Assignment → 0

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CS512 - FALL24

PART A → VECTOR OPERATIONS

Given:

$$p = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \quad q = \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix} \quad r = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

1) $3p + 2q$

$$= 3\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} + 2 \times \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times3 \\ -1\times3 \\ 4\times3 \end{bmatrix} + \begin{bmatrix} 0\times2 \\ 3\times2 \\ 5\times2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ -3 \\ 12 \end{bmatrix} + \begin{bmatrix} 0 \\ 6 \\ 10 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 3 \\ 22 \end{bmatrix}$$

2) $\hat{p}$: a unit vector in the direction of p

$$\hat{p} = \frac{p}{\|p\|} \quad \text{—— (i)}$$

$$||p|| = \sqrt{2^2 + (-1)^2 + 4^2}$$
$$= \sqrt{4+1+16}$$
$$= \sqrt{21} \qquad \text{———} \quad (ii)$$

Therefore $\hat{p} = \left(\frac{2}{\sqrt{21}}, \frac{-1}{\sqrt{21}}, \frac{4}{\sqrt{21}}\right)$

3) $|p|$ is already calculated refer equation (ii)

$\Rightarrow |p| = \sqrt{21}$

The angle of p Relative to the Positive Y-Axis,

$$\cos\theta = \frac{p \cdot j}{||p|| \cdot ||j||}$$

here j is unit vector in y-direction,

i.e., $j = [0, 1, 0]$

$$p \cdot j = 2 \cdot 0 + (-1) \cdot 1 + 4 \cdot 0$$
$$= -1$$

and $||j|| = 1$

Therefore, $\cos\theta = \frac{-1}{\sqrt{21}}$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{-1}{\sqrt{21}}\right)$$

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4) The direction cosines of p

$$\cos\alpha = \frac{2}{\sqrt{21}}$$

$$\cos\beta = \frac{p_2}{\sqrt{21}} = \frac{-1}{\sqrt{21}}$$

$$\cos\gamma = \frac{4}{\sqrt{21}}$$

5) The angle between p and q

$$\cos\theta = \frac{p \cdot q}{\|p\| \, \|q\|}$$

here, p.q is dot product
denominator is magnitude p & q

$p \cdot q = 2 \times 0 + (-1) \times 3 + 4 \times 5 = 0 - 3 + 20$
$= 17$ ——— (iii)

$\|q\| = \sqrt{0^2 + 3^2 + 5^2}$
$= \sqrt{9 + 25}$ ——— (iv)
$= \sqrt{34}$

Therefore, $\theta = \cos^{-1}\left(\frac{17}{\sqrt{21} \times \sqrt{34}}\right)$ ——— (v)

c) p.q and q.p

The dot product of two vector is usually commutative.

Therefore $p \cdot q = q \cdot p = 17$ (using eq(ii))

7) p.q using the Angle between p and q

$$p \cdot q = \|p\| \, \|q\| \cos\theta$$

$$= \sqrt{21} \times \sqrt{34} \times \left(\frac{17}{\sqrt{21} \times \sqrt{34}} \right)$$ [using equation (ii), (iv) & (v)]

$$= 17$$

8) Scalar Projection of q onto $\hat{p}$

$$\text{proj}_{\hat{p}}\, q = \frac{q \cdot \hat{p}}{\|\hat{p}\|}$$

$$\|\hat{p}\| = \sqrt{\frac{2^2 + (-1)^2 + 4^2}{(\sqrt{21})^2}}$$

$$= \sqrt{\frac{4+1+16}{21}} = \sqrt{\frac{21}{21}}$$

$$= 1$$

$$q \cdot \hat{p} = 0 \times \frac{2}{\sqrt{21}} + 3 \times \frac{-1}{\sqrt{21}} + 5 \times \frac{4}{\sqrt{21}}$$

$$\frac{17}{\sqrt{21}}$$

Therefore, $\text{Proj}_{\hat{p}}\, q = \dfrac{17}{\sqrt{21}}$

9) A vector perpendicular to p

let x be that vector, such that

$$p \cdot x = 0$$

$$\Rightarrow \begin{bmatrix}2\\-1\\4\end{bmatrix} \cdot \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = 0$$

$$\Rightarrow 2x_1 - x_2 + 4x_3 = 0$$

10) p×q and q×p (CROSS PRODUCTS)

$$p \times q = \begin{vmatrix} i & j & k \\ 2 & -1 & 4 \\ 0 & 3 & 5 \end{vmatrix}$$

$$= i\begin{vmatrix}-1 & 4\\ 3 & 5\end{vmatrix} - j\begin{vmatrix}2 & 4\\ 0 & 5\end{vmatrix} + k\begin{vmatrix}2 & -1\\ 0 & 3\end{vmatrix}$$

$$= i\left((-1)\times(5) - 4\times 3\right) - j\left(5\times 2 - 4\times 0\right) + k\left(3\times 2 - (-1)\times 0\right)$$

$$= i(-5-12) - j(10-0) + k(6-0)$$

$$= -17i - 10j + 6k$$

$$= [-17, -10, 6]$$

$q \times p = -(p \times q) = [17, 10, -6]$

11) A vector Perpendicular to Both p and q

$p \times q = [-17, -10, 6]$

$\downarrow$

cross product vector is perpendicular to both p and q

12) Linear Dependency Between p, q, r

p, q, r can be linearly dependent, if determinant of the matrix form using p, q, r is 0.

$$M = \begin{vmatrix} 2 & 0 & 1 \\ -1 & 3 & -2 \\ 4 & 5 & 2 \end{vmatrix}$$

$$= 2\begin{vmatrix} 3 & -2 \\ 5 & 2 \end{vmatrix} - 0\begin{vmatrix} -1 & -2 \\ 4 & 2 \end{vmatrix} + 1\begin{vmatrix} -1 & 3 \\ 4 & 5 \end{vmatrix}$$

$$= 2\left[(3\times 2) - (-2\times 5)\right] - 0 + 1\left[(-1\times 5) - (3\times 4)\right]$$

$$= 2\times\left[6 - (-10)\right] + 1\left[-5 - 12\right]$$

$$= 32 - 17$$

$$= 15$$

13) $p^T q$ and $p q^T$ (Transpose)

$$p^T = \begin{bmatrix} 2 & -1 & 4 \end{bmatrix}$$

$$q = \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix}$$

So, $p^T \cdot q$

$= 2\times 0 + (-1)\times 3) + 4\times 5$

$= -3 + 20$

$= 17$

Now, for $p q^T$

$$p = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \qquad q^T = \begin{bmatrix} 0 & 3 & 5 \end{bmatrix}$$

$$p \cdot q^T = \begin{bmatrix} 2\times 0 & 2\times 3 & 2\times 5 \\ -1\times 0 & -1\times 3 & -1\times 5 \\ 4\times 0 & 4\times 3 & 4\times 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 6 & 10 \\ 0 & -3 & -5 \\ 0 & 12 & 20 \end{bmatrix}$$

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Part B → MATRIX OPERATIONS

Given:

$$X = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 4 \\ 3 & 2 & -2 \end{bmatrix} \qquad Y = \begin{bmatrix} 4 & -1 & 2 \\ 3 & 0 & -3 \\ 1 & 2 & 1 \end{bmatrix}$$

$$Z = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 4 & 5 \\ 3 & 1 & 2 \end{bmatrix} \qquad S = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}$$

1) $X + 2Y$

$$= \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 4 \\ 3 & 2 & -2 \end{bmatrix} + 2\begin{bmatrix} 4 & -1 & 2 \\ 3 & 0 & -3 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 4 \\ 3 & 2 & -2 \end{bmatrix} + \begin{bmatrix} 4\times2 & -1\times2 & 2\times2 \\ 3\times2 & 0\times2 & -3\times2 \\ 1\times2 & 2\times2 & 1\times2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 4 \\ 3 & 2 & -2 \end{bmatrix} + \begin{bmatrix} 8 & -2 & 4 \\ 6 & 0 & -6 \\ 2 & 4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2+8 & 1+(-2) & 0+4 \\ -1+6 & 3+0 & 4+(-6) \\ 3+2 & 2+4 & -2+2 \end{bmatrix} = \begin{bmatrix} 10 & -1 & 4 \\ 5 & 3 & -2 \\ 5 & 6 & 0 \end{bmatrix}$$

2) XY and Y

XY

$$= \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 4 \\ 3 & 2 & -2 \end{bmatrix} \cdot \begin{bmatrix} 4 & -1 & 2 \\ 3 & 0 & -3 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times4+1\times3+0\times1 & 2\times-1+1\times0+0\times2 & 2\times2+1\times-3+0\times1 \\ -1\times4+3\times3+4\times1 & -1\times-1+3\times0+4\times2 & -1\times2+3\times-3+4\times1 \\ 3\times4+2\times3+(-2)\times1 & 3\times-1+2\times0+(-2)\times2 & 3\times2+2\times(-3)+(-2)\times1 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & -2 & 1 \\ 9 & 9 & -7 \\ 16 & -7 & -2 \end{bmatrix}$$

Y.X

$$= \begin{bmatrix} 4 & -1 & 2 \\ 3 & 0 & -3 \\ 1 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 4 \\ 3 & 2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 4\times2+(-1)\times(-1)+2\times3 & 4\times1+(-1)\times3+2\times2 & 4\times0+(-1)\times4+2\times-2 \\ 3\times2+0\times-1+(-3)\times3 & 3\times1+0\times3+(-3)\times(2) & 3\times0+0\times4+(-3)(-2) \\ 1\times2+2\times-1+1\times3 & 1\times1+2\times3+1\times2 & 1\times0+2\times4+1\times-2 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & 5 & -8 \\ -3 & -3 & 6 \\ 3 & 9 & 6 \end{bmatrix}$$

3) $(XY)^T$ and $Y^T X^T$

$$(XY)^T = \begin{bmatrix} 11 & -2 & 1 \\ 9 & 9 & -7 \\ 16 & -7 & -2 \end{bmatrix}^T$$

$$= \begin{bmatrix} 11 & 9 & 16 \\ -2 & 9 & -7 \\ 1 & -7 & -2 \end{bmatrix}$$

$$Y^T X^T = \begin{bmatrix} 4 & -1 & 2 \\ 3 & 0 & -3 \\ 1 & 2 & 1 \end{bmatrix}^T \cdot \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 4 \\ 3 & 2 & -2 \end{bmatrix}^T$$

$$= \begin{bmatrix} 4 & 3 & 1 \\ -1 & 0 & 2 \\ 2 & -3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -1 & 3 \\ 1 & 3 & 2 \\ 0 & 4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 4\times2+3\times1+1\times0 & 4\times-1+3\times3+1\times4 & 4\times3+3\times2+1\times-2 \\ -1\times2+0\times1+2\times0 & -1\times-1+0\times3+2\times4 & -1\times3+0\times2+2\times-2 \\ 2\times2+(-3)\times1+1\times0 & 2\times-1+(-3)(3)+1\times4 & 2\times3+(-3)\times2+1\times-2 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 9 & 16 \\ -2 & 9 & -7 \\ 1 & -7 & -2 \end{bmatrix}$$

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4) $|x|$ and $|z|$

$$|x| = \begin{vmatrix} 2 & 1 & 0 \\ -1 & 3 & 4 \\ 3 & 2 & -2 \end{vmatrix}$$

$$= 2\begin{vmatrix} 3 & 4 \\ 2 & -2 \end{vmatrix} - 1\begin{vmatrix} -1 & 4 \\ 3 & -2 \end{vmatrix} + 0\begin{vmatrix} -1 & 3 \\ 3 & 2 \end{vmatrix}$$

$$= 2\left[(3\times-2)-(4\times2)\right] - 1\left(\left((-1)\times(-2)-(4\times3)\right)\right) + 0\times0$$

$$= 2\times-14 - 1\times-10 + 0$$

$$= -28+10$$

$$= -18$$

$$|z| = \begin{vmatrix} 2 & 0 & -1 \\ 1 & 4 & 5 \\ 3 & 1 & 2 \end{vmatrix}$$

$$= 2\begin{vmatrix} 4 & 5 \\ 1 & 2 \end{vmatrix} - 0\begin{vmatrix} 1 & 5 \\ 3 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 1 & 4 \\ 3 & 1 \end{vmatrix}$$

$$= 2\left[(4\times2)-(5\times1)\right] - 0 + (-1)\left[(1\times1)-(3\times4)\right)$$

$$= 2\times3 + (-1)\times(-11)$$

$$= 6+11$$

$$= 17$$

5) The matrix (either X, Y, Z) in which the row vectors form an orthogonal set.

For X,

$R1 = (2,1,0)$

$R2 = (-1,3,4)$

$R3 = (3,2,-2)$

Lets compute Dot Product,

$R1 \cdot R2 = 2\times-1 + 1\times3 + 0\times4 = 1$ (Not Orthogonal)

$R2 \cdot R3 = -1\times3 + 3\times2 + 4\times-2 = -5$ (Not Orthogonal)

$R1 \cdot R3 = 2\times3 + 1\times2 + 0\times-2 = 8$ (" ")

2 vectors are orthogonal if they are perpendicular to each other, i.e, the dot product of two vector is 0.

For Y,

$R1 = (4,-1,2)$

$R2 = (3,0,-3)$

$R3 = (1,2,1)$

$R1 \cdot R2 = 4\times3 + (-1)\times0 + 2\times(-3) = 6$ (Not orthogonal)

$R2 \cdot R3 = 3\times1 + 0\times2 + (-3)\times1 = 0$ (orthogonal)

$R1 \cdot R3 = 4\times1 + (-1)\times2 + 2\times1 = 4$ (Not orthogonal)

For Z,

$R1 = (2,0,-1)$

$R2 = (1,4,5)$

$R3 = (3,1,2)$

$R1 \cdot R2 = 2\times1 + 0\times4 + (-1)\times5 = -3$ (not orthogonal)

$R2 \cdot R3 = 1\times3 + 4\times1 + 5\times2 = 17$ (not orthogonal)

$R1 \cdot R3 = 2\times3 + 0\times1 + (-1)\times2 = 4$ (not orthogonal)

So, we have only R2 and Row3 in matrix Y which forms an orthogonal set.

6) X^{-1} and Y^{-1}

$$X^{-1} = \frac{1}{|X|} \operatorname{adj}(X)$$

we already know that $|X| = -18$

A matrix is only invertible if its determinant is non-zero.

For adjugate of X, i.e., $\operatorname{adj}(X)$, we need to find cofactor matrix of X

$$\text{Cofactor matrix of } X = \begin{bmatrix} \begin{vmatrix} 3 & 4 \\ 2 & -2 \end{vmatrix} & -\begin{vmatrix} -1 & 4 \\ 3 & -2 \end{vmatrix} & \begin{vmatrix} -1 & 3 \\ 3 & 2 \end{vmatrix} \\ -\begin{vmatrix} 1 & 0 \\ 2 & -2 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 3 & -2 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} \\ \begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ -1 & 4 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ -1 & 3 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} [3\times-2 - 4\times2] & -[(-1)\times(-2) - 4\times3] & [-1\times2 - 3\times3] \\ -[1\times(-2) - 0\times2] & [2\times-2 - 0\times3] & -[2\times2 - 1\times3] \\ [1\times4 - 0\times3] & -[2\times4 - 0\times(-1)] & [3\times2 - 1\times(-1)] \end{bmatrix}$$

$$= \begin{bmatrix} -14 & 10 & -11 \\ 2 & -4 & -1 \\ 4 & -8 & 7 \end{bmatrix}$$

adj(x) is transpose of co-factor matrix

$$adj(x) = \begin{bmatrix} -14 & 10 & -11 \\ 2 & -4 & -1 \\ 4 & -8 & 7 \end{bmatrix}^T$$

$$= \begin{bmatrix} -14 & 2 & 4 \\ 10 & -4 & -8 \\ -11 & -1 & 7 \end{bmatrix}$$

Therefore,

$$x^{-1} = \frac{1}{-18} \begin{bmatrix} -14 & 2 & 4 \\ 10 & -4 & -8 \\ -11 & -1 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-14}{-18} & \frac{2}{-18} & \frac{4}{-18} \\ \frac{10}{-18} & \frac{-4}{-18} & \frac{-8}{-18} \\ \frac{-11}{-18} & \frac{-1}{-18} & \frac{7}{-18} \end{bmatrix}$$

$$= \begin{bmatrix} 7/9 & -1/9 & -2/9 \\ -5/9 & 2/9 & 4/9 \\ 11/18 & 1/18 & -7/18 \end{bmatrix}$$

$$Y^{-1} = \frac{1}{|Y|} \operatorname{adj}(Y)$$

$$|Y| = \begin{vmatrix} 4 & -1 & 2 \\ 3 & 0 & -3 \\ 1 & 2 & 1 \end{vmatrix}$$

$$= 4\begin{vmatrix} 0 & -3 \\ 2 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 3 & -3 \\ 1 & 1 \end{vmatrix} + 2\begin{vmatrix} 3 & 0 \\ 1 & 2 \end{vmatrix}$$

$$= 4\left[0\times1 - (-3)\times2\right] - (-1)\left(3\times1 - (-3)\times1\right] + 2\left[3\times2 - 0\times1\right]$$

$$= 4\times6 + 1\times6 + 2\times6 \quad = 6(4+1+2) = 6\times7 = 42$$

Since $|Y| \neq 0$, Y is invertible

Lets compute $\operatorname{adj}(Y)$ by finding the co-factor matrix

$$\text{Co-factor of } Y = \begin{bmatrix} \begin{vmatrix} 0 & -3 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 3 & -3 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 0 \\ 1 & 2 \end{vmatrix} \\ -\begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 4 & 2 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 4 & -1 \\ 1 & 2 \end{vmatrix} \\ \begin{vmatrix} -1 & 2 \\ 0 & -3 \end{vmatrix} & -\begin{vmatrix} 4 & 2 \\ 3 & -3 \end{vmatrix} & \begin{vmatrix} 4 & -1 \\ 3 & 0 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} +6 & -6 & 6 \\ 5 & 2 & -9 \\ 3 & +18 & 3 \end{bmatrix}$$

$$adj(y) = \begin{bmatrix} +6 & -6 & 6 \\ 5 & 2 & -9 \\ 3 & +18 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} +6 & 5 & 3 \\ -6 & 2 & +18 \\ 6 & -9 & 3 \end{bmatrix}$$

Therefore $$y^{-1} = \frac{1}{42} \begin{bmatrix} +6 & 5 & 3 \\ -6 & 2 & +18 \\ 6 & -9 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} +6/42 & 5/42 & 3/42 \\ -6/42 & 2/42 & +18/42 \\ 6/42 & -9/42 & 3/42 \end{bmatrix}$$

$$= \begin{bmatrix} +1/7 & 5/42 & 1/14 \\ -1/7 & 1/21 & +3/7 \\ 1/7 & -3/14 & 1/14 \end{bmatrix}$$

7) Z^{-1}

$$Z^{-1} = \frac{1}{|Z|} adj(z)$$

we already calculate $|Z| = 17 \neq 0$, so its invertible

$$\text{Co-factor of } Z = \begin{bmatrix} \begin{vmatrix} 4 & 5 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 5 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 4 \\ 3 & 1 \end{vmatrix} \\ -\begin{vmatrix} 0 & -1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & -1 \\ 4 & 5 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 1 & 5 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 1 & 4 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 13 & -11 \\ -1 & 7 & -2 \\ 4 & -11 & 8 \end{bmatrix}$$

$$\therefore \text{Adj}(Z) = \begin{bmatrix} 3 & 13 & -11 \\ -1 & 7 & -2 \\ 4 & -11 & 8 \end{bmatrix}^T$$

$$= \begin{bmatrix} 3 & -1 & 4 \\ 13 & 7 & -11 \\ -11 & -2 & 8 \end{bmatrix}$$

$$\text{Therefore, } Z^{-1} = \frac{1}{17} \begin{bmatrix} 3 & -1 & 4 \\ 13 & 7 & -11 \\ -11 & -2 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 3/17 & -1/17 & 4/17 \\ 13/17 & 7/17 & -11/17 \\ -11/17 & -2/17 & 8/17 \end{bmatrix}$$

8) The product Xs

$$X \cdot s = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 4 \\ 3 & 2 & -2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$$

$$\Rightarrow X \cdot s = \begin{bmatrix} 2 \times -1 + 1 \times 4 + 0 \times 0 \\ -1 \times -1 + 3 \times 4 + 4 \times 0 \\ 3 \times -1 + 2 \times 4 + (-2) \times 0 \end{bmatrix}$$

$$\Rightarrow X \cdot s = \begin{bmatrix} 2 \\ 13 \\ 5 \end{bmatrix}$$

9) The scalar projection of the rows of X onto the vector s (with s normalized)

Let first normalize s,

$$\hat{s} = \frac{1}{|s|} s$$

$$|s| = \sqrt{(-1)^2 + 4^2 + 0^2}$$

$$= \sqrt{17}$$

$$\therefore \hat{s} = \frac{1}{\sqrt{17}} \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{17} \\ 4/\sqrt{17} \\ 0 \end{bmatrix}$$

The scalar projection of a vector a onto a unit vector b is given by,

scalar projection = a·b

So, here we have each row of X onto $\hat{s}$,

Row 1 of X,

$$Rx_1 = [2,1,0]$$

$$\text{Scalar projection} = [2\ 1\ 0] \cdot \begin{bmatrix} -1/\sqrt{17} \\ 4/\sqrt{17} \\ 0 \end{bmatrix}$$

$$= 2 \times -1/\sqrt{17} + 1 \times \frac{4}{\sqrt{17}} = 2/\sqrt{17}$$

Row 2 of X,

$$Rx_2 = [-1,3,4]$$

$$\text{Scalar projection} = [-1,3,4] \begin{bmatrix} -1/\sqrt{17} \\ 4/\sqrt{17} \\ 0 \end{bmatrix}$$

$$= -1 \times \frac{-1}{\sqrt{17}} + 3 \times \frac{4}{\sqrt{17}} = \frac{13}{\sqrt{17}}$$

Row 3 of X, $Rx_3 = [3,2,-2]$

$$\text{Scalar projection} = [3\ 2\ -2] \begin{bmatrix} -1/\sqrt{17} \\ 4/\sqrt{17} \\ 0 \end{bmatrix}$$

$$= 3 \times \frac{-1}{\sqrt{17}} + 2 \times \frac{4}{\sqrt{17}} + 0 = \frac{5}{\sqrt{17}}$$

10) The vector projection of the rows of X onto the vector s (with s normalized)

The vector projection of the vector a onto b (where b is normalized) is given by:

$$\text{vector projection} = (a \cdot b)\, b$$

We can use scalar projection as $(a \cdot b)$ for that row

Row 1 of X:

$$\text{Vector projection} = \left(\frac{2}{\sqrt{17}}\right)\begin{bmatrix} -1/\sqrt{17} \\ 4/\sqrt{17} \\ 0 \end{bmatrix} = \begin{bmatrix} -2/17 \\ 8/17 \\ 0 \end{bmatrix}$$

Row 2 of X

$$\text{Vector projection} = \left(\frac{13}{\sqrt{17}}\right)\begin{bmatrix} -1/\sqrt{17} \\ 4/\sqrt{17} \\ 0 \end{bmatrix} = \begin{bmatrix} -13/17 \\ 52/17 \\ 0 \end{bmatrix}$$

Row 3 of X.

$$\text{Vector projection} = \left(\frac{5}{\sqrt{17}}\right)\begin{bmatrix} -1/\sqrt{17} \\ 4/\sqrt{17} \\ 0 \end{bmatrix} = \begin{bmatrix} -5/17 \\ 20/17 \\ 0 \end{bmatrix}$$

11) The linear combination of the columns of X using the elements of S

It is given by,

$s_1 \cdot$ column 1 of X + $s_2 \cdot$ col 2 of X + $s_3 \cdot$ col 3 of X

$$= (-1)\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + 4\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 4 \\ -2 \end{bmatrix}.$$

$$= \begin{bmatrix} (-2 + 4 + 0) \\ (1) + 12 + 0 \\ -3 + 8 + 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 13 \\ 5 \end{bmatrix}$$

12) The solution t for equation $Yt = s$

$\rightarrow t = Y^{-1}s$

we know $Y^{-1} = \begin{bmatrix} +1/7 & 5/42 & 1/14 \\ -1/7 & 1/21 & +3/7 \\ 1/7 & -3/14 & 1/14 \end{bmatrix}$ $\quad s = \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$

Therefore, $Y^{-1} \cdot s$ =

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$$= \begin{bmatrix} +1/7 \times -1 + \frac{5}{42} \times 4 + \frac{1}{14} \times 0 \\ -1/7 \times -1 + \frac{1}{21} \times 4 + \left(\frac{-1}{7}\right) \times 0 \\ \frac{1}{7} \times -1 + \left(\frac{-3}{14}\right) \times 4 + \frac{1}{14} \times 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-6+20}{42} \\ \frac{3+8}{21} \\ \frac{-2-12}{14} \end{bmatrix} = \begin{bmatrix} 0.333 \\ 0.523 \\ -1 \end{bmatrix}$$

13) The solution t for the equation $Zt = s$

$$t = z^{-1} \cdot s$$

we have $$z^{-1} = \begin{bmatrix} 3/17 & -1/17 & 4/17 \\ 13/17 & 7/17 & -11/17 \\ -11/17 & -2/17 & 8/17 \end{bmatrix} \qquad s = \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$$

$$\therefore z^{-1} \cdot s = \begin{bmatrix} \frac{3}{17} \times -1 + \left(\frac{-1}{17}\right) \times 4 + 0 \\ \frac{13}{17} \times -1 + \frac{7}{17} \times 4 + 0 \\ \frac{-11}{17} \times -1 + \left(-\frac{2}{17}\right) \times 4 + 0 \end{bmatrix} = \begin{bmatrix} -7/17 \\ 15/17 \\ 3/17 \end{bmatrix}$$

Test # C

EIGEN VALUES & EIGENVECTORS

1) The eigenvalues and corresponding eigenvectors of M

Given:

$$M = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \quad N = \begin{bmatrix} 5 & -3 \\ -3 & 6 \end{bmatrix} \quad P = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}.$$

$$\det(M - \lambda I) = 0$$

$$\det(M - \lambda I) = \begin{bmatrix} 3-\lambda & 2 \\ -1 & 4-\lambda \end{bmatrix}$$

$$= (3-\lambda)(4-\lambda) - (-1)\times 2$$

$$\Rightarrow (3-\lambda)(4-\lambda) + 2 = 0$$

$$\Rightarrow 12 - 3\lambda - 4\lambda + \lambda^2 + 2 = 0$$

$$\Rightarrow \lambda^2 - 7\lambda + 14 = 0$$

we know that
$(x+y)^2 = x^2 + y^2 + 2xy$
and
$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Therefore we have

$$\lambda = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(14)}}{2(1)}$$

$$= \frac{7 \pm \sqrt{49 - 56}}{2}$$

$$= \frac{7 \pm \sqrt{-7}}{2}$$

So, we have

$$\lambda_1 = \frac{7 + \sqrt{-7}}{2} \quad \text{and} \quad \lambda_2 = \frac{7 - \sqrt{-7}}{2}$$

$$= \frac{7 + i\sqrt{7}}{2} \qquad\qquad = \frac{7 - i\sqrt{7}}{2}$$

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For $\lambda_1 = \frac{7+i\sqrt{7}}{2}$

$(M - \lambda I) = 0$

$$\Rightarrow \left(\begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} - \begin{bmatrix} \frac{7+i\sqrt{7}}{2} & 0 \\ 0 & \frac{7+i\sqrt{7}}{2} \end{bmatrix} \right) v = 0$$

$$\Rightarrow \begin{bmatrix} 3 - \frac{7+i\sqrt{7}}{2} & 2-0 \\ -1-0 & 4 - \frac{7+i\sqrt{7}}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} \frac{-1-i\sqrt{7}}{2} & 2 \\ -1 & \frac{1-i\sqrt{7}}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

we have two equations,

$$\frac{-1-i\sqrt{7}}{2} x + 2y = 0 \quad \text{— (i)}$$

$$-x + \frac{1-i\sqrt{7}}{2} y = 0 \quad \text{— (ii)}$$

$$\Rightarrow x = \frac{-4y}{-1-i\sqrt{7}}$$

$$= \frac{4y}{1+i\sqrt{7}} \times \frac{(1-i\sqrt{7})}{(1-i\sqrt{7})}$$

$$= \frac{4y(1-i\sqrt{7})}{1^2-(i\sqrt{7})^2} = \frac{4y(1-i\sqrt{7})}{8} = \frac{y}{2}(1-i\sqrt{7})$$

So, eigen vector corresponding to λ_1 is,

$$v_1 = \begin{bmatrix} \frac{1-i\sqrt{7}}{2} \\ 1 \end{bmatrix}$$

and the eigenvector corresponding to λ_2 is,

$$v_2 = \begin{bmatrix} \frac{1+i\sqrt{7}}{2} \\ 1 \end{bmatrix}$$

2) The dot product between the eigenvectors of M

$$V_1 \cdot V_2$$

$$= \begin{bmatrix} \frac{1-i\sqrt{7}}{2} \\ 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1+i\sqrt{7}}{2} \\ 1 \end{bmatrix}$$

$$= \left(\frac{1-i\sqrt{7}}{2} \right) \left(\frac{1+i\sqrt{7}}{2} \right) + (1)(1)$$

$$= \frac{(1)^2 - (i\sqrt{7})^2}{2 \cdot 2} + 1$$

$$= \frac{\cancel{8}}{\cancel{4}} + 1$$

$$= 3$$

3) The dot product between the eigenvector of N

we have $N = \begin{bmatrix} 5 & -3 \\ -3 & 6 \end{bmatrix}$

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$$\det(N - \lambda I) = 0$$

$$\Rightarrow \begin{vmatrix} 5-\lambda & -3 \\ -3 & 6-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (5-\lambda)(6-\lambda) - (-3)(-3) = 0$$

$$\Rightarrow 30 - 5\lambda - 6\lambda + \lambda^2 - 9 = 0$$

$$\Rightarrow \lambda^2 - 11\lambda + 21 = 0$$

So we have eigen values are

$$\lambda = \frac{-(-11) \pm \sqrt{(-11)^2 - 4.1.21}}{2.1}$$

$$= \frac{11 \pm \sqrt{37}}{2}$$

$$\therefore \lambda_1 = \frac{11 + \sqrt{37}}{2} \qquad \lambda_2 = \frac{11 - \sqrt{37}}{2}$$

$$= 8.54 \qquad = 2.45$$

So, we have eigenvectors

For λ_1,

$$N - \lambda_1 I = \begin{bmatrix} 5 - 8.54 & -3 \\ -3 & 6 - 8.54 \end{bmatrix}$$

$$= \begin{bmatrix} -3.54 & -3 \\ -3 & -2.54 \end{bmatrix}$$

$$\begin{bmatrix} -3.54 & -2 \\ -3 & -2.54 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = 0$$

So, we have,

$$\Rightarrow \quad -3.54x_1 - 3y_1 = 0 \quad \text{— (i)}$$

$$\Rightarrow \quad -3x_1 - 2.54y_1 = 0 \quad \text{— (ii)}$$

$$x_2 = \frac{3y_2}{2.54} = 1.2\, y_2$$

$$v_1 = \begin{bmatrix} 0.84 \\ 1 \end{bmatrix}$$

For $\lambda_2 = 2.45$,

$$N - \lambda_2 I$$

$$= \begin{bmatrix} 5 - 2.45 & -3 \\ -3 & 6 - 2.45 \end{bmatrix}$$

$$= \begin{bmatrix} 2.55 & -3 \\ -3 & 3.55 \end{bmatrix}$$

eigenvector,

$$\begin{bmatrix} 2.55 & -3 \\ -3 & 3.55 \end{bmatrix} V_2 = 0$$

$$\Rightarrow \begin{bmatrix} 2.55 & -3 \\ -3 & 3.55 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = 0$$

we have $\quad 2.55x_2 - 3y_2 = 0$

$$-3x_2 + 3.55y_2 = 0$$

we have $x_2 = \frac{3y_2}{2.55} \approx 1.18 y_2$

$$V_2 = \begin{bmatrix} 1.18 \\ 1 \end{bmatrix}$$

Dot Product of $V_1 \cdot V_2$

$= (0.84)(1.18) + (1)(1)$

$= 1.99$

$\simeq 2$

4) The Property of the Eigenvectors of N and Reason for it

∴ Symmetric matrices have orthogonal eigenvectors due to spectral theorem.

Therefore, if N is symmetric, then its eigenvectors are orthogonal.

So, here, Property used: Orthogonality of Eigen Vectors

The eigenvectors of the symmetric matrix

$N = \begin{bmatrix} 5 & -3 \\ -3 & 6 \end{bmatrix}$ are orthogonal to each other.

5) The values of a Trivial Solution t to the Equation

$Pt = 0$

$$P = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}$$

For A trivial solution, your equation $Pt = 0$ is simply $t = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore$ A trivial solution is the zero vector and multiplying any matrix by it will result in zero vector.

6) The values of Two Non-trivial solution t to the Equation $Pt = 0$

$$Pt = 0$$

$$\Rightarrow \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x + 4y \\ 4x + 8y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We have $2x + 4y = 0$

$4x + 8y = 0$

$\Rightarrow 2(2x + 4y) = 0$ — same

$\Rightarrow 2x + 4y = 0$

$\Rightarrow x = -2y$

$$t = \begin{bmatrix} -2y \\ y \end{bmatrix} = y \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

So, we have,

$$t_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$t_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

7) The only solution t to the Equation $Mt = 0$ and the Reason for having a Single solution.

$$M = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix}$$

$Mt = 0$

$$\Rightarrow \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3x + 2y \\ -x + 4y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

we have 2 equation

$3x + 2y = 0$

$-x + 4y = 0 \quad \Rightarrow x = 4y$ —(i)

$3x + 2y = 0$

$\Rightarrow 3(4y) + 2y) = 0$

$\Rightarrow 14y = 0$

$\Rightarrow y = 0$

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If $y = 0$, then $x = 4y = 4(0) = 0$

so, only solution $t = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Reason → M is non-singular (invertible matrix), i.e., $|M| \neq 0$.

$\therefore$ $Mt = 0$ only has trivial solution $t = 0$.

For invertible matrix, there are no non-trivial solutions.

Part D → Neural Networks Basics

1) Given:

$x_1 = 0.5$ $w_1 = 0.4$

$x_2 = -0.8$ $w_2 = -0.6$

$x_3 = 0.3$ $w_3 = 0.9$

$b = 0.2$

o/p before applying any activation function is –

$$z = w_1x_1 + w_2x_2 + w_3x_3 + b$$
$$= (0.4 \times 0.5) + (-0.6 \times 0.8) + (0.9 \times 0.3) + 0.2$$
$$= (0.2) + 0.48 + 0.27 + 0.2$$
$$\Rightarrow z = 1.15 \quad \text{—— (i)}$$

2) Applying the Sigmoid Activation function,

It is given by,

$$\sigma(z) = \frac{1}{1+e^{-z}}$$

using eq(i),

$$\sigma(1.15) = \frac{1}{1+e^{-1.15}}$$

$$= \frac{1}{1+0.316}$$

$$\simeq 0.759$$

3) Output of the Neuron After applying ReLU Activation Function is,

$$ReLU(z) = max(0, z)$$

$$\Rightarrow ReLU(1.15) = max(0, 1.15) \quad \text{using equation(i)}$$

$$= 1.15$$

4) Given:

$$W = \begin{bmatrix} 0.4 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

$$b = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}$$

$$x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Therefore, o/p z of the layer before activation

$$z = w.x + b$$

$$= \begin{bmatrix} 0.4 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix}.$$

$$= \begin{bmatrix} 0.4 \times 1 + 0.3 \times 2 \\ 0.2 \times 1 + 0.7 \times 2 \end{bmatrix} + \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + 0.1 \\ 1.6 - 0.1 \end{bmatrix} = \begin{bmatrix} 1.1 \\ 1.5 \end{bmatrix}$$

Applying ReLU,

$$ReLU(z) = \begin{bmatrix} max(0, 1.1) \\ max(0, 1.5) \end{bmatrix} = \begin{bmatrix} 1.1 \\ 1.5 \end{bmatrix}$$

5) Output z of using sigmoid function by feeding output from previous layer

So,

Given, $w = [0.5, -0.3]$

$b = 0.1$

i/p from previous layer $= \begin{bmatrix} 1.1 \\ 1.5 \end{bmatrix}$

$\therefore$ o/p before activation,

$$z = w.x + b$$

$$= [0.5 \quad -0.3] \cdot \begin{bmatrix} 1.1 \\ 1.5 \end{bmatrix} + 0.1$$

$$= (0.5 \times 1.1) + (-0.3 \times 1.5) + 0.1$$

$$= 0.55 - 0.45 + 0.1 = 0.2$$

Applying the sigmoid function

$$\sigma(z) = \frac{1}{1+e^{-0.2}}$$

$$= 0.55$$

6) Given a neural network with

Hidden layer → weights W_h, bias b_h, output z

output layer → weight W_o, bias b_o, and output y

(After activation)

input $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Loss function : L

∴ the gradient of the Loss L w.r.to x_1, using chain rule, i.e,

$$\frac{\partial L}{\partial x_1} = \frac{\partial L}{\partial y} \times \frac{\partial y}{\partial z} \times \frac{\partial z}{\partial x_1} \longrightarrow$$

derivative of hidden layer o/ps w.r.t i/p component x_1.

here, derivative of the loss w.r.to output layer output

derivative of the o/p layer output w.r.to hidden layer o/ps

Part F → Gradient Calculation

Given:

$$f(x) = 2x^2 - 1$$

$$g(x) = 3x^2 + 4$$

$$h(x,y) = x^2 + y^2 + xy$$

1)

$$f'(x) = \frac{d}{dx} f(x)$$

$$\left[\frac{d}{dx} x^n = nx^{n-1} \right]$$

$$= \frac{d}{dx} (2x^2 - 1)$$

$$= 2 \times 2x + 0$$

$$= 4x$$

$$f''(x) = \frac{d}{dx} f'(x)$$

$$= \frac{d}{dx} 4x$$

$$= 4$$

2)

$$\frac{\partial h}{\partial x} = \frac{\partial}{\partial x} (x^2 + y^2 + xy)$$

$$= 2x + y$$

$$\frac{\partial h}{\partial y} = 2y + x$$

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3) Gradient Vector $\nabla h(x,y)$

$$\nabla h(x,y) = \left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y} \right)$$

$$= (2x+y,\ 2y+x)$$

4) $\frac{d}{dx} f(g(x))$

→ Using the chain rule,

here, $f(x) = 2x^2 - 1$

$g(x) = 3x^2 + 4$

So, $f(g(x))$

$= 2(3x^2+4)^2 - 1$

$$\frac{d}{dx} f(g(x))$$

$$= f'(g(x)) \cdot g'(x)$$ (using chain rule)

$$g'(x) = \frac{d}{dx}(3x^2+4)$$

$$= 3 \times 2x + 0$$

$$= 6x$$

$$f'(g(x)) = \frac{d}{dx}\left[2(3x^2+4)^2 - 1\right]$$

$$= 4(3x^2+4)$$

$$\therefore \frac{d}{dx} f(g(x)) = 4(3x^2+4) \times 6x$$

$$= \underline{\underline{24x(3x^2+4)}}$$

Now without using chain rule :-

$$f(g(x))$$

$$= 2(3x^2+4)^2 - 1$$

$$= 2(9x^4 + 16 + 2\times 3\times x^2 \times 4) - 1$$

$$= 2(9x^4 + 16 + 24x^2) - 1$$

$$\Rightarrow f(g(x)) = 18x^4 + 48x^2 + 31$$

$$\frac{d}{dx} f(g(x)) = \frac{d}{dx}(18x^4 + 48x^2 + 31)$$

$$= 18\times 4x^3 + 48\times 2x + 0$$

$$= 72x^3 + 96x$$

$$= \underline{\underline{24x(3x^2+4)}}$$

---- x ----