Luther College **CoCalc Module 5**
Math 200 - Fall 2024 **Due Nov 20**
Dr. Johnson

In this Module, we will visually explore the connection between eigenvectors/eigenvalues of a matrix A and orbits under the map $T(\mathbf{x}) = A\mathbf{x}$.

Create a file named Module5Assignment.sagews in your CoCalc Module5 folder. Use commenting to organize your work so that each question is clearly labeled, relevant questions are answered in the CoCalc file, and context is provided for each problem. You are welcome to work in pairs, but submit your own assignment, which will be graded directly from your CoCalc folder.

Part 1: Real Eigenvalues

(1) Let
- $A_1 = \begin{bmatrix} .3 & .3 \\ .2 & .8 \end{bmatrix}$.
- $A_2 = \begin{bmatrix} 4 & -5 \\ 10/3 & -13/3 \end{bmatrix}$,
- $A_3 = \begin{bmatrix} 1.1 & .8 \\ 0 & .5 \end{bmatrix}$,

(a) For each matrix A_1, A_2, A_3, find all eigenvalues and give a basis for the eigenspace for each eigenvalue.

(b) For each matrix A_i, define (list) a basis $\mathcal{B}_i$ of specific eigenvectors for $\mathbb{R}^2$?

(c) Define $\mathbf{x}_0 = [-4, 3]$, $\mathbf{y}_0 = [3, 2]$, $\mathbf{z}_0 = [1, -1]$. Find the coordinates of each vector with respect to your bases $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$. (There are 9 coordinate vectors you need to find).

(2) We generally define the orbit of a vector $\mathbf{p}_0$ under the map T to be the sequence of points $(\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \ldots)$ where $\mathbf{p}_{n+1} = T(\mathbf{p}_n)$. For a matrix map $T(\mathbf{x}) = A\mathbf{x}$, we get the formula $\mathbf{p}_n = A^n\mathbf{p}_0$.

(a) Use the eigenvalues, eigenvectors, and coordinates you've computed to **make a prediction** for the long-term behavior of the orbits of $\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0$, under each of the maps $T_i(\mathbf{x}) = A_i\mathbf{x}$.

Do the orbit converge to a particular vector? Do they approach a cycle between two or more vectors? Do they become unbounded? Do different things happen for different initial conditions? Hint:

$$A^n(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1\lambda_1^n\mathbf{v}_1 + c_2\lambda_2^n\mathbf{v}_2$$

for eigenvalue/eigenvector pairs λ_i , $\mathbf{v}_i$.

(b) Create an image describing the orbits of $\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0$ that contain:
- The terminal points of vectors $\mathbf{x}_k, \mathbf{y}_k, \mathbf{z}_k$ (in standard position) for $k = 0, \ldots, N$ for an appropriately large N to illustrate the relevant orbit behavior. Use different colors for each orbit.
- Line segments representing the eigenspaces E_{λ_1}, E_{λ_2} appropriate to the scale of the relevant orbits.

(c) **Reflection:** What do you observe in your images? Do your images match your predictions in part (2a)? Did you observe anything unexpected? What changes do you have in appropriately scaling your images / eigenspaces? How did the sizes of the eigenvalues impact your orbits? How are the eigenspaces related to orbits? Did all orbits behave the same for a given matrix map?

Part 2: Complex Eigenvalues

(3) Sometimes matrices have complex eigenvalues $\lambda = a + bi$ where a, b are real numbers and $i = \sqrt{-1}$. These complex eigenvalues also have complex eigenvector pairs. Because complex roots of polynomials (like the characteristic polynomial) always come in conjugate pairs:

$$\text{if } \lambda = a + bi \text{ is an eigenvalue, so is its conjugate pair } \bar{\lambda} = a - bi.$$

We define the modulus (magnitude) of a complex number $|a+bi| = \sqrt{a^2 + b^2}$. If $z = a+0i$ is a real number, then the modulus is the same as the absolute value! We get the analogous properties for $z = a + bi$ and $w = c + di$:

- $|z| \geq 0$
- $|z| = |\bar{z}|$
- $|zw| = |z||w|$
- $|z + w| \leq |z| + |w|$

(4) Let

- $A_4 = \begin{bmatrix} 3.3 & -4 \\ 2 & -2.3 \end{bmatrix}$,
- $A_5 = \begin{bmatrix} 8 & -10 \\ 5 & -6 \end{bmatrix}$,
- $A_6 = \begin{bmatrix} 5 & -6 \\ 3 & -3.4 \end{bmatrix}$.

(a) For each matrix A_i, $i = 4, 5, 6$, compute their (complex) eigenvalues. Compute the magnitude of each eigenvalue.

(b) For each matrix A_i, $i = 4, 5, 6$, create a single image containing at least the first 50 points of the orbits of $\mathbf{x}_0 = [-4, 3]$, $\mathbf{y}_0 = [3, 2]$, $\mathbf{z}_0 = [1, -1]$ (use different colors for each orbit).

(c) **Reflection:** What did you notice about the orbits for matrix maps with complex eigenvalues? What did they all share in common? Did any orbits converge to a single vector? Did any become unbounded? How did the magnitude of the eigenvalues seem to affect the long-term behavior of orbits? (Hint: What happens to $|\lambda^n| = |\lambda|^n$ as $n \to \infty$?)