what does that include?) but as the category of all the groups implemented in our favoured universe of sets. So our category Grp can be the same as Mac Lane's category Grp. Likewise for the other typical examples of sweepingly inclusive categories we looked at.

That was presentationally convenient. But do we really want to be restrictive and assume that categories all live in the universe of pure sets? Not if we conceive of category theory as potentially offering a framework for a more democratic way of organizing the mathematical universe and providing an alternative to imperialistic set-theoretic reductionism. So, if we are thinking along *those* lines, there will be reason to prefer a Type I definition of categories which allows us to be more ecumenical about the nature of the gadgets of a category, and like Awodey (2010, p. 5) we can then insist that "A category is anything that satisfies [our Type I] definition. ... I want to emphasize that the objects do not have to be sets" and the arrows too need not be implemented as sets.[7] Perhaps it will turn out and be important that categories that aren't too large can always be modelled, implemented or represented in a universe of sets: but shouldn't that be a *result*, not something required built in by definition?

26.3 Sizes of categories

Whichever type of definition we adopt, we will still need for some purposes to distinguish differently sized categories.

Now, a moment ago I called a category locally small if there is only a set's worth of arrows between any pair of its objects. But what if we don't want to understand this as meaning that those arrows actually form a set, meaning a pure set living in Set as typically conceived? What is it then to have a 'set's worth' of arrows?

Back when talking about the existence of limits in Chapter 20, we defined a finite limit as being taken over objects which can be indexed by a finite set of indices. And a 'small' (but perhaps infinite) limit is one taken over objects which can be indexed by some (perhaps infinite) set, where again indexing is a matter of there being a one-to-one function between the objects and the elements of the

[7]In fact, Mac Lane himself, in a short Appendix written for the second edition of *Categories for the Working Mathematician* rather changes his official position, now writing

> We have described a category in terms of sets, as a set of objects and a set of arrows. However, categories can be described directly – and they can then be used as a possible foundation for all of mathematics, thus replacing the use in such a foundation of the usual Zermelo-Fraenkel axioms for set theory. Here is the direct description: ...

And what follows is a Type I definition.

indexing set. So now let's use the same idea of indexing again, in a way you'll also find in e.g. Crole (1993, p. 61). Then we have:

Definition 106. A category is *finite* iff it has finitely many objects and finitely many arrows, i.e. its objects and its arrows can both be indexed by finite sets.

A category is *small* iff it has only a set's worth of objects and a set's worth of arrows, i.e. its objects and its arrows can both be indexed by sets (which can be of any size).

A category is *locally small* if, for any of its objects A and B, there is only a set's worth of arrows from A to B, i.e. in each case the arrows from A to B can be indexed by some set $\triangle$

Though as I remarked in another context (§11.3), 'small' here is something of a joke, since the relevant indexing sets can be huge!

27 Functors introduced

As announced at the beginning of the last chapter, we are now going to be looking at structure-preserving maps or morphisms between categories. Indeed functors, to use the proper term, are the central topic of the whole of this Part of these Notes. So let's immediately pin down the notion we want.

27.1 Functors defined

(a) A category has two kinds of data, its objects and its arrows. A functor F mapping the category C to the category D will therefor need to have two components, F_{ob} that operates on objects, and F_{arw} that operates on arrows. And we want the actions of these components to fit together – when the F_{arw} sends an arrow f living in C to an arrow $F_{arw}(f)$ living in D, F_{ob} should send the source and target of f to the source and target of $F_{arw}(f)$. Hence we'll say:

Definition 107. A *functor* $F\colon \mathsf{C} \to \mathsf{D}$ between categories C and D comprises the following data:

(1) An operation or mapping F_{ob} whose value for the C-object A is some D-object we can represent as $F_{ob}(A)$.

(2) An operation or mapping F_{arw} whose value for the C-arrow f from A to B is a D-arrow $F_{arw}(f)$ from $F_{ob}(A)$ to $F_{ob}(B)$.

For brevity, dropping subscripts, we can write $F_{ob}(A)$ simply as $F(A)$ or even just as FA.

And we can write $F_{arw}(f)\colon F_{ob}(A) \to F_{ob}(B)$ simply as $F(f)\colon F(A) \to F(B)$ or even just as $Ff\colon FA \to FB$. △

Perhaps annoyingly, it seems entirely conventional to use the same style of lettering both to denote functors between categories and to denote objects in categories. I'll follow convention but will try to keep things clear by defaulting to the likes of F, G, H for functors, while deploying early-alphabet or late-alphabet letters $A, B, C, \ldots, X, Y, Z$ for objects.

OK: but there's more. If a functor F is to preserve at least the most basic categorial structure, its component mappings must obey two obvious conditions.

First, F must map identity arrows to identity arrows. So F_{arw} sends 1_A, the identity arrow on A, to the identity arrow on $F_{ob}(A)$, namely $1_{F_{ob}(A)}$, or 1_{FA} for short.

Second, F needs to respect composition. That is to say, since the commutative diagram $\begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, "g \circ f"'] & & C \end{tikzcd}$ in C gets sent by F to $\begin{tikzcd} & FB \arrow[dr, "Fg"] & \\ FA \arrow[ur, "Ff"] \arrow[rr, "F(g \circ f)"'] & & FC \end{tikzcd}$ in D, the second diagram should also commute. Hence we need:

Definition 107 (cont'd). F must satisfy the following conditions:

Preserving identity arrows: for any C-object A, $F(1_A) = 1_{FA}$;

Respecting composition: for any C-arrows f, g such that their composition $g \circ f$ exists, $F(g \circ f) = Fg \circ Ff$.[1] $\triangle$

These conditions on F are often called *functoriality.* It will emerge in §27.5 why these are said, more specifically, to be the conditions for a *covariant* functor.

(b) Notation. In presenting a functor $F\colon \mathsf{C} \to \mathsf{D}$, we need to fix what F does to C-objects and what it does to C-arrows. We could schematically display that in a diagram like this, with the right-hand side spelt out (cf. Riehl 2017, p. 19):

$$\begin{tikzcd} \mathsf{C} \arrow[r, "F"] & \mathsf{D} \\ X \arrow[d, "f"] \arrow[r, mapsto] & FX \arrow[d, "Ff"] \\ {} \arrow[r, mapsto, phantom] & {} \\ Y \arrow[r, mapsto] & FY \end{tikzcd}$$

But on balance I prefer a slightly more economical display of the following shape (assuming it is clear from the context what the source and target categories are):

$$\begin{array}{lrcl} F\colon & X & \longmapsto & FX \\ & f\colon X \to Y & \longmapsto & Ff\colon FX \to FY. \end{array}$$

(c) To fix ideas, here's a simple result which will be important later and which you should pause to check:

Theorem 128. *(1) For any category C there is an identity functor $1_{\mathsf{C}}\colon \mathsf{C} \to \mathsf{C}$ which sends C's objects and arrows to themselves.*

(2) Suppose there exist functors $F\colon \mathsf{C} \to \mathsf{D}$, $G\colon \mathsf{D} \to \mathsf{E}$. Then there is also a composite functor $G \circ F\colon \mathsf{C} \to \mathsf{E}$ with the following data:[2]

[1] Or more explictly, for any C-arrows f, g such that $g \circ_{\mathsf{C}} f$ exists, $F(g \circ_{\mathsf{C}} f) = Fg \circ_{\mathsf{D}} Ff$. But usually, we will continue to follow the inexplicit practice of letting local context make it clear *which* composition operation must be in play.

[2] We are assuming that maps of the kind which constitute the components of functors compose-as-maps in the usual way that maps do! Then what we want to check is that functors that are built out of composable maps will themselves compose in such a way as to result in another functor.

(i) A mapping $(G \circ F)_{ob}$ *which sends a* C*-object* A *to the* E*-object to* $G(F(A))$.
(ii) A mapping $(G \circ F)_{arw}$ *which sends a* C*-arrow* $f\colon A \to B$ *to the* E*-arrow* $G(F(f))\colon\ G(F(A)) \to G(F(B))$.

Further, such composition of functors is associative. □

Note: to reduce clutter, we will later allow ourselves to write simply 'GF' for the composite functor rather than '$G \circ F$'. And then, dropping brackets, GF sends A to GFA and sends $f\colon A \to B$ to $GFf\colon GFA \to GFB$.

27.2 Some elementary examples of functors

(a) Continue to assume, as we did at the outset, that inclusive categories like Mon, Grp, Top and the rest live happily together in some suitably capacious arena of sets forming a category Set understood in not-too-unconventional a way.

So we can think, for example, of a monoid belonging to Mon as a set of objects suitably equipped with a binary operation and a distinguished element. And then our first example of a functor nicely illustrates a broad class of cases.

(F1) There is a functor F : Mon → Set with the following data:

(1) F_{ob} which sends the monoid $(M, *, e)$ to its carrier set M.

(2) F_{arw} which sends $f\colon (M, *, e) \to (N, \star, d)$, i.e. a structure-respecting homomorphism mapping elements of M to elements of N, to the same map thought of simply as a set-function $f\colon M \to N$.

Or in our less wordy shorthand:

$$F\vdots \quad \begin{array}{rcl} (M, *, e) & \longmapsto & M \\ f\colon (M, *, e) \to (N, \star, d) & \longmapsto & f\colon M \to N \end{array}$$

So all F does is 'forget' about the structure carried by the collection of objects in a monoid. It's evidently a functor, a *forgetful functor* for short.

There are equally forgetful functors from other categories of structured sets to the bare underlying sets. For example, there is the functor $F\colon$ Top → Set that sends topological spaces to their sets of points and sends continuous maps to themselves as set functions, forgetting about the topological structure.

Of course, such amnesiac functors are not in themselves very exciting! It will turn out, however, that they can be the boring members of so-called adjoint pairs of functors, where they are married to very much more interesting companions. But that's a topic for later.

To continue for a moment with the forgetful theme:

(F2) Recall: if C is a category, and X is a C-object, the slice category C$/X$'s objects are all the (A, f), for some C-object A and C-arrow $f\colon A \to X$; and C$/X$'s arrows between (A, f) and (B, g) are, economically defined, the C-arrows $j\colon A \to B$ such that $g \circ j = f$.

Then there is another kind of forgetful functor, $F\colon \mathsf{C}/X \to \mathsf{C}$, which sends a C/X-object (A, f) back to A, and sends a C/X-arrow $j\colon (A, f) \to (B, g)$ back to the original arrow $j\colon A \to B$. Or in short

$$F\colon \quad \begin{array}{rcl} (A, f) & \longmapsto & A \\ j\colon (A, f) \to (B, g) & \longmapsto & j\colon A \to B \end{array}$$

For example, take the slice category FinSet/X_n which we met in §6.3, which is the category of finite sets whose members are coloured from a palette of n colours. The forgetful functor $F\colon \mathsf{FinSet}/X_n \to \mathsf{FinSet}$ simply forgets about the colourings of a set S provided by functions $f\colon S \to X_n$.

(F3) There are somewhat less forgetful functors, such as the functor from Rng to Grp that sends a ring to the additive group it contains, ignoring the rest of the ring structure. Or take the functor from Grp to Mon, that remembers about the associative multiplicative structure and units but forgets about about inverses.

(F4) There is a functor $F\colon \mathsf{Set} \to \mathsf{Rel}$ which sends sets and triples (domain, graph, codomain) thought of as objects and arrows belonging to Set to the same items thought of as objects and arrows in Rel, i.e. forgetting that the graphs are functional.

(b) Alongside functors that forget (some of the) structure put on structured collections of objects, there are also functors which simply obliterate some distinctions between objects or between arrows.

(F5) Suppose we take a category C, preserve all the objects, but thin down the arrows, to give a pre-order category D where there is exactly one arrow from A to B in D whenever there at least one arrow from A to B in C. Then there is evidently a 'thinning' functor $F\colon \mathsf{C} \to \mathsf{D}$ which maps objects to themselves and sends any arrow in C with source A and target B to the unique arrow in D with that source and target.

(F6) An extreme case: there is a functor $\Delta_X\colon \mathsf{J} \to \mathsf{C}$ which picks an object X from the category C, and then does the following for any J-object A and any J-arrow $j\colon A \to B$:

$$\Delta_X\colon \quad \begin{array}{rcl} A & \longmapsto & X \\ j\colon A \to B & \longmapsto & 1_X\colon X \to X. \end{array}$$

In particular,

(F7) For any chosen object X in C there is a corresponding functor – overloading notation, we can call it simply $X\colon \mathsf{1} \to \mathsf{C}$ – which sends the object of the one-object category $\mathsf{1}$ to the object X, and sends the sole arrow of $\mathsf{1}$ to 1_X.

And as you'd expect,

(F8) Suppose S is a subcategory of C in the sense of §6.1. Then there is an inclusion functor $F\colon \mathsf{S} \to \mathsf{C}$ which sends objects and arrows in S to the same items in C.

(c) Let's now have two examples of a different kind. Recall how we can cook up categories from monoids and posets in particularly simple ways: we'll see now that functors between such derived categories are also familiar mappings.

(F9) Take monoids $(M, *, e)$ and $(N, \star, d)$ and consider the corresponding categories M and N in the sense of §4.7.

So M has a single object $\bullet_{\mathsf{M}}$ (whatever you like), and its arrows are simply elements of M, where the composition in M of the arrows m_1 and m_2 is just $m_1 * m_2$, and the identity arrow is the identity element of the monoid, e.

Likewise, of course, N has a single object $\bullet_{\mathsf{N}}$, and arrows are elements of N, where the composition of the arrows n_1 and n_2 is just $n_1 \star n_2$, and the identity arrow is the identity element of the monoid, d.

So now we see that a functor $F : \mathsf{M} \to \mathsf{N}$ will need to do the following:

i. F must send $\bullet_{\mathsf{M}}$ to $\bullet_{\mathsf{N}}$.
ii. F must send the identity arrow e to the identity arrow d.
iii. F must send m_1 composed with m_2 (i.e. $m_1 * m_2$) to Fm_1 composed with Fm_2 (i.e. $Fm_1 \star Fm_2$).

Apart from the trivial first condition, that just requires F to be a monoid homomorphism. So any homomorphism between two monoids induces a corresponding functor between the corresponding monoids-as-categories.

(F10) Take the posets $(S, \preccurlyeq)$ and $(T, \sqsubseteq)$ considered as categories S and T.

So the objects of S are the member of S again, and the arrows of S are pairs $\langle a, b\rangle$ such that $a \preccurlyeq b$, with composition for S defined by $\langle b, c\rangle \circ \langle a, b\rangle = \langle a, c\rangle$. Similarly of course for T.

It is easy to check that a monotone function $f\colon S \to T$ (i.e. a function such that $a \preccurlyeq b$ implies $f(a) \sqsubseteq f(b)$) induces a functor $F\colon \mathsf{S} \to \mathsf{T}$ which sends an S-object a to the T-object $f(a)$, and sends an S-arrow, i.e. a pair $\langle a, b\rangle$ where $a \preccurlyeq b$, to the T-arrow $\langle f(a), f(b)\rangle$.

(d) Finally in this section, a rather more interesting observation:

(F11) Now take the group $G = (G, *, e)$ and consider it as a category G – see §7.8. And suppose $F\colon \mathsf{G} \to \mathsf{Set}$ is a functor.

Then F must send G's unique object $\bullet$ to some set X. And F must send a G-arrow $m\colon \bullet \to \bullet$ (that's just a member m of G) to a function $F(m)\colon X \to X$. Functoriality requires that $F(e) = 1_X$ and $F(m * m') = F(m) \circ F(m')$. But those are just the conditions for F to constitute a group action of G on X.

Conversely, a group action of G on X amounts to a functor from G to Set.

27.3 Products, exponentials, and functors

(a) Let's have a couple more examples. First,

(F12) Assume C has all products, and C is any object in the category. Then there is a functor $- \times C \colon \mathsf{C} \to \mathsf{C}$, which works as follows:

$$\begin{array}{rrcl} - \times C \colon & X & \longmapsto & X \times C \\ & f \colon X \to Y & \longmapsto & f \times 1_C \colon X \times C \to Y \times C. \end{array}$$

Similarly there is a functor

$$\begin{array}{rrcl} C \times - \colon & X & \longmapsto & C \times X \\ & f \colon X \to Y & \longmapsto & 1_C \times f \colon C \times X \to C \times Y. \end{array}$$

To confirm the functoriality of $- \times C$, we need to show in particular that $(- \times C)(g \circ f)$ equals $(- \times C)g \circ (- \times C)f$. But by definition this is just the claim $g \circ f \times 1_C = (g \times 1_C) \circ (f \times 1_C)$, which follows from Theorem 52. Similarly for the twin functor.

(b) Just as taking a product with a fixed object is functorial, so is exponentiation by a fixed object. In other words – assuming we are working in a category C with exponentials and products – we can again take a fixed object C and we will expect there to be a functor we can notate as $(-)^C \colon \mathsf{C} \to \mathsf{C}$ whose object-component sends any C-object X to X^C, where that is the object component of the exponential (X^C, ev).

But how will the other component work, the component that sends an arrow $f \colon X \to Y$ to a suitable arrow $f^C \colon X^C \to Y^C$ and satisfies the functoriality conditions?

Well, take the two exponentials (X^C, ev_X) and (Y^C, ev_Y) – and I'll now subscript 'ev' arrows like this when we have more than one in play and we need to keep them distinct. Then the following diagram commutes, where $\overline{\overline{f \circ ev_X}}$ is the unique exponential transpose of $f \circ ev_X$, as defined in Defn. 70:[3]

$$\begin{tikzcd} X^C \times C \arrow[r, "ev_X"] \arrow[d, dashed, "\overline{\overline{f \circ ev_X}} \times 1_B"'] & X \arrow[d, "f"] \\ Y^C \times C \arrow[r, "ev_Y"'] & Y \end{tikzcd}$$

This is just a straight application of the definition, given that (Y^C, ev_Y) is an exponential. So in this way, for fixed C, alongside the association between the objects X and X^C, there is a natural association between the arrows $f \colon X \to Y$ and $\overline{\overline{f \circ ev_X}} \colon X^C \to Y^C$. And we might rather hope that these associations combine to give us a functor.

In other words, the following is hopefully true:

[3] In Part I, I use the common device of a tilde, as in $\tilde{f}$, in denoting an exponential transpose. But wide tildes sweeping over longer expressions become, to my eyes, unacceptably ugly – so here and elsewhere I'll substitute double overlining.

(F13) Assume C has all exponentials, and that C is a C-object. Then there is a corresponding exponentiation functor $(-)^C : \mathsf{C} \to \mathsf{C}$ where

$$(-)^C\colon \quad \begin{array}{rcl} X & \longmapsto & X^C \\ f\colon X \to Y & \longmapsto & \overline{\overline{f \circ ev}}\colon X^C \to Y^C. \end{array}$$

And for once, hope is rewarded!

Proof. To confirm functoriality, we need to show that $(-)^C$ does indeed preserve identities and respect composition.

The first of these is easy. $(1_X)^C$ is by definition $\overline{\overline{1_X \circ ev}}\colon X^C \to X^C$, so we have

$$\begin{tikzcd}
X^C \times C \arrow[r, "ev"] \arrow[d, dashed, "(1_X)^C \times 1_C"'] & X \arrow[d, "1_X"] \\
X^C \times C \arrow[r, "ev"'] & X
\end{tikzcd}$$

But evidently, an arrow $1_{X^C} \times 1_C$ on the left would also make the diagram commute. So by the uniqueness requirement that there is a single filling for $- \times 1_C$ which makes the square commute, $(1_X)^C = 1_{X^C}$, as required.

Second, we need to show that given arrows $f\colon X \to Y$ and $g\colon Y \to Z$, then $(g \circ f)^C = g^C \circ f^C$.

Consider the following diagram where the top square, bottom square, and (outer, bent) rectangle commute:

$$\begin{tikzcd}
X^C \times C \arrow[r, "ev_X"] \arrow[d, dashed, "f^C \times 1_C"] \arrow[dd, dashed, bend right=60, "(g \circ f)^C \times 1_C"'] & X \arrow[d, "f"'] \\
Y^C \times C \arrow[r, "ev_Y"] \arrow[d, dashed, "g^C \times 1_C"] & Y \arrow[d, "g"'] \\
Z^C \times C \arrow[r, "ev_Z"'] & Z
\end{tikzcd}$$

But now note that, by Theorem 52, $(g^C \times 1_C) \circ (f^C \times 1_C) = (g^C \circ f^C) \times 1_C$. Hence $(g^C \circ f^C) \times 1_C$ is in fact another arrow that makes a commuting outer rectangle.

So again by the requirement that there is a unique filling for $- \times 1_C$ which makes that outer rectangle commute, $(g \circ f)^C = g^C \circ f^C$. $\square$

27.4 A 'free' functor from Set to Mon

(a) Next, we are going to define a functor going in the reverse direction to the forgetful functor in (F1), i.e. a functor $F : \mathsf{Set} \to \mathsf{Mon}$.

There are of course utterly trivial ways of doing this. For example just pick a monoid M living in Mon. Then there is a constant functor we could call $!_M : \mathsf{Set} \to \mathsf{Mon}$ which sends every set X to M and sends every set-function $f\colon X \to Y$ to the identity homomorphism $1_M\colon M \to M$.

But it is instructive to try to come up with something rather less boring! So, consider how we might methodically send sets to monoids, but this time *making as few assumptions as we possibly can* about which monoid a given set gets mapped to.

(b) First we need a preliminary point, about one more functor:

(F14) There is a list functor $List\colon \mathsf{Set} \to \mathsf{Set}$, where $List_{ob}$ sends a set X to $List(X)$, the set of all finite lists or sequences of elements of X, including the empty one.[4] And $List_{arw}$ sends a function $f\colon X \to Y$ to the function $List(f)\colon List(X) \to List(Y)$ which sends the list $x_0{}^\frown x_1{}^\frown x_2{}^\frown \ldots{}^\frown x_n$ to $fx_0{}^\frown fx_1{}^\frown fx_2{}^\frown \ldots{}^\frown fx_n$ (where $^\frown$ symbolizes concatenation).

It is trivial to check that this is indeed a functor.

(c) OK: start with a set X we are going to send to a corresponding monoid. Since we are making no more assumptions than we need to, we'll have to take the objects in X as providing us with an initial supply of objects for building our monoid, the monoid's *generators*. We now need to equip our incipient monoid with a two-place associative function $*$. But we are assuming as little as we can about $*$ too, so we don't know that applying it keeps us inside the original set of generators X. So X will need to be expanded to a set M that contains not only the original members of X, e.g. $x, y, z, \ldots$, but also all the possible 'products', i.e. everything like $x*x$, $x*y$, $y*x$, $y*z$, $x*y*x$, $x*y*x*z$, $x*x*y*y*z\ldots$, etc., etc. We know, however, that since $*$ is associative, we needn't distinguish between e.g. $x*(y*z)$ and $(x*y)*z$.

But even taking all those products is not enough, for (in our assumption-free state) we don't know whether any of the resulting elements of M will act as an identity for the $*$-function. To get a monoid, then, we need to throw into M some unit 1.

Since we are making as few assumptions as we can, we can't assume either that any of the products in M are equal (at least once we have multiplied out occurrences of the identity element), or that there are any other objects in M other than those generated from the unit and members of X.

Now, here's a neat way to model the resulting monoid generated from the set X in this assumption-free way. Represent a monoid element (such as $x*x*y*y*z$) just as a *finite list of members of* X (such as $x^\frown x^\frown y^\frown y^\frown z$), so M gets represented by $List(X)$ as in (F14) and we model the $*$-function by simple concatenation. The identity element will then be modelled by the null list $\emptyset$. The resulting monoid $(List(X), {}^\frown, \emptyset)$ is often simply called *the free monoid* on X – though perhaps it is better to say it is a standard exemplar of a monoid freely constructed from X.

[4]So think of lists as implemented in Set in some standard way; the details won't matter.

Which all goes to motivate the following construction:

(F15) There is a 'free' functor $F : \mathsf{Set} \to \mathsf{Mon}$ with the following data:

i. F_{ob} sends the set X to the monoid $(List(X), ^\frown, \emptyset)$.

ii. F_{arw} sends the arrow $f \colon X \to Y$ to $List(f)$ as in (F14) again, where this is treated as an arrow from $(List(X), ^\frown, \emptyset)$ to $(List(Y), ^\frown, \emptyset)$.

It is again trivial to check that F is indeed a functor.

(d) We can generalize. There are similar functors that send sets to other *freely generated* structures on the set. For example there is a functor from Set to Ab which sends a set X to the freely generated abelian group on X (which is in fact the direct sum of X-many copies of $(\mathbb{Z}, +, 0)$ – the integers $\mathbb{Z}$ with addition forming the paradigm free abelian group on a single generator). But we need not concern ourselves with the further details of such cases.

27.5 Duality and contravariance

(a) We now need to introduce an important new idea.

Functors of the kind we've met so far send arrows $f \colon A \to B$ to arrows $Ff \colon FA \to FB$. Call these *covariant* functors, and compare:

Definition 108. A *contravariant functor* $F \colon \mathsf{C} \to \mathsf{D}$ between categories C and D comprises the following data:

(1) A mapping F_{ob} whose value for the C-object A is some D-object $F(A)$.

(2) A mapping F_{arw} whose value for the C-arrow $f \colon A \to B$ is a D-arrow $F(f) \colon FB \to FA$. (NB the contrary directions of the arrows!)

And this data satisfies the two axioms:

Preserving identity arrows: for any C-object A, $F(1_A) = 1_{F(A)}$;

Respecting composition: for any C-arrows f, g such that their composition $g \circ f$ exists, $F(g \circ f) = Ff \circ Fg$. (NB the order of the compositions!) △

(b) Let's immediately have a couple of examples of naturally arising contravariant functors.

(F16) The covariant powerset functor $P \colon \mathsf{Set} \to \mathsf{Set}$ maps a set X to its powerset $\mathcal{P}(X)$ and maps a set-function $f \colon X \to Y$ to the function which sends $U \in \mathcal{P}(X)$ to its f-image $f[U] = \{f(x) \mid x \in U\} \in \mathcal{P}(Y)$. (Check that really is a functor!)

That functor has contravariant twin $\overline{P} \colon \mathsf{Set} \to \mathsf{Set}$ which again maps a set to its powerset, and this time maps a set-function $f \colon X \to Y$ to the function which sends $U \in \mathcal{P}(Y)$ to its inverse image $f^{-1}[U] \in \mathcal{P}(X)$ (where $f^{-1}[U] = \{x \mid f(x) \in U\}$). (Check that this works too!)

(F17) Take FVect, the category whose objects are the finite dimension vector spaces over the reals, and whose arrows are linear maps between spaces.

Now recall, the dual space of given finite-dimensional vector space V over the reals is V^*, the set of all linear functions $f\colon V \to \mathbb{R}$ (where this set is equipped with vectorial structure in the obvious way). V^* has the same dimension as V (so, a fortiori, is also finite dimensional and belongs to FVect). We'll construct a dualizing functor $D\colon \mathsf{FVect} \to \mathsf{FVect}$, where D_{ob} sends a vector-space to its dual.

So how is our functor's component D_{arw} going to act on arrows in the category FVect? Take the spaces V, W and consider any linear map $f\colon V \to W$. Then, on the dual spaces, there will be a corresponding map $(- \circ f) : W^* \to V^*$ which sends a function $g\colon W \to \mathbb{R}$ to $g \circ f\colon V \to \mathbb{R}$. This suggests what we want the action of the component D_{arw} to be: it will send a linear map f to the functional $(- \circ f)$.

It is readily checked that these components D_{ob} and D_{arw} do give us a contravariant functor.

(c) A general point about contravariant functors, which (although almost trivial) is probably important enough to highlight as a theorem. Given our definition,

Theorem 129. *$F\colon \mathsf{C} \to \mathsf{D}$ is a* contravariant *functor from C to D if and only if $F\colon \mathsf{C}^{op} \to \mathsf{D}$ with the same data is a functor in the original* covariant *sense.*

Proof. Suppose $F\colon \mathsf{C} \to \mathsf{D}$ is a contravariant functor. Then we know that it is the case that

(1) the object-mapping F_{ob} sends the C^{ob}-object A to some D-object $F(A)$,

just because the C^{ob}-object A is none other than the C-object A. Further,

(2) the arrow-mapping F_{arw} sends the C^{op}-arrow $f\colon A \to B$ to the D-arrow $F(f)\colon\ FA \to FB$,

since the C^{op}-arrow $f\colon A \to B$ is none other than the C-arrow $f\colon B \to A$. Moreover, F_{arw} preserves identity arrows on C^{ob}-objects, and respects (covariant) composition: for any C^{op}-arrows f, g such that their composition $g \circ f$ exists, $F(g \circ f) = Fg \circ Ff$. Being more explicit about the composition operator for once, the point is that if $g \circ_{\mathsf{C}^{op}} f$ in C^{op} exists, then by definition it is $f \circ_{\mathsf{C}} g$ in C. And we know that the contravariant F sends $f \circ_{\mathsf{C}} g$ to $Fg \circ_{\mathsf{D}} Ff$.

And all that simply adds up to our contravariant F having the data and satisfying the conditions to be a standard, covariant, functor from C^{op} to D. □

So whenever we are tempted to talk of a contravariant functor from the category C, we *could* always talk of the same data in its guise as a covariant functor from C^{op}. But I don't think that's the best way to keep things clear, and will often take contravariant functors as nature intended!

However I do adopt one common convention: from now on, unqualified talk of a functor should be read as referring by default to a covariant one.

Finally, for future use, another mini-theorem:

Theorem 130. *If two contravariant functors compose, the result is a covariant functor.* □

Exercise: check that, and explain how it can be that contravariant functors don't compose to give a contravariant functor while covariant functors compose to give a covariant functor, even though contravariant functors have the same data as covariant functors.

28 What do functors preserve?

A functor $F\colon \mathsf{C} \to \mathsf{D}$ sends each C-object A to its image FA and sends each C-arrow $f\colon A \to B$ to its image $Ff\colon FA \to FB$. These resulting images assemble into an overall image or representation of the category C living in the category D. But how good a representation do we get in the general case? What features of C get carried over by a functor?

28.1 Images assembled by a functor needn't be categories

First an important general observation worth highlighting as a theorem.

The image of a group G under a group homomorphism $f\colon G \to H$ will itself be a group, a subgroup of H (see Theorem 5). By contrast,

Theorem 131. *The image of* C *in* D *assembled by a functor* $F\colon \mathsf{C} \to \mathsf{D}$ *need not be a subcategory of* D.

Proof. A toy example establishes the point. Let C be the four-object category we can diagram as

$$A \longrightarrow B \qquad\qquad C \longrightarrow D$$

and let D be the three-object category

$$\begin{tikzcd} X \arrow[r] \arrow[rr, bend left] & Y \arrow[r] & Z \end{tikzcd}$$

(where we omit the identity arrows from our diagrams). Suppose F_{ob} sends A to X, both B and C to Y, and D to Z; and let F_{arw} send identity arrows to identity arrows, and send the arrows $A \to B$ and $C \to D$ respectively to $X \to Y$ and $Y \to Z$. Trivially F with those components is functorial. But the image of C under F is not a category (and so not a subcategory of D), since it contains the arrows $X \to Y$ and $Y \to Z$ but not their composition. $\square$

28.2 Preserving and reflecting

(a) We next introduce a pair of standard notions for describing the actions of functors:

Definition 109. Suppose $F\colon \mathsf{C} \to \mathsf{D}$ and P is some property of objects or arrows or some combination. Then

(1) F *preserves* P iff, for any relevant C-data D, if D has property P, so does the result of applying F to D, $F(D)$ for short.
(2) F *reflects* P iff, for any C-data D, if $F(D)$ has property P, so does D.

We will also say that F preserves (reflects) Xs if F preserves (reflects) the property of being an X. △

For example, to say that a functor preserves commutative diagrams – and they all do! – is to say that if some objects/arrows have the property of forming a commutative diagram in a category C, then the result of applying some functor $F\colon \mathsf{C} \to \mathsf{D}$ to those objects and arrows always gives us another commutative diagram in D.

(b) So what properties of arrows in particular do and don't get preserved or reflected by functors? First, a negative result:

Theorem 132. *Functors do not necessarily preserve or reflect monomorphisms and epimorphisms.*

Proof: functions needn't preserve monics. A toy example makes the point. Recall 2, the two-object category which we can diagram on the left (omitting identity arrows), and let C be the cofork-shaped category on the right, where $f \neq g$ but $k \circ f = f \circ g$ (again omitting identity arrows).

$$V \xrightarrow{j} W \qquad\qquad X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{k} Z$$

Trivially, the non-identity arrow j on the right is monic in 2. Equally trivially, the arrow k on the right is not monic in C. And so the inclusion functor $F\colon 2 \to \mathsf{C}$ which sends V to Y, sends W to Z, and sends j to k, doesn't preserve monics. □

Are there less artificial examples? Well, we'll find that lots of nice functors preserve limits like pullbacks, and Theorem 144 will tell us that such functors *do* preserve monos. But for an example 'in nature', so to speak, it can be shown that the functor mentioned in §33.2 which sends a group to its abelianization doesn't preserve monomorphisms.

Proof: functions needn't preserve epics. We might note that if F is a functor from C to D, it also serves as a functor from C^{op} to D^{op}. So if a functor F doesn't preserve monics in C (and it needn't) it won't preserve epics in C^{op}.

However, there is a more natural example we can rely on to make the point that functors needn't preserve epics. In §7.5, Ex. (3) we saw that the inclusion map $i_M\colon (\mathbb{N}, +, 0) \to (\mathbb{Z}, +, 0)$ in Mon is epic. But plainly the inclusion map $i_S\colon \mathbb{N} \to \mathbb{Z}$ in Set is not epic (as it isn't surjective). Therefore the forgetful functor $F\colon \mathsf{Mon} \to \mathsf{Set}$ maps an epic map (i_M) to a non-epic one (i_S), so does not preserve epics. □

Proof: functions needn't reflect monics or epics. For an easy example of a functor which need not reflect monics or epics, consider a collapse functor which maps C to $\mathbf{1}$, thereby sending arrows of all sorts to the trivially monic and epic identity arrow on the sole object of $\mathbf{1}$. □

(c) Now for a more positive result:

Theorem 133. *Functors preserve right inverses, left inverses, and isomorphisms. But functors do not necessarily reflect those.*

Proof. We show functors preserve right inverses. Suppose $F\colon \mathsf{C} \to \mathsf{D}$ is a functor and the arrow $f\colon A \to B$ is a right inverse in the category C. Then for some arrow g, $g \circ f = 1_A$. Hence $F(g \circ f) = F(1_A)$. By functoriality, that implies $F(g) \circ F(f) = 1_{FA}$. So $F(f)$ is a right inverse in the category D.

Similarly, left inverses are preserved. And putting the two results together shows that isomorphisms are preserved.

For the negative result, just consider again the collapse functor sending C to $\mathbf{1}$. The only arrow in $\mathbf{1}$, the identity arrow, is trivially an isomorphism (and so a left and right inverse). The C-arrows sent to it will generally not be. □

So functors needn't reflect isomorphisms: but there is a special term for those which do:

Definition 110. A functor F is *conservative* iff it reflects all isomorphisms. △

28.3 Faithful, full, and essentially surjective functors

(a) The moral of the previous section is that in the general case a functor may not preserve or reflect very much. We are obviously going to be interested, then, in looking at some special kinds of functor which *do* preserve and/or reflect more.

Let's start by defining analogues for the notions of injective and surjective functions. First, as far as their behaviour on *arrows* is concerned, the useful notions for functors turn out to be these:

Definition 111. A functor $F\colon \mathsf{C} \to \mathsf{D}$ is *faithful* iff given any C-objects A, B, and any pair of parallel arrows $f, g\colon A \to B$, then if $F(f) = F(g)$, then $f = g$.

F is *full* (that's the standard term) iff given any C-objects A, B, then for any D-arrow $g\colon FA \to FB$ there is an arrow $f\colon A \to B$ such that $g = Ff$.

F is *fully faithful*, some say, iff it is full and faithful. △

Note, a faithful functor needn't be, overall, injective on arrows. For suppose C is in effect two copies of D, and F sends each copy faithfully to D: then F sends two copies of an arrow to the same image arrow. However, for each particular pair of objects A and B, a faithful functor is injective from the arrows $A \to B$ to the arrows $FA \to FB$. Likewise, a full functor needn't be, overall, surjective on arrows: but it is locally surjective from the arrows $A \to B$ to the arrows $FA \to FB$.

(b) Second, in connection with the way functors treat *objects*, the notions worth highlighting are these:

Definition 112. A functor $F\colon \mathsf{C} \to \mathsf{D}$ is *essentially injective on objects* iff $FC \cong FC'$ implies $C \cong C'$, for any C-objects C and C'.

More importantly: a functor $F\colon \mathsf{C} \to \mathsf{D}$ is *essentially surjective on objects* (e.s.o.) iff for any D-object D, there is a C-object C such that $FC \cong D$. △

Plain injectivity on objects (requiring that $FC = FC'$ implies $C = C'$) is less interesting, given that we usually only care, categorially speaking, about the identity of objects up to isomorphism. Likewise plain surjectivity on objects (requiring, for every object D in D, an object C such that $FC = D$) is less interesting, given that we similarly won't care whether D has extra non-identical-but-isomorphic copies of objects.

In fact, caring about whether objects are actually identical as opposed to isomorphic is often jokingly said to be 'evil' as far as category theory is concerned – a point we'll return to in §34.5.

(c) Some examples:

(1) The forgetful functor $F : \mathsf{Mon} \to \mathsf{Set}$ is faithful, as F sends a set-function which happens to be a monoid homorphism to itself, so different arrows in Mon get sent to different arrows in Set. But the functor is not full: there will be lots of arrows in Set that don't correspond to a monoid homomorphism.

(2) The forgetful functor $F\colon \mathsf{Ab} \to \mathsf{Grp}$ is faithful. And this one is full, because any objects C, C' in Ab (in other words, any two abelian groups) also live in Grp, and a group homomorphism between them as an arrow in Grp is also an abelian group homomorphism in Ab. But lots of groups aren't abelian, so F is not essentially surjective on objects.

(3) Take the category $\mathsf{2}^*$ with exactly two isomorphic objects $\bullet$ and $\star$, whose four arrows are the identity arrows and inverse arrows each way between the two objects. Then the only possible functor $F\colon \mathsf{2}^* \to \mathsf{1}$ is trivially faithful (there are no parallel arrows in $\mathsf{2}^*$) and trivially full. But note that it is not injective on objects.

(4) The 'thinning' functor (F5) from §27.2, $F\colon \mathsf{C} \to \mathsf{D}$, is full but not faithful unless C is already a pre-order category. But it will be e.s.o.

(5) Suppose M and N are the categories that correspond to the monoids $(M, *, e)$ and $(N, \star, d)$. And let f be a monoid homomorphism between those monoids which is surjective but not injective. Then the functor $F\colon \mathsf{M} \to \mathsf{N}$ corresponding to f is full but not faithful.

(6) You might be tempted to say that the 'total collapse' functor $\Delta_1 : \mathsf{Set} \to \mathsf{1}$ (which ends every set to the sole object of $\mathsf{1}$, and every set-function to identity arrow of $\mathsf{1}$) is full but not faithful. But it isn't full. Take A, B in Set to be respectively some singleton and the empty set. There is a trivial identity map in $\mathsf{1}$, $1\colon \Delta_1 A \to \Delta_1 B$; but there is no arrow in Set from A to B.

(7) An inclusion functor $F\colon \mathsf{S} \to \mathsf{C}$ is faithful; if S is a full subcategory of C, then the inclusion map is fully faithful, but usually not e.s.o.

(8) Consider again the free functor (F15) from §27.4. It sends different set functions $f, g\colon X \to Y$ to different functions $Ff, Fg\colon \mathit{List}(X) \to \mathit{List}(Y)$ (if f and g give different values when applied to the object x, then Ff and Fg will give different values applied to the corresponding list whose sole object is x). So F is faithful.

Now consider a singleton set 1. This gets sent by F to the free monoid with a single generator – which is tantamount to $N = (\mathbb{N}, +, 0)$. The sole set-function from 1 to itself, the identity function, gets sent by F to the identity monoid homomorphism on N. But there are other monoid homomorphisms from N to itself, e.g. $n \mapsto 2n$. So F is not full.

(d) Now some more about how faithful or fully faithful functors behave.

Being full is being locally surjective on arrows, and compositions of surjective functions are surjective; being faithful is being locally injective, and compositions of injective functions are injective. Hence

Theorem 134. *The composition of full functors is full and the composition of faithful functors is faithful.* □

Next, we note

Theorem 135. *A faithful functor $F\colon \mathsf{C} \to \mathsf{D}$ reflects monomorphisms and epimorphisms.*

Proof. Suppose Ff is monic, and suppose $f \circ g = f \circ h$. Then $F(f \circ g) = F(f \circ h)$, so by functoriality $Ff \circ Fg = Ff \circ Fh$, and since Ff is monic, $Fg = Fh$. Since F is faithful, $g = h$. Hence f is monic. Dually for epics. □

Theorem 136. *If a functor is fully faithful it reflects right inverses and left inverses, and hence is conservative.*

Proof. Suppose $F\colon \mathsf{C} \to \mathsf{D}$ is a fully faithful functor, and let Ff be a right inverse, with f an arrow in C with source A. Since F is full, Ff must be the right inverse of Fg for some arrow g in C. So $Fg \circ Ff = 1_{FA}$, whence $F(g \circ f) = 1_{FA} = F(1_A)$. Since F is faithful, it follows that $g \circ f = 1_A$, and f is a right inverse.

Dually, F reflects left inverses, and combining the two results shows that F reflects isomorphisms, i.e. is conservative. □

Note, however, that the reverse of the last result is not true. A functor can reflect isomorphisms without being fully faithful. An example is the forgetful functor $F\colon \mathsf{Mon} \to \mathsf{Set}$. This is faithful but not full. But it is conservative because if the set function Ff is an isomorphism, so is the monoid homomorphism f, because a monoid homomorphism is an isomorphism if and only if its underlying function is one too.

Finally, we have seen that fully faithful functors need neither be injective on objects nor essentially surjective on objects (and hence not plain surjective). However,

Theorem 137. *A fully faithful functor* is *essentially injective on objects.*

Proof. If $F\colon \mathsf{C} \to \mathsf{D}$ is full then if $FC \cong FC'$, i.e. if there is an isomorphism $g\colon FC \xrightarrow{\sim} FC'$, then there is some $f\colon C \to C'$ such that $g = Ff$. But then, by the previous theorem, if F is faithful as well, it reflects isomorphisms, so f is an isomorphism, witnessing that $C \cong C'$. □

28.4 An example from topology

(a) Let's descend from airy generalities to a particular case. We'll look at a classic example from topology. And it is easy enough to get a glimmer of what's going on even if you know almost nothing about the setting. You just need the idea of the fundamental group of a topological space (at a point), roughly as follows.

Given a space and a chosen base point in it, consider all the directed paths that start at this base point then wander around and eventually loop back to their starting point. Such directed loops can be "added" together in an obvious way: you traverse the "sum" of two loops by going round the first loop, then round the second. Every loop has an "inverse" (you go round the same path in the opposite direction). Two loops are considered homotopically equivalent if one can be continuously deformed into the other. We can take, then, the set of all such equivalence classes of loops – so-called homotopy equivalence classes – and define "addition" for these classes in the obvious derived way. This set, when equipped with addition, evidently forms a group: it is the *fundamental group* for that particular space, with the given basepoint. (Though for many spaces, the nature of the group is independent of the basepoint.)

Suppose, therefore, that Top_* is the category of pointed topological spaces: an object in the category is a topological space X equipped with a distinguished base point x_0, and the arrows in the category are continuous maps that preserve basepoints.

Then here's our new example of a functor:

(F3) There is a functor $\pi\colon \mathsf{Top}_* \to \mathsf{Grp}$ with the following data

i. π sends a pointed topological space (X, x_0) – i.e. X with base point x_0 – to the fundamental group $\pi(X, x_0)$ of X at x_0.

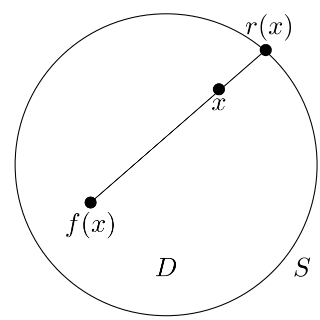
ii. π sends a basepoint-preserving continuous map $f\colon (X, x_0) \to (Y, y_0)$ to a corresponding group homomorphism $f_*\colon \pi(X, x_0) \to \pi(Y, y_0)$.

To explain: f will send a continuous loop based at x_0 to a continuous loop based at y_0. And since f is continuous, it can be used to send a continuous deformation of a loop in (X, x_0) to a continuous deformation of a loop in (Y, y_0). And that induces a corresponding association f_* between the homotopy equivalence classes of (X, x_0) and (Y, y_0), and this will respect the group structure of adding and reversing loops. Moreover, a composition of continuous maps f, g will give rise to a composition of corresponding homomorphisms f_*, g_*. OK – that's a bit arm-waving. But, that it should be enough to persuade that π is indeed functorial.

(b) Here, then, is a nice application. We'll prove Brouwer's famed Fixed Point Theorem:

Theorem 138. *Any continuous map of the closed unit disc to itself has a fixed point.*

Proof Suppose, for reductio, that there is a continuous map f on the two-dimensional disc D (considered as a topological space) *without* a fixed point, i.e. such that for any point x in D, we always have $f(x) \neq x$.



Let the boundary of the disc be the circle S (again considered as a topological space). Then we can define a map that sends the point x in D to the point $r(x)$ on S at which the straight line starting from the point $f(x)$ and passing through x intersects the boundary of the disc.

This map sends a point on the boundary to itself. Pick a boundary point to be the base point of the pointed space D_* and also of the pointed space S_*, then our map induces a map $r\colon D_* \to S_*$. Moreover, this map is evidently continuous (intuitively: nudge a point x and since f is continuous that just nudges $f(x)$, and hence the ray from $f(x)$ through x is only nudged, and the point of intersection with the boundary is only nudged). And r is a left inverse of the inclusion map $i\colon S_* \to D_*$ in Top_*, since $r \circ i = 1$.

Functors preserve left inverses by Theorem 133, so $\pi(r)$ will be a left inverse of $\pi(i)$, which means that $\pi(i)\colon \pi(S_*) \to \pi(D_*)$ is an injection by Theorem 16.

But that's impossible. $\pi(S_*)$, the fundamental group of S_*, is equivalent to the group of integers under addition (think of looping round a circle, one way or another, n times – each positive or negative integer corresponds to a different path). While $\pi(D_*)$, the fundamental group of D_*, is just a one element group (for every loop in the disk D_* can be smoothly shrunk to a point). And there is no injection between the integers and a one-element set! $\square$

(c) What, if anything, do we gain from putting the proof in category-theoretic terms? It might be said: the proof crucially depends on facts of algebraic topology – continuous maps preserve homotopic equivalences in a way that makes π a functor, and the fundamental groups of S^* and D^* are respectively the group of integers and the trivial group. And we could run the whole proof without actually mentioning categories at all.

Of course. Still, what we've done is to very clearly demarcate those bits of the proof that depend on topic-specific facts of topology and those bits which depend on general proof-ideas about functoriality and about kinds of maps (inverses, injections), ideas which are thoroughly *portable* to other contexts. And *that* arguably counts as a real gain in understanding.

28.5 An afterword on the idea of concrete categories

(a) In §4.6, I mentioned that categories like Mon and Preord whose objects are sets-equipped-with-some-structure and whose arrows are structure-respecting-set-functions are often called concrete categories. As we also saw right at the outset, lots of categories are *not* concrete in this intuitive sense – for example, neither a monoid-as-category nor a pre-ordered-collection-as-category will count.

Well, now that we have the notion of a faithful functor in play, I guess I should mention a conventional official definition:

Definition 113. A *concrete category* is a pair (C, U) such that C is a category and $U \colon \mathsf{C} \to \mathsf{Set}$ is a faithful functor.

A category C is *concretizable* if there exists a faithful functor $U \colon \mathsf{C} \to \mathsf{Set}$.

A paradigm case is then indeed provided by a category like Mon equipped with the forgetful functor U which sends a monoid to its underlying set and sends a monoid homomorphism to its underlying set-function.

(b) So far, so good. However, it is easy to see that this new definition doesn't capture the original intuitive notion of a concrete category as a category of structured sets.

For example, suppose we take some objects P (non-sets, and not too many!) pre-ordered by the relation $\preccurlyeq$. Then, as noted in §4.4, there is a corresponding category P whose objects are just P again, and which has a single arrow from a particular object q to an object r if and only if $q \preccurlyeq r$. Now, we can presumably index these objects by associating them one-by-one with pure sets. Let p' be the index of the object p. We can then define a functor $F \colon \mathsf{P} \to \mathsf{Set}$ by stipulating

1. F_{ob} sends a P-object q to the set $P_{\leqslant q}$ of p' such that $p \leqslant q$,
2. F_{arw} sends a P-arrow $q \to r$ to the inclusion function $i \colon P_{\leqslant q} \hookrightarrow P_{\leqslant r}$.

It is trivial to check that this *is* a functor, and that it is faithful.

But this makes P equipped with F officially count as a concrete category according to our new-fangled definition: yet the likes of P are just the sort of example that we originally *contrasted* with concrete categories in our intuitive sense. And it gets worse. We can similarly show that *any* small category whose objects and arrows can be indexed by sets can be made concrete in our new sense.

(c) What to do? Should we aim to find a revised definition which sticks closer to that original intuitive idea of concrete categories as being actually built out of suitably equipped sets? Perhaps not. For category theory is centrally concerned with the *structural* properties of structures, not what they are 'made of'. So perhaps our Defn. 113 does home in on the categorially significant idea here. And while all categories that aren't too big do count as concretizable on this definition, it still draws an interesting distinction among the very large categories. For example, while Set is trivially concrete, it can be proved that the category hTop is not concretizable, where that is the category of topological spaces whose arrows are whole classes of maps which can be continuously deformed into each other. But following this theme any further would take us too far from the elementary focus of these notes. So we must let the topic rest here.[1]

[1] The highly non-trivial result about hTop is due to Freyd (1970). I note that the elementary texts by e.g. McLarty (1992), Goldblatt (2006), Simmons (2011), and Leinster (2014) perhaps sensibly avoid the topic of concreteness altogether!

29 Functors, diagrams, and limits

As we have seen, a functor $F\colon \mathsf{J} \to \mathsf{C}$ will, just in virtue of its functoriality, preserve/reflect some minimal aspects of the categorial structure of J as it sends objects and arrows into C. And if the functor has properties like being full or faithful it will preserve/reflect more.

We now want to ask: how do things stand with respect to preserving/reflecting products, equalizers, quotients and the like? – more generally, we want to know about preserving/reflecting limits and colimits.

29.1 Diagrams redefined as functors

We start with some important redefinitions. For now we have the notion of a functor in play, we can pin down the notion of a diagram, and the notion of a limit over a diagram (or colimit under a diagram), in a particularly neat way.

(a) First, diagrams. The idea is this:

(i) A functor $D\colon \mathsf{J} \to \mathsf{C}$ will send the objects and arrows of J to some corresponding objects and arrows sitting inside C. So we can think of D as generating in C a *diagram* in the sense introduced rather loosely in §5.1 and then refined a little in §18.2.

(ii) This diagram might not be a faithful representation of J – because distinct arrows of J might get diagrammed by a single C arrow, and likewise for objects.

(iii) Still, we could say that – like a geographical map which doesn't distinguish some features but still pictures something of the shape of the terrain – the induced diagram in C retains something of the shape of the original category J.

And this is enough to motivate some absolutely standard terminology:

Definition 114. Given a category C, and a category J, a functor $D\colon \mathsf{J} \to \mathsf{C}$, is said to be a *diagram of shape* J *in* C.[1] $\triangle$

[1] Yes, the functor with source J which creates a diagram which retains something of the shape of J is said itself to *be* a diagram and the diagram is said, without qualification, to *have* shape J. We can learn to live with this idiom!

(b) To go along with this redefinition of diagrams as functors, there are entirely predictable redefinitions of cones and limit cones (we'll leave co-cones and colimits to look after themselves). We just modify in obvious ways the definitions we met earlier in §18.1, 18.2.

So, look again at Defn. 75: this told us that a cone with vertex C over a diagram (old sense!) with objects D_j is a suite of arrows $c_j : C \to D_j$ where, for any arrow $d \colon D_k \to D_l$ in the diagram, the triangle formed with the arrows c_k and c_l commutes, so $c_l = d \circ c_k$. And look again at Defn. 76: this told us that, among the cones over a given diagram (old sense!), (L, λ_j) is a limit cone if, for any other cone (C, c_j), there is a unique arrow $u \colon C \to L$ such that, for every c_j, we have $c_j = \lambda_j \circ u$.

These old definitions now naturally generate the following two-part definition for cones and limit cones over diagrams in our new sense:

Definition 115. Suppose we are given a category C, together with a diagram-as-functor $D \colon \mathsf{J} \to \mathsf{C}$. By definition, D sends a J-arrow $j \colon K \to L$ to the C-arrow $D(j) \colon D(K) \to D(L)$. Then:

(1) *A cone over D* in C has an object C as vertex and an arrow $c_J : C \to D(J)$ for each J-object J, subject to the following condition: for any J-arrow $j \colon K \to L$, we have $c_L = D(j) \circ c_K$ in C. We again use (C, c_J) to denote such a cone.

(2) *A limit cone over D* is a cone (L, λ_J) such that for every cone (C, c_J) over D, there is a unique arrow $u \colon C \to L$ such that, for all J-objects J, $c_J = \lambda_J \circ u$. △

(c) How does our old talk of diagrams and limits relate to our new talk? Two quick general points:

(1) Evidently, not every diagram-in-C in the original sense of §5.1 corresponds to a diagram-as-functor. There's a trivial reason. A diagram of shape J in C will always need to carry over the required identity arrows on all the objects in J to identity arrows on all their images. But a diagram-in-a-category as we first defined it doesn't need to have identity arrows on all (or indeed any) of its objects.

(2) Still, the lack of a straight one-to-one correspondence between diagrams in the two senses makes no difference when thinking about limits.

A limit (new sense) over the diagram $D \colon \mathsf{J} \to \mathsf{C}$ will of course be a limit (old sense) over the D-image of J living in C.

Conversely, suppose (L, λ_j) is a limit cone over some diagram D (in the original sense of diagram). Then by Theorem 77, (L, λ_j) is also a limit over the reflexive, transitive closure of D because *every* cone over D is equally a cone over its closure. But we noted that we can think of this closure as a subcategory of C, call it J. So now take the inclusion functor $D_I : \mathsf{J} \to \mathsf{C}$. Then, by our new definition, (L, λ_j) becomes a limit cone over the diagram-as-functor $D_I : \mathsf{J} \to \mathsf{C}$.

In short, limits in the old and new senses come to the just same, and we can from now on follow the widely-adopted line of taking our revised definitions of diagrams and their limits as the basic one.[2]

(d) Our key specific examples of limits earlier were terminal objects, products, equalizers, and pullbacks. In our original discussion in §18.2, we saw these to be respectively limits over the following diagrams (in the old sense):

(1) the empty diagram,

(2) a two-object no-arrow diagram,

(3) a two-object two-parallel-arrow diagram, and

(4) a three-object, two-arrow corner diagram.

Now thinking of the diagrams and limits in the new way, terminal objects, products, equalizers, and pullbacks become limit cones over respectively the following kinds of diagrams-as-functors:

(1′) a diagram of the shape of the empty category (we allowed the limiting case of an empty category in §4.1, and here's a case where we see why this is a convenient policy).

(2′) a diagram of the shape of the discrete two-object category $\overline{2}$: $\begin{tikzcd} \bullet \arrow[loop above] & \star \arrow[loop above] \end{tikzcd}$

(3′) a diagram of the shape of the category $\begin{tikzcd} \bullet \arrow[loop left] \arrow[r, shift left] \arrow[r, shift right] & \star \arrow[loop right] \end{tikzcd}$.

(4′) a diagram of shape of the category $\begin{tikzcd} \bullet \arrow[loop left] \arrow[r] & \star \arrow[loop above] & \diamond \arrow[l] \arrow[loop right] \end{tikzcd}$

(e) Finally, an obvious bit of terminology for future use:

Definition 116. A category C *has all limits of shape* J iff for every functor $D\colon \mathsf{J} \to \mathsf{C}$, the category has a limit cone over D. △

29.2 Preserving limits

(a) Let's start with another definition, extending the notion of preservation we met in §28.1: we say a functor preserves limits if it sends limits of a given shape to limits of the same shape (and preserves colimits if it sends colimits to colimits; but I won't keep mentioning the dual case). More carefully,

Definition 117. A functor $F\colon \mathsf{C} \to \mathsf{D}$ *preserves the limit* (L, λ_J) over $D\colon \mathsf{J} \to \mathsf{C}$ iff $(FL, F\lambda_J)$ is a limit over $F \circ D\colon \mathsf{J} \to \mathsf{D}$. △

But limits, we know, are only unique up to isomorphism. So let's quickly check that a functor will treat all limits over a given diagram the same way:

[2]See e.g. Borceux (1994, p. 56), Leinster (2014, p. 118) and Riehl (2017, §3.1).

Theorem 139. *If $F\colon \mathsf{C} \to \mathsf{D}$ preserves a limit over the diagram $D\colon \mathsf{J} \to \mathsf{C}$, it preserves* all *limits over that diagram.*

Proof. Suppose (L, λ_J) is a limit cone over $D\colon \mathsf{J} \to \mathsf{C}$. Then, by the argument of Theorem 74, if (L', λ'_J) is another such cone, there is an isomorphism $f : L' \to L$ in C such that $\lambda'_J = \lambda_j \circ f$.

Suppose now that F preserves (L, λ_J) so $(FL, F\lambda_J)$ is a limit cone over $F \circ D$. Then F will send (L', λ'_J) to $(FL', F\lambda'_J)$, i.e. $(FL', F\lambda_J \circ Ff)$. But then this factors through $(FL, F\lambda_J)$ via the $Ff\colon FL' \to FL$, and Ff is an isomorphism (remember, functors preserve isomorphisms). Hence, by Theorem 75, $(FL', F\lambda'_J)$ is also a limit over $F \circ D$. In other words, F preserves (L', λ'_J) too. □

(b) We have just been talking about preserving limits over one particular given diagram. Next, let's say

Definition 118. A functor $F\colon \mathsf{C} \to \mathsf{D}$ *preserves all limits of shape* J *in* C iff, for *any* diagram $D\colon \mathsf{J} \to \mathsf{C}$, then F preserves the limits over D. △

We should note immediately however that preserving all limits of one shape can go along with preserving none of another shape.

A toy example establishes the point. Take the two posets $(\{0, 1, 2, 3, 4, 5\}, \leqslant)$ and $(\mathbb{N}, \leqslant)$ thought of as categories. There is a trivial inclusion functor I from the first category to the second. This functor preserves all limits of the shape $\overline{2}$, i.e. all products (recall that the product of two elements in a poset, when it exists, is their greatest lower bound). But the inclusion functor doesn't preserve limits of the shape of the null category, i.e. doesn't map a terminal object to a terminal object (5 is terminal in the first category, but $5 = I(5)$ is not terminal in the second).

However, some functors do preserve limits more generally. Let's say, for example, that

Definition 119. A functor $F\colon \mathsf{C} \to \mathsf{D}$ *preserves all finite limits* iff it preserves limits of shape J whenever J is finite (i.e. has only finitely many objects and arrows). △

Then we have the following sample theorem:

Theorem 140. *The forgetful functor $F\colon \mathsf{Mon} \to \mathsf{Set}$ preserves all finite limits.*

(c) We'll prove that last theorem in a moment, after first establishing a more general result:

Theorem 141. *If C is finitely complete, and a functor $F\colon \mathsf{C} \to \mathsf{D}$ preserves terminal objects, binary products and equalizers, then F preserves all finite limits.*

Proof. Assume C is finitely complete as defined in Defn. 84 – so for any finite diagram in the original sense, C has a limit cone over it. Then it will evidently be finitely complete in the sense that for any finite diagram in the new sense, i.e. any functor $D\colon \mathsf{J} \to \mathsf{C}$ with J finite, then C has a limit cone (C, c_J) over D.

Then by the argument in the proof of Theorem 88, there will also be a limit cone (C', c'_J) over the diagram D constructed from equalizers and finite products. And by the general uniqueness-up-to-isomorphism result for limit cones, (C', c'_J) is isomorphic to (C, c_J).

Now, by assumption, F preserves terminal objects, binary products and equalizers, so F will send the construction (C', c'_J) to a construction (FC', Fc'_J) which will similarly be constructed from terminal objects, binary products and equalizers in a way making it a limit cone over $F \circ D \colon \mathsf{J} \to \mathsf{D}$.

But F preserves isomorphisms, so F also sends the isomorphism in C between (C, c_J) and (C', c'_J) to an isomorphism in D between (FC, Fc_J) and (FC', Fc'_J). But what is isormorphic to a limit cone over $F \circ D$ is itself a limit cone over $F \circ D$. Hence (FC, Fc_J) is indeed a limit cone, showing that F preserves that limit. □

And that established, we can now return to the previously stated theorem:

Proof. First, the forgetful functor $F \colon \mathsf{Mon} \to \mathsf{Set}$ evidently sends a terminal object in Mon, a one-object monoid, to its underlying singleton set, which is terminal in Set. So F preserves terminal objects (limits of the null shape).

Second, the same functor sends a product $(M, *, e) \times (N, \star, d)$ in Mon to its underlying set of pairs of objects from M and N, which is a product in Set. So the forgetful F also preserves products..

Third, we saw in §15.1, Ex. (2), the equalizer of two parallel monoid homomorphisms $(M, *, e) \overset{f}{\underset{g}{\rightrightarrows}} (N, \star, d)$ is $(E, \cdot)$ equipped with the inclusion map $E \to M$, where E is the set on which f and g agree. Which means that the forgetful functor takes the equalizer of f and g as monoid homomorphisms to their equalizer as set functions. So F preserves equalizers.

So: we know from Theorem 91 that Mon is finitely complete, and have now seen that the forgetful functor $F \colon \mathsf{Mon} \to \mathsf{Set}$ preserves terminal objects, binary products and equalizers. Hence F in fact preserves all the finite limits in Mon. □

(d) We've been concentrating on finite limits because in many contexts this is the interesting case. But we can go further, as we already briefly noted in §20.5.

Recall, a category J is small if (roughly) it doesn't have too many objects or arrows to form sets. Then we can say, predictably,

Definition 120. A functor $F \colon \mathsf{C} \to \mathsf{D}$ *preserves all small limits* iff it preserves limits of shape J whenever J is small. △

And we just note again that our proof of Theorem 88 still goes through even when dealing with non-finite diagrams – and if we assume everything is set-sized, then the argument could still be dressed up as set-theoretically respectable. So, we will be able to beef up Theorem 140 to give

Theorem 142. *The forgetful functor $F \colon \mathsf{Mon} \to \mathsf{Set}$ preserves all small limits.*

But we perhaps needn't go into more details.

29.3 A few small challenges!

(a) To help fix ideas, pause to prove the following three simple results about preservation and limits.

First, let's reinforce the point that functors can preserve some limits while not preserving others. With 2 your favourite two-object set,

Theorem 143. *Let $Q\colon \mathsf{Set} \to \mathsf{Set}$ send any set X to the set $X \times 2$, and send any arrow $f\colon X \to Y$ to $f \times 1_2\colon X \times 2 \to Y \times 2$. Then Q is a functor, and preserves equalizers but not binary products.*

Now for a sample result showing how preservation results can be inter-related:

Theorem 144. *If a functor preserves pullbacks then it preserves monomorphisms. Dually, if it preserves pushouts it preserves epimorphisms.*

Thirdly, note:

Theorem 145. *The forgetful functor $F\colon \mathsf{Mon} \to \mathsf{Set}$ does* not *preserve colimits.*

(b) Take the three ingredients of the first theorem in turn:

Proof that Q is a functor. We need to confirm, in particular, that $Q(f \circ g) = Qf \circ Qg$ – in other words $(f \circ g) \times 1_2 = (f \times 1_2) \circ (g \times 1_2)$. But that follows from Theorem 52 and the trivial fact that $1_2 \circ 1_2 = 1_2$. □

Proof that Q doesn't preserve products. Suppose e.g. that X, Y are singletons, then evidently $Q(X \times Y) = (X \times Y) \times 2 \ncong (X \times 2) \times (Y \times 2) = QX \times QY$. But if $Q(X \times Y) \ncong QX \times QY$, Q doesn't preserve products. □

Proof that Q preserves equalizers. The equalizer of $f, g\colon X \to Y$ is essentially E, the subset of X on which f and g take the same value. And the equalizer of the parallel arrows $Qf, Qg\colon QX \to QY$ is the subset of $X \times 2$ on which $f \times 1_2$ and $g \times 1_2$ take the same value, which will be $E \times 2$, i.e. QE. So indeed Q preserves equalizers. □

For the next theorem we need to show:

Preserving pullbacks implies preserving monomorphisms. We use Theorem 82. This tells us that if $f\colon X \to Y$ in C is monic then it is part of a pullback square, as on the left:

$$\begin{tikzcd} X \arrow[r, "1_X"] \arrow[d, "1_X"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} FX \arrow[r, "1_{FX}"] \arrow[d, "1_{FX}"'] & FX \arrow[d, "Ff"] \\ FX \arrow[r, "Ff"'] & FY \end{tikzcd}$$

Now suppose $F\colon \mathsf{C} \to \mathsf{D}$ sends pullback squares to pullback squares. Then the square on the right is also a pullback square. Therefore, by Theorem 82 again, Ff is monic too.

Duality gives the other half of Theorem 144. □

Proof that the forgetful $F\colon \mathsf{Mon} \to \mathsf{Set}$ *doesn't preserve colimits.* Take the simplest kind of colimit – initial objects (i.e. colimits under diagrams-as-functors with the 'shape' of the empty category). Then note that a one-object monoid is initial in Mon; but its underlying singleton set is not initial in Set. □

We might usefully also note that the forgetful F does not preserve coproducts either – essentially because coproducts in Mon can be larger than coproducts in Set. Recall our discussion in §10.7 of coproducts in Grp : similarly, $F(M \oplus N)$, the underlying set of a coproduct of monoids M and N, is (isomorphic to) the set of finite sequences of alternating non-identity elements from M and N. Contrast $FM \oplus FN$, which is just the disjoint union of the underlying sets.

Our example generalizes, by the way. A forgetful functor from a category of structured sets to Set typically preserves finite limits but does not preserve all colimits.

29.4 Reflecting limits

(a) Here's a companion definition to set alongside the definition of preserving limits, together with a couple of general theorems:

Definition 121. A functor $F\colon \mathsf{C} \to \mathsf{D}$ *reflects limits of shape* J iff, given a cone (C, c_J) over a diagram $D\colon \mathsf{J} \to \mathsf{C}$, then if (FC, Fc_J) is a limit cone over $F \circ D\colon \mathsf{J} \to \mathsf{D}$, (C, c_J) is already a limit cone over D.

Reflecting colimits is defined dually. △

Theorem 146. *Suppose* $F\colon \mathsf{C} \to \mathsf{D}$ *is fully faithful. Then* F *reflects limits.*

Proof. Suppose (C, c_J) is a cone over a diagram $D\colon \mathsf{J} \to \mathsf{C}$, and (FC, Fc_J) is a *limit* cone over $F \circ D\colon \mathsf{J} \to \mathsf{D}$. We need to show that (C, c_J) must already be a limit cone too.

Now take any other cone (B, b_J) over D. F sends this to a cone (FB, Fb_J) which must uniquely factor through the limit cone (FC, Fc_J) via some $u\colon FB \to FC$ which makes $Fb_J = Fc_J \circ u$ for each $J \in \mathsf{J}$. Since F is full and faithful, $u = Fv$ for some unique $v\colon B \to C$ such that $b_J = c_J \circ v$ for each J. So (B, b_J) factors uniquely through (C, c_J). Which shows that (C, c_J) is a limit cone. □

Theorem 147. *Suppose* $F\colon \mathsf{C} \to \mathsf{D}$ *preserves finite limits. Then if* C *is finitely complete and* F *reflects isomorphisms, then* F *reflects finite limits.*

Proof. Since C is complete there exists a limit cone (B, b_J) over any diagram $D\colon \mathsf{J} \to \mathsf{C}$ (where J is finite), and so – since F preserves limits – (FB, Fb) is a limit cone over $F \circ D\colon \mathsf{J} \to \mathsf{D}$.

Now suppose that there is a cone (C, c_J) over D such that (FC, Fc_J) is another limit cone over $F \circ D$. Now (C, c_J) must uniquely factor through (B, b_J) via a map $f\colon C \to B$. Which means that (FC, Fc_J) factors through (FB, Fb) via Ff. However, since these are by hypothesis both limit cones over $F \circ D$, Ff

must be an isomorphism. Hence, since F reflects isomorphisms, f must be an isomorphism. So (C, c_J) must be a limit cone by Theorem 75. $\square$

(b) Since Mon is finitely complete, and the forgetful functor $F\colon \mathsf{Mon} \to \mathsf{Set}$ preserves limits, and reflects isomorphisms the last theorem shows that

(1) The forgetful functor $F\colon \mathsf{Mon} \to \mathsf{Set}$ reflects all limits. Similarly for some other forgetful functors from familiar categories of structured sets to Set.

However, be careful! For we also have ...

(2) The forgetful functor $F\colon \mathsf{Top} \to \mathsf{Set}$ which sends topological space to its underlying set *preserves* all limits but does not *reflect* all limits.

Here's a case involving binary products. Suppose X and Y are a couple of spaces with a coarse topology, and let Z be the space $FX \times FY$ equipped with a finer topology. Then, with the obvious arrows, $X \leftarrow Z \rightarrow Y$ is a wedge to X, Y but not the limit wedge in Top: but $FX \leftarrow FX \times FY \rightarrow FY$ is a limit wedge in Set.

Given the previous theorem, we can conclude that $F\colon \mathsf{Top} \to \mathsf{Set}$ doesn't reflect isomorphisms. Which is also something we can show directly. (Consider the continuous bijection from the half-open interval $[0, 1)$ to S^1. Think of this bijection as a topological map f; then f is not a homeomorphism in Top. However, treating the bijection as a set-function, i.e. as Ff, it *is* an isomorphism in Set.)

30 Functors and comma categories

In the last chapter, equipped with the notion of a functor, we backtracked and reworked our old ideas of diagrams and of limits over/under diagrams. In this chapter, we use functors to define the new idea of a comma category. This construction is mainly for occasional future use; but we can more immediately use it to backtrack again and rework the ideas of slice categories and arrow categories. We also get a rather nice result about free monoids.

30.1 Comma categories defined

(a) Suppose that we start with three categories $\mathsf{A}, \mathsf{B}, \mathsf{C}$ and a pair of functors involving them, $S\colon \mathsf{A} \to \mathsf{C}$ and $T\colon \mathsf{B} \to \mathsf{C}$. We are going to use these to build a new category.

Now, our two functors give us a way of indirectly connecting an object A in A to an object B in B, namely by looking at their respective images SA and TB and considering C-arrows $f\colon SA \to TB$ between them. (So the functor S provides the *S*ource for the connecting arrow f and the functor T the *T*arget – hence our chosen labels for the functors!)

We are going to define a category involving just such indirect connections. But if its objects are to comprise an A-object A, a B-object B, together with a C-arrow $f\colon SA \to TB$, what could be the arrows in our new category? Suppose we have two triples (A, B, f), (A', B', f'); an arrow between them will presumably involve arrows $a\colon A \to A'$ and $b\colon B \to B'$. But note that these two arrows are sent by S and T respectively to the arrows $Sa\colon SA \to SA'$ and $Tb\colon TB \to TB'$ in C; and we will want these two C-arrows to interact appropriately with the other C-arrows f and f'.

That thought prompts the following suggested definition:

Definition 122 (?). Given functors $S\colon \mathsf{A} \to \mathsf{C}$ and $T\colon \mathsf{B} \to \mathsf{C}$, then the *comma category* $(S \downarrow T)$ is the category with the following data:

(1) The objects of $(S \downarrow T)$ are triples (A, B, f) where A is an A-object, B is a B-object, and $f\colon SA \to TB$ is an arrow in C.

(2) An arrow of $(S \downarrow T)$ from (A, B, f) to (A', B', f') is a pair (a, b), where $a\colon A \to A'$ is an A-arrow, $b\colon B \to B'$ is an B-arrow, and the following diagram commutes:

$$\begin{tikzcd} SA \arrow[r, "f"] \arrow[d, "Sa"] & TB \arrow[d, "Tb"] \\ SA' \arrow[r, "f'"] & TB' \end{tikzcd}$$

(3) The identity arrow on the object (A, B, f) is the pair $(1_A, 1_B)$.

(4) Composition in $(S \downarrow T)$ is induced by the composition laws of A and B, thus: $(a', b') \circ (a, b) = (a' \circ_{\mathsf{A}} a, b' \circ_{\mathsf{B}} b)$. △

So at any rate runs the usual definition.[1] The label 'comma category', by the way, comes from an unhappy earlier notation '(S, T)' – the notation has long been abandoned but the name has stuck.

(b) However, as with our initial attempt at a definition of a slice category back in §6.3, there is a snag. Suppose there are two C-arrows $f_1, f_2 \colon SA \to TB$; then we have two distinct $(S \downarrow T)$-objects (A, B, f_1) and (A, B, f_2). But our definition so far would assign these distinct objects the same identity arrow in $(S \downarrow T)$, namely the pair $(1_A, 1_B)$. However, on our original definition of a category, distinct objects can't share an identity arrow.

So our account of $(S \downarrow T)$ doesn't quite work as it stands as a kosher definition of a category. What to do? In some way, we want to make an arrow from (A, B, f) and (A', B', f') into a triple $(a, b, ?)$ where the mystery ingredient tells us something about the third components of the objects involved. Well, why not just choose the commutative squares which are in the story anyway! And then the identity arrow on (A, B, f_1) will involve a different third component to the identity arrow on the distinct object (A, B, f_2). Hooray!

However, it would be an annoying complication to write this into the official story. From now on, we'll in fact cheat a tiny bit, as we did with slice categories. When we talk of comma categories, we'll continue to talk of an arrow from (A, B, f) and (A', B', f') as if it is simply a suitable pair of arrows which (when hit with S and T) give rise to a commuting square. Since that story gives us the square anyway, no information at all is lost by if we don't beef up the pair into a triple including that square. Allowing ourselves that pinch of salt, we can count $(S \downarrow T)$ as a category.

30.2 Three types of comma category

But why on earth we should be bothering with such an apparently contorted construction?

Well, for a start, the notion of a comma category in fact nicely generalizes a number of simpler constructions. And indeed, we have already met two comma categories in thin disguise.

[1] See for example Adámek et al. (2009, p. 46), Barr and Wells (1985, p. 13), Riehl (2017, p. 22), Simmons (2011, p. 86).

(a) First take the minimal case where $\mathsf{A} = \mathsf{B} = \mathsf{C}$, and where both S and T are the identity functor on that category, 1_{C} (as introduced in Theorem 128). Then,

(1) The objects in this category $(1_{\mathsf{C}} \downarrow 1_{\mathsf{C}})$ are triples $(X, Y, X \xrightarrow{f} Y)$ for X, Y both C-objects.

(2) An arrow in $(1_{\mathsf{C}} \downarrow 1_{\mathsf{C}})$ from $(X, Y, X \xrightarrow{f} Y)$ to $(W, Z, Y \xrightarrow{g} Z)$ is a pair of C-arrows $j \colon X \to W$, $k \colon Y \to X$ such that this square commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "j"'] & Y \arrow[d, "k"] \\ W \arrow[r, "g"] & Z \end{tikzcd}$$

So the only difference between $(1_{\mathsf{C}} \downarrow 1_{\mathsf{C}})$ and the arrow category $\mathsf{C}^{\to}$ which we defined in §6.4 is that we have now 'decorated' the objects of $\mathsf{C}^{\to}$, which were plain C-arrows $f \colon X \to Y$, with explicit assignments of their sources and targets as C-arrows, to give triples $(X, Y, X \xrightarrow{f} Y)$. Hence $(1_{\mathsf{C}} \downarrow 1_{\mathsf{C}})$ and $\mathsf{C}^{\to}$, although not strictly identical, evidently come to the just same.

Of course, we should be able to do a bit better than limply say the two categories 'come to just the same'. And we will do! But we can be content with slightly more casual arm-waving for the moment.

(b) Let's secondly take another special case, this time one where $\mathsf{A} = \mathsf{C}$, while $\mathsf{B} = \mathsf{1}$ (the category with a single object $\star$ and the single identity arrow $1_\star$). So in place of the functor $S \colon \mathsf{A} \to \mathsf{C}$ we can have the identity functor 1_{C}. And in place of the functor $T \colon \mathsf{B} \to \mathsf{C}$ there will be some functor $X \colon \mathsf{1} \to \mathsf{C}$ which sends $\star$ to some individual C-object which we'll also call X and sends $1_\star$ to 1_X (see §27.2, Ex. (F7)). Our definition will now grind out the following specification for the relevant comma category:

(1) The objects of $(1_{\mathsf{C}} \downarrow X)$ will be triples $(A, \star, f)$, where A is any C-object and f any C-arrow from A to X.

(2) And a $(1_{\mathsf{C}} \downarrow X)$-arrow between $(A, \star, A \xrightarrow{f} X)$ and $(B, \star, B \xrightarrow{g} X)$ will be a pair $(j, 1_\star)$, with $j \colon A \to B$ an arrow such this square commutes:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"'] & X \arrow[d, "1_X"] \\ B \arrow[r, "g"] & X \end{tikzcd}$$

But of course that square is trivially equivalent to this triangle

$$\begin{tikzcd} A \arrow[dr, "f"] \arrow[dd, "j"'] & \\ & X \\ B \arrow[ur, "g"'] & \end{tikzcd}$$

And *that* should look rather familiar! In fact, we've ended up with something tantamount to a slice category C/X. The only differences being that (i) instead of the original slice category's objects, i.e. pairs (A, f), in the category $(1_\mathsf{C} \downarrow X)$ we now have corresponding 'decorated' pairs $(A, \star, f)$. And (ii) instead of the slice category's arrows $j\colon A \to B$ satisfying a certain condition, in the category $(1_\mathsf{C} \downarrow X)$ we have corresponding 'decorated' arrows $(j, 1_\star)$, with j still satisfying the same condition.

Hence we can again say: the comma category $(1_\mathsf{C} \downarrow X)$ and the slice category C/X in some good sense amount to the same. Exactly similarly, of course, the comma category $(X \downarrow 1_\mathsf{C})$ and the co-slice category X/C amount to the same.

(c) Let's add a third illustrative case, mainly for future use. This time we don't suppose $\mathsf{A} = \mathsf{C}$, and we make no special assumptions about the functor $S\colon \mathsf{A} \to \mathsf{C}$. However, we again put $\mathsf{B} = \mathsf{1}$ and take the second functor used in constructing our comma category to be some $X\colon \mathsf{1} \to \mathsf{C}$ which sends the unique object $\star$ of $\mathsf{1}$ to some individual C-object X and sends $1_\star$ to 1_X.

Turning the handle again, our definition of a comma category now grinds out this:

(1) The objects of $(S \downarrow X)$ will be triples $(A, \star, f)$, where A is any A-object and f is any C arrow from $SA \to X$.

(2) And a $(S \downarrow X)$-arrow between $(A, \star, SA \xrightarrow{f} X)$ and $(A', \star, SA' \xrightarrow{f'} X)$ will be a pair $(j, 1_\star)$, with $j\colon A \to A'$ an arrow in A such this square commutes in C:

$$\begin{tikzcd} SA \arrow[r, "f"] \arrow[d, "Sj"] & X \arrow[d, "1_X"] \\ SA' \arrow[r, "f'"] & X \end{tikzcd}$$

But as with our last example, the $\star$ component of objects and the $1_\star$ component of arrows are just coming along for the ride, doing no real work. And our commuting square is equivalent to a triangle. So we might as well say, more snappily,

(1′) The objects of $(S \downarrow X)$ are pairs (A, f), where A is any A-object and f is any C-arrow from $SA \to X$.

(2′) And a $(S \downarrow X)$-arrow between $(A, SA \xrightarrow{f} X)$ and $(A', SA' \xrightarrow{f'} X)$ will be an A-arrow $j\colon A \to A'$ such this triangle commutes in C:

$$\begin{tikzcd} SA \arrow[d, "Sj"] \arrow[dr, "f"] & \\ & X \\ SA' \arrow[ur, "f'"'] & \end{tikzcd}$$

Likewise, for a companion definition, now assume instead that $\mathsf{A} = 1$ and take our two functors to be some $X\colon 1 \to \mathsf{C}$ and $T\colon \mathsf{B} \to \mathsf{C}$; then we can say – jumping straight to the snappy version –

(1′′) The objects of $(X \downarrow T)$ will be pairs (B, f), where B is any B-object and f is any C-arrow from X to TB.

(2′′) And a $(X \downarrow T)$-arrow between $(B, X \xrightarrow{f} TB)$ and $(B', X \xrightarrow{f'} TB')$ will be an B-arrow $j\colon B \to B'$ such this triangle commutes in C:

$$\begin{tikzcd} & TB \arrow[d, "Tj"] \\ X \arrow[ur, "f"] \arrow[r, "f'"'] & TB' \end{tikzcd}$$

(d) A minor terminological point. Officially, a comma category $(S \downarrow T)$ is defined in terms of two functors, S and T. When we defined e.g. $(S \downarrow X)$ the 'X' still denotes a *functor*, a functor $X\colon 1 \to \mathsf{C}$ which picks out a particular object X. But of course, it would make no difference if we took the 'X' in notations like $(S \downarrow X)$ to officially denote an *object*, the one which gives rise to the corresponding functor. I mention this as you'll find the notation being interpreted that way, and find corresponding definitions of the likes of $(S \downarrow X)$ which consequently only need to explicitly mention one functor, in this case S.

30.3 An application: free monoids again

We can now make a connection between the idea of a free monoid (which we met in §27.4) and a certain comma category.

Start with the categories 1, Mon, and Set; take the functor $X\colon 1 \to \mathsf{Set}$ which picks out a particular set X, and the forgetful functor $F\colon \mathsf{Mon} \to \mathsf{Set}$. Applying the definition at the end of the last section, we get a comma category $(X \downarrow F)$:

(1) The objects of $(X \downarrow F)$ are pairs of a monoid $(M, *, e)$ and a function f in Set from X to $F(M, *, e)$, i.e. a function $f\colon X \to M$.

(2) An $(X \downarrow F)$-arrow from the pair $(M, *, e)$ with f to the pair $(N, \star, d)$ with g is a homomorphism $j\colon (M, *, e) \to (N, \star, d)$, such that $g = Fj \circ f$ in Set.

Then we have the following result:

Theorem 148. *Equip the free monoid on X, $L_X = (List(X), ^\frown, \emptyset)$, with the function $l\colon X \to List(X)$ which sends an element x of X to the one-element list with x as sole element. Then $(L_X, \emptyset)$ is an initial object of $(X \downarrow F)$.*

Proof. Take a monoid $(N, \star, d)$, call it $N_\star$ for short, and let $g\colon X \to N$ be a set function. We need to show that there is a unique $(X \downarrow F)$-arrow from (L_X, l) to $(N_\star, g)$. In other words, there exists a unique monoid homomorphism $j\colon L_X \to N_\star$ such that $g = Fj \circ l$.

For existence, let j send the empty list $\emptyset$ in $List(X)$ to d, the unit of $N_\star$, and send a one-element list x from $List(X)$ to $g(x)$. Extend the function to all members of $List(X)$ by putting $j(x_1{}^\frown x_2{}^\frown \ldots{}^\frown x_n) = j(x_1) \star j(x_2) \star \ldots \star j(x_n)$. Then j is easily seen to be a monoid homomorphism; and by construction, $g = Fj \circ l$.

For uniqueness, suppose k is another monoid homomorphism $k\colon L_X \to N_\star$ such that $g = Fk \circ l$. Then, being a homomorphism, k needs to send the empty list to the unit of $N_\star$. And because $g = Fk \circ l$, k has to agree with j on single elements of X, both sending an element x to $g(x)$. Hence

$$\begin{aligned}
k(x_1{}^\frown x_2{}^\frown \ldots{}^\frown x_n) &= k(x_1) \cdot k(x_2) \cdot \ldots \cdot k(x_n) \\
&= j(x_1) \cdot j(x_2) \cdot \ldots \cdot j(x_n) \\
&= j(x_1{}^\frown x_2{}^\frown \ldots{}^\frown x_n).
\end{aligned}$$

Whence j and k must agree on all members of $List(X)$. $\square$

We originally defined the idea of a free monoid on some given objects by a concrete construction. Now we see that we can define it, up to isomorphism of course, by a unique mapping property, as the initial object in a certain category. That's a really rather neat result!

31 Hom-functors

This chapter introduces the notion of a *hom-functor*, a type of functor which will turn out play a starring role in some later chapters (featuring essentially, for example, in the famed Yoneda Lemma). After defining two kinds of hom-functor, we show that, unlike the general run of functors, one sort behaves particularly nicely with limits.

31.1 Two kinds of hom-functor

(a) We are going to be focusing for moment on categories which are locally small *and* which live in our universe of sets. Let's temporarily call these *locally small** categories. In these cases we can unproblematically talk about the hom-sets whose members are the C-arrows from A to B. Later we will want to widen our discussion to cover locally small categories more generally (cf. §§26.2, 26.3).

And for brevity's sake we'll now introduce some standard and slightly snappier notation to use from now on:

Definition 123. We will denote the hom-set $Hom_{\mathsf{C}}(A, B)$ of C-arrows from A to B simply by $\mathsf{C}(A, B)$.

(b) So take a locally small* category C and choose a fixed C-object A. Then as we vary X through the objects in C, we get varying hom-sets $\mathsf{C}(A, X)$. In other words, there is a function which sends X in C to the corresponding hom-set $\mathsf{C}(A, X)$ living in Set.

Question: can we treat this function on C-*objects* as the first component of a *functor*, call it $\mathsf{C}(A, –)$, from C to Set? Well, how could we fix a second component of the functor to deal with the C-*arrows*?

By definition, such a component is going to need to send an arrow $j\colon X \to Y$ in C to a Set-function from $\mathsf{C}(A, X)$ to $\mathsf{C}(A, Y)$. And the most obvious candidate for the latter function is the one we can notate as $j \circ –$, i.e. the function that maps any $h\colon A \to X$ in $\mathsf{C}(A, X)$ to $j \circ h\colon A \to Y$ in $\mathsf{C}(A, Y)$.

Note, assuming that an arrow $h\colon A \to X$ exists (there doesn't have to be one!), the arrow $j \circ h\colon A \to Y$ has to be in $\mathsf{C}(A, Y)$ – because C is a category which by hypothesis contains h and j and hence contains their composite.

We should check:

Theorem 149. *If* C *is a locally small category, and* A *is one of its objects, then there is a (covariant) functor* $\mathsf{C}(A,–)$ *from* C *to* Set *which operates like this:*

$$\begin{array}{lrcl} \mathsf{C}(A,–)\colon & X & \longmapsto & \mathsf{C}(A,X) \\ & j\colon X \to Y & \longmapsto & j \circ –\colon \mathsf{C}(A,X) \to \mathsf{C}(A,Y). \end{array}$$

Proof. Temporarily write $\mathsf{C}(A,–)$ as simply F for short. Then Fj applied to a member g of $\mathsf{C}(A,X)$ yields $j \circ h$: in short, $Fj(h) = j \circ h$.

To confirm functoriality, note first that $F1_X(h) = h$ for any $h\colon A \to X$. Hence $F1_X$ is the function which maps any member of $\mathsf{C}(A,X)$ to itself, making it the unique identity function for $\mathsf{C}(A,X)$, i.e. $1_{F(X)}$.

Second note that for any h, $F(j \circ k)(h) = (j \circ k) \circ h = j \circ (k \circ h) = F(j)(k \circ h) = F(j)F(k)(h) = (Fj \circ Fk)(h)$ Hence $F(j \circ k) = Fj \circ Fk$. □

(c) Now, start again from the hom-set $\mathsf{C}(A,B)$ but this time keep B fixed: then as we vary X through the objects in C, we again get varying hom-sets $\mathsf{C}(X,B)$. Which generates a function which sends an object X in C to an object $\mathsf{C}(X,B)$ in Set.

To turn *this* into some kind of functor $\mathsf{C}(–,B)$, we need again to add a component to deal with C-arrows. Which will need to send $j\colon X \to Y$ in C to some function between $\mathsf{C}(X,B)$ and $\mathsf{C}(Y,B)$. But this time, if we are going to use the same sort of idea as before, to get functions to compose properly, things need to go the other way about. $\mathsf{C}(–,B)$ will have to send an arrow $j\colon X \to Y$ to the function we can notate as $– \circ j$ which maps any arrow $h\colon Y \to B$ in $\mathsf{C}(Y,B)$ to $h \circ j\colon X \to B$ in $\mathsf{C}(X,B)$.

We can leave it as an exercise to check:

Theorem 150. *If* C *is a locally small category, and* B *is one of its objects, then there is a (contravariant) functor* $\mathsf{C}(–,A)$ *from* C *to* Set *which operates like this:*

$$\begin{array}{lrcl} \mathsf{C}(–,B)\colon & X & \longmapsto & \mathsf{C}(X,B) \\ & j\colon Y \to X & \longmapsto & – \circ j\colon \mathsf{C}(X,B) \to \mathsf{C}(Y,B). \end{array}$$ □

Or noting the point made at the end of §27.5, you might prefer to put it this way:

Theorem 150*. *If* C *is a locally small category, and* B *is one of its objects, then there is a (covariant) functor* $\mathsf{C}(–,A)$ *from* C^{op} *to* Set *which operates like this:*

$$\begin{array}{lrcl} \mathsf{C}(–,B)\colon & X \text{ (in } \mathsf{C}^{op}) & \longmapsto & \mathsf{C}(X,B) \\ & j\colon X \to Y \text{ (in } \mathsf{C}^{op}) & \longmapsto & – \circ j\colon \mathsf{C}(X,B) \to \mathsf{C}(Y,B). \end{array}$$ □

And yes, the second clause here is right: we need to combine the C-arrow h from $\mathsf{C}(X,B)$ with another C-arrow, and in its guise as a C-arrow j does go from Y to X. So $h \circ j$ is correctly composed.

(d) A note on notation. The use of a blank in the notation '$\mathsf{C}(A,–)$' invites an obvious handy shorthand: instead of writing e.g. '$\mathsf{C}(A,–)_{arw}(j)$' or '$\mathsf{C}(A,–)j$' to indicate the result of the component of the functor which acts on arrows applied to the function f, we will more snappily write '$\mathsf{C}(A,j)$'. Similarly for the dual case.

31.2 Points of view

The hom-functor $\mathsf{C}(A,-)$ encapsulates A's view of its world – or so we might put it. For any X, the functor outputs the set of arrows from A to X, i.e. the various ways that A 'sees' X. Now start from any other object A'; the corresponding hom-functor $\mathsf{C}(A',-)$ likewise encapsulates A''s view of its world.

A natural issue arises. The views of the world from distinct objects A and A' won't be the same: but the perspectives should coherently fit together. There should be a story to be told about how going from A to A' transforms the view, depending how A and A' themselves relate.

Dropping the metaphor, we want a story about the relations between the hom-functors $\mathsf{C}(A,-)$ and $\mathsf{C}(A',-)$. But this is going to need a general story about 'transformations' between functors, and we haven't got one yet: this will be the business for the next two chapters (we then return to the story about hom-functors in particular in Chapter 37).

Similarly, of course, for a hom-functor like $\mathsf{C}(-,B)$. This encapsulates how B is seen by its world. For given any object X, the functor outputs the set of arrows from X to B, i.e. the various ways that B is seen by X. Another natural issue arise, the dual of the one before. The ways that the world sees B and B' won't be the same; but there should be a story to be told about how the views fit together, depending on how B and B' relate. More on this too in due course.

31.3 Hom-functors preserve limits

Here's an important result about hom-functors which we *can* prove now:

Theorem 151. *Suppose that* C *is a locally small* category. Then the covariant hom-functor* $\mathsf{C}(A,-)\colon \mathsf{C} \to \mathsf{Set}$*, for any* A *in the category* C*, preserves all limits that exist in* C*.*

Proof. We'll first check that $\mathsf{C}(A,-)\colon \mathsf{C} \to \mathsf{Set}$ sends a cone over the diagram $D\colon \mathsf{J} \to \mathsf{C}$ to a cone over $\mathsf{C}(A,-) \circ D\colon \mathsf{J} \to \mathsf{Set}$.

A cone has a vertex C, and arrows $c_J\colon C \to DJ$ for each J-object J, where for any $d\colon J \to K$ in J, so for any $Dd\colon DJ \to DK$, $c_K = Dd \circ c_J$.

Now, acting on objects, $\mathsf{C}(A,-)$ sends C to $\mathsf{C}(A,C)$ and sends DJ to $\mathsf{C}(A,DJ)$. And acting on arrows, $\mathsf{C}(A,-)$ sends $c_J\colon C \to DJ$ to the set function $c_J \circ -$ (the function which takes $g\colon A \to C$ and outputs $c_J \circ g\colon A \to DJ$). And it sends $Dd\colon DJ \to DK$ to the set-function $Dd \circ -$ (the function which takes $h\colon A \to DJ$ and outputs $Dd \circ h\colon A \to DK$). So, in summary, the functor $\mathsf{C}(A,-)$ sends the triangle on the left to the one on the right:

$$\begin{tikzcd} & C \arrow[dl, "c_J"'] \arrow[dr, "c_K"] & \\ DJ \arrow[rr, "Dd"'] & & DK \end{tikzcd} \quad\Rightarrow\quad \begin{tikzcd} & \mathsf{C}(A,C) \arrow[dl, "c_J \circ -"'] \arrow[dr, "c_K \circ -"] & \\ \mathsf{C}(A,DJ) \arrow[rr, "Dd \circ -"'] & & \mathsf{C}(A,DK) \end{tikzcd}$$

Assuming $c_K = Dd \circ c_J$, we have $c_K \circ - = (Dd \circ c_J) \circ - = (Dd \circ -) \circ (c_J \circ -)$.[1] Hence, if the triangle on the left commutes, so does the triangle on the right. Likewise for other such triangles. Which means that if (C, c_J) is a cone over D, then $(\mathsf{C}(A, C), c_J \circ -)$ is indeed a cone over $\mathsf{C}(A, -) \circ D$.

So far, so good! It remains, then, to show that in particular $\mathsf{C}(A, -)$ sends limit cones to limit cones.

Suppose, then, that (L, λ_J) is a limit cone in C over D. The functor $\mathsf{C}(A, -)\colon \mathsf{C} \to \mathsf{Set}$ sends the left-hand commuting diagram below to the commuting triangle at the bottom of the right-hand diagram. And we now suppose that (M, m_J) is any other cone over the image of D:

$$\begin{tikzcd}
 & L \arrow[dl, "\lambda_J"'] \arrow[dr, "\lambda_K"] & \\
DJ \arrow[rr, "Dd"'] & & DK
\end{tikzcd}
\qquad
\begin{tikzcd}
 & M \arrow[ddl, "m_J"'] \arrow[ddr, "m_K"] & \\
 & \mathsf{C}(A, L) \arrow[dl, "\lambda_J \circ -"] \arrow[dr, "\lambda_K \circ -"'] & \\
\mathsf{C}(A, DJ) \arrow[rr, "Dd \circ -"'] & & \mathsf{C}(A, DK)
\end{tikzcd}$$

Hence $m_K = (Dd \circ -) \circ m_J$.

Now remember that M lives in Set: so take a member x. Then $m_J(x)$ is a particular arrow in $\mathsf{C}(A, DJ)$, in other words $m_J(x)\colon A \to DJ$. Likewise we have $m_K(x)\colon A \to DK$. But $m_K(x) = Dd \circ m_J(x)$. Which means that for all d the outer triangles on the left below commute and so $(A, m_J(x))$ is a cone over D. And this must factor uniquely through an arrow $u(x)$ as on the left:

$$\begin{tikzcd}
 & A \arrow[ddl, "m_J(x)"'] \arrow[ddr, "m_K(x)"] \arrow[d, "u(x)"] & \\
 & L \arrow[dl, "\lambda_J"] \arrow[dr, "\lambda_K"'] & \\
DJ \arrow[rr, "Dd"'] & & DK
\end{tikzcd}
\qquad
\begin{tikzcd}
 & M \arrow[ddl, "m_J"'] \arrow[ddr, "m_K"] \arrow[d, "u"] & \\
 & \mathsf{C}(A, L) \arrow[dl, "\lambda_J \circ -"] \arrow[dr, "\lambda_K \circ -"'] & \\
\mathsf{C}(A, DJ) \arrow[rr, "Dd \circ -"'] & & \mathsf{C}(A, DK)
\end{tikzcd}$$

Hence $u(x)$ is an arrow from A to L, i.e. an element of $\mathsf{C}(A, L)$. So consider the map $u\colon M \to \mathsf{C}(A, L)$ which sends any x in M to $u(x)$. Since $m_J(x) = \lambda_J \circ u(x)$ for each x, $m_J = (\lambda_J \circ -) \circ u$. And since this applies for each object J, (M, m_J) factors through the image of the cone (L, λ_J) via u.

Suppose there is another map $v\colon M \to \mathsf{C}(A, L)$ such that we also have each $m_J = (\lambda_J \circ -) \circ v$. Then again take an element x in M: then $m_J(x) = \lambda_J \circ v(x)$. So again, $(A, m_J(x))$ factors through (L, λ_J) via $v(x)$ – which, by the uniqueness of factorization through limits, means that $v(x) = u(x)$. Since that obtains for all x in M, $v = u$. Hence (M, m_j) factors uniquely through the image of (L, λ_J).

[1] For the second equality think: composing a function with c_J-followed-by-Dd is the same as composing-with-c_J and then following that by composing-with-Dd.

Since (M, m_J) was an arbitrary cone, we have therefore proved that the image of the limit cone (L, λ_J) is also a limit cone. □

(a) What is the dual of Theorem 151? We have two dualities to play with: limits vs colimits and covariant functors vs contravariant functors.

Two initial observations. First, a covariant hom-functor need not preserve colimits such as initial objects. For example, take the hom-functor $\mathsf{Grp}(A, -)$. In Grp the initial object 0 is also the terminal object, so for any group A, $\mathsf{Grp}(A, 0)$ is a singleton, which is not initial in Set.

Second, contravariant hom-functors from a category C can't preserve either limits or colimits in C, because contravariant functors reverse arrows.

The dual result we *can* get is this:

Theorem 152. *Suppose that* C *is a locally small* category. Then the contravariant hom-functor* $\mathsf{C}(-, A)\colon \mathsf{C} \to \mathsf{Set}$*, for any* A *in the category* C*, sends a colimit of shape* J *to a limit of that shape.* □

Yes, that's right: contravariant functors send colimits to limits. We can leave the proof as a merry dualizing challenge.

31.4 Hom-functors, redefined

I want to turn now from talking about the special case of locally small* categories to talking about locally small categories more generally.

Recall that Defn. 106 tells us that a category C is locally small in the general sense if, for any of its objects A and B, there is only a set's worth of C-arrows from A to B, which we glossed as meaning that in each case these arrows from A to B can be indexed by some set. So the arrows themselves don't have to live in Set; what does live there is a set containing one-for-one representatives for those arrows.

Now, we will want some notation for that indexing set, which we'll still call a hom-set. So why not recycle the notation $\mathsf{C}(A, B)$? Then, whatever the nature of C itself, the hom-set $\mathsf{C}(A, B)$ will happily be living in Set (it is just that, in the general case, the set won't contain the C-arrows themselves, just representatives for them).

And suppose we now adopt one of those happy further abuses of notation which can helpfully smooth our way. So if $f\colon A \to B$ is a C-arrow, one of the arrows from A to B indexed by the hom-set $\mathsf{C}(A, B)$, then we'll denote its index in that set by f again, leaving context to disambiguate.

But then, with *that* understanding, we can just replace 'small*' with plain 'small' in Theorems 151 and 152, and everything will still work. Which is neat. And shows that, after all, we needn't fuss too much about whether we think of the arrows in a locally small category C as forming a set that lives in Set or as forming a collection which can be faithfully represented by a set living in Set. A story about hom-functors from C to Set preserving limits, etc., can be spun either way.

32 Natural isomorphisms

> Category theory is an embodiment of Klein's dictum that it is the maps that count in mathematics. If the dictum is true, then it is the functors between categories that are important, not the categories. And such is the case. Indeed, the notion of category is best excused as that which is necessary in order to have the notion of functor. But the progression does not stop here. There are maps between functors, and they are called natural transformations.
> (Freyd 1965, quoted in Marquis 2008.)

Natural transformations between functors – and more specifically, natural isomorphisms – were there from the very start. The founding document of category theory is the paper by Samuel Eilenberg and Saunders Mac Lane 'General theory of natural equivalences' (Eilenberg and Mac Lane 1945). But the pivotal idea had already been introduced, three years previously, in a paper on 'Natural isomorphisms in group theory', before the categorial framework was invented in order to provide a proper setting for the account (Eilenberg and Mac Lane 1942). These natural isomorphisms and the more general natural transformations will be our main topic for the next couple of chapters.

32.1 Natural isomorphisms between functors defined

Suppose we have a pair of functors $F, G\colon \mathsf{C} \to \mathsf{D}$. Then each of the functors projects the objects and arrows of C into D, giving us two images of C within D. We might in part have:

$$\begin{tikzcd}
 & & & FA \arrow[rr, "Ff"] \arrow[dr, "Fg"'] & & FB \\
A \arrow[rr, "f"] \arrow[dr, "g"'] & & B \arrow[ur, dashed, "F"] \arrow[ddr, dashed, "G"'] & & FC \arrow[ur, "Fh"'] & \\
 & C \arrow[ur, "h"'] & & & & \\
 & & & GA \arrow[rr, "Gf"] \arrow[dr, "Gg"'] & & GB \\
 & & & & GC \arrow[ur, "Gh"'] &
\end{tikzcd}$$

in C in D

Now, in general, the images of C projected by F and G could be significantly different. But suppose there is a suite ψ of isomorphisms in D, $\psi_A\colon FA \xrightarrow{\sim} GA$, $\psi_B\colon FB \xrightarrow{\sim} GB$, $\psi_C\colon FC \xrightarrow{\sim} GC$, etc., ensuring that $FA \cong GA$, $FB \cong GB$, $FB \cong GB$, etc. And suppose too that all the squares like these in D commute:

$$\begin{tikzcd}
FA \arrow[rr, "Ff"] \arrow[dr, "Fg"'] \arrow[dd, "\psi_A"'] & & FB \arrow[dd, "\psi_B"] \\
& FC \arrow[ur, "Fh"'] \arrow[dd, "\psi_C"] & \\
GA \arrow[rr, "Gf"] \arrow[dr, "Gg"'] & & GB \\
& GC \arrow[ur, "Gh"'] &
\end{tikzcd}$$

Then that suite ψ of isomorphisms, nicely interacting with arrows like Ff and Gf, means that the G-image of C does behave just like copy of the F-image (a faithful copy except perhaps in collapsing isomorphic objects together). So, stretching our terminology a bit, we might say that in this case, F and G are themselves isomorphic.

Which all goes to motivate the following standard definition – or rather, it's a pair of definitions, one for each flavour of functor:

Definition 124. Let C and D be categories, let $F, G\colon \mathsf{C} \to \mathsf{D}$ be covariant functors (respectively, contravariant functors), and suppose that for each C-object C there is a D-isomorphism $\psi_C\colon FC \xrightarrow{\sim} GC$. Then ψ, the family of arrows ψ_C, is said to be a *natural isomorphism* between F and G if for every arrow $f\colon A \to B$ (respectively, $f\colon B \to A$, note the reversal!) in C the following *naturality square* commutes in D:

$$\begin{tikzcd}
FA \arrow[r, "Ff"] \arrow[d, "\psi_A"] & FB \arrow[d, "\psi_B"] \\
GA \arrow[r, "Gf"] & GB
\end{tikzcd}$$

In this case, we write $\psi\colon F \xRightarrow{\sim} G$, and the ψ_C are said to be components of ψ. If there is such a natural isomorphism, the functors F and G will be said to be naturally isomorphic, and we write $F \cong G$. $\triangle$

Let's have an immediate toy example. In §27.3, (F12) introduced us to the product functors $(- \times C)\colon \mathsf{C} \to \mathsf{C}$ and $(C \times -)\colon \mathsf{C} \to \mathsf{C}$.

These functors 'do the same', outputting products which are isomorphic to each other. So intuitively the functors ought to be 'isomorphic' on any nice definition of isomorphism for functors. And they are.

Let ψ_A be the isomorphism that takes a pair involving an object from A and switches the components around. Then the following square commutes for any $f\colon A \to B$:

$$\begin{tikzcd}
(- \times C)_{ob}(A) \arrow[r, "(- \times C)_{arw}(f)"] \arrow[d, "\psi_A"] & (- \times C)_{ob}(B) \arrow[d, "\psi_B"] \\
(C \times -)_{ob}(A) \arrow[r, "(C \times -)_{arw}(f)"] & (C \times -)_{ob}(B)
\end{tikzcd}$$

because by the definition in (F12) that is just the square

$$\begin{tikzcd}
A \times C \arrow[r, "f \times 1_C"] \arrow[d, "\psi_A"] & B \times C \arrow[d, "\psi_B"] \\
C \times A \arrow[r, "1_C \times f"] & C \times B
\end{tikzcd}$$

which commutes because both routes send a pair $\langle a, c\rangle$ round to $\langle c, fa\rangle$.

Hence assembling all the components ψ_A gives us the desired natural isomorphism $\psi\colon (- \times C) \xRightarrow{\sim} (C \times -)$.

32.2 Some basic properties

Theorem 153. *Suppose F, G, H are covariant functors from C to D:*

(1) There is an identity natural isomorphism $1_F\colon F \xRightarrow{\sim} F$.

(2) Given natural isomorphisms $\psi\colon F \xRightarrow{\sim} G$ and $\chi\colon G \xRightarrow{\sim} H$, there is a composite natural isomorphism $\chi \circ \psi\colon F \xRightarrow{\sim} H$, and composition is associative.

(3) Any $\psi\colon F \xRightarrow{\sim} G$ has an inverse $\psi^{-1}\colon G \xRightarrow{\sim} F$ such that $\psi^{-1} \circ \psi = 1_F$ and $\psi \circ \psi^{-1} = 1_G$.

(4) If $F \cong G$ then F is faithful if and only if G is.

(5) If $F \cong G$ then F is full if and only if G is.

(6) If $F \cong G$ then F is essentially surjective on objects if and only if G is.

(1) to (3) tell us that natural isomorphisms do behave like isomorphisms. While (4) to (6) illustrate that naturally isomorphic functors share properties in predictable ways. Do pause to find the (unchallenging!) proofs.

Proof: there's an identity natural isomorphism. The following diagram evidently commutes for any C-arrow $f\colon A \to B$:

$$\begin{tikzcd}
FA \arrow[r, "Ff"] \arrow[d, "1_{FA}"] & FB \arrow[d, "1_{FB}"] \\
FA \arrow[r, "Ff"] & FB
\end{tikzcd}$$

So we have a natural isomorphism $1_F\colon F \xRightarrow{\sim} F$, where the components $(1_F)_A$ of the isomorphism are the identity arrows 1_{FA}. $\square$

Proof: natural isomorphisms compose. Given $\psi\colon F \xRightarrow{\sim} G$ and $\chi\colon G \xRightarrow{\sim} H$, the squares below commute for any C-arrow $f\colon A \to B$:

$$\begin{tikzcd}
FA \arrow[r, "Ff"] \arrow[d, "\psi_A"] \arrow[dd, bend right=60, "(\chi\circ\psi)_A"'] & FB \arrow[d, "\psi_B"'] \arrow[dd, bend left=60, "(\chi\circ\psi)_B"] \\
GA \arrow[r, "Gf"] \arrow[d, "\chi_A"] & GB \arrow[d, "\chi_B"'] \\
HA \arrow[r, "Hf"] & HB
\end{tikzcd}$$

So, let's put $(\chi \circ \psi)_A = \chi_A \circ \psi_A$ for every A (more isomorphisms, of course). Then, with those components, it is immediate that $(\chi \circ \psi)\colon F \xRightarrow{\sim} H$.

The associativity of composition for natural isomorphisms is then inherited from the associativity of composition for their components. □

Proof: natural isomorphisms have inverses. The commuting naturality squares which show that $\psi\colon F \xRightarrow{\sim} G$ tell us that (for any $f\colon A \to B$) we have $\psi_B \circ Ff = Gf \circ \psi_A$. The components of ψ being isomorphisms have inverses. So

$$\psi_B^{-1} \circ (\psi_B \circ Ff) \circ \psi_A^{-1} = \psi_B^{-1} \circ (Gf \circ \psi_A) \circ \psi_A^{-1}$$

whence $\psi_B^{-1} \circ Gf = Ff \circ \psi_A^{-1}$. But that gives us all the commuting squares showing that $\psi^{-1}\colon G \xRightarrow{\sim} F$, if we put $(\psi^{-1})_A = \psi_A^{-1}$.

It is then immediate that as we want $\psi^{-1} \circ \psi = 1_F$ and $\psi \circ \psi^{-1} = 1_G$. □

Next, we need only prove one direction of each of the following three biconditional results. So:

Proof: $F \cong G$ implies that if F is faithful, so is G. By the naturality square, for any $f\colon A \to B$, $Gf = \psi_B \circ Ff \circ \psi_A^{-1}$. Hence if $Gf = Gg$, then $\psi_B \circ Ff \circ \psi_A^{-1} = \psi_B \circ Fg \circ \psi_A^{-1}$, whence (composing with ψ_B^{-1} and ψ_A) we have $Ff = Fg$.

By definition F is faithful iff given any pair of parallel arrows $f, g\colon A \to B$, then if $Ff = Fg$ then $f = g$. So, given $F \cong G$ and F is faithful, then if $Gf = Gg$ we have $Ff = Fg$ and so $f = g$, making G faithful. □

Proof: $F \cong G$ implies that if F is full so is G. Recall, F is full just if for any $k\colon FA \to FB$ there is an arrow $f\colon A \to B$ such that $k = Ff$.

Now, trivially, this commutes:

$$\begin{tikzcd}
FA \arrow[rr, "\psi_B^{-1}\circ g\circ\psi_A"] \arrow[d, "\psi_A"] & & FB \arrow[d, "\psi_B"] \\
GA \arrow[rr, "g"] & & GB
\end{tikzcd}$$

So if F is full, there is an arrow f such that $Ff = \psi_B^{-1} \circ g \circ \psi_A$, and hence $g = \psi_B \circ Ff \circ \psi_A^{-1} = Gf$ (with the second equation as before). So G is full too. $\square$

Proof: $F \cong G$ implies that if F is e.s.o., then G is too. Let D be any D-object. If $F\colon \mathsf{C} \to \mathsf{D}$ is e.s.o., then there is a C-object C such that $D \cong FC$. But if $F \cong G$, $FC \cong GC$, and then by transitivity, we also get $D \cong GC$. Which makes G e.s.o. $\square$

Challenge: think through what the analogue of Theorem 153 is for contravariant functors.

32.3 Why 'natural'?

But why call what we've defined a *natural* isomorphism? There's a mathematical back-story which I alluded to in the preamble of the chapter and which I should now pause to explain, using one of Eilenberg and Mac Lane's own examples. (Hopefully, even if your grip on the theory of vector spaces is a bit shaky, the general drift of the example should be reasonably clear.)

(a) Consider a finite dimensional vector space V over the reals $\mathbb{R}$, and the corresponding dual space V^* of linear functions $g\colon V \to \mathbb{R}$.

It is elementary to show that V is isomorphic to V^* (there's a bijective linear map between the spaces). Proof sketch: take a basis $B = \{v_1, v_2, \ldots, v_n\}$ for the space V. Define the functions $v_i^*\colon V \to \mathbb{R}$ by putting $v_i^*(v_j) = 1$ if $i = j$ and $v_i^*(v_j) = 0$ otherwise. Then $B^* = \{v_1^*, v_2^*, \ldots, v_n^*\}$ is a basis for V^*, and then the linear function $\varphi_B\colon V \to V^*$ generated by putting $\varphi_B(v_i) = v_i^*$ is an isomorphism. QED

Note, however, that the isomorphism we have arrived at here depends on the initial choice of basis B. And no choice of basis B is more 'natural', no more 'canonical', than any other. So no one of the isomorphisms $\varphi_B\colon V \to V^*$ of the kind just defined is to be especially preferred.

To get a sharply contrasting case, now consider V^{**} the double dual of V, i.e. the space of functionals $h\colon V^* \to \mathbb{R}$ (so each h takes any linear function $V \to \mathbb{R}$ as input and outputs a corresponding real in $\mathbb{R}$).

Suppose we select a basis B for V, define a derived basis B^* for V^* as we just did, and then use this new basis in turn to define a basis B^{**} for V^{**} by repeating the same little trick. Then we can construct an isomorphism from V to V^{**} by mapping the elements of B to the corresponding elements of B^{**}. However, and this is the key observation, *we don't have to go through any such palaver of initially choosing a basis.* Suppose we simply define $\psi_V : V \to V^{**}$ as acting on an element $v \in V$ to give as output the functional $\psi_V(v)\colon V^* \to \mathbb{R}$ which sends a function $g\colon V \to \mathbb{R}$ to the value $g(v)$: in short, we set $\psi_V(v)(g) = g(v)$. It is a simple result that ψ_V is an isomorphism (we can rely on the fact that V is finite-dimensional). And obviously we get *this* isomorphism independently of any arbitrary choice of basis.

Interim summary: it is very tempting to say that the isomorphisms of the kind we described between V and V^* are not particularly 'natural': they are cooked up on the basis(!) of some arbitrary choices. By contrast there *is* a 'natural' isomorphism between V and V^{**}, generated by a procedure that doesn't involve any arbitrary choices.

Now, there are many other cases where we might similarly want to contrast intuitively 'natural' or 'canonical' maps with more arbitrarily cooked-up maps between structured objects. So a question arises: can we give a general account of what makes for naturality here? Eilenberg and Mac Lane were aiming to provide such a story.

(b) To continue with our example, the nice isomorphism $\psi_V : V \xrightarrow{\sim} V^{**}$ only depended on the fact that V is a finite dimensional vector space over the reals. Which implies that our construction will work in exactly same way for any other such vector space W, so we get a corresponding isomorphism $\psi_W : W \xrightarrow{\sim} W^{**}$. Now, we will expect such naturally constructed isomorphisms to respect the relation between a structure-preserving map f between the spaces V and W and its double-dual correlate map between V^{**} to W^{**}. Putting that more carefully, we want the following informal diagram to commute, whatever vector spaces we take and for any linear map $f\colon V \to W$,

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\psi_V"] & W \arrow[d, "\psi_W"] \\ V^{**} \arrow[r, "DD(f)"] & W^{**} \end{tikzcd}$$

where $DD(f)$ is the double-dual correlate of f.

Recall, back in §27.5 (F17), we saw that the (contravariant) dualizing functor D which sends a vector space V to its dual V^* will send an arrow $f\colon V \to W$ to $D(f)\colon W^* \to V^*$, where $D(f)$ takes a member of W^* such as $g\colon W \to \mathbb{R}$ and outputs $g \circ f$ which is a member of V^*.

What about the double-dualizing functor DD which sends a vector space V to its double dual V^{**}? Where should it send an arrow $f\colon V \to W$? To $DD(f)\colon V^{**} \to W^{**}$, where $DD(f)$ takes a member of V^{**} such as $h\colon V^* \to \mathbb{R}$ and outputs $h \circ D(f)$ which is a member of W^{**}. (It is readily checked that this is indeed functorial.)

Thus understood, our square does indeed commute. By either route, a vector v in V gets sent to the functional living in W^{**} which sends a function $k\colon W \to \mathbb{R}$ to the value $k(f(v))$. Think about it![1]

[1] OK, if you insist. The top route sends v to $f(v)$; but then $\psi_W(f(v))$ by definition outputs the functional which sends a function $k\colon W \to \mathbb{R}$ to $k(f(v))$. For the bottom route, $\psi_V(v)$ by definition outputs the functional h which sends any function $j\colon V \to \mathbb{R}$ to $j(v)$. Now we need to hit that functional with DD, and that gives as another functional with takes any $k\colon W \to \mathbb{R}$, applies Df which gives us $k \circ f\colon V \to \mathbb{R}$, and then applies h to that which sends it to the output $(k \circ f)(v) = k(f(v))$ again.

(c) So far, so good. Now let's pause to consider why there *can't* be a similarly 'natural' way of setting up isomorphisms from vector spaces V to their duals V^*. (The isomorphisms we mentioned which are based on an arbitrary choice of basis aren't natural: but we want to show that there is no other way of getting a 'natural' isomorphism.)

Suppose then that there were a construction which gave us an isomorphism $\varphi_V : V \xrightarrow{\sim} V^*$ which again does not depend on information about V other than that it has the structure of a finite dimensional vector space. So again we will want the construction to work the same way on other such vector spaces, and to be preserved by structure-preserving maps between the spaces. This time, therefore, we will presumably want the following diagram to commute for any structure-preserving f between vector spaces (note, however, that we have to reverse an arrow for things to make any sense, given our definition of the contravariant functor D):

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\varphi_V"] & W \arrow[d, "\varphi_W"] \\ V^* & W^* \arrow[l, "D(f)"] \end{tikzcd}$$

Hence $D(f) \circ \varphi_W \circ f = \varphi_V$. But by hypothesis, the φs are isomorphisms; so in particular φ_V has an inverse. So we have $(\varphi_V^{-1} \circ D(f) \circ \varphi_W) \circ f = 1_V$. Therefore f has a left inverse. But it is obvious that in general, a linear map $f : V \to W$ need not have a left inverse. Hence there can't in general be isomorphisms φ_V, φ_W making that diagram commute.

(d) We started off by saying that, intuitively, there's always a 'natural', intrinsic, isomorphism between a (finite dimensional) vector space and its double dual, one that depends only on their structures as vector spaces. And we've now suggested that this intuitive idea can be reflected by saying that a certain diagram involving such isomorphisms always commutes, for any choice of vector spaces and structure-preserving maps between them.

We have also seen that we can't get analogous always-commuting diagrams for the case of isomorphisms between a vector space and its dual – which chimes with the intuition that the obvious examples are *not* 'natural' isomorphisms.

So this gives us a promising way forward: characterize 'naturality' here in terms of the availability of a family of isomorphisms which make certain diagrams commute.

But now a key move. The claim that the diagram

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\psi_V"] & W \arrow[d, "\psi_W"] \\ V^{**} \arrow[r, "DD(f)"] & W^{**} \end{tikzcd}$$

always commutes can be indeed be recast as a claim about two functors. For we have been talking about the category FVect (of finite-dimensional spaces over the reals and the structure-preserving maps between them), and about the functor $DD\colon \mathsf{FVect} \to \mathsf{FVect}$ which takes a vector space to its double dual, and maps each arrow between vector spaces to its double-dual correlate as explained. But of course there is also a trivial functor $1\colon \mathsf{FVect} \to \mathsf{FVect}$ that maps each vector space to itself and each FVect-arrow to itself. So we can re-express the claim that the last diagram commutes as follows: for every arrow $f\colon V \to W$ in FVect, there are isomorphisms ψ_V and ψ_W in FVect such that *this* diagram commutes:

$$\begin{tikzcd} 1(V) \arrow[r, "1(f)"] \arrow[d, "\psi_V"] & 1(W) \arrow[d, "\psi_W"] \\ DD(V) \arrow[r, "DD(f)"] & DD(W) \end{tikzcd}$$

In other words, in the terms of the previous section, *the suite of isomorphisms* ψ_V *provide a natural isomorphism* $\psi\colon 1 \xRightarrow{\cong} DD$.

(e) In sum: our claim that there is an intuitively 'natural' isomorphism between two *spaces*, a vector space and its double dual, now becomes reflected in the claim that there is an isomorphism in our official sense between two *functors*, the identity and the double-dual functors from the category FVect to itself. Hence the aptness of calling the latter isomorphism between functors a *natural* isomorphism.

We will return at the end of the chapter to consider the question whether we can generalize from our example of vector spaces and claim that in very many (most? all?) cases, intuitively 'natural' isomorphisms between widgets and wombats can be treated officially as natural isomorphisms between suitable functors.

32.4 More examples of natural isomorphormisms

(a) We now have one important case to hand. Many more are to be had. For example, Riehl (2017, p. 26) gives a nice example of a classic representation theorem relating – as we can now say – the category of compact Hausdorff spaces and the category of Banach spaces and continuous maps, a theorem which amounts to a claim of natural isomorphism between relevant functors. But it would take us too far afield to chase down the details. So here, let's stick to much more elementary examples.

(1) Given a group $G = (G, *, e)$ we can define its mirror-image or opposite $G^{op} = (G, *^{op}, e)$, where $a *^{op} b = b * a$.

We can also define a functor $Op\colon \mathsf{Grp} \to \mathsf{Grp}$ which sends a group G to its opposite G^{op}, and sends an arrow f in the category, i.e. a group homomorphism $f\colon G \to H$, to $f^{op}\colon G^{op} \to H^{op}$ where $f^{op}(a) = f(a)$ for all a in G. f^{op} so defined is indeed a group homomorphism, since

$$f^{op}(a *^{op} b) = f(b * a) = f(b) * f(a) = f^{op}(a) *^{op} f^{op}(b)$$

Claim: there is a natural isomorphism $\psi\colon 1 \overset{\sim}{\Longrightarrow} Op$ (where 1 is the trivial identity functor in Grp). Which is as it should be, since the functors produce isomomorphic results.

Proof. We need to find a family of isomorphisms $\psi_G, \psi_H, \ldots$ in Grp such that the following diagram commutes for any homomorphism $f\colon G \to H$:

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\psi_G"] & H \arrow[d, "\psi_H"] \\ G^{op} \arrow[r, "f^{op}"] & H^{op} \end{tikzcd}$$

Now, since taking the opposite *between* groups involves reversing the order of multiplication and taking inverses *inside* a group in effect does the same, let's put $\psi_G(a) = a^{-1}$ for any G-element a, and likewise for ψ_H, etc. It is easy to check that each ψ_G is a group isomorphism, and that they assemble to give a natural isomorphism. ☐

(2) Recall from §27.4 the functor $List\colon \mathsf{Set} \to \mathsf{Set}$ which sends a set X to the set of finite lists of members of X. One natural isomorphism from this functor to itself is the identity isomorphism $1\colon List \overset{\sim}{\Longrightarrow} List$. But there is also another natural isomorphism $\rho\colon List \overset{\sim}{\Longrightarrow} List$, whose component $\rho_X\colon List(X) \to List(X)$ acts on a list of X-elements to reverse their order.

(3) Now for a slightly more interesting example, this time involving contravariant functors from Set to Set.

First, recall from §27.5 the contravariant powerset functor $\overline{P}\colon \mathsf{Set} \to \mathsf{Set}$ which maps a set X to its powerset $\mathcal{P}(X)$, and maps a set-function $f\colon Y \to X$ to the function $Inv(f)\colon \mathcal{P}(X) \to \mathcal{P}(Y)$ which sends $U \subseteq X$ to its inverse image $f^{-1}[U] \subseteq Y$.

And let now C be the hom-functor $\mathsf{Set}(-, 2)$, where 2 is some nice two-element set which we can think of as $\{true, false\}$. So C sends a set X to $\mathsf{Set}(X, 2)$, i.e. the set of functions from X to 2: and C sends an arrow $f\colon Y \to X$ to the function $- \circ f\colon \mathsf{Set}(X, 2) \to \mathsf{Set}(Y, 2)$ (i.e. the function which sends an arrow $g\colon X \to 2$ to the arrow $g \circ f\colon Y \to 2$).

Claim: $\overline{P} \cong C$.

Proof. We need to find a family of isomorphisms $\psi_X, \psi_Y, \ldots$ in Set such that the following diagram always commutes, for any $f\colon Y \to X$:

$$\begin{tikzcd} \overline{P}X \arrow[r, "\overline{P}f"] \arrow[d, "\psi_X"] & \overline{P}Y \arrow[d, "\psi_Y"] \\ CX \arrow[r, "Cf"] & CY \end{tikzcd} \iff \begin{tikzcd} \mathcal{P}(X) \arrow[r, "Inv(f)"] \arrow[d, "\psi_X"] & \mathcal{P}(Y) \arrow[d, "\psi_Y"] \\ \mathsf{Set}(X, 2) \arrow[r, "- \circ f"] & \mathsf{Set}(Y, 2) \end{tikzcd}$$

Well, for any X, let ψ_X send the set $U \in \mathcal{P}(X)$ to its characteristic function – i.e to the function which sends an element of X to *true* iff it is in U. ψ_X is evidently bijective and hence an isomorphism in Set. And it is easy to see that our diagram will always commute. Both routes sends a set $U \in \mathcal{P}(X)$ to the function which sends y in Y to *true* iff $fy \in U$.

Here the natural isomorphism reflects the non-arbitrary association of subsets with characteristic functions. □

(4) Next, we take a certain pair of (covariant) functors $U, V : \mathsf{Grp} \to \mathsf{Set}$. Here U is simply the forgetful functor which sends a group G to its underlying set $\underline{G}$, and sends a homomorphism to the underlying set-function. While V is the hom-functor $\mathsf{Grp}(Z, –)$, where Z is the group of integers under addition. So, by definition, V sends an object, i.e. a group G, to the set of group homomorphisms from Z to G. And V sends an arrow $f : G \to G'$ to the function we notate $f \circ –$, i.e. the function which sends a homomorphism $h : Z \to G$ to the homomorphism $f \circ h : Z \to G'$. Claim: $U \cong V$.

Proof. Note first that a group homomorphism from $Z = (\mathbb{Z}, 0, +)$ to $G = (\underline{G}, e, \cdot)$ is entirely determined by fixing where 1 goes. For 0 has to go to the identity element e; and if 1 goes to the element a, every sum $1+1+1+\ldots+1$ has to go to the corresponding $a \cdot a \cdot a \cdot \ldots \cdot a$, with inverses going to inverses. Which means that there is a set-bijection ψ_G from an element a of $\underline{G}$ to the corresponding member of $\mathsf{Grp}(Z, G)$, the homomorphism which sends 1 to a.

It is then immediate that the required naturality square

$$\begin{tikzcd} UG \arrow[r, "f"] \arrow[d, "\psi_G"] & UG' \arrow[d, "\psi_{G'}"] \\ VG \arrow[r, "f \circ -"] & VG' \end{tikzcd}$$

commutes for any $f : G \to G'$, with either route taking us from an element a in G to the unique homomorphism from Z to G' which sends 1 to fa. □

(5) Recall from §16.1 that there is an obvious bijection between two-place set functions from A and B to C and one-place functions from A to functions-from-B-to-C. That is to say, there is a natural way of constructing an isomorphism between the hom-sets $\mathsf{Set}(A \times B, C)$ and $\mathsf{Set}(A, C^B)$. And the point applies more generally to the hom-sets of a locally small category with exponentials: $\mathsf{C}(A \times B, C)$ will be isomorphic to $\mathsf{C}(A, C^B)$.

So consider the two contravariant functors $\mathsf{C}(– \times B, C)$ and $\mathsf{C}(–, C^B)$. The object and arrow components of the first of these functors work as follows:

$$\begin{array}{rcl} X & \longmapsto & \mathsf{C}(X \times B, C) \\ j : X \to Y & \longmapsto & – \circ (j \times 1_B) : \mathsf{C}(Y \times B, C) \to \mathsf{C}(X \times B, C). \end{array}$$

In other words, $\mathsf{C}(– \times B, C)$ applied to $j : X \to Y$ gives the function which sends an arrow $f : Y \times B \to C$ to the arrow $f \circ (j \times 1_B) : X \times B \to C$.

(Check that this does define a functor!) While our second functor is just a hom-functor of the familiar kind, which works as follows:

$$\begin{array}{rcl} X & \longmapsto & \mathsf{C}(X, C^B) \\ j\colon X \to Y & \longmapsto & - \circ j\colon \mathsf{C}(Y, C^B) \to \mathsf{C}(X, C^B). \end{array}$$

Now, our two functors systematically send any object X to isomorphic outputs, and 'do the simple, obvious, thing' to arrows $j\colon X \to Y$. So we might expect them to be naturally isomorphic functors. Which they are.

Proof. We need to find a suite of isomorphisms $\psi_X, \psi_Y, \ldots$ such that for every $j\colon X \to Y$ in C, the following diagram commutes in Set, with the direction of arrows dictated by the contravariance of the functors:

$$\begin{tikzcd}
\mathsf{C}(X \times B, C) \arrow[d, "\psi_X"] & \mathsf{C}(Y \times B, C) \arrow[l, "- \circ (j \times 1_B)"'] \arrow[d, "\psi_Y"] \\
\mathsf{C}(X, C^B) & \mathsf{C}(Y, C^B) \arrow[l, "- \circ j"']
\end{tikzcd}$$

But the obvious isomorphism we know about between $\mathsf{C}(X \times B, C)$ and $\mathsf{C}(X, C^B)$ is the one given in the chapter about exponentials by Theorem 68: so let's try putting ψ_X, for a given X, to be the function which sends an arrow $f\colon X \times B \to C$ to its exponential transpose $\overline{\overline{f}}\colon X \to C^B$.

Then our square will commute if for any $g\colon Y \times B \to C$,

$$\overline{\overline{g}} \circ j = \overline{\overline{g \circ (j \times 1_B)}}.$$

But that's true, for consider the following diagram:

$$\begin{tikzcd}
X \times B \arrow[d, "j \times 1_B"] \arrow[dr, "f \circ (j \times 1_B)"] \arrow[dd, dashed, bend right=60, "\overline{\overline{f \circ (j \times 1_B)}} \times 1_B"'] & \\
Y \times B \arrow[r, "g"] \arrow[d, dashed, "\overline{\overline{g}} \times 1_B"] & C \\
B^C \times B \arrow[ur, "ev"'] &
\end{tikzcd}$$

The top-right triangle trivially commutes. The bottom-right triangle commutes by the definition of the existential transpose. So the composite vertical arrow $(\overline{\overline{g}} \circ j) \times 1_B$ gives us a commuting large triangle together with the arrows $f \circ (j \times 1_B)$ and ev. But by definition, $\overline{\overline{f \circ (j \times 1_B)}} \times 1_B$ is the unique arrow making that triangle commute. So, as we needed to show, it must be the case that $\overline{\overline{g}} \circ j = \overline{\overline{g \circ (j \times 1_B)}}$. $\square$

(6) We'll make use of that last particular result later, in §37.5. But our proof also illustrates a general fact. We can have functors that systematically

'do the same thing, up to isomorphism', and hence *ought* to be naturally isomorphic; but often, showing this they really *are* isomorphic can be annoyingly fiddly.

For another example, working in a locally small category with exponentials, take first the hom-functor $\mathsf{C}(A \times B, -)$. And compare this with the functor we can notate

$$\mathsf{C}(A, -^B) \vdots \qquad \begin{array}{rcl} X & \longmapsto & \mathsf{C}(A, X^B) \\ f\colon X \to Y & \longmapsto & \overline{\overline{f \circ ev}} \circ -\colon \mathsf{C}(A, X^B) \to \mathsf{C}(A, Y^B). \end{array}$$

Two challenges. First, why is the given operation on arrows the natural way to define the arrow component of a functor which operates on objects as described? (Hint: compare §27.3, (F13).) Second, noting that the two functors systematically send given objects to isomorphic outputs, we might conjecture that they are naturally isomorphic: prove that they are. (Hint: the natural isomorphism you need isn't new, but you'll want to prove something of the form $\overline{\overline{f \circ ev}} \circ \overline{\overline{g}} = \overline{\overline{f \circ g}}$, where $g\colon A \times B \to X$.)

(7) Let's have one more example, very much simpler but illustrating an important general point.

Take again the category FVect whose objects are the finite dimension vector spaces over the reals, and whose arrows are linear maps between spaces. Then, for any V, let $\sigma^s_V\colon V \to V$ be the linear function which maps any vector v to its scalar product with the real number $s \neq 0$, i.e. to sv, and let $f\colon V \to V$ be any linear map from a space V to itself. Let $1\colon \mathsf{FVect} \to \mathsf{FVect}$ be the identity functor which sends a vector space to itself, and any arrow between spaces to itself.

Just by the definition of linearity, the square on the left of course nicely commutes:

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\sigma^s_V"'] & V \arrow[d, "\sigma^s_V"] \\ V \arrow[r, "f"] & V \end{tikzcd} \qquad \Rightarrow \qquad \begin{tikzcd} 1(V) \arrow[r, "1(f)"] \arrow[d, "\sigma^s_V"'] & 1(V) \arrow[d, "\sigma^s_V"] \\ 1(V) \arrow[r, "1(f)"] & 1(V) \end{tikzcd}$$

And hence, by definition of the functor 1, the square on the right (in fact, the same square) commutes. But each σ^s_V is a bijection for $s \neq 0$, so is an isomorphism. Which means that σ^s (assembled from the components σ^s_V) is a natural isomorphism from the functor 1 to itself. And note we get a different such isomorphism for each choice of the non-zero scaling factor s. That's not at all surprising – it simply reflects the fact that vector spaces that differ by a scaling factor can be thought of as in effect being the same, and in a natural way.

But the example illustrates a more general moral: that there can be multiple natural isomorphisms between given functors – in fact, even infinitely many.

32.5 Another basic property of isomorphic functors

Parts (4) to (6) of Theorem 153 tell us that if the functors F and G are naturally isomophic, then one of them is full (faithful, essentially surjective) if and only the other is. Here, for future reference, is another crucial respect in which naturally isomorphic functors behave in the same way:

Theorem 154. *Suppose the functors $F, G\colon \mathsf{C} \to \mathsf{D}$ are naturally isomorphic. Then if F preserves a given limit so does G.*

Proof. We mostly just need to apply definitions.

So let (L, π_J) be a limit cone for $D\colon \mathsf{J} \to \mathsf{C}$. Then by definition, this diagram commutes in C for any $f\colon J \to K$ in J:

$$\begin{tikzcd}
 & L \arrow[dl, "\pi_J"'] \arrow[dr, "\pi_K"] & \\
DJ \arrow[rr, "Df"'] & & DK
\end{tikzcd}$$

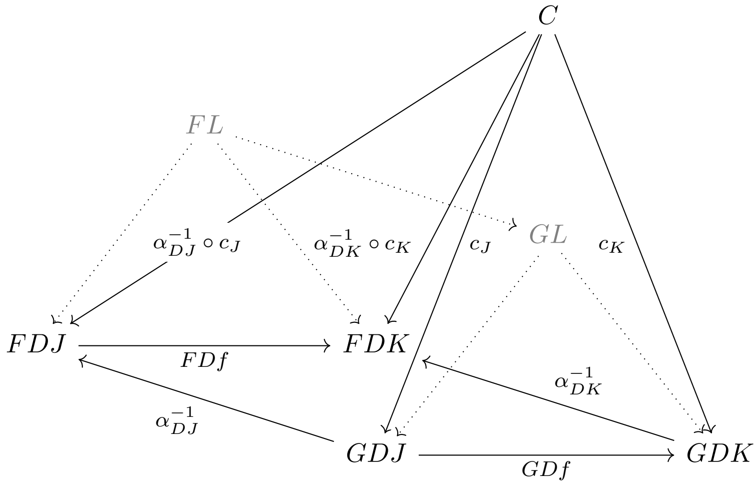
By definition, F and G both map this triangle into D, giving the two commuting triangles below. And the assumed natural isomorphism $\alpha\colon F \overset{\sim}{\Longrightarrow} G$ by definition gives us *three* naturality squares, making a commuting prism in D:

$$\begin{tikzcd}
 & FL \arrow[ddl, "F\pi_J"'] \arrow[ddr, "F\pi_K"] \arrow[drr, "\alpha_L"] & & & \\
 & & & GL \arrow[ddl, "G\pi_J"'] \arrow[ddr, "G\pi_K"] & \\
FDJ \arrow[rr, "FDf"] \arrow[drr, "\alpha_{DJ}"'] & & FDK \arrow[drr, "\alpha_{DK}"] & & \\
 & & GDJ \arrow[rr, "GDf"'] & & GDK
\end{tikzcd}$$

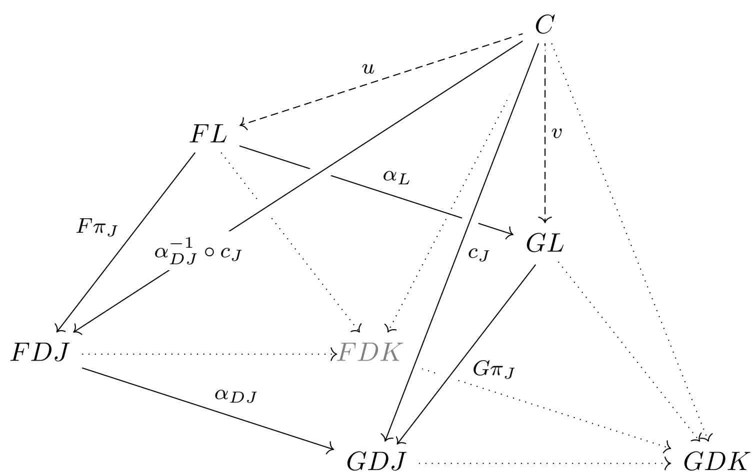
Now consider any cone (C, c_J) over the functor $G \circ D\colon \mathsf{J} \to \mathsf{D}$. Being part of a cone, each new triangle like the one below commutes by definition:

$$\begin{tikzcd}
 & & & C \arrow[dddd, "c_J"', shift right] \arrow[ddddr, "c_K"] & \\
 & FL \arrow[ddl, dotted] \arrow[ddr, dotted] \arrow[drr, dotted] & & & \\
 & & & GL \arrow[ddl, dotted] \arrow[ddr, dotted] & \\
FDJ \arrow[rr, dotted] \arrow[drr, dotted] & & FDK \arrow[drr, dotted] & & \\
 & & GDJ \arrow[rr, "GDf"'] & & GDK
\end{tikzcd}$$

Further, using the commuting base square of the relevant prisms, we can extend each leg c_J of the cone by composition with the corresponding α_{DJ}^{-1} to get a cone $(C, \alpha_{DJ}^{-1} \circ c_J)$ over FD, making further triangles like this:



OK: now *suppose for the sake of argument that F preserves the limit* (L, π_J). Then by definition $(FL, F\pi_J)$ is a limit cone over FD. Which means that our cone $(C, \alpha_{DJ}^{-1} \circ c_J)$ over FD must factor through this limit cone via a unique $u\colon C \to FL$, so e.g. $\alpha_{DJ}^{-1} \circ c_J = F\pi_j \circ u$. So all this, for example, commutes:



But it is easy to check – chasing arrows round the diagram, using the sloping sides of the prism – that it is also the case that the cone (C, c_J) over GD factors through $(GL, G\pi_J)$ via $v = \alpha_L \circ u$. For example, we have

$$c_J = \alpha_{DJ} \circ \alpha_{DJ}^{-1} \circ c_J = \alpha_{DJ} \circ F\pi_j \circ u = G\pi_j \circ \alpha_L \circ u = G\pi_j \circ v$$

And further (C, c_J) can't factor through a distinct v': or else there would be a distinct $u' = \alpha_L^{-1} \circ v'$ which makes everything commute, which is impossible by the uniqueness of u.

Hence, in sum, any (C, c_J) factors through $(GL, G\pi_J)$ via a unique v, and therefore $(GL, G\pi_J)$ is a limit cone. Therefore, as we wanted to show, G *also preserves the limit* (L, π_J). $\square$

32.6 Unnatural and natural isomorphisms between objects

To finish the chapter, we return to some more general considerations about isomorphic functions and isomorphic objects.

(a) Suppose we have functors $F, G\colon \mathsf{C} \to \mathsf{D}$; and let $A, B, C, \ldots$ be objects in C. Then there will be objects $FA, FB, FC \ldots$ and $GA, GB, GC \ldots$ in D. And in some cases these will be pairwise isomorphic, so that we have $FA \cong GA$, $FB \cong GB$, $FC \cong GC$ ....

One way this can happen, as we have just been emphasizing, is if there is a natural isomorphism between the functors F and G. But it is also important to stress that it can happen in other, 'unnatural', ways. Let's have a couple more examples, first a toy example to hammer home the point of principle, then a standard illustrative case which is worth thinking through:

(1) Suppose C is a category with exactly one object A, and two arrows, the identity arrow 1_A, and a distinct arrow f, where $f \circ f = f$. And now consider two functors, the identity functor $1_\mathsf{C}\colon \mathsf{C} \to \mathsf{C}$, and the functor $F\colon \mathsf{C} \to \mathsf{C}$ which sends the only object to itself, and sends both arrows to the identity arrow.

Then, quite trivially, we have $1_\mathsf{C}(A) \cong F(A)$ for the one and only object in C. But there isn't a natural isomorphism between the functors, because by hypothesis $1_A \neq f$, and hence the square

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "1_A"] & F(A) \arrow[d, "1_A"] \\ 1_\mathsf{C}(A) \arrow[r, "1_\mathsf{C}(f)"] & 1_\mathsf{C}(A) \end{tikzcd}, \text{ which is simply } \begin{tikzcd} A \arrow[r, "1_A"] \arrow[d, "1_A"] & A \arrow[d, "1_A"] \\ A \arrow[r, "f"] & A \end{tikzcd},$$

cannot commute.

(2) For a more interesting example, we'll work in the category of finite sets and *bijections* between them which I'll locally call simply F for short.

There is a functor $Sym\colon \mathsf{F} \to \mathsf{F}$ which (i) sends a set A in F to the set of permutations on A (treating permutation functions as sets, this is a finite set), and (ii) sends a bijection $f\colon A \to B$ in F to the bijection that sends the permutation p on A to the permutation $f \circ p \circ f^{-1}$ on B. Note: if A has n members, there are $n!$ members of the set of permutations on A.

There is also a functor $Ord\colon \mathsf{F} \to \mathsf{F}$ which (i) sends a set A in F to the set of total linear orderings on A (you can identify an order-relation with a set, so we can think of this too as a finite set), and (ii) sends a bijection $f\colon A \to B$ in F to the bijection $Ord(f)$ which sends a total order on A to the total order on B where $x <_A y$ iff $f(x) <_B f(y)$. Again, if A has n members, there are also $n!$ members of the set of linear orderings on A.

Now, for any object A of F, $Sym(A) \cong Ord(A)$ (since they are equinumerous finite sets). But there cannot be a natural isomorphism ψ between the functors Sym and Ord. For suppose otherwise, and consider the functors acting on a bijection $f\colon A \to A$. Then the following naturality square would have to commute:

$$\begin{tikzcd} Sym(A) \arrow[r, "Sym(f)"] \arrow[d, "\psi_A"] & Sym(A) \arrow[d, "\psi_A"] \\ Ord(A) \arrow[r, "Ord(f)"] & Ord(A) \end{tikzcd}$$

Consider then what happens to the identity permutation i in $Sym(A)$: it gets sent by $Sym(f)$ to $f \circ i \circ f^{-1} = i$. So the naturality square would tell us that $\psi_A(i) = Ord(f)(\psi_A(i))$. But that in general won't be so – suppose f swaps around elements, so $Ord(f)$ is not the 'do nothing' identity map.

Think of it this way: yes, $Sym(A)$ and $Ord(A)$ are isomorphic; but there is no privileged, canonical, way of setting up an isomorphism between them. In a summary slogan: pointwise isomorphism doesn't entail natural isomorphism.

(b) We are, however, going to be particularly interested in cases where we have $FA \cong GA$ as a result of a natural isomorphism between functors. And so we can introduce the following key definition:

Definition 125. Two objects in a category D are said to be *naturally isomorphic (in A)* if they are the images FA and GA of the same object A under a couple of naturally isomorphic functors $F, G\colon \mathsf{C} \to \mathsf{D}$. △

The definition mentions one specific object A in C; but there is an implicit generality here. For if $FA \cong GA$ naturally in A, F is naturally isomorphic to G, so there is also a component of the natural isomorphism at A' for any object A' in C, making $FA' \cong GA'$ naturally in A'.

Let's have some quick examples:

(1) A toy case. The products $A \times C$ and $C \times A$ are of course isomorphic, and intuitively the isomorphism here arises in an entirely natural way in a category C with all products (we don't have to make arbitrary choices to set it up). And so we will want to show that $A \times C$ and $C \times A$ are naturally isomorphic in A, in the sense just defined.

But that follows immediately from the result in §32.1 that the functors $- \times C$ and $C \times -$ are naturally isomorphic.

(2) More excitingly, we have seen that $V \cong DDV$ naturally in V in FVect: that is the message of §32.3.

(3) Likewise, $UG \cong \mathsf{Grp}(Z, G)$ naturally in G: that is the message of §32.4 (4).

(4) And for fourth case, given a category C with exponentials, $\mathsf{C}(A \times B, C) \cong \mathsf{C}(A, C^B)$ both naturally in A and naturally in C: that is the combined message of §32.4 (5) and (6). Challenge: use similar arguments to show that the isomorphism is natural in B too.

We will see more examples of naturally isomorphic objects in due course. But let's finish this section with a mini-theorem:

Theorem 155. *Given functors $F, G, H \colon \mathsf{C} \to \mathsf{D}$, an object A in C, and a functor $K \colon \mathsf{B} \to \mathsf{C}$, then*

(1) if $FA \cong GA$ and $GA \cong HA$, both naturally in A, then $FA \cong HA$ naturally in A.

(2) if $FA \cong GA$ naturally in A, then $FKB \cong GKB$ naturally in B.

Proof. For (1), just recall that natural isomorphisms vertically compose – see Theorem 32.2.

For (2), note that, if there is a natural isomorphism α between F and G, then (by 'whiskering', see Theorem 158) there is a natural isomorphism between FK and GK, whose component at B is α_{KB}, making $FKB \cong GKB$ naturally in B. □

32.7 An 'Eilenberg/Mac Lane Thesis'?

Let's return to the question we left hanging at the end of §32.3. Can we generalize from e.g. our example of a vector space and its double dual, and say that whenever we have a 'natural' or 'canonical' isomorphism between widgets and wombats (i.e. one that doesn't depend on arbitrary choices of co-ordinates, or the like), this can be seen as resulting from a natural isomorphism between suitable associated functors in the was we've just defined? Let's call the claim that we *can* generalize like this the 'Eilenberg/Mac Lane Thesis'.

I choose the label to be reminiscent of the Church/Turing Thesis that we all know and love, which asserts that every algorithmically computable function (in an informally characterized sense) is in fact recursive/Turing computable/lambda computable. A certain intuitive concept, this Thesis claims, in fact picks out the same functions as certain (provably equivalent) sharply defined concepts.

What kind of evidence do we have for the Church/Turing Thesis? Two sorts: (1) 'quasi-empirical', i.e. no unarguable clear exceptions have ever been found, and (2) conceptual, as in for example Turing's own efforts to show that when we reflect hard on what we mean by algorithmic computation we get down to the sort of operations that a Turing machine can emulate, so a computable function just ought to be Turing computable.

Leaving aside the conceptual route to the Thesis, the 'quasi-empirical' evidence in this case is so overwhelming that in fact we are allowed to appeal to the

Church/Turing Thesis as a labour-saving device: if we can give an arm-waving sketch of an argument that a certain function is algorithmically computable, we are allowed to assume that it is indeed recursive/Turing computable/lambda computable without doing the hard work of e.g. defining a Turing maching to compute it.

We now seem to have on the table another Thesis of the same general type: an informal intuitive concept of a canonical or natural isomorphism, the Eilenberg/Mac Lane Thesis claims, in effect picks out the same isomorphisms as a certain sharply defined categorial concept.

Evidence? We would expect again two sorts. (1*) ‘quasi-empirical’, a lack of clear exceptions, and maybe (2*) conceptual, an explanation of why the Thesis just ought to be true.

It is, however, not clear exactly how things stand evidentially here, and the usual textbook discussions of natural isomorphisms usually don’t pause to do much more than give a few examples.

More really needs to be said. We therefore can’t suppose that the new Eilenberg/Mac Lane Thesis is so secure that we can cheerfully appeal to it in the same labour-saving way as the old Church/Turing Thesis. In other words, even if (i) intuitively an isomorphism between objects seems to be set up in a very ‘natural’ way, without appeal to arbitrary choices, and (ii) we can readily massage the claim of an isomorphism into a claim about at least pointwise isomorphism of relevant functors, we really need to pause to work through a proof if we are to conclude that in fact (iii) there is a natural isomorphism here in the official categorial sense. Annoying: for as we have already seen, such proofs can be a bit tedious.

33 Natural transformations

We typically think of isomorphisms categorially as special cases of some wider class of morphisms – they are the morphisms which have two-sided inverses. OK: so we'll want to think of natural isomorphisms between functors as special cases of . . . . Well, what?

33.1 Natural transformations defined

(a) The generalized notion of morphisms between functors that we need is perhaps obvious enough. As before, we'll give a pair of definitions, one for each flavour of functor:

Definition 126. Let C and D be categories, let $F, G \colon \mathsf{C} \to \mathsf{D}$ be covariant functors (respectively, contravariant functors), and suppose that for each C-object C there is a D-arrow $\alpha_C \colon FC \to GC$. Then α, the family of arrows α_C, is a *natural transformation* between F and G if for every $f \colon A \to B$ (respectively $f \colon B \to A$, note the reversal!) in C the following naturality square commutes in D:

$$\begin{tikzcd}
FA \arrow[r, "Ff"] \arrow[d, "\alpha_A"] & FB \arrow[d, "\alpha_B"] \\
GA \arrow[r, "Gf"] & GB
\end{tikzcd}$$

In this case, we write $\alpha : F \Rightarrow G$, and the α_C are said to be components of α. △

So, to compare: natural isomorphisms are the special case of natural transformations where the components are themselves all isomorphisms. And looking at the diagrams in §32.1, we see that a natural transformation between functors $F, G \colon \mathsf{C} \to \mathsf{D}$ again sends an F-image of (some or all of) C to its G-image in a way which respects some of the internal structure of the original – perhaps now collapsing even non-isomorphic objects together, but at least preserving composition of arrows.

(b) On notation: different styles of arrows can be found in use for talking about natural transformations, but Greek letters are almost universally used for their names.

A common diagrammatic convention will prove useful. When we have two functors $F, G\colon \mathsf{C} \to \mathsf{D}$, together with a natural transformation $\alpha\colon F \Rightarrow G$, we can neatly represent the whole situation thus:

$$\begin{tikzcd} \mathsf{C} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "G"'{name=L}] & \mathsf{D} \arrow[Rightarrow, from=U, to=L, "\alpha"] \end{tikzcd}$$

(c) Suppose now that we have three functors $F, G, H\colon \mathsf{C} \to \mathsf{D}$, together with two natural transformations $\alpha\colon F \Rightarrow G$, and $\beta\colon G \Rightarrow H$. Then we can simply generalize what we said about composing natural isomorphisms in §32.2 to apply to all natural transformations. In other words, we can compose $\alpha\colon F \Rightarrow G$, and $\beta\colon G \Rightarrow H$ to give a natural transformation $\beta \circ \alpha\colon F \Rightarrow H$ defined componentwise by putting $(\beta \circ \alpha)_A = \beta_A \circ \alpha_A$ for all objects A in C. Composing transformations $\begin{tikzcd} \mathsf{C} \arrow[r, bend left=60, "F"{name=U}] \arrow[r, "G"{name=M, description}] \arrow[r, bend right=60, "H"'{name=L}] & \mathsf{D} \arrow[Rightarrow, from=U, to=M, "\alpha"] \arrow[Rightarrow, from=M, to=L, "\beta"] \end{tikzcd}$ in this way to get $\begin{tikzcd} \mathsf{C} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "H"'{name=L}] & \mathsf{D} \arrow[Rightarrow, from=U, to=L, "\beta \circ \alpha"] \end{tikzcd}$ is rather predictably called *vertical composition*, and as with natural isomorphisms, vertical composition is associative. We'll meet a companion notion of horizontal composition shortly.

So, summarizing, we have:

Theorem 156. *Suppose F, G, H are covariant functors from C to D:*

1. *There is an identity natural transformation (isomorphism!) $1_F\colon F \Rightarrow F$.*
2. *Given natural transformations $\alpha\colon F \Rightarrow G$ and $\beta\colon G \Rightarrow H$, there is a composite natural transformation $\beta \circ \alpha\colon F \Rightarrow H$, and composition is associative.*

(d) Since it makes sense to compose natural transformations, we can talk of a natural transformation $\alpha\colon F \Rightarrow G$ as being an isomorphism in the sense of having a two-sided inverse (so there is some $\beta\colon G \Rightarrow F$ such that $\beta \circ \alpha = 1_F$ and $\alpha \circ \beta = 1_G$, where 1_F and 1_G are as in §32.2). We then have a predictable result:

Theorem 157. *A natural transformation is an isomorphism (in the sense of having a two-sided inverse) if and only if a natural isomorphism.* □

Proof of 'if'. Suppose $\alpha\colon F \xRightarrow{\sim} G$ is a natural isomorphism between the parallel functors $F, G\colon \mathsf{C} \to \mathsf{D}$, in the sense of Defn. 124. So for any $f\colon A \to B$, the naturality square

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\alpha_A"] & FB \arrow[d, "\alpha_B"] \\ GA \arrow[r, "Gf"] & GB \end{tikzcd}$$

commutes. But if $\alpha_B \circ Ff = Gf \circ \alpha_A$, then $Ff \circ \alpha_A^{-1} = \alpha_B^{-1} \circ Gf$ (relying on the fact that the components of α have inverses). Which makes this diagram always commute for any $f \colon A \to B$:

$$\begin{tikzcd} GA \arrow[r, "Gf"] \arrow[d, "\alpha_A^{-1}"] & GB \arrow[d, "\alpha_B^{-1}"] \\ FA \arrow[r, "Ff"] & FB \end{tikzcd}$$

Whence $\alpha^{-1} \colon G \xRightarrow{\sim} F$, where α^{-1} is assembled from the components α_A^{-1} etc. And then trivially $\alpha^{-1} \circ \alpha = 1_F$ and $\alpha \circ \alpha^{-1} = 1_G$. Which gives α a two-sided inverse. □

Proof of 'only if'. Suppose the natural transformation $\alpha \colon F \Rightarrow G$ has an inverse α^{-1}, so $\alpha^{-1} \circ \alpha = 1_F$, and $\alpha \circ \alpha^{-1} = 1_G$ But composition of natural transformations is defined component-wise, so this requires for each component that $\alpha_X^{-1} \circ \alpha_X = 1_{FX}$, $\alpha_X \circ \alpha_X^{-1} = 1_{GX}$. Therefore each component of α has an inverse, so is an isomorphism, and hence α is a natural isomorphism. □

33.2 Some examples

(a) Let's have a couple of initial toy examples of natural transformations which aren't isomorphisms:

(1) Suppose D has a terminal object 1, and let $F \colon \mathsf{C} \to \mathsf{D}$ be any functor. Then there is also a parallel functor $T \colon \mathsf{C} \to \mathsf{D}$ which sends every C-object to D's terminal object 1, and every C-arrow to the identity arrow on the terminal object.

Claim: there is a natural transformation $\alpha \colon F \Rightarrow T$.

Proof. We need a suite of D-arrows $\alpha_A \colon FA \to TA$ (one for each A in C) which makes the following commute for any $f \colon A \to B$ in C, remembering that $TA = 1$:

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\alpha_A"] & FB \arrow[d, "\alpha_B"] \\ 1 \arrow[r, "1_1"] & 1 \end{tikzcd}$$

But there is only one candidate for each of α_A, α_B, etc.: it has to be the unique arrow from its source to the terminal object. And then the diagram must commute because all arrows from FA to 1 are equal. □

And now here's a companion claim: there is no natural transformation $\alpha \colon T \Rightarrow F$. Essentially, that's because there is no canonical way of choosing where a candidate transformation should send the T-image of an object in C.

Proof. If there were a natural transformation, there would need to be a suite of arrows $\gamma_A \colon TA \to FA$, i.e. $\gamma_A \colon 1 \to FA$, which makes the following commute for any $f \colon A \to B$ in C:

$$\begin{tikzcd} 1 \arrow[r, "1_1"] \arrow[d, "\gamma_A"] & 1 \arrow[d, "\gamma_B"] \\ FA \arrow[r, "Ff"] & FB \end{tikzcd}$$

But γ_A will pick out a particular element a of FA, and a particular element b of FB. And obviously there is no guarantee that Ff (for each and any f) always sends a to the particular target b. □

(2) Recall the functor $List \colon \mathsf{Set} \to \mathsf{Set}$ where $List_{ob}$ sends a set A to the set of finite lists of members of A and $List_{arw}$ sends a set-function $f \colon A \to B$ to the map from $List(A)$ to $List(B)$ that sends a list $a_0 {}^\frown a_1 {}^\frown a_2 {}^\frown \ldots {}^\frown a_n$ to $fa_0 {}^\frown fa_1 {}^\frown fa_2 {}^\frown \ldots {}^\frown fa_n$. Claim: there is a natural transformation $\alpha \colon 1 \Rightarrow List$, where 1 is the trivial identity functor $1 \colon \mathsf{Set} \to \mathsf{Set}$.

Proof. We need a suite of functions α_A which make the following commute for any $f \colon A \to B$ in C:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha_A"] & B \arrow[d, "\alpha_B"] \\ List(A) \arrow[r, "List(f)"] & List(B) \end{tikzcd}$$

For any non-empty A, put α_A to be the function which sends an element of A to the length-one list containing just that element (and sends the empty set to the empty list). We are immediately done. □

What about going in the opposite direction? Can there be a natural transformation from the functor *List* to the trivial functor 1? We'd need a uniform way of choosing from a list in $List(A)$ a specific member of A. Well, how about the function γ_A that chooses the first element of a list in $List(A)$? Won't that make the square always commute?

Nice try! But the snag is that even if A is non-empty, $List(A)$ will contain the empty list – and there is no canonical way of recovering a specific member of A from the empty list. And that thought can be parlayed into a proof that there is no natural transformation from *List* to 1.

(b) Now for a case with rather more significance, but which still requires only relatively elementary ideas.

(3) We are going to set up two functors from CRng, the category of rings where multiplication commutes, to our familiar friend Mon.

One is the functor $F\colon \mathsf{CRng} \to \mathsf{Mon}$ which just forgets about the ring structure other than multiplication.

The other is the functor we can notate M_n which sends a ring R to the monoid of $n \times n$ matrices with elements from R (with matrix multiplication as the monoid operation, and the unit diagonal matrix as the monoid unit). How does M_n operate on a ring homomorphism $f\colon R \to S$? It applies f to each component of a matrix with elements from R to give us a matrix with elements from S. And since f respects multiplication in R, $M_n(f)\colon M_n(R) \to M_n(S)$ will respect multiplication between matrices, i.e. will be a monoid homomorphism.

Now the claim is that there is a nice natural transformation $M_n \Rightarrow F$. How come? We need a suite of arrows α such that, for every ring homomorphism $f\colon R \to S$, the naturality squares commute:

$$\begin{tikzcd}
M_n(R) \arrow[r, "M_N(f)"] \arrow[d, "\alpha_R"] & M_n(S) \arrow[d, "\alpha_S"] \\
F(R) \arrow[r, "F(f)"] & F(S)
\end{tikzcd}$$

OK, what's a nice canonical way of mapping an $n \times n$ matrix of elements of R to an element of $F(R)$ which is (of course) simply an element of R? How do we go in a natural way from matrices of reals, say, to a real? Take a determinant! So let's put α_R the function det_R that sends matrices $M_n(R)$ to their determinants.

And then everything will nicely commute. A ring homomorphism $f\colon R \to S$ respects both addition and multiplication. So if we take a matrix M_R of R-elements and hit all the elements with f to get a matrix of M_S of S-elements, it is immediate that $f(det_R(M_R)) = det_S(M_S)$.

In short, then, we have a natural transformation $det\colon M_n \Rightarrow F$.

(c) I'll briefly mention two more cases of natural transformations which aren't isomorphisms and which have mathematical significance (but feel quite free to skip if the background ideas are unfamiliar):

(4) For those who know just a bit more group theory, consider the abelianization of a group G. Officially, this is the quotient of a group by its commutator subgroup $[G, G]$ (but you can think of it as the 'biggest' Abelian group A for which there is a surjective homomorphism from G onto A). There is then a functor Ab which sends a group G to its abelianization $Ab(G)$, and sends an arrow $f\colon G \to H$ to the arrow $Ab(f)\colon Ab(G) \to Ab(H)$ defined in a fairly obvious way.

We therefore have a pair of functors, $\mathsf{Grp} \underset{Ab}{\overset{1}{\rightrightarrows}} \mathsf{Grp}$, and we can then check that the following diagram always commutes,

$$
\begin{tikzcd}
G \arrow[r, "f"] \arrow[d, "\alpha_G"] & H \arrow[d, "\alpha_H"] \\
Ab(G) \arrow[r, "Ab(f)"] & Ab(H)
\end{tikzcd}
$$

where α_G sends G to $G/[G, G]$. So we have a natural transformation, but not usually a natural isomorphism, between the functors 1 and Ab.

(5) For those who know rather more topology, we can mention two important functors from topological spaces to groups. One we've met before in §28.4, namely the functor $\pi\colon \mathsf{Top}_* \to \mathsf{Grp}$ which sends a space with a basepoint to its fundamental group at the base point. The other functor $H\colon \mathsf{Top} \to \mathsf{AbGrp}$ sends a space to the abelian group which is its first homology group (I'm going to try to explain that here!). Now these functors aren't yet parallel functors between the same categories. But we can define a functor $H'\colon \mathsf{Top}_* \to \mathsf{Grp}$ which first forgets base points of spaces, then applies H, and then forgets that the relevant groups are abelian. I simply record that it is a very important fact of topology that, in our categorial terms, there is natural transformation from π to H'.

(d) Can we generalize from these sorts of examples, and motivate a wider 'Eilenberg Mac Lane Thesis' to the effect that, whenever we have a particularly natural sort of construction taking widgets to wombats we can view it as a natural transformation between suitable functors?

This looks a notably less plausible claim than the more restricted Thesis about isomorphisms which we met in §32.7, just because the notion of a natural construction seems quite permissive. In fact, there seem to be clear counterexamples to the wider Thesis.

For example, take the construction that forms the centre of a group (the abelian subgroup of the elements which commute with all the other members of the group). That's surely a natural enough construction. But it is quite easy to show that there can't be a *functor* from Grp to Ab that takes groups to their centres and then behaves functorially on arrows.[1] And that stymies the possibility of thinking of group centres in terms of a suitable natural transformation between functors.

[1] An argument for enthusiasts. Suppose A is the free group on one generator a, and B is the free group on two generators a, b. The centre of A, $Z(A)$, is the abelian group A itself. The centre of B, $Z(B)$, is the trivial one-object group 1. Consider the homomorphism $f\colon A \to B$ generated by sending a to a, and the homomorphism $g\colon B \to A$ generated by sending both a and b to a. Note that $g \circ f = 1_A$. Suppose there were a functor Grp to Ab which on objects sends groups G to their centres $Z(G)$. How would it act on homomorphisms? It would need to send $f\colon A \to B$ to $Ff\colon Z(A) \to Z(B)$, i.e. $Ff\colon A \to 1$ (which sends everything to the one object of the trivial group) and send $g\colon B \to A$ to $Ff\colon Z(B) \to Z(A)$, i.e. $Fg\colon 1 \to A$. So $Fg \circ Ff$ is the map $!_A$ which sends everything in A to its unit element. But then we have

$$F1_A = F(g \circ f) = Fg \circ Ff = !_A \neq 1_{Z(A)}$$

So F is not functorial after all, not always sending identity maps to identity maps.

So a universal thesis here would overshoot. But yes, many cases of natural constructions *can* be treated categorially as natural transformations, and that's enough to make the notion of key interest.

33.3 Horizontal composition of natural transformations

(a) We have seen how to compose natural transformations 'vertically'. We can, however, also put things together *horizontally* in various ways.

First, there is so-called *whiskering*(!) where we combine a single functor with a natural transformation between functors to get a new natural transformation. So, for example, consider what happens when we 'add a whisker' on the left of a diagram for a natural transformation.

The situation $\begin{tikzcd} \mathsf{C} \arrow[r, "F"] & \mathsf{D} \arrow[r, bend left, "J"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathsf{E} \end{tikzcd}$ gives rise to $\begin{tikzcd} \mathsf{C} \arrow[r, bend left, "J \circ F"] \arrow[r, bend right, "K \circ F"'] \arrow[r, phantom, "\Downarrow\beta F"] & \mathsf{E} \end{tikzcd}$ where the component of βF at A is the component of β at FA – i.e. $(\beta F)_A = \beta_{FA}$.[2] Why does this hold? Consider the function $Ff\colon FA \to FB$ in D (where $f\colon A \to B$ is in C). Now apply the functors J and K, and since β is a natural transformation we get the commuting 'naturality square'

$$\begin{tikzcd} J(FA) \arrow[r, "J(Ff)"] \arrow[d, "\beta_{FA}"] & J(FB) \arrow[d, "\beta_{FB}"] \\ K(FA) \arrow[r, "K(Ff)"] & K(FB) \end{tikzcd}$$

and we can read that as giving a natural transformation between $J \circ F$ and $K \circ F$.

Likewise, adding a whisker on the right,

the situation $\begin{tikzcd} \mathsf{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathsf{D} \arrow[r, "J"] & \mathsf{E} \end{tikzcd}$ gives rise to $\begin{tikzcd} \mathsf{C} \arrow[r, bend left, "J \circ F"] \arrow[r, bend right, "J \circ G"'] \arrow[r, phantom, "\Downarrow J\alpha"] & \mathsf{E} \end{tikzcd}$

where the component of $J\alpha$ at X is $J(\alpha_X)$.

For future use, by the way, we should note the following mini-result:

Theorem 158. *Whiskering a natural isomorphism yields a natural isomorphism.*

Proof. Retaining the same notation as above, but now taking α and β to be isomorphisms, we saw that 'post-whiskering' α by J to get $J\alpha$ yields a transformation whose components are $J\alpha_X$, and since functors preserve isomorphisms, these components are all isomorphisms, hence so is $J\alpha$.

[2]The suggestive notation 'β_F' is quite often preferred to 'βF'.

While 'pre-whiskering' β by F to get βF yields a transformation whose components are (some of the) components of β and therefore are isomorphisms, hence again so is βF. □

(b) Second, we can *horizontally compose* two natural transformations in the following way:

$$\text{We take}\quad \begin{tikzcd} \mathsf{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathsf{D} \arrow[r, bend left, "J"] \arrow[r, bend right, "K"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathsf{E} \end{tikzcd} \quad\text{and get}\quad \begin{tikzcd} \mathsf{C} \arrow[r, bend left, "J \circ F"] \arrow[r, bend right, "K \circ G"'] \arrow[r, phantom, "\Downarrow\beta * \alpha"] & \mathsf{E}. \end{tikzcd}$$

How do we define $\beta * \alpha$? Take an arrow $f\colon A \to B$ living in C and form this naturality square for α:

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\alpha_A"] & FB \arrow[d, "\alpha_B"] \\ GA \arrow[r, "Gf"] & GB \end{tikzcd} .$$

Applying the functor J to the ingredients of that diagram, we get

$$\begin{tikzcd} J(FA) \arrow[r, "J(Ff)"] \arrow[d, "J(\alpha_A)"] & J(FB) \arrow[d, "J(\alpha_B)"] \\ J(GA) \arrow[r, "J(Gf)"] & J(GB) \end{tikzcd}$$

which also commutes. And since $Gf\colon GA \to GB$ is a map in D, and β is a natural transformation between $\mathsf{D} \underset{K}{\overset{J}{\rightrightarrows}} \mathsf{E}$, this too must commute:

$$\begin{tikzcd} J(GA) \arrow[r, "J(Gf)"] \arrow[d, "\beta_{GA}"] & J(GB) \arrow[d, "\beta_{GB}"] \\ K(GA) \arrow[r, "K(Gf)"] & K(GB) \end{tikzcd}$$

Gluing together those last two commutative diagrams one above the other gives a natural transformation from $J \circ F$ to $K \circ G$, if we set the component of $\beta * \alpha$ at X to be $\beta_{GX} \circ J\alpha_X$.

Three remarks:

(1) That definition for $\beta * \alpha$ looks surprisingly asymmetric. But note that applying J to the initial naturality square for α and then pasting the result above a naturality square for β, we could have similarly applied K to the initial naturality square and pasted the result below another naturality

square for β, thus showing that we can alternatively define the natural transformation $J \circ F$ to $K \circ G$ as having the components $K\alpha_X \circ \beta_{FX}$. So symmetry is restored: we get equivalent accounts which mirror each other.

(2) We can think of whiskering as a special case of the horizontal composition of two natural transformations where one of them is the identity natural transformation. For example $\begin{tikzcd} \mathsf{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathsf{D} \arrow[r, bend left, "J"] \arrow[r, bend right, "J"'] \arrow[r, phantom, "\Downarrow 1_J"] & \mathsf{E} \end{tikzcd}$ produces $\begin{tikzcd} \mathsf{C} \arrow[r, bend left, "J \circ F"] \arrow[r, bend right, "J \circ G"'] \arrow[r, phantom, "\Downarrow 1_J * \alpha"] & \mathsf{E} \end{tikzcd}$, and the component of $1_J * \alpha$ at X is an identity composed with $J\alpha_X$. So this is the same as taking the left-hand natural transformation and simply whiskering with J on the right.

(3) We could now go on to consider the case of horizontally composing a couple of pairs of vertical compositions – and show that it comes to the same if we construe the resulting diagram as the result of vertically composing a couple of horizontal compositions. In symbols

$$(\delta \circ \gamma) * (\beta \circ \alpha) = (\delta * \beta) \circ (\gamma * \alpha)$$

But we won't now pause over this – take it as a challenge to draw some diagrams and prove it!

33.4 Cones as natural transformations

(a) A *natural transformation* is defined as a suite of arrows from various sources, with each pair of arrows making certain diagrams commute. But now compare: we earlier defined a *cone* as again essentially a suite of arrows – but this time all from the same source, the apex of the cone – with each pair of arrows making certain diagrams commute. Which suggests that we might be able to treat cones as special cases of natural transformations. And we can, giving us a perhaps tidier characterization than Defn. 115. Let's finish the chapter by showing how.

(b) Recall, Defn. 114 (re)defines a diagram of shape J in the category C as a functor $D\colon \mathsf{J} \to \mathsf{C}$.

Recall too that among such functors there are functors $\Delta_X\colon \mathsf{J} \to \mathsf{C}$, i.e. a 'collapse' functor which picks an object X in C, sends every J-object to X, and sends every J-arrow to 1_X (see §27.2 (F6)).

And now let's ask: what does it take for there to be a natural transformation $\alpha\colon \Delta_X \to D$, for some given $D\colon \mathsf{J} \to \mathsf{C}$?

By the definition of a natural transformation, the following must commute for any J-arrow $j\colon K \to L$:

$$\begin{tikzcd} \Delta_X K \arrow[r, "\Delta_X j"] \arrow[d, "\alpha_K"'] & \Delta_X L \arrow[d, "\alpha_L"] \\ DK \arrow[r, "Dj"'] & DL \end{tikzcd} = \begin{tikzcd} X \arrow[r, "1_X"] \arrow[d, "\alpha_K"'] & X \arrow[d, "\alpha_L"] \\ DK \arrow[r, "Dj"'] & DL \end{tikzcd} = \begin{tikzcd} & X \arrow[dl, "\alpha_K"'] \arrow[dr, "\alpha_L"] & \\ DK \arrow[rr, "Dj"'] & & DL \end{tikzcd}$$

Which makes the α_J (where J runs over objects in J) the legs of a cone over D with a vertex X!

Conversely, the legs of any cone over D with a vertex X assemble into a natural transformation $\alpha\colon \Delta_X \to D$.

So that means that cones (austerely thought of as simply suites of arrows[3]) can indeed be regarded as certain natural transformations. This observation is worth summing up as a theorem:

Theorem 159. *If D is a diagram of shape J in C, then a cone over D with vertex X is a natural transformation from Δ_X to D (where $\Delta_X\colon \mathsf{J} \to \mathsf{C}$ is the collapse-to-X functor).* □

[3]See §18.1, fn.1

34 Equivalent categories

Now we have defined natural transformations, and more particularly natural isomorphisms, we are in a position to characterize what it is for categories to be *equivalent* – where equivalent categories 'come to the same', in a good intuitive sense.

However, we first define a stronger notion, and see why it is *too* strong for some purposes.

34.1 Isomorphic categories

(a) When first introducing functors in §27 we quickly met Theorem 128 which (1) tells us thatfor any category C there is an identity functor $1_{\mathsf{C}} \colon \mathsf{C} \to \mathsf{C}$ which sends C's objects and arrows to themselves, and (2) also tells us that functors $F \colon \mathsf{C} \to \mathsf{D}$ and $G \colon \mathsf{D} \to \mathsf{E}$ always compose in an obvious way to give a functor $GF \colon \mathsf{C} \to \mathsf{E}$.

Hence it makes perfect sense to say that a functor might have a two-sided inverse, and so we get a predictable definition:

Definition 127. A functor $F \colon \mathsf{C} \to \mathsf{D}$ is an *isomorphism*, in symbols $F \colon \mathsf{C} \xrightarrow{\sim} \mathsf{D}$, iff it has a two-sided inverse – meaning that there is a functor $G \colon \mathsf{D} \to \mathsf{C}$ where $GF = 1_{\mathsf{C}}$ and $FG = 1_{\mathsf{D}}$. △

And let's just check that we get sensible results with this definition, e.g.

Theorem 160. *If $F \colon \mathsf{C} \xrightarrow{\sim} \mathsf{D}$ is an isomorphism, it is fully faithful and essentially surjective on objects.*

Proof. Take parallel arrows in C, namely $f, g \colon A \to B$. Supposing $Ff = Fg$, then $GFf = GFg$ – where G is F's inverse. So $1_{\mathsf{C}}f = 1_{\mathsf{C}}g$ and hence $f = g$. Therefore F is faithful.

Suppose next we are given an arrow $h \colon FA \to FB$. Put $f = Gh$. Then $Ff = FGh = 1_{\mathsf{D}}h = h$. So every such h in D is the image under F of the associated arrow f in C. So F is full.

Finally take any D-object D, and using F's inverse G again, map that to the C-object GD. Then $F(GD) = D \cong D$. So F is e.s.o. □

The converse doesn't hold, however. We will soon find examples of functors which are fully faithful and e.s.o. but not isomorphisms.

(b) Now that we have the notion of a functor-as-isomorphism in play, this readily prompts a further definition:

Definition 128. The categories C and D are *isomorphic*, in symbols $\mathsf{C} \cong \mathsf{D}$, iff there is an isomorphism $F\colon \mathsf{C} \xrightarrow{\sim} \mathsf{D}$. △

It is immediate that $\cong$ is an equivalence relation, as required. So let's go straight to some examples of isomorphic categories.

(1) Take the toy categories with different pairs of objects which we can diagram as e.g.

$$\circlearrowleft a_1 \longrightarrow b_1 \circlearrowright \qquad\qquad \circlearrowleft a_2 \longrightarrow b_2 \circlearrowright$$

Plainly they are isomorphic (and indeed there is a unique isomorphic functor that sends the first to the second in the obvious way)! And if we don't care about distinguishing copies of structures that are related by a unique isomorphism, then we'll count these as the same in a strong sense. Which to that extent warrants our earlier talk about *the* category $\mathsf{2}$ – e.g. in §4.4, Ex. (C6).

(2) Recall two earlier definitions, of Set_* and $1/\mathsf{Set}$.

The objects of the category of pointed sets Set_* are pairs (X, x), comprising a non-empty set X and x a chosen base point in X. And an arrow $f\colon (X, x) \to (Y, y)$ is a set-function $f\colon X \to Y$ such that $fx = y$.

As for the coslice category $1/\mathsf{Set}$, its objects are all pairs of the form $(X, \vec{x}\colon 1 \to X)$ for any $X \in \mathsf{Set}$ (and note, since there is no arrow $\vec{x}\colon 1 \to \emptyset$, X can't be empty here). The arrows from $(X, \vec{x}\colon 1 \to X)$ to $(Y, \vec{y}\colon 1 \to Y)$ are just the set-functions $j\colon X \to Y$ such that $j \circ \vec{x} = \vec{y}$.

Now we said when introducing the construction in §6.3 that $1/\mathsf{Set}$ looks to be in some strong sense 'the same as' the category Set_* of pointed sets. And indeed the categories are isomorphic, as we'll now show.

So take the function F_{ob} from objects in $1/\mathsf{Set}$ to objects Set_* which sends $(X, \vec{x}\colon 1 \to X)$ to the pointed set (X, x) where x is the value of the function $\vec{x}$ for its sole argument. And take F_{arw} to send a set function $j\colon X \to Y$ such that $j \circ \vec{x} = \vec{y}$ to the same function treated as an arrow $j\colon (X, x) \to (Y, y)$ which must then preserve base points. It is trivial to check that F is a functor $F\colon 1/\mathsf{Set} \to \mathsf{Set}_*$.

In the other direction, we can define a functor $G\colon \mathsf{Set}_* \to 1/\mathsf{Set}$ which acts on objects by sending (X, x) to $(X, \vec{x})$, where $\vec{x}\colon 1 \to X$ of course maps 1 to the point x, and acts on arrows by sending a basepoint-preserving set-function from X to Y to itself.

And it is immediate that these two functors F and G are inverse to each other. Hence, as claimed, $\mathsf{Set}_* \cong 1/\mathsf{Set}$.

(3) We can very similarly show, e.g., that the comma category $(1_{\mathsf{C}} \downarrow X)$ from §30.2 is isomorphic to the slice category C/X. But again, that's too easy to be very interesting!

So for something giving us just a bit more to chew on, consider Boolean algebras and the two standard ways of presenting them. In categorial terms, there is a category Bool whose objects are algebras $(B, \neg, \wedge, \vee, 0, 1)$ constrained by the familiar Boolean axioms, and whose arrows are homomorphisms that preserve algebraic structure. But then there is also a category BoolR whose objects are Boolean rings, i.e. rings $(R, +, \times, 0, 1)$ where $x^2 = x$ for all $x \in R$, and whose arrows are ring homomorphisms.

There are familiar ways of marrying up Boolean algebras with corresponding rings and vice versa. Thus if we start from $(B, \neg, \wedge, \vee, 0, 1)$, take the same carrier set and distinguished objects, put

(i) $x \times y =_{\mathrm{def}} x \wedge y$,

(ii) $x + y =_{\mathrm{def}} (x \vee y) \wedge \neg(x \wedge y)$ (exclusive 'or'),

then we get a Boolean ring. And if we apply this same process to two algebras B_1 and B_2, it is elementary to check that it will carry a homomorphism of algebras $f_a \colon B_1 \to B_2$ to a corresponding homomorphism of rings $f_r \colon R_1 \to R_2$.

We can equally easily go from rings to algebras, by putting

(i) $x \wedge y =_{\mathrm{def}} x \times y$,

(ii) $x \vee y =_{\mathrm{def}} x + y + (x \times y)$

(iii) $\neg x =_{\mathrm{def}} 1 + x$.

Note that going from a algebra to the associated ring and back again takes us back to where we started.

In summary, without going into any more details, we can in this way define a functor $F \colon$ Bool $\to$ BoolR, and a functor $G \colon$ BoolR $\to$ Bool which are inverses to each other. So, as we'd surely have expected, the category Bool is isomorphic to the category BoolR.

34.2 Intuitively equivalent but non-isomorphic categories

(a) We've found some nice cases of categories that intuitively 'come to the same' and which are more or less easily seen to be isomorphic in the official sense of Defn. 128. So far, so good.

However, we can also readily give examples of pairs of categories that again seem morally to 'come to the same' but which *aren't* isomorphic. To introduce a first example, let's think for a moment about partial functions.

In the general theory of computation, there is no getting away from the central importance of the notion of a partial function. But how should we treat partial functions in logic? Suppose the partial computable function $\varphi \colon \mathbb{N} \to \mathbb{N}$ outputs no value for n (the algorithm defining φ doesn't terminate gracefully for input n). Then the term '$\varphi(n)$' apparently lacks a denotation. *But in standard first-order logic, all terms are assumed to denote* – a sentence with a non-denoting term, on the standard semantics, will lack a truth-value. What to do, if we want to formalize our theory?

One option is to bite the bullet, formally allow non-denoting terms, and then go in for some logical revisionism to cope with the truth-value gaps which come along with them.

Another option is to follow Frege and stipulate that apparently empty terms are in fact not empty at all but denote some special rogue object (so there are, strictly speaking, no empty terms and no truth-value gaps, and hence we can preserve standard logic).

So on this second option, presented with what we might naively think of as a partial function $\varphi\colon \mathbb{N} \to \mathbb{N}$, we now treat this as officially being a *total* function – namely $f\colon \mathbb{N} \cup \{\star\} \to \mathbb{N} \cup \{\star\}$, where $\star$ is any convenient non-number, and where $f(n) = \varphi(n)$ when $\varphi(n)$ takes a numerical value and f takes the value $\star$ otherwise. If you like, you can think of $\star$ as coding 'not numerically defined'. On this option, then, since our functions are all officially total, they don't generate non-denoting terms, and we can preserve our standard logic without truth-value gaps.

There is a lot more to be said: but while the debate about the best logical treatment of partial functions is the sort of thing that might grip some philosophically-minded logicians, it really does seem of very little general mathematical interest. *And that's exactly the current point.* From a mathematical point of view there surely isn't anything much to choose between the two options. We can think of a world of genuinely *partial* numerical functions $\varphi\colon \mathbb{N} \to \mathbb{N}$, or we can equally think of a corresponding world of *total* functions $f\colon \mathbb{N} \cup \{\star\} \to \mathbb{N} \cup \{\star\}$, with the distinguished point $\star \notin \mathbb{N}$, and $f(\star) = \star$. Take your pick!

More generally, on a larger scale, we can think of a category Pfn whose objects are sets X and whose arrows are (possibly) *partial* functions between them. And there is also the category $\mathsf{Set}_\star$ of pointed sets whose objects are sets with a distinguished base point, and whose arrows are (total) set-functions which preserve base points. And mathematically, shouldn't these come to the same?

However, we can now easily show:

Theorem 161. $\mathsf{Set}_\star$ *is not isomorphic to* Pfn.

We can remark that there *is* an obvious functor $F\colon \mathsf{Set}_\star \to \mathsf{Pfn}$. F sends a pointed set (X, x) to the set $X \setminus \{x\}$, and sends a base-point preserving total function $f\colon (X, x) \to (Y, y)$ to the partial function $\varphi\colon X \setminus \{x\} \longrightarrow Y \setminus \{y\}$, where $\varphi(x) = f(x)$ if $f(x) \in Y \setminus \{y\}$, and is undefined otherwise. But, nice though this is, F isn't an isomorphism (it could send distinct (X, x) and (X', x') to the same target object).

Again, there is a whole family of functors from Pfn to $\mathsf{Set}_\star$ which take a set X and add an element not yet in X to give as an expanded set with the new object as a basepoint. Here's a way of doing this in a uniform way without making arbitrary choices for each X. Define $G\colon \mathsf{Pfn} \to \mathsf{Set}_\star$ as sending a set X to the pointed set $X_* =_{\mathrm{def}} (X \cup \{X\}, X)$, remembering that in standard set theories $X \notin X$! And then let G send a partial function $\varphi\colon X \to Y$ to the total basepoint-preserving function $f\colon X_* \to Y_*$, where $f(x) = \varphi(x)$ if $\varphi(x)$ is defined and $f(x) = Y$ otherwise. G is a natural enough choice, but isn't an isomorphism (it isn't surjective on objects).

Still, those observations don't yet rule out there being *some* pair of functors between $\mathsf{Set}_\star$ and Pfn which are mutually inverse. However, simple cardinality considerations show that there can't be any such pair:

Proof. A functor which is an isomorphism from Pfn to $\mathsf{Set}_\star$ must, among other things, send isomorphisms living in Pfn one-to-one to isomorphisms living in $\mathsf{Set}_\star$, so should preserve the cardinality of isomorphism classes. But the isomorphism class of the empty set in Pfn has just one member, while there is no one-membered isomorphism class in $\mathsf{Set}_\star$. So there can't be an isomorphism between the categories. □

(b) Let's have a rather quicker but deeper example. Take again the category FVect whose objects are finite-dimensional vector spaces over the reals and whose arrows are linear maps. And now consider the category Mat whose objects are natural numbers (representing the dimension of a vector space) and whose arrows $M: m \to n$ are the $m \times n$ matrices with real-number entries (representing linear maps between spaces). Composition of arrows in Mat is the usual matrix multiplication and the identity arrow on n is the $n \times n$ identity matrix.[1]

Now, it is an entirely familiar thought that linear algebra can be equivalently done either abstractly with linear maps or concretely with matrices – and we move between the two styles as context makes convenient. So in an important sense, the corresponding categories FVect and Mat ought to 'come to the same'.

Well, there are indeed natural enough functors going in each direction. Take a basis for each space: then there is a functor $J: \mathsf{FVect} \to \mathsf{Mat}$ which sends a space to its dimension and sends a linear map $f: V \to W$ to the corresponding matrix representing f with respect to the chosen bases. And there is a functor $K: \mathsf{Mat} \to \mathsf{FVect}$ which sends n to $\mathbb{R}^n$ treated as a vector space, and sends an $m \times n$ matrix to the linear map from $\mathbb{R}^m$ to $\mathbb{R}^n$ which the matrix represents with respect to the standard bases for those spaces. But neither of those functors is an isomorphism, and again cardinality considerations show that there can't be one. Objects m and n in Mat are isomorphic only if they are the same; but there can be isomorphic vector spaces in FVect which are not identical (remember we've only fixed the scalars as being reals: the vectors of an n-dimensional space can vary ad lib).

34.3 Equivalent categories

(a) We've now seen two examples of pairs of categories which are intuitively mathematically equivalent in some strong sense but which aren't isomorphic (according to the natural definition of isomorphism for categories).

[1] We can either ban the zero-dimensional case, or allow it and give an ad hoc treatment. The details don't matter.

We did, however, in the first case note an obvious choice of functors $F\colon \mathsf{Set}_\star \to \mathsf{Pfn}$ and $G\colon \mathsf{Pfn} \to \mathsf{Set}_\star$. Now, since G adds an external base point to a set X, and then F simply removes it again (and the functors do the minimum necessary on arrows), the composite functor $FG\colon \mathsf{Pfn} \to \mathsf{Pfn}$ is in fact the identity functor on Pfn. But what about the reverse composition, GF. It can't be the identity functor on $\mathsf{Set}_\star$, or else we'd have isomorphic functors after all. However, GF *does* map $\mathsf{Set}_\star$ to itself in a very simple way. Officially, GF sends the pointed set (X, x) to $(X_x \cup \{X_x\}, X_x)$ where $X_x = X \setminus \{x\}$. *But that's just to say that GF takes a pointed set (X, x) and methodically replaces the base point with something not already in X.* And GF treats arrows to fit. So, applied to some objects and arrows, although GF doesn't take us back to where we started, it should systematically give us back an isomorphic copy. Which suggests that we can expect that GF is naturally isomorphic the identity functor $\mathsf{Set}_\star$, i.e. $GF \cong 1_{\mathsf{Set}_\star}$.

What about our second example involving FVect and Mat. In this case, the composite KJ will take us from an n-dimensional space V to $\mathbb{R}^n$. But since all n-dimensional spaces are isomorphic, although KJ isn't the identity on FVect, its systematic operation again suggests that KJ is naturally isomorphic to the identity, $KJ \cong 1_{\mathsf{FVect}}$. Similarly, depending on the basis for $\mathbb{R}^n$ chosen in defining J, the composite JK needn't be the identity either. It sends an object n to itself; but on arrows, it can systematically send matrices to their image under a change of basis. Still, given its methodical operation, we might still expect that $JK \cong 1_{\mathsf{Mat}}$.

(b) Reflection on these case suggests, then, the following weakening of the definition of isomorphism between categories:

Definition 129. Categories C and D are *equivalent*, in symbols $\mathsf{C} \simeq \mathsf{D}$, iff there are functors $F\colon \mathsf{C} \to \mathsf{D}$ and $G\colon \mathsf{D} \to \mathsf{C}$ which are pseudo-inverses, i.e. $GF \cong 1_{\mathsf{C}}$ and $FG \cong 1_{\mathsf{D}}$.

It can be safely left as an easy exercise to check that $\simeq$ really is an equivalence relation!

(c) We could now pause to upgrade our arm-waving arguments a moment ago to give a direct proof that Pfn and $\mathsf{Set}_\star$ are indeed equivalent in this sense, and likewise for FVect and Mat.

But in fact we won't do this. Rather, we'll first prove a general result which yields an alternative characterization of equivalence which can often be much easier to apply:[2]

Theorem 162. *Assuming a sufficiently strong choice principle, a functor $F\colon \mathsf{C} \to \mathsf{D}$ is part of an equivalence between C and D iff F is faithful, full and essentially surjective on objects.*

[2] Emily Riehl (2017, p. 31) says that she used to set this theorem as homework. You are hereby challenged to prove it before reading on!

Proof. First suppose F is part of an equivalence between C and D, so that there is a functor $G\colon \mathsf{D} \to \mathsf{C}$, where $GF \cong 1_{\mathsf{C}}$ and $FG \cong 1_{\mathsf{D}}$. Then:

(i) For any arrow $f\colon A \to B$ in C, then by hypothesis, the following square commutes (where η is a natural isomorphism between the identity functor 1_{C} and the composite GF),

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\eta_A"] & B \arrow[d, "\eta_B"] \\ GFA \arrow[r, "GFf"] & GFB \end{tikzcd}$$

and hence $\eta_B^{-1} \circ GFf \circ \eta_A = f$. And similarly of course, for any other $g\colon A \to B$, we have $\eta_B^{-1} \circ GFg \circ \eta_A = g$. It immediately follows that if $Ff = Fg$ then $f = g$, i.e. F is faithful. A companion argument, interchanging the roles of C and D, shows that G too is faithful.

(ii) Suppose we are given an arrow $h\colon FA \to FB$, then put $f = \eta_B^{-1} \circ Gh \circ \eta_A$. But we know that $f = \eta_B^{-1} \circ GFf \circ \eta_A$. So it follows that $GFf = Gh$, and since G is faithful, $h = Ff$. So every such h in D is the image under F of some arrow f in C. So F is full.

(iii) Recall, $F\colon \mathsf{C} \to \mathsf{D}$ is e.s.o. iff for any D in D we can find some isomorphic object FC, for C in C. But we know that there is, by assumption, a natural isomorphism $\epsilon\colon FG \Rightarrow 1_{\mathsf{D}}$. So we have a component isomorphism $\epsilon_D\colon FGD \xrightarrow{\sim} D$, therefore putting $C = GD$ gives the desired result showing that F is e.s.o.

Now for the argument in the other direction. Suppose, then, that $F\colon \mathsf{C} \to \mathsf{D}$ is faithful, full and e.s.o. We need to construct (iv) a corresponding functor $G\colon \mathsf{D} \to \mathsf{C}$, and then a pair of natural isomorphisms (v) $\epsilon\colon FG \Rightarrow 1_{\mathsf{D}}$ and (vi) $\eta\colon 1_{\mathsf{C}} \Rightarrow GF$:

(iv) By hypothesis, F is e.s.o., so by definition every D-object D is isomorphic in D to FC, for some C in C. Hence – and *here* we are invoking an appropriate choice principle – for any given D, we can choose an object C such that there is an isomorphism $\epsilon_D\colon FC \xrightarrow{\sim} D$ in D.

Now define $G_{ob}\colon \mathsf{D} \to \mathsf{C}$ as sending an object D to the chosen C from C (so $GD = C$, and $\epsilon_D\colon FGD \xrightarrow{\sim} D$).

To get a functor, we need the component G_{arw} to act suitably on an arrow $g\colon D \to E$. Now, note

$$FGD \xrightarrow{\epsilon_D} D \xrightarrow{g} E \xrightarrow{\epsilon_E^{-1}} FGE$$

and since F is full and faithful, there must be some unique $f\colon GD \to GE$ which F sends to the composite $\epsilon_E^{-1} \circ g \circ \epsilon_D$. Put $G_{arw}g = f$.

Note, for use in a moment, that this means that (*) given any $g\colon D \to E$, $FGg = \epsilon_E^{-1} \circ g \circ \epsilon_D$, so the left square below commutes, as does the right square (**) for any $f\colon A \to B$ in C:

$$\begin{tikzcd} FGD \arrow[r, "FGg"] \arrow[d, "\epsilon_D"] & FGE \arrow[d, "\epsilon_E"] \\ D \arrow[r, "g"] & E \end{tikzcd} \qquad \begin{tikzcd} FGFA \arrow[r, "FGFf"] \arrow[d, "\epsilon_{FA}"] & FGFB \arrow[d, "\epsilon_{FB}"] \\ FA \arrow[r, "Ff"] & FB \end{tikzcd}$$

We now need to check that G, *with components* G_{ob}, G_{arw}, *is indeed a functor.* So we need to show that G (a) preserves identities and (b) respects composition:

For (a), note that $G_{arw}1_D = e$ where e is the unique arrow from GD to GD such that $Fe = \epsilon_D^{-1} \circ 1_D \circ \epsilon_D = 1_{FGD}$. So $e = 1_{GD}$.

For (b) we need to show that, given D-arrows $g\colon D \to E$ and $h\colon E \to J$, $G(h \circ g) = Gh \circ Gg\colon D \to J$. But note that using (*) we have

$$\begin{aligned} FG(h \circ g) = \epsilon_J^{-1} \circ h \circ g \circ \epsilon_D &= (\epsilon_J^{-1} \circ h \circ \epsilon_E) \circ (\epsilon_E^{-1} \circ g \circ \epsilon_D) \\ &= FG(h) \circ FG(g) = F(G(h) \circ G(g)) \end{aligned}$$

Hence, since $FG(h \circ g) = F(G(h) \circ G(g))$ and F is faithful, $G(h \circ g) = G(h) \circ G(g)$, so G is indeed a functor.

(v) The commuting of the naturality square in (*) for any g shows that components ϵ_D assemble into a natural transformation $\epsilon\colon FG \xRightarrow{\sim} 1_{\mathsf{D}}$.

(vi) Note next that we have an isomorphism $\epsilon_{FA}^{-1}\colon FA \xrightarrow{\sim} FGFA$. As F is full and faithful, $\epsilon_{FA}^{-1} = F(\eta_A)$ for some unique $\eta_A\colon A \xrightarrow{\sim} GFA$. Since F is fully faithful it is conservative, i.e. reflects isomorphisms (by Theorem 136), hence η_A is also an isomorphism. Also, the naturality diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\eta_A"] & B \arrow[d, "\eta_B"] \\ GFA \arrow[r, "GFf"] & GFB \end{tikzcd}$$

always commutes for any arrow $f\colon A \to B$ in C. Why? Because

$$\begin{aligned} &F(\eta_B \circ f) = F\eta_B \circ Ff = \epsilon_{FB}^{-1} \circ Ff = \\ &\qquad FGFf \circ \epsilon_{FA}^{-1} = FGFf \circ F\eta_A = F(GFf \circ F\eta_A) \end{aligned}$$

where the middle step applies the naturality square (**). But if $F(\eta_B \circ f) = F(GFf \circ F\eta_A)$ then since F is faithful, $\eta_B \circ f = GFf \circ F\eta_A$. Hence the η_A are the components of our desired natural isomorphism $\eta\colon 1_{\mathsf{C}} \xRightarrow{\sim} GF$.

So we are done! □

(d) Our theorem enables us now to very quickly prove the following two equivalence claims without any more hard work:

Theorem 163. $\mathsf{Pfn} \simeq \mathsf{Set}_\star$

Proof. Define the functor $G\colon \mathsf{Pfn} \to \mathsf{Set}_\star$ as before. It sends a set X to a set $X_* =_{\text{def}} X \cup \{X\}$ with basepoint X, and sends a partial function $f\colon X \to Y$ to the total function $f_*\colon X_* \to Y_*$, where for $f_*(x) = f(x)$ if $f(x)$ is defined and $f_*(x) = Y$ otherwise.

G is faithful, as it is easily checked that it sends distinct functions to distinct functions. And it is equally easy to check that G is full, i.e. given any basepoint preserving function between sets X_* and Y_*, there is a partial function f which G sends to it.

But G is essentially surjective on objects. For every pointed set in $\mathsf{Set}_\star$ – i.e. every set which can be thought of as the union of a set X with $\{*\}$ where $*$ is an additional basepoint element (not in X) – is isomorphic in $\mathsf{Set}_\star$ to the set $X \cup \{X\}$ with X as basepoint.

Hence G is part of an equivalence between Pfn and $\mathsf{Set}_\star$. □

Theorem 164. $\mathsf{FVect} \simeq \mathsf{Mat}$

Proof. Take the defined functor $J\colon \mathsf{FVect} \to \mathsf{Mat}$. It's faithful as different linear maps get different matrix representations. It's full because every matrix corresponds to a linear map. It's trivially surjective on objects. Hence J is part of an equivalence between FVect and Mat. □

34.4 Why equivalence is the categorially nicer notion

Let's have another simpler but instructive example. Recall FinSet is the category of finite sets and functions between them. And $\mathsf{FinOrdn}$ is category of finite von Neumann ordinals and functions between them. We then have:

Theorem 165. $\mathsf{FinOrd} \simeq \mathsf{FinSet}$

Proof. $\mathsf{FinOrdn}$ is a full subcategory of FinSet, so the inclusion functor F is fully faithful. F is also essentially surjective on objects: for take any object in FinSet, which is some n-membered set: that is in bijective correspondence (and hence isomorphic in FinSet) with the finite ordinal n. Hence F is part of an equivalence, and $\mathsf{FinOrd} \simeq \mathsf{FinSet}$. □

And the interesting question here is how should we regard this last result. We saw that defining equivalence of categories in terms of isomorphism would be *too strong*, as it rules out our treating Pfn and $\mathsf{Set}_\star$ as in effect equivalent. But now we've seen that defining equivalence of categories as in Defn. 34.3 makes the seemingly very sparse category FinOrd equivalent to the seemingly much more abundant FinSet. Is that a strike against the definition of equivalence, showing it to be *too weak*?

It might help to think of an even simpler toy example. Consider the two categories which we can diagram respectively as follows (with the diagram on the right intended to commute):

$$\begin{tikzcd} \bullet \arrow[loop right] \end{tikzcd} \qquad\qquad \begin{tikzcd} \bullet \arrow[loop left] \arrow[r, shift left] & \star \arrow[l, shift left] \arrow[loop right] \end{tikzcd}$$

On the left, we have the category $\mathsf{1}$; on the right we have a two-object category $\mathsf{2!}$ with arrows in *both* directions between the objects, and given the diagram commutes, those two arrows are inverse to each other and hence are isomorphisms. These two categories are plainly *not* isomorphic, but they *are* equivalent. For one of the obvious inclusion functors $\mathsf{1} \hookrightarrow \mathsf{2!}$ is full and faithful, and it is trivially essentially surjective on objects as each object in the two-object category is isomorphic to the other.

What this second toy example highlights is that our equivalence criterion counts categories as amounting to the same when (putting it very roughly) one is just the same as the other padded out with new objects and just enough arrows to make the new objects isomorphic to some old objects.

But on reflection that's fine. Taking a little bit of the mathematical world and bulking it out with copies of the structures it already contains and isomorphisms between the copies won't, for many (most? nearly all?) purposes, give us a real enrichment. Therefore a criterion of equivalence of categories-as-mathematical-universes that doesn't care about surplus isomorphic copies is what we typically need. Hence the results that $\mathsf{1} \simeq \mathsf{2!}$ and $\mathsf{Finord} \simeq \mathsf{FinSet}$ are arguably welcome features, not bugs, of our account of equivalence.

34.5 Skeletons and evil

(a) Given that two categories can be regarded as being equivalent in an important sense even if one is bulked out with isomorphic extras, shouldn't the usual sort of concern for Bauhaus elegance and lack of redundancy in fact lead us to privilege categories which are as skeletal as possible? Let's say:

Definition 130. The category S is a *skeleton* of the category C if S is a full subcategory of C which contains exactly one object from each class of isomorphic objects of C. A category is *skeletal* if it is a skeleton of some category.

For a toy example, suppose C is a category arising from a pre-order – as in §4.4 (C4). Then any skeleton of C will be a poset category. (Check that!)

Theorem 166. *If S is a skeleton of the category C then $\mathsf{S} \simeq \mathsf{C}$.*

Proof. The inclusion functor $\mathsf{S} \hookrightarrow \mathsf{C}$ is fully faithful, and by the definition of S is essentially surjective on objects. So we can apply Theorem 162. □

Theorem 167. *If R and S are skeletal categories, then equivalence implies isomorphism, so if $\mathsf{R} \simeq \mathsf{S}$ then $\mathsf{R} \cong \mathsf{S}$.*

Proof. By Theorem 162, there must be a functor $F\colon \mathsf{R} \to \mathsf{S}$ which is fully faithful and essentially surjective.

Since S is skeletal, being essentially surjective implies that F is surjective on objects.

F is also injective on objects. For suppose for R-objects C and D, $FC = FD = X$. Since there is an identity C-arrow $1_X\colon FC \to FD$, and F is full, there must be an R-arrow $f\colon C \to D$ such that $Ff = 1_X$. Likewise there must be an R-arrow $g\colon D \to C$ such that $Fg = 1_X$. So $F(g \circ f) = 1_X \circ 1_X = 1_X = F(1_C)$. Hence, since F is faithful, $g \circ f = 1_C$. Similarly $f \circ g = 1_D$. Therefore f and g are isomorphisms between C and D and hence (since R is skeletal) $C = D$.

Since F is bijective on objects, full and faithful, it follows that it is also bijective on arrows. So it is an isomorphism between R and S. □

(b) So how about this for a programme? Take our favoured initial universe of categories, whatever that is. But now slim it down by taking skeletons. Then work with these. And we can now forget bloated non-skeletal categories. And forget too about the notion of equivalence and revert to using the simpler notion of isomorphism, because equivalent skeletal categories are in fact isomorphic. What's not to like?

Well, the trouble is that hardly any categories that occur in the wild (so to speak) are skeletal. And slimming down has to be done by appeal to an axiom of choice. Indeed the following statements are each equivalent to a version of the axiom of choice:

(1) Any category has a skeleton.

(2) A category is equivalent to any of its skeletons

(3) Any two skeletons of a given category are isomorphic.

The choice of a skeleton is therefore usually quite artificial – there typically won't be a canonical choice. So any gain in simplicity from concentrating on skeletal categories would be bought at the cost of having to adopt 'unnatural', non-canonical, choices of skeletons. Given that category theory is supposed to be all about natural patterns already occurring in mathematics, this perhaps isn't going to be such a brilliant trade-off after all.

(c) Categorial notions that are not invariant under equivalence are sometimes said to be 'evil'. So being skeletal is evil. So too is being small:

Theorem 168. *Smallness is not preserved by categorial equivalence.*

In other words, we can have C a small category, $\mathsf{C} \simeq \mathsf{D}$, yet D not small. This is a simple corollary of our observation in §34.3 that if we take a category, inflate it by adding lots of objects and just enough arrows to ensure that these objects are isomorphic to the original objects, then the augmented category is equivalent to the one we started with. For an extreme example, start with the one-object category $\mathbf{1}$, i.e. $\bullet \circlearrowleft$ (that's small)! Now add as new objects e.g. every ordinal, and as new arrows an identity arrow for each set, and also for every new set X a pair of arrows $\bullet \rightleftarrows X$ which compose to give identities. Then we get a new pumped-up category $\mathbf{1}^+$ (which is certainly not small). But $\mathbf{1}^+ \simeq \mathbf{1}$.

There is, however, a companion positive result

Theorem 169. *Local smallness* is *preserved by categorial equivalence.*

Proof. An equivalence $\mathsf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathsf{D}$ requires F and G to be full and faithful functors. So in particular, for any D-objects D, D', there are the same number of arrows between them as between the C-objects GD, GD'. So that ensures that if C has only a set's worth of arrows between any pair of objects, the same goes for D. □

35 Categories of categories

We have seen how structured whatnots equipped with structure-respecting maps between them can be assembled into categories. But we have also now seen that categories too can have structure-respecting maps between them, i.e. functors. So can data of *these* two sorts be assembled into further categories?

Yes indeed. Quite unproblematically, there are at least some *categories of categories.*

Going up another level, functors too have can structure-respecting maps between *them*, i.e. natural transformations. And once again, data of these two sorts can also be assembled into further categories, *functor categories.*

Of these two new ideas, it's the second one which is going to be important for us. In this chapter, I say just a little about the first.

35.1 A definition, and some tame categories of categories

Theorem 128 tells us that for any category there is an identity functor sending that category to itself, and tells us that if there are (covariant!) functors $F\colon \mathsf{C} \to \mathsf{D}$ and $G\colon \mathsf{D} \to \mathsf{E}$, then they can be composed to give us a (covariant) functor $G \circ F\colon \mathsf{C} \to \mathsf{E}$, and composition is associative.

So the following definition makes good sense:

Definition 131. A category of categories comprises two sorts of data:

(1) *Objects*: some categories, $\mathsf{C}, \mathsf{D}, \mathsf{E}, \ldots,$

(2) *Arrows*: some functors, $F, G, H, \ldots,$ between those categories,

where the arrows (i) include the identity functor on each category, and (ii) also include $G \circ F$ for each included composable pair F and G (where F's target is G's source). △

And we can immediately give some tame examples:

(1) Trivially, there is a category of categories whose sole object is your favourite category C and whose sole arrow is the identity functor 1_{C}.

(2) Equally trivially, there is a category of categories with just two objects, the categories Mon and Set, and whose arrows are the identity functors

from each object to itself, together with the forgetful functor from Mon to Set.

(3) We noted that every monoid can be thought of as itself being a category, and functors between monoids-as-categories are just monoid homomorphisms – see §4.4 (C3) and §27.2 (F9). So any category of monoids can be regarded as a category of categories. In particular, that goes for Mon.

Which is enough to establish the point of principle, that there are at least *some* elementary examples of categories of categories. But how far can we go?

35.2 A universal category?

(a) What about the extreme case? Is there a universal mega-category comprising *all* categories and the functors between them?

Yes, according to some. Thus Tom Leinster cheerfully says "there is a category CAT whose objects are categories and whose maps are functors", and – in case you are in any doubt that this is supposed to be universal – he later says again "the category of *all* categories and functors is written as CAT" (Leinster 2014, p. 18 and p. 77, his emphasis).

But isn't there a problem of a familiar kind here? Suppose we say:

Definition 132. A category is *normal* iff it is not one of its own objects. △

Typical categories we've met are surely normal in this sense. And this much is surely clear:

Theorem 170. *There is no category whose objects are all and only the normal categories.*

Proof. Suppose that there is a category N whose objects are just the normal categories. Now ask, is N normal? If it is, then it is one of the objects of N, so N is not normal. So N can't be normal. But then it is not one of the objects of N, so N is normal after all. Contradiction. □

But arguably, it seems, we can then go on to conclude that

Theorem 171 (?). *There is no category* CAT *of all categories.*

For if there were such an inclusive mega-category, we could separate out from it a subcategory containing just the normal categories, contrary to the previous theorem.

(b) Of course, that argument just re-runs, in our new environment, Russell's proof that there can be no set of all normal sets (i.e. no set of all the sets which are not members of themselves), together with the familiar further step taking us to the conclusion that there is no universal set (because if there were, we could use a separation principle to carve out from it the subset of all normal sets).

Now, to keep ourselves honest, we should note that in the set-theoretic context we can in fact resist the second step, if we are willing to radically restrict our separation principle and develop a non-standard set theory.[1] And I suppose we could perhaps similarly try to resist the move from Theorem 170 to Theorem 171 by trying to restrict when we are allowed to carve out subcategories – though it is quite difficult to see a principled way of doing this (other than one which leads to a non-standard set theory too).

But let's not get further entangled with the Russellian line of argument here. For I think that there is a rather more basic problem with the idea of a category of all categories.

(c) Right back in §3.4, I suggested that if we think of a group as just some objects equipped with a binary operation obeying the right conditions, where we don't put any restriction on the kind of objects involved, then it is very far from clear that talk of 'all' groups will locate a determinate fixed totality. Which is why I doubted that there is a determinate all-inclusive mega-category of all groups and their homomorphisms.

Well, doesn't the same go for categories? – at least on our Type I definition which places no restrictions at all on the sorts of entities that can be the objects and arrows of a category (see §26.2 again). If we are not circumscribing in advance the universe where categories live – in some favoured universe of sets, say – what good reason is there to suppose that there *is* a definite totality of categories in our generous sense?

I rather suspect, then, that the idea of a determinate category of *all* categories is, for rather boring reasons, a non-starter.

35.3 Cat, Cat and CAT

When discussing groups we in effect said: 'OK: let's not fret about whether there is a category of *all* groups, whatever that might mean. Instead, let's focus on what happens in some determinate arena which we hope is rich enough to implement copies of all the groups springing up out there in wild (or at least all the ones that we might care out) – and then it can make sense to talk of a category Grp of all the groups living *there* in that arena and the homomorphisms between *them*'.

Can we make a parallel move here for categories?

Again, then, let's not fret about whether there is a category of *all* categories, whatever that might mean. Instead, let's focus on what happens in some determinate arena – and ok, let's plump for a universe of sets – which we might hope is rich enough to implement at least copies of the categories we want initially want to think about.

So now consider the following three notions of increasing scope:

[1] See, for example, Forster (1995) for a classic discussion of deviant set theories like Quine's NF, which allows a universal set at the cost of restricting separation.

Definition 133. Cat is the category whose objects are the small categories living in our favoured universe of sets and whose arrows are the functors between them.

Cat is the category whose objects are the locally small categories living in our favoured universe of sets and whose arrows are the functors between them.

CAT is the category whose objects are the categories living in our favoured universe of sets and whose arrows are the functors between them. △

Arguably *these* definitions are unproblematic. Or at least, Russellian problems don't come back to bite us again.

First, a discrete category (with just identity arrows) only has as many arrows as objects. Which implies that the discrete category on any set is small. But that in turn implies that there are at least as many small categories as there are sets. Hence the category Cat of small categories has at least as many objects as there are sets, and hence is itself determinately *not* small. Since Cat is unproblematically *not* small, no paradox arises for Cat as it did for the putative category of normal categories.

Second, take a one-element category 1, which is certainly locally small. Then a functor from 1 to the locally small Set will just map the object of 1 to some particular set: and there will be as many distinct functors $F\colon 1 \to \mathsf{Set}$ as there are sets. In other words, arrows from 1 to Set in Cat are too many to be mapped one-to-one to a set. Hence Cat is determinately *not* locally small. So again no Russellian paradox arises in this case either.

And as for our re-defined CAT, we can take a Russellian argument just to show that such a category can't itself live in our original chosen universe of sets, and won't include itself as an object.

But having offered these definitions, we'll find that we won't – at least in these introductory-level Notes – have real occasion to use them. I only mentioned them because you'll find analogues elsewhere. So let's move on.

36 Functor categories

At the very beginning of the last chapter, we noted that the functors between two categories together with the natural transformations between those functors can taken together give us the data for a new sort of category. This chapter develops that idea.

36.1 Functor categories officially defined

Theorem 156 tells us that (1) for any functor $F\colon \mathsf{C} \to \mathsf{D}$, there is an identity natural transformation $1_F\colon F \Rightarrow F$ and that (2) given parallel functors $F, G, H\colon \mathsf{C} \to \mathsf{D}$, then if there are natural transformations $\alpha\colon F \Rightarrow G$ and $\beta\colon G \Rightarrow H$ then there is a composite natural transformation $\beta \circ \alpha\colon F \Rightarrow H$, where composition is associative.

So, lo and behold, the following definition must be in good order!

Definition 134. $[\mathsf{C}, \mathsf{D}]$, the *functor category* from C to D, is the category whose objects are all the (covariant) functors $F\colon \mathsf{C} \to \mathsf{D}$, with the natural transformations between them as arrows. △

We can also talk about a category of contravariant functors from C to D, though in fact is more usual to talk about a category $[\mathsf{C}^{op}, \mathsf{D}]$ of covariant functors instead.[1]

So, for example, $[\mathsf{C}, \mathsf{Set}]$ is the category whose objects are functors from C to Set, and – at least assuming C is locally small – these functors will include the hom-functors $\mathsf{C}(A, -)$ for all the C-objects A. In fact these hom-functors, along with the natural transformations between them form a full subcategory of $[\mathsf{C}, \mathsf{Set}]$. Likewise, the hom-functors $\mathsf{C}(-, A)$ form a full subcategory of $[\mathsf{C}^{op}, \mathsf{Set}]$.

36.2 Three simple examples

We'll be looking at more significant examples again later; but it might help to fix ideas to start by working through three challenges (the first two are very easy):

[1] By the way, the laconic square-bracket notation '$[\mathsf{C}, \mathsf{D}]$' is standard. But there is a neat alternative: just as 'D^C' is used to denote the (set of) functions from C to D, so 'D^{C}' is often used to denote the (category of) functors from C to D.

(1) Take the discrete category $\overline{2}$, which comprises just two objects we'll dub A and B together with their identity arrows. What is the functor category $[\overline{2}, \mathsf{C}]$?

(2) Recall now the category $\mathsf{2}$, which has two objects and one arrow between them. Omitting identity arrows, we can diagram this as $A \to B$. What is the functor category $[\mathsf{2}, \mathsf{C}]$?

(3) Now take the category $\mathsf{2}^+$, which again has two objects and just two parallel arrows between them. Omitting identity arrows, we can diagram this (there is a hint here!) as $E \underset{t}{\overset{s}{\rightrightarrows}} V$. What is the functor category $[\mathsf{2}^+, \mathsf{Set}]$?

(a) The objects of $[\overline{2}, \mathsf{C}]$ are functors $F\colon \overline{2} \to \mathsf{C}$. But note we can choose *any* pair of objects from C that we like to be $F_{ob}A$ and $F_{ob}B$. Then, so long as we put $F_{arw}1_A = 1_{FA}$ and $F_{arw}1_B = 1_{FB}$, the components F_{ob} and F_{arw} will make a functor. Which means there is a simple bijection between objects of our functor category and pairs of C-objects.

What about the arrows of $[\overline{2}, \mathsf{C}]$? An arrow between the parallel functors $F, G\colon \overline{2} \to \mathsf{C}$ is a natural transformation α with components

$$\begin{tikzcd} FA \arrow[d, "\alpha_A"] & FB \arrow[d, "\alpha_B"] \\ GA & GB \end{tikzcd}$$

And since there are no arrows in $\overline{2}$ between A and B, there is nothing more needed to complete a naturality square in C. So there are no constrains on those components of α. Hence a natural transformation from F to G, an arrow of $[\overline{2}, \mathsf{C}]$, is simply *any* pair of C-arrows, one from FA to GA, and one from FB to GB.

Putting things together, the objects of our category $[\overline{2}, \mathsf{C}]$ are (in effect) just pairs of C-objects; and an arrow between such pairs is a pair of C-arrows (one between the first members of the pairs, one between the second members). But that makes our category (up to isomorphism) the category $\mathsf{C} \times \mathsf{C}$ – where products of categories are defined as in Defn. 22.

(b) Now what about the functor category $[\mathsf{2}, \mathsf{C}]$?

An object in this category is a functor $F\colon \mathsf{2} \to \mathsf{C}$. But note that we can again choose *any* pair of objects we like from C to be $F_{ob}A$ and $F_{ob}B$ so long as there is at least one arrow between them. Then, so long as we put $F_{arw}1_A = 1_{FA}$ and $F_{arw}1_B = 1_{FB}$, and let F_{arw} send the unique arrow from A to B to some arrow f from $F_{ob}A$ to $F_{ob}B$ (any one!), the components F_{ob} and F_{arw} will make a functor. Which means that this time there is a simple bijection between the objects of $[\mathsf{2}, \mathsf{C}]$ and C-arrows.

And what about the arrows in our new category? A natural transformation from F to the parallel functor G will have as components any two C-arrows, j, k, which makes this a commutative square:

$$\begin{tikzcd} FA \arrow[r, "f"] \arrow[d, "j"] & FB \arrow[d, "k"] \\ GA \arrow[r, "g"] & GB \end{tikzcd}$$

Thus the arrows of the new category are exactly pairs of C-arrows which make our relevant diagram commute.

So in sum, $[2, \mathsf{C}]$ is (or strictly speaking, is isomorphic to) the arrow category $\mathsf{C}^{\rightarrow}$ we met in Defn. 27.

(c) What does a functor F from the mini-category 2^{+}, i.e. $E \overset{s}{\underset{t}{\rightrightarrows}} V$, to the category Set actually do?

On objects, F sends E to a set FE, and sends V to a set FV. And it sends the arrow $s \colon E \to V$ to a function $Fs \colon FE \to FV$ and the arrow $t \colon E \to V$ to a function $Ft \colon FE \to FV$. So we can look at the data which F picks out in Set like this. There is a set FE, call its members 'edges', and each 'edge' in FE gets assigned a 'vertex', i.e. a member of the set of FV as 'source' by Fs, and also gets assigned a vertex as 'target' by Fs. In other words, 2^{+}'s image under F is a *directed graph*.

OK now we know that the objects of the functor category $[2^{+}, \mathsf{Set}]$ do. What about the arrows in this category – in other words, what is a natural transformation $F \Rightarrow G$?

By definition, we need a pair of components, φ_E and φ_V, which make the following two diagrams commute:

$$\begin{tikzcd} FE \arrow[r, "Fs"] \arrow[d, "\varphi_E"] & FV \arrow[d, "\varphi_V"] \\ GE \arrow[r, "Gs"] & GV \end{tikzcd} \qquad \begin{tikzcd} FE \arrow[r, "Ft"] \arrow[d, "\varphi_E"] & FV \arrow[d, "\varphi_V"] \\ GE \arrow[r, "Gt"] & GV \end{tikzcd}$$

But that makes φ_E and φ_V the components of a graph homomorphism – compare (C25) in §4.7.

So we can think of the functor category $[2^{+}, \mathsf{Set}]$ as being in in effect the category Graph of graphs and graph homomorphisms.

36.3 Categories of cones again

We've just seen that three kinds of functor category turn out to be (up to isomorphism) categories we've met before. This section is another warm-up exercise, less directly relating functor categories to the already familiar.

(a) Three reminders. First, a functor $D \colon \mathsf{J} \to \mathsf{C}$, is said to be a diagram of shape J in C. So the objects of the functor category $[\mathsf{J}, \mathsf{C}]$ are just the diagrams of shape J in C – hence we can think of that functor category as a category of diagrams.

Second, recall that living in $[\mathsf{J}, \mathsf{C}]$ there will, in a particular, be a functor Δ_X which picks an object X from the category C, and sends every object of J to X, and sends every arrow in J to 1_X.

Third, we proved Theorem 159 which tells us that a natural transformation from $\Delta_X : \mathsf{J} \to \mathsf{C}$ to $D\colon \mathsf{J} \to \mathsf{C}$ can be thought of as a cone in C over the diagram D.

And with those facts at the front of our minds again, let's consider the following definition:

Definition 135. The functor $\Delta_{\mathsf{J}}\colon \mathsf{C} \to [\mathsf{J}, \mathsf{C}]$ sends a C-object X to the functor $\Delta_X : \mathsf{J} \to \mathsf{C}$ and sends a C-arrow $f\colon X \to Y$ to the natural transformation from Δ_X to Δ_Y whose every component is simply f again. △

We need to check, of course, that Δ_{J} really *is* a functor. It does the right *types* of things, sending a C-object to a $[\mathsf{J}, \mathsf{C}]$-object, and sending a C-arrow to a $[\mathsf{J}, \mathsf{C}]$-arrow. So the crucial thing is to show the last part of our definition does indeed characterize a natural transformation from Δ_X to Δ_Y.

For this, we just note that for every $d\colon K \to L$ in J, the required naturality square on the left is in fact none other than the trivially commuting square on the right:

$$\begin{tikzcd} \Delta_X K \arrow[r, "\Delta_X d"] \arrow[d, "f"] & \Delta_X L \arrow[d, "f"] \\ \Delta_Y K \arrow[r, "\Delta_Y d"] & \Delta_Y L \end{tikzcd} \qquad\qquad \begin{tikzcd} X \arrow[r, "1_X"] \arrow[d, "f"] & X \arrow[d, "f"] \\ Y \arrow[r, "1_Y"] & Y \end{tikzcd}$$

(b) Given the functor $\Delta_{\mathsf{J}}\colon \mathsf{C} \to [\mathsf{J}, \mathsf{C}]$ and an object D in $[\mathsf{J}, \mathsf{C}]$ there will be a comma category $(\Delta_{\mathsf{J}} \downarrow D)$. Applying the definition of such a category in §30.2, we get the following:

(1) An object of $(\Delta_{\mathsf{J}} \downarrow D)$ is a pair of an object C in C, and an arrow $c\colon \Delta_{\mathsf{J}} C \to D$ in $[\mathsf{J}, \mathsf{C}]$, i.e. a natural transformation from Δ_C to D. But the components of such a natural transformation are just the legs c_J of a cone over D with vertex C. So an object (C, c) of our comma category is in effect just a cone (C, c_J) over D.

(2) An arrow of $(\Delta_{\mathsf{J}} \downarrow D)$ from (C, c) to (C', c') is a C-arrow $f\colon C \to C'$ such that $c = c' \circ \Delta_{\mathsf{J}} f$, which tells us that for each J, $c_J = c'_J \circ f$. Which is just the condition for f to be an arrow between cones (C, c_J) and (C', c'_J) in the category of cones over D, as characterized in Defn. 77.

Hence, in sum, $(\Delta_{\mathsf{J}} \downarrow D)$ is just the category of cones over D, and then a limit over D is a terminal object of this category. Which is rather neat (another of those examples of how the gadgetry of category theory tightly interlocks).

It will, by this stage of the game, be no additional surprise to learn that $(D \downarrow \Delta_{\mathsf{J}})$ is the category of cocones under D!

36.4 Limit functors

(a) We work for a while with the assumption that C has all limits of shape J, i.e. every diagram $D\colon \mathsf{J} \to \mathsf{C}$ a limit.

Then we can aim to define a functor $Lim\colon [\mathsf{J}, \mathsf{C}] \to \mathsf{C}$ whose object component sends a diagram D living in the functor category $[\mathsf{J}, \mathsf{C}]$ to the vertex $Lim\, D$ for some chosen limit cone over D in C.[2]

Note however that we do need to do some choosing here! This functor is not entirely 'naturally' or 'canonically' defined: in the general case, limits over D are only unique up to isomorphism, so we will indeed have to select a particular limit object $Lim\, D$ to be the value of our functor for input D.

And now, to get a kosher functor, we need to suitably define Lim's component which acts on arrows. This must send an arrow in $[\mathsf{J}, \mathsf{C}]$, i.e. a natural transformation $\alpha\colon D \Rightarrow D'$ between the $[\mathsf{J}, \mathsf{C}]$ objects D and D', to an arrow in C from $Lim\, D$ to $Lim\, D'$. How can it do this in a, well, natural way?

By hypothesis there are limit cones over D and D', respectively $(Lim\, D, \lambda_J)$ and $(Lim\, D', \lambda'_J)$. So now take any arrow $d\colon K \to L$ living in J and consider the following diagram:

$$\begin{tikzcd}
& Lim\, D \arrow[dl, "\lambda_K"'] \arrow[dr, "\lambda_L"] \arrow[dd, dashed, "u_\alpha"] & \\
D(K) \arrow[rr, "D(d)"] \arrow[dd, "\alpha_K"'] & & D(L) \arrow[dd, "\alpha_L"] \\
& Lim\, D' \arrow[dl, "\lambda'_K"'] \arrow[dr, "\lambda'_L"] & \\
D'(K) \arrow[rr, "D'(d)"] & & D'(L)
\end{tikzcd}$$

The top triangle commutes because $(Lim\, D, \lambda_J)$ is a limit. The lower square commutes by the naturality of α. Therefore the outer pentangle commutes and so, generalizing over objects J in J, $(Lim\, D, \alpha_J \circ \lambda_J)$ is a cone over D'. But then *this* cone must factor uniquely through D''s limit cone $(Lim\, D', \lambda'_J)$ via some unique $u_\alpha\colon Lim\, D \to Lim\, D'$.

The map $\alpha \mapsto u_\alpha$ is then a plausible candidate for Lim's action on arrows; and indeed this assignment is fairly easily checked to yield a functor. In summary then:

Definition 136. Assuming every diagram D of shape J has a limit in C, a functor Lim from the functor category $[\mathsf{J}, \mathsf{C}]$ to C

1. sends an object D in $[\mathsf{J}, \mathsf{C}]$ to the vertex $Lim\, D$ of some chosen limit cone $(Lim\, D, \lambda_J)$ over D

[2] It is common to use the notation '$\underset{\leftarrow \mathsf{J}}{Lim}$' for this functor: but plain 'Lim' is neater when J is fixed by context.

2. sends an arrow $\alpha\colon D \Rightarrow D'$ in $[\mathsf{J}, \mathsf{C}]$ to the arrow $u_\alpha\colon \mathit{Lim}\, D \to \mathit{Lim}\, D'$ where for all J in J, $\lambda'_J \circ u_\alpha = \alpha_J \circ \lambda_J$. △

(b) Limit functors of this kind will play a central starring role in the coming chapters. But for the moment, we will just note a couple of simple theorems. First, the diagram above can be recycled to show

Theorem 172. *Assuming limits of the relevant shape exist then, if we have a natural isomorphism $D \cong D'$, $\mathit{Lim}\, D \cong \mathit{Lim}\, D'$.*

Proof. Because we now have an natural isomorphism $D \cong D'$, we can show as above both that there is a unique $u\colon \mathit{Lim}\, D \to \mathit{Lim}\, D'$ and symmetrically that there is a unique $u'\colon \mathit{Lim}\, D' \to \mathit{Lim}\, D$. These compose to give us map $u' \circ u\colon \mathit{Lim}\, D \to \mathit{Lim}\, D$ which must be $1_{\mathit{Lim}\, D}$ by the now familiar argument (the limit cone with vertex $\mathit{Lim}\, D$ can factor through itself by both $u' \circ u$ and $1_{\mathit{Lim}\, D}$, but there is by hypothesis only one way for the limit cone to factor through itself). Likewise, $u \circ u' = 1_{\mathit{Lim}\, D'}$. So u is an isomorphism. □

Theorem 173. *Suppose that C has all limits of shape J. Then for any $D\colon \mathsf{J} \to \mathsf{C}$ which the functor $F\colon \mathsf{C} \to \mathsf{D}$ preserves,*

$$F(\mathit{Lim}\, D) \cong \mathit{Lim}(F \circ D).$$

In brief: F commutes with Lim.

Proof. If F preserves a limit cone over $D\colon \mathsf{J} \to \mathsf{C}$ with vertex $\mathit{Lim}\, D$, then F sends that limit cone to a limit cone over $F \circ D$ with vertex $F(\mathit{Lim}\, D)$. But that vertex must be isomorphic to the vertex of any other limit cone over $F \circ D$. So in particular it must be isomorphic to whatever has been chosen to be $\mathit{Lim}(F \circ D)$. □

36.5 Hom-functors from functor categories

(a) Suppose we have a functor category $[\mathsf{C}, \mathsf{D}]$. Its arrows, by definition, are natural transformations. And the collection of natural transformations from the functor $F\colon \mathsf{C} \to \mathsf{D}$ to $G\colon \mathsf{C} \to \mathsf{D}$, will be the hom-set $\mathit{Hom}_{[\mathsf{C},\mathsf{D}]}(F, G)$.[3]

We will repeatedly meet such hom-sets. So, leaving context to supply the ambient category $[\mathsf{C}, \mathsf{D}]$, it will be useful to have the following notation, which is both snappier and more memorable:

Definition 137. $\mathit{Nat}(F, G)$ will denote the set of natural transformations from F to G. △

Now, where there are hom-sets, there are hom-functors. Here they are – we just apply the definitions built into Theorems 149 and 150:

[3] Or might the collection the too large to be a set, strictly speaking? Perhaps. But let's not fret about such issues of size right now: it will turn out that when we do later want to talk about such hom-sets, it will be in contexts where they are safely set-sized!

Definition 138. $Nat(F,-)\colon [\mathsf{C},\mathsf{D}] \to \mathsf{Set}$ is the covariant functor which

(1) sends a $[\mathsf{C},\mathsf{D}]$-object such as the functor $G\colon \mathsf{C} \to \mathsf{D}$ to the corresponding set $Nat(F,G)$; and
(2) sends a $[\mathsf{C},\mathsf{D}]$-arrow such as a natural transformation $\gamma\colon G \Rightarrow H$ to the function $\alpha \circ -$, which takes an arrow α from $Nat(F,G)$ and returns the arrow $\gamma \circ \alpha$ from $Nat(F,H)$.

And likewise, $Nat(-,G)\colon [\mathsf{C},\mathsf{D}] \to \mathsf{Set}$ is the contravariant functor which

(1) sends a $[\mathsf{C},\mathsf{D}]$-object such as the functor $F\colon \mathsf{C} \to \mathsf{D}$ to the corresponding set $Nat(F,G)$; and
(2) sends a $[\mathsf{C},\mathsf{D}]$-arrow such as a natural transformation $\gamma\colon E \Rightarrow F$ to the function $- \circ \gamma$, which takes an arrow α from $Nat(F,G)$ and returns the arrow $\alpha \circ \gamma$ from $Nat(E,G)$. △

But do check carefully that that all makes sense. We'll need those hom-functors later!

(b) A quick additional point. Start again with the functor category $[\mathsf{C},\mathsf{D}]$ and this time also pick an object A in C. Then there is an evaluation functor that looks at what is in $[\mathsf{C},\mathsf{D}]$ and evaluates it at A:

Definition 139. The functor $eval_A\colon [\mathsf{C},\mathsf{D}] \to \mathsf{D}$ sends any functor $F\colon \mathsf{C} \to \mathsf{D}$ to FA and sends any natural transformation $\alpha\colon F \Rightarrow G$ to $\alpha_A\colon FA \to GA$. △

It is an easy exercise to check that $eval_A$ really is functorial. Again, we'll need this later.

37 The Yoneda Embedding

"The Yoneda lemma is perhaps the single most used result in category theory" (that's from Steve Awodey 2010, p. 185). Again, the Lemma "is arguably the most important result in category theory" (that's from Emily Riehl 2017, p. 57). It is difficult, though, to illustrate such claims at the level of these notes – for the really interesting applications of the Lemma call on too much mathematical background to happily explore here. Still, given the Lemma's claimed importance, I should at least try to say *something* introductory.

Now, it has also been said that "the level of abstraction in the Yoneda Lemma means that many people find it quite bewildering" (that's from Tom Leinster 2000, p. 1). So here is the challenge, then: to make the Lemma as unbewildering as possible. I'll take things in gentle stages, over two chapters.

What happens in this chapter? We met hom-functors in Chapter 31. We then introduced natural transformations in Chapter 33. So now we bring things together and look at natural transformations between hom-functors. And very quickly we will arrive at what we can think of as a restricted version of the Yoneda Lemma. We will also meet the related Yoneda Embedding theorem, and the Yoneda Principle. These basic technicalities, I hope you'll be able to agree, are quite straightforward. At the end of the chapter, I say something about their significance.

37.1 Natural transformations between hom-functors

(a) So, down to business! Revisit §31.1 and recall that there are two flavours of hom-functor. Here is the covariant hom-functor from C to Set:

$$\mathsf{C}(A,-)\colon \quad \begin{array}{rcl} X & \longmapsto & \mathsf{C}(A,X) \\ j\colon X \to Y & \longmapsto & j \circ -\colon \mathsf{C}(A,X) \to \mathsf{C}(A,Y). \end{array}$$

Then we have its contravariant dual:

$$\mathsf{C}(-,A)\colon \quad \begin{array}{rcl} X & \longmapsto & \mathsf{C}(X,A) \\ j\colon X \to Y & \longmapsto & - \circ j\colon \mathsf{C}(Y,A) \to \mathsf{C}(X,A). \end{array}$$

Recall too from §31.2 that we can think of a functor like $\mathsf{C}(A,-)$ as encapsulating how A sees its world – and an obvious question arises about how A's view meshes with the view from another object B. Dropping the metaphor, we want a story about how a hom-functor $\mathsf{C}(A,-)$ can be related to the hom-functor

$\mathsf{C}(B, –)$ by some natural transformation(s). We'll also want a story about natural transformations between hom-functors of the second flavour too. I will spell out the first story, leaving its companion as an exercise in dualizing (you just need to keep a beady eye on the direction of arrows).

(b) Take a locally small category C. And let's think how can we construct a natural transformation α from the hom-functor $\mathsf{C}(A, –)\colon \mathsf{C} \to \mathsf{Set}$ to the hom-functor $\mathsf{C}(B, –)\colon \mathsf{C} \to \mathsf{Set}$.

By definition, if α is to be a natural transformation, its components must be such that the following diagram commutes, for any given C-arrow $j\colon X \to Y$:

$$\begin{tikzcd}
\mathsf{C}(A, X) \arrow[r, "\mathsf{C}(A, j)"] \arrow[d, "\alpha_X"] & \mathsf{C}(A, Y) \arrow[d, "\alpha_Y"] \\
\mathsf{C}(B, X) \arrow[r, "\mathsf{C}(B, j)"] & \mathsf{C}(B, Y)
\end{tikzcd}$$

Here $\mathsf{C}(A, j)$ is standard shorthand for the result of applying the functor $\mathsf{C}(A, –)$ to the C-arrow j. And that result, we've just reminded ourselves, is a function operating on C-arrows, namely $j \circ –$ which sends an arrow $g\colon A \to X$ to the arrow $j \circ g\colon A \to Y$. Similarly, $\mathsf{C}(B, j)$ is the function $j \circ –\colon \mathsf{C}(B, X) \to \mathsf{C}(B, Y)$

Now, by definition, the component α_X has got to send any arrow $g\colon A \to X$ to some arrow $h\colon B \to X$. And the very easiest way of doing this is to fix on some arrow $f\colon B \to A$, and get α_X to send an arrow $g\colon A \to X$ to the arrow $g \circ f\colon B \to X$.

So put α_X to be the function we can notate $– \circ f\colon \mathsf{C}(A, X) \to \mathsf{C}(B, X)$. And now let's try defining α_Y etc. similarly. In other words, take the same $f\colon B \to A$ and, for any Y, set $\alpha_Y\colon \mathsf{C}(A, Y) \to \mathsf{C}(B, Y)$ to be the function we can again notate $– \circ f$ that sends an arrow $h\colon A \to Y$ to the composite $h \circ f\colon B \to Y$..

And lo and behold, this easy first guess at suitable α_X, α_Y, etc., makes our diagram commute! Take any arrow $g\colon A \to X$ in $\mathsf{C}(A, X)$, and chase it round the diagram in both directions, and we end up with the same result. Thus:

$$\begin{tikzcd}
g\colon A \to X \arrow[r, mapsto, "j \circ -"] \arrow[d, mapsto, "- \circ f"] & j \circ g\colon A \to Y \arrow[d, mapsto, "- \circ f"] \\
g \circ f\colon B \to X \arrow[r, mapsto, "j \circ -"] & j \circ g \circ f\colon B \to Y
\end{tikzcd}$$

Generalizing: for any X, Y, and j, our first diagram always commutes when α's components are defined like α_X. So α is a natural transformation.

Let's sum this up, and also introduce some new notation:

Theorem 174. *Suppose C is a locally small category, and $\mathsf{C}(A, –)$, $\mathsf{C}(B, –)$ are hom-functors (for objects A, B in C).*

Then, given an arrow $f\colon B \to A$, there exists a corresponding natural transformation which we will now notate $\mathsf{C}(f, –)\colon \mathsf{C}(A, –) \Rightarrow \mathsf{C}(B, –)$, where for each

X, the component $\mathsf{C}(f,–)_X : \mathsf{C}(A,X) \to \mathsf{C}(B,X)$ sends an arrow $g\colon A \to X$ to $g \circ f\colon B \to X$. ☐

(c) An additional remark: if f in our theorem is an isomorphism, then each component of our natural transformation $(– \circ f)$ has an inverse (i.e. $– \circ f^{-1}$), so is an isomorphism. Therefore the induced transformation $\mathsf{C}(f,–)$ will be a natural isomorphism.

Which is as it should be. If $f\colon B \to A$ is an isomorphism, then the members of the hom-sets $\mathsf{C}(A,X)$ and $\mathsf{C}(B,X)$ will line up nicely one-to-one, and so the corresponding hom-functors $\mathsf{C}(A,–)$ and $\mathsf{C}(B,–)$ will behave in exactly parallel ways, and so there should be a suitable isomorphism between the functors.

(d) To check understanding and for future use, show the following:

Theorem 175. *Given a locally small category C including objects A, B, C, and given C-arrows $f\colon B \to A$ and $g\colon C \to B$, then*

(1) $\mathsf{C}(f \circ g,–) = \mathsf{C}(g,–) \circ \mathsf{C}(f,–)$.

(2) $\mathsf{C}(f,–)_A 1_A = f$.

(3) $\mathsf{C}(1_A,–) = 1_{\mathsf{C}(A,–)}$.

Not that this is much of a challenge! – we just have to apply definitions. So:

Proof of (1). By the definition of $\mathsf{C}(f \circ g,–)$, a component $\mathsf{C}(f \circ g,–)_X$ sends any arrow $k\colon A \to X$ to $k \circ (f \circ g)$. However, $\mathsf{C}(f,–)_X$ sends k to $= k \circ f$, and $\mathsf{C}(g,–)_X$ sends that on to $(k \circ f) \circ g$. Hence, component by component, $\mathsf{C}(f \circ g,–)$ acts in the same way as $\mathsf{C}(g,–) \circ \mathsf{C}(f,–)$, which makes them the same natural transformation. ☐

Proof of (2). By definition $\mathsf{C}(f,–)_A$ sends any $k\colon A \to A$ to $k \circ f\colon B \to A$. So in particular it sends 1_A to f. ☐

Proof of (3). By definition, $\mathsf{C}(1_A,–)_X$ sends any $k\colon A \to X$ to $k \circ 1_A\colon A \to X$ – i.e. it sends k to itself.

And what is $1_{\mathsf{C}(A,–)}$? I haven't said! But the notation indicates the identity arrow on the object $\mathsf{C}(A,–)$, where the relevant category must be a functor category including such hom-functors as objects. But then the identity object on such an object will be the identity natural transformation from $\mathsf{C}(A,–)$ to itself. So what does the X-component of that identity natural transformation do to an arrow such as $k\colon A \to X$? It will send the arrow to itself. Which shows that the X-components of $\mathsf{C}(1_A,–)$ and $1_{\mathsf{C}(A,–)}$ agree on their actions; and that holds for all X and so the transformations are identical. ☐

(e) The result (2) has an immediate corollary, which we can add to our earlier main theorem:

Theorem 174 (cont'd). *If $f, f'\colon B \to A$ are distinct arrows, then the corresponding natural transformations $\mathsf{C}(f,–)$ and $\mathsf{C}(f',–)$ are also distinct.*

Proof. We know from the result just proved that

$$\mathsf{C}(f,–)_A 1_A = f \neq f' = \mathsf{C}(f',–)_A 1_A$$

Hence the A-components of $\mathsf{C}(f,–)$ and $\mathsf{C}(f',–)$ can't be the same, hence the natural transformations can't overall be the same either. □

(f) So far, we haven't had to think hard at all! We asked 'what does it take to get a natural isomorphism between hom-functors', immediately spotted *one* trick that will work, and very easily proved some results about it. The obvious next question to ask is: can *all* possible natural transformations between the hom-functors $\mathsf{C}(A,–)$ and $\mathsf{C}(B,–)$ be produced by the same trick, i.e. are they all generated from arrows $f\colon B \to A$ in the way described in Theorem 174?

If a natural transformation $\alpha\colon \mathsf{C}(A,–) \Rightarrow \mathsf{C}(B,–)$ is already given as being of the form $\mathsf{C}(f,–)$ for some $f\colon B \to A$, then we know in this case that $f = \mathsf{C}(f,–)_A 1_A = \alpha_A 1_A$. It would be nice if the same idea always works. So here's a hopeful conjecture:

Theorem 176. *Suppose* C *is a locally small category, and consider the hom-functors* $\mathsf{C}(A,–)$ *and* $\mathsf{C}(B,–)$*, for objects* A, B *in* C*. Then if there is a natural transformation* $\alpha\colon \mathsf{C}(A,–) \Rightarrow \mathsf{C}(B,–)$*, there is a unique arrow* $f\colon B \to A$ *such that* $\alpha = \mathsf{C}(f,–)$*, namely* $f = \alpha_A(1_A)$*.*

And that is indeed right. We just have to think about what happens when we chase 1_A round a naturality square involving the component α_A (what else?). Where does 1_A live? – in the hom-set $\mathsf{C}(A,A)$. So the obvious thing to do is look again at the sort of square we met before, but now putting $X = A$. Off we go:

Proof. Since α is a natural transformation, the following diagram in particular must commute, for any X and any $j\colon A \to X$,

$$\begin{tikzcd}
\mathsf{C}(A,A) \arrow[r, "\mathsf{C}(A,j)"] \arrow[d, "\alpha_A"] & \mathsf{C}(A,X) \arrow[d, "\alpha_X"] \\
\mathsf{C}(B,A) \arrow[r, "\mathsf{C}(B,j)"] & \mathsf{C}(B,X)
\end{tikzcd}$$

And here, recalling the definitions, $\mathsf{C}(A,j)$ is the map that (among other things) sends an arrow $h\colon A \to A$ to the arrow $j \circ h\colon A \to X$, and $\mathsf{C}(B,j)$ sends an arrow $k\colon B \to A$ to the arrow $j \circ k\colon B \to X$.

Chase that identity arrow 1_A round the diagram from the top left to bottom right nodes. The top route sends it to $\alpha_X(j)$. The bottom route sends it to $j \circ (\alpha_A(1_A))$, which equals $\mathsf{C}(\alpha_A(1_A),–)_X(j)$ (check how we set up the notation in Theorem 174).

Since our square always commutes we have $\alpha_X(j) = \mathsf{C}(\alpha_A(1_A),–)_X(j)$, for all objects X and for all arrows $j\colon A \to X$. Thus the X-components of α and $\mathsf{C}(\alpha_A(1_A),–)$ agree on their application to all arrows $j\colon A \to X$, hence must be the same. Since X was arbitrary, that means all the components of α and $\mathsf{C}(\alpha_A(1_A), –)$ are the same.

So those natural transformations are identical, which is just what we need for the existence part of our theorem – we've found an f, i.e. $\alpha_A(1_A)$, such that $\alpha = \mathsf{C}(f, –)$. Look at it this way: fixing just one bit of data – about what (the relevant component of) α does to 1_A – fixes the whole natural transformation by the requirement that the naturality squares all commute.

Finally, suppose both f and f' are such that $\alpha = \mathsf{C}(f, –) = \mathsf{C}(f', –)$. Then by Theorem 175 (2)

$$f = \mathsf{C}(f, –)_A(1_A) = \mathsf{C}(f', –)_A(1_A) = f'$$

which shows f's uniqueness. □

(g) The theorems so far in this section have been about covariant hom-functors. Predictably, there are dual results for contravariant hom-functors $\mathsf{C} \to \mathsf{Set}$ (or equivalently, covariant hom-functors $\mathsf{C}^{op} \to \mathsf{Set}$).

Here's a summary theorem, whose proof can be left as a routine exercise in dualization – just pay attention to the direction of arrows:

Theorem 177. *Suppose C is a locally small category, and $\mathsf{C}(–, A)$, $\mathsf{C}(–, B)$ are contravariant hom-functors (for objects A, B in C). Then:*

1. *If there exists an arrow $f\colon A \to B$, there is a natural transformation $\mathsf{C}(–, f)\colon \mathsf{C}(–, A) \Rightarrow \mathsf{C}(–, B)$, where for each X, the component $\mathsf{C}(–, f)_X\colon \mathsf{C}(X, A) \to \mathsf{C}(X, B)$ sends an arrow $j\colon X \to A$ to $f \circ j\colon X \to B$.*
2. *$\mathsf{C}(–, g \circ f) = \mathsf{C}(–, g) \circ \mathsf{C}(–, f)$.*
3. *Different arrows $f, f'\colon A \to B$ give rise to different corresponding natural transformations $\mathsf{C}(–, f)$, $\mathsf{C}(–, f')$.*
4. *If there is a natural transformation $\alpha\colon \mathsf{C}(–, A) \Rightarrow \mathsf{C}(–, B)$, there is a unique arrow $f\colon A \to B$ such that $\alpha = \mathsf{C}(–, f)$, namely $f = \alpha_A(1_A)$.* □

37.2 The Restricted Yoneda Lemma

(a) We now have all we need to prove the Restricted Yoneda Lemma (that's my non-standard label: its rationale becomes clear in the next chapter). The key idea is simply this: in proving Theorems 174 and 176, we have shown that the arrows $B \to A$ of a locally small category C line up one-to-one with natural transformations $\mathsf{C}(A, –) \Rightarrow \mathsf{C}(B, –)$.

Recall the handy notation introduced in Defn. 137: $Nat(F, G)$ denotes the collection of natural transformations from F to G. Then, in other words, what we have shown is that *there is a bijection between the hom-set $\mathsf{C}(B, A)$ and the collection $Nat(\mathsf{C}(A, –), \mathsf{C}(B, –))$*. (And ah-ha! – since we are dealing with a locally small category, $\mathsf{C}(B, A)$ is by assumption set-sized; therefore, since the collection $Nat(\mathsf{C}(A, –), \mathsf{C}(B, –))$ is the same size, we can happily treat that as a set too.)

Likewise Theorem 177 tells us that its arrows $A \to B$ line up one-to-one with natural transformations $\mathsf{C}(-, A) \Rightarrow \mathsf{C}(-, B)$. In other words, there is a bijection between the hom-set $\mathsf{C}(A, B)$ and the set $Nat(\mathsf{C}(-, A), \mathsf{C}(-, B))$.

But bijections between sets, of course, count as isomorphisms in Set. So we have established the following key theorem:

Theorem 178 (The Restricted Yoneda Lemma). *Suppose* C *is a locally small category, and* A, B *are objects of* C*. Then* $Nat(\mathsf{C}(A, -), \mathsf{C}(B, -)) \cong \mathsf{C}(B, A)$ *and* $Nat(\mathsf{C}(-, A), \mathsf{C}(-, B)) \cong \mathsf{C}(A, B))$. □

(b) It is worth spelling out the justification for our theorem in a slightly different style.

Fix on the objects A and B. Then we've shown that there is a function $\mathcal{X}_{AB}$ with source $\mathsf{C}(B, A)$ and target $Nat(\mathsf{C}(A, -), \mathsf{C}(B, -))$, which sends an arrow $f\colon B \to A$ to $\mathsf{C}(f, -)$. And there is a function $\mathcal{E}_{AB}$ in the reverse direction, from $Nat(\mathsf{C}(A, -), \mathsf{C}(B, -))$ to $\mathsf{C}(B, A)$, which sends a natural transformation $\alpha\colon \mathsf{C}(A, -) \Rightarrow \mathsf{C}(B, -)$ to $\alpha_A(1_A)$.

Then we immediately have:

(1) Given any $f\colon B \to A$,

$$(\mathcal{E}_{AB} \circ \mathcal{X}_{AB})f = \mathcal{E}_{AB}(\mathsf{C}(-, f)) = \mathsf{C}(-, f)_A(1_A) = f.$$

where the last identity is from Theorem 175. But f was arbitrary. Whence $\mathcal{E}_{AB} \circ \mathcal{X}_{AB} = 1$ (that's the identity on $\mathsf{C}(B, A)$).

(2) Given any $\alpha\colon \mathsf{C}(-, A) \Rightarrow \mathsf{C}(-, B)$,

$$(\mathcal{X}_{AB} \circ \mathcal{E}_{AB})\alpha = \mathcal{X}_{AB}(\alpha_A(1_A)) = \mathsf{C}(\alpha_A(1_A), -) = \alpha$$

where the last identity is from Theorem 176. But α was arbitrary. Whence $\mathcal{X}_{AB} \circ \mathcal{E}_{AB} = 1$ (that's the identity on $Nat(\mathsf{C}(-, A), \mathsf{C}(-, B))$).

Having a two-sided inverse, $\mathcal{X}_{AB}$ is therefore an isomorphism, and we have half our last theorem again.

The proof of the other half is dual. There is a function $\mathcal{Y}_{AB}$, with source $\mathsf{C}(A, B)$ and target $Nat(\mathsf{C}(-, A), \mathsf{C}(-, B))$, which sends an arrow $f\colon A \to B$ to $\mathsf{C}(-, f)$. And there is a function $\mathcal{E}_{AB}$ from $Nat(\mathsf{C}(-, A), \mathsf{C}(-, B))$ to $\mathsf{C}(A, B)$, which sends a natural transformation α to $\alpha_A(1_A)$. As before, we can show these two functions are inverses. And so $\mathcal{Y}_{AB}$ is also an isomorphism, and we have the other half our theorem.

37.3 The Yoneda Embedding, the Yoneda Principle

(a) A moment ago we fixed objects A and B and defined $\mathcal{X}_{AB}$ as a map from arrows $B \to A$ to natural transformations $\mathsf{C}(A, -) \Rightarrow \mathsf{C}(B, -)$.

But there was nothing special about the objects A and B here. So we can in fact think of a more general operation on arrows $\mathcal{X}$ which, now for *any* arrow f at all living in C, sends it to a corresponding natural transformation $\mathsf{C}(f,–)$.

Similarly, we can define an operation on objects which we will also label $\mathcal{X}$ that takes any C-object A and sends it to the corresponding hom-functor $\mathsf{C}(A,–)$.

That double use of '$\mathcal{X}$' for an operation on objects and operation on arrows promises that the two components assemble into a functor! Which they do.

For consider: hom-functors like $\mathsf{C}(X,–)$ are objects of the functor category $[\mathsf{C},\mathsf{Set}]$. And natural transformations like $\mathsf{C}(f,–)$ are arrows in that same category. So, the $\mathcal{X}$ operation on objects and the $\mathcal{X}$ operation on arrows are at least of just the right types to be components of a contravariant functor $\mathcal{X}\colon \mathsf{C} \to [\mathsf{C},\mathsf{Set}]$ (contravariant, of course, because an arrow $f\colon B \to A$ is sent to a natural transformation $\mathsf{C}(f,–)\colon \mathsf{C}(A,–) \Rightarrow \mathsf{C}(B,–)$).

And we can easily confirm that the two key conditions for functoriality are indeed satisfied. First, identities are preserved:

$$\mathcal{X}(1_A) = \mathsf{C}(1,A,–) = 1_{\mathsf{C}(A,–)} = 1_{\mathcal{X}(A)}.$$

Second, composition is respected. In other words, for any composable f, g in C,

$$\mathcal{X}(g \circ f) = \mathsf{C}(g \circ f,–) = \mathsf{C}(–,f) \circ \mathsf{C}(–,g) = \mathcal{X}(f) \circ \mathcal{X}(g),$$

reversing the order of composition as is required for a contravariant function.

So to summarize this important result, and state its dual:

Theorem 179. *For any locally small category* C*, there is a contravariant functor* $\mathcal{X}\colon \mathsf{C} \to [\mathsf{C},\mathsf{Set}]$ *which operates as follows:*

$$\begin{array}{rrcl} \mathcal{X}\colon & A & \longmapsto & \mathsf{C}(A,–) \\ & f\colon B \to A & \longmapsto & \mathsf{C}(f,–)\colon \mathsf{C}(A,–) \Rightarrow \mathsf{C}(B,–). \end{array}$$

Dually, there is a covariant functor $\mathcal{Y}\colon \mathsf{C} \to [\mathsf{C}^{op},\mathsf{Set}]$ *which works like this:*

$$\begin{array}{rrcl} \mathcal{Y}\colon & A & \longmapsto & \mathsf{C}(–,A) \\ & f\colon A \to B & \longmapsto & \mathsf{C}(–,f)\colon \mathsf{C}(–,A) \Rightarrow \mathsf{C}(–,B). \end{array}$$

□

(b) It is immediate that the functors $\mathcal{X}$ and $\mathcal{Y}$ behave nicely:

Theorem 180. $\mathcal{X}$ *and* $\mathcal{Y}$ *are fully faithful and injective on objects.*

Proof. Let's work through the second case. By definition, $\mathcal{Y}$ is full just in case, for any C-objects A, B, and any natural transformation $\alpha\colon \mathsf{C}(–,A) \Rightarrow \mathsf{C}(–,B)$, there is an arrow $f\colon A \to B$ in C such that $\alpha = \mathcal{Y}f = \mathsf{C}(–,f)$. Which is given in Theorem 177.

By definition, $\mathcal{Y}$ is faithful just in case, for any C-objects A, B, and any pair of arrows $f, g\colon A \to B$ in C, then if $\mathsf{C}(–,f) = \mathsf{C}(–,g)$ then $f = g$. But that also follows immediately from Theorem 177.

So the only new claim is that $\mathcal{Y}$ is injective on objects, meaning that if $\mathcal{Y}(A) = \mathcal{Y}(B)$ then $A = B$. Suppose then that we are given $\mathcal{Y}(A) = \mathcal{Y}(B)$, i.e. $\mathsf{C}(–,A) = \mathsf{C}(–,B)$. Then for any object C we'll have $\mathsf{C}(C,A) = \mathsf{C}(C,B)$. But that can't be so if $A \neq B$, since hom-sets on different pairs of objects must be disjoint – as we've set things up, no arrow $g\colon C \to A$ can equal some $h\colon C \to B$, having distinct targets. $\square$

The situation, then, is this. The functor $\mathcal{Y}$ injects a one-to-one copy of the C-objects into the objects of the functor category $[\mathsf{C}^{op}, \mathsf{Set}]$; and then it fully and faithfully matches up the arrows between C-objects with arrows between the corresponding objects in $[\mathsf{C}^{op}, \mathsf{Set}]$. In other words:

Theorem 181 (Yoneda Embedding). *The image of* C *under the functor* $\mathcal{Y}$ *is an isomorphic copy of* C*, embedded inside the functor category* $[\mathsf{C}^{op}, \mathsf{Set}]$ *as a full sub-category.* $\square$

Note: it is customary to emphasize this result rather than its obvious dual involving $\mathcal{X}$. To be sure, $\mathcal{X}$ embeds a copy of C^{op} in the functor category $[\mathsf{C}, \mathsf{Set}]$ – but we are more likely to be interested in finding copies of the category C we start off with rather than a copy of its opposite.

(c) For the last of our initial trio of Yoneda results, here's an important corollary of what's gone before:[1]

Theorem 182 (Yoneda Principle). *For any objects* A, B *in the locally small category* C*,* $A \cong B$ *iff* $\mathcal{X}A \cong \mathcal{X}B$ *and also iff* $\mathcal{Y}A \cong \mathcal{Y}B$*.*

Proof. I'll prove the second claim, with the dual result left as an exercise.

Suppose $A \cong B$. Then there is an isomorphism $f\colon A \xrightarrow{\sim} B$. So there is a natural transformation $\mathsf{C}(-, f)\colon \mathsf{C}(-, A) \Rightarrow \mathsf{C}(-, B)$, which by the remark after Theorem 174 is a natural isomorphism. So in our alternative notation, $\mathcal{Y}f\colon \mathcal{Y}A \overset{\sim}{\Longrightarrow} \mathcal{Y}B$. Hence $\mathcal{Y}A \cong \mathcal{Y}B$.

Now suppose $\mathcal{Y}A \cong \mathcal{Y}B$. So there exists a natural isomorphism $\alpha\colon \mathsf{C}(-, A) \overset{\sim}{\Longrightarrow} \mathsf{C}(-, B)$. By Theorem 177, α is $\mathsf{C}(-, f)$ for some $f\colon A \to B$, i.e. is $\mathcal{Y}f$. But $\mathcal{Y}$ is fully faithful. So Theorem 136 tells us that since $\mathcal{Y}f$ is an isomorphism, so is f. Hence $A \cong B$.

That shows $A \cong B$ iff $\mathcal{Y}A \cong \mathcal{Y}B$. $\square$

37.4 Yoneda meets Cayley

Now, as I said in the preamble, the really interesting applications of the results we have so far come too far downstream to explore at this point. But in the rest of this chapter I can perhaps usefully make some elementary remarks about their significance.

[1] The association of all three with the name of the Japanese mathematician Nobuo Yoneda (1930–1996) is perhaps *rather* tenuous.

For what it is worth, the story goes that a key idea was in a 1954 paper by Yoneda on homology. And then Saunders Mac Lane and Yoneda met in Paris the same year, were talking (at the Gare du Nord!) about Yoneda's paper and his ideas, and Mac Lane baptized the general Yoneda Lemma. Though it seems that the first explicit appearance of the Lemma in print, in the restricted form, is in a 1960 paper by Grothendieck (which may well have been an independent rediscovery).

(a) We have seen before that it can be instructive to test-drive categorial ideas on toy cases such as ordered collections of objects considered as categories or monoids considered as categories. Let's take the second sort of case.

More specifically, let's look at those monoids which are groups, considered as one-object categories all of whose arrows are isomorphisms (see §7.8). To spell that out, take a group comprising some objects G (no more than a set's worth, for present purposes) equipped with a suitable binary operation $\star$ and with the object e distinguished as the group identity. Then the corresponding category G has the following data:

(1) The sole object of G: choose whatever object you like, dub it '$\bullet$'.

(2) The arrows of G: any object g of the group counts as an arrow $g\colon \bullet \to \bullet$. The composite $h \circ g$ of the two arrows $h, g\colon \bullet \to \bullet$ is $h \star g\colon \bullet \to \bullet$. And the identity arrow $1_\bullet$ is the group identity e.

G is locally small since its sole potential hom-set $\mathsf{G}(\bullet, \bullet)$ is the set whose members are the group objects G (or are set-indices for those objects, but let's not get fussy).

We can therefore apply our Yoneda Embedding results, Theorems 179 and 180. Consider then the version which tells us that there is a fully faithful functor $\mathcal{Y}$ which embeds G into the category $[\mathsf{G}^{op}, \mathsf{Set}]$, where

(1) for the G-object $\bullet$, $\mathcal{Y}\bullet = \mathsf{G}(-, \bullet)$.
(2) For any G-arrow $g\colon \bullet \to \bullet$, $\mathcal{Y}g = \mathsf{G}(-, g)\colon \mathsf{G}(-, \bullet) \Rightarrow \mathsf{G}(-, \bullet)$.

And what does that mean?

Note first that the arrows $\mathcal{Y}g$ (one for each $g \in$ G) form a group. Functoriality gives the required associativity and ensures $\mathcal{Y}e$ behaves as the identity element; further $\mathcal{Y}g \circ \mathcal{Y}g^{-1} = \mathcal{Y}(g \circ g^{-1}) = \mathcal{Y}e$ and likewise $\mathcal{Y}g^{-1} \circ \mathcal{Y}g = \mathcal{Y}e$, so we have the required group inverses.

And what are the elements of this group? By definition, $\mathsf{G}(-, g)$ sends an arrow $x\colon \bullet \to \bullet$ to $g \circ x\colon \bullet \to \bullet$. in other words, $\mathcal{Y}g$ acts by sending any group-object x to the group-object $g \star x$. That gives us a one-to-one permutation of G – because if $g \star x = g \star x'$ then $g^{-1} \star g \star x = g^{-1} \star g \star x'$ and hence $x = x'$.

In sum, G's isomomorphic image under the Yoneda functor $\mathcal{Y}$ is a category which has a single object, and whose arrows form a group of permutations of the objects G. And *that* is just a group of permutations of the objects G treated as category.

So we can put it, then, like this. Yoneda tells us that any group $(\mathrm{G}, \star, e)$, when thought of as a category, is isomorphic to a group of permutations of its objects G (a subgroup of *all* G's permutations), when thought of as a category. But – now deleting the reference to categories – *that's just Cayley's theorem in group theory.*

(b) Let's not get over-excited! It would be quite misleading to say that we have arrived at a new, distinctively categorial, proof of Cayley's theorem. For the key part of the argument above is that the permutation functions $g \mapsto g \star x$, for the various $x \varepsilon$ G, form a group – and that's the essence of the usual elementary proof of Cayley's theorem. So what new insight might we get out of the categorial detour?

Well, compare our earlier treatment of products. Pre-categorially, we are familiar with a bunch of constructions which look, intuitively, to involve more or less the same idea – consider, for example, forming a product of two groups, forming the product of two lattices, taking the meet of two elements *in* a lattice, forming a logical product of two propositions in logic, and so on. Then the nice thing about the categorial account of products is that it enables is to see all these and more as instances of the very same construction. Similarly here. There are a variety of pre-categorial results that are intuitively in the same ballpark as Cayley's theorem, telling us about how structures of one kind can be isomorphically embedded into other structures. There are some algebraic cousins, such as e.g. the result that a ring can be embedded into the endomorphism ring of its underlying abelian group. Then there are results such as that a partial ordering of objects can be mirrored in a collection of subsets of those objects ordered by inclusion. It turns out that the Yoneda embedding theorem applied to relevant categories reveals such results as again instances of the very same construction. And that will be an insight worth having.

37.5 "Friendship is everything"

(a) Some remarks now about the Yoneda Principle (as it is often called).

The theorem is technically helpful, because in some cases it is easy to spot a strategy for showing the relevant $\mathcal{Y}A$ and $\mathcal{Y}B$ are isomorphic; and then we can use that isomorphism to conclude that A is isomorphic to B.

Here's an example. Back in Chapter 17, Theorem 71 asserted that, for all A, B, C in a Cartesian closed category C, we have:

(1) If $B \cong C$, then $A^B \cong A^C$,

(2) $(A^B)^C \cong A^{B\times C}$,

(3) $(A \times B)^C \cong A^C \times B^C$.

I gave a brute force proof of (1), but merely sketched out the lines of an analogous proof of (2). We can now return to spell out a proof in a bit more detail.

Assume C is a Cartesian closed category which is locally small. Theorem 68 tells us that (*) there is a bijection between arrows $X \to Z^Y$ and $X \times Y \to Z$. Hence we have

$$\begin{aligned}\mathsf{C}(X, (A^B)^C) &\cong \mathsf{C}(X \times C, A^B)\\ &\cong \mathsf{C}((X \times C) \times B, A)\\ &\cong \mathsf{C}(X \times (B \times C), A)\\ &\cong \mathsf{C}(X, A^{(B\times C)})\end{aligned}$$

where we apply (*) three times, and the remaining line depends on the isomorphism of the multi-products $(X \times C) \times B$ and $X \times (B \times C)$.

Now, not only are these isomorphisms, but intuitively they should each be provable in without making special assumptions about X (or indeed about A, B, or C either), and without having to make arbitrary choices along the way. So our isomorphisms ought morally be natural in X.

So that means we ought to be able to show $\mathsf{C}(–, (A^B)^C) \cong \mathsf{C}(–, A^{(B\times C)})$, or in other words, $\mathcal{Y}(A^B)^C \cong \mathcal{Y}A^{(B\times C)}$. And then an application of the Yoneda Principle gives us $(A^B)^C \cong A^{B\times C}$, as desired.[2]

(b) However, there was some arm-waving in the middle of the argument there! For how do we *prove* that those four isomorphisms in the displayed chain really are natural in X? Well, for the applications of (*), note the proof in §32.4 (5) that the functors $\mathsf{C}(–, Z^Y)$ and $\mathsf{C}(– \times Y, Z)$ are naturally isomorphic. And the other step where we re-order products involves can be shown to involve a natural isomorphism too (compare our first example in §32.1).

OK: we *can* fill in the details, then. But it is perhaps worth also pointing out that the work done in giving those details and proving the naturality-in-X of each of the isomorphisms in the chain will in fact give us all the ingredients we need for a direct proof that $(A^B)^C \cong A^{B\times C}$ along the lines sketched in Chapter 17, a proof which never mentions Yoneda!

(c) Putting technicalities aside, there is something much more interesting about the Yoneda Principle: it reflects a certain categorial perspective.

Recall, we can think of arrows from various objects X to a given object A as probing A from various perspectives. In special cases, probing from a limited selection of objects will reveal everything we need to know about A. For example, in the category Set, probing just from a singleton $\{\star\}$ is enough, as the members of a set A are bijectively associated with arrows $\{\star\} \to A$. In general, however, we will need to probe from more sources: for example, in Grp, probes from a singleton source can only hit a group's identity element. But if we probe from *all* other objects X, that must suffice to fix the object A up to isomorphism – in other words, if we take all the arrows $X \to A$, for varying X, you get all the categorially relevant information about A. Why so?

Well, as we noted before, the functor $\mathsf{C}(–, A)$ wraps up all that perspectival information, telling you for each X what the corresponding hom-set of arrows $X \to A$ comprises. Then the Yoneda Principle tells us that if we have pinned down $\mathsf{C}(–, A)$ (i.e. $\mathcal{Y}A$) up to isomorphism – pinned down how A is seen by its world – then we have pinned down A up to isomorphism. For $\mathsf{C}(–, A) \cong \mathsf{C}(–, B)$ indeed implies $A \cong B$.

As we might say, then: you know an object, categorially speaking, by knowing how its friends see it ... Friendship, as Jane Austen remarked, is everything.[3]

[2] Challenge: can you outline a proof of (3) along similar lines?

[3] Or here's another folklore image. If you think of the objects of a category as particles and the arrows as ways to smash one particle into another, then the Yoneda Principle tells us that if you know all about the interactions when you smash various particles X into the mystery particle A, you know everything there is to know about A.

38 The Yoneda Lemma

In the previous chapter, we proved what I called the Restricted Yoneda Lemma and showed that the Yoneda Embedding functor is indeed an embedding. And the proofs of those initial results are, I hope, not at all bewildering. It is only a bit of an exaggeration to say that we asked what the natural transformations between two hom-functors might look like, and then followed our noses, doing the obvious things. And in fact the theorems obtained so far are often the ones that are actually needed when a result is proved 'by the Yoneda Lemma'.

Still, having got this far, let's press on and prove the full-power, unrestricted, Yoneda Lemma. What does this involve?

38.1 Onwards to the full Yoneda Lemma!

Assume once more that C is locally small. Then here again is one half of our restricted Theorem 178:

> Let F be the hom-functor $\mathsf{C}(B,-)\colon \mathsf{C} \to \mathsf{Set}$. Then there is an isomorphism between $Nat(\mathsf{C}(A,-),F)$ and FA.

Now, to get from that to the full Yoneda Lemma takes two more stages:

(1) *Generalizing on F.* We look again at the ingredients of the proof of the restricted version and ask 'Did we essentially depend on the fact that the second functor in the story, now notated simply 'F', was actually a hom-functor $\mathsf{C}(B,-)$ for some B?'

Inspection reveals that we didn't. So we in fact have the more general result that for *any* functor $F\colon \mathsf{C} \to \mathsf{Set}$, and any C-object A, there is an isomorphism between $Nat(\mathsf{C}(A,-),F)$ and FA.

(2) *Confirming it's all natural.* Our proof of this general result – like the proof of the original Restricted Lemma – provides a recipe for constructing the required isomorphism that doesn't involve any arbitrary choices, and doesn't depend on any special features of A or F.

In an *intuitive* sense, then, we've constructed a natural isomorphism between the objects $Nat(\mathsf{C}(A,-),F)$ and FA. And so hopefully we should be able to show that these objects are naturally isomorphic in the official *categorial* sense of Defn. 125.

In short, we will get from the Restricted Yoneda Lemma to the full-dress Yoneda Lemma by generalizing a construction, and then recasting in category-theoretic terms our intuitive judgement of the naturality of our construction. Neither stage involves anything conceptually very difficult. It is forgivable to skip the proof details, though you'll probably want to grasp the general strategy for each stage.

And needless to say, it's a two-for-one deal: once we have done all the work of strengthening one half of the Restricted Lemma, we can leave strengthening the dual half as an exercise.

38.2 The generalizing move

(a) We continue working in a locally small category C. And let's restate some of what we already know, using 'F' to abbreviate '$\mathsf{C}(B, –)$':

(i) F sends a C-object A to the set $FA = \mathsf{C}(B, A)$, and there is a bijection between elements of FA and natural transformations $\mathsf{C}(A, –) \Rightarrow F$ – this bijection sends $f\colon B \to A$ in FA to the transformation whose X-component maps an arrow $g\colon A \to X$ to $g \circ f\colon B \to X$.

(ii) F sends a C-arrow $g\colon A \to X$ to a function Fg, where this takes a C-arrow $f\colon B \to A$ to the C-arrow $g \circ f\colon B \to X$. In short, $Fg(f) = g \circ f$. (That's just from Theorem 149, re-lettered.)

(iii) Hence, putting (i) and (ii) together, we have: there's a bijection which sends an element f in FA to the natural transformation whose X-component maps $g\colon A \to X$ to $Fg(f)$.

But note: (iii) makes sense for *any* functor $F\colon \mathsf{C} \to \mathsf{Set}$. For FA, where A is a C-object, is a set. And Fg, where g is a C-arrow will be a Set-arrow, i.e. a (total!) set-function. The function Fg can be applied to any element f of the set FA.

(b) The idea in (iii) enables us prove the following generalization of half of the Restricted Lemma (again the label is non-standard):[1]

Theorem 183 (The Core Yoneda Lemma). *For any object A of the locally small category C, and any functor $F\colon \mathsf{C} \to \mathsf{Set}$, $Nat(\mathsf{C}(A, –), F) \cong FA$.*

Proof, step 1: Defining $\mathcal{X}_{AF}$, a candidate bijection. Pick an element f (in the general case, *not* a function!) from the set FA.

And taking up the idea in (iii), define $\alpha^f_X\colon \mathsf{C}(A, X) \to FX$ as the function that maps any C-arrow $g\colon A \to X$ to $Fg(f)$. Similarly, define $\alpha^f_Y\colon \mathsf{C}(A, Y) \to FY$ as the function that maps any C-arrow $h\colon A \to Y$ to $Fh(f)$; and so on.

Then it is immediate that a diagram like the following commutes for any $j\colon X \to Y$:

[1] Some call this result the Yoneda Lemma, plain and simple. See e.g. Adámek et al. (2009, p. 88), Barr and Wells (1985, p.26), Grandis (2018, p. 44). But it is more usual to take the Lemma to be the full result we get to as our Theorem 186.

$$\begin{tikzcd}
\mathsf{C}(A,X) \arrow[r, "\mathsf{C}(A,j)"] \arrow[d, "\alpha^f_X"] & \mathsf{C}(A,Y) \arrow[d, "\alpha^f_Y"] \\
FX \arrow[r, "Fj"] & FY
\end{tikzcd}$$

The upper route takes a C-arrow $g\colon A \to X$ to $j \circ g\colon A \to Y$. And α^f_Y sends that on to $F(j \circ g)(f)$ which equals $Fj((Fg(f))$ by functoriality. While the lower route takes g to $Fg(f)$ to $Fj(Fg(f))$. So we get the same result either way.

Hence, as defined, the components $\alpha^f_X, \alpha^f_Y, \ldots$ assemble into a natural transformation $\alpha^f\colon \mathsf{C}(A,–) \Rightarrow F$. Great!

So we have brought into play a nice function $\mathcal{X}_{AF}\colon FA \to \mathit{Nat}(\mathsf{C}(A,–), F)$ which sends an element f to α^f. It just remains to show that this function is bijective (as was its analogue in §37.2). $\square$

Proof step 2: Showing $\mathcal{X}_{AF}$ is surjective. We want to prove that every natural transformation $\alpha\colon \mathsf{C}(A,–) \Rightarrow F$ is some α^f generated by an element f in FA. Given the proof of Theorem 176, we know exactly how to do that.

We start by noting that, given α is a natural transformation, the following diagram in particular must commute, for any Y and any $j\colon A \to Y$:

$$\begin{tikzcd}
\mathsf{C}(A,A) \arrow[r, "\mathsf{C}(A,j)"] \arrow[d, "\alpha_A"] & \mathsf{C}(A,Y) \arrow[d, "\alpha_Y"] \\
FA \arrow[r, "Fj"] & FY
\end{tikzcd}$$

Now chase the identity arrow 1_A round the diagram from the top left to bottom right nodes. The top route sends it first to j and then on to $\alpha_Y(j)$. The bottom route sends it to $Fj(\alpha_A(1_A))$ – which, by definition, equals $\alpha^f_Y j$ for $f = \alpha_A(1_A)$. Since this holds for any j, we have $\alpha_Y = \alpha^f_Y$. But Y was arbitrary, so the equality holds for all components, therefore $\alpha = \alpha^f$ when $f = \alpha_A(1_A)$. $\square$

Proof step 3: Showing $\mathcal{X}_{AF}$ is injective. We use the same pattern of argument as for Theorem 174 (cont'd), except that where we previously used the fact that $\mathsf{C}(f,–)_A 1_A = f$, we now use the fact that $\alpha^f_A(1_A) = f$. And why is that a fact? By definition, α^f_A sends an arrow $g\colon A \to A$ to $Fg(f)$. So $\alpha^f_A(1_A)$ yields $F1_A(f)$. But the functoriality of F ensures that $F1_A$ is an identity function.

So if $f \neq f'$ we have $\alpha^f_A(1_A) = f \neq f' = \alpha^{f'}_A(1_A)$, and hence $\alpha^f \neq \alpha^{f'}$ $\square$

Which establishes the Core Lemma, and we've got a bijection $\mathcal{X}_{AF}$ between sets, showing they are isomorphic.[2]

[2]It is common enough for Theorem 183 to be stated and proved straight off, from a cold start, without any of the scene-setting from the previous chapter. Presented like that, the proof is still easy, but (to my mind) the steps can seem disconcertingly to be just pulled out of a hat. Our slow lead-up hopefully eliminates any sense of mystery!

Of course, there's a companion dual result which can safely be left as an exercise:[3]

Theorem 183 (cont'd). *For any object A of the locally small category C, and any functor $F\colon \mathsf{C}^{op} \to \mathsf{Set}$, $Nat(\mathsf{C}(-,A),F) \cong FA$.* $\square$

(c) Let's take up an idea from the discussion after Theorem 178. We'll show that $\mathcal{X}_{AF}\colon FA \to Nat(\mathsf{C}(A,-),F)$ has a two-sided inverse $\mathcal{E}_{AF}\colon Nat(\mathsf{C}(A,-),F) \to FA$ where that is the function which sends a natural transformation $\alpha\colon \mathsf{C}(A,-) \Rightarrow FA$ to the element $\alpha_A(1_A)$. Proceeding more or less as before,

(1) Given any element f of FA,

$$(\mathcal{E}_{AF} \circ \mathcal{X}_{AF})f = \mathcal{E}_{AF}\alpha^f = \alpha^f_A(1_A) = f.$$

But f was arbitrary. Whence $\mathcal{E}_{AF} \circ \mathcal{X}_{AF} = 1$ (that's the identity on FA).

(2) Given any $\alpha\colon \mathsf{C}(-,A) \Rightarrow F)$,

$$(\mathcal{X}_{AF} \circ \mathcal{E}_{AF})\alpha = \mathcal{X}_{AF}(\alpha_A(1_A)) = \alpha^{\alpha_A(1_A)} = \alpha$$

(for the last identity, see the end of the Proof step 2 above). But α was arbitrary. Whence $\mathcal{X}_{AF} \circ \mathcal{E}_{AF} = 1$ (that's the identity on $Nat(\mathsf{C}(A,-),F)$).

Having a two-sided inverse, $\mathcal{X}_{AF}$ is therefore (as we know!) an isomorphism – as is its inverse $\mathcal{E}_{AF}$, a point we'll need in a moment.

There is of course a dual story to be told about an isomorphism $\mathcal{Y}_{AF}\colon FA \to Nat(\mathsf{C}(-,A),F)$ which sends an element f of FA to a suitable $\alpha^f\colon \mathsf{C}(-,A) \Rightarrow F$. But I'll leave it as another challenge to fill in the details.

38.3 Making it all natural

(a) So where have we got to?

To concentrate again on half the story, Theorem 183 tells us that – when C is a locally small category, A is any object in that category, and $F\colon \mathsf{C} \to \mathsf{Set}$ is some functor – then FA is isomorphic to $Nat(\mathsf{C}(A,-),F)$. I called that the Core Yoneda Lemma. And the earlier Restricted Lemma, Theorem 178, is what get when we restrict to the cases where F has the form $\mathsf{C}(B,-)$ for some B.

Now, our proof of Theorem 183 didn't depend on any special facts about A or F, and didn't depend on any arbitrary choices. So, at least in an *intuitive* sense, we have found a natural isomorphism. But when we find an intuitively natural isomorphism between objects, what I called the Eilenberg/Mac Lane

[3] I'll state this dual in the conventional way, in terms of a covariant functor $F\colon \mathsf{C}^{op} \to \mathsf{Set}$. But you may well find it easier to keep track of which arrows go in which direction, and avoid dancing between C and C^{op}, if you set out the proof thinking in terms of the equivalent contravariant $F\colon \mathsf{C} \to \mathsf{Set}$.

Thesis in §32.7 enjoins us to see this as generated by a natural isomorphism in the *categorial* sense, an isomorphism between functors.

And so this is going to be our next move, the final stage of the argument taking us to the full Yoneda Lemma. We want to show how the intuitively natural isomorphism we've found between between the objects FA and $Nat(\mathsf{C}(A,–)), F)$ can be seen as arising, in fact in two ways, from natural isomorphisms between functors.

(b) A quick observation before continuing, however.

At the very beginning of the previous chapter, I noted Awodey's remark about the Yoneda lemma being perhaps the most used result in category theory. And taking a look at e.g. a characteristic pair of important monographs on category theory which go rather beyond entry level, Borceux (1994) and Mac Lane and Moerdijk (1992), we do indeed find multiple invocations of one or other of the interrelated Yoneda results.

However, I think it is correct to say that the appeal is frequently either to our easy Yoneda Embedding result, Theorem 181, or to the simple existence of some isomorphism of the kind reported in the Core Theorem 183. The further fact that such an isomorphism between objects can officially be seen as arising from a natural isomorphism between functors is often not needed.

(c) Down to work again. Here's a straight application of our earlier Defn. 125:

Definition 140. Two objects in Set are said to be *naturally isomorphic in A* if they are the images FA and GA of the same object A under a couple of naturally isomorphic functors $F, G \colon \mathsf{C} \to \mathsf{Set}$. △

So to show that FA and $Nat(\mathsf{C}(A,–), F)$ are naturally isomorphic in A, we need to find a functor $G \colon \mathsf{C} \to \mathsf{Set}$ such that $GA = Nat(\mathsf{C}(A,–), F)$, and G is naturally isomorphic to F.

But now recall two functors we already know about:

(1) Theorem 179 defined the contravariant functor $\mathcal{X} \colon \mathsf{C} \to [\mathsf{C}, \mathsf{Set}]$ which sends an object A to the hom-functor $\mathsf{C}(A,–)$ (and sends a C arrow $f \colon A \to B$ to the natural transformation $\mathsf{C}(f,–) \colon \mathsf{C}(B,–) \Rightarrow \mathsf{C}(A,–)$).

(2) Defn. 138 also gives us a contravariant functor, $Nat(–, F) \colon [\mathsf{C}, \mathsf{Set}] \to \mathsf{Set}$. This sends a functor $\mathsf{C}(A,–) \colon \mathsf{C} \to \mathsf{Set}$ to the set $Nat(\mathsf{C}(A,–), F)$ (and sends a natural transformation $\mathsf{C}(f,–) \colon \mathsf{C}(B,–) \Rightarrow \mathsf{C}(A,–)$ to the corresponding function which takes a natural transformation $\alpha \colon \mathsf{C}(A,–) \Rightarrow F$ and outputs $\alpha \circ \mathsf{C}(f,–) \colon \mathsf{C}(B,–) \Rightarrow F$).

Hence, if we put $G = Nat(–, F) \circ \mathcal{X}$ we'll get, as we wanted, a covariant functor $G \colon \mathsf{C} \to \mathsf{Set}$ (because contravariant functors compose to covariant functor by the mini-Theorem 130), and as required $GA = Nat(\mathsf{C}(A,–), F)$.

Then, to prove GA and FA are naturally isomorphic in A we need to show the following:

Theorem 184. *Let* C *be a locally small category, and* F *a functor* $F\colon \mathsf{C} \to \mathsf{Set}$. *Then the functors* $G = Nat(–, F) \circ \mathcal{X}$ *and* F *are naturally isomorphic.*

And it wouldn't be absurd to set this as a challenge to prove for yourself. You just need to write down a potential naturality square of the kind we need. But what will be the components of the candidate natural isomorphism taking us from the likes of GA (i.e. $Nat(\mathsf{C}(A,–), F)$) to FA? $\mathcal{E}_{AF}$ as defined in the previous section of course. Then you just need to prove that the square commutes. Try it before reading on!

Proof. Given some arrow $j\colon A \to B$, consider the following square:

$$\begin{tikzcd} GA = Nat(\mathsf{C}(A,–), F) \arrow[r, "Gj"] \arrow[d, "\mathcal{E}_{AF}"] & GB = Nat(\mathsf{C}(B,–), F) \arrow[d, "\mathcal{E}_{BF}"] \\ FA \arrow[r, "Fj"] & FB \end{tikzcd}$$

Take any $\alpha\colon \mathsf{C}(A,–) \Rightarrow F$ in GA. Then we have:

(1) $\mathcal{E}_{BF} \circ Gj\alpha = \mathcal{E}_{BF}(\alpha \circ \mathsf{C}(j,–) = (\alpha \circ \mathsf{C}(j,–))_B(1_B) = \alpha_B(\mathsf{C}(j,–))_B(1_B)) = \alpha_B(j)$.

(2) But also $Fj \circ \mathcal{E}_{AF}(\alpha) = Fj \circ \alpha_A(1_A) = \alpha_B \circ \mathsf{C}(A,j)(1_A) = \alpha_B(j)$ (for the middle equation we note that $Fj \circ \alpha_A = \alpha_B \circ \mathsf{C}(A,j)$ by a naturality square for α).

So our diagram will always commute, and hence there is a natural isomorphism $\mathcal{E}_F\colon G \Rightarrow F$ with components $(\mathcal{E}_F)_A = \mathcal{E}_{AF}$ for each A in C, and we are done. □

(d) That captures in categorial terms the intuition that the isomorphism between FA and $Nat(\mathsf{C}(A,–), F)$ depends in a natural way on A. Now for the companion intuition that it depends in a natural way on F too. Keeping A fixed, we want to prove $Nat(\mathsf{C}(A,–), F) \cong FA$ naturally in F.

Now, I haven't actually said what it is for such an isomorphism to be natural in a functor like F (as opposed to an object like A). But the idea is the predictable one. We want to show that our isomorphism arises again from a natural isomorphism between two functors, in this case between functors we might initially notate as $Nat(\mathsf{C}(A,–), –)$ and $–A$, when applied to F.

And in fact, we have fleetingly met the *second* of these functors in a different notation in Defn. 139: it is the functor $eval_A\colon [\mathsf{C}, \mathsf{Set}] \to \mathsf{Set}$ which sends any functor $F\colon \mathsf{C} \to \mathsf{Set}$ to FA and sends any natural transformation $\alpha\colon F \Rightarrow G$ to $\alpha_A\colon FA \to GA$.

While what about the *first* functor, $Nat(\mathsf{C}(A,–), –)$? It is another covariant functor from $[\mathsf{C}, \mathsf{Set}] \to \mathsf{Set}$. (Why so? Looking at the explanation in §36.5 and relettering, $Nat(J, –)$ defined in terms of a functor $J\colon \mathsf{E} \to \mathsf{F}$ is a hom-functor from $[\mathsf{E}, \mathsf{F}]$ to Set. So yes, in the present case $Nat(\mathsf{C}(A,–), –)$ defined in terms of the functor $\mathsf{C}(A,–)\colon \mathsf{C} \to \mathsf{Set}$ will give us a hom-functor from $[\mathsf{C}, \mathsf{Set}]$ to Set!)

So we want to prove the following:

Theorem 185. *Let* C *be a locally small category. Then* $Nat(\mathsf{C}(A,-),-)$ *and* $eval_A$ *are naturally isomorphic.*

Again, here's a challenge: prove that before reading on ...

Proof. Given any $\gamma\colon F \Rightarrow G$, consider the following diagram,

$$\begin{tikzcd} Nat(\mathsf{C}(A,-),F) \arrow[r, "Nat(\mathsf{C}(A,-)\text{,}\gamma)"] \arrow[d, "\mathcal{E}_{AF}"] & Nat(\mathsf{C}(A,-),G) \arrow[d, "\mathcal{E}_{AG}"] \\ eval_A(F) = FA \arrow[r, "eval_A(\gamma)"] & eval_A(G) = GA \end{tikzcd}$$

Take any $\alpha\colon \mathsf{C}(A,-) \Rightarrow F$, and recall that $Nat(\mathsf{C}(A,-),\gamma)$ sends α to $\gamma \circ \alpha$. Then we have:

(1) $\mathcal{E}_{AG} \circ Nat(\mathsf{C}(A,-),\gamma)(\alpha) = \mathcal{E}_{AG}(\gamma \circ \alpha) = (\gamma \circ \alpha)_A(1_A) = \gamma_A(\alpha_A(1_A))$.

(2) But also $eval_A(\gamma) \circ \mathcal{E}_{AF}(\alpha) = \gamma_A(\alpha_A(1_A))$.

Hence the diagram always commutes. Therefore there is a natural isomorphism $\mathcal{E}_A\colon K \Rightarrow eval_A$ with components $(\mathcal{E}_A)_F = \mathcal{E}_{AF}$ for each F from $[\mathsf{C}, \mathsf{Set}]$. $\square$

38.4 Putting everything together

Our last two theorems have duals – which I'll leave you to state and prove. But taking those as read, we can now combine all the ingredients from the last three theorems ...

Cue drum-roll!

... and at last we get the fully caffeinated Lemma:

Theorem 186 (Yoneda Lemma). *For any locally small category* C*, object* A *in* C*, and covariant functor* $F\colon \mathsf{C} \to \mathsf{Set}$*,* $Nat(\mathsf{C}(A,-),F) \cong FA$*, both naturally in* A *and naturally in* F*.*

Likewise for any contravariant functor $F\colon \mathsf{C} \to \mathsf{Set}$ *(equivalently, covariant functor* $F\colon \mathsf{C}^{op} \to \mathsf{Set}$*),* $Nat(\mathsf{C}(-,A),F) \cong FA$*, both naturally in* A *and naturally in* F*.* $\square$

... So we are done!

... And that wasn't *too* bewildering, was it?[4]

[4] To aid comparison with other presentations, two quick notational/terminological points. First, where I have written $\mathsf{C}(A,-)$ and $\mathsf{C}(-,A)$ for, respectively, the covariant and contravariant hom-functors, you will often find others writing simply h^A and h_A.

Second, for reasons I won't go into here, a contravariant functor from C to Set, i.e. a functor $F\colon \mathsf{C}^{op} \to \mathsf{Set}$, is standardly called a *presheaf* on C. And then the presheaves on C (as objects) together with the natural transformations between them (as arrows) form the presheaf category on C, often denoted simply $\widehat{\mathsf{C}}$ where we wrote $[\mathsf{C}^{op}, \mathsf{Set}]$. So in this notation, the Yoneda embedding is a functor $\mathcal{Y}\colon \mathsf{C} \to \widehat{\mathsf{C}}$.

39 Representables and universal elements

Hom-functors have nice properties like preserving limits (see e.g. Theorem 151). Other functors which are naturally isomorphic to hom-functors will share these nice properties (see e.g. Theorem 154). It is an obvious next move, then, to think a little about this wider class of well-behaved functors and about their contravariant companions.

39.1 Representable functors

(a) We need to fix some (perhaps not entirely predictable) terminology.

Definition 141. Let $F : \mathsf{C} \to \mathsf{Set}$ be a (covariant) set-valued functor. If, for some C-object A there is a natural isomorphism $\psi\colon F \overset{\sim}{\Longrightarrow} \mathsf{C}(A, -)$, then (A, ψ) is said to be a *representation* of F, with A its *representing object*.

If F has a representation, i.e. is naturally isomorphic to a hom-functor, it is said to be *representable*.

Dually, let F now be a contravariant set-valued functor. If there is a natural isomorphism $\psi\colon F \overset{\sim}{\Longrightarrow} \mathsf{C}(-, A)$, then (A, ψ) is again said to be a representation of F, with A its representing object. And F is representable if it has a representation. △

It is immediate, by the way, that representing objects are unique up to isomorphism:

Theorem 187. *If $F\colon \mathsf{C} \to \mathsf{Set}$ is represented by both A and B, then $A \cong B$.*

Proof. If we have $\mathsf{C}(A, -) \cong F \cong \mathsf{C}(B, -)$ then, in the notation of Theorem 182, $\mathcal{X}A \cong \mathcal{X}B$ and hence $A \cong B$ by the Yoneda Principle. □

39.2 Two elementary examples

(a) Quite trivially, hom-functors themselves are representables. But are there other kinds of example?

First, a toy example. Take the trivial identity functor between Set and Set:

$$\begin{array}{rrcl} 1_{\mathsf{Set}}\colon & X & \longmapsto & X \\ & f\colon X \to Y & \longmapsto & f\colon X \to Y. \end{array}$$

Is this representable?

Well, if it is to be representable by a covariant hom-functor $\mathsf{Set}(S, –)$ for some some S, we would need a suite of isomorphism ψ_X, ψ_Y, ..., such that this square always commutes for any f:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\psi_X"] & Y \arrow[d, "\psi_Y"] \\ \mathsf{Set}(S, X) \arrow[r, "f \circ -"] & \mathsf{Set}(S, Y) \end{tikzcd}$$

But how can we ensure that the members of X are in bijection with the arrows from S to X? By taking S to be some singleton $\{\bullet\}$ (any will do). Then, if we put ψ_X to be the function which sends $x \in X$ to the corresponding arrow $\vec{x}\colon \{\bullet\} \to X$, and similarly for ψ_Y, etc., our square will trivially commute. So the ψ_X are components of a natural transformation $\psi\colon 1_{\mathsf{Set}} \overset{\sim}{\Longrightarrow} \mathsf{Set}(\{\bullet\}, –)$.

Which gives us the following mini-theorem:

Theorem 188. *The identity functor* $1_{\mathsf{Set}}\colon \mathsf{Set} \to \mathsf{Set}$ *is representable, and is represented by a singleton* $\{\bullet\}$.

(b) Let's next return to the very first functor we met back in §27.2, the forgetful functor $F\colon \mathsf{Mon} \to \mathsf{Set}$ which sends any monoid living in Mon to its underlying set, and sends a monoid homomorphism to the same function thought of as an arrow in Set.

Suppose $M = (\underline{M}, *, e)$ and $M' = (\underline{M}', \star, d)$, so underlining indicates the underlying set of the monoid, then in our shorthand:

$$F\colon \quad \begin{array}{rcl} M & \longmapsto & \underline{M} \\ f\colon M \to M' & \longmapsto & f\colon \underline{M} \to \underline{M}' \end{array}$$

And now let's ask: is there a representing object, i.e. a monoid R, such that the hom-functor $\mathsf{Mon}(R, –)$ is naturally isomorphic to the forgetful F?

Applying the usual definition, the hom-functor $\mathsf{Mon}(R, –)$ sends a monoid M to $\mathsf{Mon}(R, M)$. And it sends a monoid homomorphism $f\colon M \to M'$ to the set-function $f \circ –$ which sends an arrow $g\colon R \to M$ in $\mathsf{Mon}(R, M)$ to the arrow $f \circ g\colon R \to M'$ in $\mathsf{Mon}(R, M')$. And if this functor $\mathsf{Mon}(R, –)$ is to be naturally isomorphic with the forgetful functor F, there will have to be an isomorphism ψ with a component at each monoid M such that, for any $f\colon M \to M'$ in Mon, the following diagram commutes in Set:

$$\begin{tikzcd} \underline{M} \arrow[r, "f"] \arrow[d, "\psi_M"] & \underline{M}' \arrow[d, "\psi_{M'}"] \\ \mathsf{Mon}(R, M) \arrow[r, "f \circ -"] & \mathsf{Mon}(R, M') \end{tikzcd}$$

Now, for this to work, we certainly need to choose a representing monoid R such that (for any monoid M) there is a bijection between M and $\mathsf{Mon}(R, M)$. And presumably, for the needed generality, R will have to be a monoid without too much distinctive structure. That severely limits the possible options.

First shot: take the simplest such 'boring' monoid, the one-element monoid 1. But a moment's reflection shows that this can't work as a candidate for R (typically $\underline{M}$ has many members, $\mathsf{Mon}(1, M)$ can have only one, so there won't be an isomorphism between them).

Second shot: take the next simplest unstructured monoid, the free monoid with a single generator ('free' in the sense that there are no further conditions on the monoid other than it *is* a monoid and every element bar the identity is the result of applying the monoid operations as many times as we like to the generating element). We can usefully think of this monoid as $N = (\mathbb{N}, +, 0)$ whose generator is 1, and whose every element is a sum of 1s. Now consider a homomorphism from N to M. $0 \in \mathbb{N}$ has to be sent to the identity element e in M. And once we also fix where $1 \in \mathbb{N}$ gets sent to, namely some $m \in \underline{M}$, that determines where every element of $\mathbb{N}$ goes (since every non-zero $\mathbb{N}$ element $1+1+1+\ldots+1$ will be sent to a corresponding $\underline{M}$-element $m * m * m * \ldots * m$).

So consider $\psi_M \colon M \to \mathsf{Mon}(N, M)$ which maps m to the unique homomorphism $\overline{m} \colon N \to M$ which sends $1 \in \mathbb{N}$ to $m \in \underline{M}$. ψ_M is evidently bijective – each homomorphism from N to M is some $\overline{m}$ for one and only one m in M. Hence ψ_M is an isomorphism in Set.

And now it is easily seen that our diagram always commutes, for any homomorphism $f \colon M \to M'$. Chase an element $m \in \underline{M}$ round the diagram. The route via the north-east node gives us $m \longmapsto fm \longmapsto \overline{fm}$, the other route gives us $m \longmapsto \overline{m} \longmapsto f \circ \overline{m}$. But $f \circ \overline{m} = \overline{fm}$. Why? Because $f \circ \overline{m}$ is the composite map which takes e.g. $3 \in \mathbb{N}$ to the result of f applied to $m * m * m$, i.e. takes 3 to $fm \star fm \star fm$ – but that's what $\overline{fm}$ does.

Since the diagram always commutes, this means in turn that the maps ψ_M assemble into a natural isomorphism $\psi \colon F \xRightarrow{\sim} \mathsf{Mon}(N, -)$. Hence, in summary:

Theorem 189. *The forgetful functor $F \colon \mathsf{Mon} \to \mathsf{Set}$ is representable, and is represented by N, the free monoid on one generator.*

Being representable, so naturally isomorphic to a hom-functor, it follows that the forgetful F preserves limits (by Theorems 151 and 154). But we knew that already (from Theorem 140).

(c) These last two theorems are simple to understand; but what is not quite so easy to spot is their conceptual significance.

Take our second example. The forgetful functor $F \colon \mathsf{Mon} \to \mathsf{Set}$ may forget about the monoid operation on the underlying sets, but it remembers about the sets and about the functions underlying the homomorphisms. So we'd expect an isomorphic functor to similarly preserve information.

Now, as we put it before, we can think of a hom-functor $\mathsf{C}(A, -)$ as encapsulating A's view of the category it lives in. So a hom-functor $\mathsf{Mon}(R, -)$ isomorphic to the forgetful functor must specify a viewpoint R from which we can recover the same information about underlying sets and functions. Which is indeed the case when $R = N = (\mathbb{N}, +, 0)$.

For then $\mathsf{Mon}(N, -)$ applied to a monoid M gives us all the homomorphisms $N \to M$, so taking in turn the value of each homomorphism for 1 gives us back all the objects of $\underline{M}$. And $\mathsf{Mon}(N, -)$ applied to a homomorphism $f\colon M \to M'$ gives us gives us back a function $f \circ -$ which, for any M-element m, takes the homomorphism $\overline{m}\colon N \to M$ and returns $\overline{fm}\colon N \to M'$. But this function $\overline{m} \mapsto \overline{fm}$ induces a map $\overline{m}(1) \mapsto \overline{fm}(1)$ which is of course the set-function $f\colon \underline{M} \to \underline{M}'$.

So in short, we can think of N as giving us a sort of universal vantage point from which we can view the underlying sets and functions of the monoids in Mon.

39.3 More examples of representables

(a) Unsurprisingly, there are analogous representation theorems for other forgetful functors. For instance, although we won't pause over the proofs, we have:

Theorem 190. *(1) The forgetful functor $F\colon \mathsf{Grp} \to \mathsf{Set}$ is representable, and is represented by Z, the group of integers under addition.*

(2) The forgetful functor $F\colon \mathsf{Ab} \to \mathsf{Set}$ is representable, and is also represented by Z.

(3) The forgetful functor $F\colon \mathsf{Vect} \to \mathsf{Set}$ (where Vect is the category of vector spaces over the reals) is representable, and is represented by R, the reals treated as a vector-space.

(4) The forgetful functor $F\colon \mathsf{Top} \to \mathsf{Set}$ is representable, and is represented by the one-point topological space, call it S_0.

To comment on the only last of these, we simply note that a trivial continuous function with domain S_0 into a space S in effect picks out a single point of S, so the set of arrows $\mathsf{Top}(S_0, S)$ is indeed in bijective correspondence with the set of points FS.

(b) Given such examples, you might be tempted to conjecture that *all* such forgetful functors into Set are representable. But not so. Consider FinGrp, the category of finite groups. Then

Theorem 191. *The forgetful functor $F\colon \mathsf{FinGrp} \to \mathsf{Set}$ is not representable*

Proof. Suppose a putative representing group R has r members, and take any group G with $g > 1$ members, where g is coprime with r. Then it is well known that the only group homomorphism from R to G is the trivial one that sends everything to the identity in G. But then the underlying set of G can't be in bijective correspondence with $\mathsf{FinGrp}(R, G)$ as would be required for a naturality square proving that R represented F. $\square$

(c) Let's take another pair of examples. We first need to recall definitions from §27.5:

(i) The (covariant) *powerset functor* $P\colon \mathsf{Set} \to \mathsf{Set}$ maps a set X to its powerset $\mathcal{P}(X)$ and maps a set-function $f\colon X \to Y$ to the function which sends $U \in \mathcal{P}(X)$ to its image $f[U] \in \mathcal{P}(Y)$.

(ii) The *contravariant powerset functor* $\overline{P}\colon \mathsf{Set} \to \mathsf{Set}$ again maps a set to its powerset, and maps a set-function $f\colon Y \to X$ to the function which sends $U \in \mathcal{P}(X)$ to its inverse image $f^{-1}[U] \in \mathcal{P}(Y)$.

Theorem 192. *The contravariant powerset functor $\overline{P}$ is represented by the set $2 = \{0, 1\}$; but the covariant powerset functor P is not representable.*

Proof. As yet, we don't have any general principles about representables and non-representables which we can invoke to prove theorems such us this. So again we just need to labour through by applying definitions and seeing what we get.

If the contravariant functor $\overline{P}$ is to be representable, then there must be a representing set R and a natural isomorphism ψ with components such that, for all set functions $f\colon Y \to X$, the following diagram always commutes:

$$\begin{tikzcd}
\overline{P}X \arrow[r, "\overline{P}f"] \arrow[d, "\psi_X"] & \overline{P}Y \arrow[d, "\psi_Y"] \\
\mathsf{Set}(X, R) \arrow[r, "\mathsf{Set}(f, R)"] & \mathsf{Set}(Y, R)
\end{tikzcd}$$

Now $\mathsf{Set}(X, R)$ is the set of set-functions from X to R, whose cardinality is $|R|^{|X|}$; and the cardinality of $\overline{P}X$, i.e. $\mathcal{P}(X)$, is $2^{|X|}$. So that forces R to be a two-membered set: so we pick the set $2 = \{0, 1\}$.

$\mathsf{Set}(X, 2)$ is then the set of characteristic functions for subsets of X, i.e. the set of functions $c_U\colon X \to \{0, 1\}$ where $c_U(x) = 1$ iff $x \in U \subseteq X$. So the obvious next move is to take $\psi_X\colon \overline{P}X \to \mathsf{Set}(X, 2)$ to be the isomorphism that sends a set $U \in \mathcal{P}(X)$ to its characteristic function c_U.

With this choice, the diagram always commutes. Chase the element $U \in \overline{P}X$ around. The route via the north-east node takes us from $U \subseteq X$ to $f^{-1}[U] \subseteq Y$ to its characteristic function. i.e. the function which maps $y \in Y$ to 1 iff $f(y) \in U$. Meanwhile, the route via the south-west node takes us first from $U \subseteq X$ to c_U, and then we apply $\mathsf{Set}(f, 2)$, which maps $c_U\colon X \to 2$ to $c_U \circ f\colon Y \to 2$, which again is the function which maps $y \in Y$ to 1 iff $f(y) \in U$. Which establishes the first half of the theorem.

For the second half of the theorem, we just note that if we try to run a similar argument for the covariant functor P, we'd need to find a representing set R' such that PX and $\mathsf{Set}(R', X)$ are always in bijective correspondence. But $\mathsf{Set}(R', X)$ is the set of set-functions from R' to X, whose cardinality is $|X|^{|R'|}$, while the cardinality of PX is $2^{|X|}$. And there is no choice of R' which will make these equal for varying X. $\square$

39.4 Universal elements

(a) To repeat, we say that pair (A, ψ) is a representation of the covariant functor $F\colon \mathsf{C} \to \mathsf{Set}$ just if $\psi\colon F \xRightarrow{\sim} \mathsf{C}(A, -)$. But in talking about a natural isomorphism between a functor into Set and a hom-functor we are of course back in Yoneda territory.

In proving the Core Yoneda Lemma, Theorem 183, we showed that there is a bijection $\mathcal{X}_{AF}$ between the members of FA and the members of $Nat(\mathsf{C}(A, -), F)$. This bijection matches up any $a \in FA$ with $\alpha^a = \mathcal{X}_{AF}(a)\colon \mathsf{C}(A, -) \xRightarrow{\sim} F$.[1] And this is the natural transformation whose X-component α^a_X sends a map $g\colon A \to X$ to $Fg(a)$.

And when is this α^a in fact a natural isomorphism? When each α^a_X is a bijection. Which means that for each $x \in FX$ there must be a unique $g\colon A \to X$ such that $Fg(a) = x$.

Which motivates introducing the following concept (though of course it doesn't yet explain the label for the concept):

Definition 142. A *universal element* of the functor $F\colon \mathsf{C} \to \mathsf{Set}$ is a pair (A, a), where $A \in \mathsf{C}$ and $a \in FA$, and where for each $X \in \mathsf{C}$ and $x \in FX$, there is a unique map $g\colon A \to X$ such that $Fg(a) = x$. △

And we now have all the ingredients in place for the following result (check this!):

Theorem 193. *A functor $F\colon \mathsf{C} \to \mathsf{Set}$ is representable by the object A iff it has a universal element (A, a).* □

The story for contravariant functors, by the way, will be exactly the same, except that the map g will go the other way about, $g\colon X \to A$.

(b) Why '*universal* element'? Because the definition invokes a universal mapping property: (A, a) is a universal element iff for every ... there is a unique map such that ....

Now, recall the case of products which we defined by a universal mapping property; in §10.3 we then showed products to be terminal objects in a category of wedges. Dually, of course, co-products are also defined by a universal mapping property; this case they are initial objects in a category of corners. Similarly in other cases, we showed that constructions defined by universal mapping properties could be identified (up to isomorphism) as the initial or terminal objects in some appropriate categories. We can play the same game here. So first let's define a suitable category:

Definition 143. $\mathsf{Elts}_\mathsf{C}(F)$, the *category of elements of the functor* $F\colon \mathsf{C} \to \mathsf{Set}$, has the following data:

(1) Objects are the pairs (A, a), where $A \in \mathsf{C}$ and $a \in FA$.

[1] I should signal a trivial change: to link up with other notation in the earlier context, we then labeled our selected element of FA 'f'. I now choose 'a' to conform with common notation in the current context.

(2) An arrow from (A, a) to (B, b) is a C-arrow $f\colon A \to B$ such that $Ff(a) = b$.
(3) The identity arrow on (A, a) is 1_A.
(4) Composition of arrows is induced by composition of C-arrows.

It is easily checked that this *is* a category. (Alternative symbolism for the category includes variations on '$\int_{\mathsf{C}} F$'.)

But why 'category of *elements*'? After all, functors don't in a straightforward sense have elements. But we can perhaps throw some light on the name as follows.

(i) Suppose first that we are given a category C whose objects *are* sets (perhaps with some additional structure on them) and whose arrows are functions between sets. Then we might be interested in a derived category which digs inside the sets which are C's objects, and looks at their elements.

Now perhaps we don't want the derived category to forget about which sets in C have which elements as members. Then a natural way to go would be to say that the objects of the derived category are all the pairs (A, a) for $A \in \mathsf{C}$, $a \in A$. And then given elements $a \in A$, $b \in B$, whenever there is a C-arrow $f\colon A \to B$ such that $f(a) = b$, we'll say that f is also an arrow from (A, a) to (B, b) in our new category. In a sense, this derived 'category of elements' unpacks what's going on inside the original category C.

(ii) However, in the general case, C's objects need not be sets so need not have elements. But a functor $F\colon \mathsf{C} \to \mathsf{Set}$ gives us a diagram of C inside Set, and of course the objects in the resulting diagram of C *do* have elements. So we can consider the category of elements of F's-diagram-of-C, which – following the template in (i) – has as objects all the pairs (FA, a) for $A \in \mathsf{C}$, $a \in FA$. And then given elements $a \in FA$, $b \in FB$, whenever there is a Set-arrow $Ff\colon FA \to FB$ such that $Ff(a) = b$, we'll say that Ff is also an arrow from (FA, a) to (FB, b) in our new category.

Now, we can streamline that. Instead of taking the objects to be pairs (FA, a) take them simply to be pairs (A, a) (but where, still, $a \in FA$). And instead of talking of the arrow $Ff\colon FA \to FB$ we can instead talk more simply of $f\colon A \to B$ (but where, still, $Ff(a) = b$). And with that streamlining – lo and behold! – we are back with the category $\mathsf{Elts}_{\mathsf{C}}(F)$, which is isomorphic to category of elements of F's-diagram-of-C, and which – as convention has it – we'll call the category of elements of F, for short.

So the construction of $\mathsf{Elts}_{\mathsf{C}}(F)$ is tolerably natural.

(c) Here is another way of thinking of this category. Let 1 be some singleton in Set; and recycling notation in the usual kind of way let 1 be the trivial functor from the one-object category $\mathsf{1}$ to Set which sends the sole object of $\mathsf{1}$ to 1. Then what is the comma category $(1 \downarrow F)$ where as before $F\colon \mathsf{C} \to \mathsf{Set}$? Applying the account of such comma categories given in §30.2, the objects of this category are pairs (A, a) where $A \in \mathsf{C}$ and $a\colon 1 \to FA$ is an arrow in Set. And the arrows of the category from (A, a) to (B, b) is a C-arrow $f\colon A \to B$ such that $b = Ff \circ a$.

But *that* is just the definition of $\mathsf{Elts_C}(F)$ except that we have traded in the requirement that a is *member* of FA for the requirement that a is an *arrow* $1 \to FA$. But as we well know by now, members of a set are in bijective correspondence with such arrows from a fixed singleton, and from a categorial perspective we can treat members as such arrows (hence our using the same label 'a' here for both). Therefore:

Theorem 194. *For a given functor $F\colon \mathsf{C} \to \mathsf{Set}$, the category $\mathsf{Elts_C}(F)$ is (isomorphic to) the comma category $(1 \downarrow F)$ where 1 is terminal in Set.*

(d) Having defined a category $\mathsf{Elts_C}(F)$ for elements of $F\colon \mathsf{C} \to \mathsf{Set}$ to live in, we can finish by asking: how do we distinguish universal elements from other elements categorially? The answer is immediate from Defn. 142, which in our new terminology says:

Theorem 195. *An object $I = (A, a)$ in $\mathsf{Elts_C}(F)$ is a universal element iff, for every object E in $\mathsf{Elts_C}(F)$ there is exactly one morphism $f\colon I \to E$, so I is initial in $\mathsf{Elts_C}(F)$.*

But initial objects are unique up to unique isomorphism. Which, recalling what isomorphisms in $\mathsf{Elts_C}(F)$ are, implies

Theorem 196. *If (A, a) and (A', a') are universal elements for $F\colon \mathsf{C} \to \mathsf{Set}$, then there is a unique C-isomorphism $f\colon A \to A'$ such that $Ff(a) = a'$.*

39.5 Limits and exponentials as universal elements

Let's finish the chapter making a couple of rather satisfying connections between categorial gadgets.

(a) Let $Cone(C, D)$ be the set of cones over some diagram D with vertex C in some given category C – and we will assume that C is small enough for $Cone(C, D)$ indeed to be a set living in Set.

We can now define a contravariant functor $Cone(–, D)\colon \mathsf{C} \to \mathsf{Set}$ as follows.

(i) $Cone(–, D)$ sends an object C to $Cone(C, D)$.

(ii) $Cone(–, D)$ sends an arrow $f\colon C' \to C$ to $Cone(f, D)\colon Cone(C, D) \to Cone(C', D)$, which takes a cone (C, c_j) and sends it to the cone $(C', c_j \circ f)$.

It is easily checked that this is indeed a functor.

We now apply the definition of universal elements, tweaked for the contravariant case (so universal elements are terminal in the relevant category of elements). Then a universal element of the functor $Cone(–, D)$ is a pair $(L, (L, l_J))$, where L is in C and (L, l_J) is in $Cone(L, D)$, the set of cones over D with vertex L. And moreover, we require that for each $C \in \mathsf{C}$ and each cone (C, c_J), there is a unique map $f\colon C \to L$ such that $Cone(f)(L, l_J) = (C, c_J)$, which requires $l_J \circ f = c_J$ for each J. But that's just to say that (L, l_J) is a limit cone! Hence:

Theorem 197. *In small enough categories, a limit cone over a diagram D is a universal element for $Cone(-, D)$.*

(b) Consider now the contravariant functor $\mathsf{C}(- \times B, C)$ which we met in §32.4 Ex. (5). This sends an object A in C to the hom-set of arrows from $A \times B$ to C. And it sends an arrow $f\colon A' \to A$ to the map $- \circ f \times 1_B$ (i.e. to the map which takes an arrow $j\colon A \times B \to C$ and yields the arrow $j \circ f \times 1_B\colon A' \times B \to C$).

Now apply the definition of universal element for the contravariant case. Then a universal element of $\mathsf{C}(- \times B, C)$ is a pair (E, ev), with E in C and ev in $\mathsf{C}(E \times B, C)$, such that for every A and every $g \in \mathsf{C}(A \times B, C)$, there is a unique $\overline{g}\colon A \to E$ such that $\mathsf{C}(- \times B, C)(\overline{g})(ev) = g$, i.e. $ev \circ \overline{g} \times 1_B = g$.

But a pair (E, ev) with those properties is exactly the exponential (C^B, ev). Hence

Theorem 198. *The exponential (C^B, ev), when it exists in C, is a universal element of $\mathsf{C}(- \times B, C)$.*

Since exponentials are therefore also terminal objects in an associated category of elements, they too have to be unique up to a unique appropriate isomorphism, giving us this time another proof of Theorem 65.

40 Galois connections

We will have more to say about functors, limits and representables and about how they interrelate after we have introduced the last really important Big Idea from category theory that we will meet in these Notes – namely, the idea of pairs of *adjoint functors* and of the *adjunctions* they form.

Now, we could just to dive straight into talk about adjoints. But that multi-faceted story can initially seem rather complex, and it is easy to get lost in the details. So the plan here is to start by looking first at a very restricted class of cases. These are the so-called Galois connections, which are in effect adjunctions between two categories which are posets. In this chapter, then, we warm up by first discussing Galois connections in an elementary, precategorial, way.

40.1 Posets: some probably unnecessary reminders

Just for the record, and to fix local notation:

Definition 144. A poset consists of a set C equipped with a partial order $\preccurlyeq$ – i.e., for all $x, y, z \in C$, (i) $x \preccurlyeq x$, (ii) if $x \preccurlyeq y$ and $y \preccurlyeq z$ then $x \preccurlyeq z$, (iii) if $x \preccurlyeq y$ and $y \preccurlyeq x$ then $x = y$. (We will, as appropriate, recruit '$\sqsubseteq$', '$\leqslant$', '$\subseteq$' as other symbols for partial orders.)

If $(C, \preccurlyeq)$ is a poset and $X \subseteq C$, then m is a *maximum* of X (with respect to the inherited order $\preccurlyeq$) iff $m \in X$ and $(\forall x \in X)\, x \preccurlyeq m$.

Suppose $C = (C, \preccurlyeq)$ and $D = (D, \sqsubseteq)$ are two posets, and the map $F\colon C \to D$ is a function between the sets C and D. Then

(1) F is *monotone* iff, for all $x, y \in C$, if $x \preccurlyeq y$ then $Fx \sqsubseteq Fy$;

(2) F is an *order-embedding* iff, for all $x, y \in C$, $x \preccurlyeq y$ just in case $Fx \sqsubseteq Fy$;

(3) F is an *order-isomorphism* iff F is a surjective order-embedding.

Partially ordered collections are deemed isomorphic when there is an order-isomorphism between them. △

Two quick comments about this:

(i) Talk of posets here is conventional. We could use plurals and non-committal, eliminable, talk of collections instead (see again §3.1). And indeed sometimes we fall into talking of posets in cases where the ordered objects in question may be too many to form a set on standard stories. But to avoid irrelevant distractions, I'll here stick to the conventional idiom (and not worry unduly about issues of 'size').

(ii) There is of course a related notion of a strict poset defined in terms of a strict partial order $\prec$, where $x \prec y$ iff $x \preccurlyeq y \wedge x \neq y$ for some partial order $\preccurlyeq$. It is just a matter of convenience whether we concentrate on the one flavour of poset or the other, and you will already be familiar with a variety of examples of 'naturally occurring' posets of both flavours.

And now a very elementary composite theorem:

Theorem 199. *(1) Maxima are unique when they exist.*

(2) Order-embeddings are injective.

(3) An order-isomorphism is bijective, and have unique inverses which are also order-isomorphisms.

(4) Monotone maps compose to give monotone maps and composition is associative. Likewise for order-embeddings and order-isomorphisms.

(5) If $F[C]$ is the image of C under F, an order-embedding $F\colon (C, \preccurlyeq) \to (D, \sqsubseteq)$ is an order-isomorphism from $(C, \preccurlyeq)$ to $(F[C], \sqsubseteq)$.

(6) Any partially ordered collection is order-isomorphic to an inclusion poset, i.e. a collection of sets ordered by inclusion.

Proof. For (1) we note that if $m, m' \in X$ are both maxima, $m' \preccurlyeq m$ and similarly $m \preccurlyeq m'$ and hence $m = m'$.

For (2) we suppose $Fx = Fy$ and hence both $Fx \sqsubseteq Fy$ and $Fy \sqsubseteq Fx$, and then note that if F is an embedding, $x \preccurlyeq y$ and $y \preccurlyeq x$, and hence $x = y$.

(3) to (5) are immediate. For (6) take $(C, \preccurlyeq)$, and for each $y \in C$, form the set π_y containing it and its $\preccurlyeq$-predecessors, so $\pi_y = \{x \in C \mid x \preccurlyeq y\}$. Let Π be set of π_y for $y \in C$. Define $F\colon (C, \preccurlyeq) \to (\Pi, \subseteq)$ by putting $Fx = \pi_x$. Then F is very easily seen to be a bijection, and also $x \preccurlyeq y$ iff $\pi_x \subseteq \pi_y$. So F is an order-isomorphism.[1] □

40.2 A first example of a Galois connection

We rather informally describe what will turn out to be an important instance of a Galois connection: and let's choose notation with an eye to smoothing the transitions to later generalizations.

Let C be the set whose members are the various sets of sentences that can be formed some suitable formal language L (the details of L won't matter too much); and let $\preccurlyeq$ simply be set-inclusion. So we can think of C as collecting together *theories* couched in the language L, with these theories then partially ordered from less specific (saying less) to more specific (saying more).

[1] Category theorist's joke: prove (6) using Yoneda.

Now let D be the set whose members are all the various sets of L-structures, so each $d \in D$ is a set of potential models for theories couched in L;[2] and this time we will take $\sqsubseteq$ to be the *converse* of inclusion. So the sets of potential models also partially ordered from less specific (more alternatives) to more specific (a narrower range).

There are then two very natural maps between the resulting 'syntactic' and 'semantic' posets:

i. $F\colon (C, \preccurlyeq) \to (D, \sqsubseteq)$ sends a theory c from C to d among D, where d is the set of models of c (i.e. d is the set containing each model on which all the sentences in c are true).

ii. $G\colon (D, \sqsubseteq) \to (C, \preccurlyeq)$ sends a set of models d to the set c containing the sentences which are true on every model in d.

Put it this way: F is the 'find the models' function. It takes a bunch of L-sentences and returns all its models, the set of L-structures where the sentences in the bunch are all true. In the other direction, G is the equally natural 'find the agreed truths' function. It takes a bunch of L-structures and returns the set of L-sentences that are true across all of those structures.

In general F and G will not be inverse to each other. But the mapping functions do interrelate in the following nice ways:

(1) F and G are monotone.

And for all $c \in C$, $d \in D$,

(2) $c \preccurlyeq GFc$ and $FGd \sqsubseteq d$,

(3) $Fc \sqsubseteq d$ iff $c \preccurlyeq Gd$.

And further

(4) $FGF = F$ and $GFG = G$.

Why so? For (1) we note that if the theory c' is more informative than c, then it will be true of a narrower range of possible models. And conversely, if d' is a narrower range of models than d, then more sentences will be true of everything in d' than are true of everything in d.

For the first half of (2) we note that if we start with a bunch of sentences c, look at the models where they are all true together, and then look at the sentences true in all those models together, we'll get back original sentences in c plus all their consequences (where consequence is defined in the obvious way in terms of preservation of truth with respect to the relevant structures).

For the other half of (2) we note that if we start from a collection of models d, find the sentences true in all of them, and then look at the models for those sentences, we must get back at least the models we started with, maybe more. (Remember, $\sqsubseteq$ is the converse of inclusion!)

[2]To sidestep issues of size, take it that L-structures all live in some big-enough set.

For (3) we note that if the models where all the sentences of c are true include all those in d then the theory c must be included in the set of sentences true in all the models in d, and vice versa.

For the first half of (4) we note that the models of a set of sentences c together with their consequences are just the models of the original c. Similarly for the other half.

So in summary: we have here a pair of posets $(C, \preccurlyeq)$, $(D, \sqsubseteq)$ and a pair of functions $F\colon C \to D$ and $G\colon D \to C$ for which conditions (1) to (4) hold. And this pair of functions F and G is then our first example of a Galois connection. In a famous Dialectica paper 'Adjointness in foundations' (1969), F. William Lawvere writes of "the familiar Galois connection between sets of axioms and classes of models, for a fixed [signature]". A set of axioms in the wide sense that Lawvere is using is just any old set of sentences from the right signature. So we've explained what Lawvere was referring to.

40.3 Galois connections defined

We now generalize. However, as we'll see in the next section, conditions (1) to (4) are not independent. The first two together imply the third and fourth, and the third by itself implies the rest. Simply because it is prettier, then, we plump in this section for a general definition just in terms of the third condition (which we relabel):

Definition 145. Suppose that $(C, \preccurlyeq)$ and $(D, \sqsubseteq)$ are two posets, and let $F\colon C \to D$ and $G\colon D \to C$ be a pair of functions such that

$$\text{(Gal)} \qquad Fc \sqsubseteq d \text{ iff } c \preccurlyeq Gd \text{ (for all } c \in C,\ d \in D).$$

Then F and G form a *Galois connection* between C and D. When this holds, we write $F \dashv G$, and F is said to be the *left adjoint* of G, and G the *right adjoint* of F.[3] △

Note: F counts as the *left* adjoint not because it here happens to be written on the left of the (symmetric!) biconditional, but because it is on the left of one of the order signs.

The first discussion of a version of such a connection $F \dashv G$ – and hence the name – is to be found in Evariste Galois's work in what has come to be known as Galois theory, a topic beyond our purview here. And there are plenty of other serious mathematical examples (e.g. from number theory, abstract algebra and topology) of two posets with a Galois connection between them. But we really don't want to get bogged down in unnecessary mathematics at this early stage; so for the moment let's just give some simple cases, to add to our informally described motivating example in the last section:

[3]Talk of adjoints here seems to have been originally borrowed from the old theory of Hermitian operators, where in e.g. a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ the operators A and A^* are said to be adjoint when we have, generally, $\langle Ax, y\rangle = \langle x, A^*y\rangle$. The formal analogy is evident.

(1) Suppose F is an order-isomorphism between $(C, \preccurlyeq)$ and $(D, \sqsubseteq)$: then F^{-1} is an order-isomorphism in the reverse direction. Take $c \in C, d \in D$: then trivially $Fc \sqsubseteq d$ iff $F^{-1}Fc \preccurlyeq F^{-1}d$ iff $c \preccurlyeq F^{-1}d$. Hence $F \dashv F^{-1}$.

(2) Take the posets $(\mathbb{N}, \leqslant)$ and $(\mathbb{Q}^+, \leqslant)$ comprising the naturals and the non-negative rationals in their standard orders. Let $I \colon \mathbb{N} \to \mathbb{Q}^+$ be the injection function which maps a natural number to the corresponding rational integer, and let $F \colon \mathbb{Q}^+ \to \mathbb{N}$ be the 'floor' function which maps a rational to the natural corresponding to its integral part. Then $I \dashv F$ is a Galois connection, with I the left adjoint. Likewise if $C \colon \mathbb{Q}^+ \to \mathbb{N}$ is the 'ceiling' function which maps a rational to the smallest integer which is at least as big, then $C \dashv I$ is a Galois connection with I the right adjoint.

(3) Let $f \colon X \to Y$ be some function between two sets X and Y. It induces a function $F \colon \mathcal{P}(X) \to \mathcal{P}(Y)$ between their powersets which sends $A \subseteq X$ to $f[A]$, and another function $F^{-1} \colon \mathcal{P}(Y) \to \mathcal{P}(X)$ which sends $B \subseteq Y$ to its pre-image under f, $F^{-1}[B] = \{x \in X \mid f(x) \in B\}$. Then $F \dashv F^{-1}$ is a Galois connection between the inclusion posets $(\mathcal{P}(X), \subseteq)$ and $(\mathcal{P}(Y), \subseteq)$.

(4) Take any poset $(C, \preccurlyeq)$, and let 1 be a one object poset such as $(\{0\}, =)$. Let $F \colon (C, \preccurlyeq) \to 1$ be the only possible function, the trivial one which sends everything to 0. Then F has a right adjoint $G \colon 1 \to (C, \preccurlyeq)$ just if it is the case that, for any $A \in C$, $FA = 0$ iff $A \preccurlyeq G0$. So F has a right adjoint just in case C has a maximum, and then G sends 1's only element to it. Dually, F has a left adjoint $E \colon \colon 1 \to (C, \preccurlyeq)$ just in case $(C, \preccurlyeq)$ has a minimum, and then the left adjoint E sends 1's only element to *that*.

(5) Let's have a (very!) simple result from topology. Recall, a topological space is standardly treated as a set X equipped with a topology, a set of 'open' sets $\mathcal{O}$ – where that is a family of subsets of X including $\emptyset$ and X and closed under unions and finite intersections. Given a space $(X, \mathcal{O})$, we can now consider two related posets, $(\mathcal{P}X, \subseteq)$ and $(\mathcal{O}, \subseteq)$.

By definition $\mathcal{O} \subseteq \mathcal{P}X$, and the inclusion function $F \colon \mathcal{O} \to \mathcal{P}X$ which sends $A \in \mathcal{O}$ to the same set as a member of $\mathcal{P}X$ is trivially monotone.

And now consider $G \colon \mathcal{P}X \to \mathcal{O}$ which sends a set $B \subseteq X$ to its (topological) interior, i.e. to the largest member of $\mathcal{O}$ which is a subset of B – the union of all the $O \in \mathcal{O}$ such that $O \subseteq B$, a union which will itself be in $\mathcal{O}$.

Then it is immediate that $F \dashv G$ – because $FA \subseteq B$ if and only if $A \subseteq GB$.

(6) For our next example we return to elementary logic. Choose a favourite logical proof-system S – it could be classical or intuitionistic, or indeed any other logic, so long as it has a normally-behaved conjunction and conditional connectives and a sensible deducibility relation. Let $\alpha \vdash \beta$ notate, as usual, that there is a formal S-proof from premiss α to conclusion β. Then let $|\alpha|$ be the equivalence class of wffs of the system interderivable with α. Take E to be set of all such equivalence classes, and put $|\alpha| \preccurlyeq |\beta|$ in E iff $\alpha \vdash \beta$. Then it is easily checked that $(E, \preccurlyeq)$ is a poset.

Now consider the following two functions between $(E, \preccurlyeq)$ and itself. Fix γ to be some S-wff. Then let F send the equivalence class $|\alpha|$ to the class $|(\gamma \wedge \alpha)|$, and let G send $|\alpha|$ to the class $|(\gamma \to \alpha)|$.

Given our normality assumption, $\gamma \wedge \alpha \vdash \beta$ if and only if $\alpha \vdash \gamma \to \beta$. Hence $|\gamma \wedge \alpha| \preccurlyeq |\beta|$ iff $|\alpha| \preccurlyeq |\gamma \to \beta|$. That is to say $F|\alpha| \preccurlyeq |\beta|$ iff $|\alpha| \preccurlyeq G|\beta|$. Hence we have a Galois connection $F \dashv G$ between $(E, \preccurlyeq)$ and itself, and in a slogan, 'Conjunction is left adjoint to conditionalization'.

(7) Our last example for the moment is another example from elementary logic. Let S now be a first-order logic, and consider the set of S-wffs with at most the variables $\vec{x}$ free.

We will write $\varphi(\vec{x})$ for a formula in this class, $|\varphi(\vec{x})|$ for the class of formulae interderivable with $\varphi(\vec{x})$, and $E_{\vec{x}}$ for the set of such equivalence classes of formulae with at most $\vec{x}$ free. Using $\preccurlyeq$ as in the last example, $(E_{\vec{x}}, \preccurlyeq)$ is a poset for any choice of variables $\vec{x}$.

We now consider two maps between the posets $(E_{\vec{x}}, \preccurlyeq)$ and $(E_{\vec{x},y}, \preccurlyeq)$. In other words, we are going to be moving between (equivalence classes of) formulae with at most $\vec{x}$ free, and (equivalence classes of) formulae with at most $\vec{x}, y$ free – where y is a new variable not among the $\vec{x}$.

First, since every wff with at most the variables $\vec{x}$ free also has at most the variables $\vec{x}, y$ free, there is a trivial map $F\colon E_{\vec{x}} \to E_{\vec{x},y}$ that sends the class of formulas $|\varphi(\vec{x})|$ in $E_{\vec{x}}$ to the same class of formulas which is also in $E_{\vec{x},y}$.

Second, we define the companion map $G\colon E_{\vec{x},y} \to E_{\vec{x}}$ that sends $|\varphi(\vec{x}, y)|$ in $E_{\vec{x},y}$ to $|\forall y \varphi(\vec{x}, y)|$ in $E_{\vec{x}}$.

Then $F \dashv G$, i.e. we have another Galois connection. For that is just to say

$$F(|\varphi(\vec{x})|) \preccurlyeq |\psi(\vec{x}, y)| \quad \text{iff} \quad |\varphi(\vec{x})| \preccurlyeq G(|\psi(\vec{x}, y)|).$$

Which just reflects the familiar logical rule that

$$\varphi(\vec{x}) \vdash \psi(\vec{x}, y) \text{ iff } \varphi(\vec{x}) \vdash \forall y \psi(\vec{x}, y),$$

so long as y is not free in $\varphi(\vec{x})$. Hence universal quantification is right-adjoint to a certain trivial inclusion operation.

And we can exactly similarly show that existential quantification is left-adjoint to the same operation.

Let's list some morals!

(i) Our first example shows that Galois connections are at least as plentiful as order-isomorphisms: and such an isomorphism will have a right adjoint and left adjoint which are the same (i.e. both are the isomorphism's inverse).

(ii) The second and fourth cases show that posets that aren't order-isomorphic can in fact still be Galois connected.

(iii) The third case shows that posets can have many Galois connections between them (as any $f\colon X \to Y$ generates a connection between the inclusion posets on the powersets of X and Y).

(iv) The fourth example gives a case where a function has both a left and a right adjoint which are different.

(v) The fourth and seventh cases give a couple of illustrations of how a significant construction (taking maxima, forming a universal quantification respectively) can be regarded as adjoint to some quite trivial operation.

(vi) The sixth example, like the third, shows that even when the relevant posets are isomorphic (in the sixth case trivially so, because they are identical!), there can be a pair of functions which aren't isomorphisms but which also go to make up a Galois connection.

(vii) And the last two examples, like the motivating example in the previous section, illustrate why Galois connections are of interest to logicians.

40.4 Galois connections re-defined

The following theorem is basic:

Theorem 200. *Suppose that $C = (C, \preccurlyeq)$ and $D = (D, \sqsubseteq)$ are posets with maps $F\colon C \to D$ and $G\colon D \to C$ between them. Then $F \dashv G$ if and only if*

(1) F and G are both monotone, and

(2) for all $c \in C$, $d \in D$, $c \preccurlyeq GFc$ and $FGd \sqsubseteq d$, and

(3) $FGF = F$ and $GFG = G$.

Proof. (*If*) Assume conditions (1) and (2) both hold. And suppose $Fc \sqsubseteq d$. Since by (1) G is monotone, $GFc \preccurlyeq Gd$. But by (2) $c \preccurlyeq GFc$. Hence by transitivity $c \preccurlyeq Gd$. That establishes one half of (Gal). We don't need (3) here. The proof of the other half is dual.

(*Only if*) Suppose (Gal) is true. Then in particular, $Fc \sqsubseteq Fc$ iff $c \preccurlyeq GFc$. Since $\sqsubseteq$ is reflexive, $c \preccurlyeq GFc$. Similarly for the other half of (2).

Now, suppose also that $c \preccurlyeq c'$. Then since we've just shown $c' \preccurlyeq GFc'$, we have $c \preccurlyeq GFc'$. But by (Gal) we have $Fc \sqsubseteq Fc'$ iff $c \preccurlyeq GFc'$. Whence, $Fc \sqsubseteq Fc'$ and F is monotone. Similarly for the other half of (1).

For (3), since for any $c \in C$, $c \preccurlyeq GFc$, and also F is monotone, it follows that $Fc \sqsubseteq FGFc$.

But the fundamental condition (Gal) yields $FGFc \sqsubseteq Fc$ iff $GFc \preccurlyeq GFc$. The r.h.s. is trivially true, so $FGFc \sqsubseteq Fc$.

By the antisymmetry of $\sqsubseteq$, then, $FGFc = Fc$. Since c was arbitrary, $FGF = F$. Similarly for the other half of (3). $\square$

This theorem means that, as already intimated at the end of §40.2, we could equally well have defined a Galois connection like this:

Definition 146 (Alternative). Suppose that $(C, \preccurlyeq)$ and $(D, \sqsubseteq)$ are two posets, and let $F\colon C \to D$ and $G\colon D \to C$ be a pair of functions such that for all $c \in C$, $d \in D$,

(1) F and G are both monotone,

(2) $c \preccurlyeq GFc$ and $FGd \sqsubseteq d$ (for all $c \in C$, $d \in D$),

(3) $FGF = F$ and $GFG = G$.

Then F and G form a Galois connection between C and D. △

Two comments about this. First, our proof of Theorem 200 shows that we needn't have explicitly given clause (3) in our alternative definition as it follows from the other two. We include it because when we move on from the case of Galois connections to discuss adjunctions more generally, again giving two definitions, we will need to explicitly mention the analogue of clause (3).

Second, note that we could replace clause (2) with the equivalent clause

(2′) (i) if $c \preccurlyeq c'$, then both $c \preccurlyeq c' \preccurlyeq GFc'$ and $c \preccurlyeq GFc \preccurlyeq GFc'$; and
(ii) if $d \sqsubseteq d'$, then both $FGd \sqsubseteq d \sqsubseteq d'$ and $FGd \sqsubseteq FGd' \sqsubseteq d'$.

For trivially (2′) implies (2); conversely (1) and (2) imply (2′). We mention this variant on our alternative definition of Galois connections for later use.

40.5 Some basic results about Galois connections

(a) We now have a pair of equivalent definitions of Galois connections, and a small range of elementary examples. In this section we start by proving a couple of theorems that show that such connections behave just as you would hope, in two different respects. First, we show that inside a Galois connection, a left adjoint uniquely fixes its right adjoint, and vice versa:

Theorem 201. *If we have Galois connections $F \dashv G$, $F \dashv G'$ between the posets $(C, \preccurlyeq)$ and $(D, \sqsubseteq)$, then $G = G'$. Likewise, if $F \dashv G$, $F' \dashv G$ are both Galois connections between the same posets, then $F = F'$.*

Proof. We prove the first part. $F \dashv G'$ implies, in particular, that for any $d \in D$, $FGd \sqsubseteq d$ iff $Gd \preccurlyeq G'd$.

But by Theorem 200, applied to the connection $F \dashv G$, we have $FGd \sqsubseteq d$. So we can infer that, indeed, $Gd \preccurlyeq G'd$.

By symmetry, $G'd \preccurlyeq Gd$. But d was arbitrary, so indeed $G = G'$. □

Careful! This theorem does not say that, for any $F\colon C \to D$ there must exist a unique corresponding $G\colon D \to C$ such that $F \dashv G$ (this isn't true as we saw in §40.3 Ex. (4)). Nor does it say that when there is a Galois connection between two posets, it is unique (our toy examples have shown that that is false too). The claim is only that, if you are given a possible left adjoint – or a possible right adjoint – there can be at most one candidate for its companion to complete a connection.

Second, we show that Galois connections combine in the nice way you might hope for:

Theorem 202. *Suppose there is a Galois connection $F \dashv G$ between the posets $(C, \preccurlyeq)$ and $(D, \sqsubseteq)$, and a connection $H \dashv K$ between the posets $(D, \sqsubseteq)$ and $(E, \subseteq)$. Then there is a Galois connection $HF \dashv GK$ between $(C, \preccurlyeq)$ and $(E, \subseteq)$.*

Proof. Take any for any $c \in C, e \in E$. Then, using the first connection, we have $Fc \sqsubseteq Ke$ iff $c \preccurlyeq GKe$. And by the second connection, we have $HFc \subseteq e$ iff $Fc \sqsubseteq Ke$.

Hence $HFc \subseteq e$ iff $c \preccurlyeq GKe$. Therefore $HF \dashv GK$. □

(b) Given that adjoint functions determine each other, we naturally seek an explicit definition of one in terms of the other. Here it is:

Theorem 203. *If $F \dashv G$ is a Galois connection between the posets $(C, \preccurlyeq)$ and $(D, \sqsubseteq)$, then*

(1) Gd = the $\preccurlyeq$maximum of $\{c \in C \mid Fc \sqsubseteq d\}$,

(2) Fc = the $\sqsubseteq$-minimum of $\{d \in D \mid c \preccurlyeq Gd\}$.

Proof. We argue for (1), leaving the dual (2) to take care of itself. Fix on an arbitrary $d \in D$ and for brevity, put $\Sigma = \{c \in C \mid Fc \sqsubseteq d\}$.

Theorem 200 tells us that (i) for any $u \in C$, $u \preccurlyeq GFu$, (ii) $FGd \sqsubseteq d$, and (iii) G is monotone. So by (ii), $Gd \in \Sigma$.

Now suppose $u \in \Sigma \subseteq C$. Then $Fu \sqsubseteq d$. By (iii), $GFu \preccurlyeq Gd$. Whence from (i), $u \preccurlyeq Gd$.

That shows Gd is both a member of and an upper bound for Σ, i.e. is a maximum for Σ. □

Recall the posets $(\mathbb{N}, \leqslant)$ and $(\mathbb{Q}^+, \leqslant)$ with the injection map $I \colon \mathbb{N} \to \mathbb{Q}^+$ and floor function $F \colon \mathbb{Q}^+ \to N$ which maps a rational to the natural corresponding to its integral part. Then we remarked before that $I \dashv F$. Now we note that $F \dashv I$ is false. Indeed, there can be no connection of the form $F \dashv G$ from Q^+ to N. For $Fq = 1$ iff $1 \leqslant q < 2$, and hence $\{q \in \mathbb{Q}^+ \mid Fq \leqslant 1\}$ has no maximum, and so there can be no right adjoint to F.

Generalizing, we should highlight the following:

Theorem 204. *Galois connections are not necessarily symmetric. That is to say, given $F \dashv G$ is a Galois connection between the posets C and D, it does not follow that $G \dashv F$ is a connection between D and C.*

40.6 Fixed points, isomorphisms, and closures

This section now explores more of the consequences of there being a Galois connection $F \dashv G$ between two posets.

(a) Theorem 200 tells us, in particular, where to find the fixed points of the composite maps GF and FG (where a fixed point of e.g. GF is of course an element c such that $GFc = c$).

Theorem 205. *Given a Galois connection $F \dashv G$ between the posets $(C, \preccurlyeq)$ and $(D, \sqsubseteq)$, then*

(1) $c \in G[D]$ iff c is a fixed point of GF; $d \in F[C]$ iff d is a fixed point of FG.

(2) $G[D] = (GF)[C]$; $F[C] = (FG)[D]$.

Proof. (1) Suppose $c \in G[D]$. Then for some $d \in D$, $c = Gd$ and hence $GFc = GFGd = Gd = c$, so c is a fixed point of GF. Conversely suppose $GFc = c$. Then c is the value of Gd for $d = Fc$, and therefore $c \in G[D]$.

Hence $c \in G[D]$ iff c is a fixed point of GF. The other half of (1) is dual.

(2) We have just seen that if $c \in G[D]$ then $c = GFc$ so $c \in (GF)[C]$. Therefore $G[D] \subseteq (GF)[C]$. Conversely, suppose $c \in (GF)[C]$, then for some $c' \in C$, $c = GFc'$; but $Fc' \in D$ so $c \in G[D]$. Therefore $(GF)[C] \subseteq G[D]$.

Hence $G[D] = (GF)[C]$. The other half of (2) is dual. □

(b) We know that a pair of posets which have a Galois connection between them needn't be isomorphic overall. But this next theorem says that they will typically contain an interesting pair of isomorphic sub-posets.

Theorem 206. *If $F \dashv G$ is a Galois connection between the posets $(C, \preccurlyeq)$ and $(D, \sqsubseteq)$, then $(G[D], \preccurlyeq)$ and $(F[C], \sqsubseteq)$ are order-isomorphic.*[4]

Proof. We show that F restricted to $G[D]$ provides the desired order isomorphism.

Note first that if $c \in G[D]$, then $Fc \in F[C]$. So F as required sends elements of $G[D]$ to elements of $F[C]$. Moreover every element of $F[C]$ is Fu for some $u \in G[D]$. For if $d \in F[C]$, then for some c, $d = Fc = FGFc = Fu$ where $u = GFc \in G[D]$.

So F restricted to $G[D]$ is onto $F[C]$. It remains to show that it is an order-embedding. We know that F will be monotone, so what we need to prove is that, if $c, c' \in G[D]$ and $Fc \sqsubseteq Fc'$, then $c \preccurlyeq c'$.

But if $Fc \sqsubseteq Fc'$, then by the monotonicity of G, $GFc \preccurlyeq GFc'$. Recall, though, that $c, c' \in G[D]$ are fixed points of GF. Hence $c \preccurlyeq c'$ as we want. □

(c) Finally, we want the idea of a closure function K on a poset which, roughly speaking, maps a poset 'upwards' to a subposet which then stays fixed under further applications of K:

Definition 147. Suppose $(C, \preccurlyeq)$ is a poset; then a *closure function* on C is a function $K: C \to C$ such that, for all $c, c' \in C$,

[4] Life is too short to fuss about notationally distinguishing the order relation $\preccurlyeq$ defined over C from that relation's restriction to $G[D]$.

(1) $c \preccurlyeq Kc$;

(2) if $c \preccurlyeq c'$, then $Kc \preccurlyeq Kc'$, i.e. K is monotone;

(3) $KKc = Kc$, i.e. K is idempotent. △

Theorem 207. *If $F \vdash G$ is a Galois connection between $(C, \preccurlyeq)$ and another poset, then GF is a closure function for C.*

Proof. We quickly check that the three conditions for closure apply. (i) is given by Theorem 200. (ii) is immediate as GF is a composition of monotone functions. And for (iii), we know that $FGF = F$, and hence $GFGF = GF$. □

40.7 Syntax and semantics briefly revisited

(a) We now note one rather characteristic way in which Galois connections can naturally arise.

Theorem 208. *Given sets X and Y, with R a binary relation between their members, form the posets $(\mathcal{P}(X), \subseteq)$ and $(\mathcal{P}(Y), \supseteq)$ and then define*

i. Define $F\colon (\mathcal{P}(X), \subseteq) \to (\mathcal{P}(Y), \supseteq)$ by putting $FA = \{b \mid (\forall a \in A)aRb\}$ for $A \subseteq X$.

ii. Similarly define $G\colon (\mathcal{P}(Y), \supseteq) \to (\mathcal{P}(X), \subseteq)$ by putting $GB = \{a \mid (\forall b \in B)aRb\}$ for $B \subseteq Y$.

Then $F \dashv G$.

Proof. We just have to prove that principle (Gal) holds, i.e. for given $A \subseteq X$, $B \subseteq Y$, then $FA \supseteq B$ iff $A \subseteq GB$.

But simply by applying definitions we see $FA \supseteq B$ iff $(\forall b \in B)(\forall a \in A)aRb$ iff $(\forall a \in A)(\forall b \in B)aRb$ iff $A \subseteq GB$. □

Let's say that a Galois connection produced in this way is *relation-generated.* Galois's original classic example was of this kind. And our original motivating example is relation-generated too. Let's briefly return to it.

(b) Let X be the set of L-sentences, let Y be the set of L-structures, and let R be the relation between $x \in X$ and $y \in Y$ such that xRy iff $y \vDash x$.

Then $(\mathcal{P}(X), \subseteq)$, and $(\mathcal{P}(Y), \supseteq)$ are just the posets from §40.2 again. Then our last theorem tells us that there is a Galois connection between these posets.

Here, from the theorem, F is defined by putting by putting $Fc = \{b \mid (\forall a \in A)b \vdash a\}$ for any $c \subseteq X$. So, as before, F is the 'find the models' function.

Similarly, from the theorem, G by putting $Gd = \{a \mid (\forall b \in d)b \vDash a\}$ for $d \subseteq Y$. So, as before, G is the 'find the agreed truths' function.

So Theorem 208 immediately gives us back the same Galois connection $F \dashv G$ that we introduced in §40.2.

(c) Now we can just turn the handle, and apply all those general theorems about Galois connections from the preceding sections to our special case of the connection between the 'syntax' poset and 'semantics' poset of §40.2, recovering the sorts of results listed at the end of that section and more.

Of course, we get no exciting logical news this way. But that's not the name of the game. The point rather is this. We take the fundamental *true-of* relation which can obtain between an L-sentence and an L-structure: this immediately generates a certain Galois connection $F \dashv G$ between two naturally ordered 'syntactic' and 'semantic' posets, and this in turn already dictates that e.g. the composite maps GF and FG will have special significance as closure operations. So we come to see some familiar old logical ideas as exemplifying essentially general order-theoretic patterns which recur elsewhere. And that's illuminating.

41 Adjoints introduced

Recall that lovely quotation from Tom Leinster which I gave at the very outset:

> Category theory takes a bird's eye view of mathematics. From high in the sky, details become invisible, but we can spot patterns that were impossible to detect from ground level. (Leinster 2014, p. 1)

Perhaps the most dramatic example of patterns that category theory newly reveals are those which involve *adjunctions*. As Mac Lane famously puts it (1997, p. vii) the slogan is "Adjoint functors arise everywhere." In the previous chapter, we have seen what, in hindsight, turns out to be a restricted version of the phenomenon. But category theory enables us to generalize radically.

41.1 Adjoint functors: a first definition

(a) Take a poset $(C, \preccurlyeq)$ and now consider the corresponding category C. So the objects of C are the members of C, and there is a (unique) C-arrow $A \to A'$ (for $A, A' \in C$) if and only if $A \preccurlyeq A'$. Similarly let D be the category corresponding to the poset $(D, \sqsubseteq)$.

Now, a Galois connection between the posets $(C, \preccurlyeq)$ and $(D, \sqsubseteq)$ is a pair of functions $F\colon C \to D$ and $g\colon D \to C$ such that

$$\text{(Gal)} \qquad FA \sqsubseteq B \text{ iff } A \preccurlyeq GB \text{ (for all } A \in C, B \in D).$$

But we know that (Gal) implies that F and G are monotone – see Theorem 200. And monotone functions between posets give rise to functors between the corresponding categories – see §27.2, Ex. (F10).

More precisely, the monotone function $F\colon C \to D$ gives rise to the functor $F\colon \mathsf{C} \to \mathsf{D}$ which sends the C-object A to the D-object FA and sends a C-arrow $A \to A'$ to the (unique) arrow D-arrow $FA \to FA'$. Similarly, $G\colon D \to C$ gives rise to a functor $G\colon \mathsf{D} \to \mathsf{C}$. So our Galois-connected *functions* F, G in opposite directions between the posets $(C, \preccurlyeq)$ and $(D, \sqsubseteq)$ give rise to a pair of *functors* F, G in opposite directions between the poset categories C and D with the following property:

> there is a unique arrow $FA \to B$ in D iff there is a corresponding unique arrow $A \to GB$ in C (for all C-objects A and D-objects B).

So (Gal) for the posets $(C, \preccurlyeq)$ and $(D, \sqsubseteq)$ becomes the following for the corresponding categories C, D: there is a bijection between the hom-sets $\mathsf{D}(FA, B)$ and $\mathsf{C}(A, GB)$, for each C-object A and D-object B. Moreover, the bijection arises *systematically* from the underlying Galois connection, in a uniform way, for any A, B. And we now know how to capture that informal claim in category-theoretic terms – that bijection will be natural in A and natural in B.

(b) Now the key new move. We generalize from the special case of poset categories in the obvious way, and also introduce some absolutely standard notation:

Definition 148. Suppose C and D are categories and $F\colon \mathsf{C} \to \mathsf{D}$ and $G\colon \mathsf{D} \to \mathsf{C}$ are functors. Then F *is left adjoint to* G *and* G *is right adjoint to* F, notated $F \dashv G$, iff

$$\mathsf{D}(F(A), B) \cong \mathsf{C}(A, G(B))$$

naturally in A and naturally in B. △

If we want notation which makes explicit where adjoint functors are going, we can write $F \dashv G\colon \mathsf{C} \to \mathsf{D}$, or can diagram the situation like this:

$$\begin{tikzcd} \mathsf{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathsf{D} \arrow[l, shift left, "G"] \end{tikzcd}$$

Note that F counts as the *left* adjoint not because it happens to appear on the left of the (symmetric!) congruence sign above, but because it is written on the left, giving the source object, in describing the relevant hom-set.

A quick comment. Here, and onwards through our discussions of adjunctions, we'll take it that there is no problem in talking about the relevant hom-*sets* – either because the categories are small enough, or because we are taking a relaxedly inclusive line on what counts as set-like collections.

(c) Let's immediately state a couple of theorems. First, recall Theorem 201 that told us that if we have Galois connections $F \dashv G$, $F \dashv G'$ between the posets $(C, \preccurlyeq)$ and $(D, \sqsubseteq)$, then $G = G'$. And likewise, if $F \dashv G$, $F' \dashv G$ are both Galois connections between the same posets, then $F = F'$. Now, we have the following rather predictable result for adjoints more generally:

Theorem 209. *If a functor has an adjoint, it is unique up to natural isomorphism. If $F \dashv G$ and $F \dashv G'$ then $G \cong G'$. If $F \dashv G$ and $F' \dashv G$ then $F \cong F'$.*

Second, recall Theorem 202 that tells us that Galois connections compose: if there is a Galois connection $F \dashv G$ between the posets $(C, \preccurlyeq)$ and $(D, \sqsubseteq)$, and a connection $H \dashv K$ between the posets $(D, \sqsubseteq)$ and $(E, \subseteq)$, then there is a Galois connection $HF \dashv GK$ between C and E. The obvious generalization again applies:

Theorem 210. *Given* $\begin{tikzcd} \mathsf{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathsf{D} \arrow[l, shift left, "G"] \end{tikzcd}$ *and* $\begin{tikzcd} \mathsf{D} \arrow[r, shift left, "H"] \arrow[r, phantom, "\perp"] & \mathsf{E} \arrow[l, shift left, "K"] \end{tikzcd}$ *, then* $\begin{tikzcd} \mathsf{C} \arrow[r, shift left, "HF"] \arrow[r, phantom, "\perp"] & \mathsf{E} \arrow[l, shift left, "GK"] \end{tikzcd}$.

But let's leave the proofs until the next chapter. First, for motivation, we want to get more examples of adjunctions onto the table; we also need to explain what the naturality requirement in our definition comes to.

41.2 Examples

For a warm-up exercise, we start with a particularly easy case:

(1) Consider any (non-empty!) category C and the one object category $\mathsf{1}$ (comprising just the object $\bullet$ and its identity arrow). There is a unique functor $F\colon \mathsf{C} \to \mathsf{1}$. Questions: when does F have a right adjoint $G\colon \mathsf{1} \to \mathsf{C}$? what about a left adjoint?

If G is to be a right adjoint, remembering that $FA = \bullet$ for any $A \in \mathsf{C}$, we require
$$\mathsf{1}(\bullet, \bullet) \cong \mathsf{C}(A, G\bullet),$$
for any A. The hom-set on the left contains just the identity arrow. And that can only be in bijection to the hom-set on the right, for each A, if there is always a *unique* arrow $A \to G\bullet$, i.e. if $G\bullet$ is terminal in C (and then the bijection is intuitively natural, as no arbitrary choices can be involved in setting it up).

In sum, F has a right adjoint $G\colon \mathsf{1} \to \mathsf{C}$ just in case G sends $\mathsf{1}$'s unique object to C's terminal object: no terminal object, no right adjoint.

Dually, F has a left adjoint if and only if C has an initial object.

This toy example reminds of what we have already seen in the special case of Galois connections, namely that a functor may or may not have a right adjoint, and independently may or may not have a left adjoint, and if both adjoints exist they may be different. But let's also note that we have here a first indication that adjunctions and limits can interact in interesting way: in this case, indeed, we could *define* terminal and initial objects for a category C in terms of the existence of right and left adjoints to the functor $F\colon \mathsf{C} \to \mathsf{1}$. We will return to this theme.

Now for a couple of more substantial examples. And to speed things along, we will continue to procede informally: we won't in this section actually prove that bijections between the relevant hom-sets in our various examples are natural in the official techincal sense, but rather we will take it as enough to find a bijection which can be evidently set up in a systematic and intuitively natural way, without arbitrary choices.

(2) Let's next consider the forgetful functor $U\colon \mathsf{Top} \to \mathsf{Set}$ which sends each topological space to its underlying set of points, and sends any continuous function between topological spaces to the same function thought of as a set-function. Questions: does this have a left adjoint? a right adjoint?

If U is to have a left adjoint $F\colon \mathsf{Set} \to \mathsf{Top}$, then for any set S and for any topological space (T, O) – with T a set of points and O a topology (a suitable collection of open sets) – we require

$$\mathsf{Top}(FS,(T,O)) \cong \mathsf{Set}(S,U(T,O)) = \mathsf{Set}(S,T),$$

where the bijection here needs to be a natural one.

Now, on the right we have the set of *all* functions $f\colon S \to T$. So that needs to be in bijection with the set of all *continuous* functions from FS to (T,O). How can we ensure this holds in a systematic way, for any S and (T,O)? Well, suppose that for any S, F sends S to the topological space (S,D) which has the discrete topology (i.e. all subsets of S count as open). It is a simple exercise to show that *every* function $f\colon S \to T$ then counts as a continuous function $f\colon (S,D) \to (T,O)$. So the functor F which assigns a set the discrete topology will indeed be left adjoint to the forgetful functor – and so by Theorem 209 will be *the* left adjoint.

Similarly, the functor $G\colon \mathsf{Set} \to \mathsf{Top}$ which assigns a set the indiscrete topology (the only open sets are the empty set and S itself) is the right adjoint to the forgetful functor U.

(3) Let's next take another case of a forgetful functor, this time the functor $U\colon \mathsf{Mon} \to \mathsf{Set}$ which forgets about monoidal structure. Does U have a left adjoint $F\colon \mathsf{Set} \to \mathsf{Mon}$. If $(M,*,e)$ is a monoid and S some set, we need

$$\mathsf{Mon}(FS,(M,*,e)) \cong \mathsf{Set}(S,U(M,*,e)) = \mathsf{Set}(S,M).$$

The hom-set on the right contains all possible functions $f\colon S \to M$. How can these be in one-one correspondence with the monoid homomorphisms from FS to $(M,*,e)$?

Arm-waving for a moment, suppose FS is some monoid with a lot of structure (over and above the minimum required to be a monoid). Then there may be few if any monoid homomorphisms from FS to $(M,*,e)$. Therefore, if there are to be *lots* of such monoid homorphisms, one for each $f\colon S \to M$, then FS will surely need to have minimal structure. Which suggests going for broke and considering the limiting case of the least structured monoid built on S. So we want the functor F to send a set S to $(\mathit{List}(S),{}^\frown,\emptyset)$ or an equivalent, the *free* monoid on S which we met back in §27.4, Ex. (F15). Recall, the objects of $(\mathit{List}(S),{}^\frown,\emptyset)$ are lists of S-elements (including the null sequence) and its monoid operation is concatenation.

There is an obvious map α which takes an arrow $f\colon S \to M$ and sends it to $\hat{f}\colon (\mathit{List}(S),{}^\frown,\emptyset) \to (M,*,e)$, where $\hat{f}$ sends the empty sequence of S-elements to the unit of M, and sends the finite sequence $x_1 * x_2 * x_3 * \ldots * x_n$ to the M-element $fx_1 * fx_2 * fx_3 * \ldots * fx_n$. So defined, $\hat{f}$ respects the unit and the monoid operation and so is a monoid homomorphism.

There is also an obvious map β which takes an arrow $g\colon (\mathit{List}(S),{}^\frown,\emptyset) \to (M,*,e)$ to the function $\hat{g}\colon S \to M$, where $\hat{g}$ sends an element $x \in S$ to $g\langle x\rangle$ (i.e. to g applied to the one-element list containing x).

Evidently α and β are inverses, so form a bijection, and their construction is quite general (i.e. can be applied to any set S and monoid $(M, *, e)$). Which establishes that, as required $\mathsf{Mon}(FS, (M, *, e)) \cong \mathsf{Set}(S, M)$.

So in sum, the free functor F which takes a set to the free monoid on that set is left adjoint to the forgetful functor U which sends a monoid to its underlying set. And by Theorem 209, this free functor is the unique left adjoint to the forgetful functor.

Our example involving monoids is actually typical of a whole cluster of cases. A left adjoint of the trivial forgetful functor from some class of algebraic structures to their underlying sets is characteristically provided by the non-trivial functor that takes us from a set to a free structure of that algebraic kind. Thus we have, for example,

(4) The forgetful functor $U\colon \mathsf{Grp} \to \mathsf{Set}$ has as a left adjoint the functor $F\colon \mathsf{Set} \to \mathsf{Grp}$ which sends a set to the free group on that set (i.e. the group obtained from a set S by adding just enough elements for it to become a group while imposing no constraints other than those required to ensure we indeed have a group).

What about *right* adjoints to our last two forgetful functors?

(5) We will later show that the forgetful functor $U\colon \mathsf{Mon} \to \mathsf{Set}$ has no right adjoint by a neat proof in §43.4. But here's a more arm-waving argument. U would have a right adjoint $G\colon \mathsf{Set} \to \mathsf{Mon}$ just in case $\mathsf{Set}(M, S) = \mathsf{Set}(U(M, *, e), S) \cong \mathsf{Mon}((M, *, e), GS)$, for all monoids $(M, *, e)$ and sets S. But this requires the monoid homomorphisms from $(M, *, e)$ to GS always to be in bijection with the set-functions from M to S. But that's not possible (consider keeping the sets M and S fixed, but changing the possible monoid operations with which M is equipped).

A similar argument shows that the forgetful functor $U\colon \mathsf{Grp} \to \mathsf{Set}$ has no right adjoint.

There are, however,examples of 'less forgetful' algebraic functors which have both left and right adjoints.

(6) Take the functor $U\colon \mathsf{Grp} \to \mathsf{Mon}$ which forgets about group inverses but keeps the monoidal structure. This has a left adjoint $F\colon \mathsf{Mon} \to \mathsf{Grp}$ which converts a monoid to a group by adding inverses for elements (and otherwise making no more assumptions that are needed to get a group). U also has a right adjoint $G\colon \mathsf{Mon} \to \mathsf{Grp}$ which rather than adding elements subtracts them by mapping a monoid to the submonoid of its invertible elements (which can be interpreted as a group).

Let's check the second of those claims. We have $U \dashv G$ so long as

$$\mathsf{Mon}(U(K, \times, k), (M, *, e)) \cong \mathsf{Grp}((K, \times, k), G(M, *, e)),$$

for any monoid $(M, *, e)$ and group $(K, \times, k)$, and in a natural way. Now we just remark that every element of $(K, \times, k)$-as-a-monoid is invertible and a monoid homomorphism sends invertible elements to invertible elements. Hence a monoid homomorphism from $(K, \times, k)$-as-a-monoid to $(M, *, e)$ will in fact also be a group homomorphism from $(K, \times, k)$ to the submonoid-as-a-group $G(M, *, e)$.

And here is another forgetful functor with a right adjoint:

(7) Recall the functor $F\colon \mathsf{Set} \to \mathsf{Rel}$ which 'forgets' that arrows are functional (see §27.2, Ex. (F4)). And now we introduce a powerset functor $P\colon \mathsf{Rel} \to \mathsf{Set}$ defined as follows:

   a) P sends a set A to its powerset $\mathcal{P}(A)$, and

   b) P sends a relation R in $A \times B$ to the function $f_R\colon \mathcal{P}(A) \to \mathcal{P}(B)$ which sends $X \subseteq A$ to $Y = \{b \mid (\exists x \in X)Rxb\} \subseteq B$.

   Claim: $F \dashv P$. Why? We observe that there is a (natural!) one-to-one correlation between a relation R in $A \times B$ and a function $f\colon A \to \mathcal{P}(B)$ where $f(x) = \{y \mid Rxy\}$ and so Rxy iff $y \in f(x)$. This gives us a natural enough bijection $\mathsf{Rel}(FA, B) \cong \mathsf{Set}(A, PB)$, for any A, B.

Now for some cases not involving forgetful functors:

(8) Suppose C is a category with exponentiation (and hence with products). Then, in a slogan, exponentiation by B is right adjoint to taking the product with B.

   To see this, we define a pair of functors from C to itself. First, there is the functor $- \times B\colon \mathsf{C} \to \mathsf{C}$ which sends an object A to the product $A \times B$, and sends an arrow $f\colon A \to A'$ to $f \times 1_B\colon A \times B \to A' \times B$.

   Second there is the functor $(-)^B\colon \mathsf{C} \to \mathsf{C}$ which sends an object C to C^B, and sends an arrow $f\colon C \to C'$ to $\overline{f \circ ev}\colon C^B \to C'^B$, as defined in §27.3, (F13).

   But $\mathsf{C}(A \times B, C) \cong \mathsf{C}(A, C^B)$ naturally in A and C (for a partial proof see §32.4(5)). Hence $(- \times B) \dashv (-)^B$.

(9) Recall Defn. 22 which defined the product of two categories. Given a category C there is a trivial diagonal functor $\Delta\colon \mathsf{C} \to \mathsf{C} \times \mathsf{C}$ which sends a C-object A to the pair $\langle A, A\rangle$, and sends a C-arrow f to the pair of arrows $\langle f, f\rangle$. What would it take for this functor to have a right adjoint $G\colon \mathsf{C} \times \mathsf{C} \to \mathsf{C}$? We'd need

$$(\mathsf{C} \times \mathsf{C})(\langle A, A\rangle, \langle B, C\rangle) \cong \mathsf{C}(A, G\langle B, C\rangle)$$

   naturally in $A \in \mathsf{C}$ and in $\langle B, C\rangle \in \mathsf{C} \times \mathsf{C}$. But by definition the left hand hom-set is $\mathsf{C}(A, B) \times \mathsf{C}(A, C)$. But then if we take G to be the product functor that sends $\langle B, C\rangle$ to the product object $B \times C$ in C we'll get an obvious natural isomorphism

$$\mathsf{C}(A, B) \times \mathsf{C}(A, C) \cong \mathsf{C}(A, B \times C).$$

   So in sum, $\Delta\colon \mathsf{C} \to \mathsf{C} \times \mathsf{C}$ has a right adjoint iff C has binary products.

(10) For topologists, let's simply mention another example of a case where the adjoint of a trivial functor is something much more substantial. The inclusion functor from KHaus, the category of compact Hausdorff spaces, into Top has a left adjoint, namely the Stone-Čech compactification functor.

41.3 Naturality

(a) We said: $F \dashv G\colon \mathsf{C} \to \mathsf{D}$ just in case (for all C-objects A and D-objects B)

$$\mathsf{D}(FA, B) \cong \mathsf{C}(A, GB)$$

holds *naturally* in A and B. Let's now be more explicit about what the official naturality requirement comes to in this case.

There will be a whole suite of bijections $\varphi_{AB}\colon \mathsf{D}(FA, B) \to \mathsf{C}(A, GB)$ (one for each A, B) – call these, taken together, the *adjunction* φ for $F \dashv G$. For naturality in A, we need to consider what happens when we vary the occupant of the A-place – keeping B fixed, we want φ_{AB}, $\varphi_{A'B}$, $\varphi_{A''B}, \ldots$, to assemble into a natural isomorphism $\varphi_B\colon \mathsf{D}(F-, B) \xRightarrow{\sim} \mathsf{C}(-, GB)$. Similarly for naturality in B, φ_{AB}, $\varphi_{AB'}$, $\varphi_{AB''}, \ldots$, keeping A fixed, should assemble into a natural isomorphism between hom-functors $\varphi_A\colon \mathsf{D}(FA, -) \xRightarrow{\sim} \mathsf{C}(A, G-)$.

Now, to take the second case, if φ_{AB}, $\varphi_{AB'}$, $\varphi_{AB''}, \ldots$ are indeed to assemble into a natural isomorphism then we require that, for any choice of B, B' and arrow $h\colon B \to B'$, the usual naturality square always commutes:

$$\begin{tikzcd}
\mathsf{D}(FA, B) \arrow[r, "\mathsf{D}(FA, h)"] \arrow[d, "\varphi_{AB}"] & \mathsf{D}(FA, B') \arrow[d, "\varphi_{AB'}"] \\
\mathsf{C}(A, GB) \arrow[r, "\mathsf{C}(A, Gh)"] & \mathsf{C}(A, GB')
\end{tikzcd}$$

But how does the covariant hom-functor $\mathsf{D}(FA, -)$ operate on $h\colon B \to B'$? As we saw in §31.1, it sends h to $h \circ -$, i.e. to that function which composes h with an arrow from $\mathsf{D}(FA, B)$ to give an arrow in $\mathsf{D}(FA, B')$. Similarly, $\mathsf{C}(A, G(-))$ will send h to $Gh \circ -$.

So consider an arrow $d\colon FA \to B$ living in $\mathsf{D}(FA, B)$. The naturality square now tells us that for any $h\colon B \to B'$, $\varphi_{B'}(h \circ d) = Gh \circ \varphi_B d$.[1]

(b) There is an additional fairly standard bit of notation which is used to indicate the action of the bijections φ_{AB} and their inverses on the relevant homsets:

Definition 149. Given $F \dashv G\colon \mathsf{C} \to \mathsf{D}$, whose adjunction is φ, then φ_{AB} sends an arrow $d\colon F(A) \to B$ to its *transpose* $\overline{d}\colon A \to G(B)$; likewise the inverse bijection φ^{-1}_{AB} sends an arrow $c\colon A \to GB$ to its transpose $\overline{c}\colon F(A) \to B$. △

[1] Let's have a helpful convention here: I'll use d, d' for arrows living in D, and hence belonging to homsets such as $\mathsf{D}(FA, B)$, while I use c, c' for arrows living in C, and hence belonging to homsets such as $\mathsf{C}(A, GB)$.

An alternative notation uses $d^\flat$ for our $\overline{d}$ and $c^\sharp$ for our $\overline{c}$, and this notation might well be thought to be preferable in principle since transposing by 'sharpening' and 'flattening' are indeed different operations. But the double use of the overlining notation is standard, and is slick. Evidently, transposing twice takes us back to where we started: $\overline{\overline{c}} = c$ and $\overline{\overline{d}} = d$.

(c) Deploying this new notation, we have shown the first part of the following theorem. And the second part follows by a dual argument, in which some arrows get reversed because the relevant hom-functors in this case are contravariant.

Theorem 211. *Given $F \dashv G \colon \mathsf{C} \to \mathsf{D}$, then*

(1) for any $d \colon FA \to B$ and $h \colon B \to B'$, $\overline{h \circ d} = Gh \circ \overline{d}$,

(2) for any $c \colon A \to GB$ and $k \colon A' \to A$, $\overline{c \circ k} = \overline{c} \circ Fk$, so $\overline{\overline{c} \circ Fk} = c \circ k$. □

There is an obvious converse to this theorem. Given functors $F \colon \mathsf{C} \to \mathsf{D}$ and $G \colon \mathsf{D} \to \mathsf{C}$ such that there is always a bijection φ_{AB} between $\mathsf{D}(FA, B)$ and $\mathsf{C}(A, GB)$ then, if conditions (1) and (2) hold with transposes defined as before, the various φ_{AB} will assemble into natural transformations, so that $\mathsf{D}(FA, B) \cong \mathsf{C}(A, GB)$ holds naturally in A and in B, and hence $F \dashv G$.

41.4 An alternative definition

Our first definition of adjunctions was inspired by our original definition of Galois connections in §40.3. But we gave an alternative definition of such connections in §40.4. This too can be generalized to give a second definition of adjunctions. In this section we show how.

(a) A Galois connection between the posets $(C, \preccurlyeq)$ and $(D, \sqsubseteq)$, according to the tweaked version of our alternative definition, comprises a pair of functions $F \colon C \to D$ and $G \colon D \to C$ such that (for any $A, A' \in C$ and $B, B' \in D$):

(1) F and G are both monotone,

(2) if $A \preccurlyeq A'$, then $A \preccurlyeq A' \preccurlyeq GFA'$ and $A \preccurlyeq GFA \preccurlyeq GFA'$,

(3) if $B \sqsubseteq B'$, then $FGB \sqsubseteq B \sqsubseteq B'$ and $FGB \sqsubseteq FGB' \sqsubseteq B'$.

As before, let C be the category corresponding to the poset $(C, \preccurlyeq)$, and recall that there is an arrow $A \to A'$ in C just when $A \preccurlyeq A'$ in the poset $(C, \preccurlyeq)$. Likewise for the category D corresponding to $(D, \sqsubseteq)$. Again as before, the monotone functions F, G between the posets give rise to functors F, G between the corresponding categories. Hence, in particular, the composite monotone function $G \circ F$ gives rise to a composite functor $GF \colon \mathsf{C} \to \mathsf{C}$, and likewise the composite monotone function $F \circ G$ gives rise to a functor $FG \colon \mathsf{D} \to \mathsf{D}$.[2]

Now, (2) corresponds to the claim that the following diagram always commutes in the poset category C, for any A, A':

[2]We'll use the compressed notation which drops the explicit composition sign for composite functors.

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A' \arrow[d] \\ GFA \arrow[r] & GFA' \end{tikzcd}$$

(We needn't label the arrows because in the poset category C arrows between objects are unique when they exist.)

So let's define η_X to be the arrows $X \to GFX$, one for each C-object X. And let $1_{\mathsf{C}}\colon \mathsf{C} \to \mathsf{C}$ be the trivial identity functor. Then our commutative diagram can be revealingly redrawn as follows:

$$\begin{tikzcd} 1_{\mathsf{C}}A \arrow[r] \arrow[d, "\eta_A"] & 1_{\mathsf{C}}A' \arrow[d, "\eta_{A'}"] \\ GFA \arrow[r] & GFA' \end{tikzcd}$$

This commutes for all A, A'. Which is just to say that the η_X assemble into a natural transformation $\eta\colon 1_{\mathsf{C}} \Rightarrow GF$.

Likewise, (3) corresponds to the claim that there is a natural transformation $\varepsilon\colon FG \Rightarrow 1_{\mathsf{D}}$.

(b) So far so good. We now have the key ingredients for an alternative definition for an adjunction between functors $F\colon \mathsf{C} \to \mathsf{D}$ and $G\colon \mathsf{D} \to \mathsf{C}$: we will require there to be a pair of natural transformations $\eta\colon 1_{\mathsf{C}} \Rightarrow GF$ and $\varepsilon\colon FG \Rightarrow 1_{\mathsf{D}}$.

However, this isn't yet quite enough. But the additional ingredients we want are again suggested by our earlier treatment of Galois connections. Recall from Theorem 200 that if $F \dashv G$ is a Galois connection between $(C, \preccurlyeq)$ and $(D, \sqsubseteq)$, then we immediately have the key identities

(4) $FGF = F$, and

(5) $GFG = G$.

By (4), $FA \preccurlyeq FGFA \preccurlyeq FA$, for any $A \in C$. Hence, now moving back from the poset to the corresponding category, in C the following diagram commutes for each A:

$$\begin{tikzcd} FA \arrow[r, "F\eta_A"] \arrow[dr, "1_{FA}"'] & FGFA \arrow[d, "\varepsilon_{FA}"] \\ & FA \end{tikzcd}$$

Or what comes to the same, in the functor category $[\mathsf{C}, \mathsf{D}]$ this next diagram commutes[3]

[3]Notational fine print: our convention has been to use single arrows to represent arrows inside particular categories, and double arrows to represent natural transformations between functors across categories. We are now dealing with natural-transformations-thought-of-as-

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "1_F"'] & FGF \arrow[d, "\varepsilon F"] \\ & F \end{tikzcd}$$

For remember 'whiskering', discussed in §33.3: the various components $F\eta_A$ assemble into a natural transformation $F\eta$, and the components ε_{FA} assemble a natural transformation εF. And then recall from §36.1 that 'vertical' composition of natural transformations between e.g. the functors $F\colon \mathsf{C} \to \mathsf{D}$ and $FGF\colon \mathsf{C} \to \mathsf{D}$ is defined component-wise. So, for each A,

$$(\varepsilon F \circ F\eta)_A = \varepsilon_{FA} \circ F\eta_A = 1_{FA} = (1_F)_A,$$

where 1_F is the natural transformation whose component at A is 1_{FA}. Since all components are equal, the left-most and right-most natural transformations in that equation are equal and our diagram commutes.

Exactly similarly, from (v) we infer that $G\varepsilon_B \circ \eta_{GB} = 1_{GB}$. In other words, the next diagram commutes in $[\mathsf{D}, \mathsf{C}]$:

$$\begin{tikzcd} G \arrow[r, "\eta G"] \arrow[dr, "1_G"'] & GFG \arrow[d, "G\varepsilon"] \\ & G \end{tikzcd}$$

(c) And *now*, let's generalize from the special case of poset categories: we can put everything together to give us our second equally standard definition for adjoint functors:

Definition 150 (Alternative). Suppose C and D are categories and $F\colon \mathsf{C} \to \mathsf{D}$ and $G\colon \mathsf{D} \to \mathsf{C}$ are functors. Then F *is left adjoint to* G *and* G *is right adjoint to* F, notated $F \dashv G$, iff

(i) there are natural transformations $\eta\colon 1_{\mathsf{C}} \Rightarrow GF$ and $\varepsilon\colon FG \Rightarrow 1_{\mathsf{D}}$ such that

(ii) $\varepsilon_{FA} \circ F\eta_A = 1_{FA}$ for all A in C, and $G\varepsilon_B \circ \eta_{GB} = 1_{GB}$ for all B in D; or equivalently such that

(ii′) the following *triangle identities* hold in the functor categories $[\mathsf{C}, \mathsf{D}]$ and $[\mathsf{D}, \mathsf{C}]$ respectively:

$$\begin{tikzcd} F \arrow[r, "F\eta"] \arrow[dr, "1_F"'] & FGF \arrow[d, "\varepsilon F"] \\ & F \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "\eta G"] \arrow[dr, "1_G"'] & GFG \arrow[d, "G\varepsilon"] \\ & G \end{tikzcd}$$

arrows-within-a-particular-functor-category. Some use double arrows for diagrams in a functor category, to remind us these are natural transformations (between functors relating some other categories); some use single arrows because these are being treated as arrows (in the functor category). I'm jumping the second way, following the majority and also getting slightly cleaner diagrams.

Note, the transformations η and ε are standardly called the *unit* and *co-unit* of the adjunction. △

(d) It remains to show that our original and our alternative definitions of adjunctions are equivalent:

Theorem 212. *For given functors $F\colon \mathsf{C} \to \mathsf{D}$ and $G\colon \mathsf{D} \to \mathsf{C}$, $F \dashv G$ holds by Defn. 148 iff it holds by Defn. 150.*

But again let's leave the details for the following chapter, so we can concentrate here on broader themes.

41.5 Isomorphism, equivalence, adjointness

We got to our second general definition of an adjunction by starting from our second definition of those special adjunctions which are Galois connections. But there is another route taking us to the same end.

For consider, first, this rephrasing of our earlier definition of an isomorphism between categories:

Definition 128* The categories C and D are isomorphic iff there are functors $F\colon \mathsf{C} \to \mathsf{D}$ and $G\colon \mathsf{D} \to \mathsf{F}$ such that we have the *identities* $1_{\mathsf{C}} = GF$ and $FG = 1_{\mathsf{D}}$. △

And then second, recall that to get our definition of equivalence of categories, we weakened the requirement that F and G are inverses to give a definition which can be rephrased like this:

Definition 129* The categories C and D are equivalent iff there are functors $F\colon \mathsf{C} \to \mathsf{D}$ and $G\colon \mathsf{D} \to \mathsf{C}$ such that there is a pair of *natural isomorphisms* $\eta\colon 1_{\mathsf{C}} \xRightarrow{\sim} GF$ and $\varepsilon\colon FG \xRightarrow{\sim} 1_{\mathsf{D}}$. △

And now third, we have weakened the required relation between F and G further to give us our new definition:

Definition 150 There is an adjunction between the categories C and D iff there is an adjoint pair of functors $F\colon \mathsf{C} \to \mathsf{D}$ and $G\colon \mathsf{D} \to \mathsf{C}$ – i.e. there is a pair of *natural transformations* $\eta\colon 1_{\mathsf{C}} \Rightarrow GF$ and $\varepsilon\colon FG \Rightarrow 1_{\mathsf{D}}$, where $\varepsilon_{FA} \circ F\eta_A = 1_{FA}$ for all A in C, and $G\varepsilon_B \circ \eta_{GB} = 1_{GB}$ for all B in D. △

Now, an isomorphism is a fortiori an equivalence. But since the pair of isomorphisms in an equivalence needn't satisfy the triangle identities, they needn't immediately give us an adjunction as they stand. However, taking such an equivalence defined as in Defn 129* and fixing one of the natural isomorphisms, we can always tinker (if necessary) with the other to get a corresponding adjunction:

Theorem 213. *If there is an equivalence between C and D constituted by a pair of functors $F\colon \mathsf{C} \to \mathsf{D}$ and $G\colon \mathsf{D} \to \mathsf{C}$ and a pair of natural isomorphisms $\eta\colon 1_{\mathsf{C}} \xRightarrow{\sim} GF$ and $\varepsilon\colon FG \xRightarrow{\sim} 1_{\mathsf{D}}$, then there is an adjunction $F \dashv G$ with unit*

η and new co-unit ε' (defined in terms of η and ε). There is also an adjunction $F \dashv G$ with co-unit ε and new unit η' (again defined in terms of η and ε).

And because the equivalence is symmetric, there is also an adjunction $G \dashv F$.

Again, we'll leave the proof of this theorem until the next chapter. But taking the theorem as given, our three notions of isomorphism, equivalence and adjunction can now be seen as indeed progressively weaker connections between categories. If we go on to drop the triangle identities from Defn. 150 we get a weaker relation still between F and G. But this turns out to be of significantly less interest, while the sort of connections made by adjunctions are of central concern, as we have already seen. We'll explore these more over the next two chapters.

42 Adjoints explored

In the preceding chapter, we gave a pair of definitions of adjoint functors, mirroring the two alternative definitions of Galois connections. I announced that the two definitions are equivalent: so the first order of business for this chapter is to prove that this is so. We'll also note two further possible definitions.

We then need to prove three other theorems which I also stated without proof in the last chapter. And we'll continue showing how to prove for adjunctions more generally an analogue of one of the other key results we established for Galois connections.

42.1 Two definitions again

(a) Suppose $F\colon \mathsf{C} \to \mathsf{D}$ and $G\colon \mathsf{D} \to \mathsf{C}$ are functors. When is F left-adjoint to G and G right-adjoint to F? When does $F \dashv G$ hold? We said:

Definition 148. $F \dashv G$ if and only if $\mathsf{D}(FA, B) \cong \mathsf{C}(A, GB)$ naturally in A in C and in B in D.

Or alternatively, we can say

Definition 150. $F \dashv G$ if and only if there are natural transformations $\eta\colon 1_{\mathsf{C}} \Rightarrow GF$ and $\varepsilon\colon FG \Rightarrow 1_{\mathsf{D}}$ such that $\varepsilon_{FA} \circ F\eta_A = 1_{FA}$ for all $A \in \mathsf{C}$, and $G\varepsilon_B \circ \eta_{GB} = 1_{GB}$ for all $B \in \mathsf{D}$.

And these definitions, we claimed, come to the same. In other words, with a bit of effort, we can derive

Theorem 212. $F \dashv G$ holds by Defn. 148 iff it holds by Defn. 150.

Proof (If). Suppose there are natural transformations $\eta\colon 1_{\mathsf{C}} \Rightarrow GF$ and $\varepsilon\colon FG \Rightarrow 1_{\mathsf{D}}$ for which the so-called triangle identities hold.

Take any d in $\mathsf{D}(FA, B)$. Then $\eta_A\colon A \to GFA$ and $Gd\colon GFA \to GB$ compose. And so we can coherently define an arrow $\varphi_{AB}\colon \mathsf{D}(FA, B) \to \mathsf{C}(A, GB)$ by putting $\varphi_{AB}\, d = Gd \circ \eta_A$.

Likewise, we can define $\psi_{AB}\colon \mathsf{C}(A, GB) \to \mathsf{D}(FA, B)$ by putting $\psi_{AB}\, c = \varepsilon_B \circ Fc$ for any $c\colon A \to GB$.[1]

[1] Recall our local convention: c for C-arrows, d for D-arrows.

We are now going to show that a pair φ_{AB} and ψ_{AB} will be mutually inverse. So take any $d\colon FA \to B$. Then

$$\begin{aligned}
\psi_{AB}(\varphi_{AB}\, d) &= \psi_{AB}(Gd \circ \eta_A) && \text{by definition of } \varphi\\
&= \varepsilon_B \circ F(Gd \circ \eta_A)) && \text{by definition of } \psi\\
&= \varepsilon_B \circ FGd \circ F\eta_A && \text{by functoriality of } F\\
&= d \circ \varepsilon_{FA} \circ F\eta_A && \text{by naturality square for } \varepsilon\\
&= d \circ 1_{FA} && \text{by triangle identity}\\
&= d
\end{aligned}$$

Hence, since d was arbitrary, $\psi_{AB} \circ \varphi_{AB} = 1$ (note, by the way, how we *did* need to appeal here to the added triangle equality, not just functoriality and the naturality of ε).

Likewise we can show $\varphi_{AB} \circ \psi_{AB} = 1$. So, the components φ_{AB} are isomorphisms.

And now keep A fixed: then, as we vary B, the various components φ_{AB} assemble into a natural isomorphism φ_A from the hom-functor $\mathsf{D}(FA, –)$ to the hom-functor $\mathsf{C}(A, G(–))$. To show this, just note that the usual sort of naturality square for hom-functors – the sort we met in §37.1 – in fact commutes for every $h\colon B \to B'$:

$$\begin{tikzcd}
\mathsf{D}(FA,B) \arrow[r, "h\ \circ\ -"] \arrow[d, "\varphi_{AB}"] & \mathsf{D}(FA,B') \arrow[d, "\varphi_{AB'}"] \\
\mathsf{C}(A,GB) \arrow[r, "Gh\ \circ\ -"] & \mathsf{C}(A,GB')
\end{tikzcd}$$

Why? Because for every f in $\mathsf{D}(FA, B)$ we have

$$\varphi_{AB'}(h \circ f) = G(h \circ f) \circ \eta_A = Gh \circ (Gf \circ \eta_A) = Gh \circ \varphi_{AB}(f)$$

which holds by the functoriality of G.

Now keep B fixed: then by a parallel argument, as we vary A, the various components φ_{AB} assemble into a natural isomorphism $\varphi_B\colon \mathsf{D}(F(–), B) \Rightarrow \mathsf{C}(–, GB)$ between the two contravariant functors.

Hence, as we wanted, $\mathsf{D}(FA, B) \cong \mathsf{C}(A, GB)$ naturally in A and in B. □

Proof (Only if). Suppose $\mathsf{D}(FA, B) \cong \mathsf{C}(A, GB)$ naturally in A and in B. We need to define a unit η and co-unit ε for the adjunction, and show that they satisfy the triangle identities.

Now, note that, as a particular case, $\mathsf{D}(FX, FX) \cong \mathsf{C}(X, GFX)$, naturally in X. So, in particular, the adjunction here sends 1_{FX} to an arrow with the right source and target that we will (in hope!) call $\eta_X\colon X \to GFX$.

We first show that the components η_X do indeed assemble into a natural transformation from 1_{C} to GF. So consider the following two diagrams:

$$\begin{tikzcd} FA \arrow[r, "Fc"] \arrow[d, "1_{FA}"] & FA' \arrow[d, "1_{FA'}"] \\ FA \arrow[r, "Fc"] & FA' \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "c"] \arrow[d, "\eta_A"] & A' \arrow[d, "\eta_{A'}"] \\ GFA \arrow[r, "GFc"] & GFA' \end{tikzcd}$$

Trivially, the diagram on the left (living in D) commutes for all $c\colon A \to A'$: i.e. $Fc \circ 1_{FA} = 1_{FA'} \circ Fc$. The bijection from D-arrows to C-arrows transposing across the adjunction must preserve identities. So $\overline{Fc \circ 1_{FA}} = \overline{1_{FA'} \circ Fc}$. But by the first of the naturality requirements in §41.3, $\overline{Fc \circ 1_{FA}} = GFc \circ \overline{1_{FA}} = GFc \circ \eta_A$. And by the other naturality requirement, $\overline{1_{FA'} \circ Fc} = \overline{\overline{\eta_{A'}} \circ Fc} = \eta_{A'} \circ f$. So we have $GFc \circ \eta_A = \eta_{A'} \circ f$ and the diagram above on the right also commutes for all c. Hence the components η_X do indeed assemble into a natural transformation.

Similarly the same bijection forming the adjunction but taken in the opposite direction sends 1_{GB} to its transpose $\varepsilon_B\colon FGB \to B$, and the components ε_B assemble into a natural transformation from FG to 1_{D}.

So now consider these two diagrams:

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "\eta_A"] & GFA \arrow[d, "1_{GFA}"] \\ GFA \arrow[r, "1_{GFA}"] & GFA \end{tikzcd} \qquad \begin{tikzcd} FA \arrow[r, "F\eta_A"] \arrow[d, "1_{FA}"] & FGFA \arrow[d, "\varepsilon_{FA}"] \\ FA \arrow[r, "1_{FA}"] & FA \end{tikzcd}$$

The diagram on the left in C trivially commutes. Transpose it via the adjunction into D, and we find that the diagram on the right must also commute. So $\varepsilon_{FA} \circ F\eta_A = 1_{FA}$ for all A in C – which gives us one of the triangle identities. The other identity we get dually. □

42.2 Another definition?

It immediately follows from what we've just shown that, if $F \dashv G\colon \mathsf{C} \to \mathsf{D}$, then the adjunction's unit $\eta\colon 1_{\mathsf{C}} \Rightarrow GF$ has the following 'universal mapping property': for any $c\colon A \to GB$ in C there is a unique associated $d\colon FA \to B$ in D such that $c = Gd \circ \eta_A$.

Why? For existence, just put $d = \psi_{AB}c$, with ψ_{AB} as before. Then

$$Gd \circ \eta_A = G\psi_{AB}c \circ \eta_A = G(\varepsilon_B \circ Fc) \circ \eta_A = G\varepsilon_B \circ GFc \circ \eta_A = \\ G\varepsilon_B \circ \eta_{GB} \circ c = 1_{GB} \circ c = c$$

where the equation at the split depends on our earlier result $GFc \circ \eta_A = \eta_{A'} \circ f$, but replacing $f\colon A \to A'$ with $c\colon A \to GB$.

For uniqueness, note that the adjunction implies any $d'\colon FA \to B$ will be equal to $\psi_{AB}c'$ for some $c'\colon A \to GB$. And if $Gd \circ \eta_A = Gd' \circ \eta_A$, then by the previous argument $c = c'$ and hence $d = d'$.

It is worth noting that we can also prove the converse here. Suppose we have functors $F\colon \mathsf{C} \to \mathsf{D}$ and $G\colon \mathsf{D} \to \mathsf{C}$, and a natural transformation $\eta\colon 1_{\mathsf{C}} \Rightarrow GF$

such that for any $d\colon A \to GB$ in C there is a unique $c\colon FA \to B$ for which $d = Gc \circ \eta_A$. Then $F \dashv G$.

Why? Define $\varphi_{AB}\colon \mathsf{D}(FA,B) \to \mathsf{C}(A,GB)$ by putting $\varphi_{AB}c = Gc \circ \eta_A$. By the same proof as for Theorem 212, when we keep A fixed, the various components φ_{AB} assemble into a natural transformation $\varphi_A\colon \mathsf{D}(FA,–) \Rightarrow \mathsf{C}(A,G–)$. And when we keep B fixed, the various components φ_{AB} assemble into a natural transformation $\varphi_B\colon \mathsf{D}(F–,B) \Rightarrow \mathsf{C}(–,GB)$. Further, by the uniqueness clause, the components φ_{AB} are bijections, so the natural transformations are indeed natural isomorphisms. Therefore $\mathsf{D}(FA,B) \cong \mathsf{C}(A,GB)$ naturally in A and in B.

Putting all this together, then, we get the first half of the following theorem – with the dual half left as an exercise:

Theorem 214. *Given functors $F\colon \mathsf{C} \to \mathsf{D}$ and $G\colon \mathsf{D} \to \mathsf{C}$, then $F \dashv G$ iff (i) there is a natural transformation $\eta\colon 1_{\mathsf{C}} \Rightarrow GF$, for which (ii) for any $c\colon A \to GB$ there is a unique $d\colon FA \to B$ such that $c = Gd \circ \eta_A$.*

And dually, we also have $F \dashv G$ iff (i′) there is a natural transformation $\varepsilon\colon FG \Rightarrow 1_{\mathsf{D}}$, for which (ii″) for any $d\colon FA \to B$ there is a unique $c\colon A \to GB$ such that $d = \varepsilon_B \circ Fc$. □

Evidently, we could have recruited either half of this companion theorem as the basis of a further alternative definition for $F \dashv G$. In fact, though rather unhelpfully to my mind, this is the *first* definition given by Awodey (2010, §9.1).

42.3 Adjunctions and comma categories

(a) Back in Chapter 30 we introduced the idea of comma categories. And at the end of that chapter, we looked at a particular case, where – changing labels – A is an object of some category C, and $G\colon \mathsf{D} \to \mathsf{C}$ is a functor, and the resulting comma category $(A \downarrow G)$ has the following data:

> The objects are pairs (B,c), where B is a D-object, and c is any C-arrow $c\colon A \to GB$.
>
> An arrow $f\colon (B,c) \to (B',c')$ is a D-arrow $f\colon B \to B'$ such that this triangle commutes in C:

$$\begin{tikzcd} & GB \arrow[d, "Gf"] \\ A \arrow[ur, "c"] \arrow[r, "c'"'] & GB' \end{tikzcd}$$

The objects of $(A \downarrow G)$ are therefore – more or less! – the members of $\mathsf{C}(A,GB)$, so we are in now familiar territory. And a moment's reflection shows we can make a neat connection:

Theorem 215. *If $F \dashv G\colon \mathsf{C} \to \mathsf{D}$ is an adjunction with unit η, then (FA,η_A) is an initial object of $(A \downarrow G)$, for any C-object A.*

Proof. We need to show that for every (B, c) in $(A \downarrow G)$ there is a unique arrow $d\colon (FA, \eta_A) \to (B, c)$.

In other words, there is a unique $d\colon FA \to B$ such that this commutes:

$$\begin{tikzcd} & GFA \arrow[d, "Gd"] \\ A \arrow[ur, "\eta_A"] \arrow[r, "c"'] & GB \end{tikzcd}$$

But that's the universal mapping property proved in the previous theorem! □

(b) We again get a converse result.

Theorem 216. *Suppose $G\colon \mathsf{B} \to \mathsf{A}$ is a functor. If the derived comma category $(A \downarrow G)$ has an initial object for every $A \in \mathsf{A}$, then G has a left adjoint.*

Proof. Suppose for each C-object A, the comma category $(A \downarrow G)$ has an initial object: and – in a spirit of hope! – let's write that initial object as (FA, η_A).

Now we can define a functor $F\colon \mathsf{C} \to \mathsf{D}$ here, which on objects, sends A to FA as just defined. And on arrows, we define F as sending an arrow $c\colon A \to A'$ to $Fc\colon FA \to FA'$ where Fc is the unique arrow making this commute:

$$\begin{tikzcd} A \arrow[r, "c"] \arrow[d, "\eta_A"] & A' \arrow[d, "\eta_{A'}"] \\ GFA \arrow[r, "GFc"] & GFA' \end{tikzcd} \qquad \text{i.e.} \qquad \begin{tikzcd} & GFA \arrow[d, "GFc"] \\ A \arrow[ur, "\eta_A"] \arrow[r, "\eta_{A'} \circ c"'] & GB \end{tikzcd}$$

For note, we need a unique Fc to make the triangle commute because (FA, η_A) is initial. To show functoriality, consider

$$\begin{tikzcd} A \arrow[r, "c"] \arrow[d, "\eta_A"] \arrow[rr, bend left, "c' \circ c"] & A' \arrow[r, "c'"] \arrow[d, "\eta_{A'}"] & A'' \arrow[d, "\eta_{A''}"] \\ GFA \arrow[r, "GFc"] \arrow[rr, bend right, "GF(c' \circ c)"'] & GFA' \arrow[r, "GFc'"] & GFA'' \end{tikzcd}$$

$F(c' \circ c)$ is by definition the unique arrow making the outer rectangle commute. But since $GFc' \circ GFc$, i.e. $G(Fc' \circ Fc)$ (by the functoriality of G) also makes that rectangle commute, we have $F(c' \circ c) = Fc' \circ Fc$.

Where have we got to? We have defined a functor F, and looking again at the previous left-hand commuting square we have shown that in fact $\eta_A, \eta_{A'}, \ldots$ assemble into a natural transformation $\eta\colon 1_{\mathsf{C}} \Rightarrow GF$. And by the assumption that (FA, η_A) is initial in $(A \downarrow G)$, we know that for every $c\colon A \to GB$ there is a unique $d\colon FA \to B$ such that $c = Gd \circ \eta_A$. So we can now just appeal to Theorem 214 to conclude that $F \dashv G$. □

(c) Theorem 203 told us how to define e.g. the left adjoint of a function in a Galois connection (if it exists). We now have a nice analogue. If a functor $\mathsf{G}\colon \mathsf{D} \to \mathsf{C}$ has a left adjoint, then there is an initial object for $(A \downarrow G)$ for all C-objects A; and then the proof of the last theorem tells us how to define G left adjoint in terms of it.

Of course, we will get dual results for everything in the section, swapping around left and right adjoints. How will the story go?

42.4 Uniqueness

(a) Let's now return to another theorem that we stated without proof in the last chapter:

Theorem 209. *If a functor has an adjoint, it is unique up to natural isomorphism. If $F \dashv G$ and $F \dashv G'$ then $G \cong G'$. If $F \dashv G$ and $F' \dashv G$ then $F \cong F'$.*

We'll just look a couple of arguments, each aimed at just one half of our target two-part result, leaving the other half to be proved similarly.

Proof sketch using Yoneda. Assume $F \dashv G\colon \mathsf{C} \to \mathsf{D}$ and $F \dashv G'\colon \mathsf{C} \to \mathsf{D}$, and we aim for $G \cong G'$.

By assumption, $\mathsf{C}(A, GB) \cong \mathsf{D}(FA, B)$ naturally in A and in B. So, in particular, there is (i) a natural isomorphism $\mathsf{C}(-, GB) \xRightarrow{\sim} \mathsf{D}(F-, B)$. Likewise, $\mathsf{D}(FA, B) \cong \mathsf{C}(A, G'B)$ again naturally in A and in B. So, in particular, there is (ii) a natural isomorphism $\mathsf{D}(F-, B) \xRightarrow{\sim} \mathsf{C}(-, G'B)$.

But Theorem 153 tells us that natural isomorphisms compose. So from (i) and (ii) we can conclude that $\mathsf{C}(-, GB) \cong \mathsf{C}(-, G'B)$. Or equivalently $\mathcal{Y}(GB) \cong \mathcal{Y}(G'B)$. And hence by the Yoneda Principle, our Theorem 182, $GB \cong G'B$. But (*) since we started off with everything natural in B, the isomorphism here will also be natural in B. So $G \cong G'$, as we want. □

To complete this proof sketch – which at least tells us why our theorem *ought* to be true! – we strictly speaking need to confirm (*).

But rather than do that, it's perhaps more fun, and adds additional insight to give a different line of proof, this time for the other half of our theorem.

Proof via relation between adjunctions and comma categories. Suppose we have both $F \dashv G\colon \mathsf{C} \to \mathsf{D}$ with unit η and $F' \dashv G\colon \mathsf{C} \to \mathsf{D}$ with unit η'.

By Theorem 215, for any C-object A, both (FA, η_A) and $(F'A, \eta'_A)$ are initial objects of $(A \downarrow G)$. So, as initial objects, there must be an isomorphism α_A between them, which by the definition of arrows in $(A \downarrow G)$ means we have an isomorphism $\alpha_A\colon FA \to F'A$ in D such that $\eta'_A = G\alpha_A \circ \eta_A$. Likewise, of course, we have an isomorphism $\alpha_{A'}\colon FA' \to F'A'$ such that $\eta'_{A'} = G\alpha_{A'} \circ \eta_{A'}$.

It is then easy to show that $\alpha_A, \alpha_{A'}, \ldots$ assemble into a natural isomorphism $\alpha\colon F \xRightarrow{\sim} F'$, and therefore – as claimed – $F \cong F'$.

OK, to complete the argument, let's give the book-keeping details. So take any arrow $c\colon A \to A'$, and form the square

$$\begin{array}{ccc} FA & \xrightarrow{Fc} & FA' \\ \big\downarrow{\scriptstyle \alpha_A} & & \big\downarrow{\scriptstyle \alpha_{A'}} \\ F'A & \xrightarrow{F'c} & F'A' \end{array}$$

We need this square to always commute to give us a natural isomorphism α. We show it commutes by checking that both (i) $\alpha_{A'} \circ Fc$ and (ii) $F'c \circ \alpha_A$ are arrows from (FA, η_A) to $(F'A', \eta'_{A'} \circ c)$ in $(A \downarrow G)$ – and hence must indeed be equal since (FA, η_A) is initial in the comma category.

But for (i) and (ii) to be arrows with the announced source and target, we need the following two triangles to commute, again by definition of what it takes to be an arrow in $(A \downarrow G)$:

$$\begin{tikzcd} & GFA \arrow[d, "G(\alpha_{A'} \circ Fc)"] \\ A \arrow[ur, "\eta_A"] \arrow[r, "\eta'_{A'} \circ c"'] & GF'A' \end{tikzcd} \qquad \begin{tikzcd} & GFA \arrow[d, "G(F'c \circ \alpha_A)"] \\ A \arrow[ur, "\eta_A"] \arrow[r, "\eta'_{A'} \circ c"'] & GF'A' \end{tikzcd}$$

(i) (ii)

For (i), we have

$$G(\alpha_{A'} \circ Fc) \circ \eta_A = G\alpha_{A'} \circ GFc \circ \eta_A = G\alpha_{A'} \circ \eta_{A'} \circ c = \eta'_{A'} \circ c$$

and for (ii), we have

$$G(F'c \circ \alpha_A) \circ \eta_A = GF'c \circ G\alpha_A \circ \eta_A = GF'c \circ \eta'_A = \eta'_{A'} \circ c$$

with the inner equations in each case appealing to earlier results. So we are done.[2] □

42.5 Adjunctions compose

Theorem 202 told us that if there is a Galois connection $F \dashv G$ between the posets $(C, \preccurlyeq)$ and $(D, \sqsubseteq)$, and a connection $H \dashv K$ between the posets $(D, \sqsubseteq)$ and $(E, \subseteq)$, then there is a Galois connection $HF \dashv GK$ between $(C, \preccurlyeq)$ and $(E, \subseteq)$.

Now we can show that, as announced in the last chapter, the same holds for adjunctions more generally:

Theorem 210. *Given* $\mathsf{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathsf{D}$ ($\perp$) *and* $\mathsf{D} \underset{K}{\overset{H}{\rightleftarrows}} \mathsf{E}$ ($\perp$), *then* $\mathsf{C} \underset{GK}{\overset{HF}{\rightleftarrows}} \mathsf{E}$ ($\perp$).

[2] I learnt the basic proof idea, as so much else, from Peter Johnstone's famed Part III course on category theory. I particularly recall this as seeming very pleasingly neat at the time! The book-keeping was left as an exercise. Thanks to Izaak van Dongen for importantly correcting my garbled notes.

Proof via homsets. Since $H \dashv K$, we have $\mathsf{E}(HFA, C) \cong \mathsf{D}(FA, KC)$, naturally in A – by the argument for Theorem 155(3)[3] – and also naturally in C.

Also, since $F \dashv G$, we have $\mathsf{D}(FA, KC) \cong \mathsf{C}(A, GKC)$, naturally in A and in C.

So by Theorem 155(2), $\mathsf{E}(HFA, C) \cong \mathsf{C}(A, GKC)$ naturally in A and in C. Hence $HF \dashv GK$ □

That was quick and easy. But there is perhaps some additional fun to be had by working through another argument:

Proof by units and co-units. Since $F \dashv G$, there are a pair of natural transformations $\eta\colon 1_{\mathsf{C}} \Rightarrow GF$ and $\varepsilon\colon FG \Rightarrow 1_{\mathsf{D}}$, satisfying the usual triangle identities.

Since $H \dashv K$, there are natural transformations $\eta'\colon 1_{\mathsf{D}} \Rightarrow KH$ and $\varepsilon'\colon HK \Rightarrow 1_{\mathsf{E}}$, again satisfying the triangle identities.

We now define two more natural transformations by composition,

$$\eta'': \quad 1_{\mathsf{C}} \xRightarrow{\ \eta\ } GF \xRightarrow{G\eta' F} GKHF$$
$$\varepsilon'': \quad HFGK \xRightarrow{H\varepsilon K} HK \xRightarrow{\ \varepsilon'\ } 1_{\mathsf{E}}$$

To show $HF \dashv GK$ it suffices to check that η'' and ε'' also satisfy the triangle identities.

Consider, then, the following diagram (dropping the double arrows):

$$\begin{tikzcd}
HF \arrow[r, "HF\eta"] \arrow[dr, "1_{HF}"'] & HFGF \arrow[r, "HFG\eta' F"] \arrow[d, "H\varepsilon F"] & HFGKHF \arrow[d, "H\varepsilon KHF"] \\
& HF \arrow[r, "H\eta' F"] \arrow[dr, "1_{HF}"'] & HKHF \arrow[d, "\varepsilon' HF"] \\
& & HF
\end{tikzcd}$$

'Whiskering' the triangle identity $\varepsilon F \circ F\eta = 1_F$ by H shows that the top left triangle commutes. And whiskering the identity $\varepsilon' H \circ H\eta' = 1_H$ on the other side by F shows that the bottom triangle commutes.

Further, the square commutes. For by either the naturality of ε or the naturality of η', the following square commutes in the functor category:

$$\begin{tikzcd}
FG \arrow[r, "FG\eta'"] \arrow[d, "\varepsilon"'] & FGKH \arrow[d, "\varepsilon KH"] \\
1 \arrow[r, "\eta'"] & KH
\end{tikzcd}$$

[3] Because by definition $\mathsf{E}(HFA, C) \cong \mathsf{D}(FA, KC)$ naturally in FA; so then put FA for KB in the proof of Theorem 155.

And whiskering again gives the commuting square in the big diagram. (Exercise: check the claims about whiskering and the naturality square.)

So the whole big diagram commutes, and in particular the outer triangle commutes. But that tells us that $\varepsilon'' HF \circ HF\eta'' = 1_{HF}$ – which is one of the desired triangle identities for η'' and ε''.

The other identity follows similarly. □

42.6 Equivalences and adjunctions again

There's one more unproved theorem left over from the previous chapter. I said:

Theorem 213. *If there is an equivalence between* C *and* D *constituted by a pair of functors* $F\colon \mathsf{C} \to \mathsf{D}$ *and* $G\colon \mathsf{D} \to \mathsf{C}$ *and a pair of natural isomorphisms* $\eta\colon 1_{\mathsf{C}} \overset{\sim}{\Longrightarrow} GF$ *and* $\varepsilon\colon FG \overset{\sim}{\Longrightarrow} 1_{\mathsf{D}}$*, then there is an adjunction* $F \dashv G$ *with unit* η *and new co-unit* ε' *(defined in terms of* η *and* ε*). There is also an adjunction* $F \dashv G$ *with co-unit* ε *and new unit* η' *(again defined in terms of* η *and* ε*).*

And because the equivalence is symmetric, there is also an adjunction $G \dashv F$*.*

In other words, we can take an equivalence defined as in Defn 129*, fix one of the natural transformations, but tinker (if necessary) with the other one so as to get an adjunction. Further we can construct an adjunction in the opposite direction.

Proof. Define the natural transformation ε by composition as follows:

$$\varepsilon\colon FG \xRightarrow{FG\gamma^{-1}} FGFG \xRightarrow{(F\eta G)^{-1}} FG \xRightarrow{\quad\gamma\quad} 1_{\mathsf{B}}$$

Since η and γ are isomorphisms, and by Theorem 158 whiskering natural isomorphisms yields another natural isomorphism, the inverses mentioned here must exist.

So we just need to establish that, with ε so defined, we get the usual triangle identities $\varepsilon_{FA} \circ F\eta_A = 1_{FA}$ for all C-objects A, and also get $G\varepsilon_B \circ \eta_{GB} = 1_{GB}$ for all D-objects B.

Therefore, firstly, for any A, we need the composite arrow (*)

$$FA \xrightarrow{F\eta_A} FGFA \xrightarrow{(FG\gamma^{-1})_{FA}} FGFGFA \xrightarrow{(F\eta G)^{-1})_{FA}} FGFA \xrightarrow{\gamma_{FA}} FA$$

to equal the identity arrow on FA (recall, the component of a 'vertical' composite of natural transformations for FA is the composite of the components of the individual transformations).

We begin be noting that, for any $A \in \mathsf{A}$, the following square commutes by the naturality of η:

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "\eta_A"'] & GFA \arrow[d, "\eta_{GFA}"] \\ GFA \arrow[r, "GF\eta_A"] & GFGFA \end{tikzcd}$$

So we have $\eta_{GFA} \circ \eta_A = GF\eta_A \circ \eta_A$. But since η_A is an isomorphism, it is epic (right-cancellable), so we have $\eta_{GFA} = GF\eta_A$ for all A. Similarly, we have $\gamma^{-1}_{FGB} = (FG\gamma^{-1})_B$ for all $B \in \mathsf{B}$.

Now consider, then, the following diagram:

$$\begin{tikzcd}
FA \arrow[r, "F\eta_A"] \arrow[d, "(\gamma^{-1})_{FA}"] & FGFA \arrow[d, "(\gamma^{-1})_{FGFA} = (FG\gamma^{-1})_{FA}"] \\
FGFA \arrow[r, "FGF\eta_A"] \arrow[d, "1_{FGFA}"] & FGFGFA \arrow[dl, "(F\eta G)^{-1}_{FA}"] \\
FGFA \arrow[d, "\gamma_{FA}"] & \\
FA &
\end{tikzcd}$$

The top square commutes, being a standard naturality square. (Fill in the schema of Defn. 126 by putting the natural transformation $\alpha = \gamma^{-1} \colon 1_{\mathsf{B}} \to FG$, and put f to be the function $F\eta_A \colon FA \to FB$.) And the triangle below commutes because $FGF\eta_A = F\eta_{GFA}$ from the equation above and $F\eta_{GFA} = (F\eta G)_{FA}$ (since $\eta_{GFA} = (\eta G)_{FA}$), so the arrows along two sides are simply inverses, and therefore compose to the identity.

The whole diagram therefore commutes. The arrows on longer circuit from top-left to bottom form the composite (*). The arrows on the direct route from top to bottom compose to the identity 1_{FA}. The composites are equal and hence we have established that the first triangle identity holds.

The second triangle identity holds by a similar argument.

Hence $F \dashv G$. And finally we note that if we put $\eta' = \gamma^{-1}$ and $\gamma' = \eta^{-1}$, and put $F' = G$, $G' = F$, the same line of proof shows that $F' \dashv G'$, and so $G \dashv F$. $\square$

43 Adjoint functors and limits

We now turn to some key results which tell us how adjoint functors interact with limits. A key bit of news is that right adjoints preserve limits: and dually, exactly as you would now expect, left adjoints preserve co-limits.

43.1 Another way of getting new adjunctions from old

We've already met one way of getting new adjunctions from old, i.e. simple composition. Finally in this chapter, we now introduce another.

Definition 151. Given a functor $F\colon \mathsf{C} \to \mathsf{D}$ and small category J, then the functor which we can notate $[\mathsf{J}, F]\colon [\mathsf{J}, \mathsf{C}] \to [\mathsf{J}, \mathsf{D}]$ sends a functor $K\colon \mathsf{J} \to \mathsf{C}$ to $F \circ K\colon \mathsf{J} \to \mathsf{D}$.

Of course, strictly speaking that's an incomplete definition. We need to specify not just how $[\mathsf{J}, F]$ acts on objects of the functor category $[\mathsf{J}, \mathsf{C}]$ (i.e. acts on functors!), but how it acts on arrows (i.e. on natural transformations!). But the needed completion, as often in defining functors, really writes itself. For what is the obvious way for $[\mathsf{J}, F]$ to act on a natural transformation from K to K' with components $\alpha_J\colon KJ \to K'J$ (for a $\mathsf{J}-object\ J$ and functors $K, K'\colon \mathsf{J} \to \mathsf{C}$)? By sending it, of course, to the natural transformation from $F \circ K$ to $F \circ K'$ with components $F\alpha_J\colon FKJ \to FK'J$. Full functoriality is then immediate.

We can now state our result about how a given adjunction between functors F and G generates a new adjunction between our new-style functors $[\mathsf{J}, F]$ and $[\mathsf{J}, G]$:

Theorem 217. *If $F \dashv G\colon \mathsf{C} \to \mathsf{D}$ then $[\mathsf{J}, F] \dashv [\mathsf{J}, G]\colon [\mathsf{J}, \mathsf{C}] \to [\mathsf{J}, \mathsf{D}]$.*

Proof sketch. Take functors $K\colon \mathsf{J} \to \mathsf{C}$, $L\colon \mathsf{J} \to \mathsf{D}$. Then take any natural transformation $\beta\colon FK \Rightarrow L$ living as an arrow in $[\mathsf{J}, \mathsf{D}]$. This has components $\beta_J\colon FKJ \to LJ$ living in $\mathsf{D}(FKJ, LJ)$. By the adjunction $F \dashv G$ these components are in a natural bijection with arrows $\alpha_J\colon KJ \to GLJ$ living in $\mathsf{C}(KJ, GLJ)$, and these assemble into a natural transformation $\alpha\colon K \Rightarrow GL$ which lives in $[\mathsf{J}, \mathsf{C}]$ (the adjunction is easily checked to associate naturality squares with naturality squares). In this way we set up a natural one-to-one correspondence between natural transformations like α and β.

So we have established that there is, naturally, a bijection

$$[\mathsf{J},\mathsf{D}]([\mathsf{J},F]K,L)\cong[\mathsf{J},\mathsf{C}](K,[\mathsf{J},G]L),$$

which proves $[\mathsf{J},F]\dashv[\mathsf{J},G]$. □

You might want to check the details as an exercise!

43.2 Limit functors as adjoints

Suppose the category C has all limits of shape J. Three observations:

(1) By Theorem 76, the cones over $D\colon\mathsf{J}\to\mathsf{C}$ with vertex C correspond one-to-one with C-arrows from C to $LimD$ (i.e. $\underset{\leftarrow\mathsf{J}}{Lim}\,D$ see §36.4).

(2) But by the remark after Theorem 33.4, the set[1] of cones over $D\colon\mathsf{J}\to\mathsf{C}$ with vertex C is the hom-set $[\mathsf{J},\mathsf{C}](\Delta(C),D)$. Here $\Delta\colon\mathsf{C}\to[\mathsf{J},\mathsf{C}]$ is the functor introduced just after that theorem, which sends an object C to the constant functor $\Delta_C\colon\mathsf{J}\to\mathsf{C}$. (For convenience, understand the cones here austerely).

(3) The set of C-arrows from C to the limit vertex $Lim\,D$ is $\mathsf{C}(C,Lim(D))$, where $Lim\colon[\mathsf{J},\mathsf{C}]\to\mathsf{C}$ is the functor introduced in §36.4, a functor that exists if C has all limits of shape J and that sends a diagram D of shape J in C to some limit object in C.

So, in summary, still assuming that C has all limits of shape J, the situation is this. We have a pair of functors $\mathsf{C}\underset{Lim}{\overset{\Delta}{\rightleftarrows}}[\mathsf{J},\mathsf{C}]$ such that

$$[\mathsf{J},\mathsf{C}](\Delta(C),D)\cong\mathsf{C}(C,Lim(D)).$$

Moreover, the isomorphism that is given in our proof of Theorem 76 arises in a natural way, without making any arbitrary choices.[2] So, we can take it that the isomorphism is natural in $C\in\mathsf{C}$ and $D\in[\mathsf{J},\mathsf{C}]$. Hence Δ has a right adjoint, and one such right adjoint is Lim.

We now argue in the opposite direction starting from the assumption that the diagram Δ has a right adjoint, call it L.

Suppose that D is a diagram $D\colon\mathsf{J}\to\mathsf{C}$. Applying Theorem 214 about a universal mapping property of adjunctions, for any arrow $c\colon\Delta(C)\to D$ in $[\mathsf{J},\mathsf{C}]$ – in other words for any cone over D with vertex C – there is a unique arrow $u\colon C\to L(D)$ in C, such that $c=\varepsilon_D\circ\Delta(u)$, where ε is the co-unit of the adjunction.

By the definition of Δ, $\Delta(u)$ is the natural transformation from Δ_C to $\Delta_{L(D)}$ with every component equal to u.

[1] We cheerfully prescind here from issues of ‘size’.

[2] Careful: there were arbitrary choices made in determining what Lim does. But once Lim is fixed, the isomorphism arises naturally.

But ε_D is the transpose of $1_L(D)$, i.e. is some arrow $\pi\colon \Delta_{L(D)} \to D$ in $[\mathsf{J}, \mathsf{C}]$, i.e. is some particular cone π over D with vertex L_D.

Taken component-wise, the equation $c = \varepsilon_D \circ \Delta(u)$ tells us that for each $J \in \mathsf{J}$, $c_J = \pi_J \circ u$. In other words any cone c factors through our cone π via the unique u. Hence the cone π with vertex $L(D)$ and projection arrows π_J is a limit cone for D. However, D was any diagram $D\colon \mathsf{J} \to \mathsf{C}$. Therefore C has all limits of shape J.

Summing up, we get the following theorem:

Theorem 218. *If category* C *has all limits of shape* J*, then* Δ *has a right adjoint, and indeed* $\Delta \dashv Lim$*. Conversely, if* Δ *has any right adjoint, then* C *has all limits of shape* J*.*

43.3 Right adjoints preserve limits

We can usefully begin by restating part of a key definition and reminding ourselves of a basic theorem:

Definition 117 A functor $G\colon \mathsf{C} \to \mathsf{B}$ *preserves limits of shape* J iff, for any diagram $D\colon \mathsf{J} \to \mathsf{C}$, if $[L, \pi_J]$ is a limit cone over D, then $[GL, G\pi_J]$ is a limit cone over $G \circ D\colon \mathsf{J} \to \mathsf{B}$.

Theorem 151 *The covariant hom-functor* $\mathsf{C}(A, –)\colon \mathsf{C} \to \mathsf{Set}$*, for any* A *in the category* C*, preserves all limits that exist in* C*.*

Now, this theorem is easily seen to imply the following:

Theorem 219. *Any set-valued functor* $G\colon \mathsf{C} \to \mathsf{Set}$ *which is a right adjoint (i.e. has a left adjoint) preserves all limits that exist in* C*.*

Proof. Suppose we have a functor F such that $F \dashv G$. Then

$$GA \cong \mathsf{Set}(1, GA) \cong \mathsf{C}(F1, A)$$

with both isomorphisms natural in A (the first relies on the familiar association in Set between elements of a set and arrows from a terminal object into that set). Hence G is naturally isomorphic to the hom-functor $\mathsf{C}(F1, –)$. But the latter preserves limits, by Theorem 151. Hence so does G, by Theorem 154. $\square$

We now show that there is in fact nothing special here about set-valued functors. We can prove quite generally:

Theorem 220. *If the functor* $G\colon \mathsf{C} \to \mathsf{B}$ *is a right adjoint (i.e. has a left adjoint), it preserves all limits that exist in* C*.*

Proof from basic principles about limits and adjoints. Suppose that G has the left adjoint $F\colon \mathsf{B} \to \mathsf{C}$; and suppose also that the diagram $D\colon \mathsf{J} \to \mathsf{C}$ has a limit cone $[L, \pi_J]$ in C.

Then $[GL, G\pi_J]$ is certainly a cone over $G \circ D$ in B. We need to show, however, that it is a *limit* cone. That is to say, we need to show that, if we take any cone $[B, b_J]$ over $G \circ D$, there is a unique $u\colon B \to GL$ such that (i) for all $J \in \mathsf{J}$, $b_J = G\pi_J \circ u$.

Well, take such a cone $[B, b_J]$ over $G \circ D$. Then, going back in the other direction, $[FB, \overline{b_J}]$ is a cone over D in C, where $\overline{b_J}\colon FB \to D_J$ is the transpose of $b_J\colon B \to GD_J$ under the adjunction.

Why is $[FB, \overline{b_J}]$ a cone? Suppose we have an arrow $d\colon D_K \to D_K$. Then by assumption, since $[B, b_J]$ is a cone over $G \circ D$, $b_K = Gd \circ b_J$. Hence $\overline{b_K} = \overline{Gd \circ b_J} = d \circ \overline{b_J}$, with the second equation by Theorem 211 (1). Which indeed makes $[FB, \overline{b_J}]$ a cone too.

And now we add that $[FB, \overline{b_J}]$ must factor through $[L, \pi_J]$ via a unique $v\colon FB \to L$ such that (ii) for all $J \in \mathsf{J}$, $\overline{b_J} = \pi_J \circ v$.

So the state of play is: we have found a unique $v\colon FB \to L$; we want to find a suitable $u\colon B \to GL$. The hopeful thought is that one will turn out to be the transpose of the other under the adjunction.

The adjunction means that $\mathsf{C}(FB, C) \cong \mathsf{B}(B, GC)$ naturally in C. Which in turn means that the following square commutes, for any $\pi_J\colon L \to D_J$:

$$\begin{tikzcd}
\mathsf{C}(FB, L) \arrow[r, "\pi_J \circ -"] \arrow[d] & \mathsf{C}(FB, D_J) \arrow[d] \\
\mathsf{B}(B, GL) \arrow[r, "G\pi_J \circ -"] & \mathsf{B}(B, GD_J)
\end{tikzcd}$$

where the vertical arrows are components of the natural transformation which sends an arrow to its transform. Chase the arrow $v\colon FB \to L$ round the diagram in both directions and we get $G\pi_J \circ \overline{v} = \overline{\pi_J \circ v}$. Therefore, using (ii), if we put $u = \overline{v}$, we indeed get as required that (i) for all $J \in \mathsf{J}$, $b_J = G\pi_J \circ u$.

It just remains to confirm u's uniqueness. Suppose that $[B, b_J]$ factors through $[GL, G\pi_J]$ by some $u' = \overline{w}$. Then for all $J \in \mathsf{J}$, $b_J = G\pi_J \circ \overline{w}$. We show as before that $\overline{b_J} = \pi_J \circ w$, whence $[FB, \overline{b_J}]$ factors through $[L, \pi_J]$ via w. By the uniqueness of factorization, $w = v$ again. □

A more compressed proof. Again, suppose that G has the left adjoint $F\colon \mathsf{B} \to \mathsf{C}$; and suppose also that the diagram $D\colon \mathsf{J} \to \mathsf{C}$ has a limit cone $[L, \pi_J]$ in C. Then, using the notation '$\mathsf{C}(X, D)$' as shorthand for the functor $\mathsf{C}(X, -) \circ D$, we have

$$\begin{aligned}
\mathsf{B}(B, GL) &\cong \mathsf{C}(FB, L) \\
&\cong \mathit{Lim}\,\mathsf{C}(FB, D) \\
&\cong \mathit{Lim}\,\mathsf{B}(B, GD) \\
&\cong \mathit{Cone}(B, GD).
\end{aligned}$$

all naturally in B. So the functor $\mathit{Cone}(-, GD)$, being naturally isomorphic to $\mathsf{B}(-, GL)$ is representable, and is represented by GL, and therefore has a universal element of the form $\langle GL, g\rangle$. But such a universal element is a limit cone with vertex GL. Hence G preserves the limit $[L, \pi_J]$. □

But compression doesn't always make for illumination, and our second proof (see Leinster 2014, p. 158; compare Awodey 2010, pp. 225–6) needs some commentary.

The first line of course comes from the adjunction, and the second from the fact that the hom-functor $\mathsf{C}(FB, -)$ preserves limits, by Theorem 151. The move from the third to the fourth line is by Theorem X *The referenced theorem, currently suppressed in an earlier chapter needs to be re-inserted!*]. And the arguments at the end about representability, universal elements and limits appeal to Theorems 193 and 197.

So that leaves the move from the second to the third line, which obviously invokes the adjunction between F and G again. We know that $\mathsf{C}(FB, X) \cong \mathsf{B}(B, GX)$ naturally in X, i.e. $\mathsf{C}(FB, -)$ is naturally isomorphic to $\mathsf{B}(B, G-)$, hence by whiskering, $\mathsf{C}(FB, -) \circ D$ is naturally isomorphic to $\mathsf{B}(B, G-) \circ D$. Now apply Theorem 172 and we can conclude that $Lim\,\mathsf{C}(FB, D) \cong Lim\,\mathsf{B}(B, GD)$.

Which all goes to combine a bunch of earlier results into a neat package: but my own feeling is that the first direct proof from the underlying principles reveals better what is really going on here.

43.4 Some examples

Right adjoints preserve limits. Dually, of course, left adjoints preserve colimits (we surely needn't pause at this stage in the game to state the duals of the theorems in the last couple of sections!). So we now mention just a few elementary examples of (co)limit preservation – and also some examples where we can argue from non-preservation to the non-existence of adjoints.

(1) Back in §29.2 we noted that the forgetful functor $U : \mathsf{Mon} \to \mathsf{Set}$ preserves limits. But we now have another proof: U has a left adjoint (by §41.2, Ex. (3)) i.e. it *is* a right adjoint, so indeed must preserve limits.

There are other examples of this kind, involving a forgetful functor $U : \mathsf{Alg} \to \mathsf{Set}$, where Alg is a category of sets equipped with some algebraic structure for U to ignore. Such a forgetful U standardly has a left adjoint, so must preserve whatever limits exist in the relevant Alg.

Further, a left-adjoint to U must preserve existing colimits in Set. But Set has *all* colimits; so that this indeed requires the left-adjoints in such cases to be rather lavish constructions (as we saw them to be).

(2) Consider exponentials again.

We noted that if C is a category with exponentiation, and hence with products, exponentiation is right adjoint to taking products: $(- \times B) \dashv (-)^B$.

Since the functor $(-)^B$ is a right adjoint, it preserves such limits as exist in C. So take in particular the functor $A : 2 \to \mathsf{C}$ (where as before 2 is the discrete two object category with objects $0, 1$). Then $A_0 \times A_1$ is the vertex of a limit over A. Hence $(A_0 \times A_1)^B$ is the vertex of a limit over $(-)^B \circ A$. But the canonical limit over that composite functor is $A_0^B \times A_1^B$. Hence $(A_0 \times A_1)^B \cong A_0^B \times A_1^B$

Since the functor $- \times B$ is a left adjoint, it preserves such colimits as exist in C. Assume C has coproducts. Then, by a similar argument, $(A_0 + A_1) \times B \cong (A_0 \times B) + (A_1 \times B)$.

(3) Take the discussion in §40.3, Ex. (7) where we looked at the Galois connection between two functions between posets of equivalence classes of wffs, with the left adjoint a trivial 'add a dummy variable' map, and the right adjoint provided by applying a universal quantifier. This carries over to an adjunction of functors between certain poset categories. Since quantification is a right adjoint, it preserves limits, and in particular preserves products, which are conjunctions in this category. Which reflects the familiar logical truth that $\forall x(Px \land Qx) \equiv (\forall x Px \land \forall x Qx)$.

(4) Claim: the forgetful functor $F\colon \mathsf{Grp} \to \mathsf{Set}$ has no right adjoint. Proof: the trivial one-object group is initial in Grp; but a singleton is not initial in Set; so there is a colimit which F doesn't preserve and it therefore cannot be a left adjoint.

(5) The proof of Theorem 132 tells us that the forgetful functor $F\colon \mathsf{Mon} \to \mathsf{Set}$ fails to preserve all epimorphisms. By Theorem 144 this implies that F doesn't preserve all pushouts, and hence doesn't preserve all colimits. Hence this forgetful functor too can't be a left adjoint. Compare the arm-waving argument to the same conclusion in §41.2. Ex. (5).

43.5 The Adjoint Functor Theorems

Right adjoints preserve limits. What about the converse? If a functor preserves limits must it be a right adjoint? Well, given some results already to hand, we can easily prove the following:

Theorem 221. *If the category* B *has all limits, and the functor* $G\colon \mathsf{B} \to \mathsf{A}$ *preserves them, then* G *is a right adjoint.*

Proof. If B has all limits and G preserves them, then for any $A \in \mathsf{A}$, $(A \downarrow G)$ has all limits (by Theorem Y *this needs to be re-instated*).

So any $(A \downarrow G)$ in particular has a limit for the big diagram-as-part-of-a-category consisting of the whole of $(A \downarrow G)$ – or in terms of diagrams-as-functors, it has a limit for the identity functor $1_{(A\downarrow G)}$. Hence by (by Theorem Y *this needs to be re-instated*), each $(A \downarrow G)$ has an initial object. Hence by Theorem 216, there is a functor $F\colon \mathsf{A} \to \mathsf{B}$ such that $F \dashv G$. $\square$

And now we see the proof, we see that the condition that B has *all* limits overshoots: the result will go through so long as B has sufficiently large limits, enough to guarantee that all the functors $1_{(A\downarrow G)}$ have a limit.

This theorem looks neat but is in fact not very useful. Having all sufficiently large limits is a hard condition to fulfil. More precisely, we have

Theorem 222. *If a category* C *has limits for diagrams over all categories of size up to the size of the collection of* C*'s arrows, then* C *has at most one arrow between any two objects.*

For example, the condition of having small limits is not satisfied by typical small categories – because, in the terminology of §4.4, a complete small category has to be a *pre-order* category.

Proof. Let J be a discrete category of the same cardinality as the set of arrows of C. Let $D\colon \mathsf{J} \to \mathsf{C}$ be the diagram which sends every object in J to B. By hypothesis, D has a limit, namely the product $\prod\limits_{J \in \mathsf{J}} D(J)$ (so this is the product of B with itself, J-many times).

Suppose there are objects $A, B \in \mathsf{C}$ with arrows $f_1, f_2\colon A \to B$ where $f_1 \neq f_2$. Simple cardinality considerations show that this further supposition leads to contradiction. Which proves the theorem.

We start by asking: how many different arrows $A \to \prod\limits_{J \in \mathsf{J}} D(J)$ are there? Theorem **??** showed that if J is the discrete two object category, then there are four such arrows. Generalizing the proof in the obvious way shows that if $|\mathsf{J}|$ is the cardinality of the objects of J, there are $2^{|\mathsf{J}|}$ different arrows from $A \to \prod\limits_{J \in \mathsf{J}} D(J)$.

Hence our suppositions imply that there is a subset of the arrows in C whose cardinality is strictly greater than the cardinality of the set of arrows in C. Contradiction. □

So, in sum, Theorem 221 is of very limited application. If we want a more widely useful result of the form 'Given such-and such conditions on the functor $G\colon \mathsf{B} \to \mathsf{A}$ and the categories it relates, then G is a right adjoint', we'll need to consider a new bunch of conditions.

And indeed there two such theorems of rather wider application: the *General Adjoint Functor Theorem* and the *Special Adjoint Functor Theorem*. However, while these results are technically very difficult, the conditions under which they apply (e.g. the so-called 'solution set condition') involve new ideas which require new motivation.[3] But the Adjoint Functor Theorems arguably sit at the boundary between basic category theory and the beginnings of more serious stuff. So given the intended limited remit of *these* Notes, this is the point at which I should probably stop for the moment.

[3]If you want to follow up the technical details, which aren't particularly difficult, I can refer you to for example Leinster (2014, pp. 159–164, 171–173) and **?**, §9.8.

Bibliography

Adámek, J., Herrlich, H., and Strecker, G., 2009. *Abstract and Concrete Categories: The Joy of Cats*. Mineola, New York: Dover Publications. URL http://www.tac.mta.ca/tac/reprints/articles/17/tr17.pdf. Originally published 1990.

Awodey, S., 2010. *Category theory*, vol. 49 of *Oxford Logic Guides*. Oxford: Oxford University Press, 2nd edn.

Barr, M. and Wells, C., 1985. *Toposes, triples, and theories*. New York: Springer-Verlag.

Borceux, F., 1994. *Handbook of Categorical Algebra 1, Basic Category Theory*, vol. 50 of *Encyclopedia of Mathematics and its Applications*. Cambridge: Cambridge University Press, Cambridge.

Crole, R. L., 1993. *Categories for Types*. Cambridge: Cambridge University Press.

Eilenberg, S. and Mac Lane, S., 1942. Natural isomorphisms in group theory. *Proceedings of the National Academy of Sciences of the United States of America*, 28: 537–543.

Eilenberg, S. and Mac Lane, S., 1945. General theory of natural equivalences. *Transactions of the American Mathematical Society*, 58: 231–294.

Forster, T., 1995. *Set Theory with a Universal Set*. Oxford: Clarendon Press, 2nd edn.

Freyd, P., 1965. The theories of functors and models. In J. W. Addison, L. Henkin, and A. Tarski (eds.), *The Theory of Models*, pp. 107–120. North-Holland Publishing Co.

Freyd, P., 1970. Homotopy is not concete. URL http://www.math.upenn.edu/~pjf/homotopy.pdf.

Goldblatt, R., 2006. *Topoi: The Categorial Analysis of Logic*. Mineola, New York: Dover Publications, revised edn. URL https://tinyurl.com/GoldblattTopoi.

Grandis, M., 2018. *Category theory and applications: a textbook for beginners*. World Scientific.

Lawvere, F. W., 1969. Adjointness in foundations. *Dialectica*, 23: 281–296.

Lawvere, F. W. and Schanuel, S. H., 2009. *Conceptual Mathematics: A first introduction to categories*. Cambridge: Cambridge University Press, 2nd edn.

Leinster, T., 2000. The yoneda lemma: what's it all about? URL https://www.maths.ed.ac.uk/~tl/categories.

Leinster, T., 2014. *Basic Category Theory*. Cambridge: Cambridge University Press. URL https://arxiv.org/abs/1612.09375.

Mac Lane, S., 1997. *Categories for the Working Mathematician*. New York: Springer, 2nd edn.

Mac Lane, S. and Moerdijk, I., 1992. *Sheaves in Geometry and Logic: A First Introduction to Topos Theory*. New York, Heidelberg, Berlin: Springer.

Marquis, J.-P., 2008. *From a Geometrical Point of View: A Study of the History and Philosophy of Category Theory*. New York: Springer.

McLarty, C., 1992. *Elementary Categories, Elementary Toposes*. Oxford: Oxford University Press.

Pareigis, B., 1970. *Categories and functors*, vol. v. 39 of *Pure and applied mathematics; a series of monographs and textbooks*. New York: Academic Press.

Pierce, B. C., 1991. *Basic category theory for computer scientists*. Cambridge, Mass.: MIT Press.

Richter, B., 2020. *Category theory for homotopy theory*. Cambridge: Cambridge University Press.

Riehl, E., 2017. *Category Theory in Context*. Aurora: Dover Modern Math Originals. Mineola, New York: Dover Publications. URL https://emilyriehl.github.io/files/context.pdf.

Roman, S., 2017. *An Introduction to the Language of Category Theory*. Compact Textbooks in Mathematics. Birkhäuser Verlag.

Schubert, H., 1972. *Categories*. New York, Heidelberg, Berlin: Springer.

Shulman, M. A., 2008. Set theory for category theory. *arXiv*. URL http://arxiv.org/abs/0810.1279.

Simmons, H., 2011. *An Introduction to Category Theory*. Cambridge: Cambridge University Press.

Spivak, D. I., 2014. *Category Theory for the Sciences*. Cambridge, MA: MIT Press.

Taylor, P., 1999. *Practical Foundations of Mathematics*. No. v. 59 in Cambridge Studies in Advanced Mathematics. Cambridge University Press.