Area of Polygons of Equivalent Side Lengths

By the Circular Ring Method

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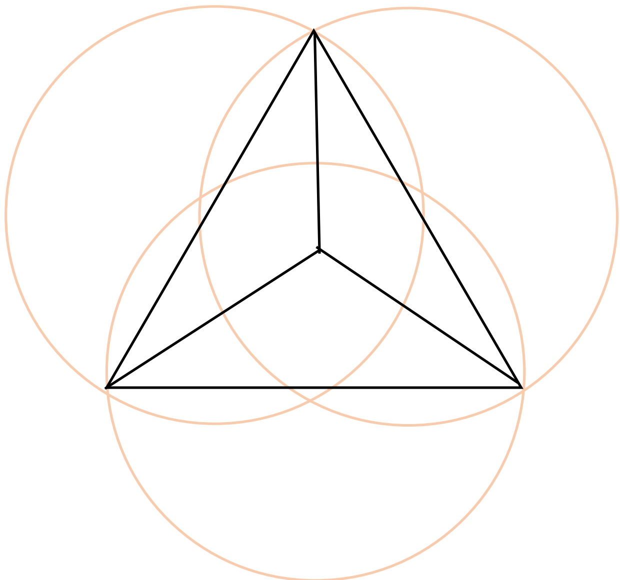
Introduction

The **Circular Ring Method** places some number of circles where the distance between the center of the circle and the center of the circular ring is identical for all considered circles. According to the following placement, the attachments of the circles connect their diameters together and forms an imaginary polygon, which is referred to as the bound. Therefore, the diameter of each circle similarly acts as the side of the imaginary polygon formed by the bound. By increasing the number of circles relevant in the formation of the circular ring, it also adds a new diameter from the new circle to join the bound, which changes the shape of the imaginary polygon. However; despite the change of the degree of polygons, the distance from the center of each circle forming the bound and the center of the circular ring always remain uniform, and this length acts as the apothem of any multi-vertex polygons. On the other hand, if all possible lines were drawn from the center of the circular ring to all existing circles of the bound, it shows that despite the change of the polygonal degree, each circle connects two lines from its circular edge towards the center of the ring, creating a triangle. Therefore, this line acts as the apothem, but also acts as the height of a single triangle divided by the number of polygonal degree (n). The main objective is to take advantage of the knowledge of triangular height and perform geometrical mathematics in order to determine a general formula for the polygonal surface area for any polygons about the degree $\mathbf{3 \leq n} < \infty$.

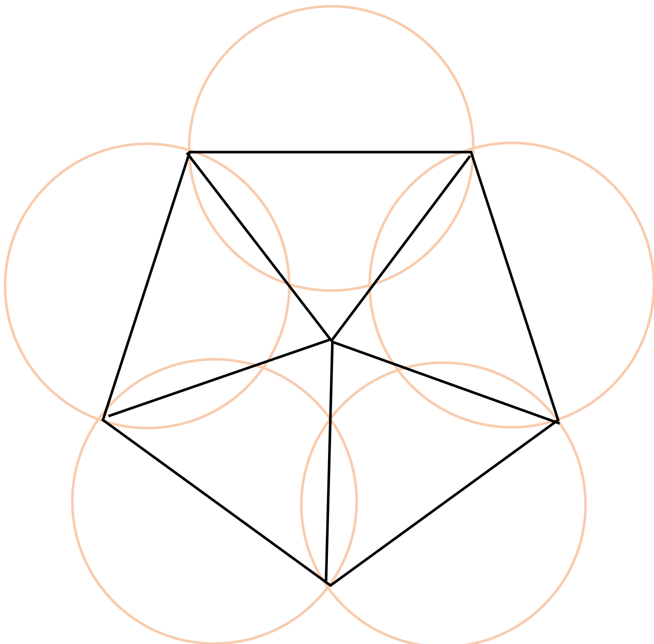
Circular Ring Method

Model A

n = 3

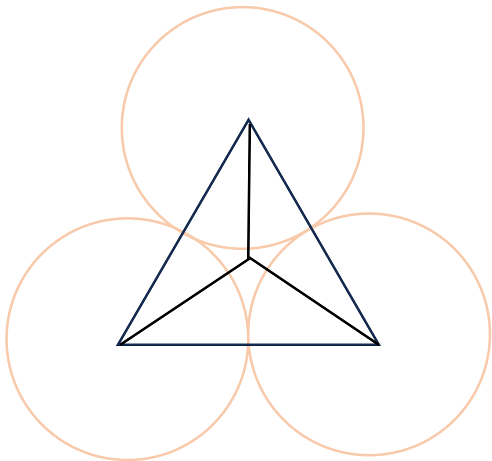


n = 5

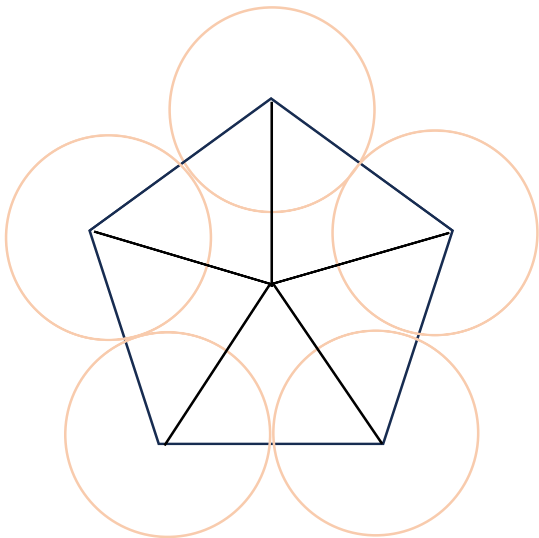


Model B

n = 3



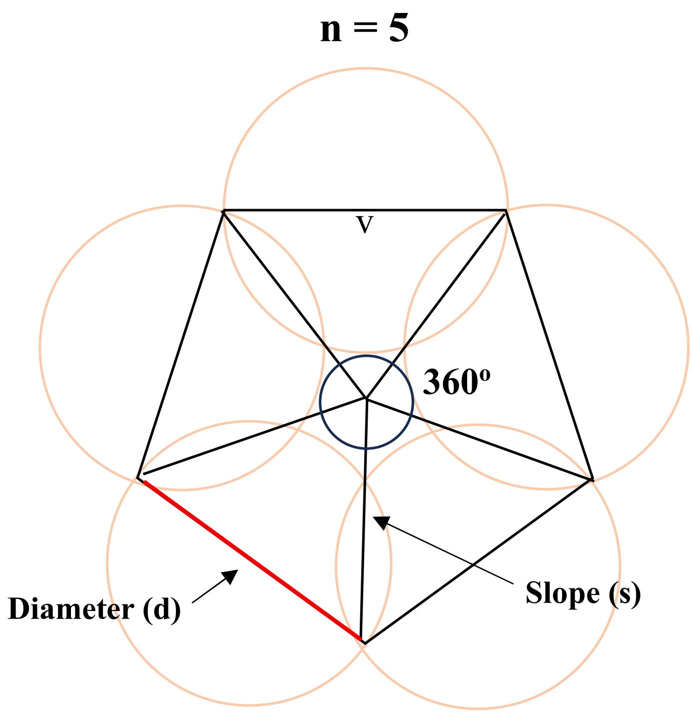
n = 5



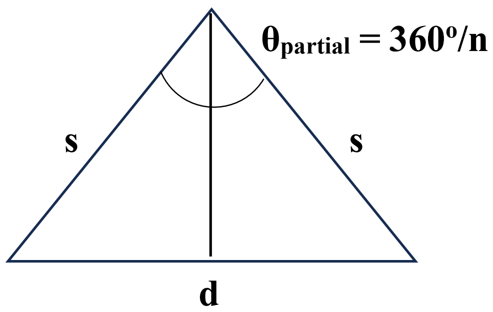
Derivation

Method #1: Sine Law

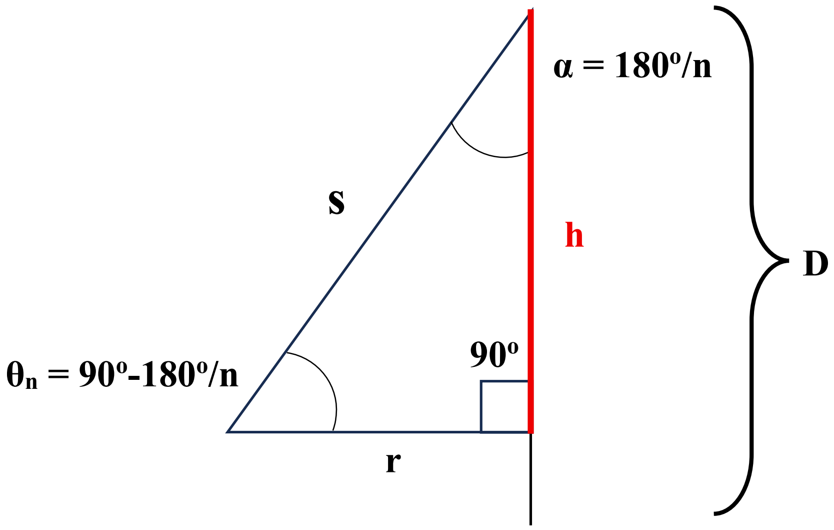
Model A



Take the polygon to be a pentagon (n = 5) and draw lines from each corner of the imaginary polygon to the center. It divides the polygon into single triangles.



Since the total inner angle of any polygon sums to 360°, the top angle of the divided triangle is 360° divided by the degree of the polygon **"n".**



In order to determine the height of a single triangle, the full triangle is divided in half; therefore, the top angle is also halved. The angle of the bottom right corner of the half-triangle is perpendicular; therefore, it is 90°. Since the sum of interior angles of a two-dimensional triangle must add up to 180°, the angle of left bottom corner can be calculated as below.

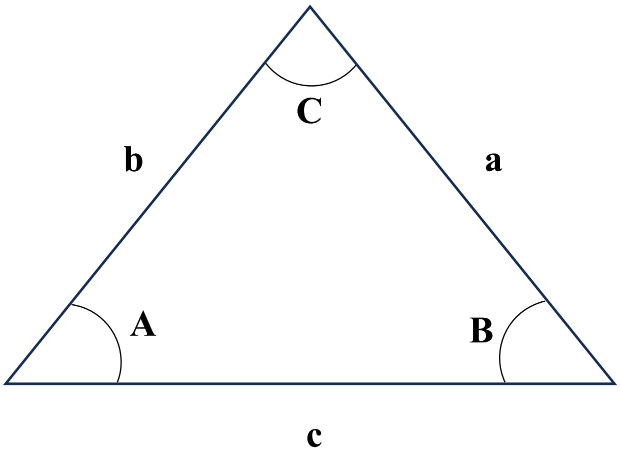
$$\theta_n = 180° - 90° - \frac{180°}{n}$$

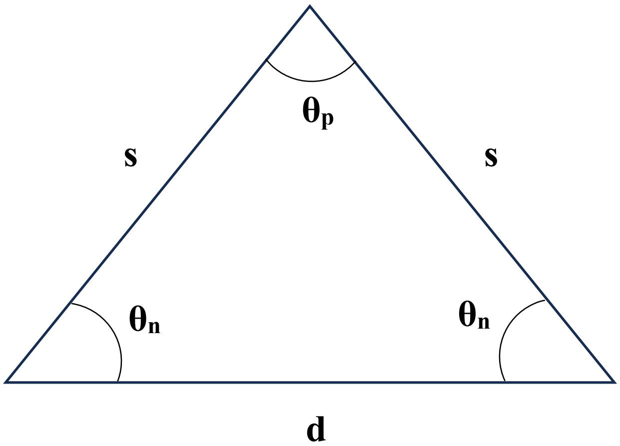
$$\boxed{\boldsymbol{\theta_n} = \left(\mathbf{90°} - \frac{\mathbf{180°}}{\boldsymbol{n}}\right)}$$

Since the height of the triangle is equivalent to the projection of **s** over a length that contains **h**, in simpler method, **h** is the horizontal component of the length **s**. Therefore, the value of **h** depends on the value of **s**.

$$\boldsymbol{h} = \mathbf{proj.}_{\boldsymbol{D}}^{\boldsymbol{s}} = \left(\frac{\boldsymbol{s} \cdot \boldsymbol{D}}{\boldsymbol{D} \cdot \boldsymbol{D}}\right)\begin{bmatrix} \boldsymbol{D_x} \\ \boldsymbol{D_y} \\ \boldsymbol{D_z} \end{bmatrix}$$

$$\boxed{\boldsymbol{h} = \boldsymbol{s}\cos(\boldsymbol{\alpha})}$$





In order to evaluate the value of the length **s**, the half triangle is recombined into the full triangle. The general angles for all the corners are determinable, while provided some degree of a polygon (**n**) and the value of **c** is defined as the diameter (**d**) of the outer circle participating in the formation of the bound. Due to Sine Law, the formula for **s** is stated as below.

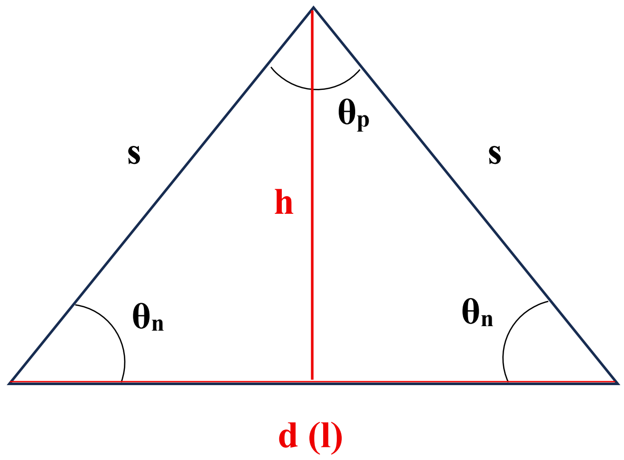
$$\frac{Sin(\theta_n)}{s} = \frac{Sin(\theta_p)}{d}$$

$$\boxed{\boldsymbol{s = d\frac{Sin(\theta_n)}{Sin(\theta_p)} = d\frac{Sin\left(90^\circ - \frac{180^\circ}{n}\right)}{Sin\left(\frac{360^\circ}{n}\right)}}}$$

$$\boldsymbol{h = s\cos(\alpha)}$$

$$h = \left(d\frac{Sin(\theta_n)}{Sin(\theta_p)}\right)cos(\alpha)$$

$$\boxed{\boldsymbol{h = d\frac{Sin(\theta_n)cos(\alpha)}{Sin(\theta_p)}}}$$



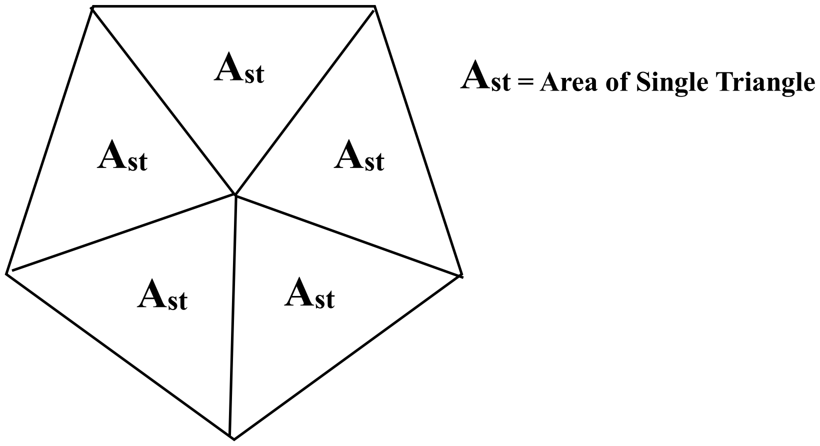
$$A_{triangle} = \frac{bh}{2}$$

$$= \frac{d\left(d\frac{Sin(\theta_n)cos(\alpha)}{Sin(\theta_p)}\right)}{2} = \frac{d^2}{2}\left(\frac{Sin(\theta_n)cos(\alpha)}{Sin(\theta_p)}\right)$$

$$\boxed{\boldsymbol{A_{single\ triangle} = \frac{l^2}{2}\left(\frac{Sin\left(90° - \frac{180°}{n}\right)cos(\frac{180°}{n})}{Sin\left(\frac{360°}{n}\right)}\right)}}$$

By determining the general formula for the height of a single triangle divided by **n**, the general formula for the area of a single triangle is derived. Since the diameter represents the side length of imaginary polygons, **d** can be expressed as **length (l)**.

n = 5



Finally, the general formula for the area of individual single triangle is stated; therefore, the total area of a polygon of any degree is simply the multiple of the single triangles. Therefore, the general formula for any equal-sided polygon ranging from **3 ≤ n** < ∞ is the **area of single triangle** multiplied by **n**, which is the number of polygon's vertices. The derived formula is expressed below.

$$A_{polygon}(n) = n\left(A_{single\ triangle}\right)$$

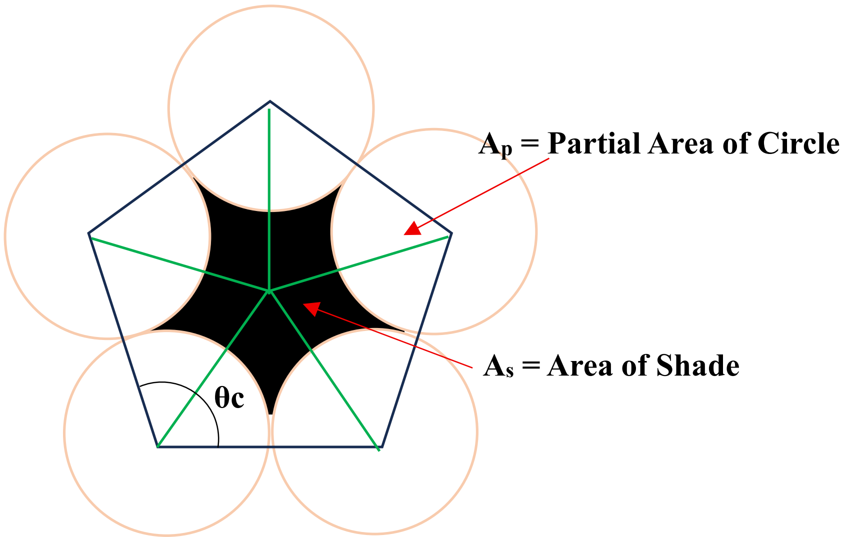
$$= n\frac{l^2}{2}\left(\frac{Sin\left(90° - \frac{180°}{n}\right)cos(\frac{180°}{n})}{Sin\left(\frac{360°}{n}\right)}\right)$$

$$\boxed{\boldsymbol{A_{polygon}(n) = \frac{nl^2}{2}\left(\frac{Sin\left(90° - \frac{180°}{n}\right)cos(\frac{180°}{n})}{Sin\left(\frac{360°}{n}\right)}\right)}}$$

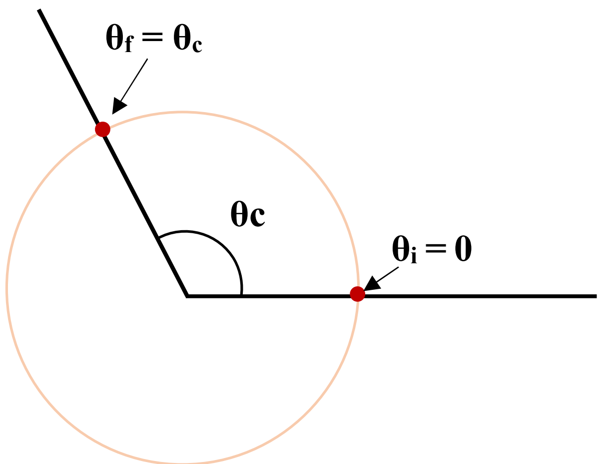
Method #2: Integration

Model B

n = 5



In theory, alternative method to evaluate the area of an equilateral polygon is to add the **summed partial area of circles** and the **area of shade**, created from the bounds of circular ring.



The total area of a circle is defined as $\mathbf{x^2 + y^2 = r^2}$; therefore, the equation reformed in terms of the **y** variable is **y = sqrt(r²-x²)**. The total area of the partial circles is the area of single partial circle multiplied by **n**.

$$A_{partial\ circle} = \int_a^b \int_c^d \sqrt{x^2 + y^2}\, \mathbf{dy\ dx}$$

$$A_{polar} = \int_0^{\theta_c} \int_0^{r} \sqrt{\cos^2\theta + \sin^2\theta}\, \mathbf{r\ dr\ d\theta}$$

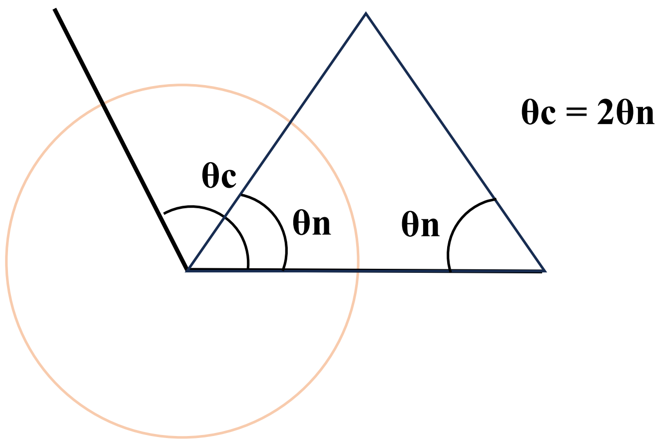
$$= \int_0^{\theta_c} \int_0^{r} \sqrt{1}\, r\, dr\, d\theta = \int_0^{\theta_c} \int_0^{r} 1\, r\, dr\, d\theta$$

$$in \rightarrow \int_0^{r} \mathbf{r\ dr} = \left(\frac{r^2}{2}\bigg|_0^r\right) = \frac{r^2}{2} - \frac{0}{2} = \frac{\mathbf{r^2}}{\mathbf{2}}$$

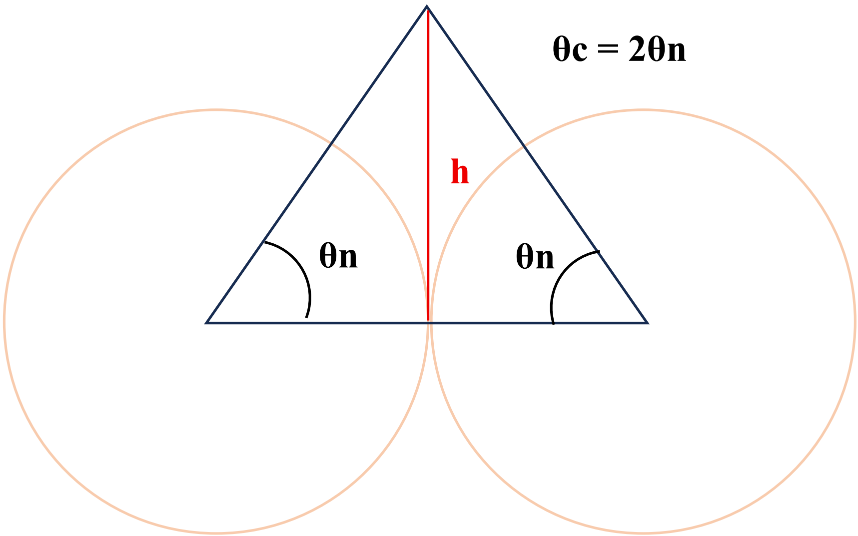
$$total \rightarrow \int_0^{\theta_c} \frac{\mathbf{r^2}}{\mathbf{2}} d\theta = \frac{r^2}{2} \int_0^{\theta_c} 1\, d\theta = \frac{r^2}{2}\left(\theta\big|_0^{\theta_c}\right)$$

$$= \frac{r^2}{2}(\theta_c - 0) = \frac{1}{2} r^2 \theta_c = \frac{1}{2} r^2 2\theta_n = r^2 \theta_n$$

$$\boxed{A_{partial\ circle} = \mathbf{r^2 \theta_{n\ (rad)}}}$$



$$\boxed{\sum \mathbf{A}_{\text{partial circle}}(n) = \mathbf{n r^2 \theta_n} = \mathbf{n r^2}\left(\frac{\pi}{2} - \frac{\pi}{n}\right) = \mathbf{n \pi r^2}\left(\frac{1}{2} - \frac{1}{n}\right)}$$



From previous derivation:

$$A_{single\ triangle} = \frac{l^2}{2}\left(\frac{Sin\left(90° - \frac{180°}{n}\right) cos(\frac{180°}{n})}{Sin\left(\frac{360°}{n}\right)}\right)$$

$$\mathrm{A}_{\text{partial circle }(2\theta_n)} = \pi r^2\left(\frac{1}{2} - \frac{1}{n}\right)$$

$$\mathrm{A}_{\text{shade of single triangle}} = \mathrm{A}_{\text{single triangle}} - \mathrm{A}_{\text{partial circle }(2\theta_n)}$$

$$= \frac{l^2}{2}\left(\frac{Sin\left(90° - \frac{180°}{n}\right) cos(\frac{180°}{n})}{Sin\left(\frac{360°}{n}\right)}\right) - \pi\left(\frac{l}{2}\right)^2\left(\frac{1}{2} - \frac{1}{n}\right)$$

The area of the shade for one divided single triangle is the difference between the **total area of a single triangle** and **the area of 2θ$_n$**. The radius **r** is the half of diameter of an imaginary polygon, which is its side length **l**. Therefore, radius **r** can be re-expressed as the **half of l**.

$$= \frac{l^2}{2}\left(\frac{Sin\left(90° - \frac{180°}{n}\right) cos(\frac{180°}{n})}{Sin\left(\frac{360°}{n}\right)}\right) - \frac{\pi l^2}{4}\left(\frac{1}{2} - \frac{1}{\mathrm{n}}\right)$$

$$= \frac{l^2}{2}\left(\frac{Sin\left(90° - \frac{180°}{n}\right) \cos\left(\frac{180°}{n}\right)}{Sin\left(\frac{360°}{n}\right)} - \frac{\pi}{2}\left(\frac{1}{2} - \frac{1}{\mathrm{n}}\right)\right)$$

$$= \frac{l^2}{2}\left(\frac{Sin\left(90° - \frac{180°}{n}\right) \cos\left(\frac{180°}{n}\right)}{Sin\left(\frac{360°}{n}\right)} - \frac{\pi}{2}\left(\frac{\mathrm{n} - 2}{2\mathrm{n}}\right)\right)$$

$$\boxed{\sum \mathbf{A}_{\mathbf{shade}}(\boldsymbol{n}) = \frac{\boldsymbol{n}\boldsymbol{l}^2}{\mathbf{2}}\left(\frac{\boldsymbol{Sin}\left(\mathbf{90°} - \frac{\mathbf{180°}}{\boldsymbol{n}}\right) \mathbf{cos}\left(\frac{\mathbf{180°}}{\boldsymbol{n}}\right)}{\boldsymbol{Sin}\left(\frac{\mathbf{360°}}{\boldsymbol{n}}\right)} - \frac{\boldsymbol{\pi}(\boldsymbol{n} - \mathbf{2})}{\mathbf{4}\boldsymbol{n}}\right)}$$

$$\mathbf{A}_{\mathbf{polygon}}(\boldsymbol{n}) = \sum \mathbf{A}_{\mathbf{shade}}(\boldsymbol{n}) + \sum \mathbf{A}_{\mathbf{partial\ circle}}(\boldsymbol{n})$$

$$= \frac{\boldsymbol{n}\boldsymbol{l}^2}{\mathbf{2}}\left(\frac{\boldsymbol{Sin}\left(\mathbf{90°} - \frac{\mathbf{180°}}{\boldsymbol{n}}\right) \mathbf{cos}\left(\frac{\mathbf{180°}}{\boldsymbol{n}}\right)}{\boldsymbol{Sin}\left(\frac{\mathbf{360°}}{\boldsymbol{n}}\right)} - \frac{\boldsymbol{\pi}(\boldsymbol{n} - \mathbf{2})}{\mathbf{4}\boldsymbol{n}}\right) + \frac{\mathbf{n}\boldsymbol{\pi}\boldsymbol{l}^2}{\mathbf{4}}\left(\frac{\mathbf{1}}{\mathbf{2}} - \frac{\mathbf{1}}{\mathbf{n}}\right)$$

$$= \frac{\boldsymbol{n}\boldsymbol{l}^2}{\mathbf{2}}\left[\left(\frac{\boldsymbol{Sin}\left(\mathbf{90°} - \frac{\mathbf{180°}}{\boldsymbol{n}}\right) \mathbf{cos}\left(\frac{\mathbf{180°}}{\boldsymbol{n}}\right)}{\boldsymbol{Sin}\left(\frac{\mathbf{360°}}{\boldsymbol{n}}\right)} - \frac{\boldsymbol{\pi}(\boldsymbol{n} - \mathbf{2})}{\mathbf{4}\boldsymbol{n}}\right) + \frac{\boldsymbol{\pi}}{\mathbf{2}}\left(\frac{\mathbf{1}}{\mathbf{2}} - \frac{\mathbf{1}}{\mathbf{n}}\right)\right]$$

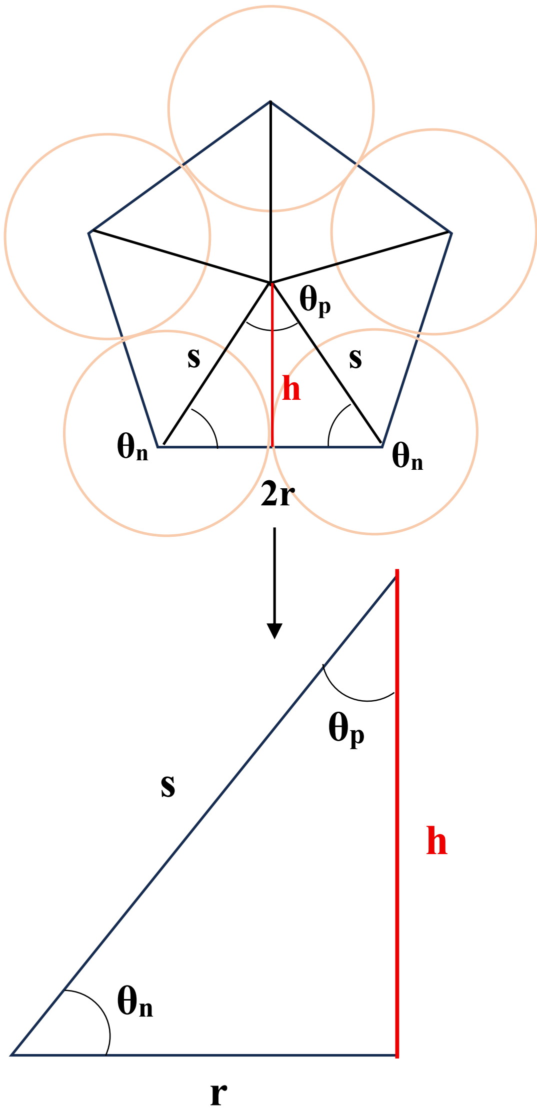
$$= \frac{\boldsymbol{n}\boldsymbol{l}^2}{\mathbf{2}}\left[\frac{\boldsymbol{Sin}\left(\mathbf{90°} - \frac{\mathbf{180°}}{\boldsymbol{n}}\right) \mathbf{cos}\left(\frac{\mathbf{180°}}{\boldsymbol{n}}\right)}{\boldsymbol{Sin}\left(\frac{\mathbf{360°}}{\boldsymbol{n}}\right)} - \cancel{\frac{\boldsymbol{\pi}(\boldsymbol{n} - \mathbf{2})}{\mathbf{4}\boldsymbol{n}}} + \cancel{\frac{\boldsymbol{\pi}(\mathbf{n} - \mathbf{2})}{\mathbf{4}\boldsymbol{n}}}\right]$$

$$\boxed{\mathbf{A}_{\mathbf{polygon}}(\boldsymbol{n}) = \frac{\boldsymbol{n}\boldsymbol{l}^2}{\mathbf{2}}\left(\frac{\boldsymbol{Sin}\left(\mathbf{90°} - \frac{\mathbf{180°}}{\boldsymbol{n}}\right) \mathbf{cos}\left(\frac{\mathbf{180°}}{\boldsymbol{n}}\right)}{\boldsymbol{Sin}\left(\frac{\mathbf{360°}}{\boldsymbol{n}}\right)}\right)}$$

Method #3: Geometry

Model B

$n = 5$



$$\mathbf{r = s\cos\theta_n} \qquad\qquad \mathbf{h = s\sin\theta_n}$$

$$s = \frac{r}{\cos\theta_n} \qquad\qquad s = \frac{h}{\sin\theta_n}$$

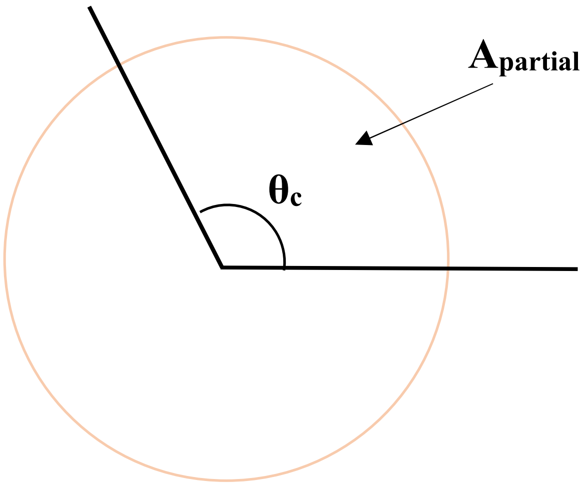
$$\frac{r}{\cos\theta_n} = \frac{h}{\sin\theta_n} \quad \rightarrow \quad \boxed{\mathbf{h = \frac{r\sin\theta_n}{\cos\theta_n} = r\tan\theta_n}}$$

$$A_{single\ triangle} = \frac{\mathbf{bh}}{\mathbf{2}}$$

$$= \frac{(\cancel{2}r)(r\tan\theta_n)}{\cancel{2}}$$

$$\boxed{A_{single\ triangle} = \mathbf{r^2 \tan\theta_n}}$$

The **radius (r)** of the bounding circle and the **height of the single triangle (h)** act as the horizontal component and vertical component of the triangle respectively. Both componental expressions isolate **s** and are set equivalent to each other in order to derive a general formula for the height of the half-triangle. From the derived expression of the triangular height, the general formula for the area of a single divided triangle by **n** is represented above.



$$\frac{\mathbf{360°}}{\mathbf{A_c}} = \frac{\boldsymbol{\theta_c}}{\mathbf{A_{partial}}}$$

The area of a complete circle covers **360°**, while the partial area of a circle only covers up to some specific $\boldsymbol{\theta_c}$. Therefore, a general proportionality formula is set in order to derive a general formula for the $\mathbf{A_{partial}}$. The complete area of a circle can be re-expressed as $\boldsymbol{\pi r^2}$.

$$\frac{360°}{\pi r^2} = \frac{\theta_c}{A_{partial}}$$

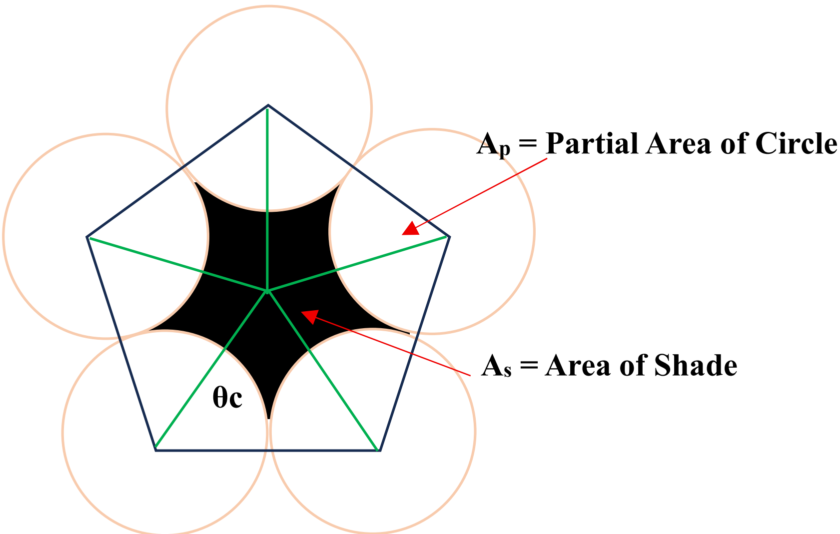
$$= \frac{\theta_c \pi r^2}{360°} = \frac{\theta_c \cancel{\pi} r^2}{2\cancel{\pi}}$$

$$= \frac{\theta_c r^2}{2} = \frac{1}{2} r^2 \theta_c$$

$$= \frac{1}{\cancel{2}} r^2 (\cancel{2}\theta_n)$$

$$\boxed{\mathbf{A_{partial\ circle} = r^2 \theta_n}}$$

n = 5



$$A_{\text{shade of single triangle}} = A_{\text{single triangle}} - A_{\text{partial}}$$

$$= r^2 \tan\theta_n - r^2\theta_n = r^2(\tan\theta_n - \theta_n)$$

$$= r^2\left(\tan\left(90° - \frac{180°}{n}\right) - \left(\frac{\pi}{2} - \frac{\pi}{n}\right)\right)$$

$$= r^2\left(\tan\left(90° - \frac{180°}{n}\right) - \pi\left(\frac{n-2}{2n}\right)\right)$$

$$\boxed{\mathbf{A_{\text{shade of single triangle}} = r^2\left(\tan\left(90° - \frac{180°}{\boldsymbol{n}}\right) - \frac{\pi(n-2)}{2n}\right)}}$$

$$\mathbf{A_{polygon}(n)} = \sum \mathbf{A_{shade}} + \sum \mathbf{A_{partial\ circle}}$$

$$= \mathrm{n}\mathrm{A}_{\text{shade}} + \mathrm{n}\mathrm{A}_{\text{partial circle}}$$

$$= \mathrm{n}\left(\mathrm{A}_{\text{shade}} + \mathrm{A}_{\text{partial circle}}\right)$$

$$= \mathrm{n}\left(\mathrm{r}^2\left(\tan\left(90^\circ - \frac{180^\circ}{n}\right) - \frac{\pi(\mathrm{n}-2)}{2\mathrm{n}}\right) + \mathrm{r}^2\theta_{\mathrm{n}}\right)$$

$$= \mathrm{n}\left(\mathrm{r}^2\left(\tan\left(90^\circ - \frac{180^\circ}{n}\right) - \frac{\pi(\mathrm{n}-2)}{2\mathrm{n}}\right) + \mathrm{r}^2\left(\frac{\pi}{2} - \frac{\pi}{\mathrm{n}}\right)\right)$$

$$= \mathrm{n}\left(\mathrm{r}^2\left(\tan\left(90^\circ - \frac{180^\circ}{n}\right) - \frac{\pi(\mathrm{n}-2)}{2\mathrm{n}}\right) + \pi\mathrm{r}^2\left(\frac{\mathrm{n}-2}{2\mathrm{n}}\right)\right)$$

$$= \mathrm{n}\left(\left(\mathrm{r}^2\tan\left(90^\circ - \frac{180^\circ}{\mathrm{n}}\right) - \frac{\pi\mathrm{r}^2(\mathrm{n}-2)}{2\mathrm{n}}\right) + \frac{\pi\mathrm{r}^2(\mathrm{n}-2)}{2\mathrm{n}}\right)$$

$$= \mathrm{n}\left(\mathrm{r}^2\tan\left(90^\circ - \frac{180^\circ}{\mathrm{n}}\right) - \cancel{\frac{\pi\mathrm{r}^2(\mathrm{n}-2)}{2\mathrm{n}}} + \cancel{\frac{\pi\mathrm{r}^2(\mathrm{n}-2)}{2\mathrm{n}}}\right)$$

$$= \mathrm{n}\mathrm{r}^2\tan\left(90^\circ - \frac{180^\circ}{\mathrm{n}}\right)$$

$$= \mathrm{n}\left(\frac{l}{2}\right)^2\tan\left(90^\circ - \frac{180^\circ}{\mathrm{n}}\right)$$

$$\boxed{\mathbf{A_{polygon}(n)} = \frac{\boldsymbol{nl^2}}{\mathbf{4}}\,\mathbf{tan}\left(\mathbf{90^\circ} - \frac{\mathbf{180^\circ}}{\mathbf{n}}\right)}$$

The general formula for the area of an equal-sided polygon is the difference between the **area of shade multiplied by n** and the **area of partial circle multiplied by n**. The following formula is purely derived from geometric relations without referring to certain laws, such as the Sine Law or advanced mathematics, such as integration.

Proof

The general formula for the area of a polygon at **nth degree** is derived by using three methods and is represented in two forms. Here, it mathematically proves that the formula of the area brought by Sine Law and Integration can be re-expressed to be exactly equivalent as the formula written from the last method.

Prove equivalence by deriving RHS to LHS:

$$\frac{\boldsymbol{nl^2}}{\mathbf{4}}\,\mathbf{tan}\left(\mathbf{90^\circ}-\frac{\mathbf{180^\circ}}{\mathbf{n}}\right)=\frac{\boldsymbol{nl^2}}{\mathbf{2}}\left(\frac{\boldsymbol{Sin}\left(\mathbf{90^\circ}-\frac{\mathbf{180^\circ}}{\boldsymbol{n}}\right)\mathbf{cos}\left(\frac{\mathbf{180^\circ}}{\boldsymbol{n}}\right)}{\boldsymbol{Sin}\left(\frac{\mathbf{360^\circ}}{\boldsymbol{n}}\right)}\right)$$

$$=\frac{nl^2}{2}\left(\frac{cos\left(\frac{180^\circ}{n}\right)\cos\left(\frac{180^\circ}{n}\right)}{Sin\left(2\cdot\frac{180^\circ}{n}\right)}\right)$$

$$=\frac{nl^2}{2}\left(\frac{cos\left(\frac{180^\circ}{n}\right)\cancel{\cos\left(\frac{180^\circ}{n}\right)}}{2\sin\left(\frac{180^\circ}{n}\right)\cancel{\cos\left(\frac{180^\circ}{n}\right)}}\right)$$

$$=\frac{nl^2}{4}\left(\frac{\cos\left(\frac{180^\circ}{n}\right)}{\sin\left(\frac{180^\circ}{n}\right)}\right)$$

$$=\frac{nl^2}{4}\left(\cot\left(\frac{180^\circ}{n}\right)\right)$$

$$=\frac{nl^2}{4}\left(-\tan\left(\frac{180^\circ}{n}\pm 90^\circ\right)\right)$$

$$\frac{nl^2}{4}\tan\left(90^\circ-\frac{180^\circ}{\mathrm{n}}\right)=\frac{nl^2}{4}\tan\left(90^\circ-\frac{180^\circ}{n}\right)$$

Therefore, the general formula of polygonal area derived using Sine Law and Integration is equivalent to the simpler formula expressed using geometry.

Conclusion

It is possible to mathematically calculate the area of a triangle, square, pentagon, etc. for equal-sided polygons by using introduced formulas. However, the objective of this derivation was set to state a single formula that could generally calculate the area of any equilateral polygons at any given number of vertices **(n)**. Three methods have been used in order to represent the corresponding general formulas of polygonal surface area, where first method borrowed an existing law, such as the Sine Law to find the general formula of the slope **s** and height **h** and determine the sum of area of single triangles. The second method preferred double integration of the circular equation $\mathbf{x^2+y^2=r^2}$, where the integral was performed in respect to **dy dx**. For simplicity, the integral in terms of **x** and **y** was converted into polar form to be in terms of **r** and **θ** and the differentials, according to Jacobian law, were as well converted to **r dr dθ**. Then, the complete integration returned a simple general formula that returns the area of a part of a full circle and it was summed with previously derived general formula of the shade in order to calculate the general area of any polygon. Lastly, the third method used pure geometry and determined the general height **h** by relating the horizontal and vertical components of a half-triangle in respect to $\boldsymbol{\theta_n}$, instead of using $\boldsymbol{\theta_p}$ just as 1st and 2nd methods. Next, the general formula of the area of a single triangle was derived using determined formula of the height. Then, a simple proportionality was set to bring a formula that determines $\boldsymbol{\theta_c}$ without integration, and the area of the shade could be determined as well. Finally, by summing the sum by **n** of the area of partial circles and area of the shades, the simplest general formula for the surface area of an equilateral polygon at any degree of **n** (number of vertices) was stated.

$$\boxed{\mathbf{A_{polygon}(n)} = \frac{nl^2}{4}\tan\left(90^\circ - \frac{180^\circ}{n}\right) = \frac{nl^2}{4}\tan\left(\frac{\pi(n-2)}{2n}\right)}$$