Introduction to Financial Engineering (W6)

Stochastic Calculus: Valuation of Financial Derivatives

1 Context

The aim of this chapter is present a discrete model to set up a basic price evolution model for S and to derive prices for CALL and PUT options. We will start by setting up a very simplistic model, the Binomial model, were prices can only rise or fall a certain amount in a discrete time step. By applying the principles of no-arbitrage, in same way we derive the Forward price, we will derive the price of any option.

The Binomial model is a very simple model yet used in pricing some very exotic derivatives or in practical circumstances were derivations under a more complex framework are not viable.

The Binomial model will serve as a first step to build up a continuous model, namely the Brownian motion, which is a well accepted model for price evolution in financial markets and used for a large amount of practical situations. The derivation of the continuous model will follow as a limiting case of the Binomial model where the time steps are decreasing towards zero.

2 Binomial Models and Random Behavior of Assets

2.1 Binomial Models

Let us consider the following setup:

- We have a stock, S, and a Call option C expiring in one day (assume time in a discrete format and time steps of one day) and strike $K = 100$.
- The stock can either rise to $S_T = 101$ or fall to $S_T = 99$ from its current value of $S_t = 100$.
- Furthermore, assume that the probability of a rise in price is $p = 0.6$ and the probability of a fall in price is $q = 1 - p = 0.4$.
- Interest rates are considered to be zero, that is $r = 0$.

Graphically the situation is as follows:

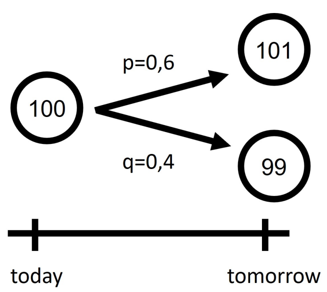


Figure 1. One step Binomial model

The main question is:

What is the option value today t?

How much is a fair price to buy or sell the above option given the underlying model for the evolution of S? In order to answer the question we need to eliminate the randomness of the experiment. As in the Forward example, there is no way a buyer and a seller will agree on their forecast of the evolution of the price of S and hence we need to derive the fair price. Essentially we will follow the same strategy as done in deriving the Forward price, i.e. find a portfolio (cleverly design) where the outcome is indifferent from the price evolution of S (in other words, it is deterministic) and then use a non-arbitrage argument to derive the price of such portfolio today.

Let us denote by Π_t the MAGIC portfolio, which consists in

- One long position in the call option with strike $K = 100$ expiring at $T =$ tomorrow, denoted by $C(t, T, K, S)$
- A short position of $\frac{1}{2}$ of the underlying stock, and let us denote it ΔS where in this particular case $\Delta = \frac{1}{2}$

Hence we are setting Π_t to be

$$\Pi_t = C(t, T, K, S) - \Delta S_t = C(t, T, K, S) - \frac{1}{2}S_t$$

For $t = T$ we know that S_T can either worth 101 or 99, but how much will Π_T worth? Well we definitely know the price of the Call option at expiry, $C(T, T, K, S)$, as it will be its payoff function:

$$\Pi_T = \begin{cases} \text{if } S_T = 101 \text{ then}, \quad C(T,T,K,S) - \frac{1}{2}S_T = \max(101-100,0) - \frac{1}{2}101 = 1 - \frac{101}{2} = \frac{-99}{2} \\ \\ \text{if } S_T = 99 \text{ then}, \quad C(T,T,K,S) - \frac{1}{2}S_T = \max(99-100,0) - \frac{1}{2}101 = 0 - \frac{99}{2} = \frac{-99}{2} \end{cases}$$

The above portfolio, Π, is indifferent on the price evolution of S_t and hence deterministic, i.e. is a known cash flow at $t = T$. We know how to discount a known cash flow tomorrow to know how much value has today with interest rates (using a non-arbitrage principle):

$$\Pi_t = \Pi_T e^{r(T-t)} = \Pi_T = \frac{-99}{2}$$

since $r = 0$. Finally

$$\frac{-99}{2} = \Pi_t = C(t, T, K, S) - \frac{1}{2}S_t = C(t, T, K, S) - \frac{100}{2} \Rightarrow C(t, T, K, S) = \frac{1}{2}$$

There are a couple of questions to be answered:

- Why is this theoretical price right?
- How did we know $\Delta = 1/2$?

The first point is answered since there is not such thing as a free lunch. Since Π is a strategy with a known outcome:

- $\boldsymbol{C(t, T, K, S) < \frac{1}{2}}$ means that $\Pi_t < \Pi_T$ and hence we can buy the CALL ans sell $\frac{1}{2}S_t$ and make a sure profit with no risk in one day of $(\Pi_T - \Pi_t)$.

- $C(t, T, K, S) > \frac{1}{2}$ we do the opposite version of the above strategy, sell the CALL and buy half stock, and make again a sure profit with no risk.

The above situations are not possible or will vanish immediately in the market, therefore there is only one possible price for the CALL option.

One the second point, we have choose Δ to hedge the option. Hedging means that we buy or sold a certain amount of the underlying position to outcome the randomness of the price evolution. In other words, we have choose Δ so that:

$$\begin{aligned} \Pi_T(101) &= \Pi_T(99) \\ C(T, T, K, 101) - \Delta 101 &= C(T, T, K, 99) - \Delta 99 \\ \max(101 - 100, 0) - \Delta 101 &= \max(101 - 100, 0) - \Delta 99 \\ 1 - \Delta 101 &= 0 - \Delta 99 \\ \frac{1}{2} = \Delta & \end{aligned}$$

The process to mitigate a derivative by buying or selling a certain quantity of the underlying stock is known as **delta-hedge**.

: Example

Stock price is $S_t = 100$ and can rise to 103 or fall to 98 in one day. How much is worth a call option today with strike $K = 100$? Assume interest rates are $r = 0$.

1. Consider the magic portfolio $\Pi_t = C(t, T, K, S) - \Delta S_t$
2. Find the Δ-hedge by solving $\Pi_T(103) = \Pi_T(98)$ ($\Delta = 0.6$)
3. Solve $\Pi_T = -58.8$
4. Apply the principle of non-arbitrage to derive

$$-58.8 = \Pi_T = \Pi_t = C(t, T, K, S) - 0.6 \times 100 \;\Rightarrow\; C(t, T, K, S) = 1.2$$

2.2 The Real and Risk-Neutral probabilities

The key question know is: did the probabilities p and q of Figure 1 play any role in the price of the call option? The answer is no; probabilities p and q are what is known as real probabilities, those are probabilities that can be inferred or estimated from real observations. Nevertheless, one could also build up another probabilities in the same space, in particular one could build up what is known as risk-neutral probabilities under some minor hypothesis.

: Risk-Neutral probabilities

Possibly one of the most difficult concepts in mathematical finance for beginners.

The difference between Real and Risk-Neutral probabilities are:

- **Real Probabilities**
  - We know how to delta-hedge and eliminate risk
  - We are risk-sensitive, we expect greater return for taking additional risk
  - Only stock prices matters, not probabilities
- **Risk-neutral Probabilities**

- We do not care about risk than can be hedge, in other words Risk-neutral probabilities are probabilities of possible future outcomes that have been adjusted for risk
- We do not estimate the probabilities of an event
- We believe that everything is price using simple expectations (recall the idea of fair price)

It is very unlikely that two parties will ever agree on Real probabilities since each one might have a different approach of computation or simply they will use different inputs. On the other hand Risk-neutral probabilities are the same for everybody since they are based on fair price.

An investor under a risk-neutral approach is indifferent whether a investment has a sure outcome or is a bet on different outcomes as long as the payoff of two investments are equal. In our example this means that you would be indifferent from owning S_t, and then not knowing whether $S_T = 101$ or $S_T = 99$, or having S_t monetary units and depositing them in a bank account to earn the interest rate r, which would be an investment without risk (a sure outcome). In other words the following two investment should be the same for a risk-neutral investor:

- **Investment A**: Large position in S_t
- **Investment B**: Cash amount S_t deposited in a bank account with interest rate r

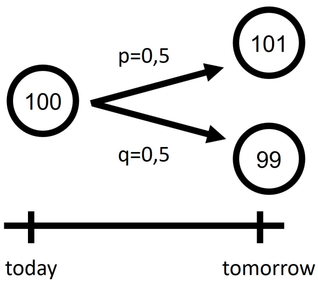
If this is the case, the outcome of A and B should be equal, i.e.

$$\mathbb{E}[S_T] = S_t e^{r(T-t)}$$

Let us consider $r = 0$ for simplicity, then the following equation must hold

$$\mathbb{E}[S_T] = 101 \times p + 99 \times (1-p) = 100 = S_t$$

and hence the risk-neutral probabilities for our binomial example are:



Now, since the investor in the risk-neutral world believes that everything should be price using simple expectations, he would price a call option in the following way:

$$C(t,T,K,S) = e^{-r(T-t)}\mathbb{E}[C(T,T,K,S)] = e^{-r(T-t)} \sum_{S_T \in \Omega} \max(x-100,0)\mathbb{P}\{x = S_T\}$$

where Ω is the space of all possible outcomes. Since $r = 0$ we end up with

$$C(t,T,K,S) = \max(101-100,0) \times 0.5 + \max(99-100,0) \times 0.5 = 0.5$$

EXACTLY the same price we found earlier. Risk neutral probabilities are different from real probabilities but give the right price using expectations:

: Risk-Neutral probabilities

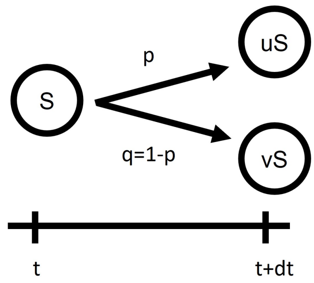
We use the wrong probabilities to get the right price using expected payoffs.

The existence of risk-neutral probability is equivalent or similar to consider a non-arbitrage framework

3 Generalized Binomial Models

We are now going to generalize the simple binomial model presented to derive a general framework to price options in multi-step processes.

Let S be an asset which can rise by $u \times S$ in one step-time, dt, or fall to $v \times S$ in the same time period, i.e. $0 < v < 1 < u$. Let us assume p is the probability of the outcome uS, then



Assume we know the value of an option V at $t + dt$, which is V^u or V^v depending on the underlying evolution (the payoff function), and want to derive the option value at t. To this end we build up the following portfolio

$$\Pi = V - \Delta S$$

Using the same notation as describe before, the value of Π at $t + dt$ can be:

$$\begin{aligned} \Pi^u_{t+dt} &= V^u - \Delta uS \\ \Pi^v_{t+dt} &= V^v - \Delta vS \end{aligned}$$

Hedging: The value Δ is the quantity that hedges the portfolio Π, that means that makes the portfolio Π_{t+dt} equal regardless of the evolution of S. In other words Δ is such that:

$$\begin{aligned} \Pi^u_{t+dt} &= \Pi^v_{t+dt} \\ V^u - \Delta uS &= V^v - \Delta vS \\ \Delta &= \frac{V^u - V^v}{(u-v)S} \end{aligned}$$

The portfolio value at expiry is then derived as

$$\begin{aligned} \Pi_{t+dt} &= \Pi^u_{t+dt} \quad \left(= \Pi^v_{t+dt}\right) \\ &= V^u - \Delta uS \quad (= V^v - \Delta vS) \\ &= V^u - \frac{u(V^u - V^v)}{u-v} \quad \left(= V^v - \frac{v(V^u - V^v)}{u-v}\right) \end{aligned}$$

Non-arbitrage: Since there is no randomness in the portfolio at expiry, the portfolio value at inception is just discounting a deterministic flow:

$$\Pi_t = \Pi_{t+dt} e^{-r(dt)}$$

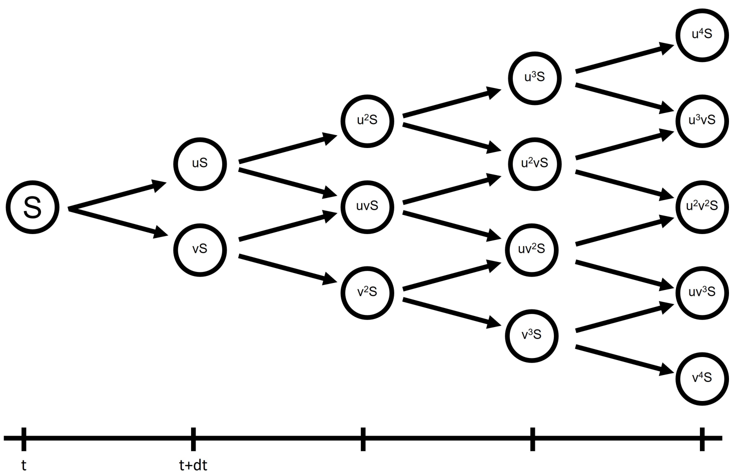
Assume for simplicity that $r = 0$, then

$$\begin{aligned}
\Pi_t &= \Pi_{t+dt} \\
V - \Delta S &= V^u - \frac{u(V^u - V^v)}{u - v} \\
V - \left(\frac{V^u - V^v}{(u - v)\cancel{S}}\right)\cancel{S} &= V^u - \frac{u(V^u - V^v)}{u - v} \\
V &= V^u + (1 - u)\frac{V^u - V^v}{u - v}
\end{aligned}$$

Finally we have derived a closed form solution to compute the value of V at t since the left hand side are all known quantities at t. Again, let us stress again that the price of V does not depend on the direction of the stock S and has nothing to do with the probability one thinks drives the stock motion. Another way to see this is think about it as hedgeable risk is worthless.

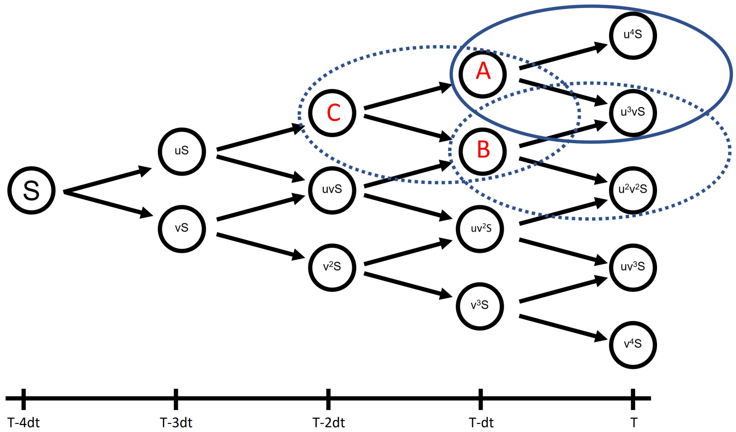
3.1 The extended Binomial Model

We can definitely extend the binomial model just described to allow multiple time steps:

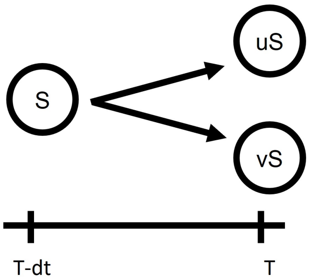


Now we need to describe the process to price options on large binomial trees. Clearly we have described how to price option in small trees, so we need to iterate recursively the process just described.

Note that each of the blue circle in the graph below:



is a small tree of the form:

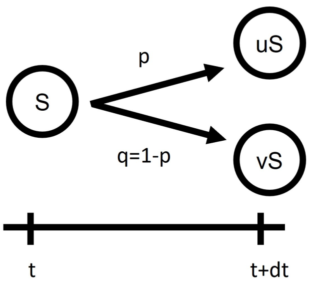


and hence we could find, following the process described in the previous section, the value of an option at points A and B. And again, if we know the values of an option in A and B, then we can compute the value of the option at C. An iterative argument will lead to compute the value of an option at $t = T - 4dt$.

4 Algebraic vs Probabilistic approach

The preceding section described an algorithm to find the price of an option under a binomial model which is a close-form algebraic expression. Since the rested in the application of a non-arbitrage principle. Nevertheless we also know that we could follow an alternative path by invoking the risk-neutral probabilities and derive the price of an option as a simple expectation; let's do it.

Recall the simple model



and assume $r = 0$ for simplicity. In a risk-neutral world the behavior of an investment in S should be the same as depositing the money in a bank:

$$\begin{aligned}
\mathbb{E}[S_{t+dt}] &= S_t e^{r(dt)} \\
puS_t + (1-p)vS_t &= S_t \ \ (r=0) \\
p &= \frac{1-v}{u-v}
\end{aligned}$$

Finally

$$V = \mathbb{E}[V_{t+dt}] = V^u p + V^v(1-p) = V^u \frac{1-v}{u-v} + V^v \frac{u-1}{u-v} = V^u + (1-u)\frac{V^u - V^v}{u-v}$$

which is exactly the same expression we found in the preceding section.

Generally speaking derivations applying non-arbitrage principle will lead to an algebraic solution of the problem while derivations that follow a risk-neutral framework will lead to solutions as discounted expectations.

We will see this very same duality effect in the next chapter when deriving a continuous model for price evolution. Option prices will be derived as solutions of a deterministic PDE (algebraic approach) or as expectations (probabilistic approach). In fact, this observation is just a particular example of the general Feynman–Kac theorem.

5 Δ-hedge

Recall the formula of Δ in the a binomial model, consider V as a function of S and notice that

$$\Delta = \frac{V^u - V^v}{(u-v)S} = \frac{V(uS) - V(vS)}{uS - vS} \cong \frac{\partial V}{\partial S}$$

And therefore by defining Π as

$$\Pi = V - \Delta S = V - \left.\frac{\partial V}{\partial S}\right|_{S_t} S$$

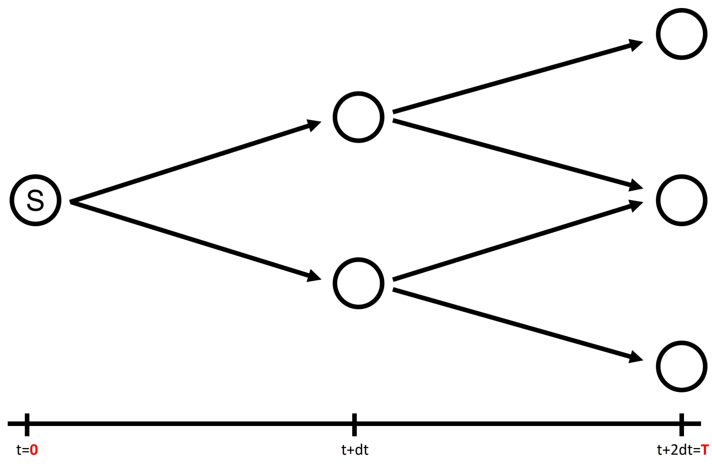
we are building a function which is indifferent from the evolution of S in the sense that:

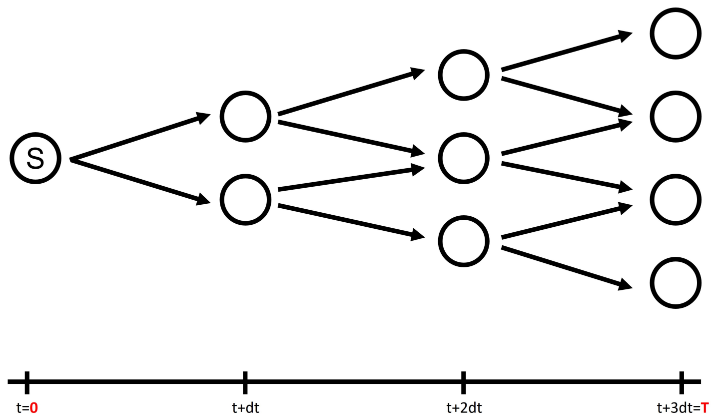
$$\frac{\partial \Pi}{\partial S} = 0$$

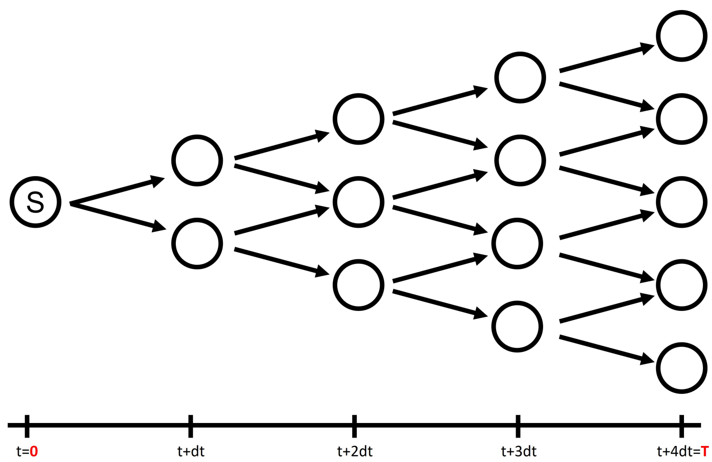
6 Recap

Note that all derivation in the chapter considered $r = 0$, obviously the same principles apply for $r \neq 0$ only formulas will be more complex.

We have presented a Binomial model and derive two alternative, yet equivalent ways, to derive the price of a derivative. The first being an algebraic derivation and the second being the discounted expected value of such derivative. Next step is to transform the discrete model to a continuous model and keep track of the two alternative ways of deriving option prices. The way to proceed will be to build increasingly dense binomial trees in a given time interval $[0, T]$:







As we let $dt \to 0$ we will be building a continuous model, namely a Brownian motion.

References

[1] Paul Willmott (2013) *Introduction to Quantitative Finance*, Wiley.