Luther College **CoCalc Module 3**
Math 200 - Fall 2024 **Due October 25**
Dr. Johnson

In this Module, we will visually explore linear transformations in $\mathbb{R}^2$ and $\mathbb{R}^3$ and connect the composition of linear maps with the product of their standard matrices!

Create a file named Module3Assignment.sagews in your CoCalc Module3 folder. Use commenting to organize your work so that each question is clearly labeled, relevant questions are answered in the CoCalc file, and context is provided for each problem. You are welcome to work in pairs, but submit your own assignment, which will be graded directly from your CoCalc folder.

Part 1 Visualizing Linear Transformations on $\mathbb{R}^2$

(1) Create a single image P that contains the plots of
  (a) the portion of the circle $x^2 + y^2 = 4$ with $x \leq \sqrt{2}$,
  (b) the line segment from $(0,0)$ to $(\sqrt{2}, \sqrt{2})$, and the line segment from $(0,0)$ to $(\sqrt{2}, -\sqrt{2})$
  (c) the circle $(x - .5)^2 + (y - 1.2)^2 = .01$

  using only parametric_plot and line commands. Hint: To parameterize a circle with radius r and center (h, k), use $x(t) = r\cos(t) + h$, $y(t) = r\sin(t) + k$. The parameter t represents the angle of a point on the circle (in polar coordinates).

(2) Create images that transform P by
  - Rotating image by $\theta = 90°$ counter-clockwise.
  - Stretches the image horizontally and compresses the image vertically by a factor of 2 (in each direction).
  - Reflect image through line $y = -x$
  - Project the image onto the line $y = -2x$.

  Clearly indicate the standard matrices used in each map.

(3) The matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is the standard matrix of the map T that creates a horizontal shearing of the plane. Let S be the map that rotates vectors by $\theta = 45°$ about the origin.
  (a) What is the standard matrix for S. Call it B.
  (b) Create a single image that contains
    - vectors $\mathbf{e}_1$, $\mathbf{e}_2$ in blue,
    - vectors $T(\mathbf{e}_1)$, $T(\mathbf{e}_2)$ in red,
    - vectors $S(T(\mathbf{e}_1))$, $S(T(\mathbf{e}_2))$ in green.
  (c) Use part (b) to compute the standard matrix for the composite map $(S \circ T)(\mathbf{x}) = S(T(\mathbf{x}))$. This map first applies T, then applies S.
  (d) Compute the products of the standard matrices AB and BA. One of these products should match the standard matrix for $S \circ T$. Describe the map that the other product represents (i.e. What map has this other product as its standard matrix?).

(4) Is $(S \circ T)(\mathbf{x}) = (T \circ S)(\mathbf{x})$? Give a short justification for your answer.

Part 2: Visualizing Linear Transformations on $\mathbb{R}^3$

(5) For a 2×3 matrix A, construct the linear map $T(\mathbf{x}) = A\mathbf{x}$. We want to explore the range of T.

(a) For each of the following matrices, display $T(P)$ for the image in part (1)? Don't forget to switch to parametric_plot3d when creating your images.

- $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$
- $A_2 = \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 2 & 0 \end{bmatrix}$
- $A_3 = \begin{bmatrix} 1 & 2 \\ -1 & -2 \\ 3 & -3 \end{bmatrix}$.

How are the images similar? Different?

(b) Create the image G of the graph of the paraboloid $z = x^2 + y^2$ using a 3-d parametric plot with $x = s\cos t$, $y = s\sin t$, $z = s^2$ for $0 \leq s, t, \leq 2$.

(c) Display $T(G)$ for each of the linear maps $T(\mathbf{x}) = A\mathbf{x}$ for each of the 3×3 matrices below:

- $A_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- $A_5 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$
- $A_6 = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$
- $A_7 = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ -1 & 1 & -1 \end{bmatrix}$

(d) Create an image H of the coordinate 3-d grid (as in Module #2) based on linear combos of the standard basis vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. Display $T(H)$ for the linear maps $T(\mathbf{x}) = A\mathbf{x}$ for each of the matrices in part (c). What do you think the range of each map is (point, line, plane, $\mathbb{R}^3$, something else)? Predict the dimension of the range of each map?

(e) For A_5, A_6, A_7, algebraically describe the range of T by determining for which vectors $\mathbf{b} = [b_1, b_2, b_3]$, the equation $T(\mathbf{x}) = \mathbf{b}$ has a solution (i.e. $\mathbf{b}$ is the output of some $\mathbf{x}$ in the domain.) Give your answer in both parametric vector form and in general form. Does your answer match your predictions from part (d)?

(f) Provide a basis for the column space of each matrix A_5, A_6, and A_7. What is the rank of each matrix. How does the column space connect to the range of the corresponding linear transformation?