Introduction to Financial Engineering (W9)

Mathematical Optimization: Portfolio Management

1 Context

The aim of this chapter is to present some applications of discounted cash flows. For investment analysts and portfolio managers much of their work will involve evaluation of present and future cash flows. On the first chapter we stated the basic rules for computing time value of money and in this reading we will touch base with some applications.

In particular we will present two techniques to asses the worthiness of a project or investment: the net present value (NPV) and the internal rate of return (IRR).

The above techniques are commonly used to decide between different investment opportunities or to value the relevance of a particular financial opportunity. For the purpose of this section we will focus on Capital Budgeting, that is the choice problem of allocating funds to relatively long range projects and investments.

Refer to Chapter 1 in [1] for further reading.

2 NPV

Net present value describes a way to characterize the value of an investment and the rule associated to the calculation a method for choosing among alternative investments. The net present value of an investment is the present value of all its cash inflows minus the present value of all its cash outflows. The net concept here represents the subtraction of costs from all potential gains. The steps of computing the NPV are then:

1. Identify all cash flows associated to the project, inflows and outflows;
2. Determine the appropriate discount rate to be used to each cash flow, r;
3. Compute the discount factor and assign the proper sign for inflows and outflows;
4. The addition of all quantities of the previous step would be the NPV;
5. The NPV rule states that:
    - If an investment has positive NPV it should be undertake;
    - For mutually exclusive projects, the investor should take the one with higher NPV.

In calculating the NPV we should use an estimate of the opportunity cost of capital as the discount rate. The opportunity cost of capital would be the investor alternative return if funds are not allocated to the investment opportunity. Therefore a company undertaking a project with positive NPV will create value for the shareholders. A negative NPV does the opposite.

The general formula to compute the NPV is:

$$\text{NPV} = \sum_{t=0}^{N} \frac{CF_i}{(1+r)^t} \tag{1}$$

where

$$\begin{aligned} CF_i &= \text{the expected net cash flow at time } t \\ N &= \text{the investment's projected life} \\ r &= \text{the discount rate or opportunity cost of capital.} \end{aligned}$$

: Example

Assume you are analyzing project X for company A, which has announced recently its intention to invest EUR 1 MM into X. Internal estimations forecast a net cashflow per year of EUR 150 K per year in perpetuity. A's cost of capital is 10%.

- State weather project X will benefit shareholders.
- Evaluate what would happen if A's cost of capital is 15%.

Solution 1

According to equation 1 we need to evaluate:

$$\text{NPV} = CF_0 + \sum_{t=1}^{\infty} \frac{CF_i}{(1+r)^t} = -1.000.000 + \sum_{t=1}^{\infty} \frac{150.000}{(1+0.1)^t} = -1.000.000 + \frac{150.000}{0.1} = 500.000 \ .$$

With an opportunity cost of 10% the NPV is a positive value and hence generates value for the shareholders.

Solution 2

Substituting the cost of capital for 15% in the above derivation will give a NPV$=0$. This means that the project does not generate value to the shareholders of the company but neither destroys it, hence undertaking the project will make the company larger but nothing else.

3 IRR

The internal rate of return represents in a single number the investment return generated by a project. By definition the internal rate of return will be the discount rate that makes the net present value equal to zero. Since it is an internal rate, we do not need more information than the project cash flows.

There is an important assumption when dealing with IRR and that is that the interpretation of the IRR is a compound rate which represents the return of the project only if we are able to reinvest all interim cash flows at exactly the IRR. Assume that the IRR of a project is 15% giving annual cash flows for 3 years, but we are unable to reinvest cash flows from year 1 and 2 with a return greater than 10%, then the overall return of the project will be lower that the IRR of 15%. The general formula to compute the IRR would be

$$\text{NPV} = \sum_{t=0}^{N} \frac{CF_i}{(1+\text{IRR})^t} = 0 \ . \tag{2}$$

The decision rule using IRR would be to accept projects and investments for which IRR is greater that the opportunity cost of capital. Going back to the previous example:

: Example

Assume you are analyzing project X for company A, which has announced recently its intention to invest EUR 1 MM into X. Internal estimations forecast a net cashflow per year of EUR 150 K per year in perpetuity. A's cost of capital is 10%.

- Calculate the IRR for project X.
- Decide whether to undertake or not the investment.

Solution 1

According to equation 2 we need to for IRR:

$$0 = CF_0 + \sum_{t=1}^{\infty} \frac{CF_i}{(1+\text{IRR})^t} = -1.000.000 + \sum_{t=1}^{\infty} \frac{150.000}{(1+\text{IRR})^t} = -1.000.000 + \frac{150.000}{\text{IRR}} \ ,$$

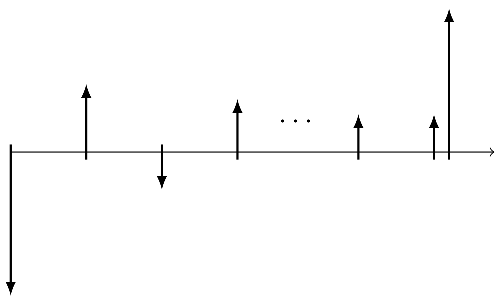
and hence IRR$= 15\%$

Solution 2

Since the opportunity cost of capital, 10%, is lower than the IRR we create value for shareholders in undertaking the project.

3.1 Solving IRR for complex projects

Sometimes investments or projects are set up in a way that outflows do not occur once and at the beginning of the project, but it can happen in several periods of times. In those cases the inflows and outflows may alternate in time as:



Solving equation (2) for a cash flow diagram as the one above, is equivalent to solve for the roots of an N polynomial with real coefficients. Now Descartes' rule of signs states that the number of strictly positive real roots (counting multiplicity) of a polynomial is equal to the number of sign changes in the coefficients. For simple cash flow model projects there is only one possible solution and no doubt about the IRR, but for more complex investments it might be the case that there are multiple positive solutions misleading the analysis.

Another point to take into consideration is that as far as IRR is concern, the reverse of cash flows will generate the same result, while clearly if one layout creates positive value the reverse will create negative value.

4 NPV vs IRR

Generally speaking IRR and NPV give the same accept or reject decision where projects are independent. But when a company cannot finance all projects and need to rank them from most profitable to least, the ranking may differ if IRR or NPV is used. The IRR and NPV rank projects differently when:

- The size or the scale of the projects differs;

- The timing of the projects' cash flow differ.

In case of conflict, we always need to follow NPV conclusions as it the one that captures the addition of value by the project.

To illustrate this situation consider the following consider these two projects with different scales:

Project	**Cash Flow at** t=0	**Cash Flow at** t=1	**IRR**	**NPV at** 8%
A	-10.000	15.000	50%	3.888,89
B	-30.000	42.000	40%	8.888,89

In terms of shareholder wealth or investment profitability, project B should be undertake before project A if both are mutually exclusive.

The same occurs when cash flow differs greatly in time frames:

Project	CF_0	CF_1	CF_2	CF_3	**IRR**	**NPV at** 8%
A	-10.000	15.000	0	0	50%	3.888,89
B	-30.000	0	0	21.220	28%	6.845,12

Again, the general rule of thumb is to act accordingly to NPV if conflict arises between both methods.

5 Recap

In this chapter we applied the concepts of present value, net present value and internal rate of return for valuing investments. These are commonly used for corporate investment and analysis or for capital budgeting problems.

NPV is a measure of the present value of all cash flows of a given project, while IRR is the discount rate at which NPV equal 0. The later can be thought as the expected return where all cashflows from the project could be reinvested at the same return.

The NPV rule for decision making is to accept all projects with positive NPV or when two projects are mutually exclusive, the one with highest NPV. The IRR can be affected by scale and timing issues with the projected cashflows.

References

[1] Paul Willmott (2013) *Introduction to Quantitative Finance*, Wiley.