Luther College | **Homework Inner Product Spaces**
Math 200 - Fall 2024 | **Due Dec 9 at 10pm**
Dr. Johnson

In this homework, you will want to compute relevant integrals and create images in CoCalc. Please name the file you've done your calculations as "InnerProductLastname.sagews".

(1) **Legendere Polynomials:** Let

$$\mathcal{B} = \left\{ 1, x, \frac{1}{2}(3x^2 - 1), \frac{1}{2}(5x^3 - 3x), \frac{1}{8}(35x^4 - 30x^2 + 3) \right\}$$

be a basis for the subspace $W = \mathbb{P}_4$ inside $C([-1, 1])$ with inner product

$$\langle\ P(x), Q(x)\ \rangle = \int_{-1}^{1} P(x) \cdot Q(x)\ dx.$$

(a) Confirm that $\mathcal{B}$ is an orthogonal basis. Is it orthonormal?

(b) For each $i = 1, 2, 3$, compute the coordinates of $\mathrm{proj}_W f_i$ with respect to the basis $\mathcal{B}$ for the following functions
- $f_1(x) = |x|$,
- $f_2(x) = \cos(\pi x)$,
- $f_3(x) = \sin(\pi x)$.

(c) For each $i = 1, 2, 3$, provide a image containing both graphs of $y = f_i(x)$, $y = \mathrm{proj}_W f_i$, each in a different color. Clearly identify each graph in your image. Note: Each projection should be the closest polynomial of degree 4 to the given function. Visually confirm these polynomials do indeed approximate the given functions.

(d) For each $i = 1, 2, 3$, calculation the distance between f_i and $W = \mathbb{P}_4$.

(2) **Trigonometric Polynomials:** Consider the basis

$$\mathcal{B} = \left\{ \sin(x), \sin(2x), \sin(3x), \ldots, \frac{1}{\sqrt{2}}, \cos(x), \cos(2x), \cos(3x), \ldots \right\}$$

for W inside $C([-\pi, \pi])$ with inner product

$$\langle\ f(x), g(x)\ \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot g(x)\ dx.$$

This basis is orthonormal using this inner product!

For a given function $f(x)$, define the Fourier Coefficients:
- $c_n = \langle\ f(x), \sin(nx)\ \rangle$ for $n \geq 1$
- $d_0 = \langle\ f(x), \frac{1}{\sqrt{2}}\ \rangle$
- $d_n = \langle\ f(x), \cos(nx)\ \rangle$ for $n \geq 1$

(a) Create a chart to record the values $c_1, \dots, c_{10}, d_0, \dots, d_{10}$ for the functions
   - $f_1(x) = x$,
   - $f_2(x) = x^2$,
   - $f_3(x) = |x|$.

(b) Identify the pattern to describe the general terms c_n, d_n to describe each function f_1, f_2, f_3 as linear combination of the basis functions:

$$f_i(x) = \sum_{n=1}^{\infty} c_n \sin(nx) + \frac{d_0}{\sqrt{2}} + \sum_{n=1}^{\infty} d_n \cos(nx).$$

This series representation is called the **Fourier series** representation of $f_i(x)$.

(c) For each $i = 1, 2, 3$, provide a image containing both graphs of $y = f_i(x)$ and its Fourier series representation, $y = \sum_{n=1}^{100} c_n \sin(nx) + \frac{d_0}{\sqrt{2}} + \sum_{n=1}^{100} d_n \cos(nx)$, each in a different color. Clearly identify each graph in your image. Visually confirm these trigonometric polynomials do indeed match the given functions.

(3) **Sums of p-series**: We want to compute the value of the following series: $\sum_{n=1}^{\infty} \frac{1}{n^2}, \sum_{n=1}^{\infty} \frac{1}{n^4}$.

(a) Compute $\langle\, x, x \,\rangle$ in two ways:
   - Directly as $\langle\, x, x \,\rangle$.
   - Indirectly using the Fourier series representation $\langle\, \sum_{n=1}^{\infty} c_n \sin(nx), \sum_{m=1}^{\infty} c_m \sin(mx) \,\rangle$.

     Hint: Use that $\sin(mx) \perp \sin(nx)$ for $m \neq n$. (Note: Because $f(x) = x$ is an odd function, $d_n = 0$ for all n.)
   - Use these calculations to show $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

(b) Compute $\langle\, x^2, x^2 \,\rangle$ in two ways:
   - Directly as $\langle\, x^2, x^2 \,\rangle$.
   - Indirectly using the Fourier series representation $\langle\, \sum_{n=1}^{\infty} d_n \cos(nx), \sum_{m=1}^{\infty} d_m \cos(mx) \,\rangle$.

     Hint: Use that $\cos(mx) \perp \cos(nx)$ for $m \neq n$. (Note: Because $f(x) = x^2$ is an even function, $c_n = 0$ for all n.)
   - Use these calculations to find a formula for $\sum_{n=1}^{\infty} \frac{1}{n^4}$.