Introduction to Financial Engineering (W7)

Stochastic Calculus: Valuation of Financial Derivatives

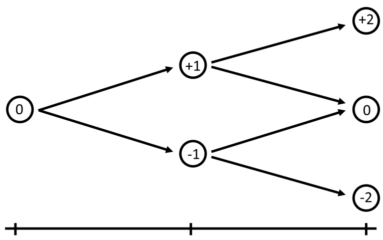
1 Context

The aim of this chapter is present a continuous model for the evolution of stock prices S. We will introduce some basic concepts of stochastic calculus and present the Black-Scholes equation. The derivation of the Black-Scholes equation was a milestone in mathematical finance worth a Nobel price in economics in 1997 for Robert C. Merton and Myron S. Scholes.

Refer to Chapter 6 in [1] for further reading.

2 Tossing coins in a binomial model

Let us start by setting up an experiment in which we toss a perfect coin and earn $+1$ if you get heads or -1 if you get tails, where each toss is independent. This experiment can be represented by a binomial tree, indeed:



Consider R_i the random variable that keeps track of the amount earned at i-th toss, then

$$\begin{aligned} \mathbb{E}[R_i] &= 0 \\ \mathbb{E}[R_i^2] &= 1 \\ \mathbb{E}[R_i R_j] &= 0\,, \ \ \forall i \neq j \end{aligned}$$

and let $S_i = \sum_{j=1}^{i} R_j$, then

$$\begin{aligned} \mathbb{E}[S_i] &= 0 \\ \mathbb{E}[S_i^2] &= i\,. \end{aligned}$$

The following properties are easily check

: Markov property

The expected value of the random variable S_i condition upon the path of S only depends on the last state S_{i-1}. The process has no memory:

$$\mathbb{E}[S_i | S_0, S_1, \ldots, S_{i-1}] = \mathbb{E}[S_i | S_{i-1}]$$

: Martingale property

Your expected winnings after any number of tosses is the amount you already have:

$$\mathbb{E}[S_i|S_{i-1}] = S_{i-1}$$

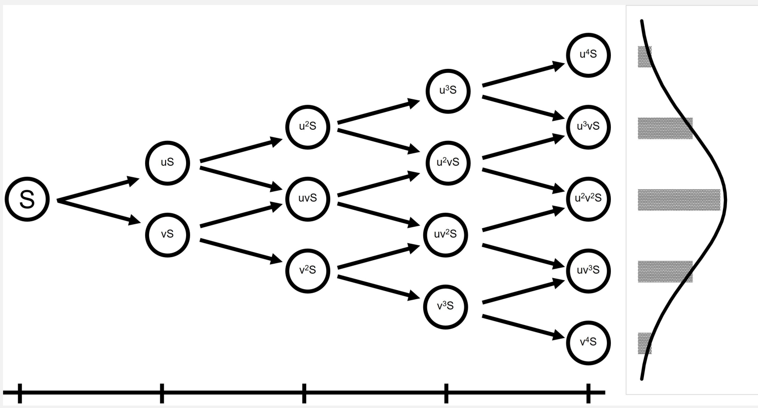
: Quadratic Variation

The expectation of the squared process is finite:

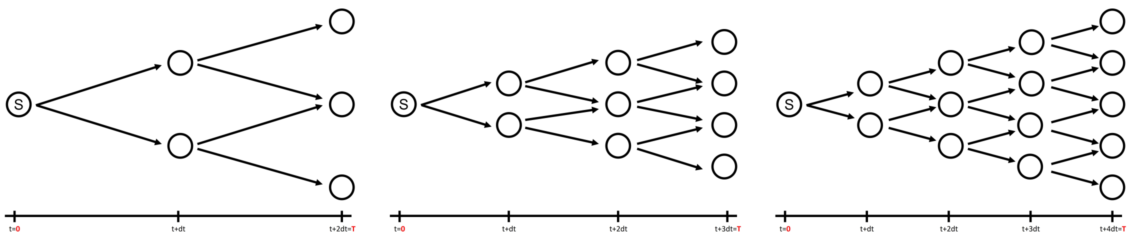
$$\mathbb{E}\left[\sum_{i=1}^{j}(S_i - S_{i-1})^2\right] = j$$

: Normality

The distribution after n iterations follows a Binominal distribution $\sim B(n, p)$, and as n increases the limit distribution is $\sim N(0, np(1-p))$



Let us change now the game slightly by playing up to time T. We will play n tossing and hence we will need to perform them every $dt = T/n$ units of time. We also modify the winning and losing amount to depend on n as $\sqrt{T/n}$. The new game sets up an increasing mesh of binomial trees in $[0, T]$ as $n \to \infty$:



Observe that

$$\begin{aligned}\mathbb{E}[S_n] &= \mathbb{E}[S_T] = 0 \\ \mathbb{E}\left[\sum_{i=1}^{j}(S_i - S_{i-1})^2\right] &= n\left(\sqrt{\frac{T}{n}}\right)^2 = T\end{aligned}$$

In particular the limiting distribution now is $\sim \sqrt{T/n}N(0, np(1-p)) \sim N(0, Tp(1-p))$

The limiting process, $\{B_t\}$, has the following properties (up to a possible scaling factor):

1. Markovian process

$$\mathbb{E}[f(B_t)|B_u,\ 0 \leq u \leq s] = \mathbb{E}[f(B_t)|B_s] \quad s < t \text{ and regular enough } f$$

2. Martingale property

$$\mathbb{E}[B_t|B_u,\ 0 \leq u \leq s] = B_s \quad s < t$$

3. Has finite Quadratic Variation

$$\lim_{dt \to 0} \sum_{t_k \leq t} |B_{t_{k+1}} - B_{t_k}|^2 = T$$

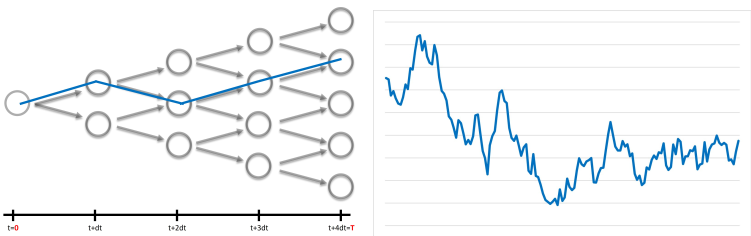
4. Normally distributed

$$B_{t+u} - B_t \sim N(0, u)$$

5. Continuous (…in some probabilistic sense)

For the purposes of this course let us take the above as a definition of a Brownian motion, although the correct characterization of the Brownian motion (also known as Weiner process) requires further concepts of stochastic calculus. In fact the first two previous properties derive from the characterization of a Brownian motion as a process with stationary and independent increments.

What we want to gather is an intuition on the behavior of a Brownian motion and hence how to approximate a realization of the path. Observe that a realization of an intermediate mesh of a binomial tree tends qualitatively to the path of a Brownian motion for large n:



3 Brownian motion and stock returns

The model we are proposing is that returns, R_t, of the stock price S_t follow a Brownian motion. In other words, we are proposing an extension of the binomial tree, in which

$$\frac{S_{t+dt} - S_t}{S_t} = R_t = \sigma\sqrt{dt}\epsilon_t$$

where $\epsilon_t \sim N(0,1)$ and $\sigma \in \mathbb{R}^+$ (in the binomial model $\sigma = \sqrt{p(1-p)}$ but we can let it be a general parameter for the model). This can also be viewed as

$$\frac{S_{t+dt} - S_t}{S_t} = R_t = \sigma(B_{t+dt} - B_t)\ . \tag{1}$$

On the other hand, if the asset S_t were not to be random it would need to earn at least the interest rate r over time, i.e. in the absence of randomness the evolution of prices will follow

$$\begin{aligned} S_{t+dt} &= S_t(1 + rdt) && (2) \\ \frac{S_{t+dt} - S_t}{S_t} &= rdt\ . && (3) \end{aligned}$$

Since we have empirical evidence of randomness in the evolution of prices we propose the combine model, equations (1) and (3), to drive the price S:

$$\begin{aligned}\frac{S_{t+dt}-S_t}{S_t} &= rdt+\sigma\sqrt{dt}\epsilon_t \\ &= rdt+\sigma(B_{t+dt}-B_t)\ .\end{aligned}$$

Let me allow to use an abuse of notation (although there is a formal derivation leading to the following equation):

$$\frac{dS_t}{S_t} = rdt+\sigma\sqrt{dt}dB_t \tag{4}$$

$$dS_t = rS_tdt+\sigma S_tdB_t\ . \tag{5}$$

The above equation (5) is a stochastic differential equation, but for us will be a nomenclature of the following discrete equation:

$$S_{t+dt}=S_t+rS_tdt+\sigma S_t\sqrt{dt}\epsilon_t\ , \tag{6}$$

which is equivalent to

$$R_t=rdt+\sigma\sqrt{dt}\epsilon_t\ .$$

If returns were to behave as the preceding equation, then their distribution would be normal, symmetric, centered in r, ... We have already show in the *Problem Sheets HW0* that these assumptions are not true, but we also have observe that returns distribution do not differ much from them qualitatively and hence we take the Brownian motion model as a good approximation of the evolution of prices.

From a practitioners point of view, Brownian motion is just the simplest model from which enhancements can be done to improve performance. For instance, by removing normality from the characterization of Brownian motion we end up with Levy processes which exhibit fatter tail distribution and can even allow jumps in their path realization.

3.1 Chain rule for stochastic processes

Let us consider the following stochastic process S_t satisfying

$$dS_t=rS_tdt+\sigma S_tdB_t$$

and let $V(t,S_t)$ be a sufficient smooth function, then

$$dV(t,S_t)=\frac{\partial V}{\partial t}(t,S_t)dt+\frac{\partial V}{\partial S}(t,S_t)dS_t+\frac{1}{2}\sigma^2S_t^2\frac{\partial^2 V}{\partial S^2}(t,S_t)dt\ .$$

The above is known as Itô's Lemma, and is nothing but the chain usual chain rule where we had add an extra final term. The proof of Itô's lemma can be found in [2].

4 Black-Scholes pricing formula

Now that we have set a model for price evolution, we can proceed as done with the binomial model and derive the price of an option $V(t,S_t)$. In order to do so we will need to built up a portfolio that is Δ-hedged. Let us consider

$$\Pi(t,S_t)=V(t,S_t)-\Delta S_t$$

and assume the underlying S_t follow the equation

$$dS_t=\mu S_tdt+\sigma S_tdB_t\ .$$

Using Itô's lemma we can derive the driving equation for Π as

$$d\Pi(t, S_t) = \frac{\partial V}{\partial t}(t, S_t)dt + \frac{\partial V}{\partial S}(t, S_t)dS_t + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2}(t, S_t)dt - \Delta dS_t \; .$$

The only terms in the above equation that are random are $\frac{\partial V}{\partial S}(t, S_t)dS_t$ and ΔdS_t, since the rest of the terms are deterministic and do not have the term dS_t which is the source of randomness. Analogously as done in the binomial model we set

$$\Delta = \frac{\partial V}{\partial S}(t, S_t) \; .$$

Therefore $d\Pi$ follows from one hand

$$d\Pi(t, S_t) = \frac{\partial V}{\partial t}(t, S_t)dt + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2}(t, S_t)dt \; , \tag{7}$$

and from the other hand it also have to follow

$$d\Pi(t, S_t) = r\Pi dt \tag{8}$$

by a non-arbitrage argument and from the fact that Π is indifferent from the evolution of S_t (is deterministic). We can now merge the two driving equation, equations (7) and (8), to get the following deterministic PDE:

$$\frac{\partial V}{\partial t}(t, S_t)dt + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2}(t, S_t)dt = r\left(V(t, S_t) - \frac{\partial V}{\partial S}(t, S_t)S_t\right)dt \; ,$$

which is more commonly written in the form:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{9}$$

and has known initial condition at $t = T$ since $V(T, x)$ is the known payoff function.

Equation (9) is the analogous of the arithmetic derivation for pricing option in binomial models since we proceeded by a non-arbitrage approach. Nevertheless there are an equivalent approach by finding a risk-neutral probability and compute the option price as a simple expectation.

5 Monte-Carlo simulation and option pricing

If we do not want to solve PDEs, we can transform the probability space of

$$dS_t = \mu S_t dt + \sigma S_t dB_t \tag{10}$$

to compute $V(t, S_t)$ as an expectation. Even though the formal derivation is out of the scope of the course, from Girsanov's Theorem (please refer to [2]) the risk neutral probability can be achieved by substituting the drift of the SDE by the risk-free rate r, i.e.

$$dS_t = rS_t dt + \sigma S_t dB_t \; .$$

Under the above model the price of the option can be computed as

$$V(t, S_t) = e^{-r(T-t)}\mathbb{E}[V(T, S_T)] \; . \tag{11}$$

One way to compute an approximation of (11) is by doing a Monte-Carlo simulation, that is

$$\mathbb{E}[V(T, S_T)] \cong \frac{1}{M}\sum_{i=1}^{M} \tilde{V}(T, S_T)_i$$

for large M and realizations $\tilde{V}(T, S_T)_i \sim V(T, S_T)$. Obviously we need to be able to simulate $V(T, S_T)$ and this can be done by evaluating the known function $V(T, x)$ over simulations of S_T and the latter can be done using an Euler discretization of (10).

5.1 Euler discretization

Let us consider the general set up of a SDE:

$$dX_t = a(X_t)dt + b(X_t)dB_t \tag{12}$$

for some regular functions a and b. The objective is to simulate an approximation path fort he process in $[0, T]$:

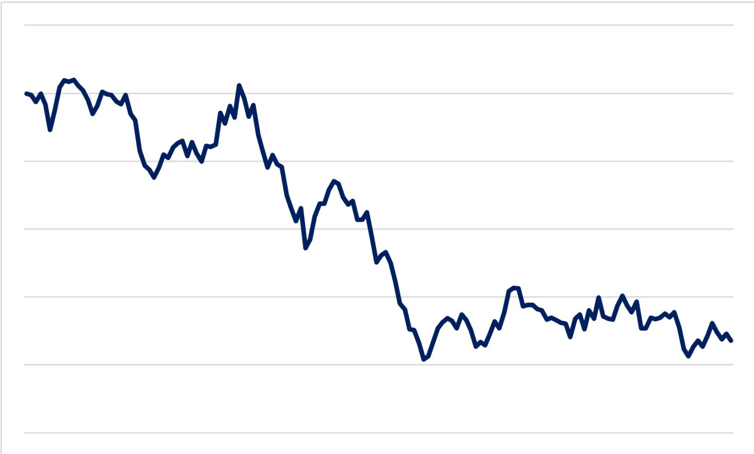


Figure 1. The path of a Brownian Motion.

Let us denote by $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$, where $t_j = j\Delta T$, $\Delta T = T/n$ and $n \in \mathbb{N}$. The Euler discretization of (12) is:

$$X_{t+\Delta T} = X_t + \int_t^{t+\Delta t} a(X_s)ds + \int_t^{t+\Delta t} b(X_s)dB_s$$

and $X_{t_0} = x_0$. For enough regular functions a and b the following holds:

$$\begin{aligned}
\int_t^{t+\Delta t} a(X_s)ds &\approx a(X_t)\int_t^{t+\Delta t} ds = a(X_t)\Delta t \\
\int_t^{t+\Delta t} b(X_s)dB_s &\approx b(X_t)\int_t^{t+\Delta t} dB_s = b(X_t)(B_{t+\Delta t} - B_t) \sim b(X_t)\sqrt{\Delta t}\epsilon_t
\end{aligned}$$

where $\epsilon_t \sim N(0, 1)$.

Summing up all together we end up with the Euler scheme algorithm

$$\begin{cases} X_{t_0} = x_0 \\ X_{t+\Delta T} = X_t + a(X_t)\Delta t + b(X_t)\sqrt{\Delta t}\epsilon_t, \qquad \epsilon_t \sim N(0,1) \end{cases} . \tag{13}$$

The piecewise linear function arising from the algorithm is a path skeleton of the Brownian motion:

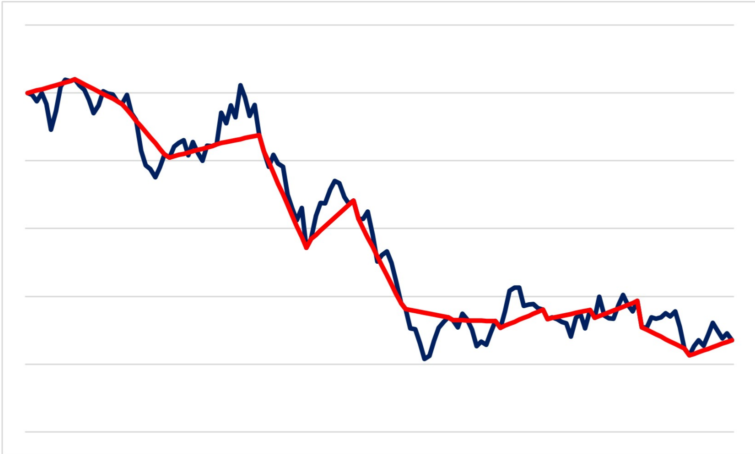


Figure 2. The red line is a piecewise linear constant approximation of the rough path of a Brownian Motion in blue.

: Pricing Options

Let us consider that we want to price a Call option under the model

$$dS_t = rS_t dt + \sigma S_t dB_t$$

Monte-Carlo algorithm:

1. Simulate a path for S applying the Euler scheme (13) with
   - $S_{t_0} = S_0$
   - $a(S) = rS$
   - $b(S) = \sigma S$

   for a given discretization mesh $N \in \mathbb{N}$
2. Apply the payoff function, V, to S_{t_N}
3. Repeat steps 1 and 2 for $M \in \mathbb{N}$ times
4. The price of the option is

$$\left(\frac{1}{M} \sum_{i=1}^{M} V(S_N^i) \right) e^{-rT} \qquad \text{as } N, M \to \infty$$

6 Recap

This chapter finalizes the module on stochastic calculus. We have completed a continuous model for stock price evolution which is widely accepted in the industry and show two equivalent ways to price option under the model. The first option is by solving the known Black-Scholes PDE and the second approach is by performing a Monte-Carlo simulation.

The advantage of Monte-Carlo simulation methods is clear in this kind of problems and are by far the most popular approach between practitioners. There are also other advantages which are not so clear in this setting but enough to mention that Monte-Carlo methods are easiest to scale than PDEs numerical methods.

References

[1] Paul Willmott (2013) *Introduction to Quantitative Finance*, Wiley.

[2] Bernt Oksendal (2005) *Stochastic Differential Equation*, Springer.