Introduction to Mobile Robotics

Error Propagation, Feature Exatraction, Extended Kalman Filter

Some slides adopted from: Wolfram Burgard, Cyrill Stachniss, Maren Bennewitz, Kai Arras and Probabilistic Robotics Book

Error Propagation: Motivation

- Probabilistic robotics is
  - **Representation**
  - → **Propagation**
  - **Reduction**

  of uncertainty

- **First-order error propagation** is fundamental for:
  Kalman filter (KF), landmark extraction, KF-based localization and SLAM

Discrete Kalman Filter (review)

- Estimates the state x of a discrete-time controlled process that is governed by the linear stochastic difference equation

$$x_t = A_t x_{t-1} + B_t u_t + \varepsilon_t$$

- with a measurement

$$z_t = C_t x_t + \delta_t$$

A_t	Matrix (nxn) that describes how the state evolves from t to t-1 without controls or noise.
B_t	Matrix (nxl) that describes how the control u_t changes the state from t to t-1.
C_t	Matrix (kxn) that describes how to map the state x_t to an observation z_t.
ε_t	Random variables representing the process and measurement noise that are assumed to be independent and normally distributed with covariance R_t and Q_t respectively.
δ_t	

Kalman Filter Algorithm

1. Algorithm **Kalman_filter**($\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$):

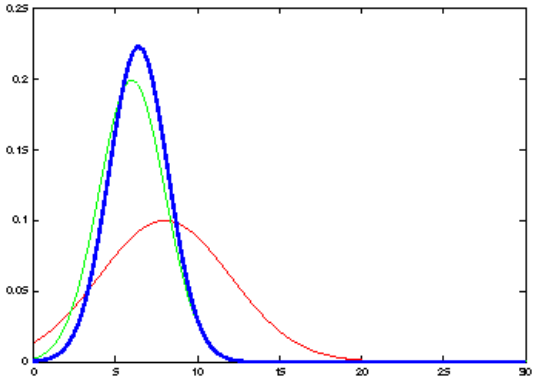
2. Prediction:
3. $\bar{\mu}_t = A_t \mu_{t-1} + B_t u_t$
4. $\bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t$

5. Correction:
6. $K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$
7. $\mu_t = \bar{\mu}_t + K_t (z_t - C_t \bar{\mu}_t)$
8. $\Sigma_t = (I - K_t C_t) \bar{\Sigma}_t$

9. Return μ_t, Σ_t

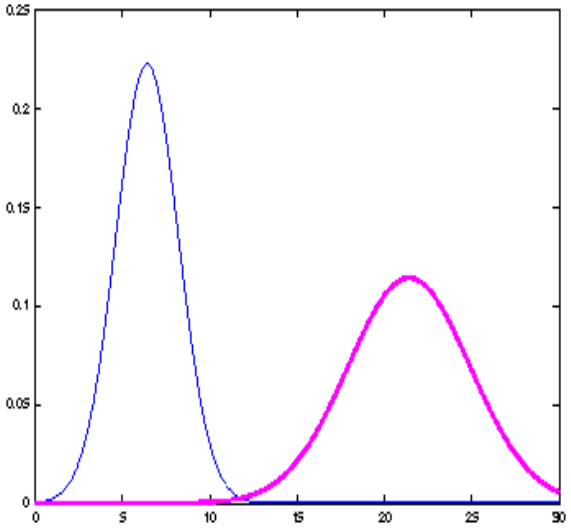
The Prediction-Correction-Cycle

Prediction

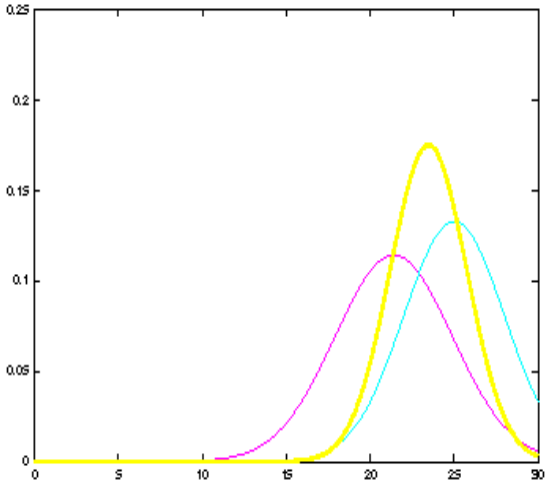


$$\overline{bel}(x_t) = \begin{cases} \overline{\mu}_t = a_t \mu_{t-1} + b_t u_t \\ \overline{\sigma}_t^2 = a_t^2 \sigma_t^2 + \sigma_{act,t}^2 \end{cases}$$

$$\overline{bel}(x_t) = \begin{cases} \overline{\mu}_t = A_t \mu_{t-1} + B_t u_t \\ \overline{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t \end{cases}$$

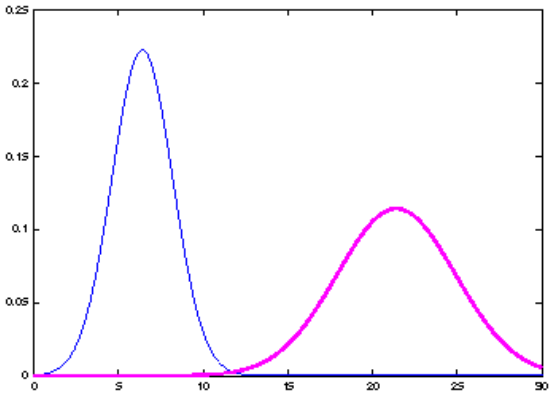


The Prediction-Correction-Cycle



$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - \bar{\mu}_t) \\ \sigma_t^2 = (1 - K_t)\bar{\sigma}_t^2 \end{cases}, K_t = \frac{\bar{\sigma}_t^2}{\bar{\sigma}_t^2 + \bar{\sigma}_{obs,t}^2}$$

$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - C_t\bar{\mu}_t) \\ \Sigma_t = (I - K_t C_t)\bar{\Sigma}_t \end{cases}, K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$$



Correction

Gaussian Distribution

Why is the Gaussian distribution everywhere?

The importance of the normal distribution follows mainly from the **Central Limit Theorem**:

- The mean/sum of a large number of independent RVs, each with finite mean and variance (ergo not e.g. uniformally distributed RVs), will be approximately **normally distributed**.
- The more RVs the better the approximation.

Nonlinear Dynamic Systems

- Most realistic robotic problems involve nonlinear functions

$$x_t = g(u_t, x_{t-1})$$

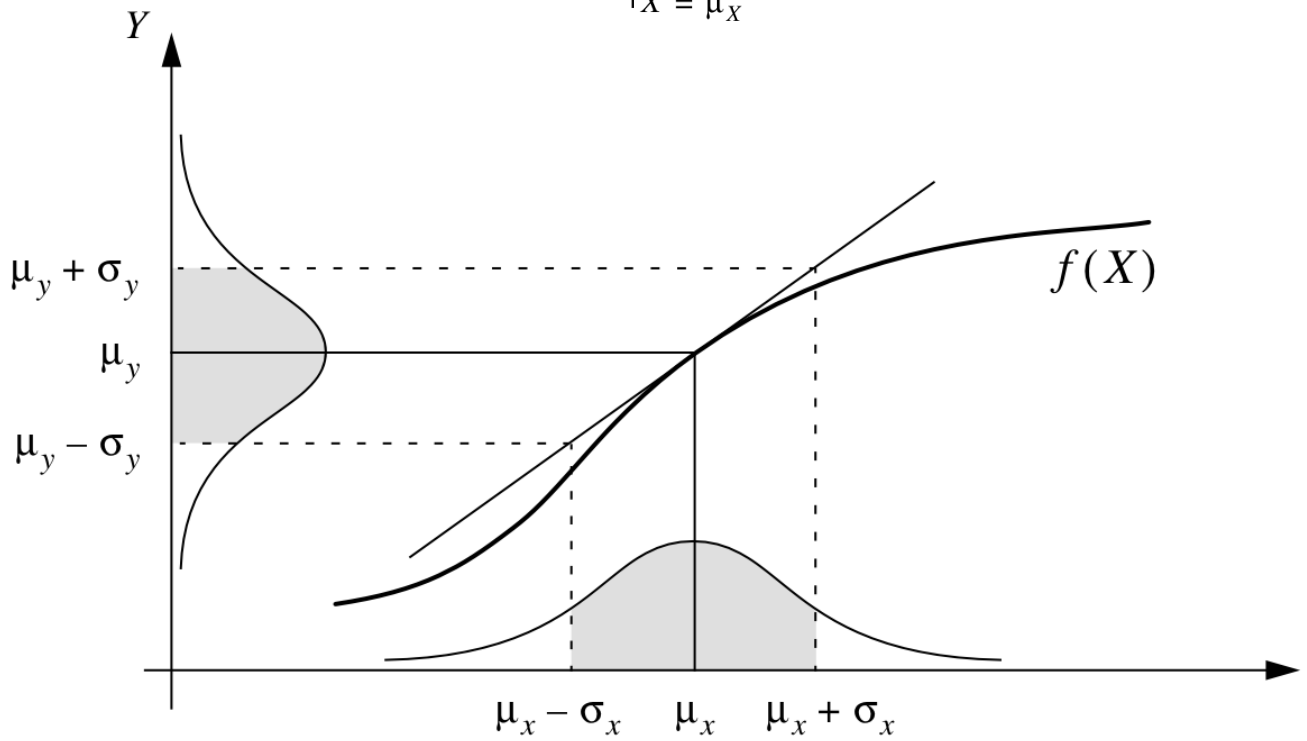
$$z_t = h(x_t)$$

- Extended Kalman filter relaxes linearity assumption

First-Order Error Propagation

Approximating $f(X)$ by a **first-order** Taylor series expansion about the point $X = \mu_X$

$$Y \approx f(\mu_X) + \left.\frac{\partial f}{\partial X}\right|_{X = \mu_X} (X - \mu_X)$$



Other Error Prop. Techniques

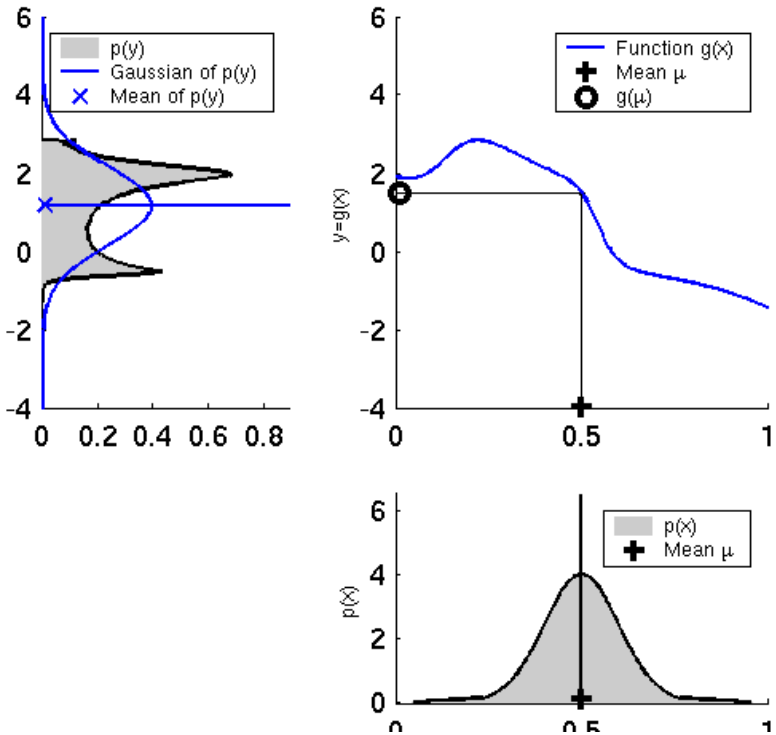
- **Second-Order Error Propagation**

Rarely used (complex expressions)

- **Monte-Carlo**

Non-parametric
representation
of uncertainties

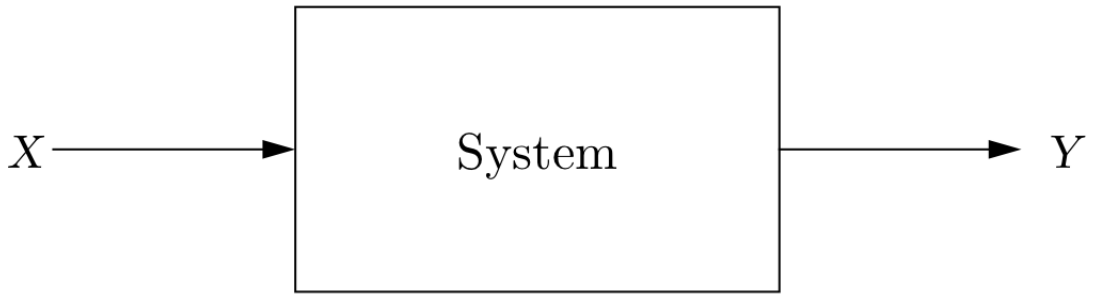
1. Sampling from $p(X)$
2. Propagation of samples
3. Histogramming
4. Normalization



First-Order Error Propagation

X,Y assumed to be Gaussian

$$Y = f(X)$$



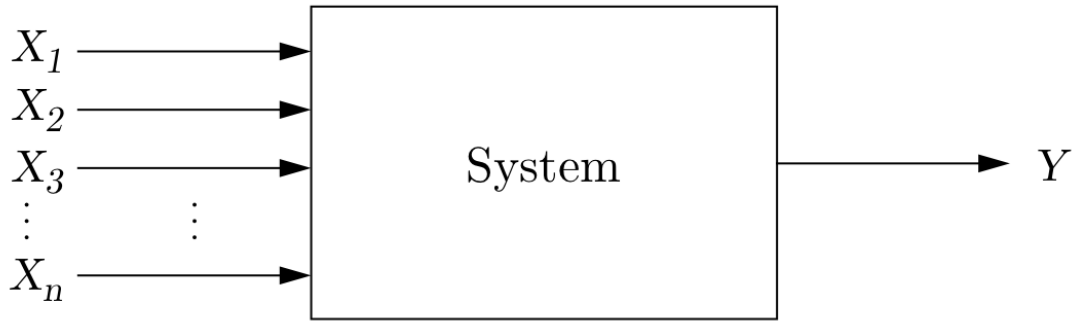
Taylor series expansion

$$Y \approx f(\mu_X) + \left.\frac{\partial f}{\partial X}\right|_{X = \mu_X} (X - \mu_X)$$

Wanted: μ_Y , σ_Y^2 (Solution on blackboard)

First-Order Error Propagation

$$Y = f(X_1, X_2, \ldots, X_n)$$



Taylor series expansion (around mean)

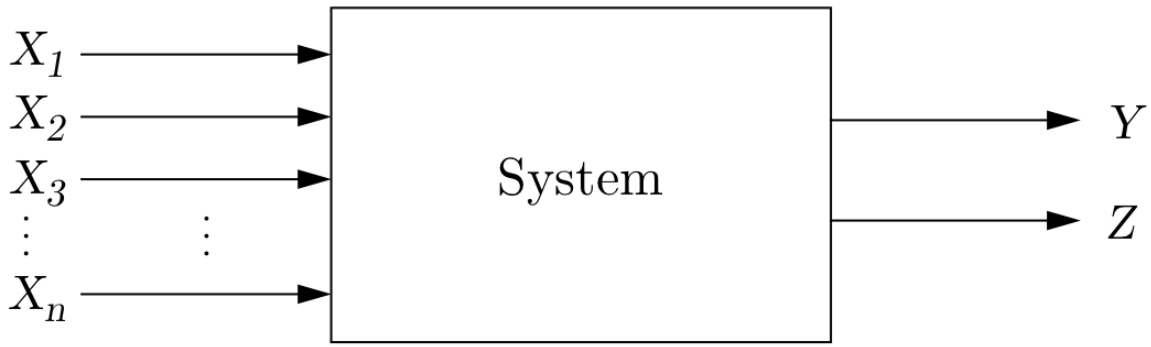
$$Y \approx f(\mu_1, \mu_2, \ldots, \mu_n) + \sum_{i=1}^{n} \left[\frac{\partial f}{\partial X_i}(\mu_1, \mu_2, \ldots, \mu_n)\right][X_i - \mu_i]$$

Wanted: μ_Y , σ_Y^2 (Solution on blackboard)

First-Order Error Propagation

$$Y = f(X_1, X_2, \ldots, X_n)$$

$$Z = g(X_1, X_2, \ldots, X_n)$$



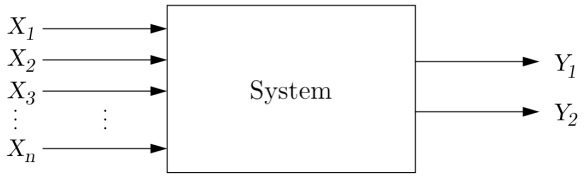
Wanted: σ_{YZ}

(Exercise)

First-Order Error Propagation

Putting things together...

$$C_X = \begin{bmatrix} \sigma_{X_1}^2 & \sigma_{X_1X_2} & \cdots & \sigma_{X_1X_n} \\ \sigma_{X_2X_1} & \sigma_{X_2}^2 & \cdots & \sigma_{X_2X_n} \\ \vdots & \vdots & & \vdots \\ \sigma_{X_nX_1} & \sigma_{X_nX_2} & \cdots & \sigma_{X_n}^2 \end{bmatrix}$$



$$C_Y = \begin{bmatrix} \sigma_{Y_1}^2 & \sigma_{Y_1Y_2} \\ \sigma_{Y_2Y_1} & \sigma_{Y_2}^2 \end{bmatrix}$$

with

$$\sigma_Y^2 = \sum_i \left(\frac{\partial f}{\partial X_i}\right)^2 \sigma_i^2 + \sum_{i \neq j} \sum \left(\frac{\partial f}{\partial X_i}\right)\left(\frac{\partial f}{\partial X_j}\right)\sigma_{ij}$$

$$\sigma_{YZ} = \sum \left(\frac{\partial f}{\partial X_i}\right)\left(\frac{\partial g}{\partial X_i}\right)\sigma_i^2 + \sum_{i \neq j} \sum \left(\frac{\partial f}{\partial X_i}\right)\left(\frac{\partial g}{\partial X_j}\right)\sigma_{ij}$$

→ “Is there a **compact form?...**”

Jacobian Matrix

- It's a **non-square matrix** $n \times m$ in general
- Suppose you have a vector-valued function $f(\mathbf{x}) = \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \end{bmatrix}$
- Let the **gradient operator** be the vector of (first-order) partial derivatives

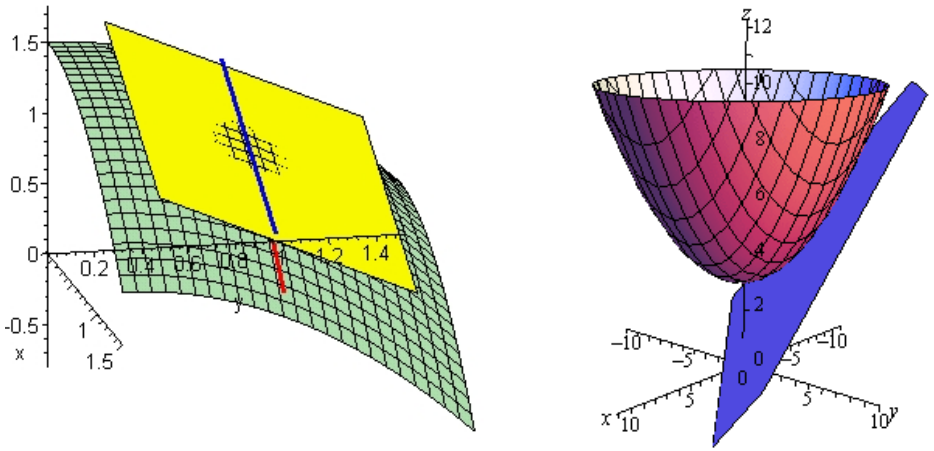
$$\nabla_{\mathbf{x}} = \begin{bmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \cdots & \frac{\partial}{\partial x_n} \end{bmatrix}^T$$

- Then, the **Jacobian matrix** is defined as

$$\mathbf{F}_{\mathbf{x}} = \begin{bmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial}{\partial x_1} & \cdots & \frac{\partial}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \\ \frac{\partial f_2}{\partial x_1} & \cdots & \frac{\partial f_2}{\partial x_n} \end{bmatrix}$$

Jacobian Matrix

- It's the orientation of the **tangent plane** to the vector-valued function at a given point

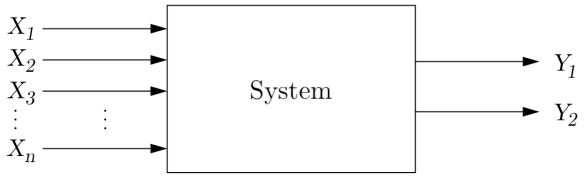


- **Generalizes the gradient** of a scalar valued function
- Heavily used for **first-order error propagation...**

First-Order Error Propagation

Putting things together...

$$C_X = \begin{bmatrix} \sigma_{X_1}^2 & \sigma_{X_1X_2} & \cdots & \sigma_{X_1X_n} \\ \sigma_{X_2X_1} & \sigma_{X_2}^2 & \cdots & \sigma_{X_2X_n} \\ \vdots & \vdots & & \vdots \\ \sigma_{X_nX_1} & \sigma_{X_nX_2} & \cdots & \sigma_{X_n}^2 \end{bmatrix}$$



$$C_Y = \begin{bmatrix} \sigma_{Y_1}^2 & \sigma_{Y_1Y_2} \\ \sigma_{Y_2Y_1} & \sigma_{Y_2}^2 \end{bmatrix}$$

with

$$\sigma_Y^2 = \sum_i \left(\frac{\partial f}{\partial X_i}\right)^2 \sigma_i^2 + \sum_{i \neq j} \sum \left(\frac{\partial f}{\partial X_i}\right)\left(\frac{\partial f}{\partial X_j}\right)\sigma_{ij}$$

$$\sigma_{YZ} = \sum \left(\frac{\partial f}{\partial X_i}\right)\left(\frac{\partial g}{\partial X_i}\right)\sigma_i^2 + \sum_{i \neq j} \sum \left(\frac{\partial f}{\partial X_i}\right)\left(\frac{\partial g}{\partial X_j}\right)\sigma_{ij}$$

→ “Is there a **compact form?...**”

First-Order Error Propagation

...Yes! Given

- Input covariance matrix C_X
- Jacobian matrix F_X

the **Error Propagation Law**

$$C_Y = F_X C_X F_X^T$$

computes the output covariance matrix C_Y

First-Order Error Propagation

Alternative Derivation in Matrix Notation

$$
\begin{aligned}
\mu_x &= E(x) \\
&= E(Au + b) \\
&= AE(u) + b \\
&= A\mu_u + b
\end{aligned}
$$

$$
\begin{aligned}
\Sigma_x &= E((x - E(x))(x - E(x))^T) \\
&= E((Au + b - AE(u) - b)(Au + b - AE(u) - b)^T) \\
&= E((A(u - E(u)))(A(u - E(u)))^T) \\
&= E((A(u - E(u)))((u - E(u))^T A^T)) \\
&= AE((u - E(u))(u - E(u))^T)A^T \\
&= A\Sigma_u A^T
\end{aligned}
$$

Derivations (2/4)

Definitions

$$\mu = \mathrm{E}(X)$$
$$\mathrm{Var}(X) = \mathrm{E}((X-\mu)^2)$$
$$\mathrm{Cov}(X,Y) = \mathrm{E}((X-\mu)(Y-\nu))$$

Rules

$$\mathrm{E}(X+c) = \mathrm{E}(X) + c$$
$$\mathrm{E}(X+Y) = \mathrm{E}(X) + \mathrm{E}(Y)$$
$$\mathrm{E}(aX) = a\mathrm{E}(X)$$

Result SISO

$$\mu_Y = f(\mu_X),$$
$$\sigma_Y = \left.\frac{\partial f}{\partial X}\right|_{X=\mu_X} \sigma_X.$$

Derivations (3/4)

Result MISO

$$\begin{aligned}
\mu_Y = E[Y] &= E[a_0 + \sum_i a_i(X_i - \mu_i)] \\
&= E[a_0] + \sum_i E[a_i X_i] - E[a_i \mu_i] \\
&= a_0 + \sum_i a_i E[X_i] - a_i E[\mu_i] \\
&= a_0 + \sum_i a_i \mu_i - a_i \mu_i \\
&= a_0 \\
\mu_Y &= f(\mu_1, \mu_2, \ldots, \mu_n)
\end{aligned}$$

$$\begin{aligned}
\sigma_Y^2 = E[(Y - \mu_Y)^2] &= E[(\sum_i a_i(X_i - \mu_i))^2] \\
&= E[\sum_i a_i(X_i - \mu_i) \sum_j a_j(X_j - \mu_j)] \\
&= E[\sum_i a_i^2(X_i - \mu_i)^2 + \sum_{i \neq i} \sum a_i a_j (X_i - \mu_i)(X_j - \mu_j)] \\
&= \sum_i a_i^2 E[(X_i - \mu_i)^2] + \sum_{i \neq j} \sum a_i a_j E[(X_i - \mu_i)(X_i - \mu_j)] \\
&= \sum_i a_i^2 \sigma_i^2 + \sum_{i \neq j} \sum a_i a_j \sigma_{ij} \\
\sigma_Y^2 &= \sum_i \left(\frac{\partial f}{\partial X_i}\right)^2 \sigma_i^2 + \sum_{i \neq j} \sum \left(\frac{\partial f}{\partial X_i}\right)\left(\frac{\partial f}{\partial X_j}\right) \sigma_{ij}
\end{aligned}$$

Derivations (4/4)

Result MIMO

$$\sigma_{YZ} = E[(Y-\mu_Y)(Z-\mu_Z)]$$

$$= E[Y \cdot Z] - E[Y]E[Z]$$

$$= E\left[\left(\mu_Y + \sum \frac{\partial f}{\partial X_i}[X_i-\mu_i]\right)\cdot\left(\mu_Z + \sum \frac{\partial g}{\partial X_i}[X_i-\mu_i]\right)\right] - \mu_Y\mu_Z$$

$$= E\left[\mu_Y\mu_Z + \mu_Z\sum \frac{\partial f}{\partial X_i}[X_i-\mu_i] + \mu_Y\sum \frac{\partial g}{\partial X_i}[X_i-\mu_i] + \sum \frac{\partial f}{\partial X_i}[X_i-\mu_i]\sum \frac{\partial g}{\partial X_i}[X_i-\mu_i]\right] - \mu_Y\mu_Z$$

$$= E[\mu_Y\mu_Z] + \mu_Z E\left[\sum \frac{\partial f}{\partial X_i}X_i - \sum \frac{\partial f}{\partial X_i}\mu_i\right] + \mu_Y E\left[\sum \frac{\partial g}{\partial X_i}X_i - \sum \frac{\partial g}{\partial X_i}\mu_i\right]$$
$$+ E\left[\sum\sum \frac{\partial f}{\partial X_i}\frac{\partial g}{\partial X_j}[X_i-\mu_i][X_j-\mu_j]\right] - \mu_Y\mu_Z$$

$$= \mu_Y\mu_Z + \mu_Z\sum \frac{\partial f}{\partial X_i}E[X_i] - \mu_Z\sum \frac{\partial f}{\partial X_i}E[\mu_i] + \mu_Y\sum \frac{\partial g}{\partial X_i}E[X_i] - \mu_Y\sum \frac{\partial g}{\partial X_i}E[\mu_i]$$
$$+ E\left[\sum \frac{\partial f}{\partial X_i}\frac{\partial g}{\partial X_i}[X_i-\mu_i]^2 + \sum_{i\neq j}\sum \frac{\partial f}{\partial X_i}\frac{\partial g}{\partial X_j}[X_i-\mu_i][X_j-\mu_j]\right] - \mu_Y\mu_Z$$

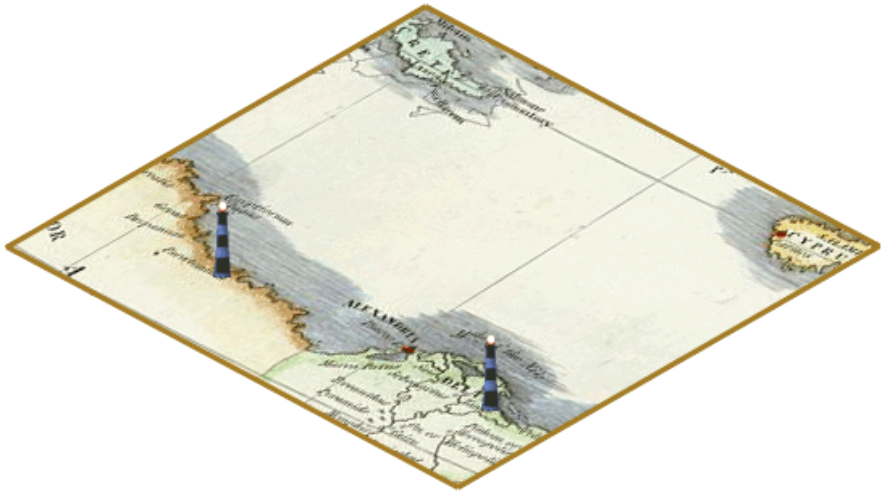
$$= \sum \frac{\partial f}{\partial X_i}\frac{\partial g}{\partial X_i}E[(X_i-\mu_i)^2] + \sum_{i\neq j}\sum \frac{\partial f}{\partial X_i}\frac{\partial g}{\partial X_j}E[(X_i-\mu_i)(X_j-\mu_j)]$$

$$\sigma_{YZ} = \sum \frac{\partial f}{\partial X_i}\frac{\partial g}{\partial X_i}\sigma_i^2 + \sum_{i\neq j}\sum \frac{\partial f}{\partial X_i}\frac{\partial g}{\partial X_j}\sigma_{ij}^2$$

Feature Extraction: Motivation

Landmarks for:

- Localization
- SLAM
- Scene analysis



Examples:

- **Lines, corners, clusters:** good for indoor
- **Circles, rocks, plants:** good for outdoor

Example: Line Extraction

Wanted: Parameter Covariance Matrix

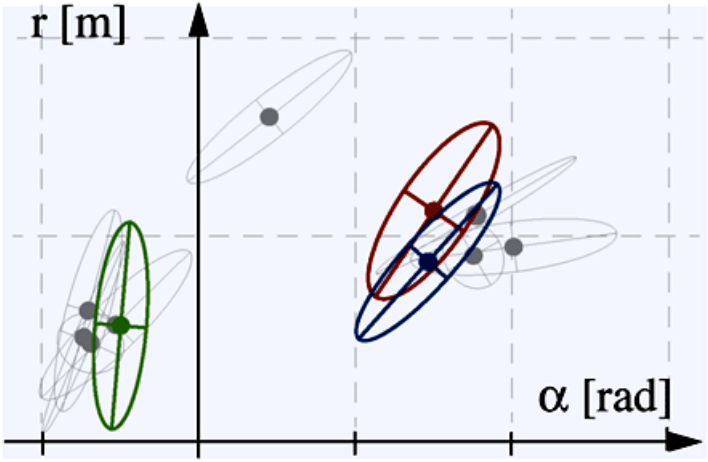
$$C_{AR} = \begin{bmatrix} \sigma_A^2 & \sigma_{AR} \\ \sigma_{AR} & \sigma_R^2 \end{bmatrix}$$

$$C_X = \begin{bmatrix} \sigma_{\rho_1}^2 & 0 & \dots & 0 \\ 0 & \sigma_{\rho_2}^2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \sigma_{\rho_n}^2 \end{bmatrix}$$

Simplified sensor model:
all $\sigma_{\theta_i}^2 = 0$, independence

$$C_{AR} = F_X C_X F_X^T$$

Result: Gaussians in the model space



Features: Properties

A feature/landmark is a **physical object** which is

- **static**
- **perceptible**
- (at least locally) **unique**

Abstraction from the raw data...

- **type** (range, image, vibration, etc.)
- **amount** (sparse or dense)
- **origin** (different sensors, map)

\+ Compact, efficient, accurate, scales well, semantics

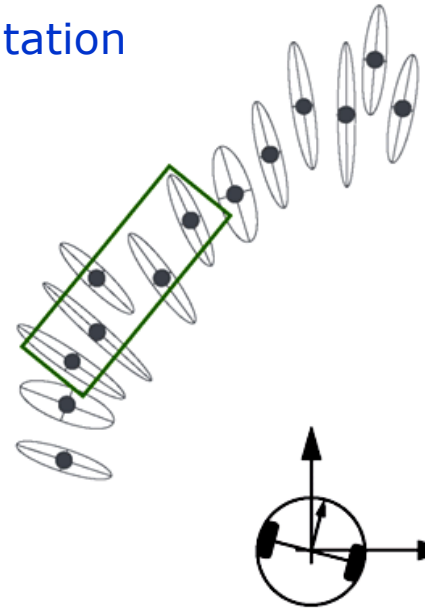
– Not general

Feature Extraction

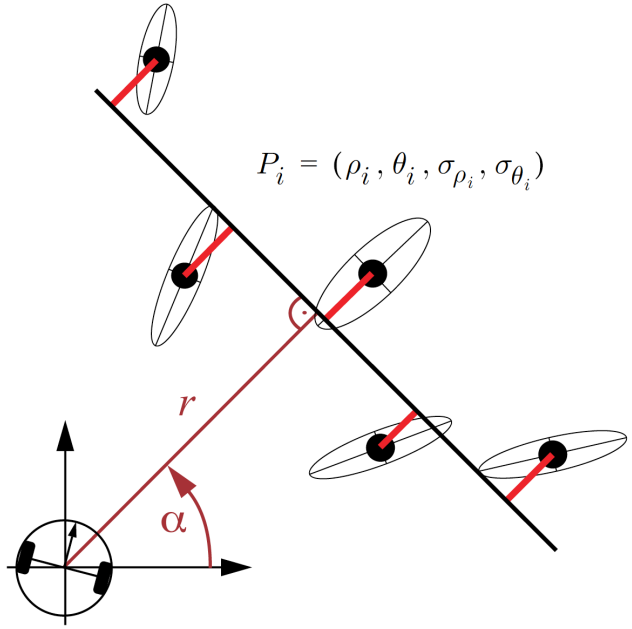
Can be subdivided into two subproblems:

- **Segmentation:** *Which* points contribute?
- **Fitting:** *How* do the points contribute?

Segmentation



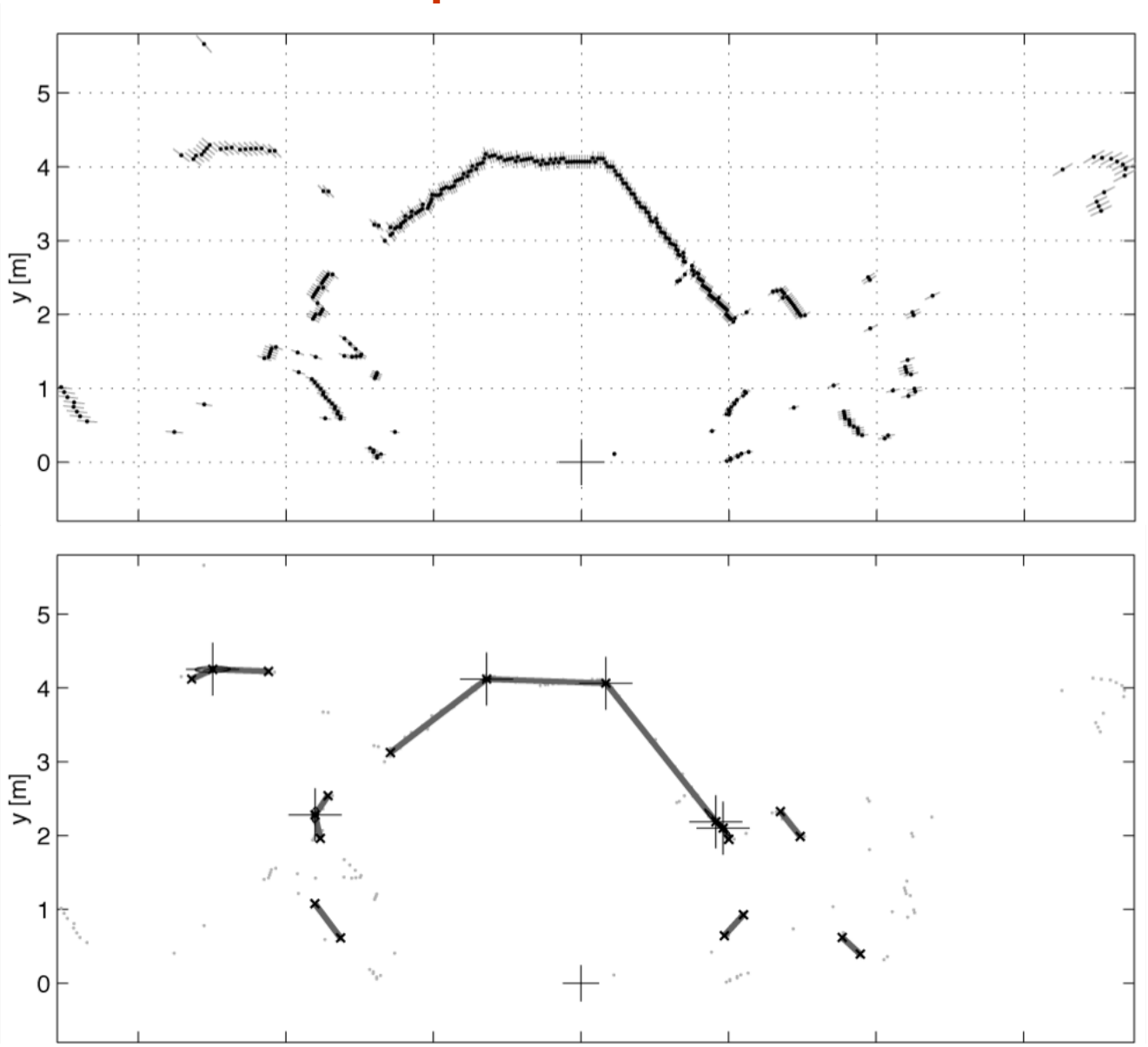
Fitting



Example: Local Map with Lines

Raw
range data

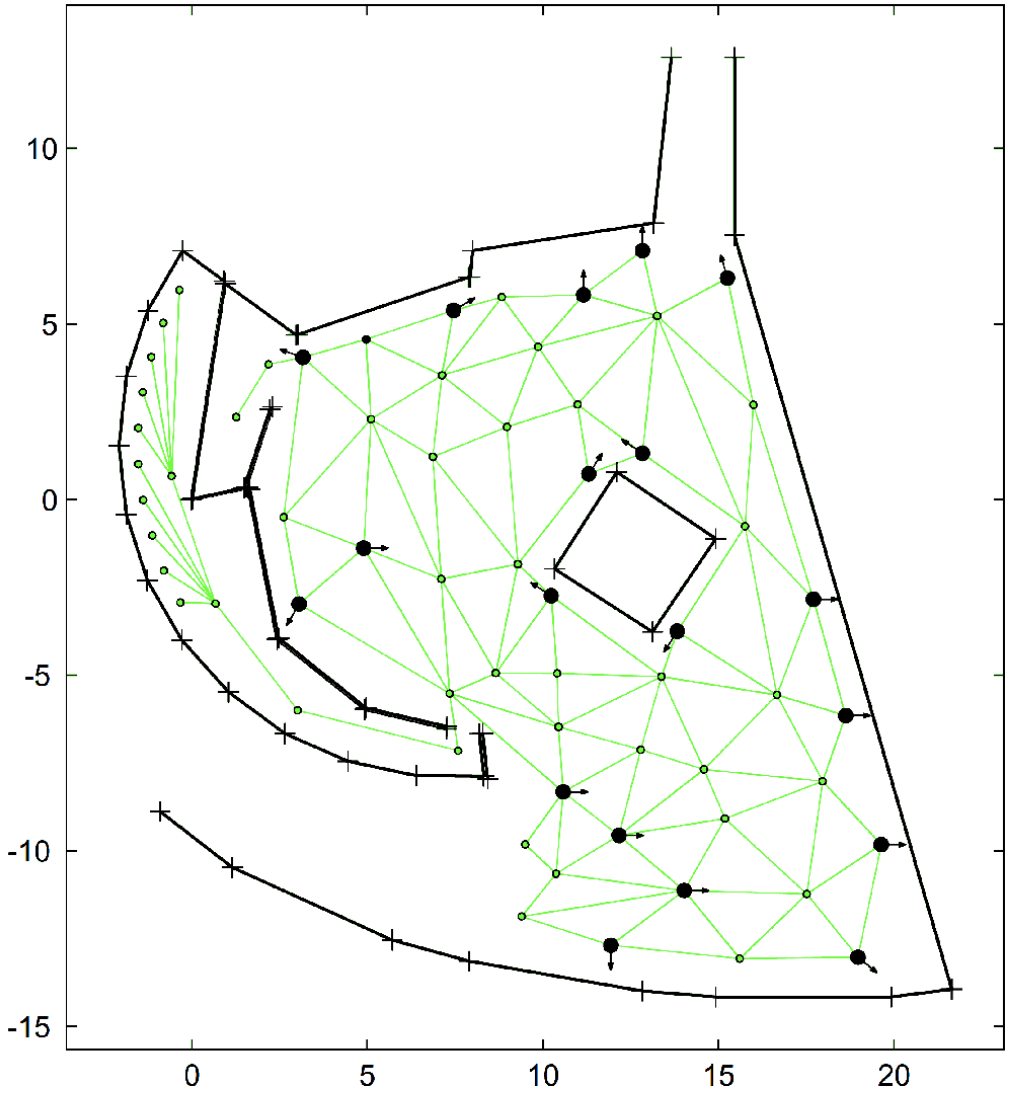
Line
segments



Example: Global Map with Lines

Expo.02 map

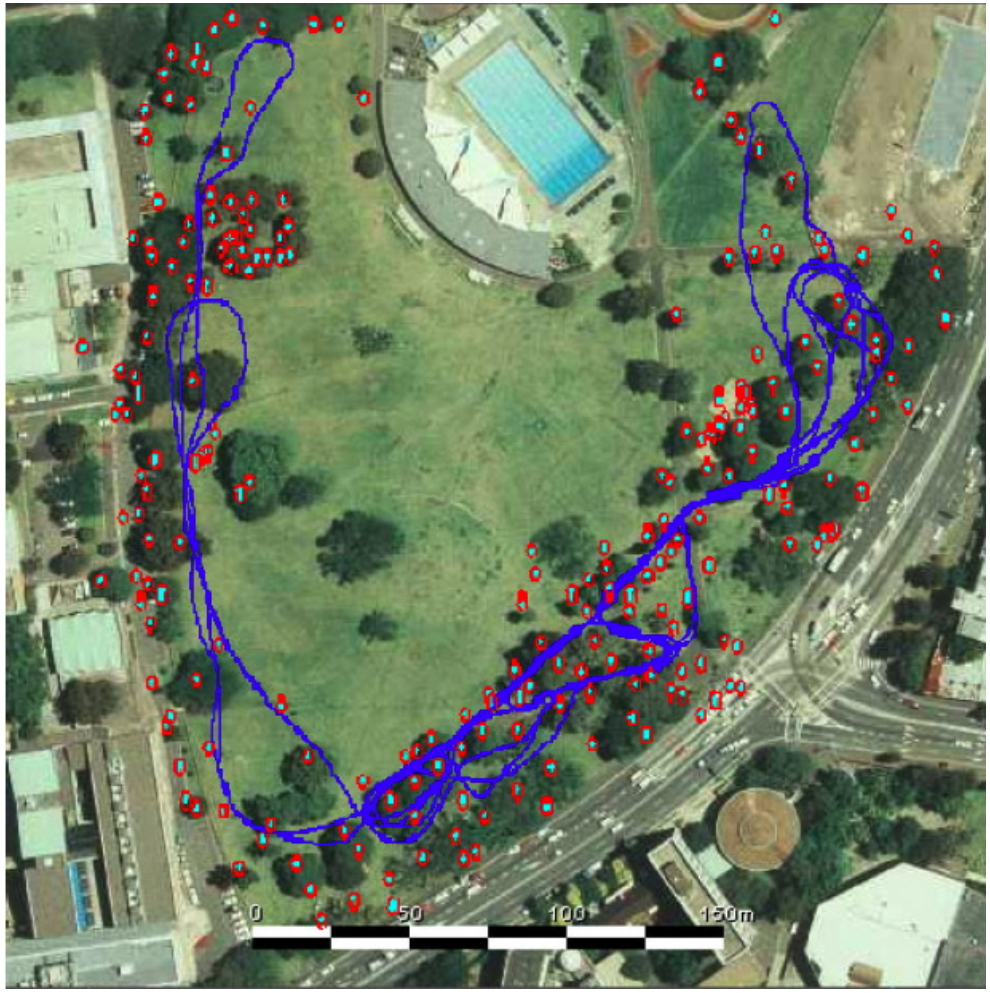
- 315 m^2
- 44 Segments
- 8 kbytes
- 26 bytes / m^2
- Localization accuracy ~1cm



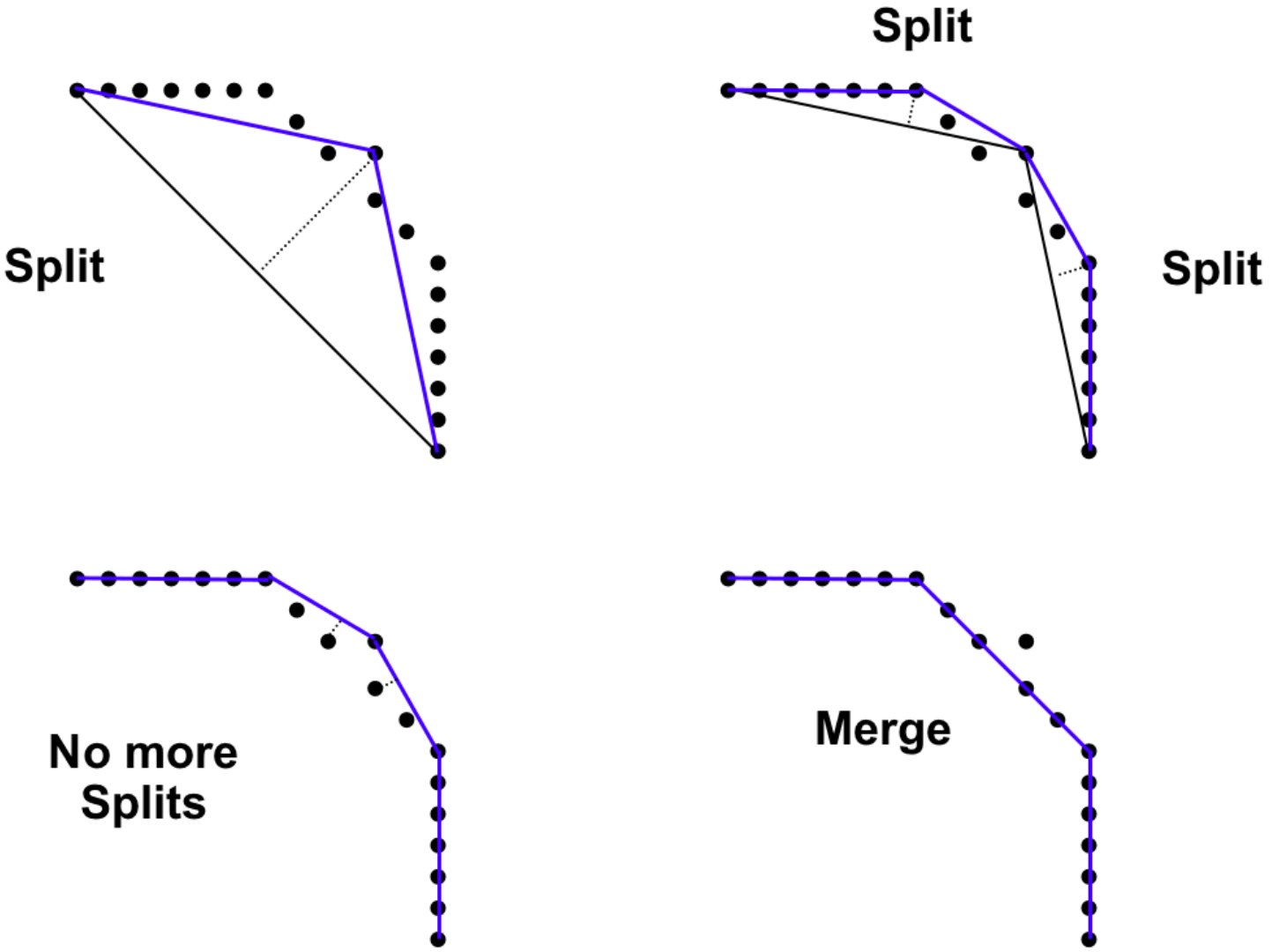
Example: Global Map w. Circles

Victoria Park, Sydney

- Trees



Split and Merge



Picture by J. Tardos

Split and Merge

Algorithm

Split

- Obtain the line passing by the two extreme points
- Find the most distant point to the line
- If distance > threshold, split and repeat with the left and right point sets

Merge

- If two consecutive segments are close/collinear enough, obtain the common line and find the most distant point
- If distance <= threshold, merge both segments

Split and Merge: Improvements

- Residual analysis before split

$$\sum_{i=P_S}^{P_E} d_i^2 > \sum_{i=P_S}^{P_B} d_i^2 + \sum_{i=P_B}^{P_E} d_i^2$$

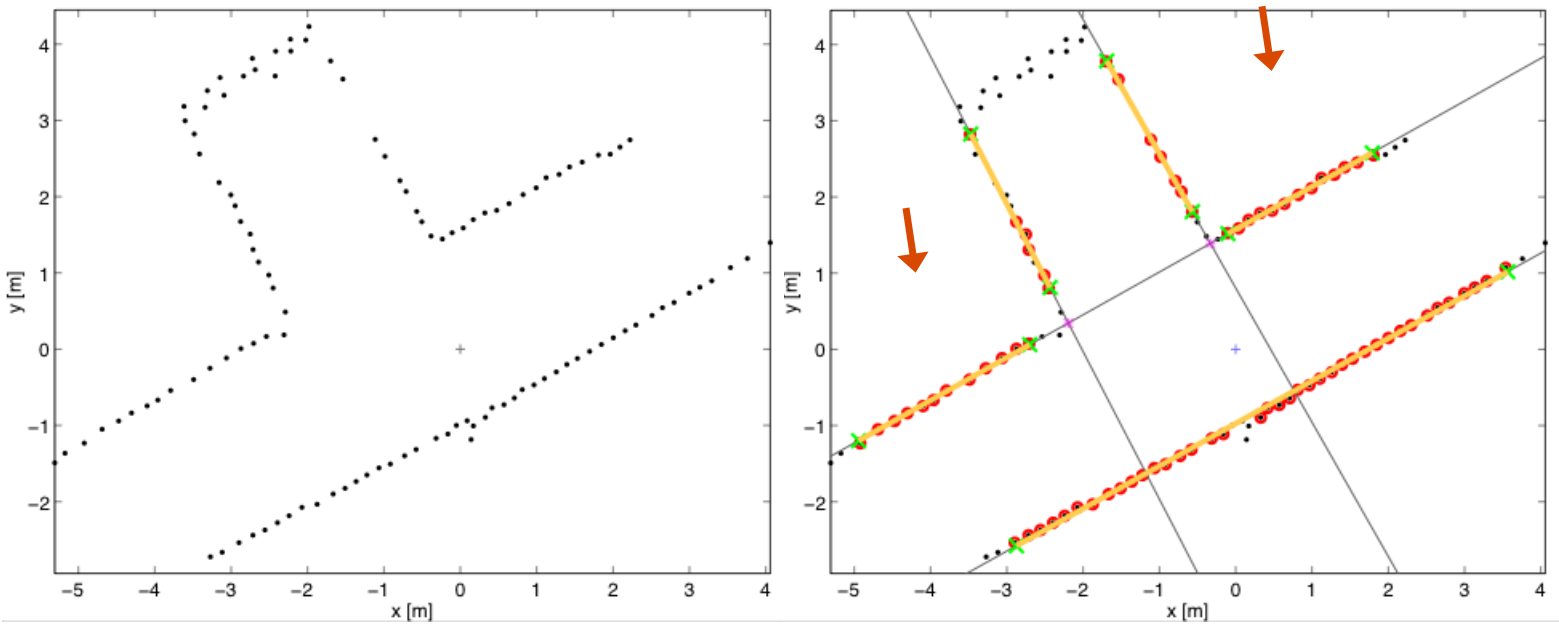
P_S, P_E, P_B : start-, end-, break-point

Split only if the break point provides a "better interpretation" in terms of the error sum

[Castellanos 1998]

Split and Merge: Improvements

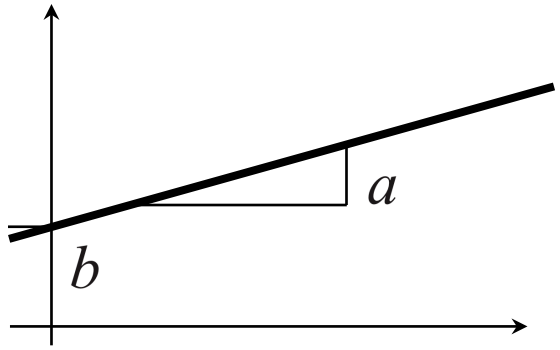
- Merge **non-consecutive** segments as a post-processing step



Line Representation

Choice of the line representation matters!

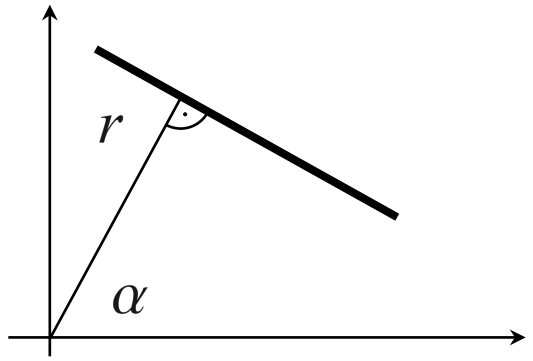
Intercept-Slope



$$y = a\,x + b$$

$$C = \begin{bmatrix} \sigma_a^2 & \sigma_{ab} \\ \sigma_{ba} & \sigma_b^2 \end{bmatrix}$$

Hessian model



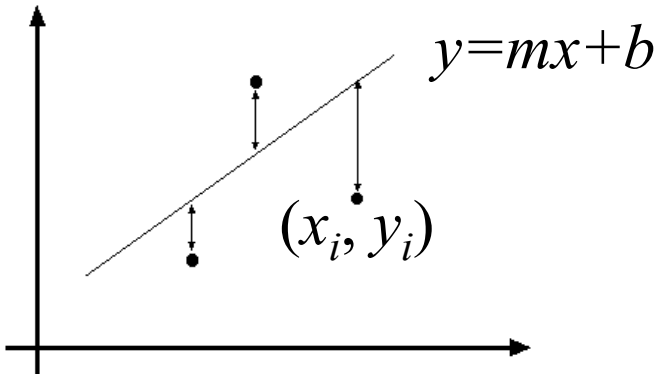
$$x\cos\alpha + y\sin\alpha - r = 0$$

$$C = \begin{bmatrix} \sigma_\alpha^2 & \sigma_{\alpha r} \\ \sigma_{r\alpha} & \sigma_r^2 \end{bmatrix}$$

Each model has advantages and drawbacks

Least squares line fitting

- Data: $(x_1, y_1), \ldots, (x_n, y_n)$
- Line equation: $y_i = m x_i + b$
- Find (m, b) to minimize



$$E = \sum_{i=1}^{n} (y_i - m x_i - b)^2$$

$$Y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \qquad X = \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \qquad B = \begin{bmatrix} m \\ b \end{bmatrix}$$

$$E = \left\| Y - XB \right\|^2 = (Y - XB)^T (Y - XB) = Y^T Y - 2(XB)^T Y + (XB)^T (XB)$$

$$\frac{dE}{dB} = 2X^T XB - 2X^T Y = 0$$

$$X^T XB = X^T Y$$

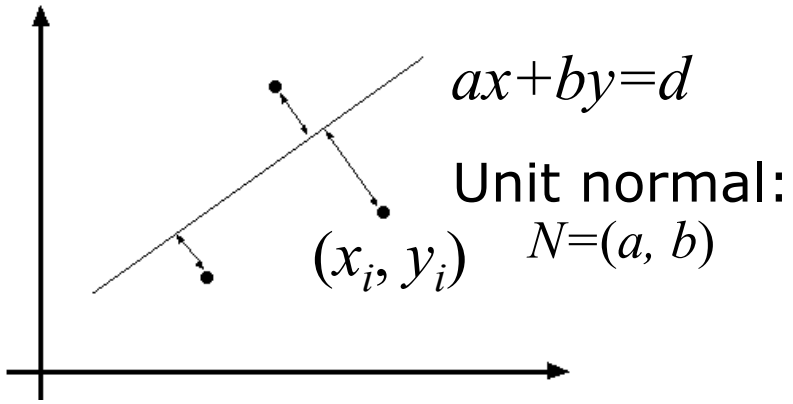
Normal equations: least squares solution to $XB = Y$

Problem with “vertical” least squares

- Not rotation-invariant
- Fails completely for vertical lines

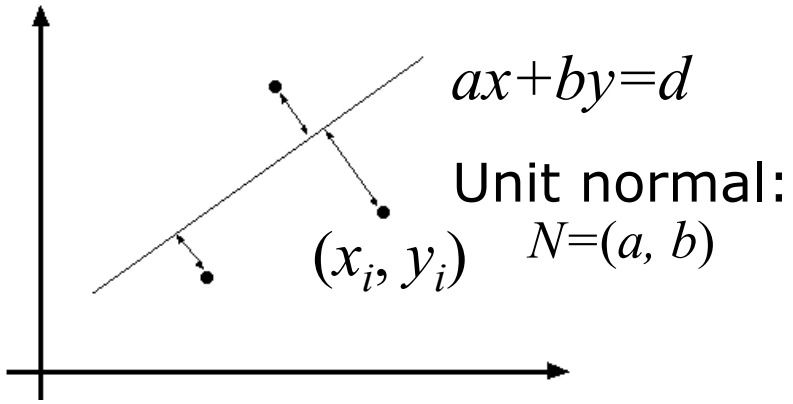
Total least squares

- Distance between point (x_i, y_i) and line $ax+by=d$ $(a^2+b^2=1)$: $|ax_i + by_i - d|$



Total least squares

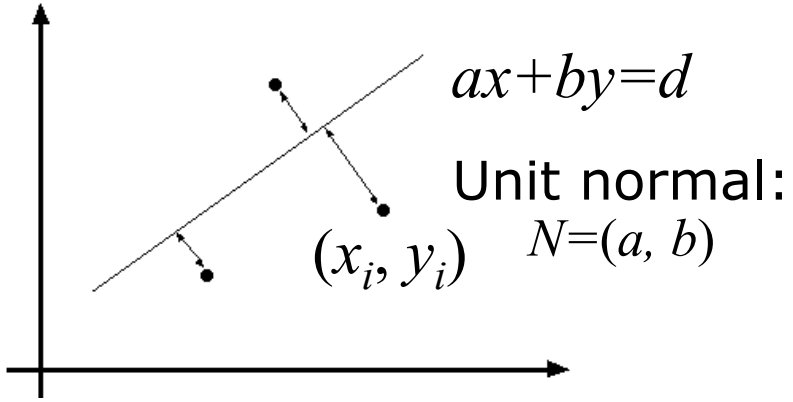
- Distance between point (x_i, y_i) and line $ax+by=d$ $(a^2+b^2=1)$: $|ax_i + by_i - d|$
- Find (a, b, d) to minimize the sum of squared perpendicular distances



$$E = \sum_{i=1}^{n} (ax_i + by_i - d)^2$$

Total least squares

- Distance between point (x_i, y_i) and line $ax+by=d$ $(a^2+b^2=1)$: $|ax_i + by_i - d|$
- Find (a, b, d) to minimize the sum of squared perpendicular distances



$$E = \sum_{i=1}^{n} (ax_i + by_i - d)^2$$

$$d = \frac{a}{n}\sum_{i=1}^{n} x_i + \frac{b}{n}\sum_{i=1}^{n} y_i = a\bar{x} + b\bar{y}$$

$$\frac{\partial E}{\partial d} = \sum_{i=1}^{n} -2(ax_i + by_i - d) = 0$$

$$E = \sum_{i=1}^{n} (a(x_i - \bar{x}) + b(y_i - \bar{y}))^2 = \left\| \begin{bmatrix} x_1 - \bar{x} & y_1 - \bar{y} \\ \vdots & \vdots \\ x_n - \bar{x} & y_n - \bar{y} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \right\|^2 = (UN)^T (UN)$$

$$\frac{dE}{dN} = 2(U^T U)N = 0$$

Solution to $(U^T U)N = 0$, subject to $\|N\|^2 = 1$: eigenvector of $U^T U$ associated with the smallest eigenvalue (least squares solution to *homogeneous linear system* $UN = 0$)

Total least squares

$$U = \begin{bmatrix} x_1 - \bar{x} & y_1 - \bar{y} \\ \vdots & \vdots \\ x_n - \bar{x} & y_n - \bar{y} \end{bmatrix}$$

$$U^T U = \begin{bmatrix} \sum_{i=1}^{n} (x_i - \bar{x})^2 & \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y}) \\ \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y}) & \sum_{i=1}^{n} (y_i - \bar{y})^2 \end{bmatrix}$$

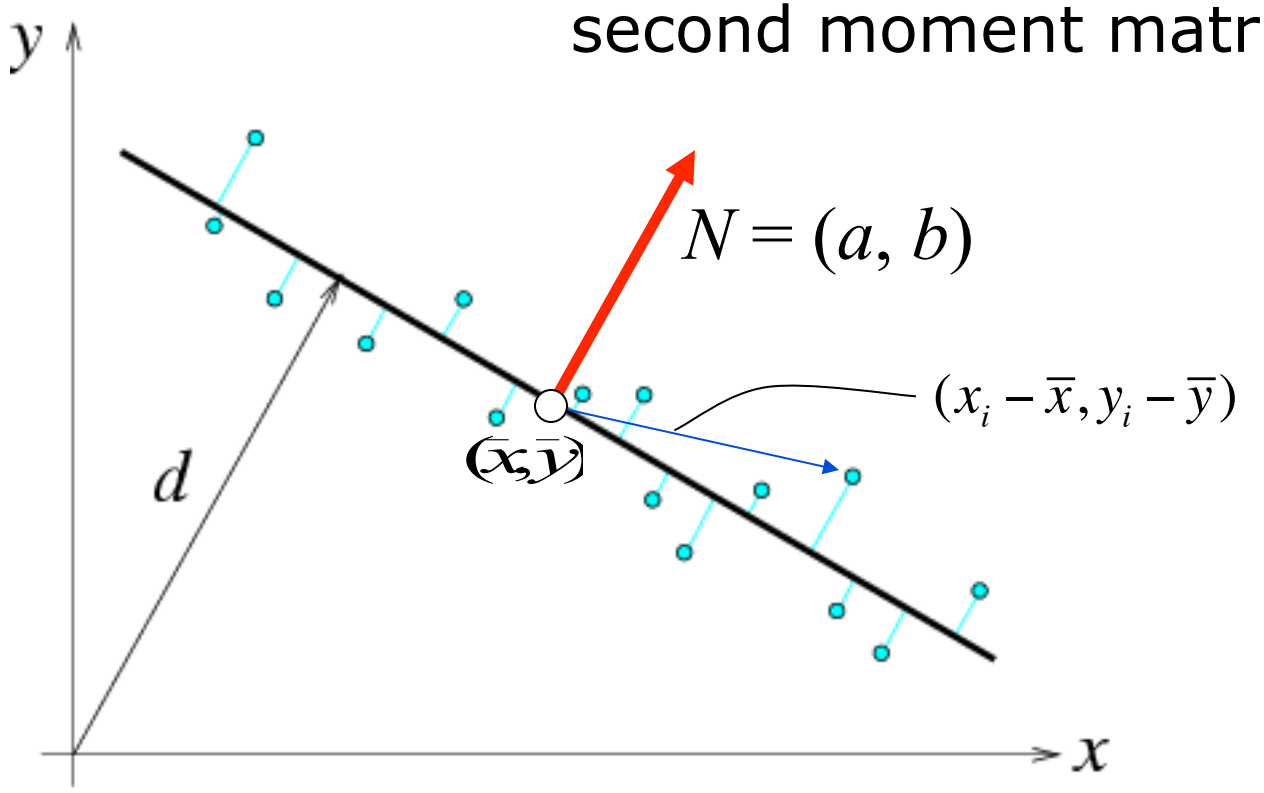
second moment matrix

Total least squares

$$U = \begin{bmatrix} x_1 - \bar{x} & y_1 - \bar{y} \\ \vdots & \vdots \\ x_n - \bar{x} & y_n - \bar{y} \end{bmatrix}$$

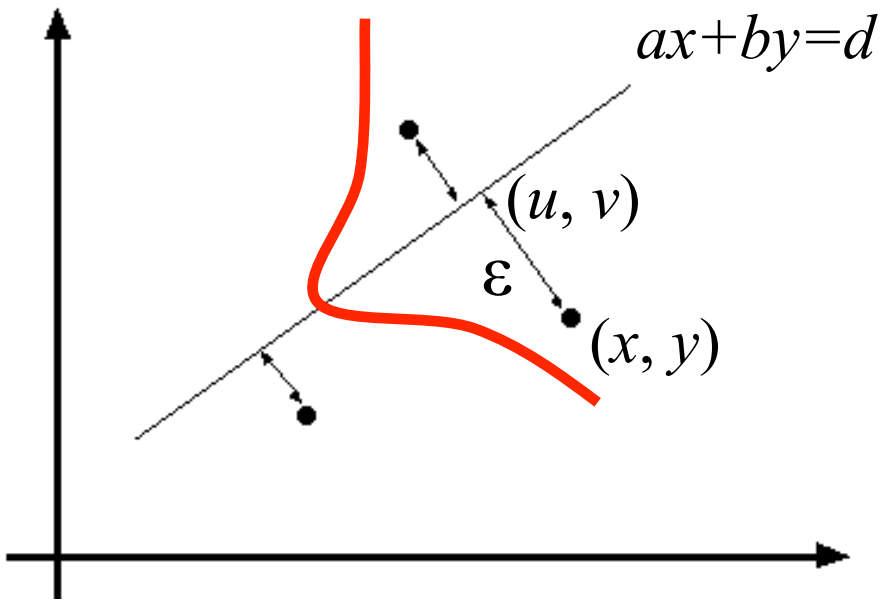
$$U^T U = \begin{bmatrix} \sum_{i=1}^{n} (x_i - \bar{x})^2 & \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y}) \\ \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y}) & \sum_{i=1}^{n} (y_i - \bar{y})^2 \end{bmatrix}$$

second moment matrix



Least squares as likelihood maximization

- **Generative model**: line points are sampled independently and corrupted by Gaussian noise in the direction perpendicular to the line



$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} + \varepsilon \begin{pmatrix} a \\ b \end{pmatrix}$$

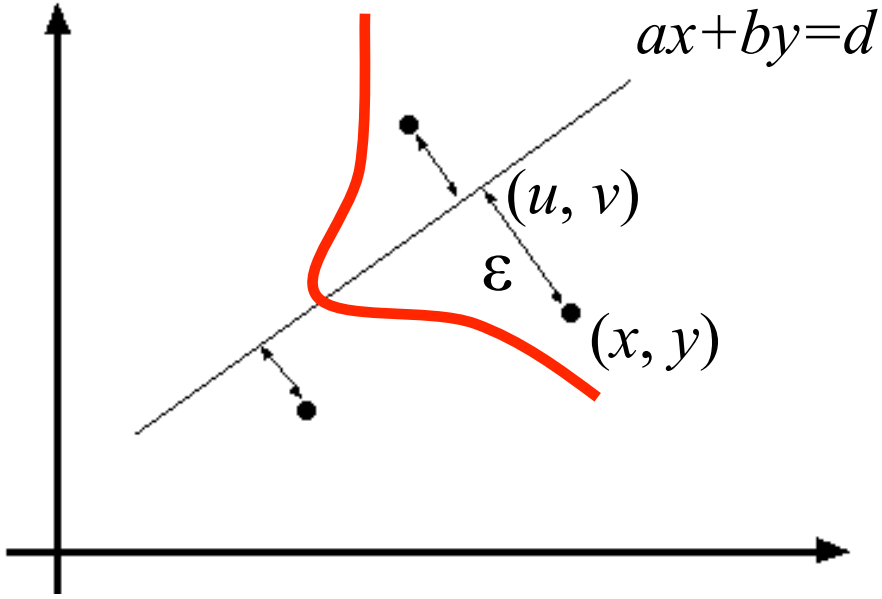
point on the line

noise: sampled from zero-mean Gaussian with std. dev. σ

normal direction

Least squares as likelihood maximization

- **Generative model**: line points are sampled independently and corrupted by Gaussian noise in the direction perpendicular to the line



$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} + \varepsilon \begin{pmatrix} a \\ b \end{pmatrix}$$

Likelihood of points given line parameters (*a*, *b*, *d*):

$$P(x_1, y_1, \ldots, x_n, y_n \mid a, b, d) = \prod_{i=1}^{n} P(x_i, y_i \mid a, b, d) \propto \prod_{i=1}^{n} \exp\left(-\frac{(ax_i + by_i - d)^2}{2\sigma^2} \right)$$

Log-likelihood: $$L(x_1, y_1, \ldots, x_n, y_n \mid a, b, d) = -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (ax_i + by_i - d)^2$$

Line fitting

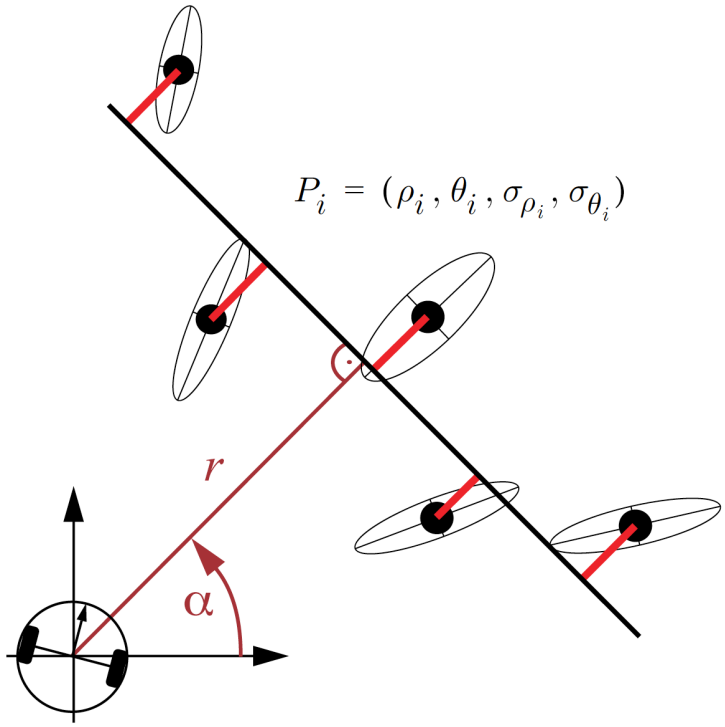
Given:

A set of n points in polar coordinates

Wanted:

Line parameters

α' r



$$\tan 2\alpha = \frac{\frac{2}{\Sigma w_i}\sum_i \sum_{<j} w_i w_j \rho_i \rho_j \sin(\theta_i + \theta_j) + \frac{1}{\Sigma w_i}\sum (w_i - \Sigma w_j) w_i \rho_i^2 \sin 2\theta_i}{\frac{2}{\Sigma w_i}\sum_i \sum_{<j} w_i w_j \rho_i \rho_j \cos(\theta_i + \theta_j) + \frac{1}{\Sigma w_i}\sum (w_i - \Sigma w_j) w_i \rho_i^2 \cos 2\theta_i}$$

$$r = \frac{\sum w_i \rho_i \cos(\theta_i - \alpha)}{\sum w_i}$$

[Arras 1997]

LSQ Estimation

Regression, Least Squares-Fitting

$$\epsilon_i = x_i \cos\alpha + y_i \sin\alpha - r$$

$$S = \sum_{i=1}^{n} \epsilon_i^{\,2}$$

Solve the non-linear equation system

$$\frac{\partial S}{\partial \alpha} = 0 \qquad \frac{\partial S}{\partial r} = 0$$

Solution (for points in Cartesian coordinates):

→ Solution on blackboard

Circle Extraction

Can be formulated as a **linear** regression problem

Given n points $\mathcal{P} = \{P_i\}_{i=1}^{n}$ with $P_i = (x_i \;\; y_i)^T$

Circle equation: $(x_i - x_c)^2 + (y_i - y_c)^2 = r_c^2$

Develop circle equation

$$x_i^2 - 2x_i x_c + x_c^2 + y_i^2 - 2y_i y_c + y_c^2 = r_c^2$$

$$(-2x_i \;\; -2y_i \;\; 1) \begin{pmatrix} x_c \\ y_c \\ x_c^2 + y_c^2 - r_c^2 \end{pmatrix} = (-x_i^2 - y_i^2)$$

Parametrization trick

Circle Extraction

Leads to **overdetermined** equation system $A \cdot x = b$

$$A = \begin{pmatrix} -2x_1 & -2y_1 & 1 \\ -2x_2 & -2y_2 & 1 \\ \vdots & \vdots & \vdots \\ -2x_n & -2y_n & 1 \end{pmatrix} \quad b = \begin{pmatrix} -x_1^2 - y_1^2 \\ -x_2^2 - y_2^2 \\ \vdots \\ -x_n^2 - y_n^2 \end{pmatrix}$$

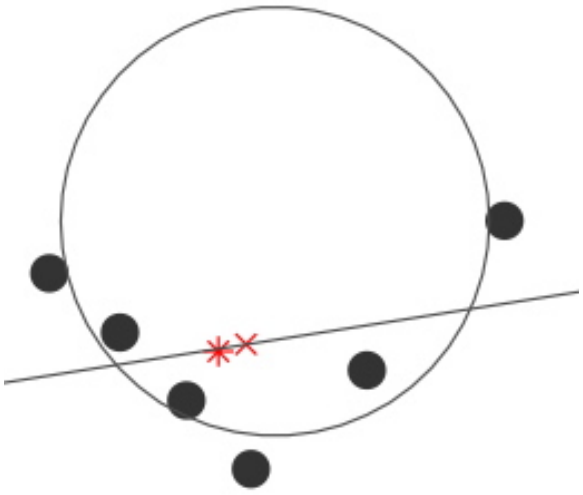
with vector of unknowns

$$x = (x_c \quad y_c \quad x_c^2 + y_c^2 - r_c^2)^T$$

Solution via **Pseudo-Inverse**

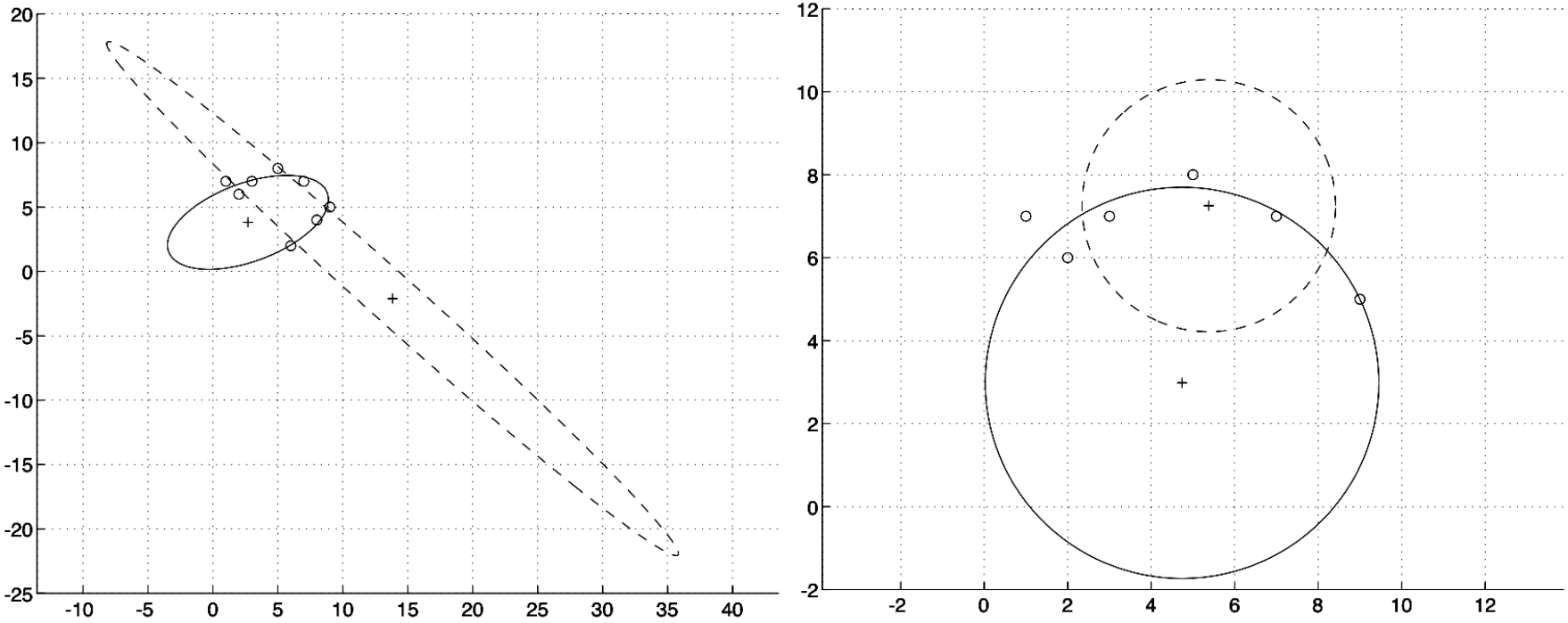
$$x = (A^T A)^{-1} A^T \cdot b$$

(assuming that *A* has full rank)



Fitting Curves to Points

Attention: Always know the errors that you minimize!



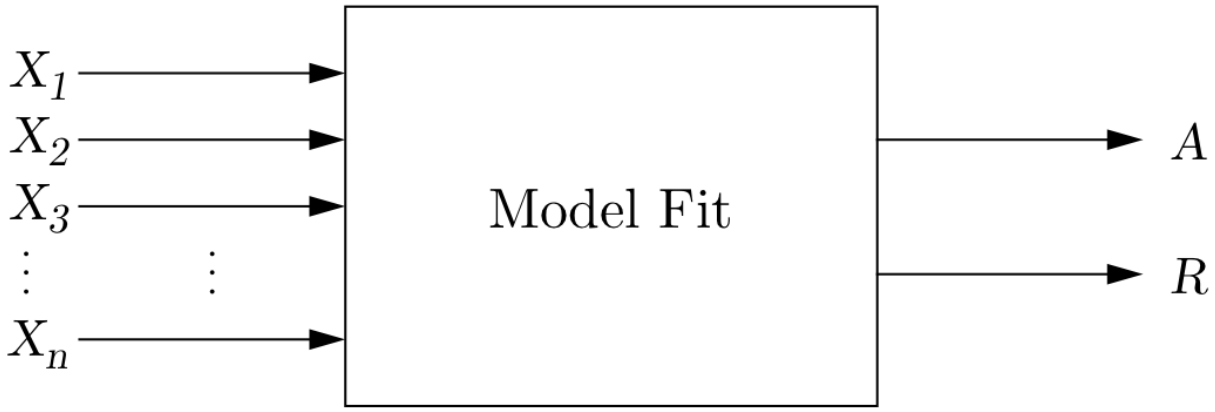
Algebraic versus **geometric** fit solutions [Gander 1994]

LSQ Estimation: Uncertainties?

How does the **input uncertainty** propagate over the fit expressions to the **output**?

$X_1, \ldots, X_n$: Gaussian input random variables

A, R : Gaussian output random variables



Example: Line Extraction

Wanted: Parameter Covariance Matrix

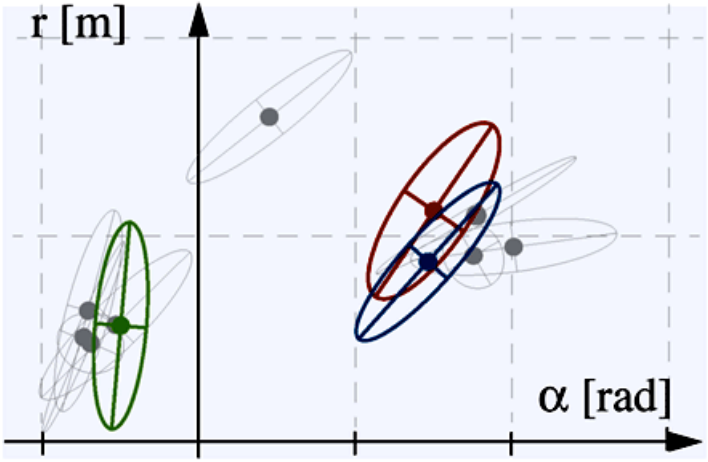
$$C_{AR} = \begin{bmatrix} \sigma_A^2 & \sigma_{AR} \\ \sigma_{AR} & \sigma_R^2 \end{bmatrix}$$

$$C_X = \begin{bmatrix} \sigma_{\rho_1}^2 & 0 & \dots & 0 \\ 0 & \sigma_{\rho_2}^2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \sigma_{\rho_n}^2 \end{bmatrix}$$

Simplified sensor model:
all $\sigma_{\theta_i}^2 = 0$, independence

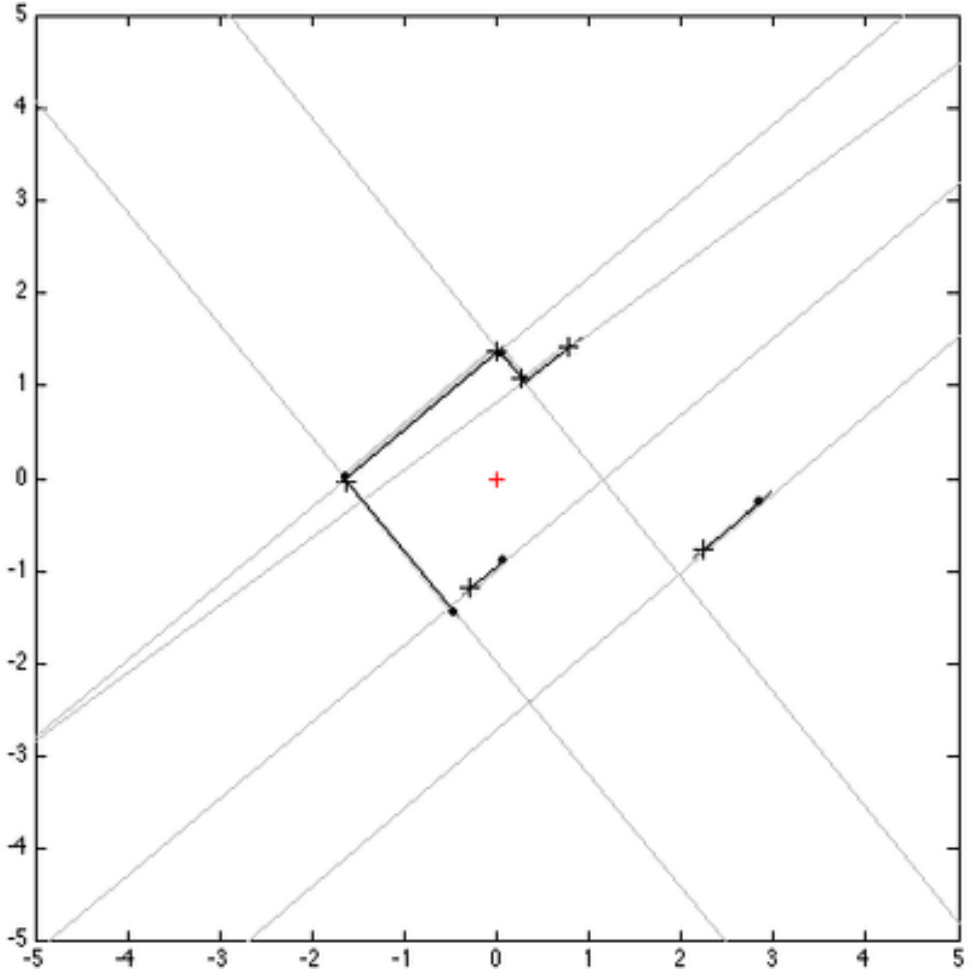
$$C_{AR} = F_X C_X F_X^T$$

Result: Gaussians in the parameter space



Line Extraction in Real Time

- Robot *Pygmalion* EPFL, Lausanne
- CPU: PowerPC 604e at 300 MHz
  Sensor: 2 SICK LMS
- Line Extraction Times: **~ 25 *ms***



Derivations (1/4)

Result: Line Fit Cartesian Coordinates

(only for *r*, α more complicated...)

$$
\begin{aligned}
&\frac{\partial S}{\partial r} = 0 \\
&\Leftrightarrow \frac{\partial}{\partial r}\{\sum \epsilon_i^2\} = \sum \frac{\partial}{\partial r}\{\epsilon_i^2\} = 2\sum \epsilon_i \frac{\partial}{\partial r}\{\epsilon_i\} \\
&\Leftrightarrow 2\sum (x_i cos\alpha + y_i sin\alpha - r)(-1) = 0 \\
&\Leftrightarrow \sum (x_i cos\alpha + y_i sin\alpha - r) = 0 \\
&\Leftrightarrow \sum x_i cos\alpha + \sum y_i sin\alpha - nr = 0 \\
&\Leftrightarrow r = 1/n \sum x_i cos\alpha + 1/n \sum y_i sin\alpha \\
&\Leftrightarrow r = \bar{x} cos\alpha + \bar{y} sin\alpha
\end{aligned}
$$

Introduction to Mobile Robotics

EKF Localization

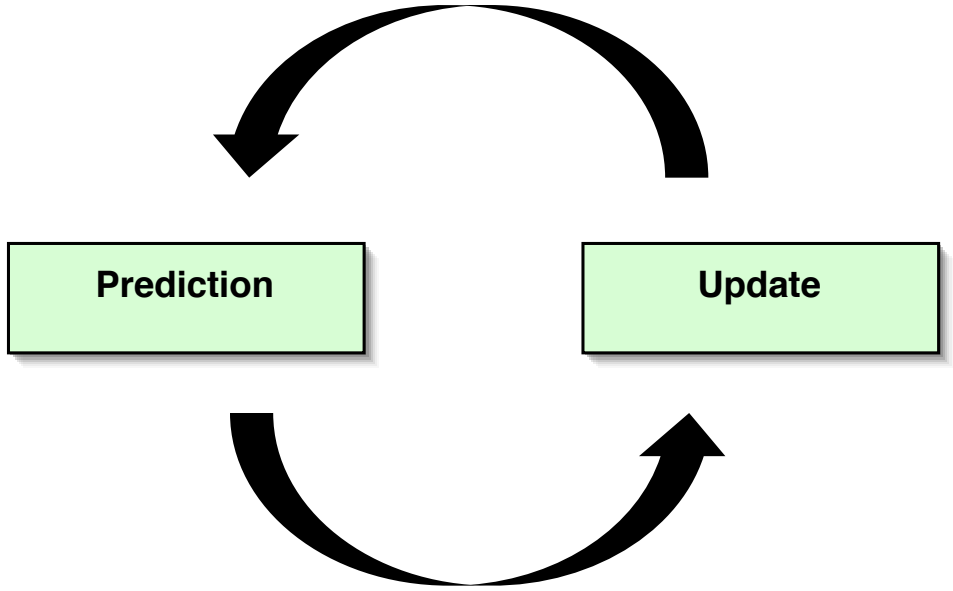
Localization

> "Using sensory information to locate the robot in its environment is the most fundamental problem to providing a mobile robot with autonomous capabilities." [Cox '91]

- **Given**
  - Map of the environment.
  - Sequence of sensor measurements.
- **Wanted**
  - Estimate of the robot's position.
- **Problem classes**
  - Position tracking
  - Global localization
  - Kidnapped robot problem (recovery)

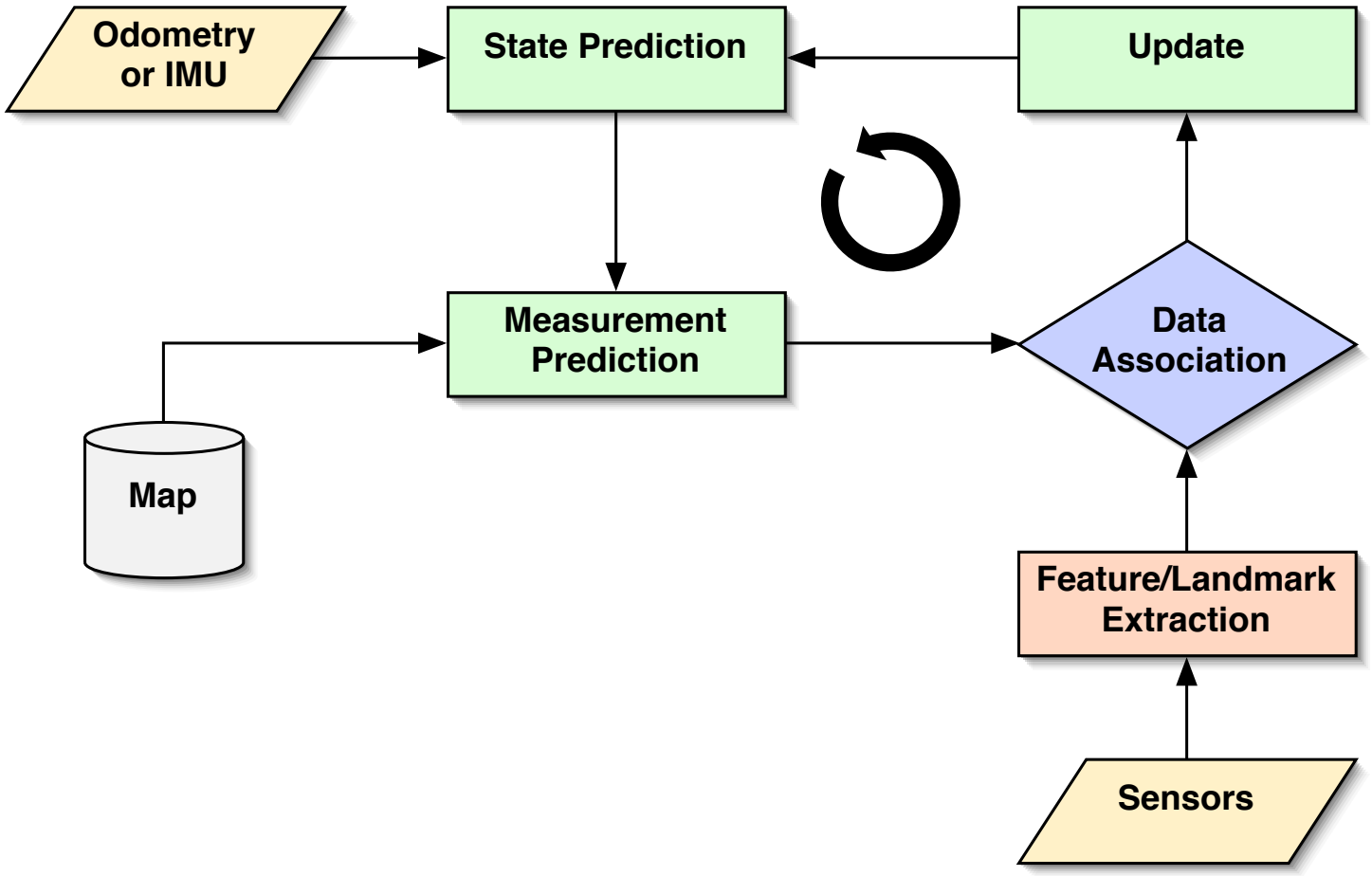
Landmark-based Localization

EKF Localization: Basic Cycle



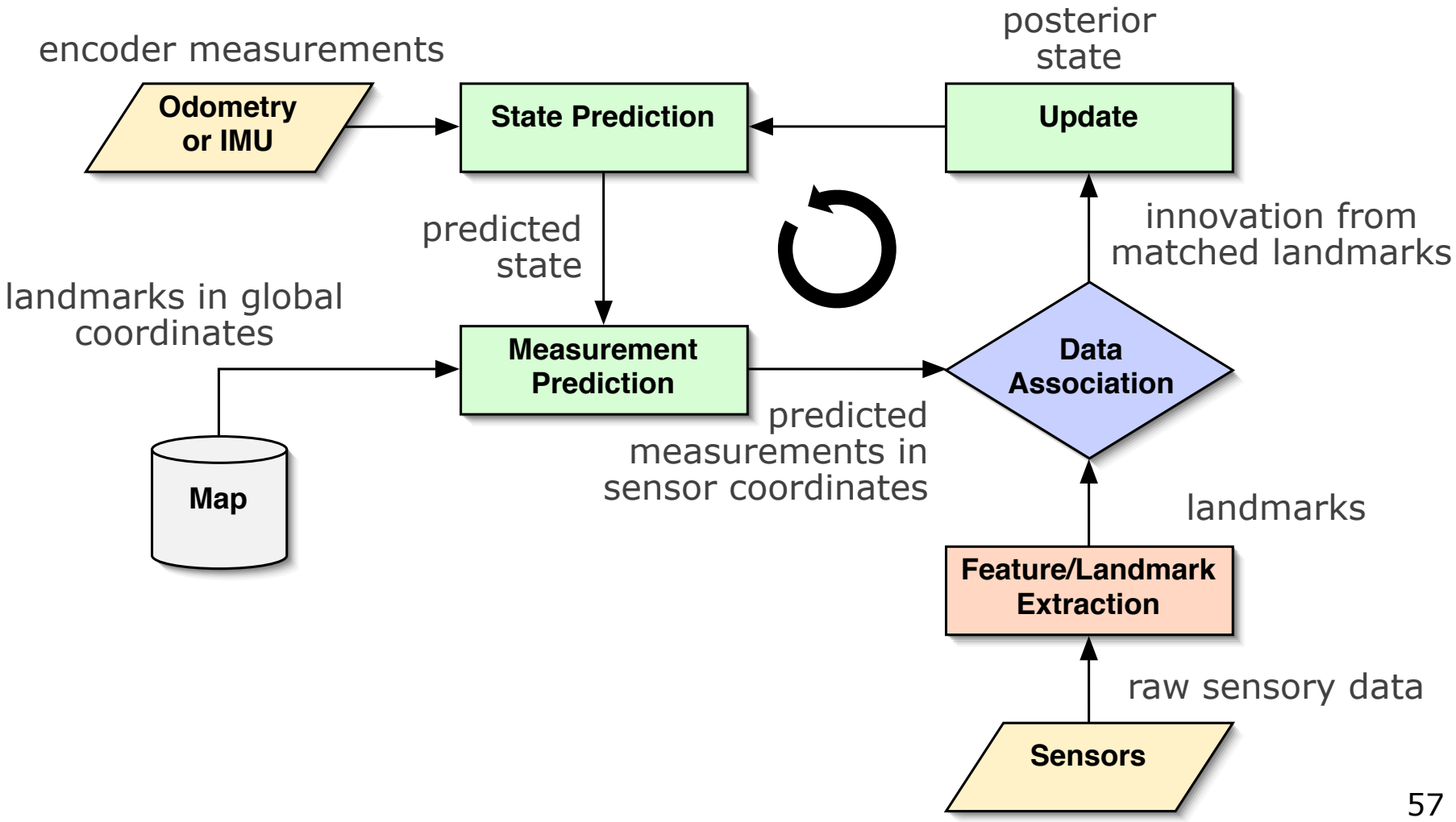
Landmark-based Localization

EKF Localization: Basic Cycle



Landmark-based Localization

EKF Localization: Basic Cycle



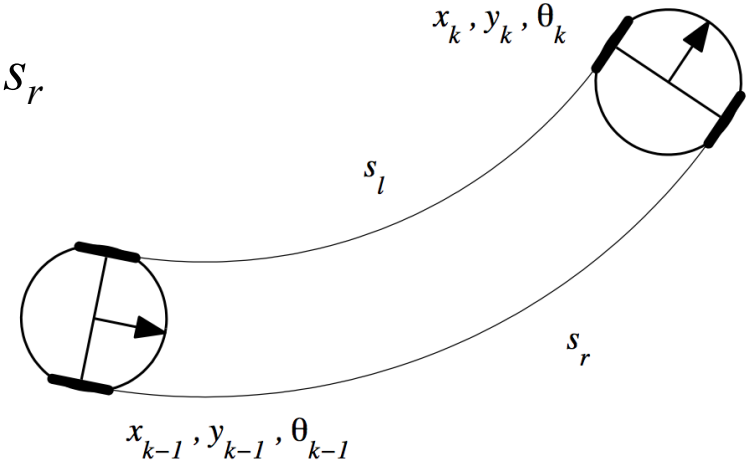
Landmark-based Localization

State Prediction (Odometry)

$$\hat{\mathbf{x}}_k = f(\mathbf{x}_{k-1}, \mathbf{u}_k)$$
$$\hat{C}_k = F_x\, C_k\, F_x^T + F_u\, U_k\, F_u^T$$

Control $\mathbf{u}_k$: wheel displacements s_l, s_r

$$\mathbf{u}_k = (s_l \;\; s_r)^T \qquad U_k = \begin{bmatrix} \sigma_l^2 & 0 \\ 0 & \sigma_r^2 \end{bmatrix}$$



Error model: linear growth

$$\sigma_l = k_l\, |s_l|$$
$$\sigma_r = k_r\, |s_r|$$

Nonlinear process model f:

$$\mathbf{x}_k = \begin{bmatrix} x_{k-1} \\ y_{k-1} \\ \theta_{k-1} \end{bmatrix} + \begin{bmatrix} \frac{b}{2}\frac{s_l+s_r}{s_r-s_l}\left(-\sin\theta_{k-1} + \sin(\theta_{k-1} + \frac{s_r-s_l}{b})\right) \\ \frac{b}{2}\frac{s_l+s_r}{s_r-s_l}\left(\cos\theta_{k-1} - \cos(\theta_{k-1} + \frac{s_r-s_l}{b})\right) \\ \frac{s_r-s_l}{b} \end{bmatrix}$$

Landmark-based Localization

State Prediction (Odometry)

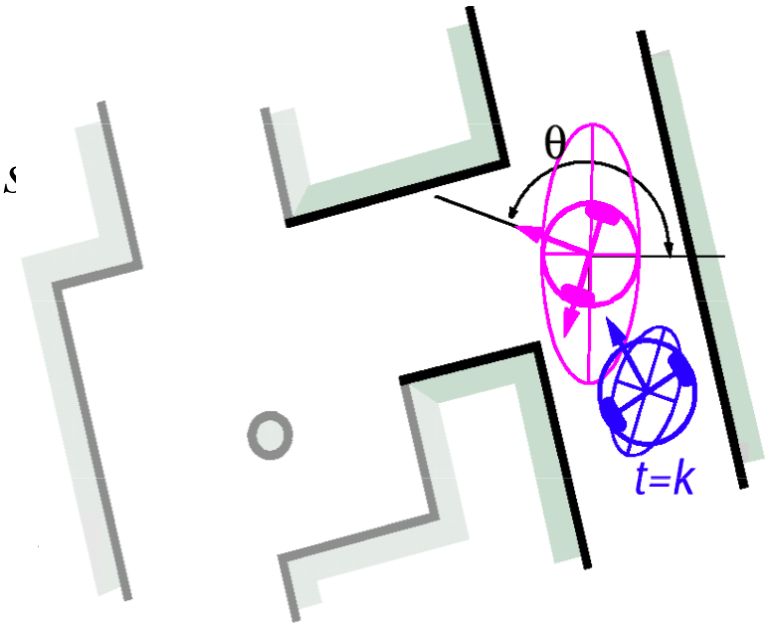
$$\hat{\mathbf{x}}_k = f(\mathbf{x}_{k-1}, \mathbf{u}_k)$$
$$\hat{C}_k = F_x\, C_k\, F_x^T + F_u\, U_k\, F_u^T$$

Control $\mathbf{u}_k$: wheel displacements s_l, s

$$\mathbf{u}_k = (s_l \;\; s_r)^T \qquad U_k = \begin{bmatrix} \sigma_l^2 & 0 \\ 0 & \sigma_r^2 \end{bmatrix}$$

Error model: linear growth

$$\sigma_l = k_l\, |s_l|$$
$$\sigma_r = k_r\, |s_r|$$



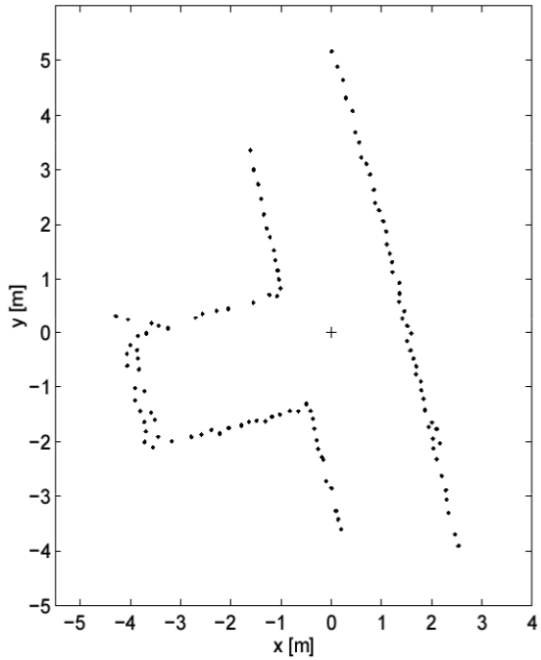
Nonlinear process model f:

$$\mathbf{x}_k = \begin{bmatrix} x_{k-1} \\ y_{k-1} \\ \theta_{k-1} \end{bmatrix} + \begin{bmatrix} \frac{b}{2}\frac{s_l+s_r}{s_r-s_l}\left(-\sin\theta_{k-1} + \sin(\theta_{k-1} + \frac{s_r-s_l}{b})\right) \\ \frac{b}{2}\frac{s_l+s_r}{s_r-s_l}\left(\cos\theta_{k-1} - \cos(\theta_{k-1} + \frac{s_r-s_l}{b})\right) \\ \frac{s_r-s_l}{b} \end{bmatrix}$$

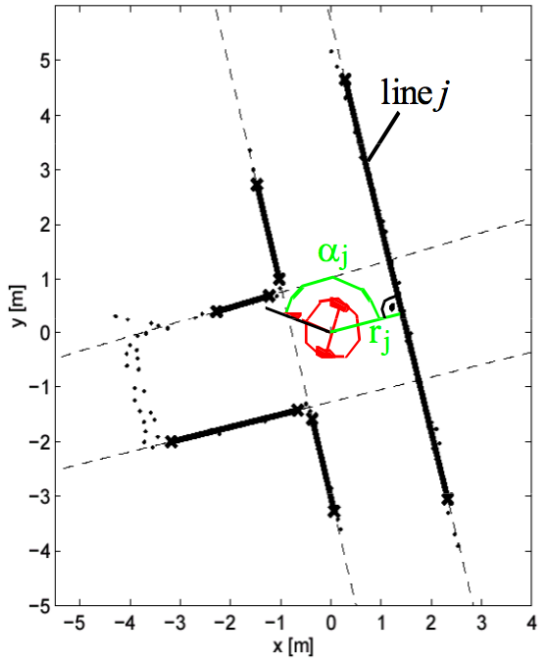
Landmark-based Localization

Landmark Extraction (Observation)

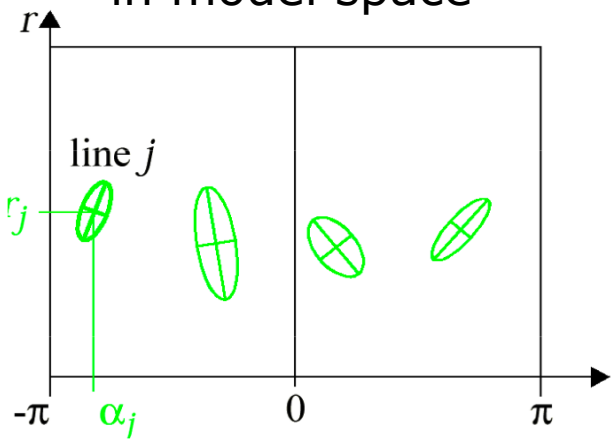
Raw laser range data



Extracted lines



Extracted lines in model space



$$\mathbf{z}_k = \begin{bmatrix} \alpha \\ r \end{bmatrix}$$

$$R_k = \begin{bmatrix} \sigma_\alpha^2 & \sigma_{\alpha r} \\ \sigma_{r\alpha} & \sigma_r^2 \end{bmatrix}$$

Hessian line model

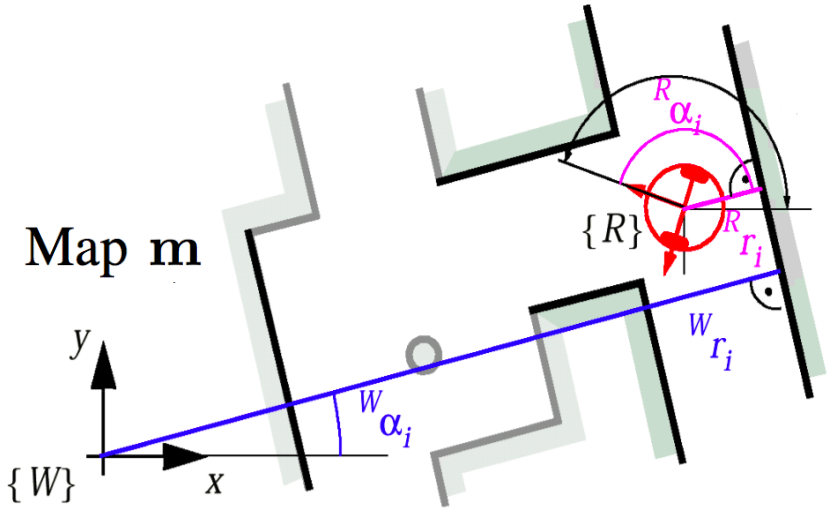
$$x\ cos(\alpha) + y\ sin(\alpha) - r = 0$$

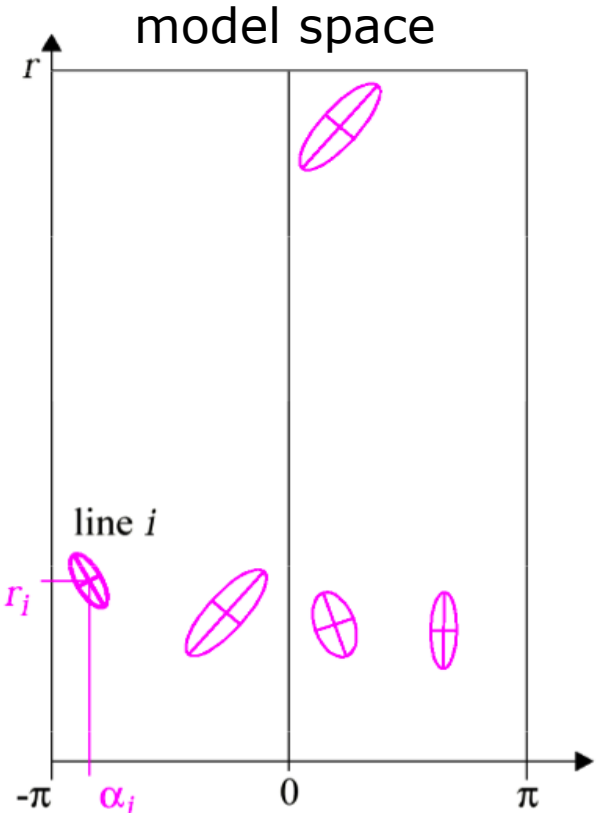
Landmark-based Localization

Measurement Prediction

- ...is a coordinate frame transform world-to-sensor
- Given the predicted state (robot pose), predicts the location $\hat{\mathbf{z}}_k$ and location uncertainty $H\,\hat{C}_k\,H^T$ of expected observations in sensor coordinates

$$\hat{\mathbf{z}}_k = h(\hat{\mathbf{x}}_k, \mathbf{m})$$





Landmark-based Localization

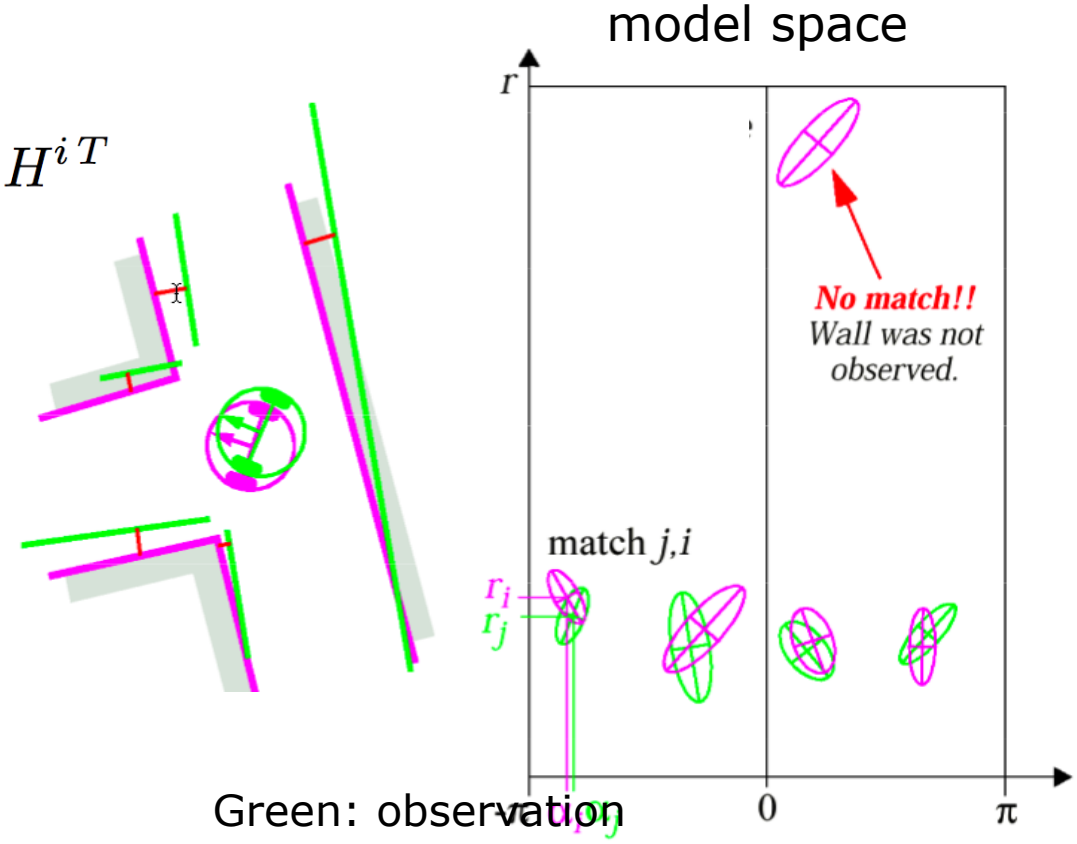
Data Association (Matching)

- Associates predicted measurements $\hat{\mathbf{z}}_k^i$ with observations $\mathbf{z}_k^j$

$$\nu_k^{ij} = \mathbf{z}_k^j - \hat{\mathbf{z}}_k^i$$
$$S_k^{ij} = R_k^j + H^i \hat{C}_k H^{iT}$$

- Innovation ν_k^{ij} and innovation covariance S_k^{ij}

- Matching on significance level alpha



Green: observation

Magenta: measurement prediction

Landmark-based Localization

Update

- Kalman gain

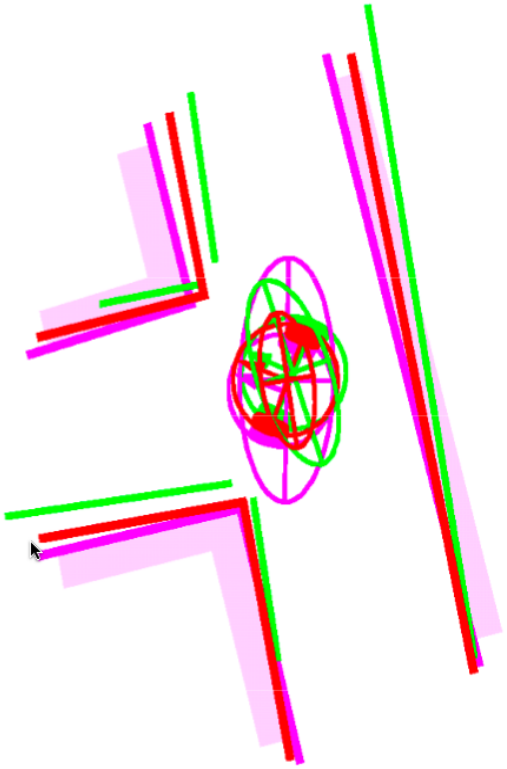
$$K_k = \hat{C}_k \, H^T S_k^{-1}$$

- State update (robot pose)

$$\mathbf{x}_k = \hat{\mathbf{x}}_k + K_k \, \nu_k$$

- State covariance update

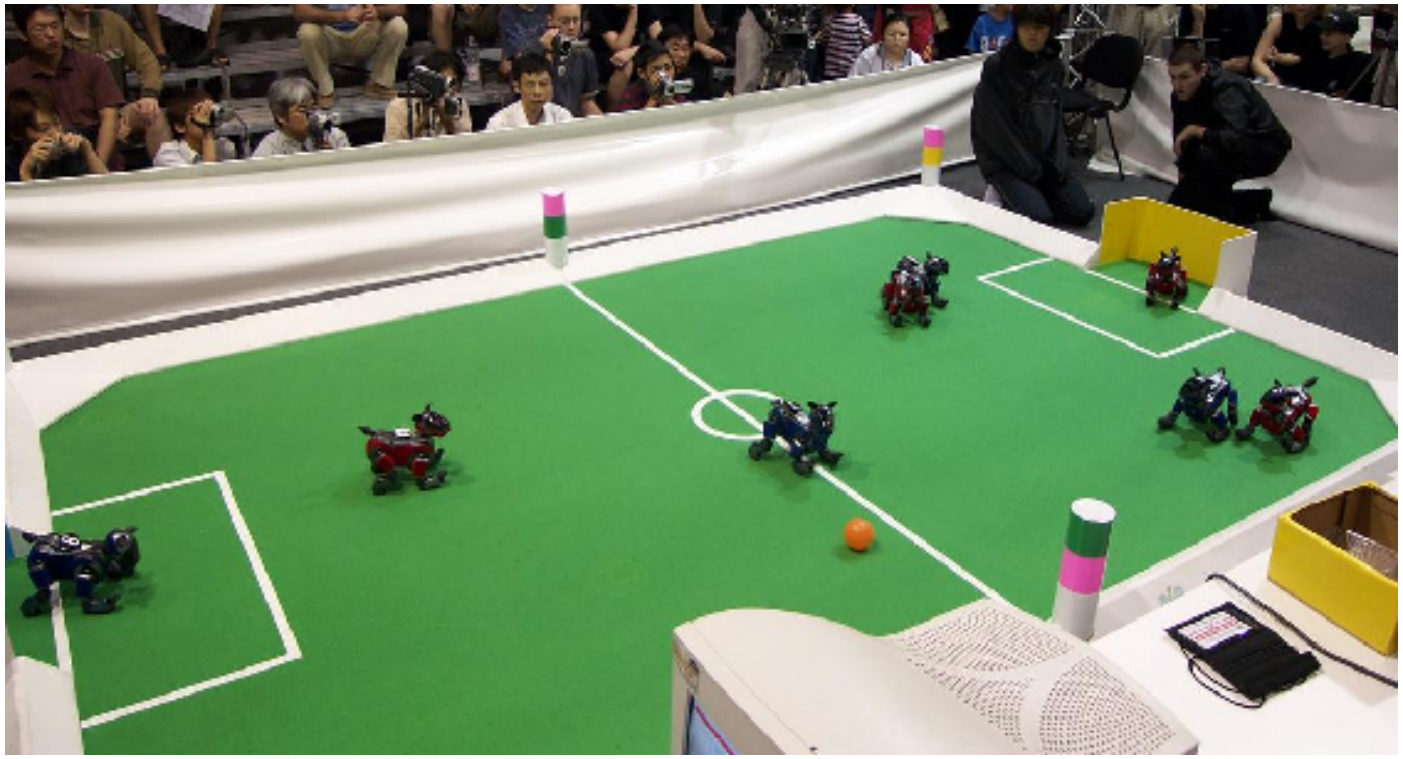
$$C_k = (I - K_k \, H) \, \hat{C}_k$$



Red: posterior estimate

Landmark-based Localization

- **EKF Localization with Point Features**



1. EKF_localization ($\mu_{t-1}, \Sigma_{t-1}, u_t, z_t, m$):

Prediction:

2. $G_t = \frac{\partial g(u_t, \mu_{t-1})}{\partial x_{t-1}} = \begin{pmatrix} \frac{\partial x'}{\partial \mu_{t-1,x}} & \frac{\partial x'}{\partial \mu_{t-1,y}} & \frac{\partial x'}{\partial \mu_{t-1,\theta}} \\ \frac{\partial y'}{\partial \mu_{t-1,x}} & \frac{\partial y'}{\partial \mu_{t-1,y}} & \frac{\partial y'}{\partial \mu_{t-1,\theta}} \\ \frac{\partial \theta'}{\partial \mu_{t-1,x}} & \frac{\partial \theta'}{\partial \mu_{t-1,y}} & \frac{\partial \theta'}{\partial \mu_{t-1,\theta}} \end{pmatrix}$ Jacobian of g w.r.t location

3. $B_t = \frac{\partial g(u_t, \mu_{t-1})}{\partial u_t} = \begin{pmatrix} \frac{\partial x'}{\partial v_t} & \frac{\partial x'}{\partial \omega_t} \\ \frac{\partial y'}{\partial v_t} & \frac{\partial y'}{\partial \omega_t} \\ \frac{\partial \theta'}{\partial v_t} & \frac{\partial \theta'}{\partial \omega_t} \end{pmatrix}$ Jacobian of g w.r.t control

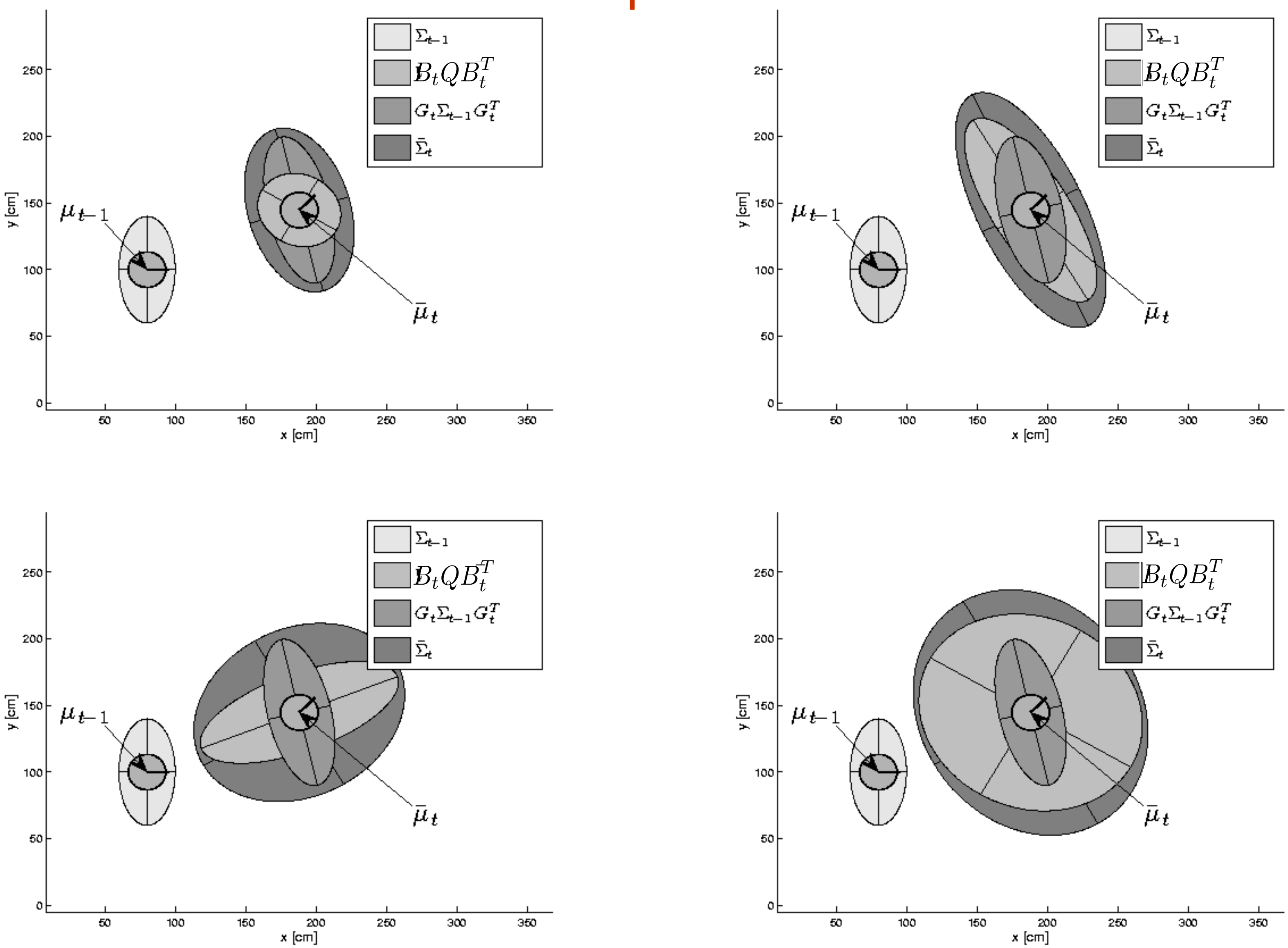
4. $Q_t = \begin{pmatrix} (\alpha_1 |v_t| + \alpha_2 |\omega_t|)^2 & 0 \\ 0 & (\alpha_3 |v_t| + \alpha_4 |\omega_t|)^2 \end{pmatrix}$ Motion noise

5. $\bar{\mu}_t = g(u_t, \mu_{t-1})$ Predicted mean

6. $\bar{\Sigma}_t = G_t \Sigma_{t-1} G_t^T + B_t Q_t B_t^T$ Predicted covariance

EKF Prediction Step

for different noise levels
left previous pose, right predicted pose



1. EKF_localization ($\mu_{t-1}, \Sigma_{t-1}, u_t, z_t, m$):

Correction:

2. $\hat{z}_t = \begin{pmatrix} \sqrt{(m_x - \bar{\mu}_{t,x})^2 + (m_y - \bar{\mu}_{t,y})^2} \\ \operatorname{atan2}(m_y - \bar{\mu}_{t,y}, m_x - \bar{\mu}_{t,x}) - \bar{\mu}_{t,\theta} \end{pmatrix}$ Predicted measurement mean

3. $H_t = \frac{\partial h(\bar{\mu}_t, m)}{\partial x_t} = \begin{pmatrix} \frac{\partial r_t}{\partial \bar{\mu}_{t,x}} & \frac{\partial r_t}{\partial \bar{\mu}_{t,y}} & \frac{\partial r_t}{\partial \bar{\mu}_{t,\theta}} \\ \frac{\partial \varphi_t}{\partial \bar{\mu}_{t,x}} & \frac{\partial \varphi_t}{\partial \bar{\mu}_{t,y}} & \frac{\partial \varphi_t}{\partial \bar{\mu}_{t,\theta}} \end{pmatrix}$ Jacobian of *h* w.r.t location

4. $R_t = \begin{pmatrix} \sigma_r^2 & 0 \\ 0 & \sigma_r^2 \end{pmatrix}$

5. $S_t = H_t \bar{\Sigma}_t H_t^T + R_t$ Innovation covariance

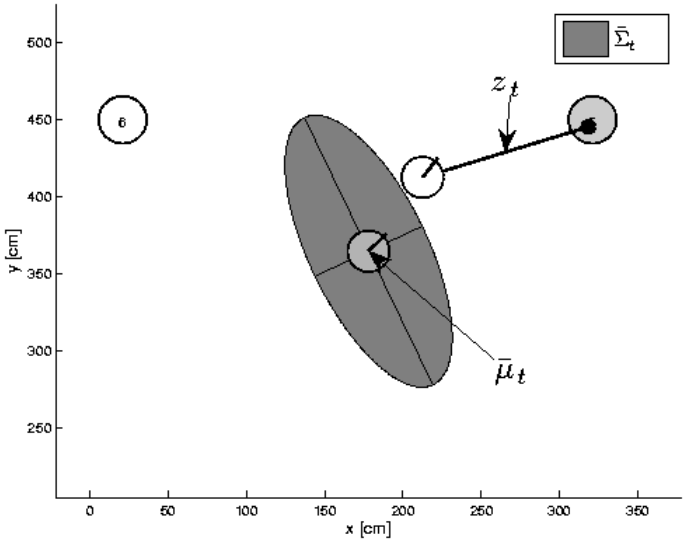
6. $K_t = \bar{\Sigma}_t H_t^T S_t^{-1}$ Kalman gain

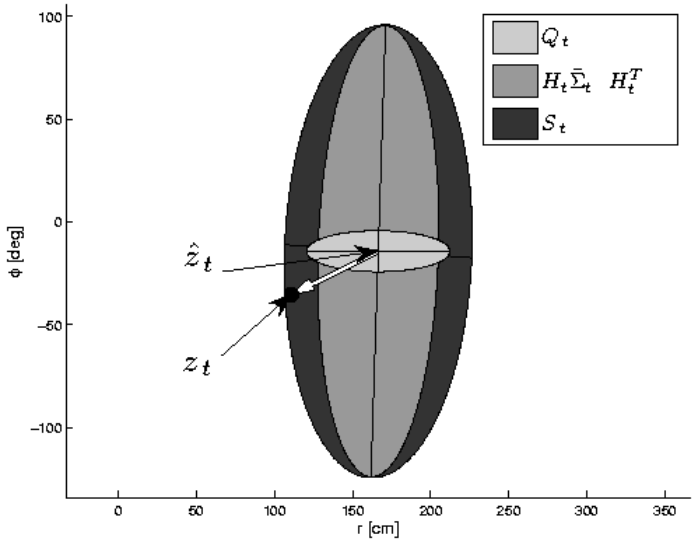
7. $\mu_t = \bar{\mu}_t + K_t(z_t - \hat{z}_t)$ Updated mean

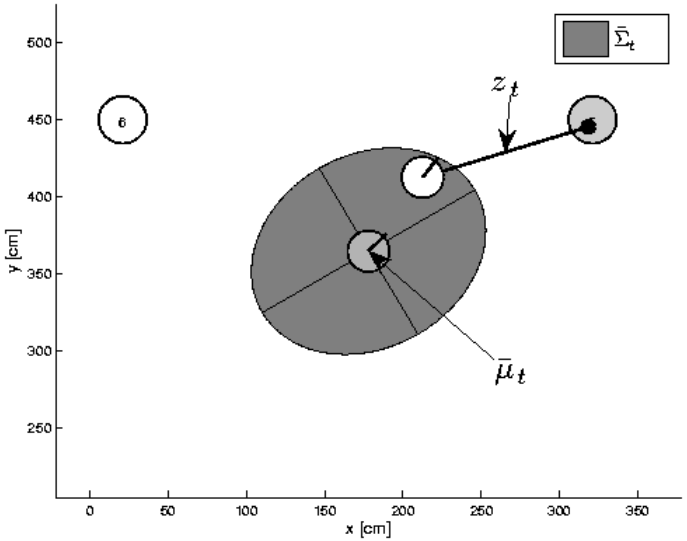
8. $\Sigma_t = (I - K_t H_t)\bar{\Sigma}_t$ Updated covariance

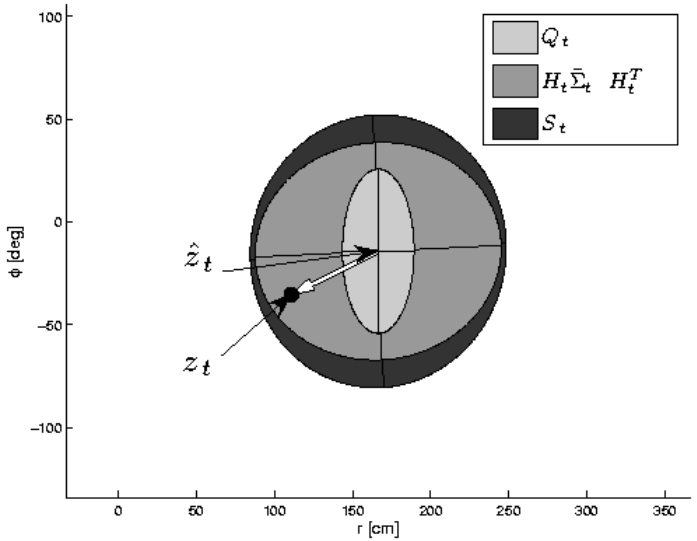
EKF Observation Prediction Step

left column predictions – white circle ground truth
right measurement – predicted and observed



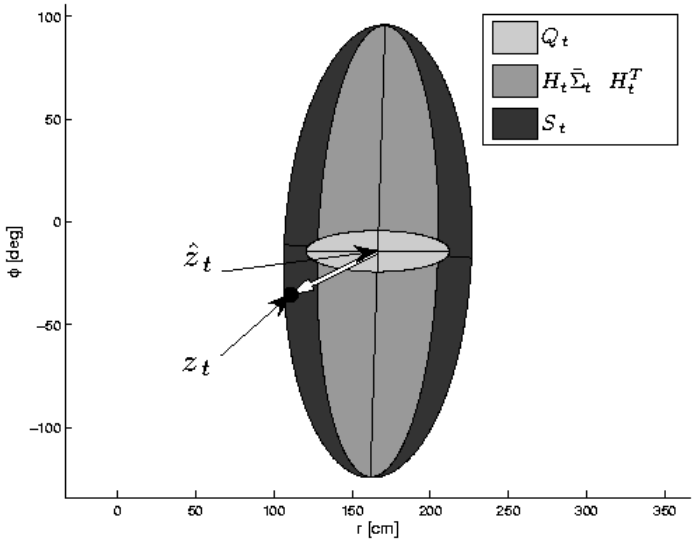


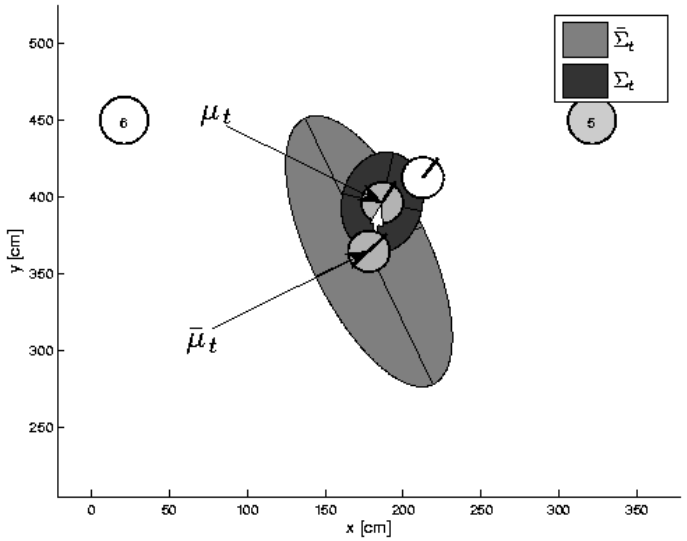


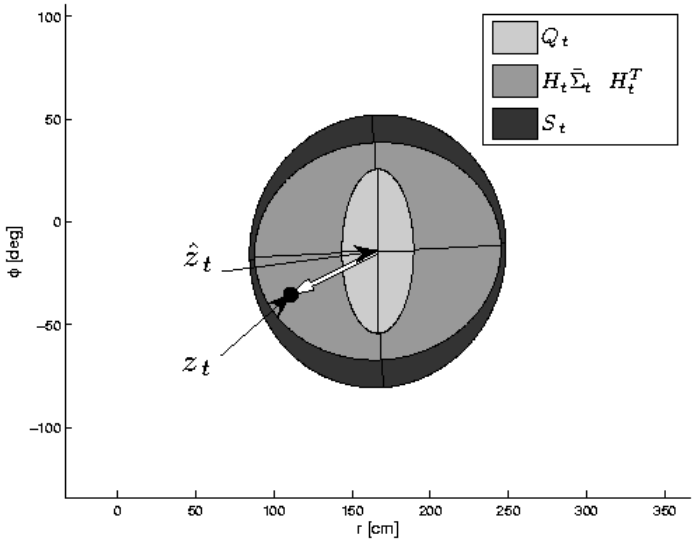


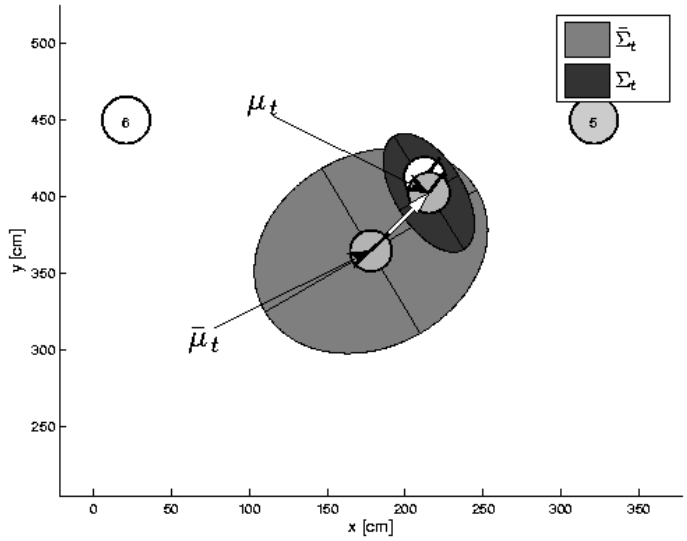
EKF Correction Step

left: given predicted and observed measurement, right corrected pose



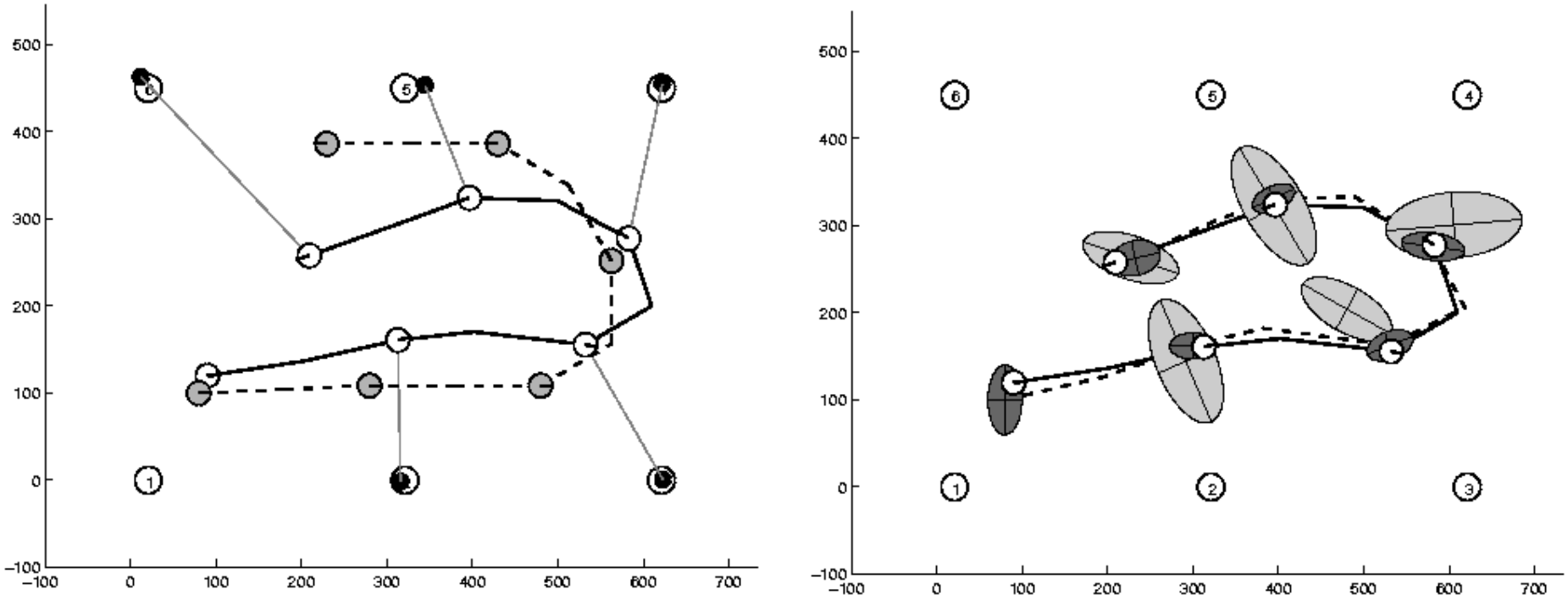






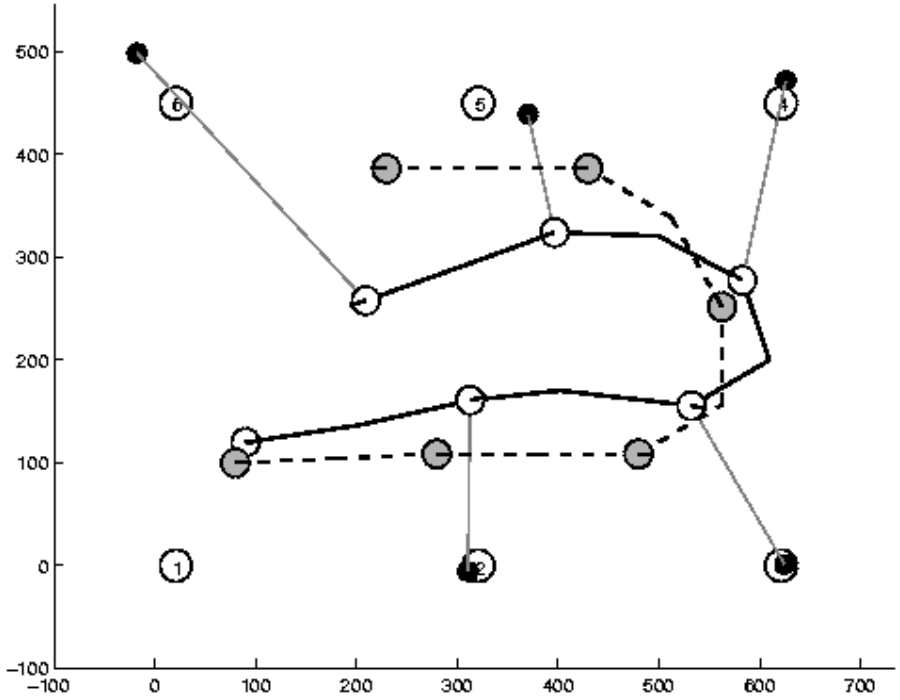
Estimation Sequence (1)

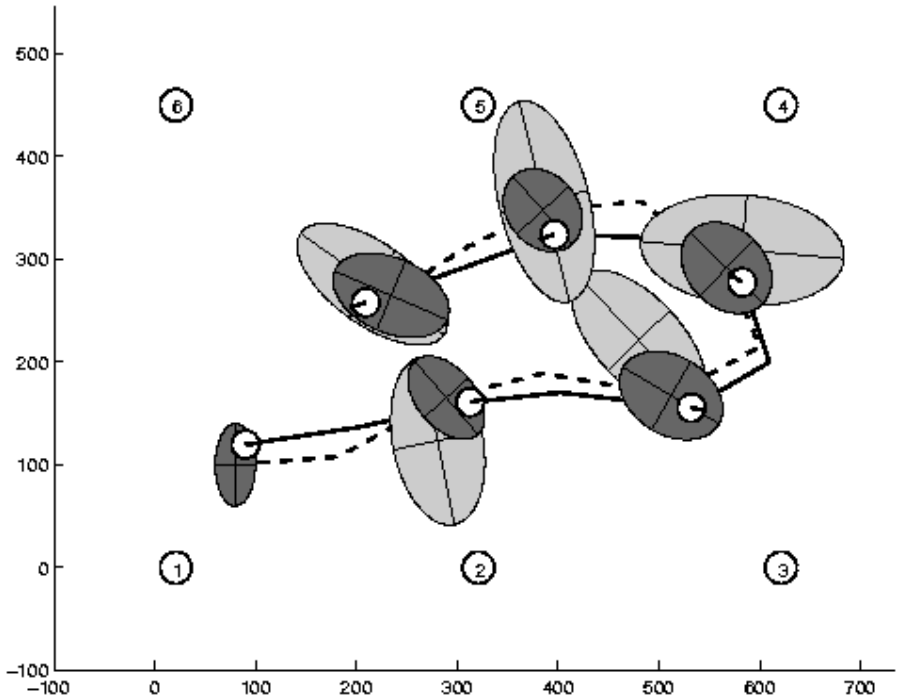
solid line: ground truth, dashed line: odometry without sensing (left), with sensing (right)



Estimation Sequence (2)

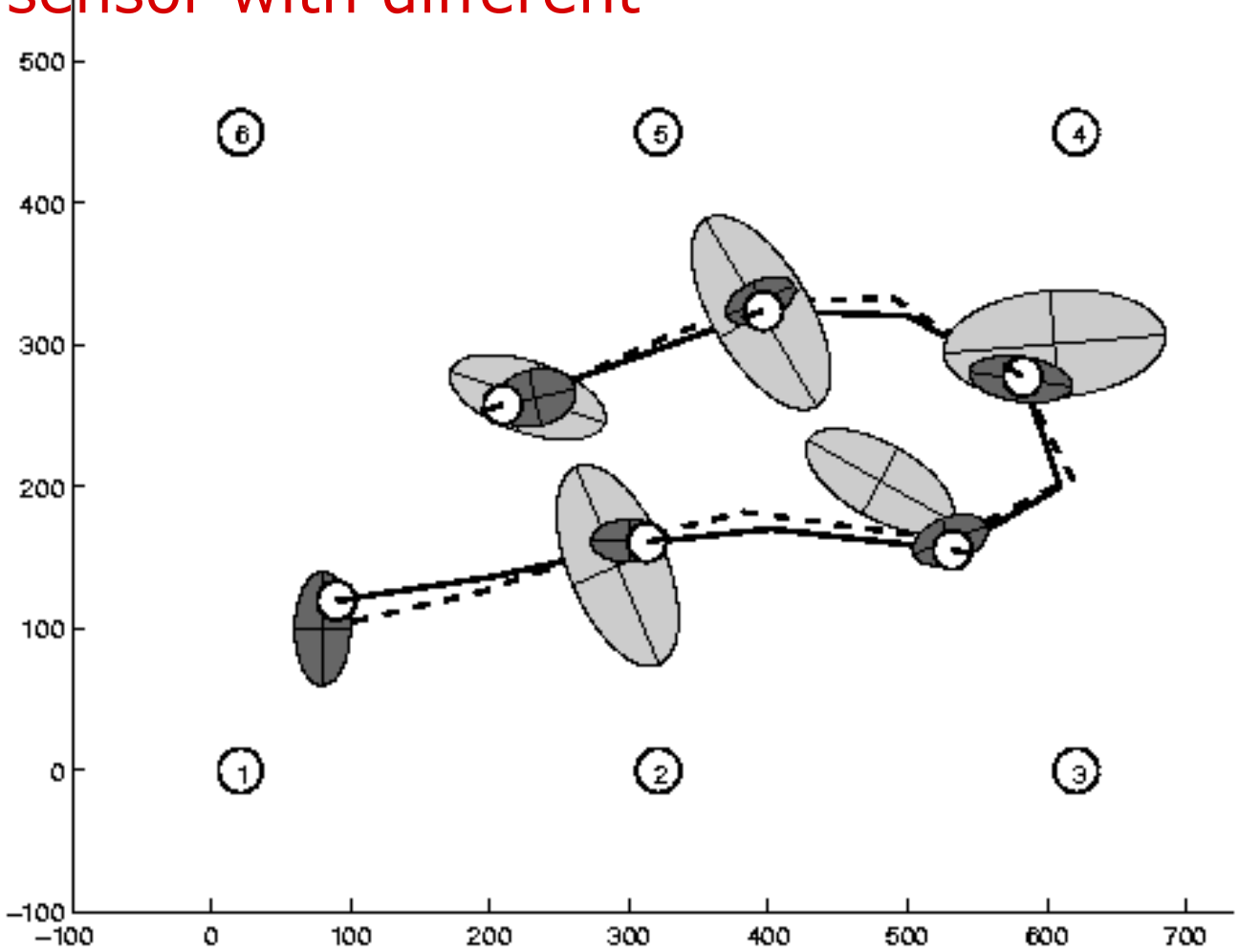
solid line: ground truth, dashed line: odometry without sensing (left), with sensing (right)





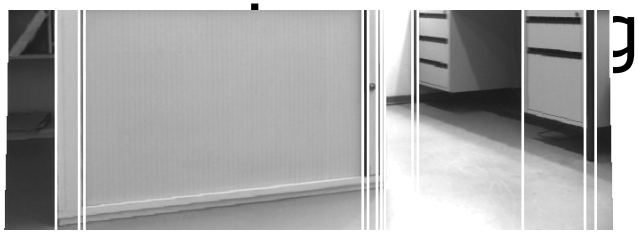
Comparison to GroundTruth

solid line: ground truth, dashed line: odometry with sensor with different

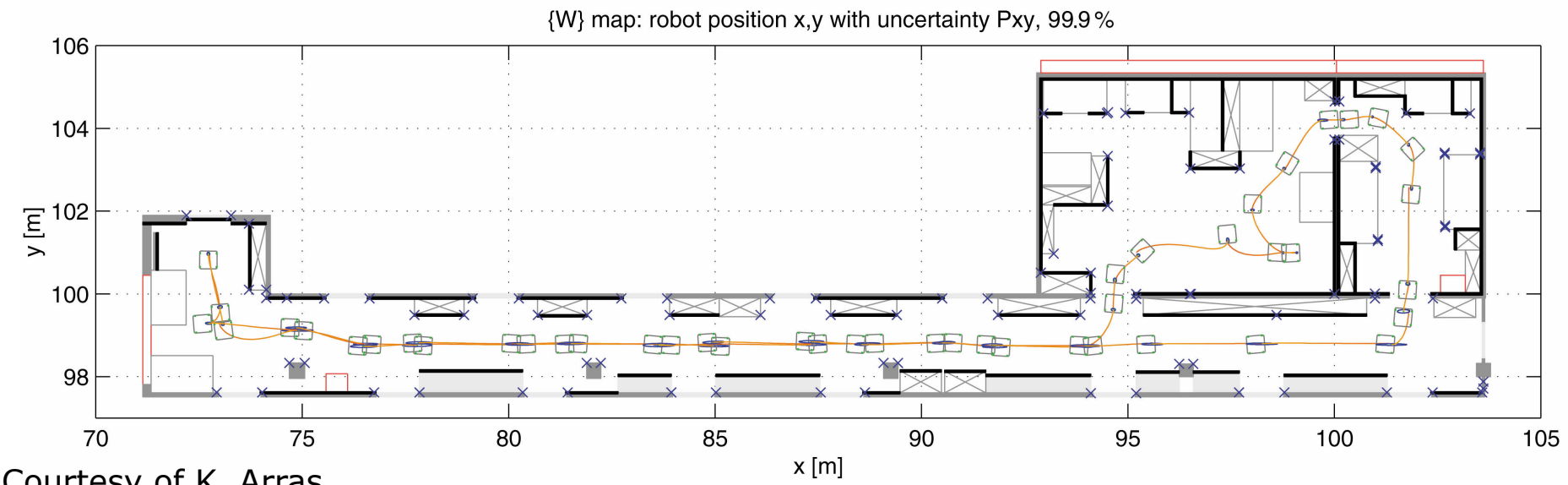


EKF Localization Example

- [Arras et al. 98]:



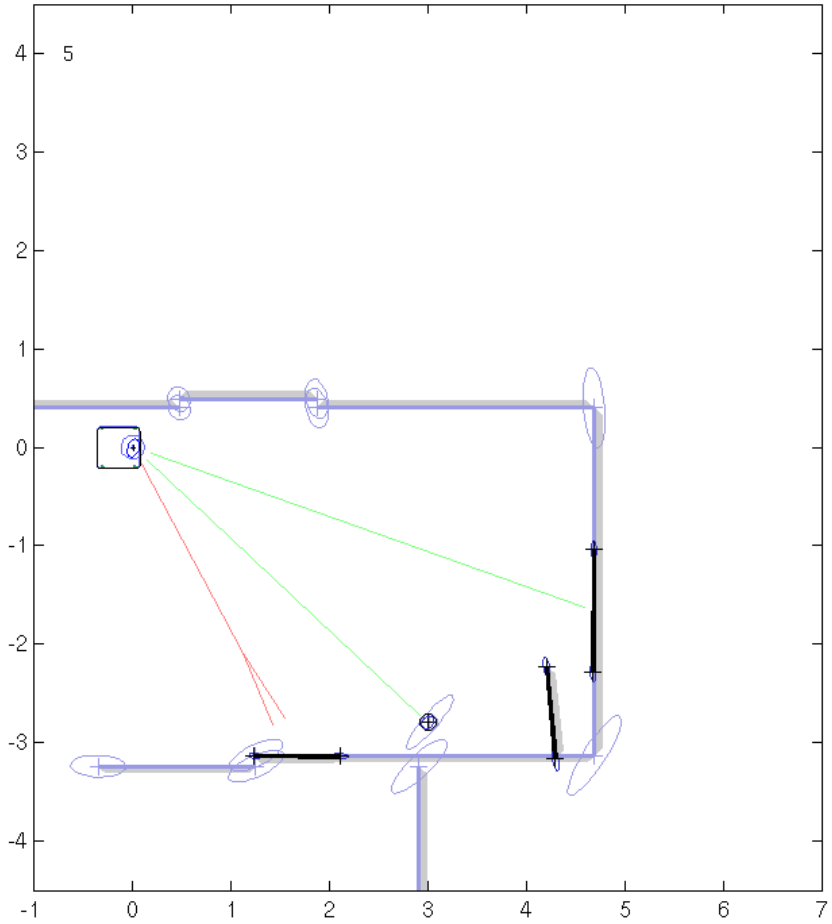
ge-finder and v



Courtesy of K. Arras

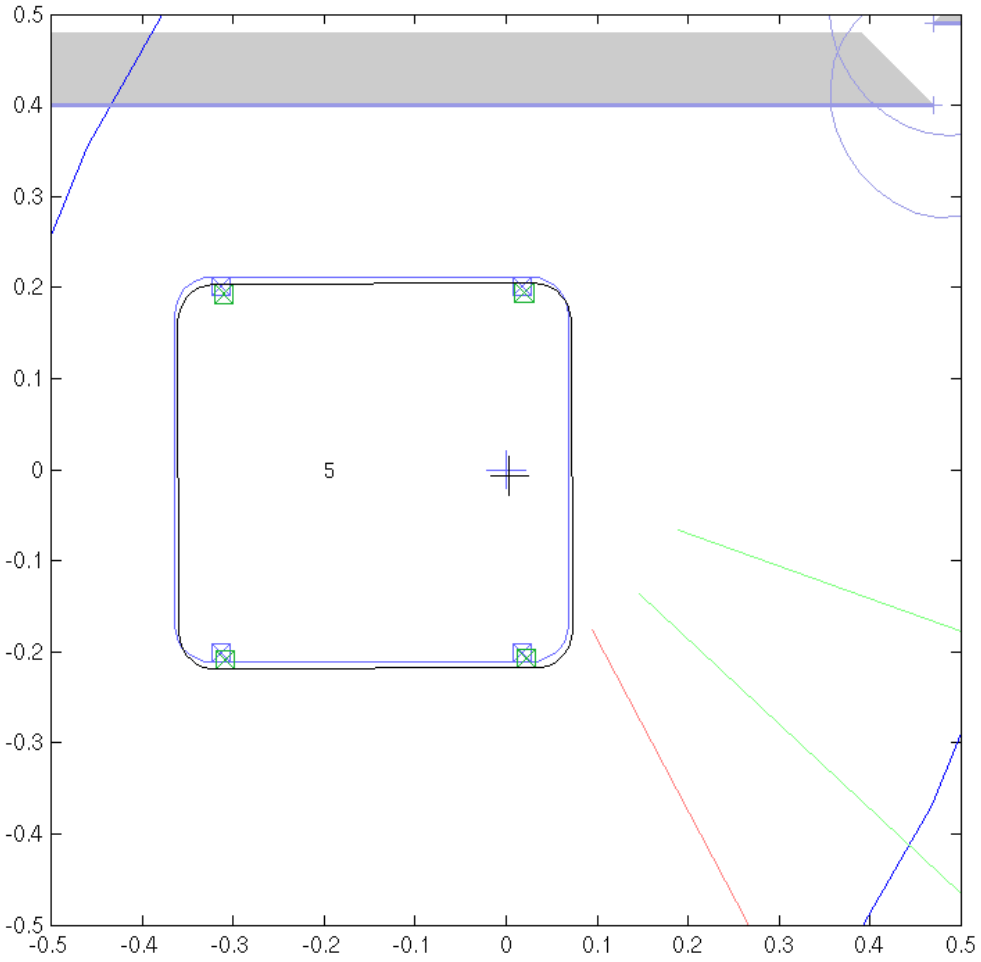
EKF Localization Example

- Line and point landmarks



EKF Localization Example

- Line and point landmarks



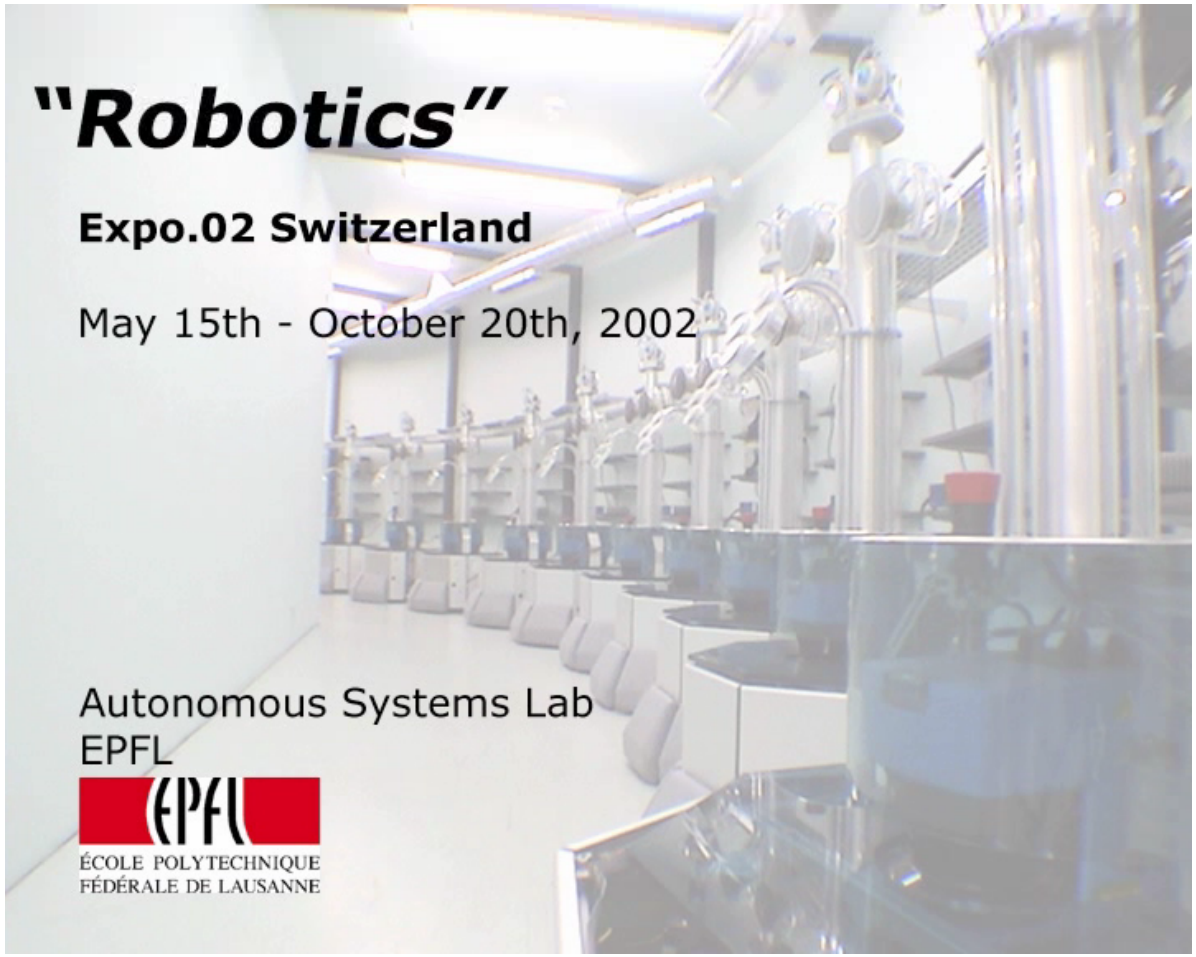
EKF Localization Example

- **Expo.02:** Swiss National Exhibition 2002
- Pavilion "Robotics"
- 11 fully autonomous robots
- tour guides, entertainer, photographer
- 12 hours per day
- 7 days per week
- 5 months

- **3,316** km travel distance
- almost **700,000** visitors
- 400 visitors per hour

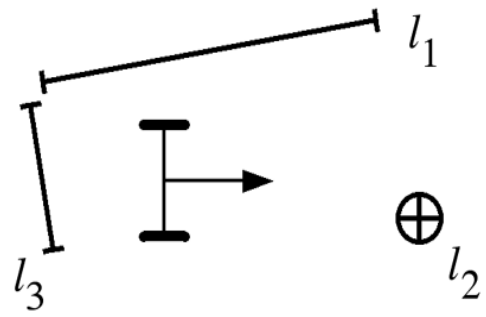
- Localization method: **Line-Based EKF**

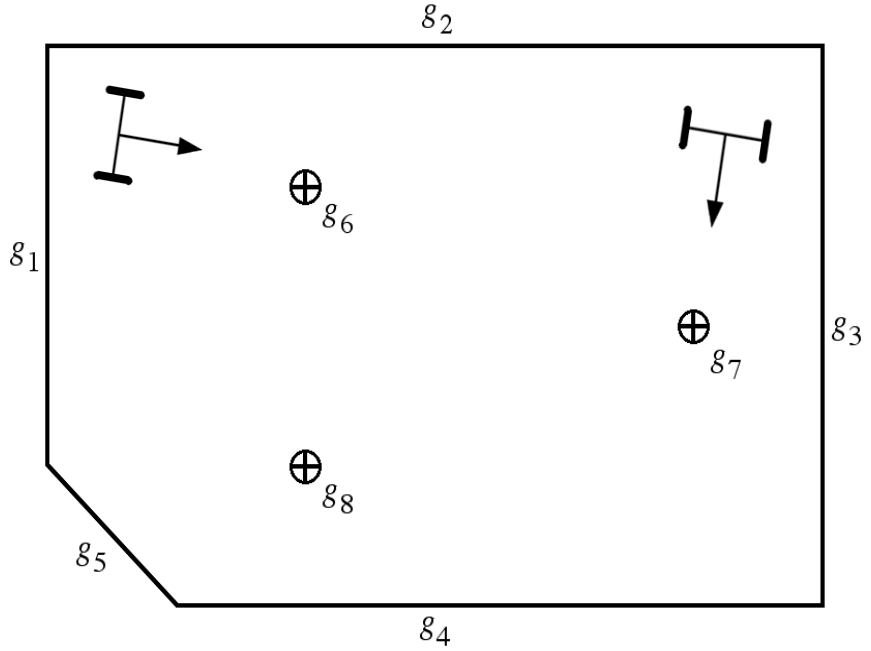
EKF Localization Example

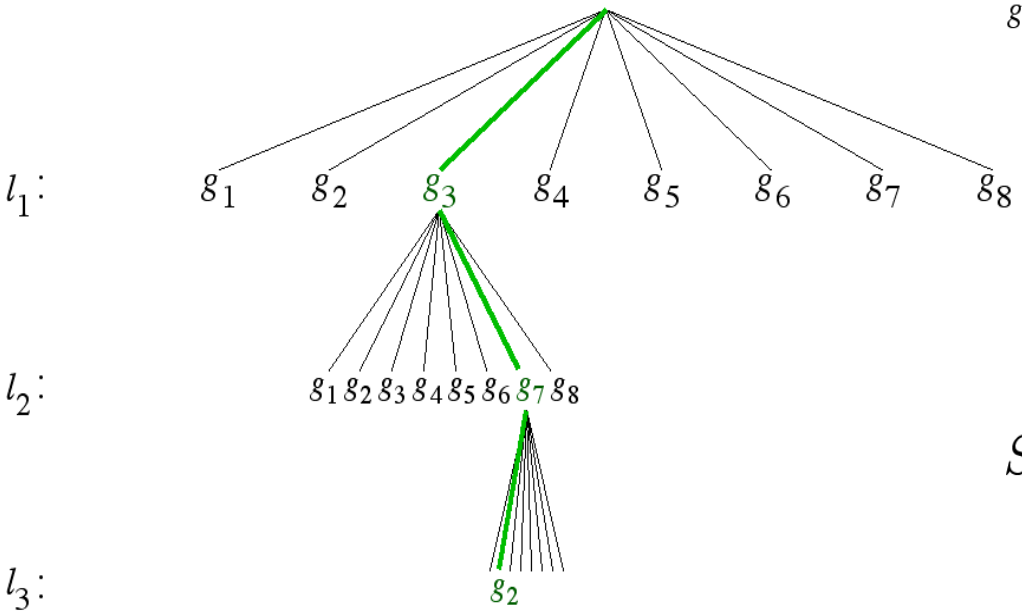


Global EKF Localization

Interpretation tree



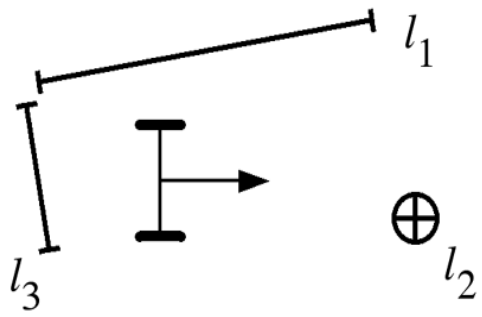


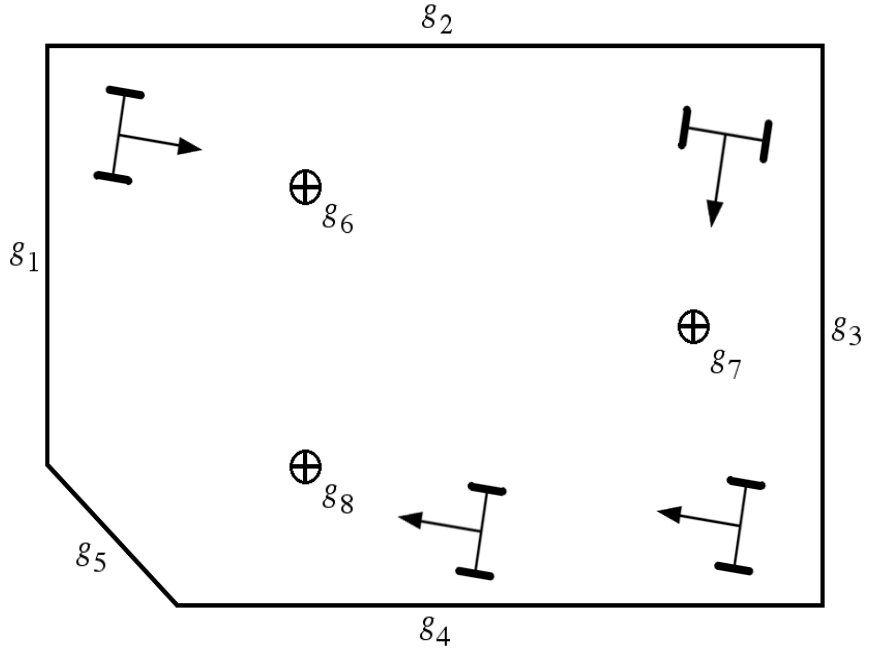


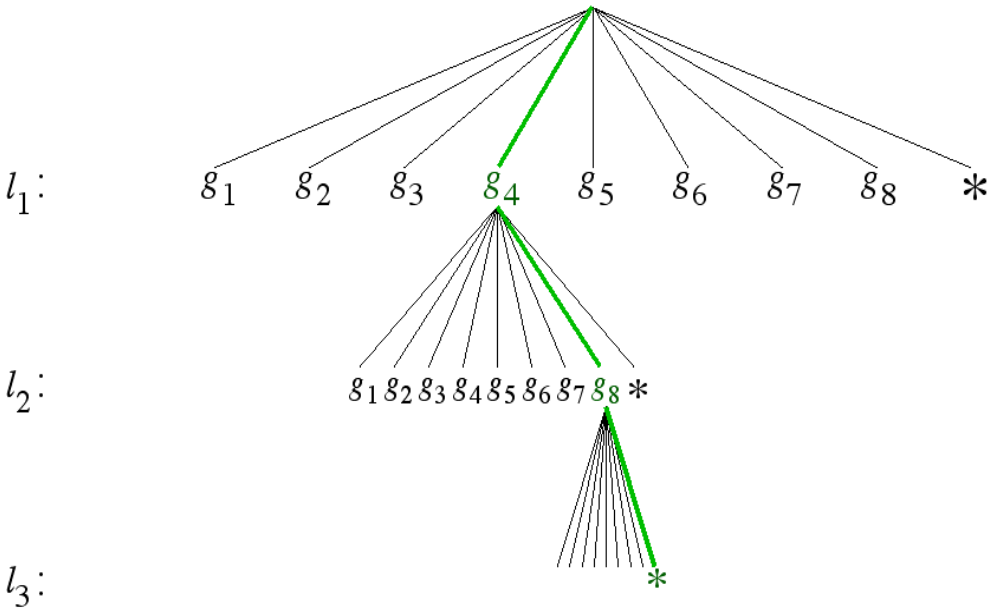
$$S_{h_2} = \{\{l_1, g_3\}, \{l_2, g_7\}, \{l_3, g_2\}\}$$

Global EKF Localization

Env. Dynamics



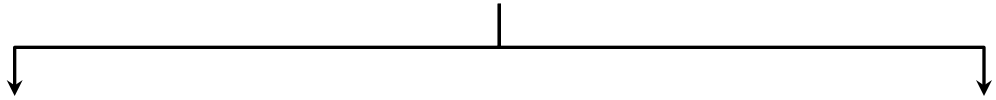




$$S_h = \{\{l_1, g_4\}, \{l_2, g_8\}, \{l_3, *\}\}$$

Global EKF Localization

Geometric constraints we can exploit



Location independent constraints

Unary constraint:
intrinsic property of feature
e.g. type, color, size

Binary constraint:
relative measure between features
e.g. relative position, angle

Location dependent constraints

Rigidity constraint:
"is the feature where I expect it given my position?"

Visibility constraint:
"is the feature visible from my position?"

Extension constraint:
"do the features overlap at my position?"

All decisions on a significance level α

Global EKF Localization

Interpretation Tree

[Grimson 1987], [Drumheller 1987], [Castellanos 1996], [Lim 2000]

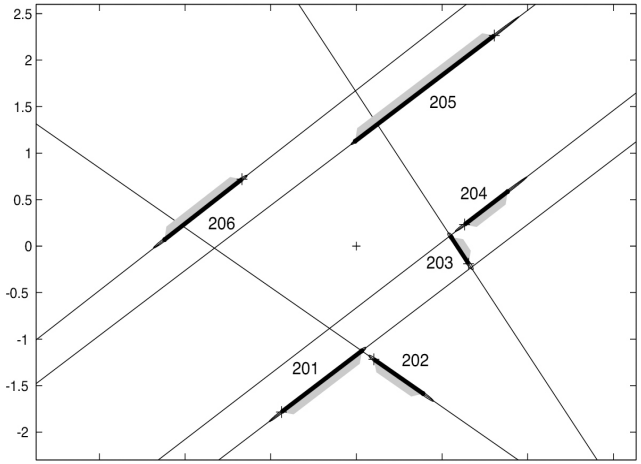
Algorithm

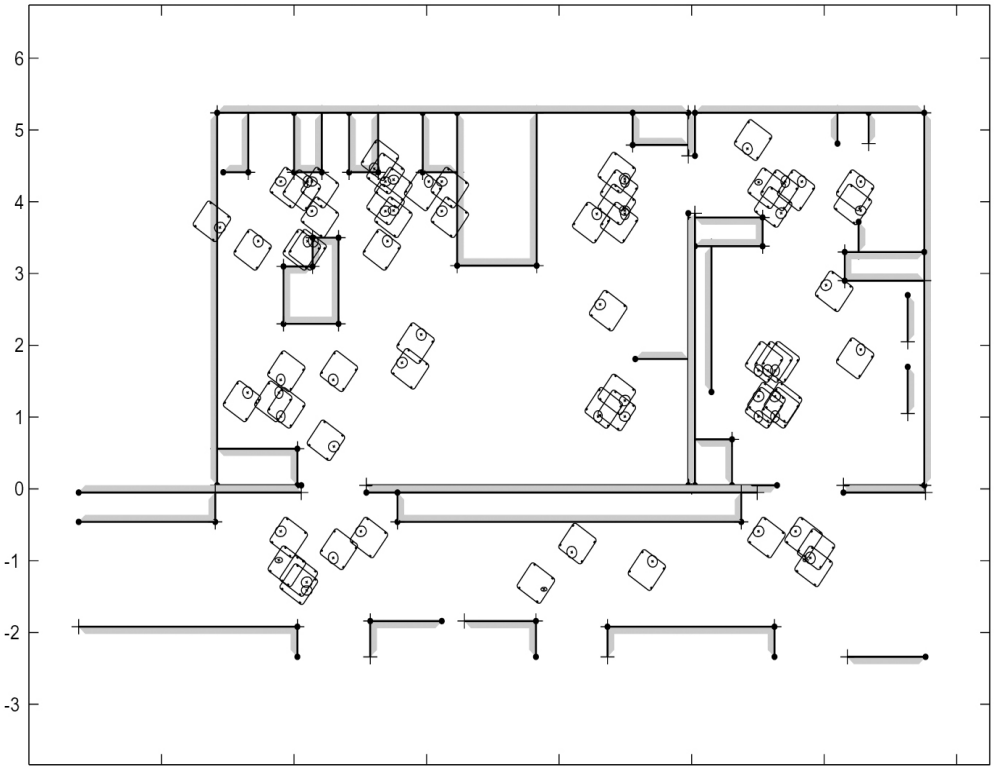
- backtracking
- depth-first
- recursive
- uses geometric constraints

```
function generate_hypotheses( h, L, G )

  H ← {}
  if L = {} then
    H ← H ∪ {h}
  else
    l ← select_observation( L )
    for g ∈ G do
      p ← {l, g}
      if satisfy_unary_constraints( p ) then
        if location_available( h ) then
A       accept ← satisfy_location_dependent_cnstr( L_h, p )
        if accept then
          h' ← h
          S_h' ← S_h ∪ {p}
          L_h' ← estimate_robot_location( S_h' )
        end
        else
B       accept ← true
        for p_p ∈ S_h while accept
          accept ← satisfy_binary_constraints( p_p, p )
        end
        if accept then
          h' ← h
          S_h' ← S_h ∪ {p}
          L_h' ← estimate_robot_location( S_h' )
          if location_available( h' ) then
            for p_p ∈ S_h while accept
              accept ← satisfy_location_dependent_cnstr( L_h', p )
            end
          end
        end
      end
      if accept then
        generate_hypotheses( h', L\{l}, G )
      end
    end
  end
  generate_hypotheses( h, L\{l}, G )
  end

return H
end
```

Global EKF Localization

Pygmalion

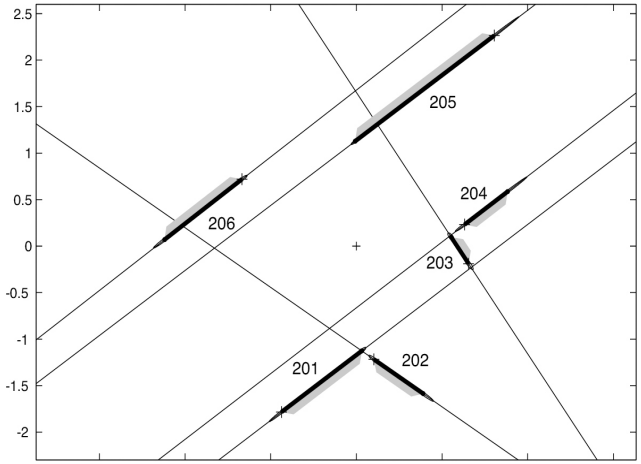


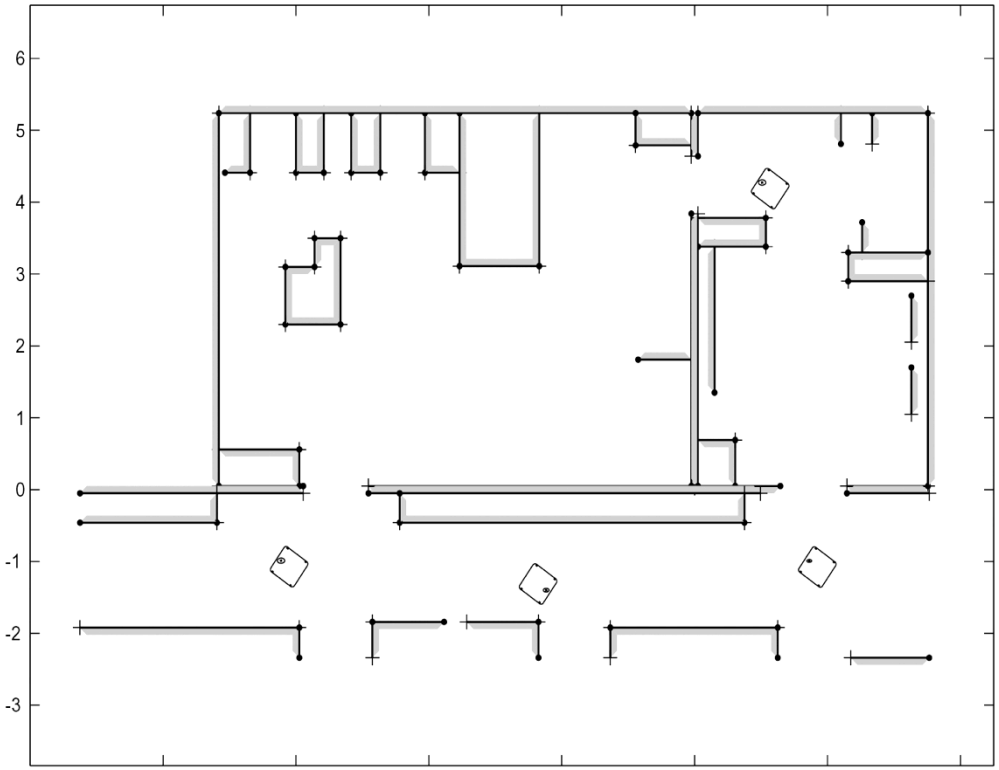


$\alpha = 0.95$, $p = 2$

Global EKF Localization

Pygmalion

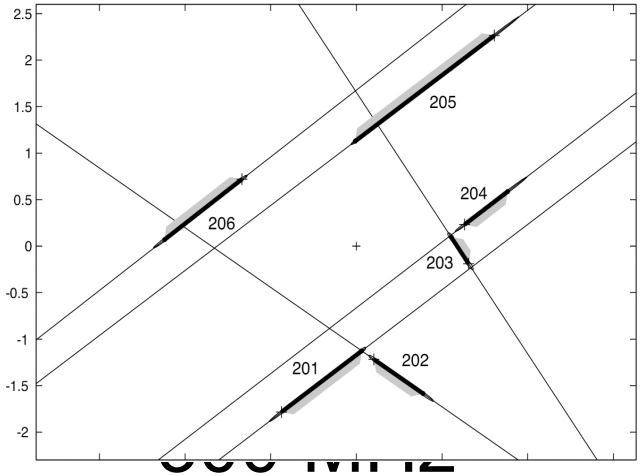


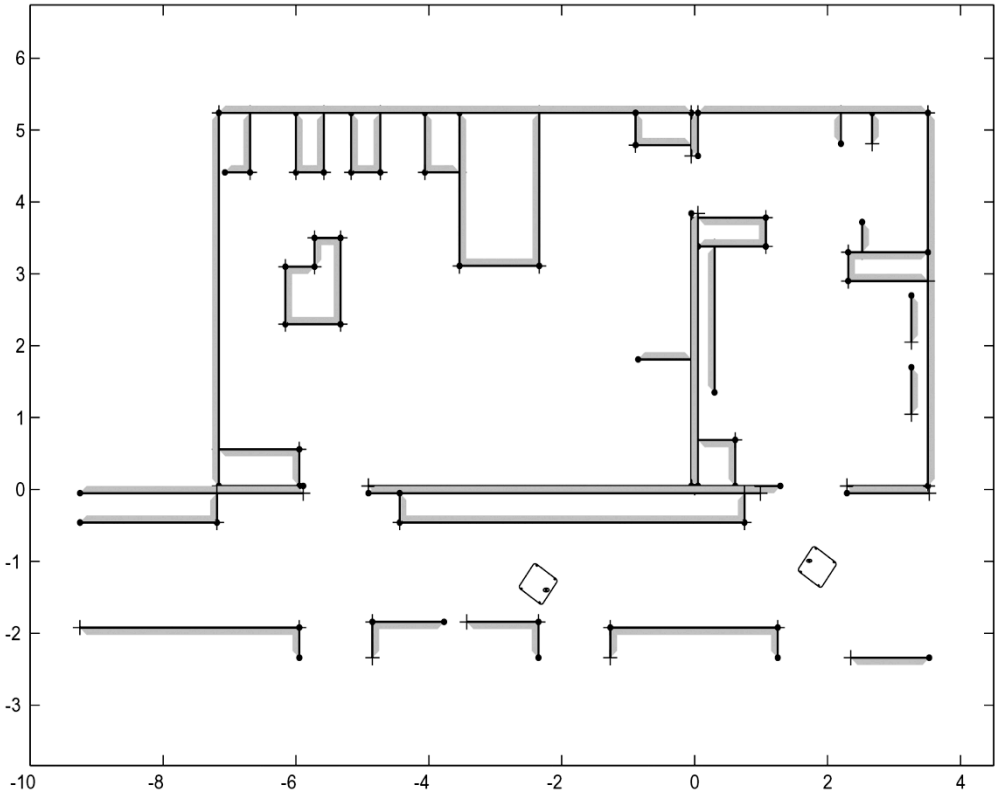


$\alpha = 0.95$, $p = 3$

Global EKF Localization

Pygmalion

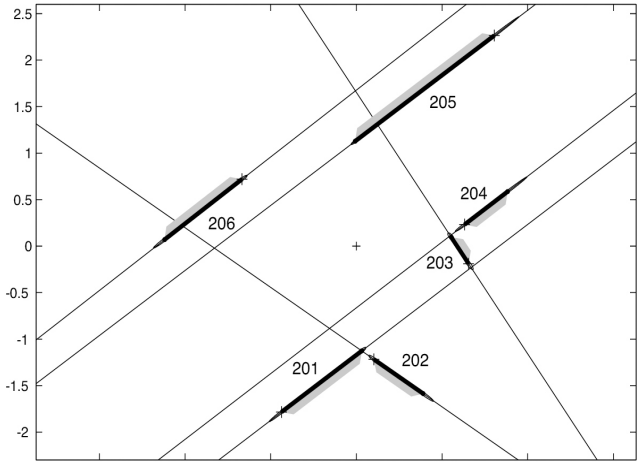


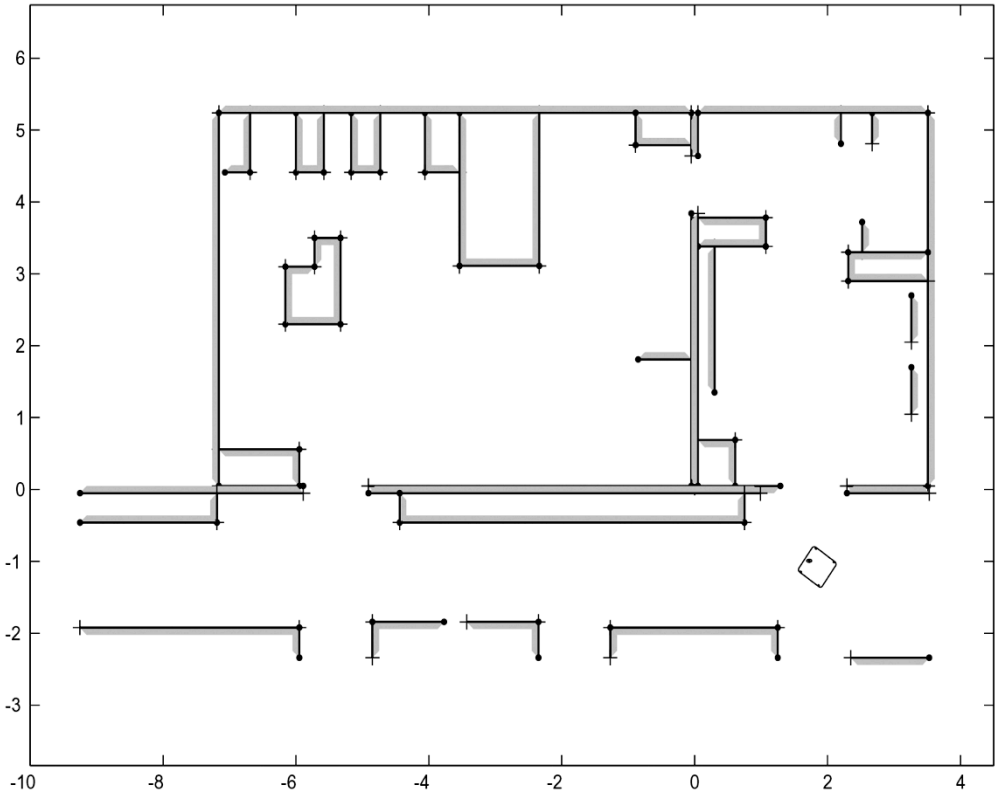


$\alpha = 0.95$, $p = 4$

Global EKF Localization

Pygmalion





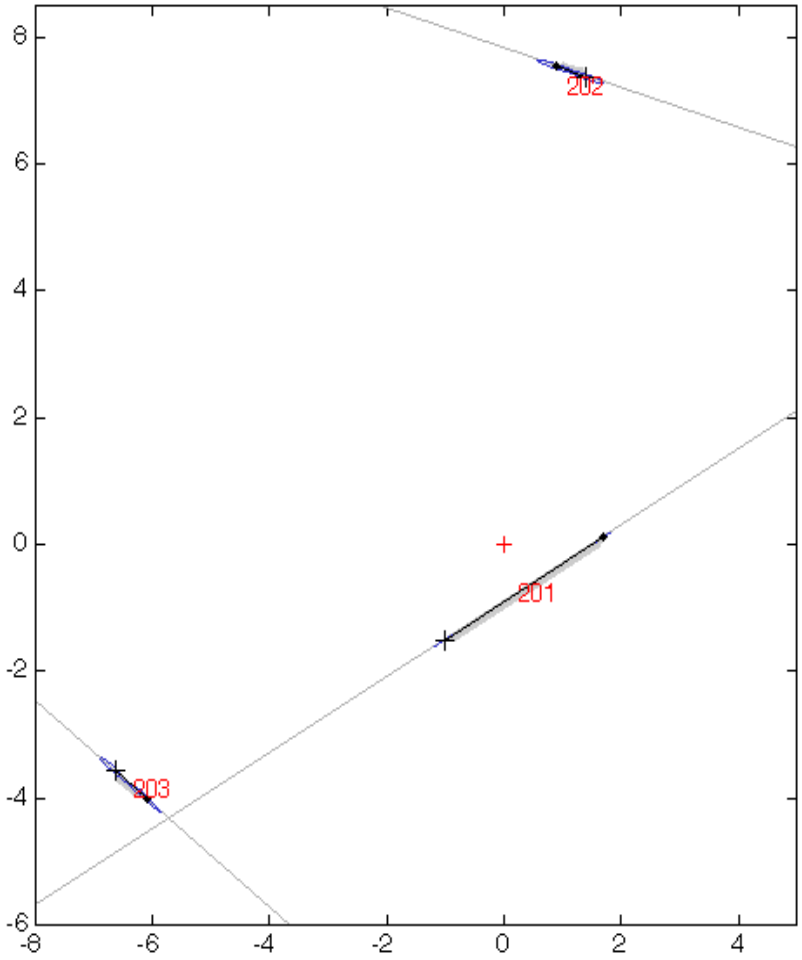
$\alpha = 0.95$, $p = 5$

t_{exe}: **633 ms** (PowerPC at 300 MHz)

Global EKF Localization

At Expo.02

05.07.02, 17.23 h



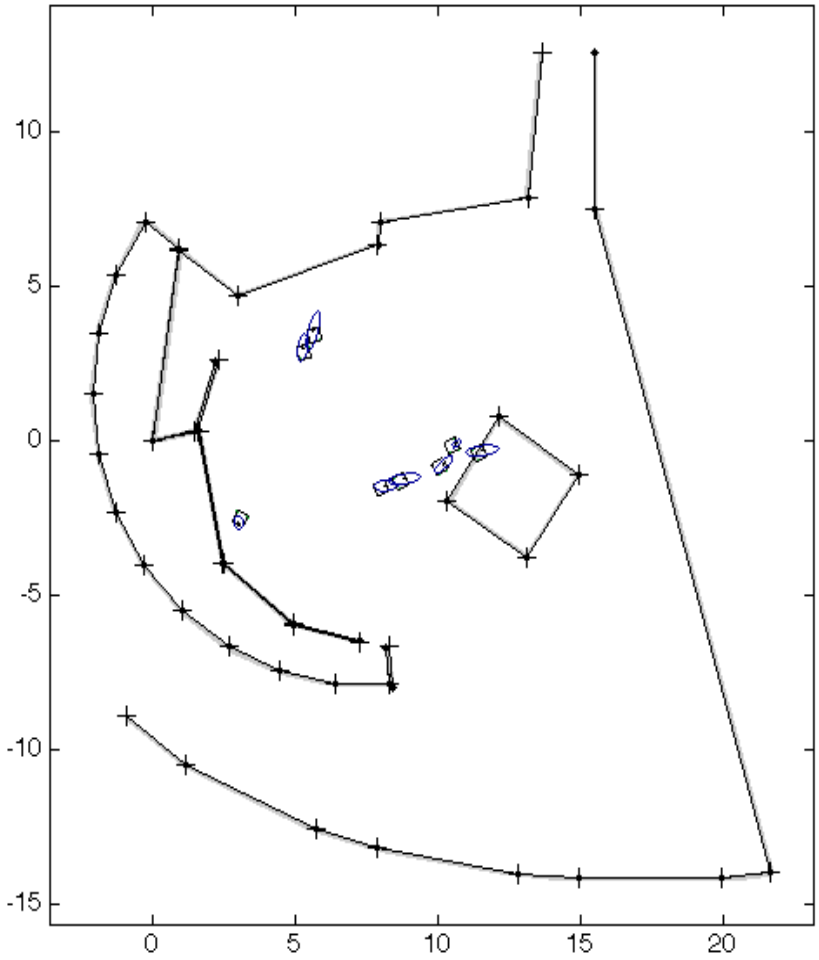
$\alpha = 0.999$

[Arras et al. 03]

Global EKF Localization

At Expo.02

05.07.02, 17.23 h



$\alpha = 0.999$

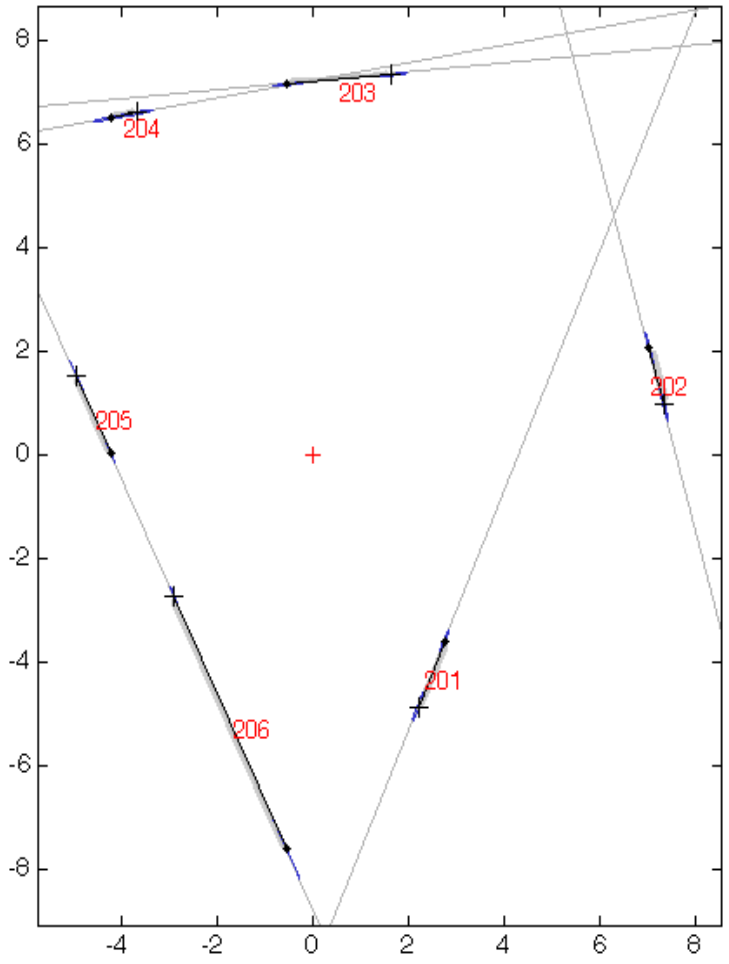
$t_{exe} = 105$ ms

[Arras et al. 03]

Global EKF Localization

At Expo.02

05.07.02, 17.32 h



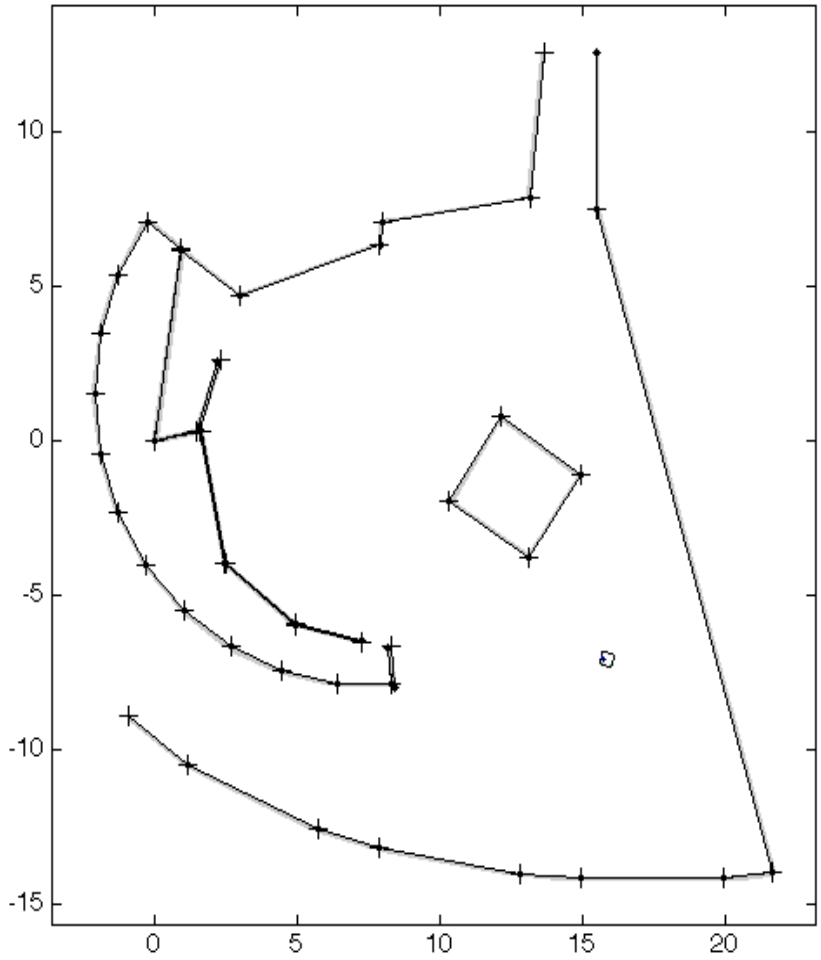
$\alpha = 0.999$

[Arras et al. 03]

Global EKF Localization

At Expo.02

05.07.02, 17.32 h



$\alpha = 0.999$

$t_{exe} = 446$ ms

[Arras et al. 03]

EKF Localization Summary

- **EKF localization** implements **pose tracking**
- Very **efficient** and **accurate** (positioning error down to subcentimeter)
- Filter divergence can cause lost situations from which the EKF **cannot recover**
- Industrial applications

- **Global EKF localization** can be achieved using interpretation tree-based data association
- Worst-case complexity is **exponential**
- **Fast** in practice for **small** maps