CSE 250A - Homework 4

4.1)

a) $\mathcal{L} = \log P(\text{data}) =$

$$\log\left(\prod_{n=1}^{N} P(X_n)\right) =$$

$$\sum_{d=1}^{2\cdot D} \sum_{n=1}^{C_d} \log(P(X=d)) =$$

$$\boxed{\sum_{d=1}^{2\cdot D} C_d \cdot \log(P_d)}$$

b) Use from part a $\qquad \sum_{d=1}^{2\cdot D} P_d - 1 = 0$

$$\mathcal{L} = C_d \cdot \log(P_d) - \lambda \cdot \left(\sum_{d=1}^{2\cdot D} P_d - 1\right)$$

$$\frac{\partial \mathcal{L}}{\partial P_d} = \frac{\partial}{\partial P_d}\left(C_d \cdot \log(P_d) - \lambda \cdot \left(\sum_{d=1}^{2\cdot D} P_d - 1\right)\right) =$$

$$\frac{C_d}{P_d} - \lambda \cdot 1 + \lambda \cdot 0 =$$

$$\frac{C_d}{P_d} - \lambda$$

$$\frac{C_d}{P_d} - \lambda = 0$$

$$\frac{C_d}{P_d} = \lambda$$

$$C_d = \lambda \cdot P_d$$

$$P_d = \frac{C_d}{\lambda}$$

Recall that $\sum_{d=1}^{2\cdot D} P_d = 1$

$$\sum_{d=1}^{2\cdot D} P_d = \sum_{d=1}^{2\cdot D} \frac{C_d}{\lambda} = 1 \longrightarrow \lambda = \sum_{d=1}^{2\cdot D} C_d$$

$$P_d = \frac{C_d}{\lambda}$$

$$\boxed{P_d = \frac{C_d}{\sum_{d=1}^{2\cdot D} C_d}}$$

$\{ \begin{matrix} -1 & \text{when } d \text{ is odd} \\ 1 & \text{when } d \text{ is even} \end{matrix}$

split the summation

c) $\sum_{d=1}^{2\cdot D} (-1)^d \cdot P_d = 0$

odd $2\cdot d - 1$
even $2\cdot d$

$$\sum_{d=1}^{2D} (-1)^d \cdot P_d =$$

$$\sum_{d=1}^{D} (-1)^{2\cdot d-1} \cdot P_{2\cdot d-1} + \sum_{d=1}^{D} (-1)^{2\cdot d} \cdot P_{2d} =$$

always -1 always 1

$$(-1)\cdot \sum_{d=1}^{D} P_{2\cdot d-1} + (1)\cdot \sum_{d=1}^{D} P_{2\cdot d} =$$

$$-\sum_{d=1}^{D} P_{2\cdot d-1} + \sum_{d=1}^{D} P_{2\cdot d} = 0$$

$$\sum_{d=1}^{D} P_{2\cdot d} = \sum_{d=1}^{D} P_{2\cdot d-1}$$

Thus, $P(X \in \{2, 4, \dots, 2\cdot D\}) = P(X \in \{1, 3, \dots, 2\cdot D - 1\})$ when $\sum_{d=1}^{2\cdot D} (-1)^d \cdot P_d = 0$

d) One of our constraints is $\sum_{d=1}^{2\cdot D} P_d = 1$. From part c, we also know that $\sum_{d=1}^{D} P_{2\cdot d} = \sum_{d=1}^{D} P_{2\cdot d-1}$. Thus, this implies that $\sum_{d=1}^{D} P_{2\cdot d} = \frac{1}{2}$ and $\sum_{d=1}^{D} P_{2\cdot d-1} = \frac{1}{2}$. From part a, we have $\sum_{d=1}^{2\cdot D} C_d \cdot \log(P_d)$. Let's split this into evens and odds too: $\sum_{d=1}^{D} C_{2\cdot d-1} \cdot \log(P_{2\cdot d-1}) + \sum_{d=1}^{D} C_{2\cdot d} \cdot \log(P_{2\cdot d})$. Overall, we want to solve for

$$\max_{P_d} \sum_{d=1}^{2\cdot D} C_d \cdot \log(P_d) \quad \text{s.t.} \quad \sum_{d=1}^{2\cdot D} P_d = 1 \quad \text{and} \quad \sum_{d=1}^{2\cdot D} (-1)^d \cdot P_d = 0$$

Rewrite this on the next page.

Using $\sum_{d=1}^{2\cdot D} (-1)^d \cdot P_d = 0$, convert

$\sum_{d=1}^{2\cdot D} P_d = 1$ into $\sum_{d=1}^{D} P_{2\cdot d-1} = \frac{1}{2}$ and $\sum_{d=1}^{D} P_{2\cdot d} = \frac{1}{2}$

Thus, we can split this maximization problem into 2 different problems.

$$\max_{P_{2\cdot d-1}} \sum_{d=1}^{D} C_{2\cdot d-1} \cdot \text{Log}(P_{2\cdot d-1}) \quad \text{s.t.} \sum_{d=1}^{D} P_{2\cdot d-1} = \frac{1}{2}$$

$$\max_{P_{2\cdot d}} \sum_{d=1}^{D} C_{2\cdot d} \cdot \text{Log}(P_{2\cdot d}) \quad \text{s.t.} \sum_{d=1}^{D} P_{2\cdot d} = \frac{1}{2}$$

Recall from part b that to maximize $\mathcal{L}$, we got $P_d = \frac{C_d}{\sum_{d=1}^{2D} C_d}$

Plug these 2 problems into the maximize equation above

$$\boxed{P_{2\cdot d-1} = \frac{C_{2\cdot d-1}}{\sum_{d=1}^{D} C_{2\cdot d-1}} \qquad P_{2\cdot d} = \frac{C_{2\cdot d}}{\sum_{d=1}^{D} C_{2\cdot d}}}$$