Assignment 1

1. Linear Regression with Gradient Descent:
   Algorithm
   Initially I read the train and test data and stored them in a NumPy array, then I normalized the random variable X for train and test data to reflect mean = 0.0 and standard deviation to 1.0.
   Using the following code : **X=((X-X.mean())/X.std())**
   Now I created a function named
   GradientDescent(X, Y, thetas=np.zeros(2), ita=0.001, JTheta=0.0):
   Here X and Y are the training data,
   thetas are all initialized to 0.0
   This function return the **thetas** after the model has been completely trained

   Inside the function I started with an infinite loop. In each iteration, I calculate the hypothesis **h(θ) = np.matmul(thetas,X),** then cost using the following equation **$J(\theta) = (1/2*m)(h(\theta)-Y)^2$**
   After this I check for the convergence criteria **$|J(\theta)_{i+1} - J(\theta)_i| < 0.00000001$,** when it turns true, simply break from the infinite loop and return the thetas current value.
   If the convergence criteria is not met then we update the thetas with the help of the following equation

$$\theta_j = \theta_j - \alpha \frac{\partial}{\partial \theta_j} J(\theta)$$

Now,

$$\frac{\partial}{\partial \theta} J_\theta = \frac{\partial}{\partial \theta} \frac{1}{2m} \sum_{i=1}^{m} \left[h_\theta(x_i) - y_i \right]^2$$

$$\frac{\partial}{\partial \theta} J_\theta = \frac{1}{m} \sum_{i=1}^{m} \left(h_\theta(x_i) - y_i \right) \cdot \frac{\partial}{\partial \theta_j} \left(\theta x_i - y_i \right)$$

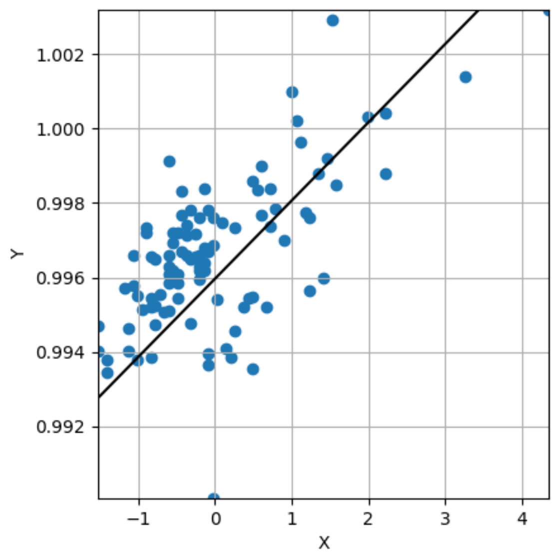
$$\frac{\partial}{\partial \theta} J_\theta = \frac{1}{m} \sum_{i=1}^{m} \left[\left(h_\theta(x_i) - y \right) x_i \right]$$

Therefore,

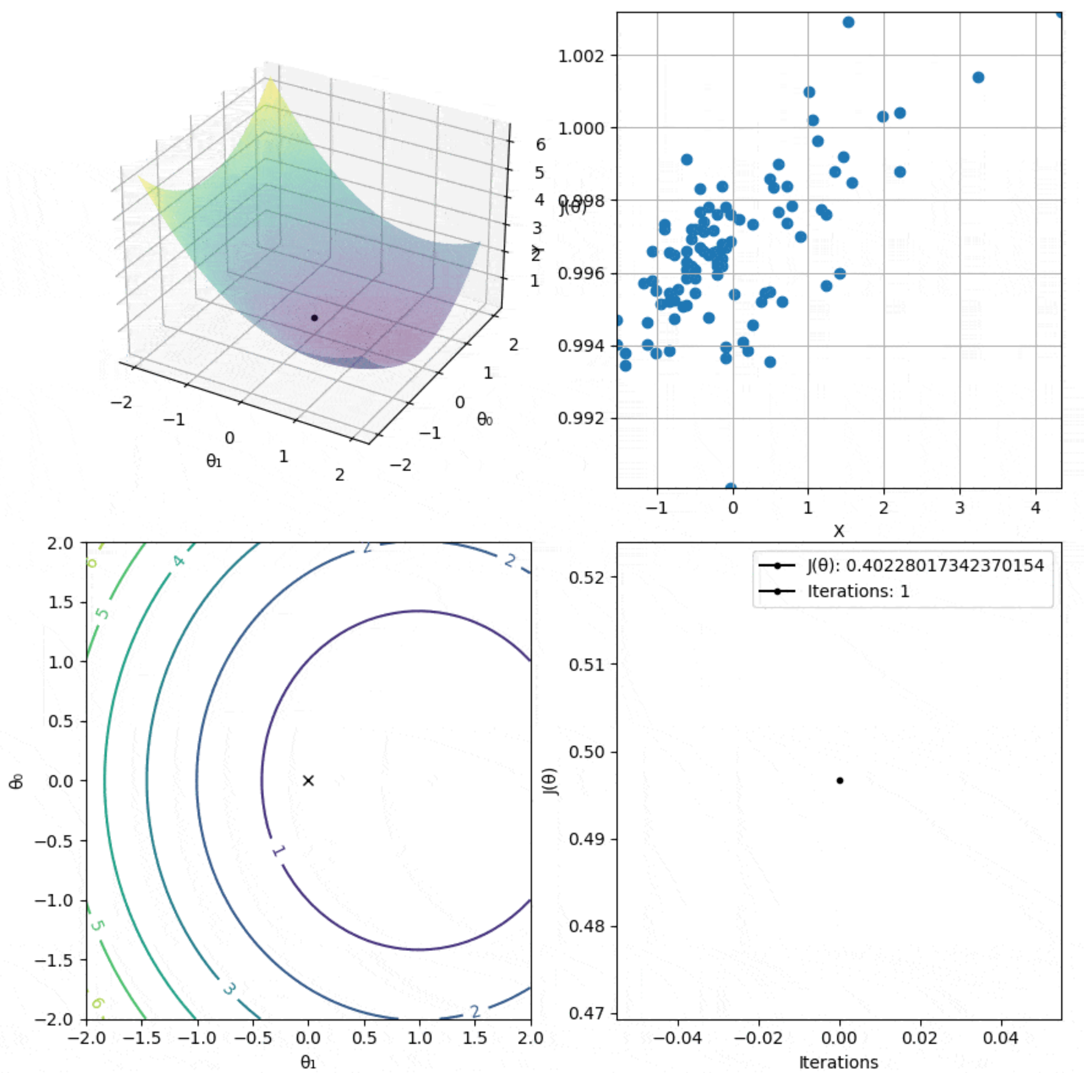
$$\theta_j := \theta_j - \frac{\alpha}{m} \sum_{i=1}^{m} \left[\left(h_\theta(x_i) - y_i \right) x_i \right]$$

1.a) Learning rate which I chose for gradient descent: **0.1**
Convergence criteria : **$|J(\theta)_{i+1} - J(\theta)_i| < 0.00000001$**
Final set of parameters : **$\theta_0 = 0.99596584$, $\theta_1 = 0.00210104$**
Final cost function value : **1.673696398896178e-06**

1.b)

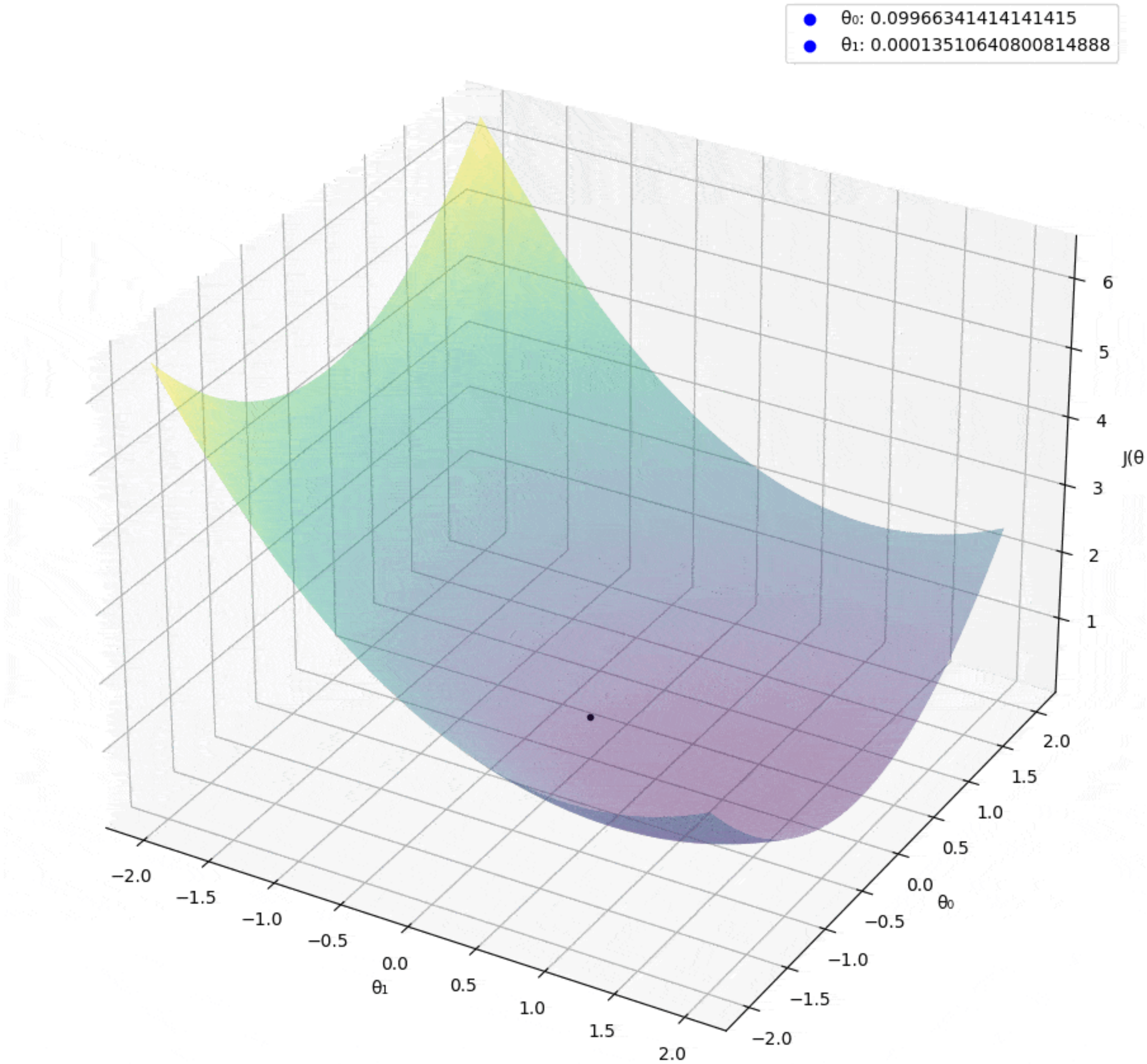


Y
X

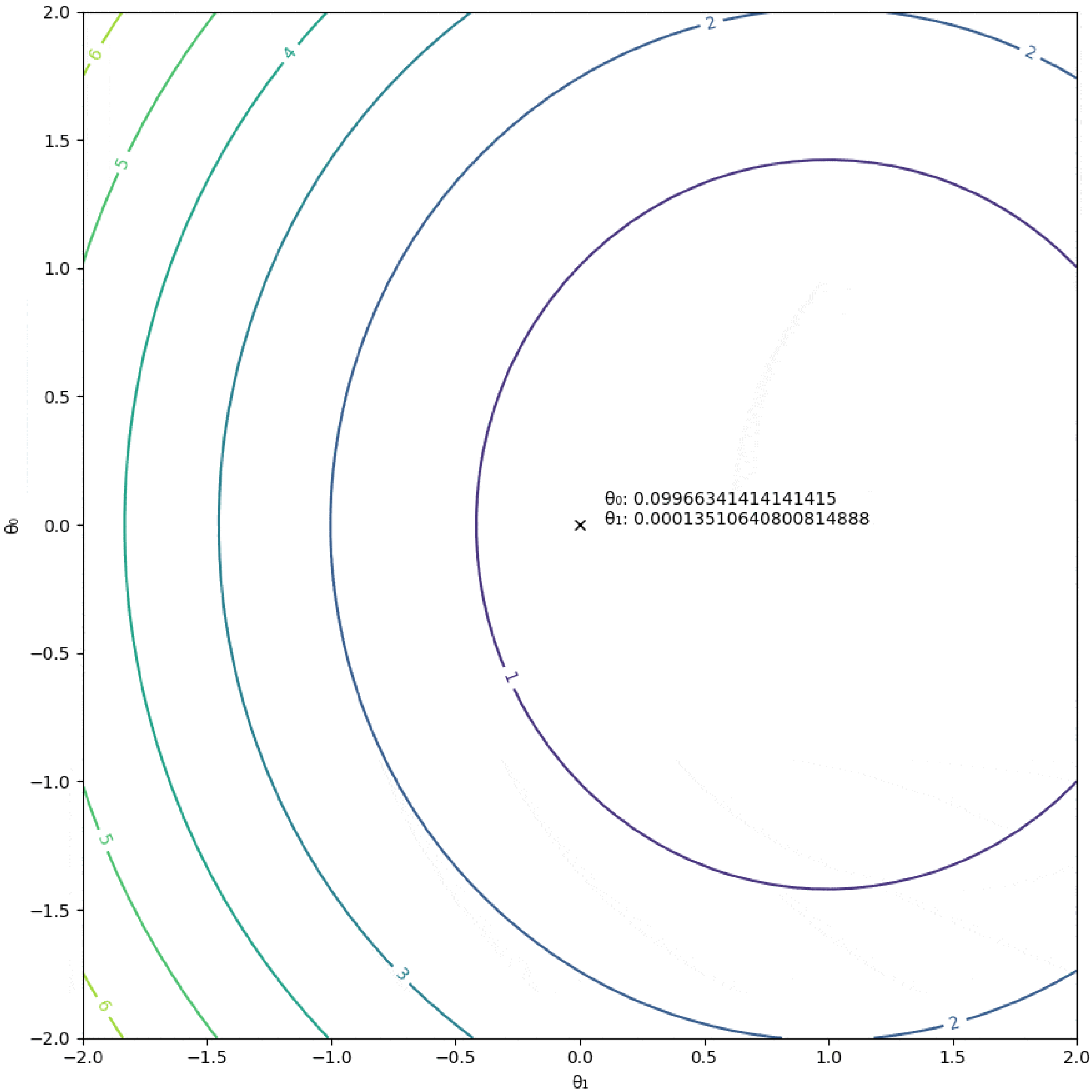


J(θ)
θ₁
θ₀
Y
X
J(θ): 0.40228017342370154
Iterations: 1
Iterations

1.c)


θ₀: 0.09966341414141415
θ₁: 0.0001351064080081488
J(θ
θ₀
θ₁
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0
1
2
3
4
5
6

1.d)


θ₀: 0.09966341414141415
θ₁: 0.00013510640800814888
θ₀
θ₁
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
1
2
3
4
5
6

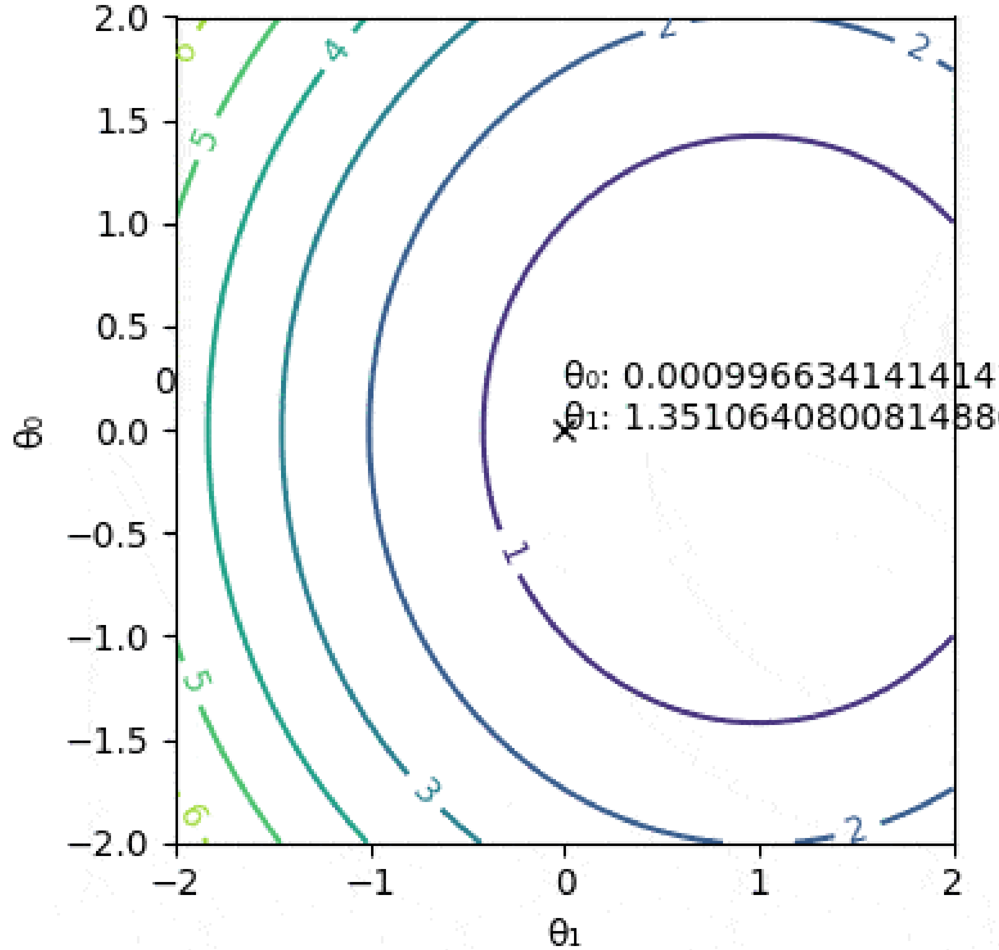
1.e) while learning data for different values for **η = [0.001, 0.025, 0.1]**, I observed that the they all safely converge however as expected lower η value required much more time and iterations.

For **η=0.001**

Executing time = 5.270089626312256 second

θ_0 =0.99346035, θ_1 = 0.00134242

Iterations = 5751

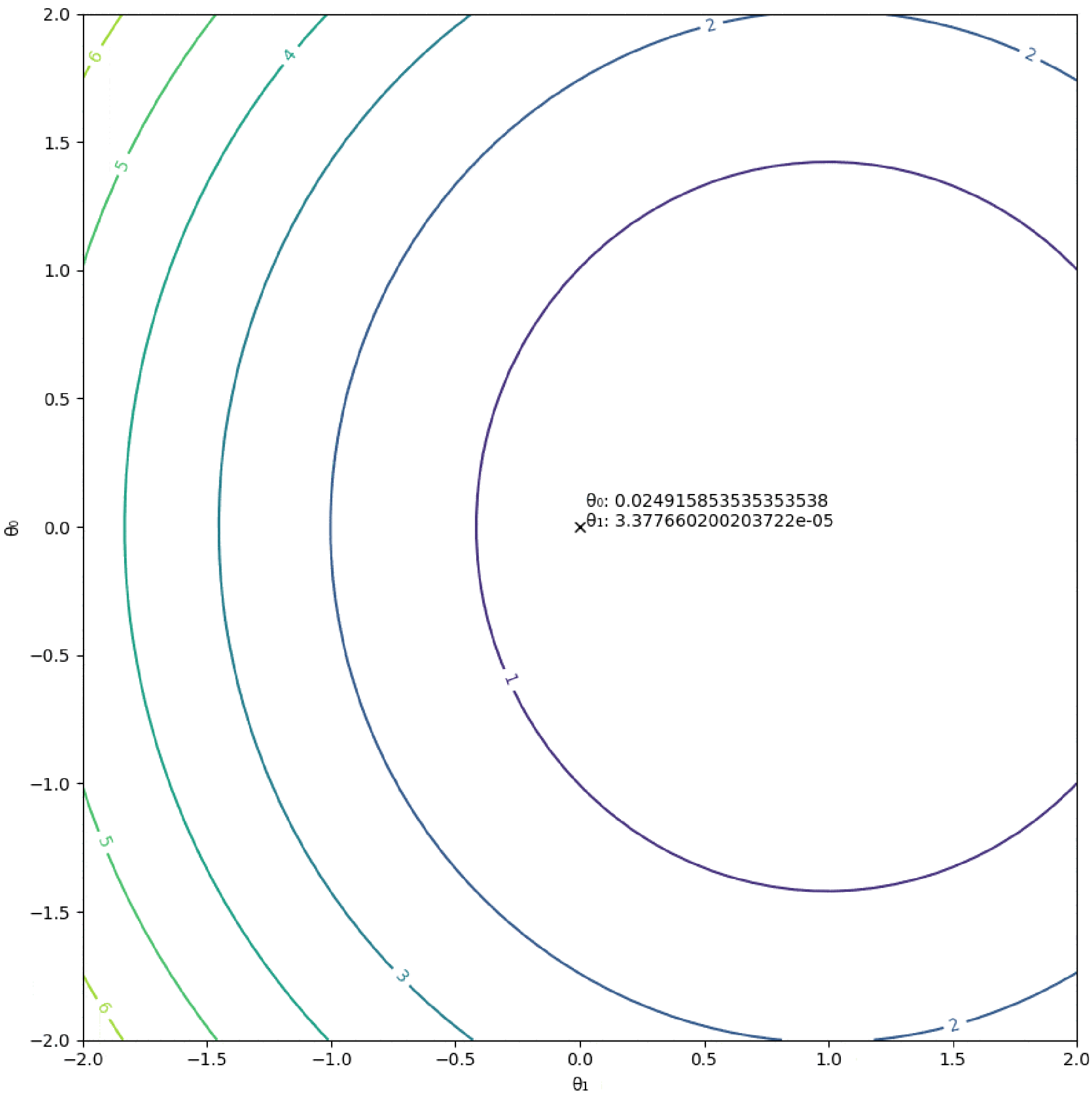


For **η=0.025**

Executing time = 0.3139345645904541 second

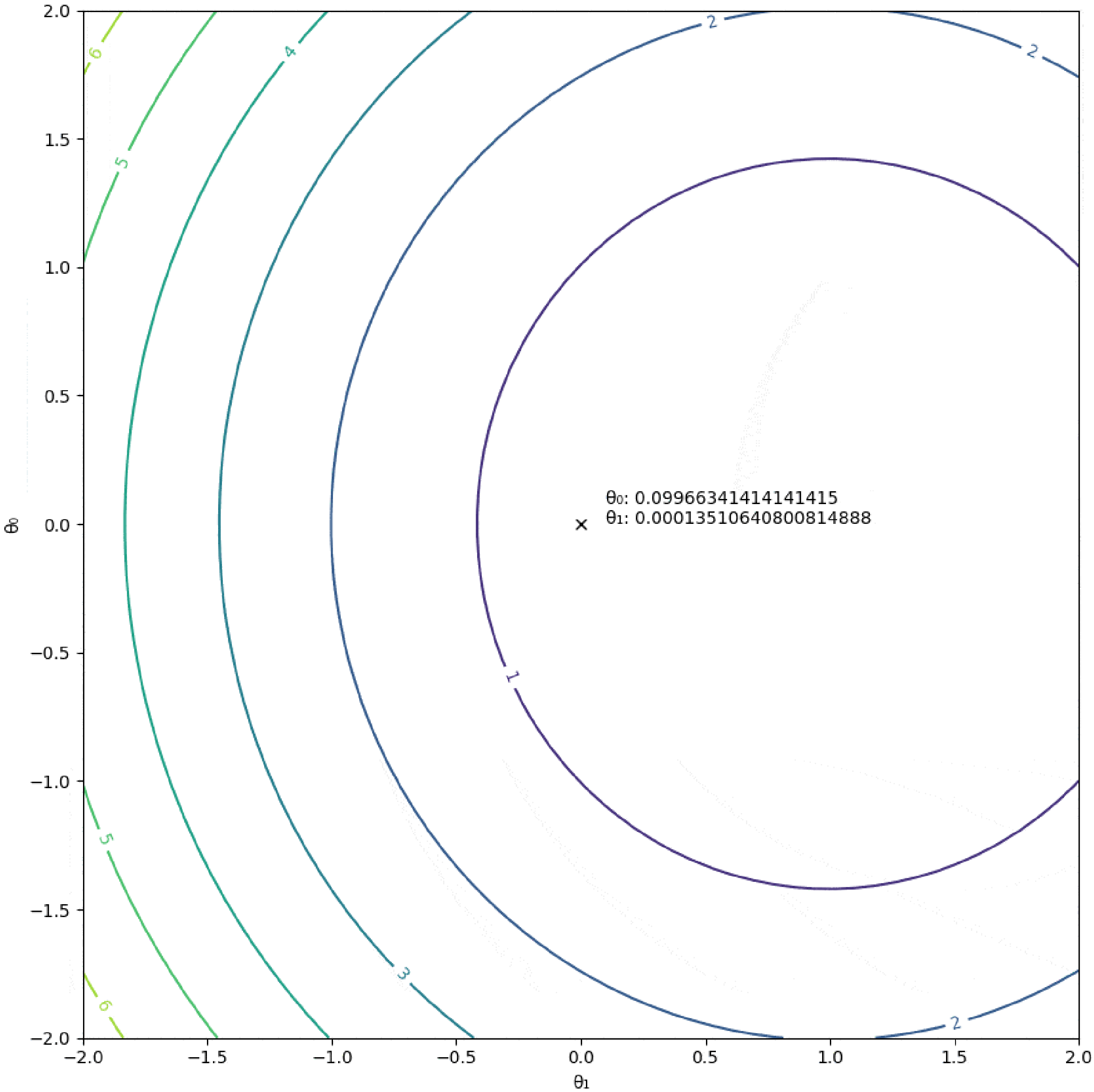
θ_0 =0.99600651, θ_1 = 0.00134605

Iterations = 292



For **η=0.1**

Executing time = 0.1031553745269775 second
θ_0 =0.99635129, θ_1 = 0.00134655
Iterations = 78



2. Linear Regression with Stochastic Gradient Descent:
   Algorithm
   As required I randomly generated 1 million samples of training data using **np.random.normal(loc=3.0, scale=2.0, size=1000000)** here loc is the distribution mean and scale is SD, just like this I sample 3 random variables **X1, X2, epsilon(error)** and using these 3 Random variable Y is generated here is **Y=3+X1+2*X2+epsilon**
   Stochastic Gradient Descent is implemented exactly like normal gradient descent in terms of JTheta calculation and parameter updation via cost minimization. Their is variation only in two aspects firstly, the parameter updation and cost calculation is performaed after every 'batch' where **0 < batch size < 1000000** and secondly, the convergence criteria, For the given question the value of **η** has been fixed to **0.001** and hence no matter the convergence criteria the batch seize is directly proportional to the execution time until convergence.

   I have experimented both ways with and without changing **η** and have publish the results in the document itself. For changing **η** following equations has been used:
   η= 1.0 / (0.5 * iterations + 10)
   I have decided the following convergence criteria for each case:
   $|J(\theta)_{i+1} - J(\theta)_i| < 0.000000001 * (batchSize + iterations)$
   Here JTheta is the cost calculated in the current iteration and PreJtheta is the cost calculated in the previous iteration just like in normal gradient descent.
   { 0.000000001 * (batchSize + iterations) } this entire expression captures the intuition that initially for any given value of batchSize the convergence criteria is much stricter so it does not converge on accident.
   Like for **batchSize=1**
   when **iterations=1** convergence criteria will be $|J(\theta)_{i+1} - J(\theta)_i| < 0.000000001$ and as iterations increase the convergence criteria will become relaxed. This also guarantees the convergence of the algorithm.
   When **iterations=100000** convergence criteria will become $|J(\theta)_{i+1} - J(\theta)_i| < 0.0001$.

2.a) Implemented in the program
2.b) Implemented in the program
2.c) Setting the **η to 0.001,** my algorithm converge in each case: **r = {1, 100, 10000, 1000000}**

For r = 1
θ_0: 2.988869576187893
θ_1: 0.9884431188592653
θ_2: 2.0305780152276656
iterations: 33304
Time taken(sec): 1.2686140537261963

For r = 100
θ_0: 3.0035321969151294
θ_1: 1.0015615552779418
θ_2: 1.9947172376572335
iterations: 29319
Time taken(sec): 5.481138467788696

For r = 10000
θ_0: 2.8498170478414764
θ_1: 1.0326582773734114
θ_2: 1.9888084690662902
iterations: 10697
Time taken(sec): 139.33497428894043

For r = 1000000
θ_0: 2.4658303876529392
θ_1: 1.2472837791713394
θ_2: 1.9167541561078172
iterations: 3295
Time taken(sec): 4937.229515552521

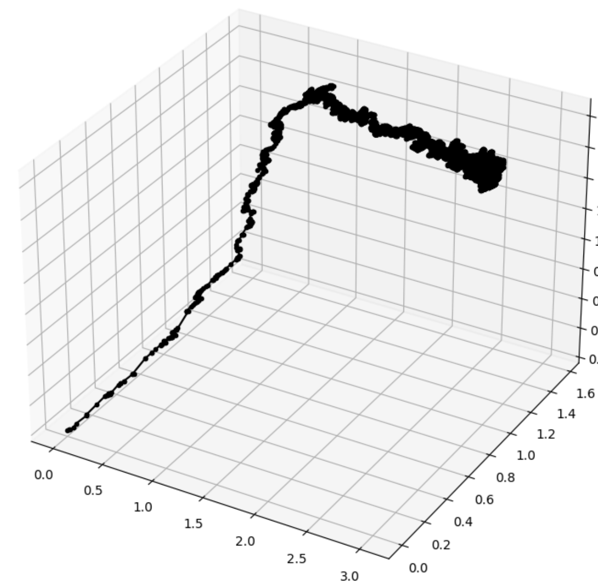
Observations
We can see that as the value of r is increasing the number of iterations are decreasing which is expected since given a fixed value of **η=0.001** each iteration will have r elements to compute cost also the time is also increasing due to the same reason.

Testing the model I trained in question **2.c** for different values of **r** on the given testing dataset **q2test.csv,** following results are obtained
Squared error for Batch of size **1** is **2.070294588876261**
Squared error for Batch of size **100** is **1.9702954712963525**
Squared error for Batch of size **10000** is **2.096354924143564**
Squared error for Batch of size **1000000** is **7.374213392531638**

2.d) Shape of moment for each case of r is relatively similar, we can observe that for smaller value of r the moment is little jittery because we are updating thetas much frequently and it does make **intuitive sense** because in any gradient descent algorithm the path followed will depend on two factors initial position of the parameters and learning rate, in out case since both are same for each case i;e, **η** = 0.001 and **θs** are all initialized to 0.0

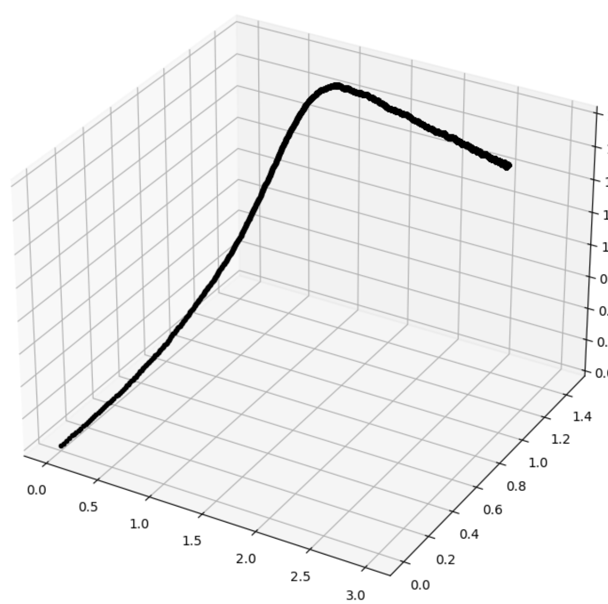
For r = 1

iterations: 33304



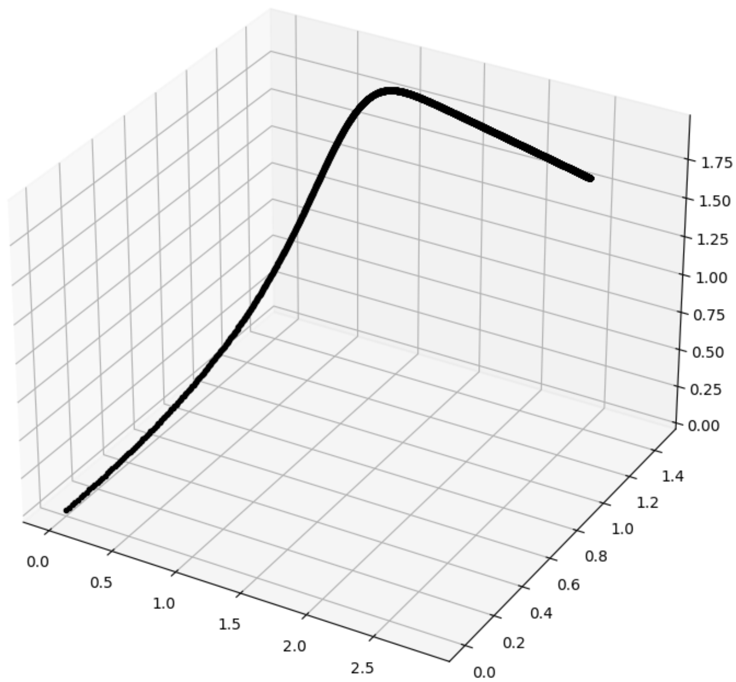
For r = 100

iterations: 29319



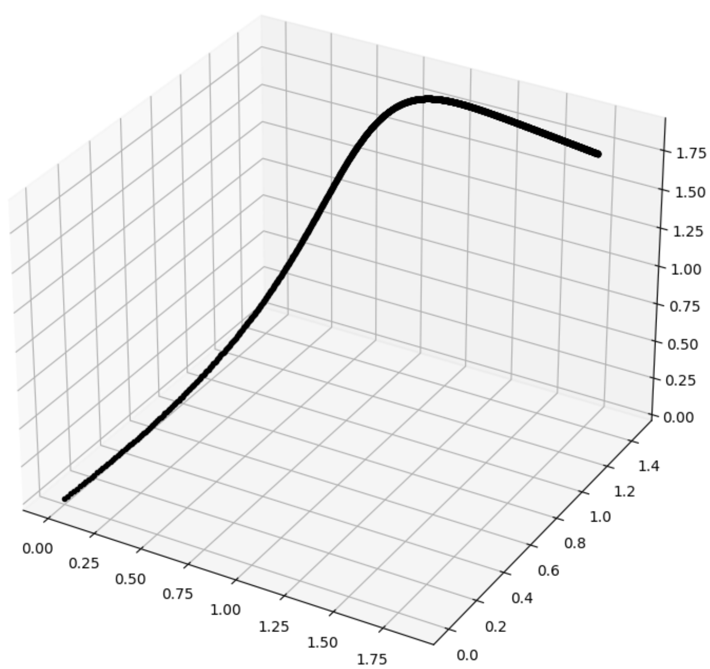
For r = 10000

iterations: 10697



For r = 1000000

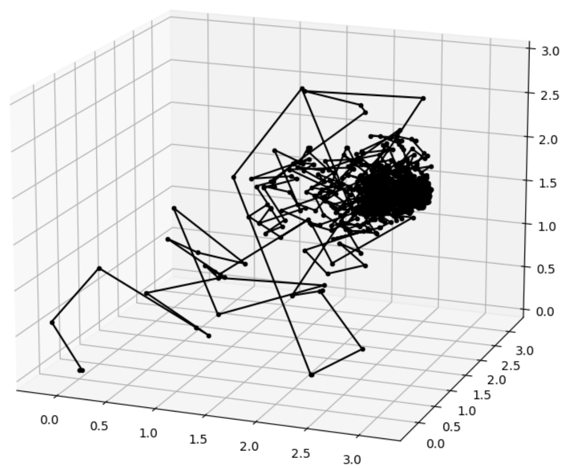
iterations: 3295



I have also experimented with the changing **η** as suggested by the Andrew NG which is changing based on the following equations **η= 1.0 / (0.5 * iterations + 10)**

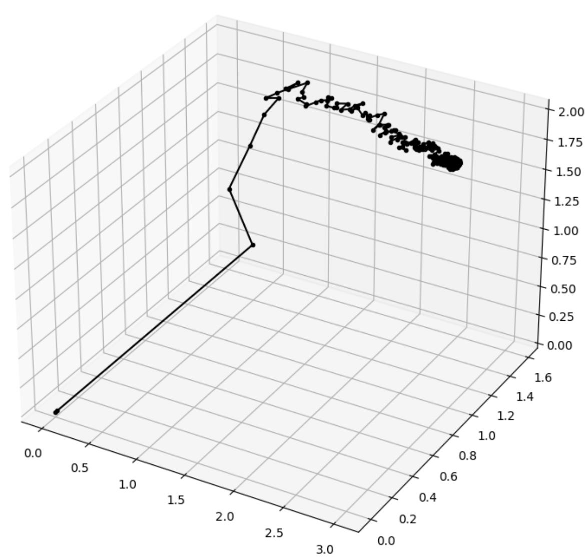
For r = 1

θ_0: 2.9990237995583433
θ_1: 0.9965381694913578
θ_2: 2.0080452480790187
iterations: 146273
Time taken: 7.635342836380005



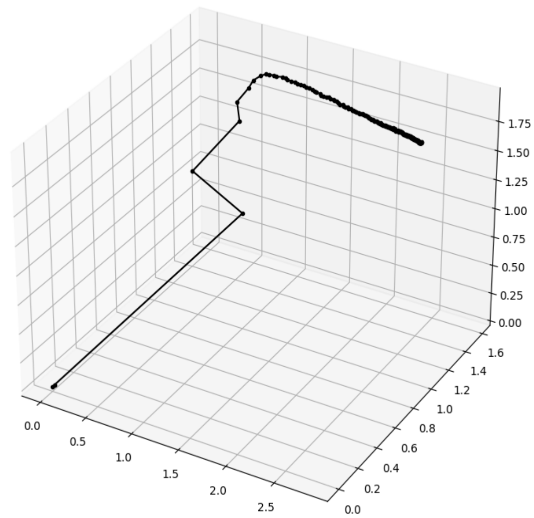
For r = 100

θ_0: 3.0006887836180436
θ_1: 0.9979751351944088
θ_2: 2.0067364600408686
iterations: 11920
Time taken: 3.058948516845703



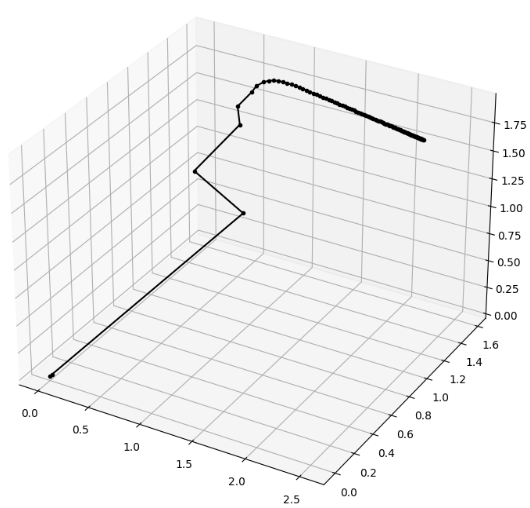
For r = 10000

θ_0: 2.8372994846309623
θ_1: 1.0359193574612098
θ_2: 1.9908785083499414
iterations: 185
Time taken: 3.4503045082092285



For r = 1000000

θ_0: 2.538104086539994
θ_1: 1.1017599439980938
θ_2: 1.966478787492758
iterations: 94
Time taken: 200.86292171478271



Squared error for Batch of size **1 is 1.9720586328921608**
Squared error for Batch of size **100** is **1.969359790473843**
Squared error for Batch of size **10000** is **2.1159349483211005**
Squared error for Batch of size **1000000** is **3.2140801969510844**

3. Logistic Regression

Algorithm

Initially I read the train and test data and stored them in a NumPy array, then I normalized the random variable X for train and test data to reflect mean = 0.0 and standard deviation to 1.0. Using the following code : **X=((X-X.mean())/X.std())**

Now I created a function named **logisticRegression()**

In the function I first calculated the hypothesis **h(θ) = 1.0/(1.0+np.exp(-Z))** by performing sigmoid squishification on Z where **Z=np.matmul(thetas,X)**

Now I calculated the log likelihood function **L(θ)** as mentioned in the question

LTheta = np.multiply(Y, np.log(hTheta)) + np.multiply((1 - Y), np.log(1 - hTheta))

Now checked the convergence criteria $|L(\theta)_{i+1} - L(\theta)_i| < 0.0000001$

When not converged, updated the **thetas** values, since It was asked to perform optimization using Newton's method, I calculated the Hessian matrix, We know that Hessian is a square matrix of secord order partial derivatives of a multinomila function.

$$\mathbf{L(\theta)} = \Sigma\ y^i \log(h(\theta)x^i) + (1- y^i) \log(1- h(\theta)x^i)$$

Partial differentiating **L(θ)** with respect to **θ** we get the following equation

$$\frac{\partial L(\theta)}{\partial \theta} = \Sigma\ x^{(i)} (y^{(i)} - h(\theta)x^{(i)})$$

Again differentiating **L'(θ)** with respect to **θ** we get

$$\frac{\partial^2 L(\theta)}{\partial \theta^2} = -\Sigma\ x^{(i)} x^{(i)T} h(\theta)x^{(i)} (1 - h(\theta)x^{(i)})$$

Finally **θs** are updating uing the following equation

$$(\theta_k)_{i+1} = (\theta_k)_i + [\frac{\partial^2 L(\theta)}{\partial \theta^2}]^{-1} \frac{\partial L(\theta)}{\partial \theta}$$

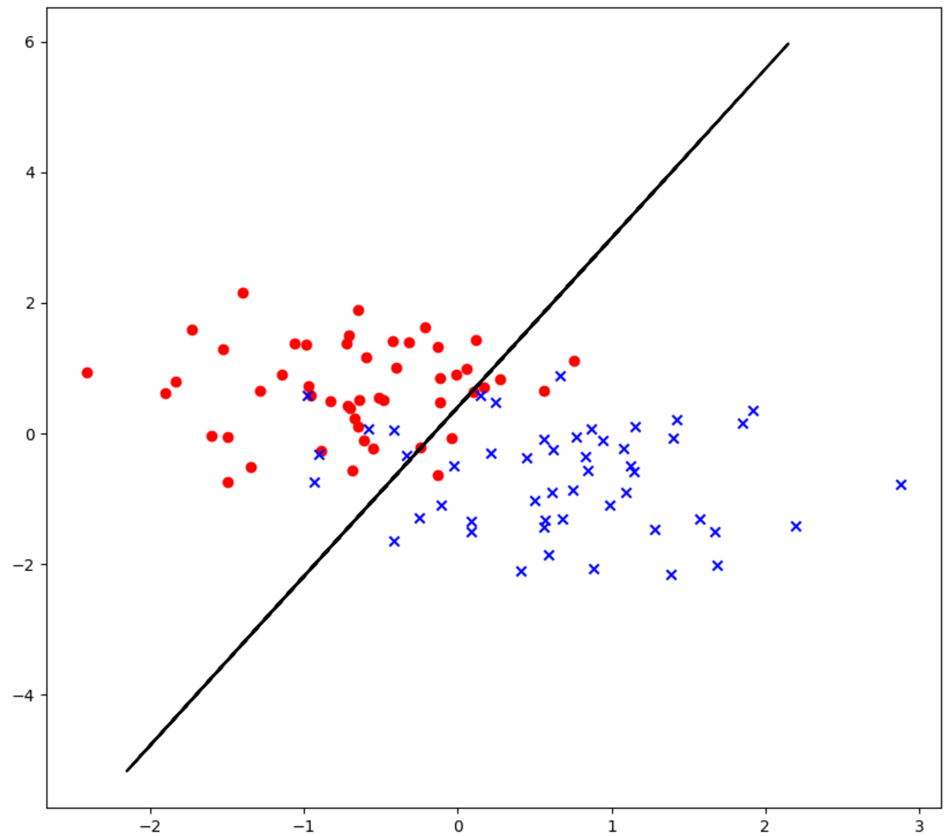
3.a) After running the algorithm it converges to the following **θ** values

θ_2 : -2.72558849

θ_1 : 2.5885477

θ_0 : 0.40125316

3.b) In the graph Red circle denotes category ‘0’ and Blue cross denotes category ‘1’



4. Gaussian Discriminant Analysis

Algorithm

Initially I read the train and test data and stored them in a NumPy array, then I normalized the random variable X for train and test data to reflect mean = 0.0 and standard deviation to 1.0. Using the following code : **X=((X-X.mean())/X.std()),** for training data I have read the categories values from **q4y.csv** and then implemented the indicator function for both when Y=**'Canada'** and when Y=**'Alaska'.**

SDA is a non iterative method for data classification and here we are trying to fit a best gaussian distribution to our data, so I just calculated gaussian parameters **(Σ, μ, π)** for each category given in training data in the code. using the following equations.

$$\pi_k = \frac{1}{m}\Sigma\, 1\{\, y^{(i)}=1\}$$

$$\mu_k = \frac{\Sigma\, \psi\{\, y^{(i)}=k\}x^{(i)}}{\Sigma\, \psi\{\, y^{(i)}=k\}}$$

$$\Sigma_k = \frac{1}{m}\Sigma\, (x^{(i)} - \mu_k).(x^{(i)} - \mu_k)^T$$

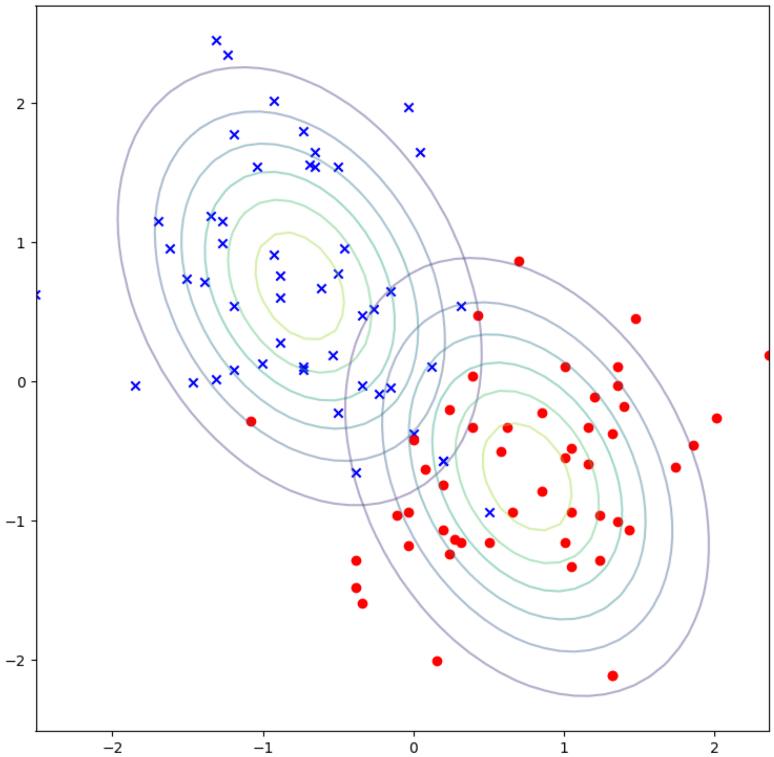
4.a) In the closed form equation with the assumption that $\Sigma_0=\Sigma_1=\Sigma$ the estimated values of μ_0, μ_1, $\Sigma_0=\Sigma_1=\Sigma$ I am getting after running the algorithm are

μ_0 = [-0.75529433, 0.68509431]

μ_1 = [0.75529433, -0.68509431]

$\Sigma_0 = \Sigma =$ [[0.38158978 -0.15486516]
[-0.15486516 0.64773717]]

4.b) Plot of distributions along with contour lines



4.c) Derivation of the equation of decision boundary

Discriminent function for the general case is given by

$$\delta_k(x) = x^T \Sigma_k^{-1} \mu_k - \frac{1}{2} \mu_k \Sigma_k^{-1} \mu_k + \ln(\pi_k)$$

Since $\Sigma_k = \Sigma$;

$$\delta_k(x) = x^T \Sigma^{-1} \mu_k - \frac{1}{2} \mu_k \Sigma^{-1} \mu_k + \ln(\pi_k)$$

$$\delta_k\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = [x, y] \underbrace{\underbrace{\begin{bmatrix} \Sigma^{-1} \end{bmatrix}}_{2\times 2} \begin{bmatrix} \mu_k^x \\ \mu_k^y \end{bmatrix}}_{\text{constant } w_k} - \underbrace{\frac{1}{2} [\mu_k^x \ \mu_k^y] \begin{bmatrix} \Sigma^{-1} \end{bmatrix} \begin{bmatrix} \mu_k^x \\ \mu_k^y \end{bmatrix}}_{\substack{\text{constant} \\ \text{let it be} \\ \omega_k \\ \text{const}}} + \ln(\pi_k)$$

0

since at boundary $\pi_0 = \pi_1 = 0.5$

At the boundary $\delta_0 = \delta_1 = 0.5$

equating δ_0 and δ_1

const

$$[x,y]\underbrace{\left[\Sigma^{-1}\right]\begin{bmatrix}\mu_0^x\\ \mu_0^y\end{bmatrix}}_{\substack{\text{constant}\\ \text{let it be}\\ w_K}} - const_0 = [x,y]\left[\Sigma^{-1}\right]\begin{bmatrix}\mu_1^x\\ \mu_1^y\end{bmatrix} - const_1$$

$$[x,y]\begin{bmatrix}w_0^x\\ w_0^y\end{bmatrix} - const_0 = [x,y]\begin{bmatrix}w_1^x\\ w_1^y\end{bmatrix} - const_1$$

$$x w_0^x + y w_0^y - const_0 = x w_1^x + y w_1^y - const_1$$

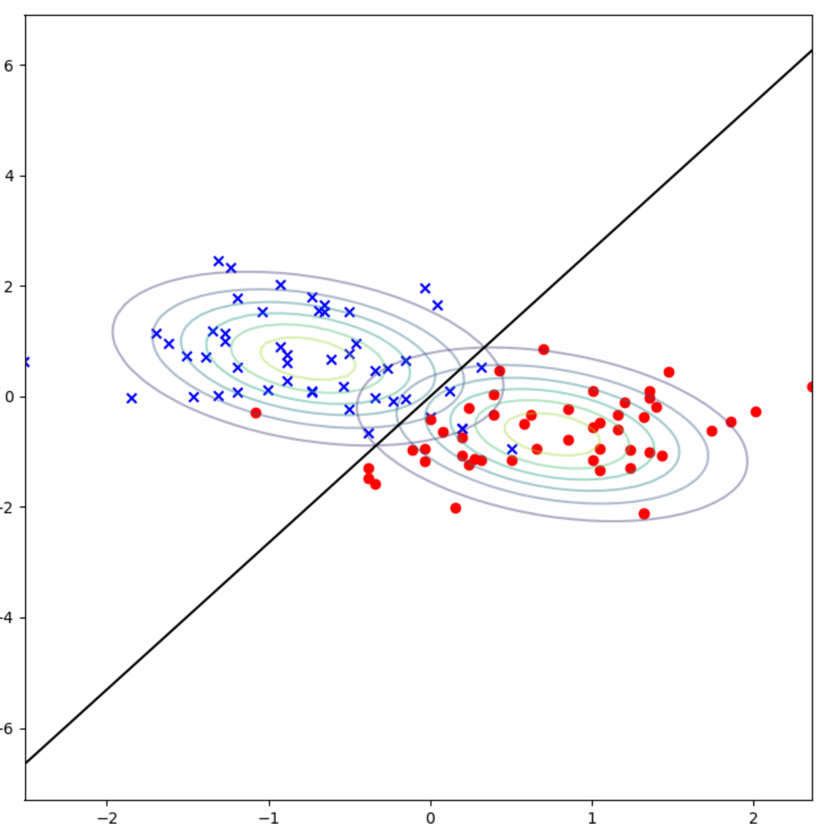
$$y(w_0^y - w_1^y) = x(w_1^x - w_0^x) \underbrace{- const_1 + const_0}_{const_K}$$

$$\boxed{y = \frac{(w_1^x - w_0^x)}{(w_0^y - w_1^y)}x + \frac{(const_0 - const_1)}{(w_0^y - w_1^y)}}$$

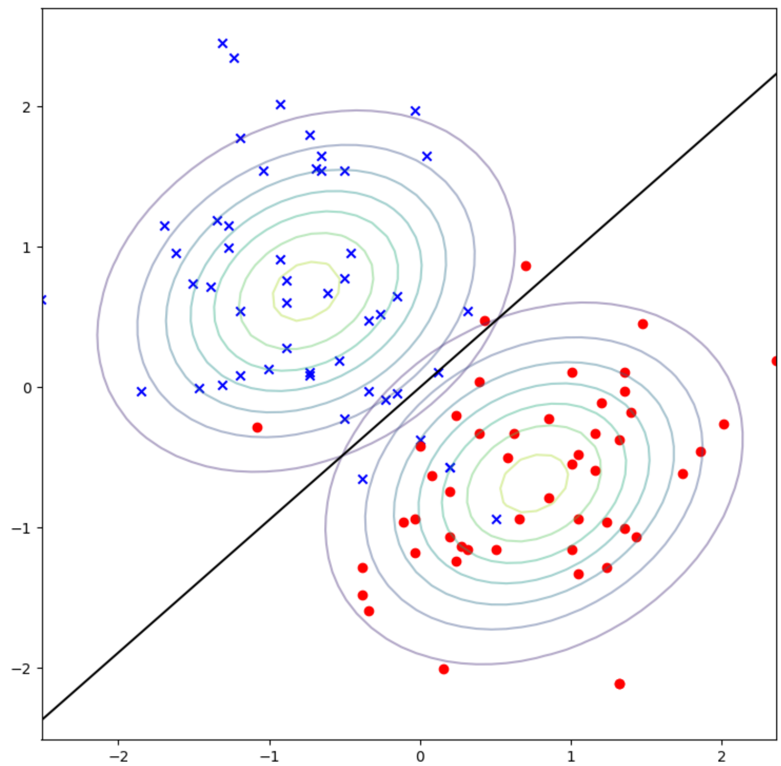
equation of Decision Boundary.

Plotting the linear decision boundary taking

$\Sigma_0 = \Sigma =$ [[0.38158978 -0.15486516]
[-0.15486516 0.64773717]]



$\Sigma_0 = \Sigma_1 = [[0.47747117\ 0.1099206\ ]$
$[0.1099206\ \ 0.41355441]]$



4.d) For Quadatic boundary separator for each category of the data we find gaussian parameters **(Σ, μ, π)** for each category

$\mu_0 = [-0.75529433, 0.68509431]$
$\mu_1 = [\ 0.75529433, -0.68509431]$

$\Sigma_0 = [[\ 0.38158978\ -0.15486516]$
$[-0.15486516\ \ 0.64773717]]$

$\Sigma_1 = [[0.47747117\ 0.1099206\ ]$
$[0.1099206\ \ 0.41355441]]$