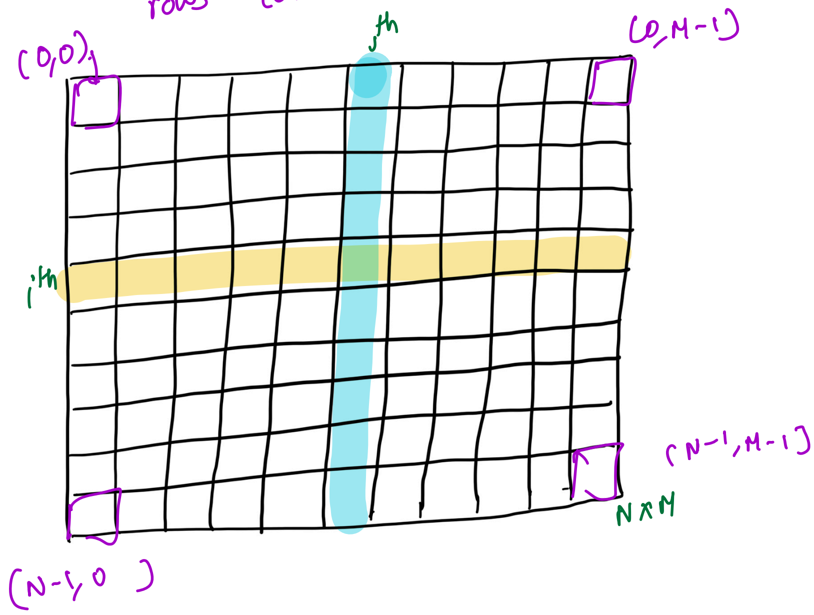
# 2D-Arrays / Matrices

int mat[4][5];

$\downarrow$ rows    $\downarrow$ cols

0 . . . . M-1




**Sub-Matrix:** A matrix obtained by removing any no. of rows or any no. of cols

$$A = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 5 & 1 & 3 & -2 & 0 \\ 1 & -4 & 0 & 3 & 2 & 5 \\ 2 & 6 & -3 & -1 & 4 & 2 \\ 3 & -3 & 1 & 3 & 7 & -3 \end{array}$$

$$\begin{bmatrix} 0 & 3 \\ -3 & -1 \end{bmatrix}$$

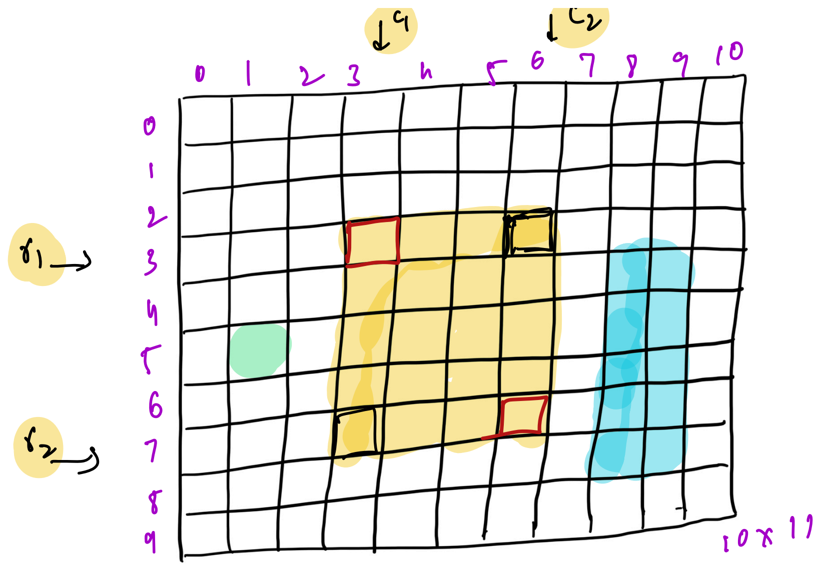
[4] ✓

Entire Matrix

How to represent a sub-matrix?

→ Top-Left

→ Bottom-Right




(3, 3) (7, 6)

(4, 8) (8, 9)

(5, 1) (5, 1)

**Question:** Submatrix Sum Queries

→ Given a 2D Matrix of dim N x M

→ Q queries, find sum of the submatrix

A:

| | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 5 | 1 | 3 | -2 | 0 |
| 1 | -4 | 0 | 3 | 2 | 5 |
| 2 | 6 | -3 | -1 | 4 | 2 |
| 3 | -3 | 1 | 3 | 7 | -3 |

N x M

Query: $(r_1, c_1)$ → Top-left, $(r_2, c_2)$ → Bottom Right

**Query:**

(0, 2) (1, 4) ⇒ 11

(2, 1) (3, 1) ⇒ -2

$Q \le 10^6$

$N, M \le 10^3$

## Brute Force

$(r_1, c_1)$ $(r_2, c_2)$

1 query: $O(N \cdot M)$

```
ans = 0;
for(i = r1;  i <= r2;  i++){
    for(j = c1;  j <= c2;  j++){
        ans += A[i][j];
    }
}
```

T.C for Q queries: $O(Q \cdot N \cdot M)$

S.C: $O(1)$

## Approach2: Prefix Sum of Rows

A =

| | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 5 | 1 | 3 | -2 | 0 |
| 1 | -4 | 0 | 3 | 2 | 5 |
| 2 | 6 | -3 | -1 | 4 | 2 |
| 3 | -3 | 1 | 3 | 7 | -3 |

$N \times M$

Pre:

| | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 5 | 6 | 9 | 7 | 7 |
| 1 | -4 | -4 | -1 | 1 | 6 |
| 2 | 6 | 3 | 2 | 6 | 8 |
| 3 | -3 | -2 | 1 | 8 | 5 |

T.C of constructing above matrix: $O(N \cdot M)$

Query: [ (0, 1) (3, 3) ]

$(r_1, c_1)$ $(r_2, c_2)$

```
for(i = r1;  i <= r2;  i++){
    sum += Pre[i][c2] - Pre[i][c1-1]
}
```

T.C per query: $O(N)$

$sum(A[i \dots j]) = pre[j] - pre[i-1]$

Total T.C for all Q queries:

$O(N \cdot M) + O(Q \cdot N)$

Approach3:

$pre[i][j]$ = sum of submatrix $(0,0) \rightarrow (i,j)$

Top-left, Bottom

A =

| | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 5 | 1 | 3 | -2 | 0 |
| 1 | -4 | 0 | 3 | 2 | 5 |
| 2 | 6 | -3 | -1 | 4 | 2 |
| 3 | -3 | 1 | 3 | 7 | -3 |

N×M

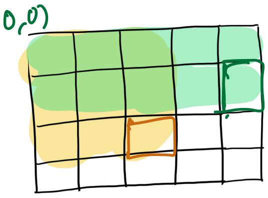
pre =

| | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 5 | 6 | 9 | 7 | 7 |
| 1 | 1 | 2 | 8 | 8 | 13 |
| 2 | | | | | |
| 3 | | | | | |

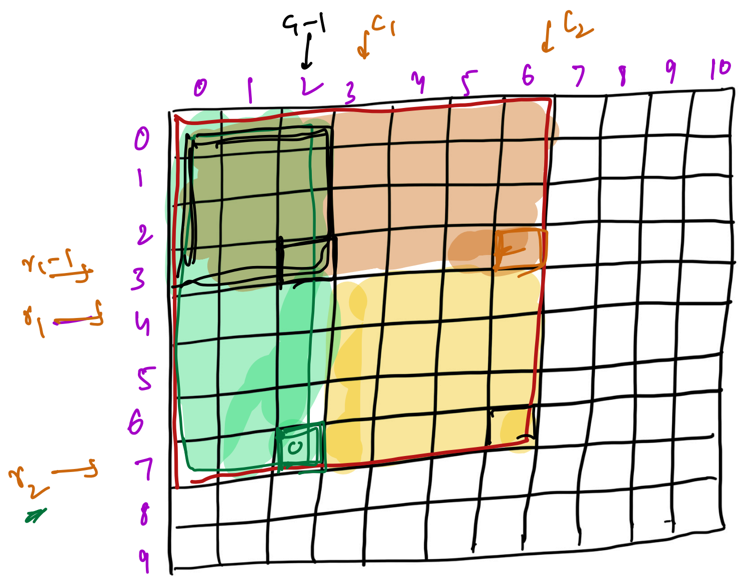
N×M ✓

$(0,0) \rightarrow (1,0)$

0,0



$(r_1, c_1)$ $(r_2, c_2)$




RED: pre[$r_2$][$c_2$]

GREEN: pre[$r_2$][$c_1 - 1$] ✓

ORANGE: pre[$r_1 - 1$][$c_2$] ✓

BLACK: pre[$r_1 - 1$][$c_1 - 1$]

YELLOW = RED − GREEN − ORANGE + BLAC

sum[ [$r_1 c_1$], [$r_2, c_2$] ] =

pre[$r_2$][$c_2$] − pre[$r_2$][$c_1 - 1$] − pre[$r_1 - 1$][$c_2$]

\+ pre[$r_1 - 1$][$c_1 - 1$]

T.C per query: O(1) ✓

Total T.C: T.C of constructing Prefix Sum Matrix

\+

O(Q)

# How to construct prefix sum matrix

Way1 :

1) Compute row-wise prefix sum
2) Compute col-wise prefix sum

| | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 5 | 1 | 3 | -2 | 0 |
| 1 | -4 | 0 | 3 | 2 | 5 |
| 2 | 6 | -3 | -1 | 4 | 2 |
| 3 | -3 | 1 | 3 | 7 | -3 |

$\xrightarrow{\text{step 1}}$ O(N·M)

Pre :

| | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 5 | 6 | 9 | 7 | 7 |
| 1 | -4 | -4 | -1 | 1 | 6 |
| 2 | 6 | 3 | 2 | 6 | 8 |
| 3 | -3 | -2 | 1 | 8 | 5 |

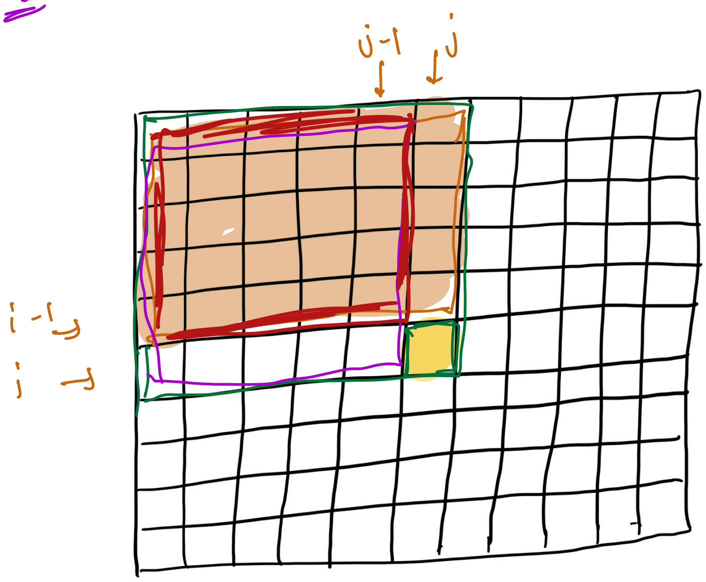
$\downarrow$ step2 O(N·M)

| | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 5 | 6 | 9 | 7 | 7 |
| 1 | 1 | 2 | 8 | 8 | 13 |
| 2 | 7 | 5 | 10 | 14 | 21 |
| 3 | 4 | 3 | 11 | 22 | 26 |

pre[i][j] = sum of submatrix [0,0] to [i,j]

Total T.C : O(N·M)

way2 :




$pre[i][j] =$

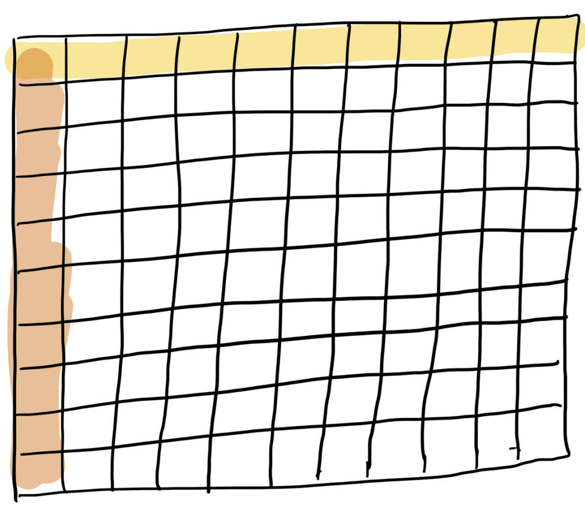
ORANGE : $pre[i-1][j]$

PURPLE : $pre[i][j-1]$

RED : $pre[i-1][j-1]$

$$pre[i][j] = pre[i-1][j] + pre[i][j-1] - pre[i-1][j-1] + A[i][j]$$

pre:



$O(N+M)$ ← // Fill the 0th row and 0th col

```
for(i=1; i<N; i++){
    for(j=1; j<M; j++){
        pre[i][j] = pre[i-1][j] + pre[i][j-1] - pre[i-1][j-1]
                    + A[i][j]
    }
}
```

$O(N \cdot M)$

T.C: $O(N \cdot M)$

T.C: $\boxed{O(N \cdot M + Q)}$

$(a > b)$

$O(a+b) \Rightarrow \boxed{O(\max(a,b))}$

$O(a+a) \Rightarrow O(\max(a,b))$

$N = 10,\ M = 10 \quad Q = 10^4$

$O(10^4)$ T.C

$N = 10^3,\ M = 10^3,\ Q = 10^6$

$O(N \cdot M + Q)$

$O(\max(N \cdot M, Q))$

## Question: Sum of all submatrices

$N \le 10^3$
$M \le 10^3$

$$\begin{bmatrix} 9 & 6 \\ 5 & 4 \end{bmatrix}_{N \times M}$$

$[9]$ → 9 $\quad$ $[6]$ → 6 $\quad$ $[5]$ → 5 $\quad$ $[4]$ → 4

$[9 \quad 6]$ → 15 $\quad$ $[5 \quad 4]$ → 9 $\quad$ $\begin{bmatrix} 9 & 6 \\ 5 & 4 \end{bmatrix}$ → 24 $\quad$ $\begin{bmatrix} 9 \\ 5 \end{bmatrix}$ → 14 $\quad$ $\begin{bmatrix} 6 \\ 4 \end{bmatrix}$ → 10

$= \boxed{96}$

## Brute Force

Consider all submatrices, find the sum of each of it.

$N \times M$

$(r_1, c_1) \quad (r_2, c_2)$

$0 \le r_1 \le r_2 \le N-1$

$0 \le c_1 \le c_2 \le M-1$

# submatrices => No. of ways of choosing $(r_1, r_2, c_1, c_2)$

No. of ways of choosing $r_1, r_2$

$r_1 = 0$ , $r_2 = 0, 1, 2 \ldots N-1$ $\Rightarrow N$

$r_1 = 1$ , $r_2 = 1, 2, 3 \ldots N-1$ $\Rightarrow N-1$

$r_1 = 2$ $\Rightarrow N-2$

$\vdots$

$r_1 = N-1$ , $r_2 = N-1$ $\Rightarrow 1$

$$1 + 2 + 3 + 4 \ldots N = \boxed{\frac{N(N+1)}{2}}$$

No. of ways choosing $c_1, c_2$

$c_1 = 0$ $c_2 = 0, 1, 2 \ldots M-1$ $\Rightarrow M$

$c_1 = 1$ $c_2 = 1, 2, 3 \ldots M-1$ $\Rightarrow M-1$

$\vdots$

$c_1 = M-1$ $c_2 = M-1$ $\Rightarrow 1$

$$\Rightarrow \frac{M(M+1)}{2}$$

No. of ways of choosing $r_1, r_2, c_1, c_2 =$

$$\left[\frac{N(N+1)}{2} \times \frac{M(M+1)}{2}\right]$$

$$= O(N^2 \cdot M^2)$$

Brute Force:

# total Submatrices = $O(N^2 \cdot M^2)$

T.C for sum of each submatrix : $O(N \cdot M)$

Total T.C : $O(N^3 \cdot M^3)$

Prefix Sum :

# total Submatrices = $O(N^2 \cdot M^2)$

T.C for sum of each submatrix : $O(1)$

Total T.C : $O(N^2 \cdot M^2)$

S.C : $O(N \cdot M)$

## Efficient Approach:

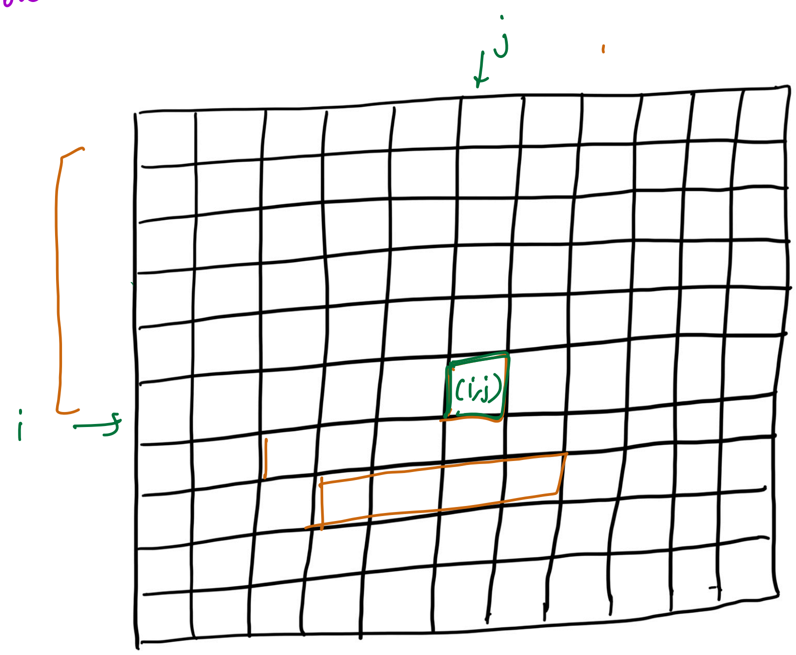
Contribution Technique:

For each element $A[i][j]$ ⇒ Find out no. of submatrices in which it's part of ($X$)

$[\ A[i][j] \cdot X\ ]$

Million Dollar Question:

How to find $X$?



$c_1 \le j$

$c_2 \ge j$

$r_1 \ge r_2$

$(r_1, c_1)$ $(r_2, c_2)$

$r_1 \le i$, $r_2 \ge i$

$[0, 1, 2, \ldots, i]$ ⇓ $(i+1)$

$[i, i+1, \ldots, N-1]$ ⇓ $(N-i)$

$c_1 \leq j$

$[0, 1, 2 \cdots j]$

$\Downarrow$

$(j+1)$

$c_2 \geq j$

$[j, j+1, j+2 \cdots M-1]$

$\Downarrow$

$(M-j)$

#submatrices in which $(i, j) \Rightarrow$

$(i+1) \times (N-i) \times (j+1)(M-j)$

```
sum = 0;
for(i = 0; i < N; i++) {
    for(j = 0; j < M; j++) {
        freq = (i+1).(N-i).(j+1).(M-j)  -j0(1)
        sum += freq x A[i][j]
    }
}
return sum;
```

T.C: $O(N \cdot M)$

S.C: $O(1)$

Quiz:

$\begin{bmatrix} & & \end{bmatrix}_{4 \times 5}$ $(1, 2)$

$N = 4, \quad M = 5, \quad i = 1, \quad j = 2$

$= [(i+1) \times (N-i) \times (j+1) \times (M-j)]$

$= 2 \times 3 \times 3 \times 3 = 54$

$\boxed{7 \text{ mins}}$

**Question:** Maximum Sum Submatrix

$\boxed{1 \le N, M \le 10^2}$

| | | | | |
|---|---|---|---|---|
| 5 | 1 | 3 | -2 | 0 |
| -4 | 0 | 3 | 2 | 5 |
| 6 | -3 | -1 | 4 | 2 |
| -3 | 1 | 3 | 7 | -3 |

Ans =

sum = 26

**Brute Force**

consider all submatrices

# submatrices: $O(N^2 \cdot M^2)$

T.C for sum: $O(N \cdot M)$

T.C: $O(N^3 \cdot M^3)$

S.C: $O(1)$

**Prefix Sum Matrix**

T.C: $O(N^2 \cdot M^2)$

S.C: $O(N \cdot M)$

$\Rightarrow$ $N, M \le 10^2$

$10^2 \cdot 10^2 \cdot 10^2 \cdot 10^2 \le 10^8$

## Efficient Solution

1D Problem: Kadane's Algo

| A = | 3 | -2 | -3 | 4 | -1 | -2 | 1 | 5 | -3 |
|---|---|---|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |

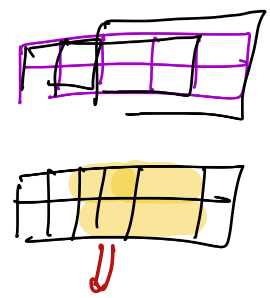
sum = ~~0~~ ~~3~~ ~~1~~ ~~-2~~ ~~0~~ ~~4~~ ~~3~~ ~~1~~ ~~2~~ ~~7~~ 4

maxSum = ~~-INF~~ ~~3~~ ~~4~~ 7

$\boxed{\text{T.C: } O(N)}$

Hint: Can we find max sum submatrix between any 2 rows

| | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | -5 | 1 | 3 | -2 | 0 |
| 1 | -4 | 0 | 3 | 2 | 5 |
| 2 | -6 | -3 | -1 | 4 | 2 |
| 3 | -3 | 1 | 3 | 7 | -3 |



| | | | | | |
|---|---|---|---|---|---|
| R0: | -5 | 1 | 3 | -2 | 0 |
| R1: | -4 | 0 | 3 | 2 | 5 |
| | -9 | 1 | 6 | 0 | 5 |

$(R0, R1, R) \Rightarrow \quad -15 \quad -2 \quad 5 \quad 4 \quad 7$

$(R2, R3) \Rightarrow \quad -9 \quad -2 \quad 2 \quad 11 \quad -1$

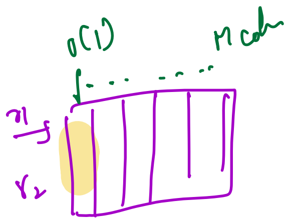
$R_1 = 0, \quad R_2 = 0, 1, 2, 3 \ldots N-1 \Rightarrow N$

$R_1 = 1, \quad R_2 = 1, 2, 3 \ldots N-1 \Rightarrow N-1$

$R_1 = 2, \quad R_2 = 2, 3 \ldots N-1 \Rightarrow N-2$

$\vdots$

$R_1 = N-1, \quad R_2 = N-1 \Rightarrow 1$




# pairs of rows $= \boxed{\frac{N(N+1)}{2}}$ }

For each pair of row

=> 1) Get the 1-D array by summing up the columns : $O(N \cdot M)$

$r_1 = 0, \quad r_2 = N-1$

2) Apply Kadane : $O(M)$

Total T.C : $\frac{N(N+1)}{2}\left[O(N \cdot M) + O(M)\right]$

$= O(N^3 \cdot M) + O(N^2 \cdot M)$

$= \boxed{O(N^3 \cdot M)}$ $\{O(N^2 \cdot M^2)\}$

## Optimization 1:

Construct column Prefix Sum: $O(N \cdot M)$

$(r_1, r_2)$ [

Step 1: $O(M)$ → $O(M) + O(M)$

Step 2: $O(M)$

$O(M) + O(M)$

Total T.C : $O(N \cdot M) + O(N^2 \times (M + M))$

$= O(N \cdot M) + O(N^2 \cdot M)$

T.C : $O(N^2 \cdot M)$

$10^4 + 10^6 \approx 10^6$

S.C : $O(N \cdot M)$

## Optimization 2:

| | | | | | |
|---|---|---|---|---|---|
| 0 | 5 | 1 | 3 | -2 | 0 |
| 1 | -4 | 0 | 3 | 2 | 5 |
| 2 | 6 | -3 | -1 | 4 | 2 |
| 3 | -3 | 1 | 3 | 7 | -3 |

Ans = [1 | 1 | 6 | 0 | 5]

[7 | -2 | 5 | 4 | 7]

M

$r_1 = 1$

[-4 | 0 | 3 | 2 | 5]

[2 | -3 | 2 | 6 | 7]

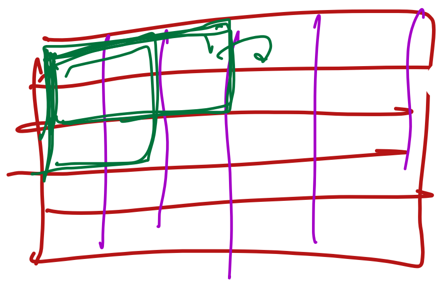
T.C : $O(N^2 \cdot M)$

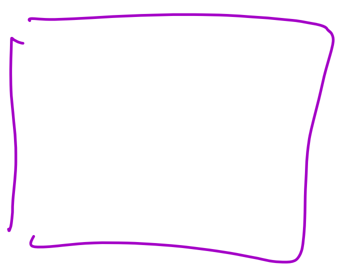
S.C : $O(M)$

$r_1 = 2$, $r_2 = 2, 3, 4$.

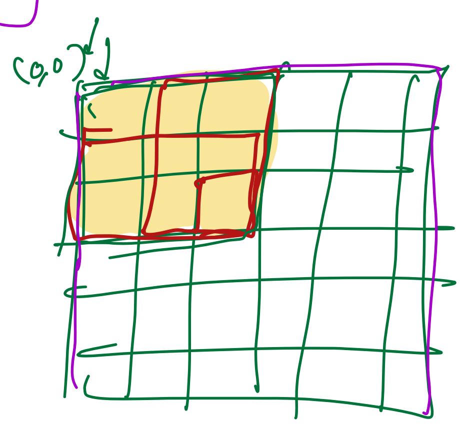
$r_1 = 0$

```
ans = -INF;
for(r1 = 0; r1 < N; r1++){
    => arr = [0, 0, 0, 0, 0]_M
    for(r2 = r1; r2 < N; r2++){
        // considering r1 ..... r2
        for(j = 0; j < M; j++){
            arr[j] += A[r2][j];
        }
        ans = max(ans, [kadane(arr)])
                       (r1, r2)
    }
}
```





if(pref[i][j] < 0)



Total # submatrices = $O(N^2 \cdot M^2)$

$O(N \cdot M)$

$(N^2 M^2 - N \cdot M)$