Combinatorics

Agenda

1) Permutations
2) Combinations
3) Properties
4) One Problem

Combinatorics deals with the selection and arrangement of objects according to some pattern and counting no. of ways it can be done.

Quiz:

Q1: T/F 2

Q2: T/F 2

Q3: T/F 2

$2 \times 2 \times 2 = \boxed{8}$

Question can be left blank.

Q1: 3

Q2: 3

Q3: 3

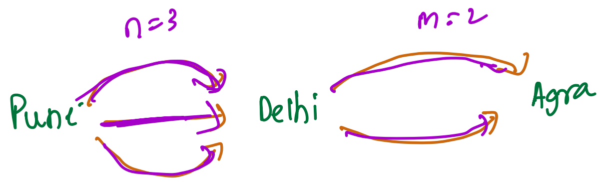
$3 \times 3 \times 3 = \boxed{27}$

Multiplicative Principle

Task A: n ways

Task B: m ways

No. of ways of doing task A & task B = n × m



2 + 2 + 2 =

2 × 3 = 6

Task A: Travel from Pune to Delhi : 3

Task B: Travel from Delhi to Agra : 2

Task A & Task B : Travel from Pune to Agra (via Delhi)

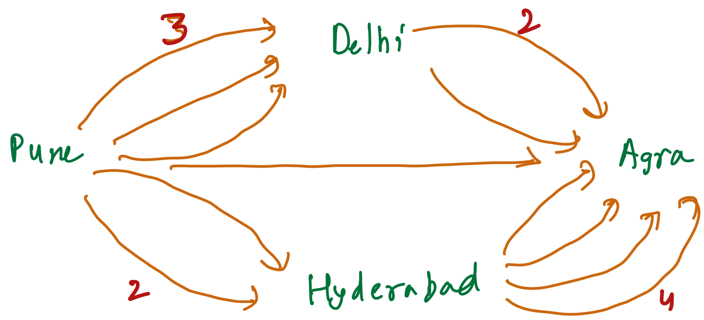
3 × 2 = $\boxed{6}$

Additive Principle

Task A: n ways

Task B: m ways

No. of ways of doing task A OR task B = n + m



Task A: Travel from Pune to Agra (via Delhi) $= 3 \times 2 = 6$

Task B: Travel from Pune to Agra (via Hyd) $= 2 \times 4 = 8$

TaskA OR TaskB: Travel from Pune to Agra

[via Hyd or via Delhi]

$6 + 8 = \boxed{14}$

Direct Route:

$6 + 8 + 1 = \boxed{15}$

AND: Multiply

OR: Add

Task1: $\boxed{2}$ (1+1) $2 \times 2 \times 2 = \boxed{8}$

Task2: 2

Task3: 2

$(1+1+1) = \boxed{3}$

Permutations

Ordered arrangement of objects

abc : abc
acb
bac
bca
cab
cba

6 permutations

abc ≠ acb

Order Matters

Question: Given a string with N distinct characters, find total no of permutations

S = "abcd"   N = 4

$\underset{0}{4}\ \underset{1}{3}\ \underset{2}{2}\ \underset{3}{1} = 4 \times 3 \times 2 \times 1 = 4!$

Task1: Place at 0th position : 4

Task2: 1 : 3

Task3: 2 : 2

Task4: 3 : 1

N ⇒ N!

Question: No of ways of arranging (unique) R characters among N characters

S = "abcde"

N = 5

R = 2

$\underset{0}{\underline{5}}\quad\underset{1}{\underline{4}}$

$$5 \times 4 = \frac{5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} = \frac{5!}{3!} \rightarrow (5-2)$$

N = 5

R = 3

$\underline{5}\quad\underline{4}\quad\underline{3}$

$$5 \times 4 \times 3 = \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = \frac{5!}{2!} \rightarrow (5-3)$$

$$\#\text{ways of arranging } R \text{ characters} = \frac{N!}{(N-R)!}$$

$^{n}P_{r}$ = #ways of arranging R characters among N characters

$$^{n}P_{R} = \frac{n!}{(N-R)!}$$

permutations

Quiz: N = 5, R = 3

$$\frac{5!}{(5-3)!} = \frac{120}{2} = 60$$

Quiz: $N=5$, $R=2$ $\frac{5!}{(5-2)!} = \frac{120}{6} = 20$

Permutations with duplicate characters

S = "a b a" (a_1, a_2)

N = 3 ⇒ 3! ⇒ 6 ?

(a_1 b a_2) ⇒ aba
(a_2 b a_1) ⇒ aba
(b a_1 a_2) ⇒ baa
(b a_2 a_1) ⇒ baa
[a_1 a_2 b] ⇒ aab
(a_2 a_1 b) ⇒ aab

3 unique Permutations

S = "abbbcd"

N = 6

Total permutations with 6 chars = 6! = 720

(b_1 b_2 b_3) ← a b_1 c d b_2 b_3
(b_1 b_3 b_2) ← a b_1 c d b_3 b_2
(b_1, b_2, b_3) ⇒ 3! = 6
(b_2 b_1 b_3) ← a b_2 c d b_1 b_3
(b_2 b_3 b_1) a b_2 c d b_3 b_1
(b_3 b_1 b_2) a b_3 c d b_1 b_2
(b_3 b_2 b_1) a b_3 c d b_2 b_1

(a b c d b b)

Ex2 :

b_1	c	b_3	a	d	b_2
b_1	c	b_2	a	d	b_3
b_2	c	b_1	a	d	b_3
b_2	c	b_3	a	d	b_1
b_3	c	b_1	a	d	b_2
b_3	c	b_2	a	d	b_1

1 permutation

Permutations = $\frac{N!}{3!}$

S = "a bbb cc" => total Permutations

N = 6 => 6! = 720

a b_1 b_2 b_3 c c

(a b bb c c) => 3!

$$\frac{6!}{3!\,2!}$$

a b b b c_1 c_2

a b b b c_2 c_1

(c_1, c_2) => 2!

$$\frac{N!}{r_1!\ r_2!\ r_3!\ \dots\ r_k!}$$

$r_i \Rightarrow$ No of times ith character appears in the string

Quiz: $s =$ " a c c b a a "

$$\frac{6!}{2!\ 3!} = \frac{\cancel{\cancel{720}}^{\cancel{120}^{60}}}{\cancel{2} \times \cancel{6}} = \boxed{60}$$

Combinations :

selection of objects (unordered)

$(a, b, c) == (b, a, c)$

S : "abcde"

$N = 5$

$r = 4$

No. of ways of selecting r chars among N chars

(abcd, abce, abde, acde, bcde)

abcd = bacd = cadb ...

Order does not matter

$$^{N}C_{r} = \frac{N!}{(N-r)!\, r!}$$

No of ways of arranging r chars among N chars = $^{n}P_{r}$

$$^{n}P_{r} = \frac{n!}{(n-r)!}$$

$S = $ "abcde"

$r = 3$

abc
acb
bac
bca
cab
cba

3!

3!

$\frac{5!}{2!}$ permutations

60 permutations

bde
bed
dbe
deb
ebd
edb

6 permutations

1 combination

$$\frac{\left(\frac{5!}{2!}\right)}{3!} = \frac{5!}{2!\,3!}$$

$$N_{C_r} = \frac{N_{P_r}}{r!} = \frac{N!}{(N-r)!\,r!}$$

$N = \{abcd\}$

$r = 2$

{ab, ac, ad, bc, bd, cd}

$$\frac{4!}{2!\cdot 2!} = \frac{24}{2\cdot 2} = \boxed{6}$$

$N = \{abcd\}$

$r = 4$

{abcd}

$$\frac{4!}{(4-0)!\,0!} = \frac{4!}{4!} = \boxed{1}$$

$N = \{abcddd\}$

$s = abcde$

$r = 1$

$\{a\}\{b\}\{c\}\{d\}\{e\}$

$\boxed{8 \text{ mins}}$

Properties

1) $^{N}C_0 = 1$ $\qquad \frac{N!}{0!\,(N-0)!} = 1$

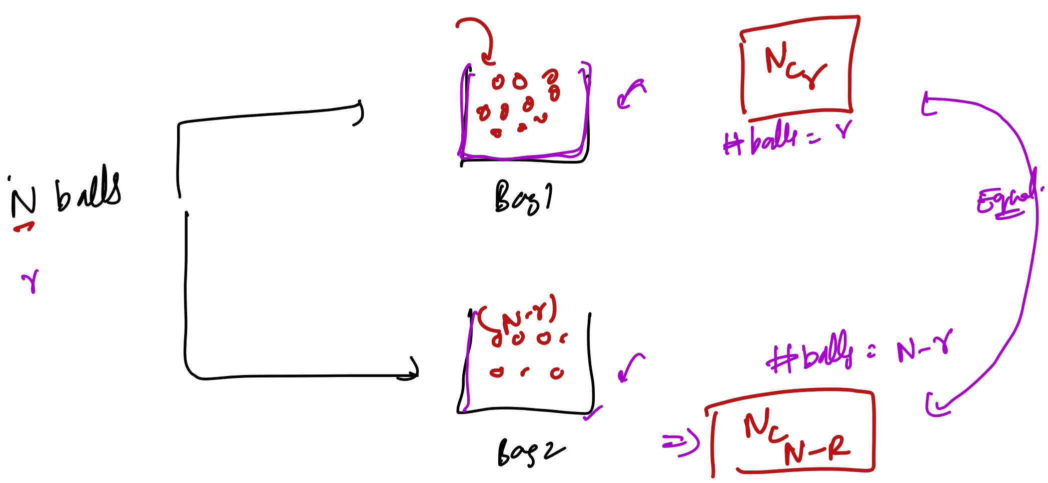
2) $^{N}C_N = 1$ $\qquad \frac{N!}{N!\,(N-N)!} = 1$

3) $^{N}C_1 = N$

4) $^{N}C_R = {}^{N}C_{N-R}$

$$\frac{N!}{R!\,(N-R)!} \qquad \frac{N!}{(N-R)!\,(N-(N-R))!}$$

$$= \frac{N!}{(N-R)!\,R!}$$



5)

$${}^{N}C_{r} = {}^{N-1}C_{R-1} + {}^{N-1}C_{R}$$

$a_1 \quad a_2 \quad a_3 \quad a_4 \quad \dots \quad a_{N-2} \quad a_{N-1} \quad a_N$

Don't Pick OR Pick

${}^{N-1}C_{R}$ ${}^{N-1}C_{r-1}$

LHS == RHS

$${}^{N}C_{R} = {}^{N-1}C_{R} + {}^{N-1}C_{R-1}$$

Homework: Prove this mathematically

Question :

$^NC_0 + {}^NC_1 + {}^NC_2 + {}^NC_3 + \cdots\cdots {}^NC_N$

$^NC_0$ → Not selecting anything

$^NC_1$ → No. of ways of selecting 1 from N

$^NC_2$ → No. of ways of selecting 2 from N

$^NC_3$ → No. of ways of selecting 3 from N

^NC_N → No. of ways of selecting N from N

$N=3$

$A = \{1, 2, 3\}$

$^3C_0 = 1 \quad \{\ \}$

$^3C_1 = 3 \quad \{1\} \ \{2\} \ \{3\}$

$^3C_2 = 3 \quad \{1,2\} \ \{1,3\} \ \{2,3\}$

$^3C_3 = 1 \quad \{1,2,3\}$

Subsets / subseqs

$\Downarrow$

2^N

Question : Given N, r, M find $^{N}C_{r} \% M$

$N, r, M \leq 10^9$
$r \leq N$

Ex1 $N=4$ $r=2$ $M=10$

$^{4}C_{2} \% 10 = 6 \% 10 = 6$

$^{N}C_{r} = \frac{N!}{r!\,(N-r)!}$

$\left(\frac{a}{b}\right) \% M = (a \% M \times b^{-1} \% M) \% M$

$^{N}C_{R} \% M = \left(\frac{N!}{r!\,(N-r)!}\right) \% M = [\,(N! \% M) \times (r!)^{-1} \% M \times (N-r)!^{-1} \% M) \% M$

$b^{-1} \bmod M \rightarrow$ b and M should be co-prime.
$\gcd(b, M) = 1$

$\rightarrow$ Assume, they are co-prime

$b^{-1} \bmod M = b^{M-2} \% M$

Prime

$\rightarrow$ $N, r < M$ is prime

$[\,(N! \% M) \times (r!)^{-1} \% M \times (N-r)!^{-1} \% M) \% M$

$\gcd(r!, M) == 1$
$\gcd((N-r)!, M)) == 1$

$r < M$ & $(N-r) < M$

$M = 23$

$r = 15 \Rightarrow \underbrace{15 \cdot 14 \cdot 13 \cdot 12 \cdots 1}$

$\gcd(r!, M) = 1$

→ $r!$ & $(N-r)!$ would never have M as a factor as they are less than M

1) $N, r < M$ ✓

2) M has to prime ✓

$$Ans = \left[(N! \% M) \times (r!)^{-1} \% M \times (N-r)!^{-1} \% M \right) \% M$$

$$a\%(b+c) = a\%b + a\%c$$ ✗

prime, perm

$N = 5, \quad R = 3 \quad M = 10$

$\boxed{N_{C_0} = 1}$, $\quad N_{C_1} = N, \quad N_{C_N} = 1$

$5_{C_3} = \overset{4}{4_{C_3}} + \overset{6}{4_{C_2}}$

$4_{C_3} = \overset{1}{3_{C_3}} + \overset{3}{3_{C_2}}$

$3_{C_2} = \underset{1}{2_{C_2}} + \underset{2}{2_{C_1}}$

$4_{C_2} = \overset{3}{3_{C_2}} + \overset{3}{3_{C_1}}$

$3_{C_2} = \underset{1}{2_{C_2}} + \underset{2}{2_{C_1}}$

$N = 5$
$r = 3$
$M = 10$

A :

N \ R	0	1	2	3	4	5
0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1	
5	1	5	0	0	5	1

$N_{C_0} = 1$
$N_{C_1} = N$
$N_{C_N} = 1$

$3_{C_2} = 2_{C_1} + 2_{C_2}$
$4_{C_2} = 3_{C_2} + 3_{C_1}$

$A[i][j] = i_{C_j} \% M$

$(a+b)\%M = (a\%M + b\%M)\%M$

$N, R \le 10^3$
$P \le 10^9$

Steps

1) Define Matrix $(N+1) \times (N+1)$
2) Fill the 0^{th} col with 1 $\Rightarrow O(N)$
3) Fill the 1^{st} col with i $\Rightarrow O(N)$
3) Fill Diagonal $\Rightarrow O(N)$
4) Iterate over remaining cells and populate the values $\Rightarrow O(N^2)$

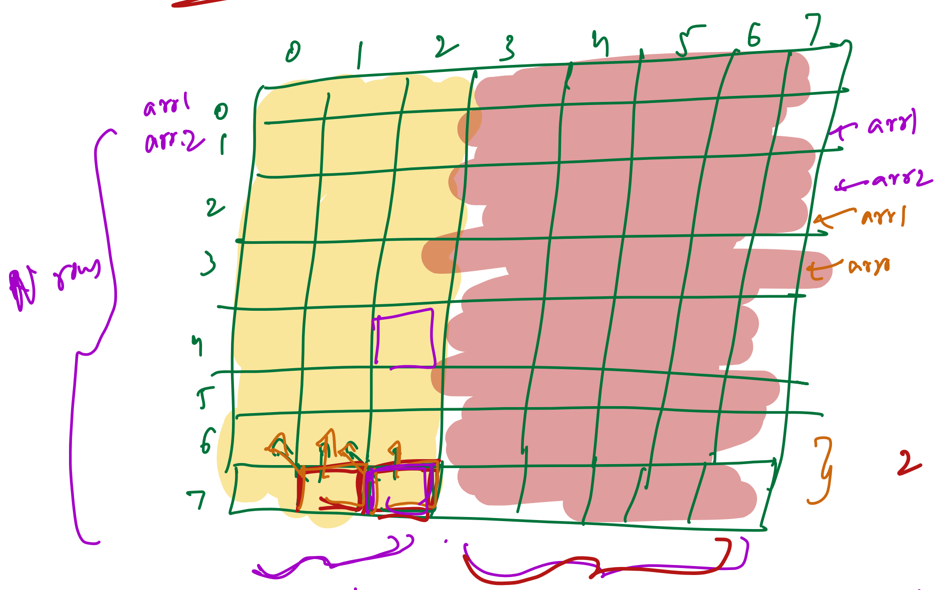
```
for(i= 0; i<=N; i++){
        A[i][0] = 1          // 0th col
        A[i][1] = i          // 1st col
        A[i][i] = 1          // Diagonal
}

for(i= 2; i<= N; i++){
    for(j = 2; j<i ; j++ ){
        A[i][j] =   (A[i-1][j] + A[i-1][j-1])%M
    }
}
```

T.C: $O(N^2)$

S.C: $O(N^2)$

Optimisation :



N = 7
R = 2

2 1-D arrays

R cols

k = 0, 1, 2

T.C : O(N·R)

S.C : O(N·R)

O(2·r) ⇒ $\boxed{O(r)}$

Nc3 Nc0 Nc1 Nc2 Nc3

Nc3

T.C of optimized sieve's Algo

```
for(i=2; i*i <= N; i++){
    if(prime[i] == True){
        for(j = i*i; j <= N; j += i){
            prime[j] = false;
        }
    }
}
```

N = 50

i = 2 ⇒ 2, 4, 6, 8 . . . . . . 50 ⇒ $\frac{N}{2} - 2$

i = 3 ⇒ 3, 6, 9 . . . . . . 48 ⇒ $\frac{N}{3} - 3$

i = 5 ⇒ 5, 10, 15, 25, 30, 35, 40, 45, 50 ⇒ $\frac{N}{5} - 5$

i =

i = $\sqrt{N}$

[1 . . . . N] ⇒ $\frac{N}{i}$

[1 . . . . i·i] ⇒ $\frac{i^2}{i} = i$

$\left[\frac{N}{i} - i\right]$

$\frac{N}{\sqrt{N}} - \sqrt{N}$

$$\#iters = \frac{N}{2} - 2 + \frac{N}{3} - 3 + \frac{N}{5} - 5 + \cdots \frac{N}{\sqrt{N}} - \sqrt{N}$$

$$= \left(\frac{N}{2} + \frac{N}{3} + \frac{N}{5} + \cdots \frac{N}{\sqrt{N}}\right) - \left(\underbrace{2+3+5+\cdots\sqrt{N}}_{2+3+4\cdots\sqrt{N}}\right)$$

$$= N\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \cdots \frac{1}{\sqrt{N}}\right) - (\;N\;)$$

?

$(1+2+3 \ldots \sqrt{N}) = \frac{\sqrt{N}(\sqrt{N}+1)}{2} \simeq N$

$\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{5} \ldots \frac{1}{N}\right) = \log(\log(N))$

$\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{5} \ldots \frac{1}{\sqrt{N}}\right) \simeq \log\log\sqrt{N}$

$O\left(\underbrace{N \cdot \log(\log(\sqrt{N}))}_{} - \underbrace{N}_{}\right)$

$= O(N \cdot \log(\log(\sqrt{N})))$

$\boxed{N \log\left(\frac{1}{2}\log_2^N\right)}$

$\log_2^{\sqrt{N}} = \log_2^{N^{1/2}} = \frac{1}{2}\log_2^N$

P.S $\Rightarrow$

Backtracking

a_1 a_2 b c_1 c_2

r = 3

5_{C_3} = $\frac{5 \times 4}{2}$ = 10

107

2

1 to 10^5

10^5

10_{C_6}

9_{C_5} 9_{C_6}

8_{C_4}

7_{C_3}

6_{C_2}

5_{C_1}

4_{C_0}