Homework Assignment – 5

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Problem 1

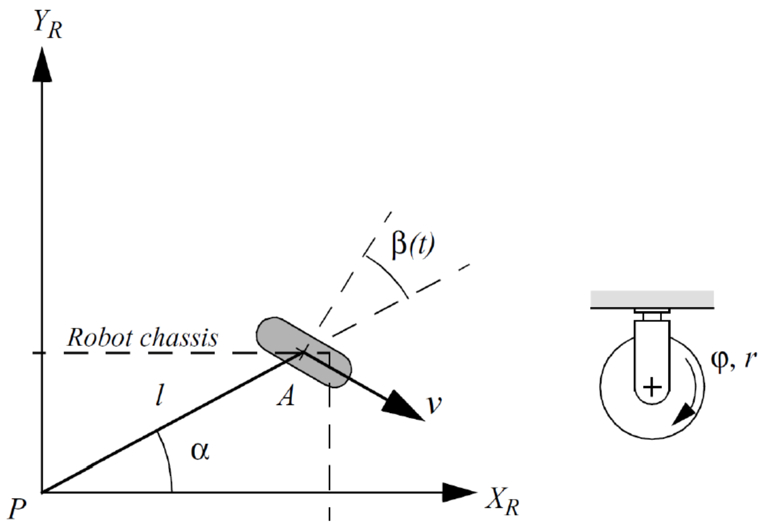

Figure 1: A fixed standard wheel.

(a)

Ans – (a)

The rolling constraint for this wheel enforces that all motion along the direction of the wheel plane must be accompanied by the appropriate amount of wheel spin so that there is pure rolling at the contact point:

$$[\sin(\alpha+\beta) \quad -\cos(\alpha+\beta) \quad (-l)\cos\beta]R(\theta)\dot{\xi}_I - r\dot{\varphi} = 0 \quad \ldots(i)$$

Here,

- The first term of the sum denotes the total motion along the wheel plane.
- The three elements of the vector on the left represent mappings from each of $\dot{x}, \dot{y}, \dot{\theta}$ to their contributions for motion along the wheel plane.

- $\boldsymbol{R(\theta)\dot{\xi}_I}$ term is used to transform the motion parameters $\boldsymbol{\dot{\xi}_I}$ that are in the global reference frame $\{X\ Y\}_I$ into the motion parameters in the local reference frame $\{X\ Y\}_R$.
- This is necessary because all other parameters in the equation α, β, l are in terms of the robot's local reference frame.
- This motion along the wheel plane must be equal, according to this constraint, to the motion accomplished by spinning the wheel $\boldsymbol{r\dot{\varphi}}$

The sliding constraint for this wheel enforces that the component of the wheel's motion orthogonal to the wheel plane must be zero:

$$[\cos(\boldsymbol{\alpha}+\boldsymbol{\beta}) \quad \sin(\boldsymbol{\alpha}+\boldsymbol{\beta}) \quad \boldsymbol{l}\sin\boldsymbol{\beta}]\boldsymbol{R}(\boldsymbol{\theta})\dot{\boldsymbol{\xi}}_{\boldsymbol{I}} = \mathbf{0} \qquad \ldots(\boldsymbol{ii})$$

Problem 1

(b)

Ans – (b)

The wheel is in a position such that $\boldsymbol{\alpha} = \mathbf{0},\ \boldsymbol{\beta} = \mathbf{0}.$ If $\theta = 0$, then the equation $[\cos(\boldsymbol{\alpha}+\boldsymbol{\beta})\sin(\boldsymbol{\alpha}+\boldsymbol{\beta})\ \boldsymbol{l}\sin\boldsymbol{\beta}]\boldsymbol{R}(\boldsymbol{\theta})\dot{\boldsymbol{\xi}}_{\boldsymbol{I}} = \mathbf{0}$ reduces to:

$$[\mathbf{1} \quad \mathbf{0} \quad \mathbf{0}]\begin{bmatrix}\mathbf{1} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{1} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{1}\end{bmatrix}\begin{bmatrix}\dot{\boldsymbol{x}}\\ \dot{\boldsymbol{y}}\\ \dot{\boldsymbol{\theta}}\end{bmatrix} = [\mathbf{1} \quad \mathbf{0} \quad \mathbf{0}]\begin{bmatrix}\dot{\boldsymbol{x}}\\ \dot{\boldsymbol{y}}\\ \dot{\boldsymbol{\theta}}\end{bmatrix} = \mathbf{0} \qquad \ldots(\boldsymbol{iii})$$

Interpretations:

- This would place the contact point of the wheel on the $X_I$with the plane of the wheel oriented parallel to Y_I.
- This constrains the component of motion along X_I to be zero and since X_I and X_R are parallel, the wheel is constrained from sliding sideways.

Problem 2

(a)

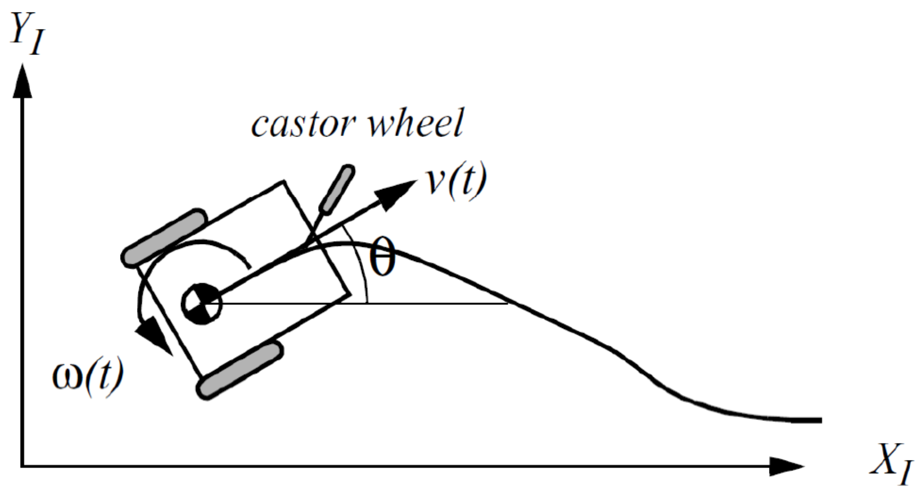

Figure 2: Differential-drive mobility system.

Ans – (a)

The fixed standard wheel's rolling constraint formula:

$$[\sin(\alpha+\beta) \quad -\cos(\alpha+\beta)\,(-l) \quad \cos\beta]R(\theta)\dot{\xi}_I - r\dot{\varphi} = 0 \qquad \dots(iv)$$

$$[\cos(\alpha+\beta) \quad \sin(\alpha+\beta) \quad l\sin\beta]R(\theta)\dot{\xi}_I = 0 \qquad \dots(v)$$

Parameters are as follows:

For the right wheel: $\alpha = -\frac{\pi}{2}, \quad \beta = \pi$

For the left wheel: $\alpha = \frac{\pi}{2}, \quad \beta = 0$

Now we can compute $J_1(\beta_s)$ and $C_1(\beta_s)$ matrices using the matrix terms from *equation* (i) and *equation* (ii) respectively. Because the two fixed standard wheels are parallel, *equation* (ii) results in only one independent equation.

$$\begin{bmatrix} J_1(\beta_s) \\ C_1(\beta_s) \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 0 & l \\ 1 & 0 & -l \end{bmatrix} \\ [0 \quad 1 \quad 0] \end{bmatrix} \qquad \dots(vi)$$

Therefore,

$$\begin{bmatrix} \boldsymbol{J_1(\beta_s)} \\ \boldsymbol{C_1(\beta_s)} \end{bmatrix} \boldsymbol{R(\theta)\xi_I^{\cdot}} = \begin{bmatrix} \boldsymbol{J_2\varphi^{\cdot}} \\ \boldsymbol{0} \end{bmatrix} \qquad \dots (\boldsymbol{vii})$$

Here,

- $R(\theta)$ is the instantaneous rotation matrix.

$$R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- $\xi_I^{\cdot}$ is the motion parameters in the global reference frame.

$$\xi_I^{\cdot} = \begin{bmatrix} x^{\cdot} \\ y^{\cdot} \\ \theta^{\cdot} \end{bmatrix}$$

- $\begin{bmatrix} J_1(\beta_s) \\ C_1(\beta_s) \end{bmatrix}$ can be retrieved from $equation\ (vi)$.

$$\begin{bmatrix} J_1(\beta_s) \\ C_1(\beta_s) \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 0 & l \\ 1 & 0 & -l \end{bmatrix} \\ [0 \quad 1 \quad 0] \end{bmatrix}$$

- $\varphi^{\cdot}$ is the wheel spin speed of the robot's wheel.

$$\begin{bmatrix} J_2\varphi^{\cdot} \\ 0 \end{bmatrix} = \begin{bmatrix} r_R\varphi_R^{\cdot} \\ r_L\varphi_L^{\cdot} \\ 0 \end{bmatrix}$$

Problem 2

(b)

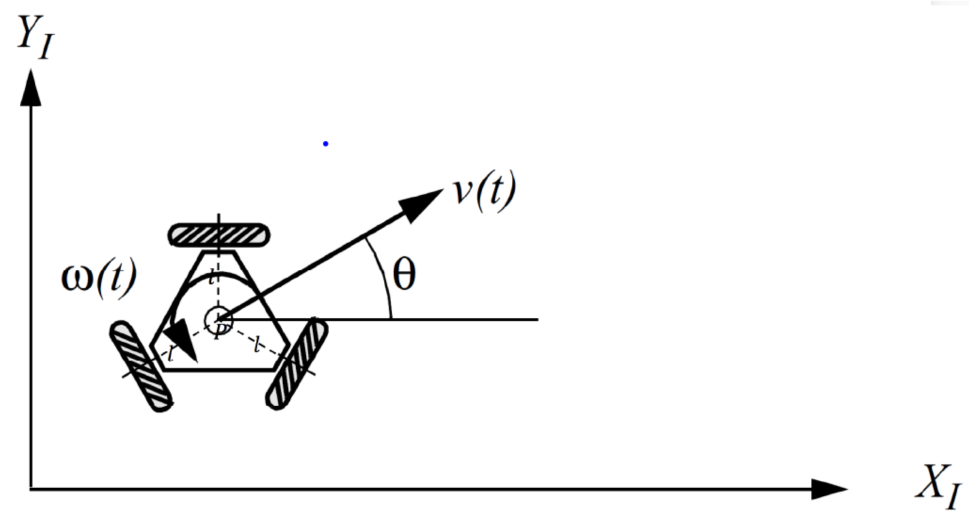

Figure 3: A three-wheel omnidrive robot.

Ans – (b)

The pose of a Swedish wheel is expressed exactly as in a fixed standard wheel, with the addition of a term $\boldsymbol{\gamma}$, representing the angle between the main wheel plane and the axis of rotation of the small circumferential roller.

The motion constraint that is derived looks identical to the rolling constraint for the fixed standard wheel in $equation\ (i)\ or\ equation(iv)$ except that the formula is modified by adding $\boldsymbol{\gamma}$ such that the effective direction along which the rolling constraint holds is along this zero component rather than along the wheel plane:

$$[\sin(\boldsymbol{\alpha}+\boldsymbol{\beta}+\gamma) \quad -\cos(\boldsymbol{\alpha}+\boldsymbol{\beta}+\gamma) \quad (-\boldsymbol{l})\cos(\boldsymbol{\beta}+\gamma)]\boldsymbol{R}(\boldsymbol{\theta})\dot{\xi}_I - \boldsymbol{r}\dot{\boldsymbol{\varphi}}\cos\gamma = \mathbf{0} \quad \ldots(\boldsymbol{viii})$$

Conditions:

- The behavior of this constraint and thereby the Swedish wheel changes dramatically as the value $\boldsymbol{\gamma}$ varies. Consider $\boldsymbol{\gamma} = \mathbf{0}$ .This represents the Swedish 90-degree wheel.
- This immediately simplifies equation (viii) to equation (i), the rolling constraints of a fixed standard wheel.

- Furthermore, $\boldsymbol{\beta = 0}$ for all wheels because the wheels are tangent to the robot's circular body.
- The value of α for each wheel is:

$$\boldsymbol{\alpha_1 = \frac{\pi}{3}, \qquad \alpha_2 = \pi, \qquad \alpha_3 = -\frac{\pi}{3}}$$

Constructing and simplifying $\boldsymbol{J_{1f}}$ using above conditions into the $\boldsymbol{equation}(\boldsymbol{viii})$:

$$\boldsymbol{J_{1f} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & -l \\ 0 & 1 & -l \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & -l \end{bmatrix} \quad \dots (ix)}$$

Therefore, the value of $\boldsymbol{\dot{\xi}_I}$ can be computed as a combination of the rolling constraints of the robot's three omnidirectional wheels:

$$\boldsymbol{\dot{\xi}_I = R(\theta)^{-1} {J_{1f}}^{-1} J_2 \dot{\varphi} \quad \dots (x)}$$

Here,

- $R(\theta)^{-1} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- $J_{1f}^{-1} = \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & -\frac{1}{\sqrt{3}} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3l} & -\frac{1}{3l} & -\frac{1}{3l} \end{bmatrix}$
- $J_2 \dot{\varphi} = \begin{bmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{bmatrix} \dot{\varphi}$