GCD: Greatest Common Divisor
or
HCF: Highest Common factor

gcd(15, 25) = 5

15: 1, 3, 5, 15
25: 1, 5, 25

gcd(12, 15) = 3

12: 1, 2, 3, 4, 6, 12
15: 1, 3, 5, 15

gcd(7, 9) = 1

7: 1, 7
9: 1, 9

gcd(7, 14) = 7

14 → Multiple of that prime

gcd(14, 9) = 1

If gcd(a, b) = 1, then a, b are co-prime

$\gcd(5, -15) = 5$

5	-15
1	-15
5	-5
	-3
	-1
	1
	3
	5
	15

$\gcd(-16, -24) = 8$

-16	-24
-16	-24
-8	-12
-4	-8
-2	-6
-1	-4
1	-3
2	-2
4	-1
8	1
16	2
	3
	4
	6
	8
	12
	24

$\gcd(a,b) = \gcd(|a|, |b|)$

$\gcd(0, 8) = 8$

0	8
1	1
2	2
3	4
4	8
5	
6	
7	
8	
⋮	
∞	

when is x a factor of 8?

$8 \% x == 0$

$x = 1, 2, 4, 8$

$0 \% x == 0$

$x = 1, 2, 3,$

gcd (5, 0) = 5

5   1, 2, 3, 4, 5, 6, ⋮, ∞

gcd (0, a) = a

Question: Find gcd (Numbers are positive)

Input : a, b

Min value : 1

Max value : min(a,b)

```
int ans;
for(i= 1; i ≤ min(a,b) ; i++){
    if( a%i==0 && b%i ==0){
        ans = i
    }
}
```

T.C: O(min (a, b))

```
if (a == 0) return b;   if(b==0) return a;
ans = max(a,b)
for(i= min(a,b) ; i ≥ 1 ; i--){
    if( a%i==0 && b%i ==0){
        return i;
    }
}
```

T.C: O(min(a,b))

a = 5
b = 0

(5, 7)
i = 5, 4, 3, 2, 1

LCM : Least Common Multiple

lcm(100, 75) = 300

100	75
200	150
300	225
400	300
	375

gcd(100, 75) = 25

$$a \times b = gcd(a, b) \times lcm(a, b)$$

$100 \times 75 = 25 \times 300$

$7500 = 7500$

$a \mid b$ → divides

a divides b

$25 \mid 100 \Rightarrow$ True $\Rightarrow 100/25 = 4$ $\quad 100 \% 25 = 0$

$10 \mid 100 \Rightarrow$ True. $\Rightarrow 100/10 =$

$6 \mid 3 \Rightarrow$ False $\Rightarrow 3/6 = 0.5$

$25 \mid 75 \Rightarrow$ True $\quad 75/25 = 3$ $\quad 75 \% 25 = 0$

a/b is an integer, then $b \mid a$

Properties

1) $\gcd(a,b) = \gcd(|a|, |b|)$

2) $\gcd(a,b) = \gcd(b,a)$ [Commutative property]

3) $\gcd(a,0) = a$

4) $\gcd(a,1) = 1$

5) $\gcd(a,b,c) = \gcd(\gcd(a,b), c)$ [Associative Property]
$\gcd(\gcd(a,c), b)$
$\gcd(\gcd(b,c), a)$

6) $\gcd(a,b) = \gcd(a, b-a)$ where $b \geq a$

$\gcd(4,5) = \gcd(4, 1) = 1$

$\gcd(3,7) = \gcd(3,4) = \gcd(3,1)$
$= \gcd(1,3) = \gcd(1, 3-1)$
$= \gcd(1,2)$
$= \gcd(1, 2-1)$
$\gcd(1,1)$
$\gcd(1,0)$

$\gcd(5,10) = \gcd(5, 10-5)$
$= \gcd(5,5) = \gcd(5,0) = 5$

$\boxed{\gcd(7,9)} \Rightarrow \boxed{\gcd(7,2)} =$
$\gcd(2,7) = \gcd(2,5) = \gcd(2,3) = \gcd(2,1)$
$= \gcd(1,2)$
$\gcd(1,1)$
$\gcd(1,0)$
1

$\gcd(12, 24) = \gcd(12, 12) \Rightarrow \gcd(12, 0) = \boxed{12}$

Proof:

Step1 $\quad \gcd(a, b) = \boxed{d}$ ✓ $\quad [b \geq a]$

$\boxed{d \mid a}$ - (6) and $d \mid b$

$\Rightarrow a = k_1 d$ - (1) $\qquad b = k_2 d$ - (2) $\quad [k_2 \geq k_1]$

$b - a = k_2 d - k_1 d = (k_2 - k_1) d$

$\boxed{d \mid b - a}$ - (7)

Step2: $\quad \gcd(a, b - a) = m$ ✓

$\boxed{m \mid a}$ - (4) & $m \mid b - a$

$a = t_1 m$ - (1) & $b - a = t_2 m$ - (2)

$\cancel{a} + b - \cancel{a} = t_1 m + t_2 m$

$b = (t_1 + t_2) m$ ✓

$\boxed{m \mid b}$ - (5)

From Eq (4), (5) $\Rightarrow$ $\boxed{m}$ is a factor of a and b } factor of $a, b = d$

greatest common factor

$\boxed{m \leq d}$

From Eq (6), (7) $\Rightarrow$ d is a factor of $a, b - a$

GCD of $a, b - a = m$ $\qquad d \leq m$

$m \leq d \rightarrow$ 1

$d \leq m \rightarrow$ 2

$\boxed{d == m}$

$gcd(small, big) = gcd(small, big - small)$

1) $gcd(a,b) = gcd(a, b-a) = gcd(a, b-a-a)$
$\qquad = gcd(a, b-a-a-a)$

$gcd(a, b-a-a \dots -a) = $

until this is less than a

$b \% a$

$gcd(a,b) = gcd(a, b\%a)$ ★ $b \geq a$

$gcd(3,20) = gcd(3,17) = gcd(3,14) = gcd(3,11) = gcd(3,8)$
$= gcd(3,5) = gcd(3,2) = gcd(2,1) = gcd(1,1) = gcd(1,0) = 1$

$gcd(3,20) = gcd(3, 20\%3)$
$= gcd(3,2) = gcd(2, 3\%2)$
$= gcd(2,1) = gcd(1, 2\%1)$
$= gcd(1,0) = 1$

$$\gcd(big, small) = \gcd(\overset{a}{small}, \overset{b}{big \% small})$$

$[0, small-1]$

$x > y$

Version 1:

4 lines

```
int gcd(a, b) {
    if (a == 0) return b;
    if (b == 0) return a;

    if (a >= b) return gcd(b, a%b);
    else return gcd(a, b%a);        // b > a
}
```

Version 2:

Always assume $a \geq b$

```
int gcd(int a, int b) {
    if (b == 0) return a;
    return gcd(b, a%b);
}
```

$\gcd(\overset{a}{6}, 30) \Rightarrow \gcd(\overset{a}{30}, \overset{b}{6}) \Rightarrow \gcd(6, 0) \Rightarrow \boxed{6}$

Euclidean Algo

Time complexity $(a \geq b)$

```
int gcd(a, b) {
    if(b == 0) return a;
    return gcd(b, a%b);
}
```

$a\%b = [0, b-1]$

Claim: $a\%b < a/2$ when $a \geq b$

$[a \geq b]$

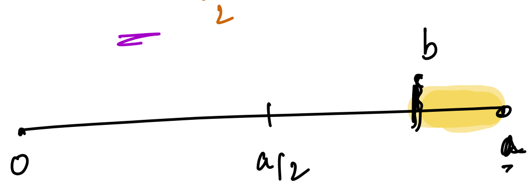
$a\%b$

- case 1: $\boxed{b < \frac{a}{2}}$, $a\%b < b < \frac{a}{2}$, $\therefore$ $\boxed{a\%b < \frac{a}{2}}$
- case 2: $\boxed{b = \frac{a}{2}}$, $a = 2b$, $2b\%b = 0$, $a\%b = 0$
- case 3: $\boxed{b > \frac{a}{2}}$

Cases: $b > \frac{a}{2}$

$gcd(a, b) = gcd(b, a-b)$

$b = [\frac{a}{2} \ldots a]$



$a - b < \frac{a}{2} < b$

$\Downarrow$

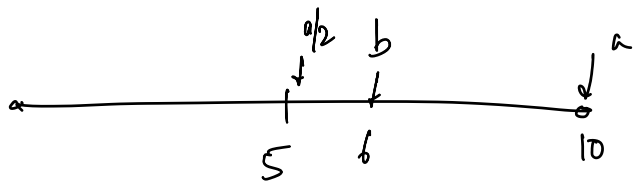
$a\%b$

$a - b = a\%b$

$$a\%b = (a - b - b - b \cdots\cdots) < b$$

$$a\%b < \frac{a}{2}$$

$$a\%b < \frac{a}{2}$$ → Always $[a \geq b]$



$$\frac{a}{2} < b$$

$$a\%b = a - b = \boxed{4} < \frac{10}{2}$$

$a > b$

```
int gcd(a, b) {
    if(b == 0) return a;
    return gcd(b, a%b);
}
```

	a	b
Step 0:	a_0	b_0
Step 1:	$a_1(b_0)$	$b_1(a_0 \% b_0) < \frac{a_0}{2}$
Step 2:	$a_2(b_1) < \frac{a_0}{2}$	$b_2(a_1 \% b_1)$
Step 3:	$a_3(b_2)$	$b_3(a_2 \% b_2) < \frac{a_2}{2}$
Step 4:	$\boxed{a_4(b_3)} < \frac{a_2}{2} < \boxed{\frac{a_0}{4}}$	$b_4(a_3 \% b_3)$
Step 5:	$a_5(b_4)$	$b_5(a_4 \% b_4) < \frac{a_4}{2}$

Step6 : $a_6(b_5) < \frac{a_4}{2} < \frac{a_0}{8}$ $b_6(a_5 \% b_5)$

$a \rightarrow \frac{a}{2} \rightarrow \frac{a}{4} \rightarrow \frac{a}{8} + \ldots \ldots 1$

$\log a$

T.C of $gcd(a,b) = O(\log(\max(a,b)))$

S.C: $O(\log(\max(a,b)))$

Recursion Stack

7mins

Question: Find GCD of factorials of the given elements

ans % (10^9+7) → Prime

$x \% 7$

$y \% 7$

A : [4 3 8 6]

$\gcd(4!, 3!, 8!, 6!) \% 10^9+7$

$4! = 24$, $3! = 6$, $8! = 40{,}320$, $6! = 720$

$1000 \times 999 = 10^6$

$N = 1000$

(10^{18})

ans % 10^9+7

$4! : 1 \cdot 2 \cdot 3 \cdot 4$

Min element → $3!$: $1 \cdot 2 \cdot 3$

$8! : 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8$

$6! : 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6$

$1 \cdot 2 \cdot 3 = \boxed{6}$

Ans : [Factorial (min Element)] % 10^9+7

Question: Given an array, check if there exists a subsequence such that gcd(subsequence) = 1

A = [6, 10, 15, 25, 24, 18]

#subseq = $\boxed{2^N}$

gcd(25, 18) = 1
gcd(25, 24) = 1

A = [3 6 9 4]

[3, 4] [3, 9, 4] (9, 4)

gcd(a, b, c) = gcd(gcd(a, b), c)

a, b, c, d

gcd(a₁, a₂, a₃, a₄) = 1

gcd(a₁, a₂, a₃, a₄, x) = 1

gcd(a₁, a₂, a₃, a₄, x₁, x₂, x₃ . . . . x₁₀) = 1

→ take GCD of entire array

If there is atleast one subsequence with GCD as 1, then gcd of entire array = 1

(a, b) : $\log(\max(a,b))$

A = 3 4 5 6 7 8

g

T.C : $O(N \times \log(\max(A)))$

S.C : $O(1)$ Excluding space occupied by gcd

Question: Given an array of N elements, delete exactly one element from the array such that gcd(array) is maximized

A: 9 18 49 12 30

gcd (a,b,c,d) = x
can we find gcd (a,b,c) using x & d ?

sum (a,b,c,d) = x
sum (a,b,c) = x − d

gcd(10, 50, 200, 75) = 5
gcd (10, 50, 200) =

$N = 10^5$

Brute Force:

Consider deleting every element

$$\underbrace{9 \quad 18}_{x} \quad 49 \quad \underbrace{12 \quad 30 \quad 60 \quad 50}_{y}$$

A = 9 18 49 12 30 60 50

gcd (x, y)

T.C: $N \times O(N \cdot \log(MAX))$

$\therefore O(N^2 \cdot \log(MAX))$

Efficient Approach :

- ✓ Prefix gcd array
- ✓ Suffix gcd array

	0	1	2	3	4
A =	9	18	49	12	30
PrefixGCD =	9	9	1	1	1
suffixGCD =	1	1	1	6	30

ans = ~~1~~ 3

gcd [prefixGCD[i-1] , suffixGCD[i+1]]

Steps

1) Construct prefix GCD $O(N \cdot \log(MAX))$
2) Construct suffix GCD $O(N \cdot \log(MAX))$
3) $O(N \cdot \log(MAX))$

$\boxed{T.C: \quad O(N \cdot \log(MAX))}$

A = [64 32 16 8 4 2]

gcd : [64 32 16 8 4 2]

A = [64 2 8 4 16 32]

[64 2 2 2 2 2] ⟹ N log(MAX)

A = [49, 2, 4, 8]

= [49, 1, 1, 1]

% Prime

$\boxed{x \% 7}$

ans % 10

10

20

30

4

1000! ⟹

ans % 2 ⟹ { 0, 1 }

ans % Big Number

{ < 10^9 }

1000! = (10^{50})

$10^{50} \% 10^9+7$

[0 . . . . 10^9+6]

Multiply of 2

Multiple of 3

Multiply

$\boxed{\text{Joining}}$ 2

A = [6, 9]

log (2 * sum)

$\boxed{\log(a+b) \neq \log(a) + \log(b)}$

5 6 11 7 3 10

gcd (max (5, 6) + gcd (max (6, 11)) + - - -

log (max (5, 6)) log (max (6, 11)), 1, - - -

log (6) + Log (11) + log (n) $\neq$

log (6 + 11 + - - -)

o (max (a, b)) = o (a + b)

log (sum of array)

$(a \cdot b) \% b =$