<u>Homework Assignment – 2</u>

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<u>Problem 1</u>

(a) Consider the following sequence of rotations.

A: Rotate by Ø about the world x-axis.

B: Rotate by Θ about the current z-axis.

C: Rotate by ψ about the world y-axis.

What is the resulting rotation matrix?

Ans – (a)

1) Rotate by Ø about the world X-axis, A = rot(X, Ø)

2) Rotate by Θ about the current Z-axis, B = rot(X, Ø) rot(Z, Θ) rot^{-1}(X, Ø)

3) Rotate by ψ about the world Y-axis, C = rot (Y, ψ)

Resulting Matrix = C.B.A

= rot (Y, ψ). rot(X, Ø) rot(Z, Θ) rot^{-1}(X, Ø).rot(X, Ø)

= rot (Y, ψ). rot(X, Ø) rot(Z, Θ)

$$= \begin{bmatrix} cos\psi & 0 & sin\psi \\ 0 & 1 & 0 \\ -sin\psi & 0 & cos\psi \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\emptyset & -sin\emptyset \\ 0 & sin\emptyset & cos\emptyset \end{bmatrix} \begin{bmatrix} cos\Theta & -sin\Theta & 0 \\ sin\Theta & cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} cos\psi & sin\psi\ sin\emptyset & sin\psi\ cos\emptyset \\ 0 & cos\emptyset & -sin\emptyset \\ -sin\psi & \cos\psi\ sin\ \emptyset & \cos\psi\ cos\emptyset \end{bmatrix} \begin{bmatrix} cos\Theta & -sin\Theta & 0 \\ sin\Theta & cos\Theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \boldsymbol{cos\psi\ \cos\Theta + \sin\psi\ \sin\emptyset\ sin\Theta} & \boldsymbol{-\cos\psi\ \sin\Theta + \sin\psi\ \sin\emptyset\ cos\Theta} & \boldsymbol{sin\psi\ cos\emptyset} \\ \boldsymbol{\cos\emptyset\ sin\ \Theta} & \boldsymbol{cos\emptyset\ \cos\Theta} & \boldsymbol{-sin\emptyset} \\ \boldsymbol{-sin\psi\ \cos\Theta + \cos\psi\ \sin\emptyset\ sin\Theta} & \boldsymbol{\sin\psi\ \sin\Theta + \cos\psi\ \sin\emptyset\ cos\Theta} & \boldsymbol{\cos\psi\ cos\emptyset} \end{bmatrix}$$

Problem 1

(b) Consider the following sequence of rotations.

A: Rotate by Ø about the world x-axis.

B: Rotate by ϴ about the world z-axis.

C: Rotate by ψ about the current x-axis.

What is the resulting rotation matrix?

Ans – (b)

1) Rotate by Ø about the world X-axis, A = rot(X, Ø)

2) Rotate by ϴ about the world Z-axis, B = rot(Z, ϴ)

3) Rotate by ψ about the current X-axis, C = rot(Z, ϴ) rot(X, Ø) rot (X, ψ) rot^{-1}(Z, ϴ) rot^{-1}(X, Ø)

Resulting Matrix = C.B.A

= rot(Z, ϴ) rot(X, Ø) rot (X, ψ) rot^{-1}(Z, ϴ) rot^{-1}(X, Ø).rot(Z, ϴ). rot(X, Ø)

= rot(Z, ϴ) rot(X, Ø) rot (X, ψ)

$$= \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\emptyset & -sin\emptyset \\ 0 & sin\emptyset & cos\emptyset \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\psi & -sin\psi \\ 0 & sin\psi & cos\psi \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta\cos\emptyset & \sin\theta\sin\emptyset \\ sin\,\theta & \cos\theta\cos\emptyset & -\cos\theta\sin\emptyset \\ 0 & \sin\emptyset & \cos\emptyset \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\psi & -sin\psi \\ 0 & sin\psi & cos\psi \end{bmatrix}$$

$$= \begin{bmatrix} \boldsymbol{cos\theta} & \mathbf{-\sin\theta\cos\emptyset}\,\boldsymbol{cos\psi} + \mathbf{\sin\theta\sin\emptyset\sin\psi} & \mathbf{\sin\theta\cos\emptyset\sin\psi + \sin\theta\sin\emptyset\cos\psi} \\ \boldsymbol{sin\theta} & \mathbf{\cos\theta\cos\emptyset\cos\psi - \cos\theta\sin\emptyset\sin\psi} & \mathbf{-}\boldsymbol{cos}\mathbf{\theta\cos\emptyset\sin\psi - \cos\theta\sin\emptyset\cos\psi} \\ \mathbf{0} & \mathbf{\sin\emptyset\cos\psi + \cos\emptyset\sin\psi} & \mathbf{-\sin\emptyset\sin\psi + \cos\emptyset\cos\psi} \end{bmatrix}$$

Problem 2

(a)

Solution – (a) **As $x \in SO(3)$, $x^T x = x x^T = I$ and det $x = 1$**

(i) I^{st} Property:

$(x_1x_2)^T (x_1x_2) = x_2^T x_1^T x_1 x_2 = x_2^T I x_2 = I$.

By using the determinant matrix multiplication property,

$det(AB) = det(A)\ det(B)$, we have: $det(x3) = det(x1x2) = det(x1)\ det(x2) = 1$.

Hence, we can conclude that $x_1 x_2 \in SO(3)$ $\forall$ $x_1,x_2 \in SO(3)$

(ii) 2^{nd} Property: Associative Property of Matrix Multiplication

$(x_1*x_2)*x_3 = x_1*(x_2*x_3)$. For $x_1,x_2,x_3 \in SO(3)$

(iii) The n x n identity matrix satisfies the 3^{rd} Property.

So, $x*I = I*x = x$ for all $x \in SO(3)$

(iv) As $x^Tx = xx^T = I$, it follows $x^T = x^{-1}$

Therefore, $SO(3)$ with the operation of matrix multiplication is a group

Problem 2

(b)

Solution – (b)

Assume, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SO(2)$

From Cramer's Rule and also $A \in SO(3)$, we have

$A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

Which means, a = d, and b = -c

Therefore,

$A = \begin{bmatrix} a & -c \\ c & a \end{bmatrix}$, With det A = 1 = a^2+c^2 , and $\theta = \tan^{-1}(\frac{c}{a})$

$\cos\theta = a$ and $\sin\theta = c$

Consequently, $\boldsymbol{\theta}$ is unique. i.e. $\boldsymbol{\theta = \tan^{-1}(\frac{c}{a})}$

Problem 3

(a)

Solution – (a)

$H^0_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $H^0_2 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$H^1_2 = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $H^2_1 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$H^2_1 = (H^1_2)^{-1}$

Problem 3

(b) Drive the forward kinematic solution for the robot shown in Fig. 2 (apply Danvit-Hartenberg convention after choosing the coordinate frames).

Solution – (b)

Using axis mentioned on the figure only.

Link	**a_i**	**α_i**	**d_i**	**θ_i**
1	0	90°	0	θ_1
2	a_2	0	0	$-\theta_2$

C	a3	0	0	$-\theta_3$
3	0	90°	0	90°
4	0	90°	0	θ_4
5	0	-90°	0	θ_5
6	0	0	d_6	θ_6

$$A1 = \begin{bmatrix} c1 & 0 & s1 & 0 \\ s1 & 0 & -c1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A2 = \begin{bmatrix} c2 & s2 & 0 & a2*c2 \\ -s2 & c2 & 0 & -a2*s2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad Ac = \begin{bmatrix} c3 & s3 & 0 & a3*c3 \\ -s3 & c3 & 0 & -a3*s3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A4 = \begin{bmatrix} c4 & 0 & s4 & 0 \\ s4 & 0 & -c4 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A5 = \begin{bmatrix} c5 & 0 & -s5 & 0 \\ s5 & 0 & c5 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A6 = \begin{bmatrix} c6 & -s6 & 0 & 0 \\ s6 & c6 & 0 & 0 \\ 0 & 0 & 1 & d6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_6^0 = [A1 \dots Ac \dots A6] = \begin{bmatrix} r11 & r12 & r13 & dx \\ r21 & r22 & r23 & dy \\ r31 & r32 & r33 & dz \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Here,

$c1 = \cos\theta_1$	$s1 = \sin\theta_1$	$c2 = \cos\theta_2$	$s2 = \sin\theta_2$
$c3 = \cos\theta_3$	$s3 = \sin\theta_3$	$c4 = \cos\theta_4$	$s4 = \sin\theta_4$
$c5 = \cos\theta_5$	$s5 = \sin\theta_5$	$c6 = \cos\theta_6$	$s6 = \sin\theta_6$

r11 = c6*(s5*(c1*c2*c3 - c1*s2*s3) + c5*(s1*s4 + c4*(c1*c2*s3 + c1*c3*s2))) + s6*(c4*s1 - s4*(c1*c2*s3 + c1*c3*s2))

r12 = c6*(c4*s1 - s4*(c1*c2*s3 + c1*c3*s2)) - s6*(s5*(c1*c2*c3 - c1*s2*s3) + c5*(s1*s4 + c4*(c1*c2*s3 + c1*c3*s2)))

r13 = c5*(c1*c2*c3 - c1*s2*s3) - s5*(s1*s4 + c4*(c1*c2*s3 + c1*c3*s2))

dx = d6*(c5*(c1*c2*c3 - c1*s2*s3) - s5*(s1*s4 + c4*(c1*c2*s3 + c1*c3*s2))) + a2*c1*c2 - a3*c1*s2*s3 + a3*c1*c2*c3

r21 = c6*(s5*(c2*c3*s1 - s1*s2*s3) - c5*(c1*s4 - c4*(c2*s1*s3 + c3*s1*s2))) - s6*(c1*c4 + s4*(c2*s1*s3 + c3*s1*s2))

r22 = - s6*(s5*(c2*c3*s1 - s1*s2*s3) - c5*(c1*s4 - c4*(c2*s1*s3 + c3*s1*s2))) - c6*(c1*c4 + s4*(c2*s1*s3 + c3*s1*s2))

r23 = c5*(c2*c3*s1 - s1*s2*s3) + s5*(c1*s4 - c4*(c2*s1*s3 + c3*s1*s2))

dy = d6*(c5*(c2*c3*s1 - s1*s2*s3) + s5*(c1*s4 - c4*(c2*s1*s3 + c3*s1*s2))) + a2*c2*s1 - a3*s1*s2*s3 + a3*c2*c3*s1

r31= - c6*(s5*(c2*s3 + c3*s2) - c4*c5*(c2*c3 - s2*s3)) - s4*s6*(c2*c3 - s2*s3)

r32 = s6*(s5*(c2*s3 + c3*s2) - c4*c5*(c2*c3 - s2*s3)) - c6*s4*(c2*c3 - s2*s3)

r33 = - c5*(c2*s3 + c3*s2) - c4*s5*(c2*c3 - s2*s3)

dz = - a2*s2 - d6*(c5*(c2*s3 + c3*s2) + c4*s5*(c2*c3 - s2*s3)) - a3*c2*s3 - a3*c3*s2