Introduction to Financial Engineering (W8)

Mathematical Optimization: Portfolio Management

1 Context

The aim of this chapter is to present the techniques in portfolio optimization, that is the type of problems a portfolio manager will face in their tasks. In this setting we will consider that we can invest a given wealth in numerous different assets and we are ask to perform the task to obtain an optimal set of assets and amounts in each asset to invest.

Harry Markovitz presented in 1952 the basic ideas of portfolio selection, i.e. to find a combination of assets that in a given period of time produces the highest possible return at the least possible risk. He received the Nobel price in 1990.

Refer to Chapter 8 in [1] and Chapter 11 and 16 in [2] for further reading.

2 The Optimization Problem

Let us introduce the optimization problem with a toy example:

> Imagine you are in a tropical island and have only USD 100 to invest. Investment opportunities are very limited, and you can only invest in umbrellas or ice cream.

The payoff or return of the investment is uncertain and depend on the weather evolution as:

Weather/Investment	**Ice cream**	**Umbrella**
Sunny	120	90
Rainy	90	120

Assume the probability of the weather being sunny or rainy is the same and lets consider the following two investment strategies:

1. Invest everything you have in either product
2. Split the investment equally between products

The first option expected return is the same whether you chose to invest all in ice cream or umbrellas and follows, for instance we choose to invest everything in ice creams, then:

$$\mathbb{E}[R] = \frac{\mathbb{P}(sunny) \times 120 + \mathbb{P}(rainy) \times 90}{100} - 1 = \frac{0.5 \times 120 + 0.5 \times 90}{100} - 1 = 5\% \ , \tag{1}$$

but there is a chance you will earn more or less than this 5% in the real situation.

On the other hand, if we split the investment equally between both products the outcome will not depend on the evolution of the weather. Indeed, you will earn USD 45 in one product and USD 60 in the other product, hence:

$$\mathbb{E}[R] = R = \frac{45 + 60}{100} - 1 = 5\% \,, \tag{2}$$

and the outcome is not random but deterministic.

Even though the outcome in the first and second situation is the same, equations (1) and (2), the difference is that the second option is riskless.

The main concept of portfolio optimization is captured in this example. Based on the correlation between investment products we can reduce the risk of the portfolio and still get the desired return.

Diversification among investments reduces risk.

We here understand risk as variance or uncertainty in the outcome of an investment.

Let us formulate the above idea by letting X and Y be two random variables with variances σ_X^2 and σ_Y^2 respectively. The variance of the a linear combination is then:

$$\mathbb{V}(\alpha X + (1-\alpha)Y) = \alpha^2\sigma_X^2 + (1-\alpha)^2\sigma_Y^2 + 2\alpha(1-\alpha)\text{Cov}(X, Y)$$

For different values of the co-variance or correlation, ρ, between X and Y we get different values for the variance of the linear combination:

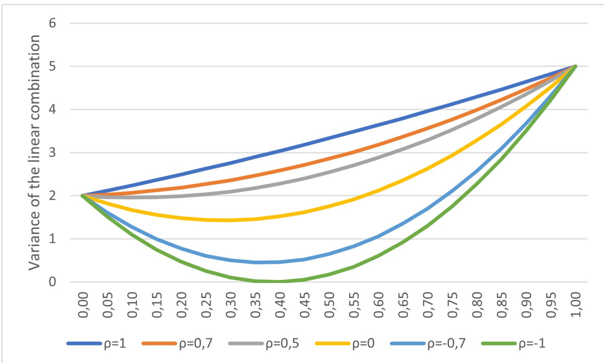

Figure 1. For $\alpha \in [0, 1]$ and different values of ρ. Note that for a given α the points in the different curves will have the same expected return and different risk. Indeed the expected return only depends on α, but not from ρ, i.e. $\mathbb{E}[\alpha X + (1-\alpha)Y] = \alpha\mathbb{E}[X] + (1-\alpha)\mathbb{E}[Y]$.

The variance of the combination can be completely eliminated if and only if the correlation between X and Y is 1 or -1 and the $\sigma_X^2 \neq \sigma_Y^2$.

Portfolio optimization is the theory that studies the correlation between assets to build up portfolios with low risk and high return.

3 Mean-Variance Model

The starting hypothesis of Markovitz theory is that:

1. Investors should consider expected returns as desirable;
2. Investors should avoid variance (risk) of returns.

This rule of thumb has a first consequence, known by investors as the importance of diversification.

Let be a set of N risky assets, say for instance stocks, and define the following portfolio

$$\mathcal{P} = \sum_{i=1}^{N} \omega_i a_i \ ,$$

where $\sum_{i=1}^{N} \omega_i = 1$ are the weights and a_i is the return of the i-th asset with expected return $R_i = \mathbb{E}[a_i]$. The condition imposed in the weights is to assure you cannot invest more than what you have, a situation which not always happens as one can invest on a leveraged position.

From the above equation the expected return the portfolio is

$$\mu = \mathbb{E}[\mathcal{P}] = \sum_{i=1}^{N} \omega_i \mathbb{E}[a_i] = \sum_{i=1}^{N} \omega_i R_i \ ,$$

and the variance follows the equation

$$\sigma^2 = \mathbb{V}(\mathcal{P}) = \sum_{i=1}^{N} \sum_{j=1}^{N} \omega_i \omega_j \sigma_{ij} \ , \quad \text{where } \sigma_{ij} = \text{Cov}(a_i, a_j) \ .$$

Consider the following extreme cases:

- **Option A:** The return of the assets are pairwise uncorrelated, i.e.

$$\text{Cov}(a_i, a_j) = \left\{ \begin{array}{cc} 0 & i \neq j \\ \sigma_i^2 & i = j \end{array} \right. .$$

  Therefore

$$\mathbb{V}(\mathcal{P}) = \sum_{i=1}^{N} \omega_i^2 \sigma_i^2 \ ,$$

  furthermore we consider the portfolio being equally weighted, that is $\omega_i = 1/N$. Hence we can simply further and get the following upper bound

$$\mathbb{V}(\mathcal{P}) = \frac{1}{N^2} \sum_{i=1}^{N} \sigma_i^2 \leq \frac{\max\{\sigma_i^2 \mid i = 1, \ldots, N\}}{N} \ .$$

  Considering plausible that there is a cap in the variance of any asset, as $N \to \infty$ the variance of the portfolio decreases to zero.

- **Option B:** Assume now that the returns of the portfolio are similarly correlated, for instance an equally weighted portfolio with $\mathbb{V}(a_i) = \sigma^2$ and correlation $0 \leq c \leq 1$. Then

$$\begin{aligned}
\mathbb{V}(\mathcal{P}) &= \sum_{i=1}^{N}\sum_{j=1}^{N} \omega_i \omega_j \sigma_{ij} \\
&= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} \sigma_{ij} \\
&= \frac{1}{N^2}\left(\sum_{i=1}^{N} \sigma_{ii} + \sum_{i \neq j} \sigma_{ij}\right) \\
&= \frac{\sigma^2}{N} + \left(1 - \frac{1}{N}\right) c\sigma^2 = \frac{1-c}{N}\sigma^2 + c\sigma^2 \, .
\end{aligned}$$

This shows the opposite situation of the previous option, as no matter how large is N, it is impossible to reduce the variance below certain threshold.

The conclusion from the above examples is:

- The greater the number of pairwise uncorrelated assets, the smaller the risk of the portfolio.
- The greater the presence of similar correlated assets, the closer is the risk of the portfolio to the structural risk of the assets.

Therefore, diversification in a mean-variance world is accomplished by considering highly uncorrelated assets in some reasonable number.

4 Feasible set of investments

Let us know formulate and derive the solution of the optimization problem. Given a portfolio N risky assets, consider the following matrices:

- vector of weights $\omega = (\omega_1, \ldots, \omega_N)^t$;
- matrix of variance-covariance $C = [\sigma_{ij}]_{1<\leq i,j \leq N}$ where $\sigma_{ij} = \text{Cov}(a_i, a_j)$;
- vector of expected returns $\mu = (\mu_1, \ldots, \mu_N)^t$ where $\mu_i = \mathbb{E}[a_i]$.

Under the above vector notation we have:

$$\begin{aligned}
\mathbb{E}[\mathcal{P}] &= \omega^t \mu \\
\mathbb{V}(\mathcal{P}) &= \omega^t C \omega \, .
\end{aligned}$$

According to Markovitz's rationale, the investor's objective is to obtain a certain level of return with the smallest amount of risk. The minimum variance optimization problem is formulated as:

Find ω^* such that

$$\omega^* = \min_{\omega}\{\omega^t C \omega\}$$

subject to

$$\omega^t \mu = r^* \quad \text{and} \quad \sum_{i=1}^{N} \omega_i = 1 \; ,$$

for a given level of expected return r^*.

This is a quadratic programming problem which can be solved using Lagrange multipliers. Before we give the analytic solution though, observe the following:

- there is no restriction on the sign of the weights ω_i, therefore we might have long and short positions in the portfolio;
- we use all our money to invest, i.e. the sum of the weights equals 1.

The Lagrangian function is then defined as:

$$\begin{aligned} L &= \underbrace{\omega^t C \omega}_{\text{objective function}} - \underbrace{\lambda_1(\omega^t \mu - r^*) - \lambda_2(\omega^t \mathbb{1} - 1)}_{\text{restrictions}} \\ &= \sum_{1 \le i,j \le N} \omega_i \omega_j \sigma_{ij} - \lambda_1 \left(\sum_{i=1}^{N} \omega_i \mu_i - r^* \right) - \lambda_2 \left(\sum_{i=1}^{N} \omega_i - 1 \right) \; , \end{aligned}$$

where $\mathbb{1} = (1, \ldots, 1)^t$. Then we need to solve for a local minimum extreme:

$$\left.\begin{array}{l} \frac{\partial L}{\partial \lambda_1} = 0 \\ \frac{\partial L}{\partial \lambda_2} = 0 \end{array}\right\} \text{these ensures the restriction are met}$$

$$\left.\frac{\partial L}{\partial \omega_i} = 0 \quad i = 1, \ldots, N \;\right\} \text{finds a local extreme for } \omega_i$$

ending up with $N + 2$ equations:

$$\left\{ \begin{array}{l} 2\sum_{i=1}^{N} \omega_i \sigma_{ij} - \lambda_1 \mu_j - \lambda_2 = 0 \qquad j = 1, \ldots, N \\ \sum_{i=1}^{N} \omega_i \mu_i - r^* = 0 \\ \sum_{i=1}^{N} \omega_i - 1 = 0 \end{array} \right.$$

which can be summarized into matrix notation as:

$$\begin{pmatrix} 2C & \mu & \mathbb{1} \\ \mu^t & 0 & 0 \\ \mathbb{1}^t & 0 & 0 \end{pmatrix} \begin{pmatrix} \omega \\ \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ r^* \\ 1 \end{pmatrix}$$

The above system is easily solve by any linear method.

This solution is termed efficient in the sense of being the portfolio with the expected return, r^*, and minimum variance. Any other portfolio with that return r^* will have a higher risk.

4.1 The efficient frontier

For a set of N assets, we consider a given objective return r^*, then the Lagrange optimization problem finds the optimal weights $\{\omega_i\}_i$ for the minimal variance portfolio σ^*.

If we plot the pair (σ^*, r^*) for different values of r^* we get the efficient frontier:

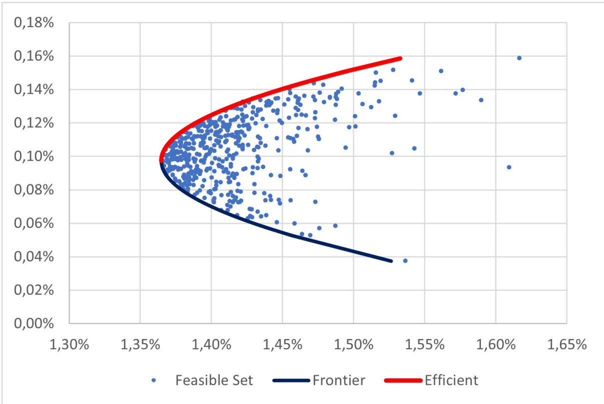

Figure 2. Feasible set of portfolios, frontier and efficient frontier solutions.

- **Feasible set**: the region inside the hyperbola is the feasible set consisting in all possible combinations of the N risky assets, i.e. all possible existing portfolios.
- **Efficient frontier**: The upper branch of the hyperbola is the efficient frontier, i.e. for a given level of risk it gives you the highest return portfolio possible.
- **Minimum variance portfolio**: The locus of the hyperbola corresponds to the minimum variance portfolio, the less risky portfolio possible.

Anything outside the hyperbola is not possible to achieve with combinations of the given N assets.

- **Lower branch**: The lower branch of the hyperbola is clearly not efficient and hence not desirable from the point of view of the investor.

5 Portfolio Optimization with one risk-free asset

To complete the theory of portfolio optimization we would like to include a risk-free asset upon the possibilities to invest; in that way we would be able to built-up portfolios with any amount of targeted risk and hence for any potential investor appetite.

Let's assume that the universe of potential investments are reduced to one risky asset and one risk-free asset

	Return μ	**Risk σ**
Risky	0.15	0.25
Risk-free	0.06	0.0

and we assume ω to be the fraction of our wealth invested in risky asset. Then

$$\begin{aligned}
\mathbb{E}[\mathcal{P}] &= \omega \times 0.15 + (1-\omega) \times 0.06 = 0.06 + 0.02\omega \\
\mathbb{V}(\mathcal{P}) &= \omega^2 \times 0.25^2 = 0.0625\omega^2 \ .
\end{aligned}$$

To decide which portion ω to invest in the risky asset one chooses:

- **Return**: The level of desired return $\mathbb{E}[\mathcal{P}]$ and solves for ω;
- **Risk**: The level of desired return $\mathbb{V}(\mathcal{P})$ and solves for ω .

The idea is that you can configure any level of risk or return that you desire and hence there is a combination for every investor. We will elaborate further on this later in the section, but note that one could choose $\omega > 1$ by means of borrowing money to achieve any desired return level.

The generalized formula for one risk-free asset and one risky asset are:

$$\begin{aligned}
\mathbb{E}[\mathcal{P}] &= \omega \times \mu_r + (1-\omega) \times \mu_{rf} && (3) \\
\mathbb{V}(\mathcal{P}) &= \omega^2 \times \sigma_r^2 \ , && (4)
\end{aligned}$$

where μ_r and σ_r^2 are the return and variation of the risky asset and μ_{rf} the return of the risk-free asset.

This seems an extraordinary simplification from our previous set-up, but in fact we are going to show that there exists a characteristic risky portfolio in the universe of all possible portfolios which will reduce the investment decision into how much wealth we want to invest in that characteristic portfolio and in the risk-free asset, just as the we have shown in the example above. We will denote the characteristic portfolio by the Tangent Portfolio.

5.1 Two risky assets and the tangent portfolio

The mathematics of considering N-risky assets and one risk-free asset is easily understood when we set $N = 2$ and imposes not further difficulties on the general setting.

Let a_1 and a_2 two risky assets and returns μ_1 and μ_2 and standard deviations σ_1 and σ_2 respectively. The metrics of any combination of a_1 and a_2 are then:

$$\begin{aligned}
\mathbb{E}[\mathcal{P}] &= \omega\mu_1 + (1-\omega)\mu_2 \\
\mathbb{V}(\mathcal{P}) &= \omega^2\sigma_1^2 + (1-\omega)^2\sigma_1^2 + 2\omega(1-\omega)\sigma_{12} \ ,
\end{aligned}$$

where σ_{12} is the covariance function of a_1 and a_2.

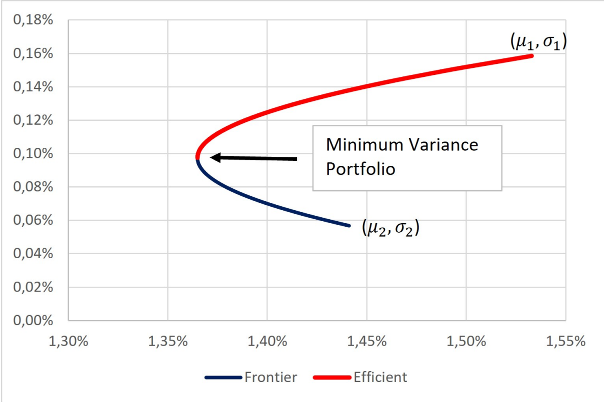

Figure 3. Portfolio combinations of two risky assets.

For each portfolio on the frontier we can combine it with a risk-free asset, F. Let us take and arbitrary portfolio A, in the feasible set of combinations of a_1 and a_2, and combine it with with the risk-free asset according to the generalized formulas (3) and (4). Note that (3) and (4) describes a straight line in a parametric form for the plane (μ, σ), which then can be represented as:

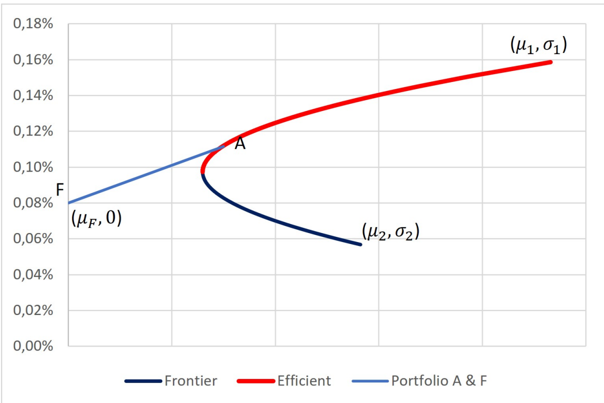

Figure 4. Portfolio combinations of A and F.

The point F in the plot represents the risk-free asset and the line from it to A represents different allocation of wealth, $\omega \in [0, 1]$.

We could built different combinations of a portfolio consisting in the risk-free asset and a point in the frontier generated by the risky assets:

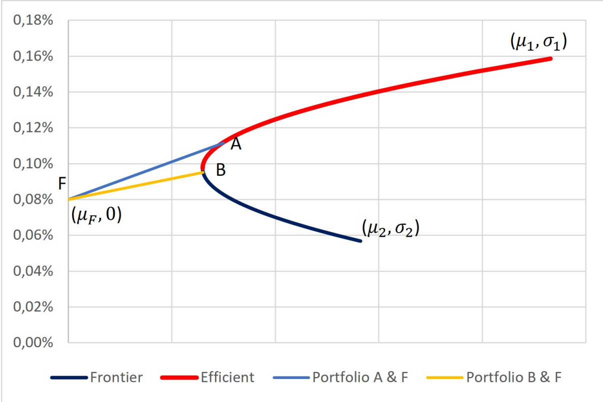

Figure 5. Portfolio combinations of A and F and B and F.

although not all combinations seem efficient. In the above example, for a given level of risk all combinations lying in the line between F and B are less efficient that the portfolios with the same risk lying in the line between F and A, because the later have more return. One may guess that there is a characteristic portfolio on the efficient frontier that is the tangent portfolio, since by definition no other combination of F and any portfolio on the efficient frontier will get better return for the same level of risk:

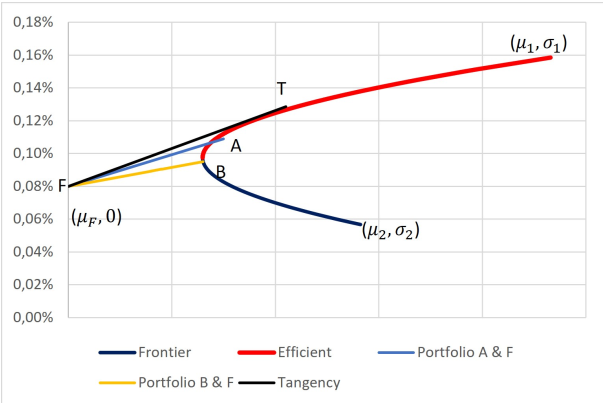

Figure 6. Tangent portfolio T.

The slope of the investments lying on the combination between the risk free asset and the risky asset is known as Sharpe's ratio:

$$\frac{\mu_r - \mu_{rf}}{\sigma_r} \ .$$

Sharpe's ratio can be thought as a risk-reward ratio, in the sense that gives you the return reward per unit of risk.

Note that with the inclusion of the risk-free asset in the universe of potential investments, all optimal portfolios turn out to be a combination of the risk free asset and a particular risky portfolio called tangent portfolio.
Therefore all optimal portfolios have the same combination of risky assets and the only parameter that can vary is the amount of wealth allocated to the risk-free asset.

5.2 Leverage investments

Sometimes the target expected return by the investor is not accomplish by any point between the line of the risk-free asset and the tangency portfolio. In such cases one needs to borrow money to invest more into the risky asset, in a position known as leveraged.

The below graph depicts the region of leverage position under the hypothesis that the risk-free rate is also the rate asked by the lender of the money. If the later is not true the line after the tangent portfolio would have a lower slope compared to the line before the tangent portfolio.

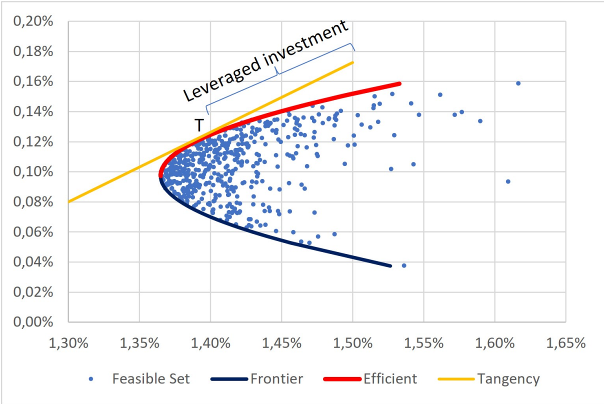

Figure 7. Leverage investments.

6 Recap

Portfolio optimization turn out to be the last step in the portfolio management process, and as presented in this chapter turn out to be a straight forward process. Nevertheless, the difficulty in this approach relies not only on the hypothesis of the model, which were discussed previously, but also on the assumption that we know or can compute expected returns and variance-covariance matrices. These quantities are far from being known and each different management team may use a different approach to estimate them. Some may use past data while other may use forecasting based on a given model or even the frequency of data used might give notable different outputs.

Next section follows another aspect of portfolio managers or managers that devote to M&A activities. The idea is to lay down the criteria for which an investment creates value or destroy it and therefore when to enter a particular investment or not.

References

[1] Argimiro Arratia (2014) *Computational Finance*, Atlantis Press.

[2] David Ruppert (2010) *Statistics and Data Analysis for Financial Engineering*, Springer.