Transpose of matrix

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Definition

The *transpose* of a matrix is simply a flipped version of the original matrix. We can transpose a matrix by switching its rows with it columns. We denote the transpose of matrix A by $\mathbf{A}^{\mathrm{T}}$, ${}^{\mathrm{T}}\mathbf{A}$, $\mathbf{A}'$, $\mathbf{A}^{\mathrm{tr}}$, ${}^{\mathrm{t}}\mathbf{A}$ or $\mathbf{A}^{\mathrm{t}}$, may be constructed by any one of the following methods:

1. Reflect **A** over its main diagonal (which runs from top-left to bottom-right) to obtain $\mathbf{A}^{\mathrm{T}}$;
2. Write the rows of **A** as the columns of $\mathbf{A}^{\mathrm{T}}$;
3. Write the columns of **A** as the rows of $\mathbf{A}^{\mathrm{T}}$.

Formally, the *i*-th row, *j*-th column element of $\mathbf{A}^{\mathrm{T}}$ is the *j*-th row, *i*-th column element of **A**:

$[\mathbf{A}^{\mathrm{T}}]_{ij} = [\mathbf{A}]_{ji}$

Definition

If $\mathbf{A}$ is an $m \times n$ matrix, then $\mathbf{A}^{\mathsf{T}}$ is an $n \times m$ matrix.

In the case of square matrices, $\mathbf{A}^{\mathsf{T}}$ may also denote the Tth power of the matrix $\mathbf{A}$. For avoiding a possible confusion, many authors use left upperscripts, that is, they denote the transpose as ${}^{\mathsf{T}}\mathbf{A}$. An advantage of this notation is that no parentheses are needed when exponents are involved: as $({}^{\mathsf{T}}\mathbf{A})^n = {}^{\mathsf{T}}(\mathbf{A}^n)$, notation ${}^{\mathsf{T}}\mathbf{A}^n$ is not ambiguous.

Matrix definitions involving transposition

- A square matrix whose transpose is equal to itself is called a *symmetric matrix*; that is, **A** is symmetric if

  $\mathbf{A}^T = \mathbf{A}$

- A square matrix whose transpose is equal to its negative is called a *skew-symmetric matrix*; that is, **A** is skew-symmetric if

  $\mathbf{A}^T = -\mathbf{A}$

- A square complex matrix whose transpose is equal to the matrix with every entry replaced by its complex conjugate (denoted here with an overline) is called a *Hermitian matrix* (equivalent to the matrix being equal to its conjugate transpose); that is, **A** is Hermitian if

  $\mathbf{A}^T = \mathbf{A}'$

Matrix definitions involving transposition

- A square complex matrix whose transpose is equal to the negation of its complex conjugate is called a *skew-Hermitian matrix*; that is, **A** is skew-Hermitian if

  $\mathbf{A}^{T} = -\mathbf{A}'$

- A square matrix whose transpose is equal to its inverse is called an *orthogonal matrix*; that is, **A** is orthogonal if

  $\mathbf{A}^{T} = \mathbf{A}^{-1}$

- A square complex matrix whose transpose is equal to its conjugate inverse is called a *unitary matrix*; that is, **A** is unitary if

  $\mathbf{A}^{T} = (\mathbf{A}^{-1})'$

Example

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2 \times 3} \xrightarrow{\text{Tranpose}} \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}_{3 \times 2}$$

Original matrix
of order 2 x 3

Transpose matrix
of order 3 x 2

Properties

Let **A** and **B** be matrices and *c* be a scalar.

1. **$(A^T)^T = A$**

   The operation of taking the transpose is an involution (self-inverse).

2. **$(A+B)^T = A^T + B^T$**

   The transpose respects addition.

3. **$(AB)^T = B^T A^T$**

   Note that the order of the factors reverses. From this one can deduce that a square matrix **A** is invertible if and only if $\mathbf{A}^T$ is invertible, and in this case we have $(\mathbf{A}^{-1})^T = (\mathbf{A}^T)^{-1}$. By induction, this result extends to the general case of multiple matrices, where we find that $(\mathbf{A}_1\mathbf{A}_2\ldots\mathbf{A}_{k-1}\mathbf{A}_k)^T = \mathbf{A}_k^T\mathbf{A}_{k-1}^T\ldots\mathbf{A}_2^T\mathbf{A}_1^T$

4. **$(cA)^T = cA^T$**

   The transpose of a scalar is the same scalar. Together with (2), this states that the transpose is a linear map from the space of $m \times n$ matrices to the space of all $n \times m$ matrices.

Properties

5. **det(A^T) = det (A)**

   The determinant of a square matrix is the same as the determinant of its transpose.

6. The dot product of two column vectors **a** and **b** can be computed as the single entry of the matrix product:

   $[a.b]^T = a^T b$

7. **$(A^T)^{-1} = (A^{-1})^T$**

   The transpose of an invertible matrix is also invertible, and its inverse is the transpose of the inverse of the original matrix. The notation $\mathbf{A}^{-T}$ is sometimes used to represent either of these equivalent expressions.

Products

If $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{A}^{\mathrm{T}}$ is its transpose, then the result of matrix multiplication with these two matrices gives two square matrices: $\mathbf{A}\mathbf{A}^{\mathrm{T}}$ is $m \times m$ and $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ is $n \times n$. Furthermore, these products are symmetric matrices. Indeed, the matrix product $\mathbf{A}\mathbf{A}^{\mathrm{T}}$ has entries that are the inner product of a row of $\mathbf{A}$ with a column of $\mathbf{A}^{\mathrm{T}}$. But the columns of $\mathbf{A}^{\mathrm{T}}$ are the rows of $\mathbf{A}$, so the entry corresponds to the inner product of two rows of $\mathbf{A}$. If p_{ij} is the entry of the product, it is obtained from rows i and j in $\mathbf{A}$. The entry p_{ji} is also obtained from these rows, thus $p_{ij} = p_{ji}$, and the product matrix (p_{ij}) is symmetric. Similarly, the product $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ is a symmetric matrix. A quick proof of the symmetry of $\mathbf{A}\mathbf{A}^{\mathrm{T}}$ results from the fact that it is its own transpose

$$(\mathbf{A}\mathbf{A}^{\mathrm{T}})^{\mathrm{T}} = (\mathbf{A}^{\mathrm{T}})^{\mathrm{T}}\mathbf{A}^{\mathrm{T}} = \mathbf{A}\mathbf{A}^{\mathrm{T}}$$

Implementation of Transpose of matrix on Computer

On a computer, one can often avoid explicitly transposing a matrix in memory by simply accessing the same data in a different order. For example, software libraries for linear algebra, such as BLAS, typically provide options to specify that certain matrices are to be interpreted in transposed order to avoid the necessity of data movement.

However, there remain a number of circumstances in which it is necessary or desirable to physically reorder a matrix in memory to its transposed ordering. For example, with a matrix stored in row-major order, the rows of the matrix are contiguous in memory and the columns are discontiguous. If repeated operations need to be performed on the columns, for example in a fast Fourier transform algorithm, transposing the matrix in memory (to make the columns contiguous) may improve performance by increasing memory locality.

Implementation of Transpose of matrix on Computer

Ideally, one might hope to transpose a matrix with minimal additional storage. This leads to the problem of transposing an $n \times m$ matrix in-place, with $O(1)$ additional storage or at most storage much less than mn. For $n \neq m$, this involves a complicated permutation of the data elements that is non-trivial to implement in-place. Therefore, efficient in-place matrix transposition has been the subject of numerous research publications in computer science, starting in the late 1950s, and several algorithms have been developed