1 The Basics

The aim of this chapter is to introduce the basic mathematical operations in finance. We will also introduce the derivation for the monthly installments of a loan or a mortgage with French amortization (for the vast majority of you, this will most likely be the most important investment in your life).

The main concepts to deal with are:

- Basic definitions
- Types of financial laws and compounding frequencies
  - Capitalization
  - Discounting
- Loans and Mortgages

Some of this concepts will not be used until chapter 3, nevertheless these are basic concepts not to be forgotten.

2 Basic Definitions

We name **financial capital** to the value of an asset at the moment it becomes available. These means that the value of an asset might not be its price today, but the expected price whenever you are entitle to have it. That also means that we cannot compare money amounts on different time frames, i.e. we cannot compare (or sum, or subtract, ...) one monetary unit today with respect to one monetary unit in one year time. Therefore we need to understand how to transform either one of them to be comparable and make operations with them.

In the above sense, **interest rate** is the return demanded for renouncing to consume an asset now, in exchange for a promise of a future consumption. If you are lending money to someone else for one month, you will ask some interest in return since you are renouncing to use this money for a period of time of one month. Plus, you are incurring in the risk that the borrower is not able to return back this money and hence you need to charge for that risk.

All interest rates have at least two components:

- **Risk free interest rate**: Reflecting the time value of money, that is purely the return which would be asked to a party without any doubt of its capacity to honor the loan. In practice, this rates are represented by the interest rate derived from public debt from very solvent countries (Germany for EUR, US for Dollars, ...)
- **Risk premium**: Reflecting the extra return demanded for assuming the risk that the promise of future payment will not be honored by that particular party. This largely depends on the specific party, the term of the contract or many other aspects. For instance, Italian debt has a risk premium over the German debt as it is assume by the investors that there is a much greater probability of bankruptcy for the Italian economy than the German one, even though this might be very low.

For simplicity in this course we will only deal with risk free rates and assume there is not a counterparty risk.

3 Financial Laws

Financial laws are models for (to value) moving money over time. If one unit of money is worth more or less than the same unit of money in one year time, how can we compare cash flows in different time periods? For instance, how can we decide what is the best investment opportunity between the following two:

- €1 today
- €1.1 in 1 year

As mention before, one way to make a certain correspondence between cash flows in different time frames is via interest rate. For instance the problem above could be solved in the following way: go to a bank and ask what would be the interest they are giving for a deposit of €1 for a year; if that is less than €0.1 then the second option above is preferred. What we just did is to compare the two options above (two cash flows in different time frames) via an interest rate. As we mention earlier, we will only consider one known interest rate, the **risk free rate**, and hence we will be able to compare any cash flows. In practice, the issue is more complex, but it is enough to say that there are some interest rate curves acceptable to be denominated risk free and are available for any one to consult.

A financial equivalence is the correspondence between two financial capitals under a given financial law:

- **Capitalization**: Moving money forward in time. Determining the final value that is equivalent to a certain initial value.
- **Discounting**: Moving money backwards in time. Determining the initial value that is equivalent to a certain final value.

As a mater of fact, some market convention apply when using these laws and slightly different formula is used when dealing with time frames less than a year (simple interest) or greater than a year (compound interest).

: Simple interest capitalization

$$C_T = C_0(1 + it)$$

where C_T is the final capital (€), C_0 is the initial capital (€), i is the interest rate annualized (%) and t is the maturity or time frame (years).

: Simple discount capitalization

$$C_0 = \frac{C_T}{(1 + it)}$$

where C_T is the final capital (€), C_0 is the initial capital (€), i is the interest rate annualized (%) and t is the maturity or time frame (years).

: Example

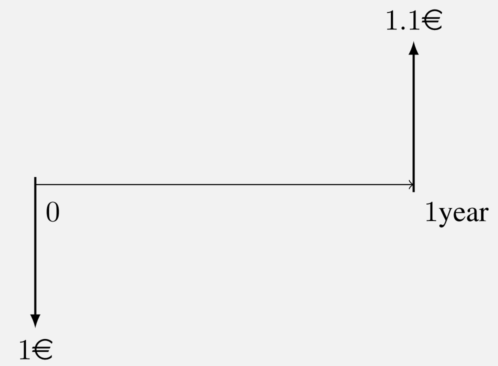
Assume a flat interest rate of 10%, what would be the expected return for a loan of €1 of maturity 1 year?

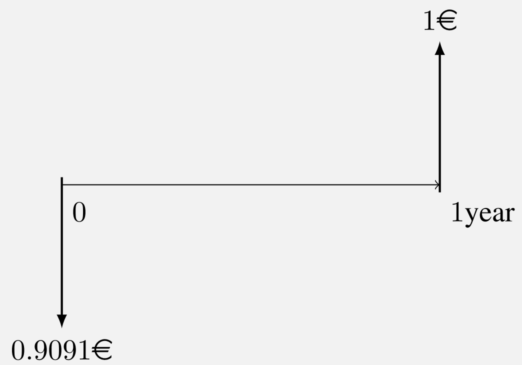
$$\left.\begin{array}{ll} C_0 & = 1€ \\ i & = 10\% \\ t & = 1\text{year} \end{array}\right\} \Rightarrow C_T = C_0(1+it) = 1.1€$$

Assume I want to get €1 in one year time, how much should I invest?

$$\left.\begin{array}{ll} C_T & = 1€ \\ i & = 10\% \\ t & = 1\text{year} \end{array}\right\} \Rightarrow C_0 = \frac{C_T}{(1+it)} = 0.9091€$$

The first set up would be the typical problem to know how much you would earn in an investment and the second situation is the regular setup for an insurance problem. For instance, insurance companies promise a certain revenues in the future base on some inflow now, therefore they would need to compute what are the payments the clients need to make now depending on the current interest rate to earn a specific amount when they retire.
Graphically we will depict the above financial equivalences as:





Arrows going downwards are money we give or invest, and arrows going upwards are money we receive.

In simple interest formulas, interest rate is only applied to initial capital, but on longer investment it is commonly assume that yo would apply interest on interest in a compounding way.

: Compounding interest capitalization

$$C_T = C_0 \left(1 + \frac{i}{n}\right)^{tn}$$

where C_T is the final capital (€), C_0 is the initial capital (€), i is the interest rate annualized (%), t is the maturity or time frame (years) and n is the frequency payment of interest in a year.

: Compounding interest discounting

$$C_0 = \frac{C_T}{\left(1 + \frac{i}{n}\right)^{tn}}$$

where C_T is the final capital (€), C_0 is the initial capital (€), i is the interest rate annualized (%), t is the maturity or time frame (years) and n is the frequency payment of interest in a year.

: Example

Assume an investment pays an interest semiannually of 5%. What would an investment of €1 make in 3 years?

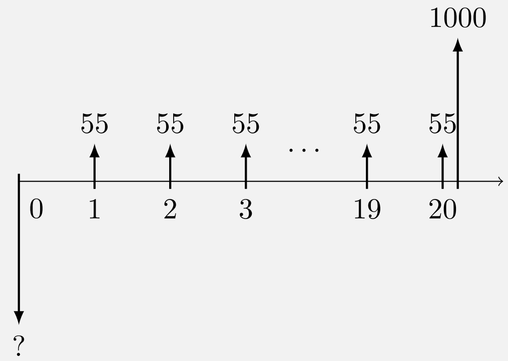
$$\left.\begin{aligned} C_0 &= 1€ \\ i &= 5\% \\ n &= 2 \\ t &= 3\text{year} \end{aligned}\right\} \Rightarrow C_T = C_0 \left(1 + \frac{i}{n}\right)^{tn} = 1.1597€$$

: Example

Consider a bond which makes regular payments plus principal at maturity with the following characteristics:

- Principal €1000
- Risk free rate 6%
- Maturity 20 years
- Annual coupon 5.5%

What would be the price today? In other words, what is the price equivalence today of the following structure:



We need to discount each cash flow to today's value:

$$\text{Price} = \sum_{i=1}^{20} \frac{55}{(1+0.06)^i} + \frac{1000}{(1+0.06)^{20}} = 942.65€$$

There is still another type of compounding interest used in pricing derivatives (we will use this method in Chapter 3), which is the continuously compounding. Lets assume that our investment produces instantaneous interest continuously, then we would have limit the frequency of annual payments tend to infinity as $n \to \infty$, which transforms the previous capitalization formula into:

$$C_T = C_0 \left(1 + \frac{i}{n}\right)^{tn} \xrightarrow[n \to \infty]{} C_0 e^{it}$$

: Continuously compounding interest capitalization

$$C_T = C_0 e^{it}$$

where C_T is the final capital (€), C_0 is the initial capital (€), i is the interest rate annualized (%) and t is the maturity.

: Continuously compounding interest discount

$$C_T = C_0 e^{-it}$$

where C_T is the final capital (€), C_0 is the initial capital (€), i is the interest rate annualized (%) and t is the maturity.

4 Loans and Mortgages

A **loan** is a financial operation in which the lender gives a sum, termed principal, to the borrower who undertakes to amortize the principal and pay interest until the whole debt has been repaid.

The bond definition does indeed fall into the above category, but as you may know a regular loan does not amortize principal in the way we have described bonds. Bonds tend to amortize the full amount of principal at the end of the term of the contract, while a loan to a retail customer is structured in a way that monthly installments include principal amortization as well as interests accrued. This is done in order to minimize the risk of the costumer being not able to repay its debts.

The way in which monthly installments of principal and interest are made constant over the life of the loan is named French amortization profile and is by far the most common loan or mortgage payment structure available.

Lets assume you take a loan of €D repayable in T years at a rate $R\%$ on a monthly basis. Every monthly installment is the sum of all interests accrued during the month on the principal outstanding at the beginning of the month and a certain amount to repay the principal. Let us summarize the installments as follows:

Month	Debt at start of the period	Interest Accrued	Capital Repayment
1	D	$D\frac{R}{12}$	x
2	$D-x$	$(D-x)\frac{R}{12}$	y
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

The monthly installment is then the sum of the *Interest Accrued* (recall that due to monthly payments frequency, the interest accrued during one month is computed via simple interest capitalization) plus the *Capital Repayment*, and since we want the monthly installment to be constant over the entire life of the loan, in particular it holds that:

$$D\frac{R}{12} + x = (D-x)\frac{R}{12} + y$$

and hence $y = \left(1 + \frac{R}{12}\right)x$. Therefore the above table can be rewritten as:

Month	Debt at start of the period	Interest Accrued	Capital Repayment
1	D	$D\frac{R}{12}$	x
2	$D-x$	$(D-x)\frac{R}{12}$	$x\left(1+\frac{R}{12}\right)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$12T$	$\vdots$	$\vdots$	$x\left(1+\frac{R}{12}\right)^{12T-1}$

Since *Capital Repayment* should add up to the initial debt we have

$$\sum_{i=0}^{12T-1} x\left(1+\frac{R}{12}\right)^i = D$$

and hence $x = \frac{D\frac{R}{12}}{\left(\left(1+\frac{R}{12}\right)^{12T}-1\right)}$. Finally to derive the monthly installment we need to add the interest accrued during the first payment period to end up with:

$$
\begin{aligned}
\text{Monthly Installment} &= D\frac{R}{12} + \frac{D\frac{R}{12}}{\left(\left(1+\frac{R}{12}\right)^{12T} - 1\right)} \\
&= \frac{D\frac{R}{12}\left(1+\frac{R}{12}\right)^{12T}}{\left(\left(1+\frac{R}{12}\right)^{12T} - 1\right)}
\end{aligned}
$$

The above formula and table construction allows you to compute not only the monthly installment, but also the interest paid during the entire life of the loan. You can even simulate a change in the interest rate R from an specific month onward (you just need to reset the problem as a new loan with the given principal outstanding at than moment and the remaining time until maturity to compute the new monthly installments).

> The above derivation is far from being the most important formula in this course, but it will certainly be the most important formula for the vast majority of the society with respect to their family economy.