AI1110 ASSIGNMENT 8

JANGA TUSHITA SHARVA (CS21BTECH11022)

***Abstract*—This document gives solution to Chapter 8, Question 8.6 from the Papoulis and Pillai Probability, Random Variables and Stochastic Processes Text Book.**

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1. Question

Consider a random variable x with density $f(x) = xe^{-x}U(x)$. Predict with 95% confidence that the next value of x will be in the interval (a, b). Show that the length $b - a$ of this interval is minimum if a and b are such that

$$f(a) = f(b) \qquad \Pr(a < x < b) = 0.95 \tag{1.0.1}$$

Find a and b.

2. Answer

We shall show that if $f(x)$ is a density with a single maximum and $\Pr(a < x < b) = \gamma$, the $b - a$ is minimum if $f(a) = f(b)$. The density $xe^{-x}U(x)$ is a special case. It suffices to show that $b - a$ is not minimum if $f(a) < f(b)$ or $f(a) > f(b)$.
We can clearly see that $f'(a) > 0$ and $f'(b) < 0$, hence we can find two constants $\delta_1 > 0$ and $\delta_2 > 0$ such that $\Pr(a + \delta_1 < x < \delta_2) = \gamma$ and $f(a) < f(a + \delta_1) < f(b + \delta_2) < f(b)$.

Case 1: $f(a) > f(b)$
We know that $\delta_1 > \delta_2$. Hence, the length of the new interval $(a + \delta_1, b + \delta_2)$ is $b - a + (\delta_2 - \delta_1)$ smaller than $b - a$.

Case 2: $f(a) < f(b)$
Here, we can form the interval $(a - \delta_1, b - \delta_2)$. Then the length of the interval becomes $b - a + (\delta_1 - \delta_2)$. Since $\delta_1 < \delta_2$, the length is less than $b - a$.
Hence for $f(a) < f(b)$ or $f(a) > f(b)$, the interval length is greater then $b - a$. Therefore the length of $b - a$ is minimum iff $f(a) = f(b)$.

3. Special Case

$$f(x) = xe^{-x} \tag{3.0.1}$$

$$\implies F(x) = 1 - e^{-x} - xe^{-x}. \tag{3.0.2}$$

$$\begin{aligned}\Pr(a < x < b) = e^{-a} + ae^{-a} \\ -e^{-b} - be^{-b} = 0.95\end{aligned} \tag{3.0.3}$$

Since f(a) = f(b),

$$ae^{-a} = be^{-b} \tag{3.0.4}$$

$$\implies e^{-a} - e^{-b} = 0.95 \tag{3.0.5}$$

$$\implies a = 0.04 \tag{3.0.6}$$

$$\implies b = 4.75 \tag{3.0.7}$$

$$\implies interval = 4.71 \tag{3.0.8}$$

If we set $0.025 = \Pr(x \leq a) = F(a)$ and $0.025 = \Pr(x > b) = 1 - F(b)$

$$\implies 0.025 = 1 - e^{-a} - ae^{-a} \tag{3.0.9}$$

$$\implies 0.025 = e^{-b} + be^{-b} \tag{3.0.10}$$

$$a = 0.242, b = 5.572 \tag{3.0.11}$$

$$interval = 5.33 \tag{3.0.12}$$

Hence proved a special case also.