Homework Assignment – 3

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Problem 1

(a)

Ans-(a)

(i) With respect to θ_1: The first joint variable is the base rotation and it is given by:

$$\theta_1 = \tan^{-1}\frac{y_c}{x_c} = Atan2(x_c, y_c)$$

(ii) With respect to θ_2 and θ_3: We consider the plane formed by the second and third links as shown in figure below:

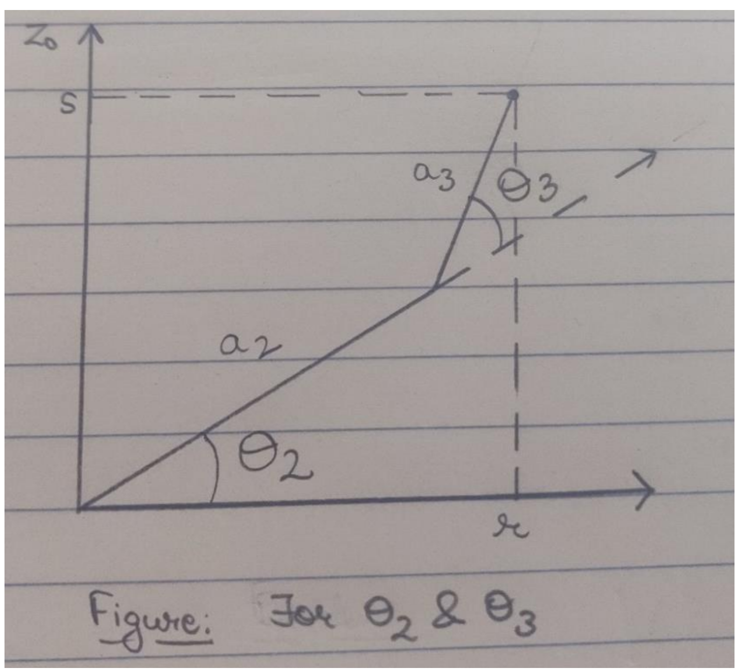

Figure: Plan formed by Link – 2 and Link – 3

The motion of second and third links is planar, we can apply the law of cosines to obtain

$$\cos\theta_3 = \frac{r^2 + s^2 - a_2^2 - a_3^2}{2a_2a_3}$$

Here, $r^2 = x_c^2 + y_c^2$ and $s = z_c - d_1$

The above equation becomes:

$$\cos\theta_3 = \frac{(x_c^2 + y_c^2) + (z_c - d_1)^2 - a_2^2 - a_3^2}{2a_2a_3}$$

Assume:

$$\cos\theta_3 = \frac{(x_c^2 + y_c^2) + (z_c - d_1)^2 - a_2^2 - a_3^2}{2a_2a_3} = T$$

Hence, θ_3 is given by

$$\theta_3 = Atan2(T, \pm\sqrt{1 - T^2})$$

Similarly, θ_2 is given by:

$$\theta_2 = Atan2(r, s) - Atan2(a_2 + a_3c_3, a_3s_3)$$

$$\theta_2 = Atan2\left(\sqrt{x_c^2 + y_c^2},\ z_c - d_1\right) - Atan2(a_2 + a_3c_3, a_3s_3)$$

Therefore,

$$\boldsymbol{\theta_1 = Atan2(x_c, y_c)}$$

$$\boldsymbol{\theta_2 = Atan2\left(\sqrt{x_c^2 + y_c^2},\ z_c - d_1\right) - Atan2(a_2 + a_3c_3, a_3s_3)}$$

$$\boldsymbol{\theta_3 = Atan2(T, \pm\sqrt{1 - T^2}), \text{ where } T = \frac{(x_c^2 + y_c^2) + (z_c - d_1)^2 - a_2^2 - a_3^2}{2a_2a_3}}$$

(b)

Ans – (b)

Referencing the above values obtained in Part – (a), when $x_c = y_c = 0$,

The new values are:

(i) θ_1 = becomes undefined. As $Atan2(0,0) \rightarrow$ undefined.

(ii) $\theta_2 = Atan2(0,\ z_c -\ d_1) - Atan2(a_2 + a_3c_3, a_3s_3)$

$= \frac{\pi}{2} - Atan2(a_2 + a_3c_3, a_3s_3)$

(iii) $\theta_3 = Atan2(T, \pm\sqrt{1 -\ T^2})$ will be same but the value of T changes.

Here T becomes,

$$T = \frac{(z_c -\ d_1)^2 - a_2^2 - a_3^2}{2a_2a_3}$$

So, for $x_c =\ y_c = 0$

θ_1 is undefined.

$$\boldsymbol{\theta_2 = \frac{\pi}{2} - Atan2(a_2 + a_3c_3, a_3s_3)}$$

$$\boldsymbol{\theta_3 = Atan2(T, \pm\sqrt{1 -\ T^2})}\text{, where } \boldsymbol{T = \frac{(z_c - d_1)^2 - a_2^2 - a_3^2}{2a_2a_3}}$$

(c)

Ans – (c)

Part – (i) If there is an off-set, $d \neq 0$, then the wrist center cannot intersect z_0.

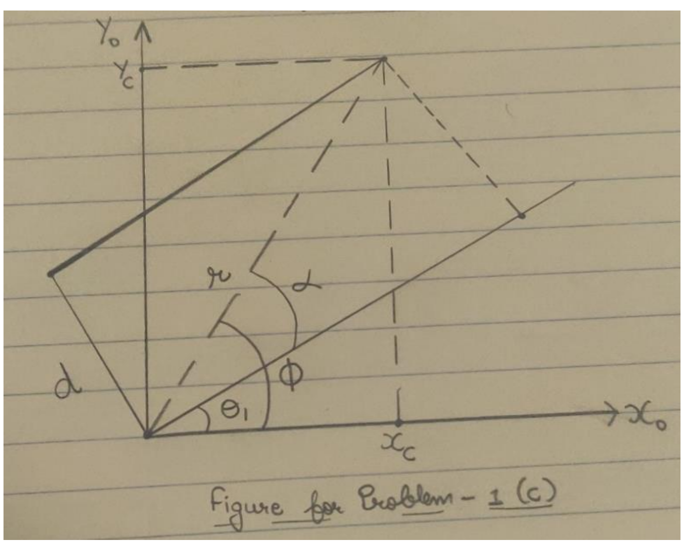

Figure: DoF with Off-set (Configuration-1)

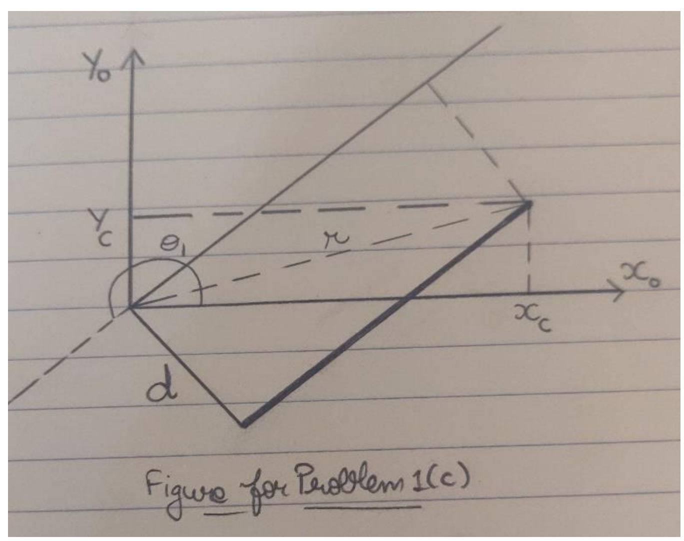

Figure: DoF with Off-set (Configuration-2)

From the above figure of configuration-1, we can see that: $\boldsymbol{\theta_1 = \emptyset - \alpha}$

Here, $\emptyset = Atan2(x_c, y_c)$ and $\alpha = Atan2(\sqrt{r^2 - d^2}, d) = Atan2(\sqrt{{x_c}^2 + {y_c}^2 - d^2}, d)$

So, the above equation becomes:

$$\theta_1 = Atan2(x_c, y_c) - Atan2(\sqrt{x_c^{\,2} + y_c^{\,2} - d^2}, d)$$

$$\boldsymbol{\theta_1 = Atan2(x_c, y_c) + Atan2(-\sqrt{x_c^{\,2} + y_c^{\,2} - d^2}, -d)}$$

Part – (ii) There will be only two solutions for θ_1.

Problem 2

(a)

Ans – (a)

$$A_i = \begin{bmatrix} cos\theta_i & -sin\theta_i cos\alpha_i & sin\theta_i cos\alpha_i & a_i cos\theta_i \\ sin\theta_i & cos\theta_i cos\alpha_i & -cos\theta_i sin\alpha_i & a_i sin\theta_i \\ 0 & sin\alpha_i & cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$A_1 = \begin{bmatrix} c1 & 0 & -s1 & 0 \\ s1 & 0 & c1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $A_2 = \begin{bmatrix} c2 & 0 & s2 & 0 \\ s2 & 0 & -c2 & 0 \\ 0 & 1 & 0 & d2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$A_4 = \begin{bmatrix} c4 & 0 & -s4 & 0 \\ s4 & 0 & c4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $A_5 = \begin{bmatrix} c5 & 0 & s5 & 0 \\ s5 & 0 & -c5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $A_6 = \begin{bmatrix} c6 & -s6 & 0 & 0 \\ s6 & c6 & 0 & 0 \\ 0 & 0 & 1 & d6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Here,

c1 = $\cos\theta_1$ s1 = $\sin\theta_1$ c2 = $\cos\theta_2$ s2 = $\sin\theta_2$ c4 = $\cos\theta_4$

s4 = $\sin\theta_4$ c5 = $\cos\theta_5$ s5 = $\sin\theta_5$ c6 = $\cos\theta_6$ s6 = $\sin\theta_6$

$T_6^{\,o} = [A_1 A_2 A_3 A_4 A_5 A_6] = \begin{bmatrix} r11 & r12 & r13 & dx \\ r21 & r22 & r23 & dy \\ r31 & r32 & r33 & dz \\ 0 & 0 & 0 & 1 \end{bmatrix}$

r11 = - c6*(c5*(s1*s4 - c1*c2*c4) + c1*s2*s5) - s6*(c4*s1 + c1*c2*s4)

r12 = s6*(c5*(s1*s4 - c1*c2*c4) + c1*s2*s5) - c6*(c4*s1 + c1*c2*s4)

r13 = c1*c5*s2 - s5*(s1*s4 - c1*c2*c4)

dx = c1*d3*s2 - d6*(s5*(s1*s4 - c1*c2*c4) - c1*c5*s2) - d2*s1

r21 = c6*(c5*(c1*s4 + c2*c4*s1) - s1*s2*s5) + s6*(c1*c4 - c2*s1*s4)

r22 = c6*(c1*c4 - c2*s1*s4) - s6*(c5*(c1*s4 + c2*c4*s1) - s1*s2*s5)

r23 = s5*(c1*s4 + c2*c4*s1) + c5*s1*s2

dy = c1*d2 + d6*(s5*(c1*s4 + c2*c4*s1) + c5*s1*s2) + d3*s1*s2

r31= s2*s4*s6 - c6*(c2*s5 + c4*c5*s2)

r32 = s6*(c2*s5 + c4*c5*s2) + c6*s2*s4

r33 = c2*c5 - c4*s2*s5

dz = c2*d3 + d6*(c2*c5 - c4*s2*s5)

(b)

Ans – (b)

First, o_j is given by the first three entries of the last column of $T_{oj} = A1 \_\_\_ Aj$.

The vector zj is given as $zj = R0jk$ where $R0j$ is the rotational part of $T0j$.

Using MATLAB function on the above condition for calculating $\boldsymbol{o_j}$,

$$o_0 = o_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad o_2 = \begin{bmatrix} -s_1 d_2 \\ c_1 d_2 \\ 0 \end{bmatrix} \qquad o_3 = o_4 = o_5 = \begin{bmatrix} c_1 s_2 d_3 - s_1 d_2 \\ s_1 s_2 d_3 + c_1 d_2 \\ c_2 d_3 \end{bmatrix}$$

$$o_6 = \begin{bmatrix} c_1 s_2 d_3 - s_1 d_2 + d_6(c_1 c_2 c_4 s_5 + c_1 c_5 s_2 - s_1 s_4 s_5) \\ s_1 s_2 d_3 + c_1 d_2 + d_6(c_1 s_4 s_5 + c_2 c_4 s_1 s_5 + c_5 s_1 s_2) \\ c_2 d_3 + d_6(c_2 c_5 - c_4 s_2 s_5) \end{bmatrix}$$

Various Z-axes are as follows:

The vector zj is given as $zj = R0jk$ where $R0j$ is the rotational part of $T0j$.

$z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ $\quad z_1 = \begin{bmatrix} -s_1 \\ c_1 \\ 0 \end{bmatrix}$ $\quad z_2 = z_3 = \begin{bmatrix} c_1 s_2 \\ s_1 s_2 \\ c_2 \end{bmatrix}$

$z_4 = \begin{bmatrix} -c_1 c_2 s_4 - s_1 c_4 \\ -s_1 c_2 s_4 + c_1 c_4 \\ s_2 s_4 \end{bmatrix}$ $\quad z_5 = \begin{bmatrix} c_1 c_2 c_4 s_5 + c_1 c_5 s_2 - s_1 s_4 s_5 \\ c_1 s_4 s_5 + c_2 c_4 s_1 s_5 + c_5 s_1 s_2 \\ c_2 c_5 - c_4 s_2 s_5 \end{bmatrix}$

The columns of the Jacobian matrix are as follows:

$[J_1] = \begin{bmatrix} z_0 * (o_6 - o_0) \\ z_0 \end{bmatrix}$ $\quad [J_2] = \begin{bmatrix} z_1 * (o_6 - o_1) \\ z_1 \end{bmatrix}$ $\quad [J_3] = \begin{bmatrix} z_2 \\ 0 \end{bmatrix}$

$[J_4] = \begin{bmatrix} z_3 * (o_6 - o_3) \\ z_3 \end{bmatrix}$ $\quad [J_5] = \begin{bmatrix} z_4 * (o_6 - o_4) \\ z_4 \end{bmatrix}$ $\quad [J_6] = \begin{bmatrix} z_5 * (o_6 - o_5) \\ z_5 \end{bmatrix}$

Final Jacobian Matrix, $J = [J_1 \ J_2 \ J_3 \ J_4 \ J_5 \ J_6]$

$$J = \begin{bmatrix} J_{11} & J_{12} & J_{13} & J_{14} & J_{15} & J_{16} \\ J_{21} & J_{22} & J_{23} & J_{24} & J_{25} & J_{26} \\ J_{31} & J_{32} & J_{33} & J_{34} & J_{35} & J_{36} \\ J_{41} & J_{42} & J_{43} & J_{44} & J_{45} & J_{46} \\ J_{51} & J_{52} & J_{53} & J_{54} & J_{55} & J_{56} \\ J_{61} & J_{62} & J_{63} & J_{64} & J_{65} & J_{66} \end{bmatrix}$$

$\boldsymbol{J_{11}} = -c1 * d2 - d6 * (c5 * s1 * s2 + c1 * s4 * s5 + c2 * c4 * s1 * s5) - d3 * s1 * s2$

$\boldsymbol{J_{21}} = d6 * (c1 * c5 * s2 - s1 * s4 * s5 + c1 * c2 * c4 * s5) - d2 * s1 + c1 * d3 * s2$

$\boldsymbol{J_{31}} = 0$ $\quad \boldsymbol{J_{41}} = 0$ $\quad \boldsymbol{J_{51}} = 0$ $\quad \boldsymbol{J_{61}} = 1$

$\boldsymbol{J_{12}} = c1 * (c2 * d3 + d6 * (c2 * c5 - c4 * s2 * s5))$

$\boldsymbol{J_{22}} = s1 * (c2 * d3 + d6 * (c2 * c5 - c4 * s2 * s5))$

$\boldsymbol{J_{32}} = -c1 * (d6 * (c1 * c5 * s2 - s1 * s4 * s5 + c1 * c2 * c4 * s5) - d2 * s1 + c1 * d3 * s2) - s1 * (c1 * d2 + d6 * (c5 * s1 * s2 + c1 * s4 * s5 + c2 * c4 * s1 * s5) + d3 * s1 * s2)$

$\boldsymbol{J_{42}} = -s1$ $\quad \boldsymbol{J_{52}} = c1$ $\quad \boldsymbol{J_{62}} = 0$

$\boldsymbol{J_{31}} = c1 * s2$ $\quad \boldsymbol{J_{32}} = s1 * s2$ $\quad \boldsymbol{J_{33}} = c2$ $\quad \boldsymbol{J_{43}} = 0$ $\quad \boldsymbol{J_{53}} = 0$ $\quad \boldsymbol{J_{63}} = 0$

$\boldsymbol{J_{14}} = d6 * s1 * s2 * (c2 * c5 - c4 * s2 * s5) - c2 * d6 * (c5 * s1 * s2 + c1 * s4 * s5 + c2 * c4 * s1 * s5)$

$$\boldsymbol{J_{24}} = c2 * d6 * (c1 * c5 * s2 - s1 * s4 * s5 + c1 * c2 * c4 * s5) - c1 * d6 * s2 * (c2 * c5 - c4 * s2 * s5)$$

$$\boldsymbol{J_{34}} = c1 * d6 * s2 * (c5 * s1 * s2 + c1 * s4 * s5 + c2 * c4 * s1 * s5) - d6 * s1 * s2 * (c1 * c5 * s2 - s1 * s4 * s5 + c1 * c2 * c4 * s5)$$

$\boldsymbol{J_{44}} = c1 * s2$ $\quad$ $\boldsymbol{J_{54}} = s1 * s2$ $\quad$ $\boldsymbol{J_{64}} = c2$

$$\boldsymbol{J_{15}} = d6 * (c1 * c4 - c2 * s1 * s4) * (c2 * c5 - c4 * s2 * s5) - d6 * s2 * s4 * (c5 * s1 * s2 + c1 * s4 * s5 + c2 * c4 * s1 * s5)$$

$$\boldsymbol{J_{25}} = d6 * (c4 * s1 + c1 * c2 * s4) * (c2 * c5 - c4 * s2 * s5) + d6 * s2 * s4 * (c1 * c5 * s2 - s1 * s4 * s5 + c1 * c2 * c4 * s5)$$

$$\boldsymbol{J_{35}} = -d6 * (c1 * c4 - c2 * s1 * s4) * (c1 * c5 * s2 - s1 * s4 * s5 + c1 * c2 * c4 * s5) - d6 * (c4 * s1 + c1 * c2 * s4) * (c5 * s1 * s2 + c1 * s4 * s5 + c2 * c4 * s1 * s5)$$

$\boldsymbol{J_{45}} = -c4 * s1 - c1 * c2 * s4$ $\quad$ $\boldsymbol{J_{55}} = c1 * c4 - c2 * s1 * s4$ $\quad$ $\boldsymbol{J_{65}} = s2 * s4$

$\boldsymbol{J_{16}} = 0$ $\quad$ $\boldsymbol{J_{26}} = 0$ $\quad$ $\boldsymbol{J_{36}} = 0$

$$\boldsymbol{J_{46}} = c1 * c5 * s2 - s1 * s4 * s5 + c1 * c2 * c4 * s5$$

$$\boldsymbol{J_{56}} = c5 * s1 * s2 + c1 * s4 * s5 + c2 * c4 * s1 * s5$$

$$\boldsymbol{J_{66}} = c2 * c5 - c4 * s2 * s5$$