Homework Assignment – 4

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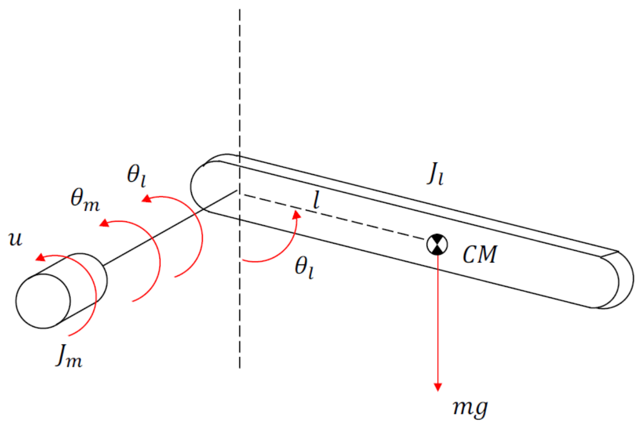
Problem 1

(a)



Ans-(a)

Given: $$\theta_m = r\theta_l \qquad \dots(i)$$

If we choose as generalized coordinate $$q = r\theta_l \qquad \dots(ii)$$

The kinetic energy of the system is given by:

$$K = \frac{1}{2}J_m\dot{\theta}_m^2 + \frac{1}{2}J_l\dot{\theta}_l^2 \qquad \dots(iii)$$

Here, J_m J_l are the rotational inertias of the motor and link, respectively.

Using equations (i) we get,

$$K = \frac{1}{2}(r^2J_m + J_l)\dot{\theta}_l^2 \qquad \dots(iv)$$

Using equations (ii) we get,

$$K = \frac{1}{2}(r^2J_m + J_l)\dot{q}^2 \qquad \dots(v)$$

The potential energy is given by:

$$P = Mgl(1 - cos\theta_l) \qquad \dots(vi)$$

$$Using\ equations\ (ii)\ we\ get,$$

$$P = Mgl(1 - cosq) \qquad \dots(vii)$$

where M is the total mass of the link and l is the distance from the joint axis to the link center of mass.

Defining, $$I = r^2 J_m + J_l \qquad \dots(viii)$$

The Lagrangian $\mathcal{L}$ is the difference of the kinetic and potential energy and is given by:

$$\mathcal{L} = K - P \qquad \dots(ix)$$

$$Using\ equations\ (v)\ and\ (vii)\ we\ get,$$

$$\mathcal{L} = \frac{1}{2}(r^2 J_m + J_l)q^{\cdot 2} - Mgl(1 - cosq) \qquad \dots(x)$$

$$Using\ equations\ (viii) we\ get,$$

$$\mathcal{L} = \frac{1}{2} I q^{\cdot 2} - Mgl(1 - cosq) \qquad \dots(xi)$$

Transforming the equation (xi) into Euler-Lagrange equation at generalized coordinate θ_l.

$$\boldsymbol{\frac{d}{dt}\frac{d\mathcal{L}}{dq} - \frac{d\mathcal{L}}{dq} = \tau_l} \qquad \dots(\boldsymbol{xii})$$

Here, τ_k is the (generalized) force associated with q.

$$\frac{d}{dt}\frac{d(\frac{1}{2}Iq^{\cdot 2} - Mgl(1 - cosq))}{dq} - \frac{d\left(\frac{1}{2}Iq^{\cdot 2} - Mgl(1 - cosq)\right)}{dq} = \tau_l \qquad \dots(xiii)$$

$$\boldsymbol{Iq^{\cdot\cdot} + Mgl \sin q = \tau_l} \qquad \dots(\boldsymbol{xiv})$$

Equation $(\boldsymbol{xiv})$ represents Euler-Lagrange equations for the motion. The generalized force $\boldsymbol{\tau_l}$ represents those external forces and torques that are not derivable from a potential function.

(b)

Ans - (b)

The generalized force $\boldsymbol{\tau_l}$ represents those external forces and torques that are not derivable from a potential function. For this example, $\boldsymbol{\tau_l}$ consists of the input motor torque $\boldsymbol{u = r\tau_m}$ ... $(\boldsymbol{xv})$, reflected to the link and (nonconservative) damping torques given by:

$$\tau_m = -B_m \dot{\theta_m} \qquad \dots (xvi)$$

$$\tau_l = -B_l \dot{\theta_l} \qquad \dots (xvii)$$

Reflecting the motor damping to the link yields,

$$\tau_l = u - B\dot{q} \qquad \dots (xviii)$$

Here, $\boldsymbol{B = rB_m + B_l}$

$$Substituting\ equation\ (xviii)\ into\ equation\ (xiv), we\ get$$

$$I\ddot{q} + Mgl\sin q = u - B\dot{q} \qquad \dots (xix)$$

Here, $\boldsymbol{I = r^2 J_m + J_l}$ from equation $(viii)$

$$\boldsymbol{I\ddot{q} + B\dot{q} + Mgl\sin q = u} \qquad \dots (\boldsymbol{xx})$$

Equation $(\boldsymbol{xx})$ represents the complete expression for the dynamics of the system.

(c)

Ans - (c)

Given Parameters:

- $r = 30$
- $J_m = J_l = 0.001$
- $M = 0.1$
- $g = 9.81$
- $l = 0.1$

- $u = 0.1$
- $Initial\ States\ (\theta_l; \dot{\theta_l}) = (0; 0)$
- $B_m = B_l \in \{0.001, 0.01, 0.1, 1\}$

Given change of variable conditions:

- $x_1 = \theta_l$
- $x_2 = \dot{\theta_l}$
- $\dot{x_1} = x_2$
- Therefore, from equation (xx), we can get

  $$\dot{x_2} = -\frac{(rB_m + B_l)x_2}{r^2 J_m + J_l} - \frac{Mgl}{r^2 J_m + J_l} \sin x_1 + u$$

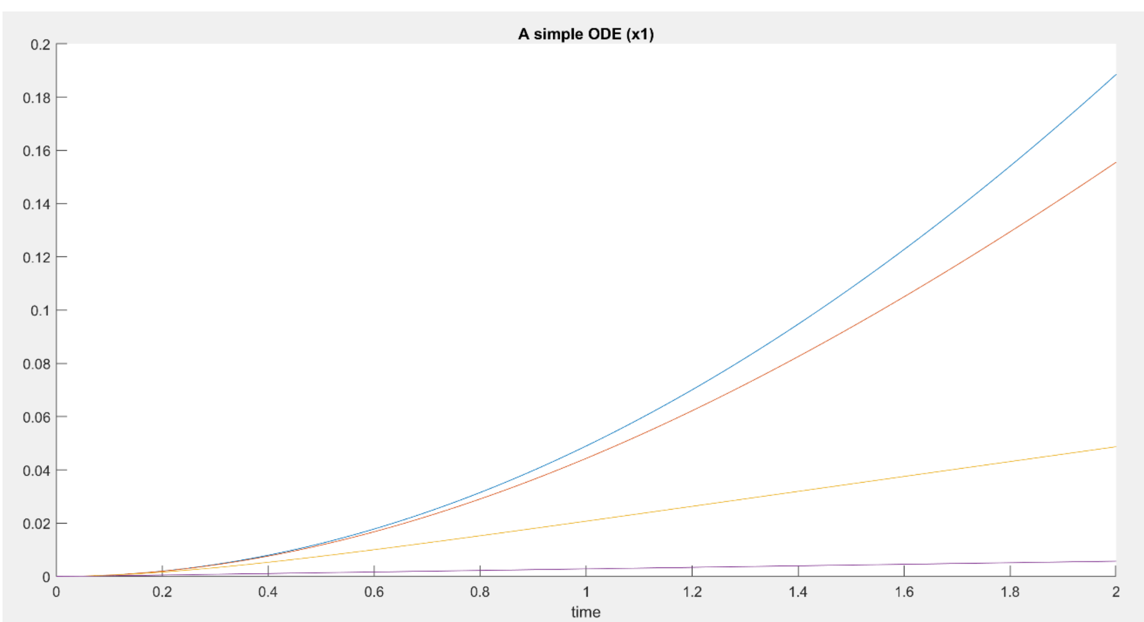
  Here, $\alpha = \frac{(rB_m + B_l)}{r^2 J_m + J_l}$ and $\beta = \frac{Mgl}{r^2 J_m + J_l}$

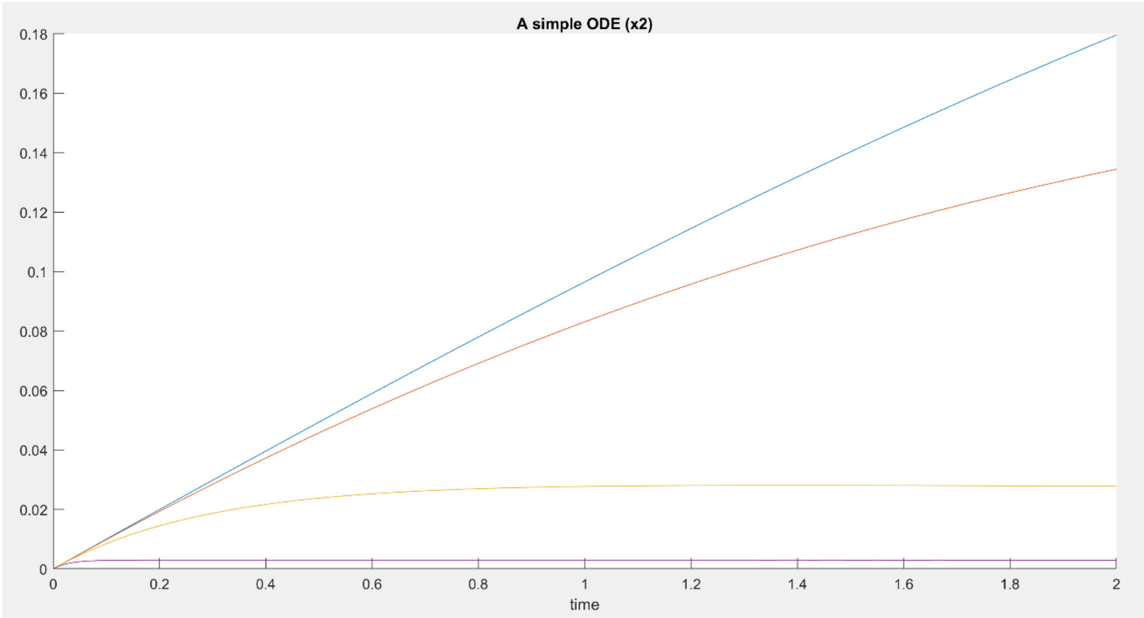
  $$\boldsymbol{\dot{x_2} = -\alpha x_2 - \beta \sin x_1 + u}$$

Results/ Conclusions:

- The response speed decreases as the B_m, B_l increases.
- Stable $\dot{\theta_l}$ is achieved at larger B_m, B_l.

Plots/ Graphs:





MATLAB Code Used for achieving the above results:

```
%% Homework Assignment - 4 by Tushar Goel** %%
% Taking reference from the example given in the assignment %
function func_a_simple_ODE()
    close all;

    % time vector
    t = 0:0.01:2;

    % initial states
    x_10 = 0;
    x_20 = 0;

    % parameters
    r = 30;
    Jm = 0.001;
    Jl = Jm;
    M = 0.1;
    g = 9.81;
    l = 0.1;
    u = 0.1;
    Bl = [0.001, 0.01, 0.1, 1];
    Bm= Bl;

    % Figure Representation and Display
    fh1 = figure('Name', 'A simple ODE(x1)');
    ah1 = axes('parent',fh1);
    hold(ah1,'on');
    xlabel(ah1,'time');
    fh2 = figure('Name', 'A simple ODE(x2)');
    ah2 = axes('parent',fh2);
    hold(ah2,'on');
    xlabel(ah2,'time');
```

```
    % Relations
    beta = M*g*l/(Jm*r^2+Jl);
    alpha_vector = (Bm*r+Bl)/(Jm*r^2+Jl);


    for i = 1:4
        alpha = alpha_vector(i);
        % solving ODE
        [t,x]=ode45(@func,t,[x_10,x_20]);
        % plotting solutions
        figure(1)
        plot(ah1,t,x(:,1));
        title('A simple ODE (x1)');
        figure(2)
        plot(ah2,t,x(:,2));
        title('A simple ODE (x2)');
    end

% function
    function dxdt=func(t,x)
        dx1 = x(2);
        dx2 = - alpha*x(2) - beta*sin(x(1)) + u;
        dxdt = [dx1;dx2];
    end
end
```