Beyond Pointwise Limits

Preserving Continuity, Integrability, and Differentiability

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February 27, 2026

Abstract

This expository paper explores the critical distinction between pointwise and uniform convergence of function sequences in Real Analysis. We delve into the definitions, properties, and implications of both types of convergence, highlighting how uniform convergence preserves continuity, integrability, and differentiability, while pointwise convergence does not necessarily do so. Through rigorous proofs, we demonstrate the significance of uniform convergence in ensuring the stability of function properties under limits, thereby providing a deeper understanding of the behavior of function sequences in mathematical analysis.

1 Introduction

We almost always read first about the sequence and series whose terms were numbers. It was only in particularly simple cases that the terms depended on a variable. In this paper we will talk about the sequences and series whose term depended on a variable, *i.e.*, those whose terms are real valued functions defined on an interval as domain. We accordingly, denote the terms by $f_n(x)$ and consider sequences and series of the form $\{f_n\}$ and $\sum f_n$ respectively.

2 Pointwise Convergence

Suppose $\{f_n\}, n = 1, 2, 3, \ldots$ is a sequence of functions, defined on an interval $[a, b]$ say I. To each point $\psi \in I$, there corresponds a sequence of numbers $\{f_n(\psi)\}$ with terms $f_1(\psi), f_2(\psi), f_3(\psi), \ldots$.

Now, let the sequence $\{f_n(\psi)\}$ converge for every $\psi \in I$.

Let $\{f_n(\psi)\}$ converge to $\{f(\psi)\}$.

Similarly, Let the sequences at all points $\psi, \eta, \zeta, \ldots$, of I converge to

$$f(\psi), f(\eta), f(\zeta), \ldots \tag{1}$$

Now, a real valued function f with domain I and range as the set defined by (1), so that its value $f(\psi)$ for every $\psi \in I$ is $\lim\{f_n(\psi)\}$

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \forall\ x \in I$$

The function f referred to the pointwise limit of the sequence $\{f_n\}$ on $[a, b]$, and the sequence $\{f_n\}$ is said to be pointwise convergent to f on $[a, b]$.

Similarly, if the series $\sum f_n$ converges for every point $x \in I$, and we defined

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \quad \forall\ x \in [a, b]$$

ε - N definition

$$\forall\ \varepsilon > 0\ \exists\ N \in \mathbb{N} \text{ such that } |f_n(x) - f(x)| < \varepsilon\ \forall\ n \geq N$$

1. The geomteric series
$$1 + x + x^2 + x^3 + \ldots$$
converges to $(1 - x)^{-1}$ in the interval $-1 < x < 1$.

   All the items are bounded without the sum being so.

2. For the series $\displaystyle\sum_{n=1}^{\infty} f_n$, where $f_n(x) = \dfrac{x^2}{(1 + x^2)^n}$

   At, $x = 0$, $f_n(0) = 0$, so that sum of the series $f(0) = 0$.

   At $x \neq 0$, it formas a geometric series with common ratio $\dfrac{1}{1 + x^2}$, so that its sum function $f(x) = 1 + x^2$.
$$f(x) = \begin{cases} 0 & , x = 0 \\ 1 + x^2 & , x \neq 0 \end{cases}$$
   Here each term of the series is continuous but the sum f is not.

3. The sequence $\{f_n\}$ where $f_n(x) = \dfrac{\sin nx}{\sqrt{n}}$,

   Here, $\displaystyle\lim_{n\to\infty} f_n(x) = f(x) = 0$
$$\therefore\ f'(x) = 0, \text{ and so } f'(0) = 0$$
   But $f_n'(x) = \sqrt{n}\cos nx$, and so $f_n'(0) = 0$. So as $n \to \infty$, $f_n'(x) \to \infty$.

Thus, at $x = 0$ the sequence $\{f_n'(x)\}$ diverges whereas the limit function $f'(x) = 0$.

4. Consider the sequence $\{f_n\}$

$$f_n(x) = nx(1-x^2)^n, \quad 0 \le x \le 1, \quad n = 1, 2, 3, \ldots$$

For $0 < x \le 1$, $\lim_{n\to\infty} f_n(x) = 0$
At $x = 0$, each $f_n(0) = 0$ so, $\lim_{n\to\infty} f_n(0) = 0$
Thus the limit $f(x) = \lim_{n\to\infty} f_n(x) = 0$ for $x \in [0, 1]$.

$$\therefore \int_0^1 f(x)\, dx = 0.$$

Again, $\displaystyle\int_0^1 f_n(x)\, dx = \int_0^1 nx(1-x^2)^n\, dx = \frac{n}{2n+2}$

$$\therefore \lim_{n\to\infty} \left\{ \int_0^1 nx(1-x^2)^n\, dx \right\} = \frac{1}{2}.$$

$$\lim_{n\to\infty} \left\{ \int_0^1 f_n\, dx \right\} \ne \int_0^1 \left[\lim_{n\to\infty} \{f_n\} \right] dx.$$

To overcome all the above problems, we need a stronger type of convergence than the pointwise convergence. This is called the uniform convergence.

3 Uniform Convergence

Definition 3.1. A sequence $\{f_n\}$ is said to converge uniformly on an interval $[a, b]$ to a function f, if for any $\varepsilon > 0$, there exists a natural number N(independent of x but dependent on ε) such that $\forall\, x \in [a, b]$,

$$|f_n(x) - f(x)| < \varepsilon, \quad \forall\, n \ge N.$$

Similarly, a series $\sum f_n$ is said to converge uniformly on $[a, b]$ if its sequence of partial sums $\{S_n\}$ defined by

$$S_n(x) = \sum_{i=1}^{n} f_i(x)$$

converges uniformly on $[a, b]$.

Thus $\sum f_n$ converges uniformly to f on $[a, b]$ if, for any $\varepsilon > 0$, there exists a natural number N such that $\forall\, x \in [a, b]$,

$$|S_n(x) - f(x)| < \varepsilon, \quad \forall\, n \ge N.$$

3.1 Tests for Uniform Convergence

For Sequence

Theorem 3.1. *Let $\{f_n\}$ be a sequence of functions such that*

$$\lim_{n\to\infty} f_n(x) = f(x), \quad x \in [a,b]$$

and let

$$M_n = \sup_{x\in[a,b]} |f_n(x) - f(x)|$$

Then $f_n \to f$ uniformly if and only if $M_n \to 0$ as $n \to \infty$.

Proof. Let $f_n \to f$ uniformly on $[a,b]$. Thus, $\forall\ \varepsilon > 0\ \exists\ N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \varepsilon \quad \forall\ n \geq N \text{ and } \forall\ x \in [a,b].$$

Therefore, we can say that the supremum of $|f_n(x) - f(x)|$ on $x \in [a,b]$ will be less than or equal to ε.

$$\therefore \sup_{x\in[a,b]} |f_n(x) - f(x)| \leq \varepsilon \quad \forall\ n \geq N.$$

$$\therefore M_n \leq \varepsilon$$

$$\text{or } M_n - 0 \leq \varepsilon \quad \forall\ n \geq N$$

$$\therefore M_n \to 0 \text{ as } n \to \infty.$$

Conversely, let $M_n \to 0$ as $n \to \infty$. So $\forall\ \varepsilon > 0\ \exists\ N \in \mathbb{N}$ such that

$$M_n < \varepsilon \quad \forall\ n \geq N$$

$$\implies \sup\{|f_n(x) - f(x)| : x \in [a,b]\} < \varepsilon \quad \forall\ n \geq N.$$

$$\therefore \varepsilon \text{ is an upper bound of } |f_n(x) - f(x)|$$

$$\therefore |f_n(x) - f(x)| < \varepsilon \quad \forall\ n \geq N \text{ and } x \in [a,b].$$

$$\therefore f_n \to f \text{ uniformly on } [a,b]$$

□

For Series

Theorem 3.2 (Weierstrass M-test)**.** *A series of functions $\sum f_n$ will converge uniformly (and absolutely) on $[a,b]$ if $\exists$ a convergent series $\sum M_n$ of positive numbers such that for all $x \in [a,b]$, $|f_n(x)| \leq M_n, \forall\ n$.*

Proof. Let $\varepsilon > 0$.
Since $\sum M_n$ is convergent, therefore by Cauchy criterion, $\exists\ N \in \mathbb{N}$ such that

$$|M_{m+1} + M_{m+2} + \cdots + M_n| < \varepsilon, \quad \forall\ n > m \geq N.$$

Hence, $\forall\ x \in [a, b]$ and $\forall\ n > m \geq N$, we have

$$|f_{m+1}(x) + f_{m+2}(x) + \cdots + f_n(x)| \leq |f_{m+1}(x)| + |f_{m+2}(x)| + \cdots + |f_n(x)| \tag{1}$$

Since $|f_n(x)| \leq M_n, \forall\ n$,

$$\therefore\ |f_{m+1}(x)| \leq M_{m+1}, |f_{m+2}(x)| \leq M_{m+2}, \ldots$$

$$\therefore\ (1) \implies |f_{m+1}(x) + f_{m+2}(x) + \cdots + f_n(x)| \leq M_{m+1} + M_{m+2} + \cdots + M_n < \varepsilon,$$

$$\forall\ n > m \geq N.$$

$\therefore$ we can say, $|f_{m+1}(x) + f_{m+2}(x) + \cdots + f_n(x)| < \varepsilon$.
$\therefore\ \sum f_n$ satisfies the Cauchy criterion.
$\therefore\ \sum f_n$ is uniformly and absolutely convergent. □

3.2 Properties of Uniform Convergence

Theorem 3.3. *If a sequence $\{f_n\}$ converges uniformly in $[a, b]$, and $x_0 \in [a, b]$ s.t.*

$$\lim_{x \to x_0} f_n(x) = a_n, \quad n = 1, 2, 3, \ldots$$

then

1. *$\{a_n\}$ converges*
2. *$\lim\limits_{x \to x_0} f(x) = \lim\limits_{n \to \infty} a_n$.*

Proof. 1. Since $\{f_n\}$ converges uniformly on $[a, b]$, therefore by Cauchy's criterion, $\forall\ \varepsilon > 0\ \exists\ N \in \mathbb{N}$ (independent of x) s.t. $\forall\ x \in [a, b]$,

$$|f_n(x) - f_m(x)| < \frac{\varepsilon}{2} \quad \forall\ n > m \geq N.$$

Keeping n and m fixed and letting $x \to x_0$, we get,

$$|a_n - a_m| \leq \frac{\varepsilon}{2} < \varepsilon \quad \forall\ n > m \geq N.$$

Thus the sequence $\{a_n\}$ converges. [By Cauchy criterion]

2. Since $\{f_n\}$ converges uniformly on $[a, b]$. Thus $\forall\ \varepsilon > 0\ \exists\ N_1 \in \mathbb{N}$ such that $\forall\ x \in [a, b]$,

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3} \quad \forall\ n \geq N_1 \tag{i}$$

Now, since we know $\{a_n\}$ converges, let it converge to some A.

$\therefore \exists\ N_2 \in \mathbb{N}$ such that $\forall\ x \in [a, b]$,

$$|a_n - A| < \frac{\varepsilon}{3} \quad \forall\ n \geq N_2 \tag{ii}$$

Let $N = \max(N_1, N_2)$.

Again, since $\lim\limits_{x\to x_0} f_n(x) = a_n\ \forall\ n$, therefore $\lim\limits_{x\to x_0} f_N(x) = a_N$. So

$$\forall\ \varepsilon > 0\ \exists\ \delta > 0 \text{ such that } \forall\ x \in [a, b],\ |x - x_0| < \delta \implies |f_N(x) - a_N| < \frac{\varepsilon}{3} \tag{iii}$$

Hence, for $|x - x_0| < \delta$, we have

$$\begin{aligned} |f(x) - A| &\leq |f(x) - f_N(x)| + |f_N(x) - a_N| + |a_N - A| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \quad \text{[From } (i), (ii) \ \& \ (iii)\text{]} \\ & |f(x) - A| < \varepsilon. \end{aligned}$$

$$\therefore \lim_{x\to x_0} f(x) = A.$$

$$\therefore \lim_{x\to x_0} f(x) = \lim_{n\to\infty} a_n$$

More Importantly,

$$\lim_{x\to x_0}\left(\lim_{n\to\infty} f_n(x)\right) = \lim_{n\to\infty}\left(\lim_{x\to x_0} f_n(x)\right)$$

□

Theorem 3.4. *If a series* $\sum_{n=1}^{\infty} f_n$ *converges uniformly to* f *in* $[a, b]$, *and* $x_0 \in [a, b]$ *such that*

$$\lim_{x\to x_0} f_n(x) = a_n, \quad n = 1, 2, 3, \ldots$$

then

1. $\sum a_n$ *converges*

2. $\lim_{x\to x_0} f(x) = \sum a_n$

Proof. 1. Since $\sum f_n$ converges uniformly on $[a,b]$. Thus by Cauchy criterion, $\forall\ \varepsilon > 0\ \exists\ N \in \mathbb{N}$ such that $\forall\ x \in [a,b]$,

$$|f_{m+1}(x) + f_{m+2}(x) + \cdots + f_n(x)| < \frac{\varepsilon}{2} \quad \forall\ n > m \geq N.$$

Keeping m and n fixed and letting $x \to x_0$, we get,

$$|a_{m+1} + a_{m+2} + \cdots + a_n| \leq \frac{\varepsilon}{2} < \varepsilon \quad \forall\ n > m \geq N.$$

$\therefore \sum a_n$ converges. [By Cauchy criterion]

2. Since, $\sum f_n$ converges uniformly to f on $[a,b]$. So $\forall\ \varepsilon > 0\ \exists\ N_1 \in \mathbb{N}$ such that $\forall\ x \in [a,b]$,

$$\left|\sum_{n=1}^{n} f_n(x) - f(x)\right| < \frac{\varepsilon}{3} \quad \forall\ n \geq N_1 \tag{i}$$

Since, we know $\sum a_n$ converges, let it converge to some A. So, similarly $\exists\ N_2 \in \mathbb{N}$ such that $\forall\ x \in [a,b]$,

$$\left|\sum_{n=1}^{n} a_n - A\right| < \frac{\varepsilon}{3} \quad \forall\ n \geq N_2 \tag{ii}$$

Let, $N = \max(N_1, N_2)$.

Again since, $\lim\limits_{x\to x_0} f_n(x) = a_n, n = 1, 2, 3, \ldots, N$. Thus, for the above $\varepsilon > 0\ \exists\ \delta > 0$ such that for $n = 1, 2, 3, \ldots, N$, we have (taking $\delta = \min\{\delta_1, \delta_2, \ldots, \delta_N\}$) such that $\forall\ x$, if $|x - x_0| < \delta$

$$|f_n(x) - a_n| < \frac{\varepsilon}{3N}$$

$$\therefore \left|\sum_{n=1}^{N} f_n(x) - \sum_{n=1}^{N} a_n\right| \leq \sum_{n=1}^{N} |f_n(x) - a_n| < N \cdot \frac{\varepsilon}{3N} = \frac{\varepsilon}{3} \tag{iii}$$

Hence for $|x - x_0| < \delta$, we have

$$|f(x) - A| \leq \left|f(x) - \sum_{n=1}^{N} f_n(x)\right| + \left|\sum_{n=1}^{N} f_n(x) - \sum_{n=1}^{N} a_n\right| + \left|\sum_{n=1}^{N} a_n - A\right|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \quad \text{[From } (i), (ii) \ \& \ (iii)\text{]}$$

$$|f(x) - A| < \varepsilon.$$

$$\therefore \lim_{x\to x_0} f(x) = A.$$

More Importantly,

$$\lim_{x\to x_0}\left(\sum_{n=1}^{\infty} f_n(x)\right) = \sum_{n=1}^{\infty}\left(\lim_{x\to x_0} f_n(x)\right)$$

$\square$

Theorem 3.5.

1. *If $\{f_n\}$ is a sequence of continuous functions on an interval $[a,b]$, and if $f_n \to f$ uniformly on $[a,b]$, then f is continuous on $[a,b]$.*

2. *If a series $\sum f_n$ converges uniformly to f in an interval $[a,b]$ and its terms f_n are continuous at a point x_0 of the interval, then the sum function f is also continuous at x_0.*

Proof. 1. Let x_0 be an arbitrary point in $[a,b]$. To prove that f is continuous at x_0, we must show that for $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta$.

Since $f_n \to f$ uniformly on $[a,b]$, for a given $\varepsilon > 0$, we can choose $N \in \mathbb{N}$ such that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3}, \quad \forall\, x \in [a,b] \text{ and } \forall\, n \geq N \tag{1}$$

In particular, at $x = x_0$:

$$|f_n(x_0) - f(x_0)| < \frac{\varepsilon}{3} \quad \forall\, n \geq N \tag{2}$$

Since f_n is continuous at x_0, there exists $\delta > 0$ such that

$$|f_n(x) - f_n(x_0)| < \frac{\varepsilon}{3} \quad \text{whenever } |x - x_0| < \delta \text{ and } n \geq N \tag{3}$$

Hence for $|x - x_0| < \delta$, we have:

$$\begin{aligned} |f(x) - f(x_0)| &= |f(x) - f_n(x) + f_n(x) - f_n(x_0) + f_n(x_0) - f(x_0)| \\ &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

$\implies f(x) \to f(x_0)$ when $x \to x_0$. Thus, f is continuous at x_0.

2. Since $\sum f_n$ converges uniformly to f on $[a,b]$, for $\varepsilon > 0$, we can choose N such that for all $x \in [a,b]$:

$$\left|\sum_{r=1}^{n} f_r(x) - f(x)\right| < \frac{\varepsilon}{3}, \quad \forall\, n \geq N \tag{1}$$

and in particular, at a point x_0 in $[a,b]$, and $n = N$:

$$\left|\sum_{r=1}^{N} f_r(x_0) - f(x_0)\right| < \frac{\varepsilon}{3} \tag{2}$$

Again, since each f_n is continuous at x_0, the sum of a finite number of functions, $\sum_{r=1}^{N} f_r$, is also continuous at $x = x_0$. Therefore for $\varepsilon > 0$, $\exists\, \delta > 0$, such that

$$\left|\sum_{r=1}^{N} f_r(x) - \sum_{r=1}^{N} f_r(x_0)\right| < \frac{\varepsilon}{3}, \quad \text{for } |x - x_0| < \delta \tag{3}$$

Hence for $|x - x_0| < \delta$, we have

$$\begin{aligned}|f(x) - f(x_0)| &\leq \left|f(x) - \sum_{r=1}^{N} f_r(x)\right| + \left|\sum_{r=1}^{N} f_r(x) - \sum_{r=1}^{N} f_r(x_0)\right| + \left|\sum_{r=1}^{N} f_r(x_0) - f(x_0)\right| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \quad [\text{using (1), (2) \& (3)}]\end{aligned}$$

$\implies f(x) \to f(x_0)$ when $x \to x_0$. i.e., the sum function f is continuous at $x = x_0$.

□

Theorem 3.6.

1. *If a sequence $\{f_n\}$ of integrable functions on $[a,b]$ converges uniformly to a function f on $[a,b]$, then f is integrable on $[a,b]$ and the sequence $\left\{\int_a^x f_n(t)\,dt\right\}$ converges uniformly to $\int_a^x f(t)\,dt$ on $[a,b]$, i.e.,*

$$\lim_{n\to\infty} \int_a^x f_n(t)\,dt = \int_a^x f(t)\,dt, \quad \forall\, x \in [a,b].$$

2. *If a series $\sum f_n$ converges uniformly to f on $[a,b]$, and each term $f_n(x)$ is integrable, then f is integrable and*

$$\int_a^x f(t)\,dt = \sum_{n=1}^{\infty} \int_a^x f_n(t)\,dt, \quad \forall\, x \in [a,b].$$

Proof. 1. Let $\varepsilon > 0$ be any number.

By the uniform convergence of the sequence, there exists an integer N such that for all $x \in [a,b]$

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3(b-a)}, \quad \forall\, n \geq N \tag{2}$$

In particular,

$$|f_N(x) - f(x)| < \frac{\varepsilon}{3(b-a)} \tag{3}$$

For this fixed N, since f_N is integrable, we choose a partition P of $[a,b]$, such that

$$U(f_N, P) - L(f_N, P) < \varepsilon/3 \tag{4}$$

From equation (3),

$$f(x) < f_N(x) + \varepsilon/3(b-a)$$

$$\implies U(f, P) < U(f_N, P) + \varepsilon/3 \tag{5}$$

Again from equation (3),

$$f(x) > f_N(x) - \varepsilon/3(b-a)$$

$$\implies L(f, P) > L(f_N, P) - \varepsilon/3 \tag{6}$$

From equations (4), (5) and (6), we get

$$\begin{aligned} U(f,P) - L(f,P) &< U(f_N,P) - L(f_N,P) + 2\varepsilon/3 \\ &< \varepsilon/3 + 2\varepsilon/3 = \varepsilon \end{aligned}$$

$\implies f$ is integrable on $[a,b]$.
We now proceed to prove relation (1).
Since the sequence $\{f_n\}$ converges uniformly to f, therefore for $\varepsilon > 0$, there exists an integer N such that for all $x \in [a,b]$,

$$|f_n(x) - f(x)| < \varepsilon/(b-a), \quad \forall\, n \geq N$$

Then for all $x \in [a,b]$ and for $n \geq N$, we have

$$\begin{aligned} \left| \int_a^x f\,dt - \int_a^x f_n\,dt \right| = \left| \int_a^x (f - f_n)\,dt \right| &\leq \int_a^x |f - f_n|\,dt \\ &< \frac{\varepsilon}{b-a}(x-a) \leq \varepsilon \end{aligned}$$

$\implies \left\{ \int_a^x f_n\,dt \right\}$ converges uniformly to $\int_a^x f\,dt$ over $[a,b]$, i.e.,

$$\int_a^x f\,dt = \lim_{n\to\infty} \int_a^x f_n\,dt, \quad \forall\, x \in [a,b]$$

More Importantly,

$$\int_a^x \left(\lim_{n\to\infty} f_n(t)\right)\, dt = \lim_{n\to\infty}\left(\int_a^x f_n(t)\, dt\right), \quad \forall\, x \in [a,b]$$

2. Let $S_n(x)$ be the n-th partial sum of the series, defined as:

$$S_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$$

Since each f_i is integrable on $[a,b]$, their finite sum $S_n(x)$ is also integrable for every $n \in \mathbb{N}$.

By the definition of a uniformly convergent series, the sequence of partial sums $\{S_n\}$ converges uniformly to the sum function f on $[a,b]$.

From the above result (the sequence version of this proof), since $\{S_n\}$ is a sequence of integrable functions converging uniformly to f:

(a) f is integrable on $[a,b]$.

(b) The integral of the limit is the limit of the integrals:

$$\int_a^x f(t)\, dt = \lim_{n\to\infty} \int_a^x S_n(t)\, dt \tag{i}$$

Now, looking at the right-hand side of (i):

$$\int_a^x S_n(t)\, dt = \int_a^x \left[\sum_{i=1}^{n} f_i(t)\right] dt$$

By the property of linearity for finite integrals:

$$\int_a^x S_n(t)\, dt = \sum_{i=1}^{n} \int_a^x f_i(t)\, dt$$

Substituting this back into equation (i):

$$\int_a^x f(t)\, dt = \lim_{n\to\infty} \sum_{i=1}^{n} \int_a^x f_i(t)\, dt$$

By the definition of an infinite series:

$$\int_a^x f(t)\, dt = \sum_{n=1}^{\infty} \int_a^x f_n(t)\, dt$$

More Importantly,

$$\int_a^x \left(\sum_{n=1}^{\infty} f_n(t)\right) dt = \sum_{n=1}^{\infty}\left(\int_a^x f_n(t)\, dt\right), \quad \forall\, x \in [a,b]$$

$\square$

Theorem 3.7.

1. *Let $\{f_n\}$ be a sequence of differentiable functions on $[a,b]$ such that it converges at least at one point $x_0 \in [a,b]$. If the sequence of differentials $\{f_n'\}$ converges uniformly to G on $[a,b]$, then the given sequence $\{f_n\}$ converges uniformly on $[a,b]$ to f and $f'(x) = G(x)$.*

2. *Let $\sum f_n$ be a series of differentiable functions on $[a,b]$ such that it converges at least at one point $x_0 \in [a,b]$. If the series of differentials $\sum f_n'$ converges uniformly to G on $[a,b]$, then the given series $\sum f_n$ converges uniformly on $[a,b]$ to f and $f'(x) = G(x)$, i.e.,*
$$\frac{d}{dx}\left[\sum_{n=1}^{\infty} f_n(x)\right] = \sum_{n=1}^{\infty} \frac{d}{dx} f_n(x)$$

Proof. 1. Let $\varepsilon > 0$ be any number. By the convergence of $\{f_n(x_0)\}$ and uniform convergence of $\{f_n'\}$, for $\varepsilon > 0$, we can choose a positive integer N such that for all $x \in [a,b]$,

$$|f_n(x_0) - f_m(x_0)| < \varepsilon/2, \quad \forall\, n > m \geq N \tag{1}$$

$$|f_n'(x) - f_m'(x)| < \varepsilon/2(b-a), \quad \forall\, n > m \geq N \tag{2}$$

Applying Lagrange's mean value theorem to the function $(f_n - f_m)$ for any two points x and t of $[a,b]$, we get for $x < \xi < t$, for all $n > m \geq N$:

$$\begin{aligned} |(f_n(x) - f_m(x)) - (f_n(t) - f_m(t))| &= |x-t||f_n'(\xi) - f_m'(\xi)| \\ &< \frac{|x-t|\varepsilon}{2(b-a)} && (3) \\ &< \varepsilon/2 && (3\text{A}) \end{aligned}$$

and

$$\begin{aligned} |f_n(x) - f_m(x)| &\leq |f_n(x) - f_m(x) - f_n(x_0) + f_m(x_0)| + |f_n(x_0) - f_m(x_0)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{[using (1) \& (3A)]} \end{aligned}$$

$\implies$ The sequence $\{f_n\}$ uniformly converges on $[a,b]$. Let it converge to f, say. For a fixed x on $[a,b]$ and for $t \in [a,b], t \neq x$, let us define

$$\phi_n(t) = \frac{f_n(t) - f_n(x)}{t-x}, n = 1,2,3,\ldots \tag{4}$$

$$\phi(t) = \frac{f(t) - f(x)}{t-x} \tag{5}$$

Since each f_n is differentiable, therefore for each n:

$$\lim_{t\to x} \phi_n(t) = f_n'(x) \tag{6}$$

$\therefore |\phi_n(t) - \phi_n(t)| = \frac{1}{|t-x|}|f_n(t) - f_n(t) - f_n(x) + f_n(x)|$

$$\begin{aligned} &= \frac{1}{|t-x|}|\{f_n(t) - f_n(x)\} - \{f_n(t) - f_n(x)\}| \\ &< \frac{\varepsilon}{2(b-a)}, \quad \forall n \geq N \quad \text{[using (3)]} \end{aligned}$$

so that $\{\phi_n(t)\}$ converges uniformly on $[a,b]$, for $t \neq x$. Since $\{f_n\}$ also converges uniformly on f, therefore from (4):

$$\lim_{n\to\infty} \phi_n(t) = \lim_{n\to\infty} \frac{f_n(t) - f_n(x)}{t-x} = \frac{f(t) - f(x)}{t-x} = \phi(t)$$

Thus $\{\phi_n(t)\}$ converges uniformly to $\phi(t)$ on $[a,b]$, for $t \in [a,b], t \neq x$. Applying the property of interchanging limits to the uniformly convergent sequence $\{\phi_n(t)\}$ and using (6), we get

$$\lim_{t\to x} \phi(t) = \lim_{t\to x} \lim_{n\to\infty} \phi_n(t) = \lim_{n\to\infty} \lim_{t\to x} \phi_n(t) = \lim_{n\to\infty} f_n'(x) = G(x)$$

$\implies \lim_{t\to x} \phi(t)$ exists, and therefore (5) implies that f is differentiable and

$$\lim_{t\to x} \phi(t) = f'(x)$$

Hence,

$$f'(x) = G(x) = \lim_{n\to\infty} f_n'(x)$$

More Importantly,

$$\frac{d}{dx}\left(\lim_{n\to\infty} f_n(x)\right) = \lim_{n\to\infty}\left(\frac{d}{dx} f_n(x)\right)$$

2. Let $S_n(x)$ be the n-th partial sum of the series of functions:

$$S_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$$

Since each f_i is differentiable on $[a, b]$, the finite sum $S_n(x)$ is also differentiable for every $n \in \mathbb{N}$, and its derivative is:

$$S_n'(x) = f_1'(x) + f_2'(x) + \cdots + f_n'(x)$$

Now we apply the conditions from the sequence version from the above result to the sequence of partial sums $\{S_n\}$:

(a) $\{S_n(x_0)\}$ converges (given that the series converges at x_0).

(b) The sequence of derivatives $\{S_n'\}$ converges uniformly to G on $[a, b]$ (given that the series of derivatives converges uniformly).

By using the above result, we conclude that:

(a) The sequence $\{S_n\}$ converges uniformly to a limit function f. Thus, the series $\sum f_n$ converges uniformly to f.

(b) The limit function f is differentiable and $f'(x) = \lim_{n\to\infty} S_n'(x)$.

Therefore:

$$f'(x) = \lim_{n\to\infty} \left[\sum_{i=1}^{n} f_i'(x) \right]$$

$$f'(x) = \sum_{n=1}^{\infty} f_n'(x)$$

Substituting $f(x) = \sum_{n=1}^{\infty} f_n(x)$, we get:

$$\frac{d}{dx} \left[\sum_{n=1}^{\infty} f_n(x) \right] = \sum_{n=1}^{\infty} \frac{d}{dx} f_n(x)$$

□

4 Conclusion

In this chapter, we have explored the concept of uniform convergence and its properties. We have seen that uniform convergence is a stronger form of convergence than pointwise convergence, and it allows us to interchange limits with various operations such as integration and differentiation. The theorems presented in this chapter provide a solid foundation for understanding how uniform convergence behaves in different contexts, and they are essential tools for analyzing sequences and series of functions in mathematical analysis.

References

[1] S.C. Malik and Savita Arora, *Mathematical Analysis*, 5th ed. New Delhi: New Age International Private Limited, 2017.