Introduction to Financial Engineering (W5)

Stochastic Calculus: Valuation of Financial Derivatives

1 Context

The aim of this chapter is to introduce you to the very basic tools used to price financial derivatives. The techniques used here are common practice in front-office departments, market risk divisions or in general which ever financial activity involved directly to trading.

Although we already know that equity markets are small compared to overall financial markets, it is convenient to introduce the following tools through the simple product of equity shares. Nevertheless the ideas explained here can be applied to a wider variety of products.

We will also simplify our exposition by restricting the scope of derivatives to what is known in the market as *plain vanilla* derivatives.

We will also introduce some amount of trading jargon which is used in financial trading activities, the concepts behind them are simple but sometimes rookies may struggle at first. This is just a way to make easy concepts complicated.

2 Products, Markets and Derivatives

2.1 Equities

Equities will be our basic financial instrument in this chapter. They are also known as stocks, shares, assets or when referred to options on equities, underlying.

: Share

Owning an equity share means partially ownership of the company and hence rights on dividends.

The following time line represents the most common process for a company to go into the market:

1. **A company is born**: When a company is founded is own by its founder who typically have capitalize the company with their own private capital. They own and run the company.
2. **A company grows up**: Founders typically make further investments or ask for loans to banks to expand the company. If it well know, the company may also issue debt which typically is bought by specialized funds on private equity.
3. **IPO (Initial Public Offering**: When a particular investment is too much for the owners to cope with, the company may go public in the stock market and rise capital against offering stocks.
   - Shareholders have a say in running the company
   - They receive part of the benefits of the company in form of dividends
4. **M&A,Takeovers, ...**: If an investor is able to buy all outstanding shares in the market, the company can become private again. This is a very rare event and usually a company ends its life by merging with another company or bought by a competitor or other stakeholder.

There are many theories about pricing formation in stock markets, among the most important ones are

- **Dividend Cash Flow**: Market price reflect consensus on future dividend forecast.
- **Residual Value**: Market price reflect consensus on company's book value, that is the residual value of all assets minus all liabilities.

The first approach tends to view the company as a living entity while the other tends to take a current static picture of it. In practice, market players use both valuation methodologies and hence market prices are formed by a mixture of different views using either of the above.

Even if we could explain all the particularities of the model, there is some randomness in price evolution arising from the different views on the input of the models. In other words, randomness in prices is a manifestation of investor's different views on the picture or behavior of the company. A dividend cash flow model is pretty simple in its mathematical foundations, but investors may take different view on the forecast of dividends for example. The different estimations may rise from different appreciations of

- Predictability: How certain are those future dividends?
- Litigation: Are there any legal process or regulation that may affect the performance of the company?
- Market Evolution: Is the position of the company in jeopardize by any competitor?
- Consumer Response: Are consumers going to change their preference?
- Geopolitical Events: Is the performance of the company affected by a potential conflict?
- ...

The following example shows the effect on market price of a news related to the outcome of a litigation process with respect to fees that banks asked to clients when setting up their mortgages. As you can see the news affected instantaneously the price of banks (Sabadell and Santander), but did not have any appreciable effect on the other two examples (Inditex and CIE (automobile component manufacturer)).

PAGARÁN EL TRIBUTO SOBRE ACTOS JURÍDICOS

Los bancos se hunden tras la 'bofetada' del Supremo con los impuestos de la hipoteca

El alto tribunal establece que es el banco y no el cliente quien debe abonar el tributo sobre Actos Jurídicos Documentados en las escrituras públicas con garantía hipotecaria

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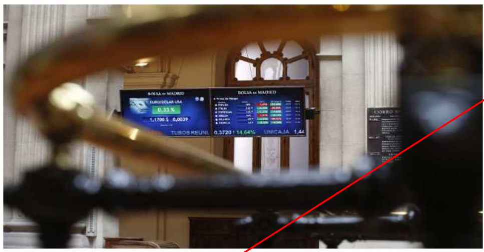

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2.2 Arbitrage and Markets

A more general paradigm for market equilibrium arises from the assumption that markets are efficient in terms of information. The paradigm is related to the information present in the price at the moment of making an investment. This hypothesis is known as Efficient Market Hypothesis (EMH).

There are 3 levels of efficiency:

- **Weak Efficiency**: Means that only the price history of the security is the available information.
- **Semi-strong Efficiency**: All public information known to the present is available.
- **Strong Efficiency**: All public and private information known to the present is available.

In practice, professional players on the market, have access to better sources of information and are also able to trade more quickly that the majority of the market and hence we cannot say, even though it might be assumed from a theoretical point of view, that the market is completely efficient.

Even though market might be a bit inefficient, it is very difficult to find what is known as arbitrage opportunities:

: Aribtrage

Is the process of trading a single product in two different markets with different prices. The trader would buy the security in the market that shows the cheapest price and sell it immediately in the market showing the highest price and hence locking a profit with no risk.

In jargon, arbitrage free market is sometimes described as: *there is not a such thing as a free lunch*, meaning that you cannot make a profit with any risk.

In fact if there is an arbitrage opportunity of security X quoting a price p_1 in market M_1 which is different than the price of the same security in market M_2 and it holds that $p_1 > p_2$ then there is an arbitrage opportunity. Once this is spotted by market participants, they will try to benefit from it by buying in M_2 and selling in M_1. Obviously, as more people is buying in M_2 the price p_2 will increase and vice-versa, as more people is selling in M_1 the price p_1 will drop and eventually $p_1 = p_2$ and the arbitrage vanishes.

The above set up is idealized, and there are situations though where the same security may trade different in two markets without arbitrage, may be because there is a different taxation in one market than the other which makes this situation profitless.

As said, it is very difficult to find arbitrage opportunities and when they appear tend to vanish instantly and hence we will consider (and use extensively in the following derivations) that the market is arbitrage free.

: Proposition

Assuming no arbitrage, it holds that:

- two portfolios with the same risk profile and same value at time T, must have the same value at $t < T$
- if A and B are two portfolios worth $V(A,T) > V(B,T)$ at T, it also holds that $V(A,t) > V(B,t)$ for $t < T$

3 Forwards and Futures

Among all type of financial structures, Forwards and Futures are the simplest ones. The objective of this section is to give a formal definition of both contracts and derive the first pricing formula of the course!

: Forward

An agreement where one party, A, promise to buy an asset, S, from another party, B, at a specific time in the future, T, at some specific price, F, agreed today, t. No money changes hands until the end of the contract and the terms of it make it an obligation for both A and B.

A Forward contract is tailor made and the terms and conditions to it are agreed by A and B. Some types of these contracts are so commonly traded that have become standardized and can be cleared in a organized market, those are Futures:

: Future

Similar to a Forward in its concept but the terms are standardized (maturity T, notional S, $\ldots$, ..) and traded in an organized exchange market. Typically settled in cash format.

Since Futures are traded in an exchange, party A will never know party B and the market itself will be in charge of liquidation of the contract. For that reason there is a process called mark to market which reduces the risk of the exchange market by setting on a daily basis the profits and loss of the Future.

: Example

Assume you have a coffee shop and currently buy coffee beans from CoffeeCo at \$5 per kg which is enough to get you a comfortable margin. However, you read from a newspaper that the cyclone season might be larger than expected and become worried that the price of the beams will increase due to low supply.

- **Forward**: You negotiate directly with CoffeeCo to buy 1.200 kg of beams at a price of \$5 per kg in 6 months time. CoffeeCo agrees to it because they have recently upgrade its farming equipment and do not worry the cyclone season. After 6 months:
  - **Cyclone season destroys plantation and coffee beams price rise to \$7 per kg**: You are obliged to buy 1.200 kg of beams at a price of \$5 per kg from CoffeeCo who is in turn obliged to sell, hence saving \$2.400 in costs due to the Forward.
  - **Cyclone season does not destroy plantation and coffee beams price drop to \$4 per kg due to increase in supply**: You are obliged to buy 1.200 kg of beams at a price of \$5 per kg from CoffeeCo who is in turn obliged to sell, hence CoffeeCo earns \$1.200 in benefit over the market price at the time due to the Forward.
- **Future**: You go to the exchange market and look at prices of futures which mature in 6 months time. Each contract is set with respect to 100 kg of beams, therefore you will need to buy 12 contracts (you do not know who sells the beams). Each day the exchange market will monitor the price evolution of the coffee beam and ask you to settle the difference (to not incur in much risk):
  1. Day 1, beam price \$6 per kg: Exchange will send you a cash flow of \$1.200
  2. Day 2, beam price \$4 per kg: Exchange will ask you a cash flow of \$2.400
  3. ...

  After 6 months:
  - **Cyclone season destroys plantation and coffee beams price rise to \$7 per kg**: The exchange will have outflow you \$2.400 and hence you will be able to buy coffee beams at market price effectively at a price of \$5 per kg.
  - **Cyclone season does not destroy plantation and coffee beams price drop to \$4 per kg due to increase in supply**: You have given the exchange an amount of \$1.200 and need to buy the beams in the market making a cost effectively price of \$5 per kg.

Main uses of both contracts are:

- **Speculation**: Taking a market view to make a profit.
- **Hedging**: Lock in price to avoid volatility in costs.

3.1 Computation of Forward price

One might think that the price of a forward contract F, the price agreed today t by the two counterparts A and B, which will be the price to buy/sell the goods in the future T has something to do with the expectation of A or B on the evolution of the price of the asset S. Clearly, as shown in the previous examples, the evolution of the price of the asset will make either A or B to get a profit or a loss, but the price cannot depend on the expectations of A or B.

In other words, you could think the price of the forward as a fair price to participate in a random game. Lets say A sales tickets to participate in a random game where you could profit or loose depending on the outcome of a fair coin, and B wants to participate. The ticket price should be the expected value, $\mathbb{E}$, of the game; otherwise if the price is higher B should not play from a rational point of view and if the price is lower then A will be bankrupt in the long term. Notice that the expected value of the game, depends on the distribution of the random behavior of the coin and not on how lucky B is feeling that day.

The sort of derivations we are going to use are examples of non-arbitrage applications.

Let us recall that a Forward contract obliges you to deliver $\$F$ at time T and to receive the underlying asset $S(T)$. Today's price of the asset is $S(t)$ and the profit or loss of the forward contract at time T is $S(T) - F$. Indeed, as shown in the example above, is $S(T) > F$ then you will be able to receive an asset S which value is $S(T)$ paying F and hence you could sell it immediately in the market and make a profit of $S(T) - F$ and vice-versa, if $S(T) < F$ you have to pay F for something you could buy in the market at a lower price and thus are making a loss of $S(T) - F$.

Let us now derive the forward price by a non-arbitrage argument. We first enter into a Forward contract which cost us nothing today, because is priced fairly, and has a profit or loss at time T of $S(T) - F$. In other words the first trade of the strategy looks like:

Position	**Value at t**	**Value at T**
Forward Contract	0	$S(T) - F$
Short Position on the Stock		
Bank Account		
$\sum$		

The second trade of the strategy is to borrow the asset $S(t)$. In financial markets you could go short a position, meaning you sell something you do not own by borrowing from someone else and giving it back afterwards. Conversely being long in a position means you own it. For simplicity we will assume that borrowing will cost us nothing. Clearly if I borrow an asset today from someone, the value of the asset today is $S(t)$, but when I returned back it will be valued $S(T)$. The negative sign below means I own the asset to someone else:

Position	**Value at t**	**Value at T**
Forward Contract	0	$S(T) - F$
Short Position on the Stock	$-S(t)$	$-S(T)$
Bank Account		
$\sum$		

Now that I have the asset $S(t)$, which I borrowed, I can sell it and get $S(t)$ monetary units which I can deposit in a bank account to earn interest at a rate r. Assume we earn interests in continuous compound form:

Position	**Value at t**	**Value at T**
Forward Contract	0	$S(T) - F$
Short Position on the Stock	$-S(t)$	$-S(T)$
Bank Account	$S(t)$	$S(t)e^{r(T-t)}$
$\sum$		

At maturity, T, we exercise the Forward contract and receive the asset S which I will give it to the person who lent me the asset at t, and hence magically $S(T)$, which is the only source of randomness, is eliminated from the equations:

Position	**Value at t**	**Value at T**
Forward Contract	0	$\cancel{S(T)} - F$
Short Position on the Stock	$-S(t)$	$-\cancel{S(T)}$
Bank Account	$S(t)$	$S(t)e^{r(T-t)}$
$\sum$		

: Magic

The strategy has been set to eliminate all kind of randomness. Parameters, $S(t)$, r, F and T are fixed and known at time t. The only unknown is $S(T)$ which is the future price of the asset.

Now let us compute the total worth value of the strategy at time t and time T:

Position	**Value at t**	**Value at T**
Forward Contract	0	$S(T) - F$
Short Position on the Stock	$-S(t)$	$-S(T)$
Bank Account	$S(t)$	$S(t)e^{r(T-t)}$
$\sum$	0	$S(t)e^{r(T-t)} - F$

The non-arbitrage setting tells that you cannot make money from nothing, i.e. if you have something that is worth zero it will be worth zero always and hence the price of the Forward contract is:

$$F = S(t)e^{r(T-t)}$$

Note that the price of the contract does not depend on the evolution of the price of the underlying asset $S(T)$ in the same way the price ticket to participate in the random game does not depend on particular outcome of the coin. In the next chapter we will see that in fact on can also derive:

$$F = S(t)e^{r(T-t)} = \mathbb{E}[S(T)]$$

but for that we will need to show a model for the evolution of $S(t)$.

Clearly, $S(t)e^{r(T-t)} > F$ cannot be since in that situation I will perform the above strategy, i.e. buying a forward, borrowing the asset and opening a bank account, and make a positive profit with any investment. If for any reason the preceding situation happens and market participant try to profit from it, they will immediately buy forward contracts which will make F to increase and the inequality to vanish. The opposite is also true, if $S(t)e^{r(T-t)} < F$ I would sell a forward contract, get a loan from a bank and buy the asset reverting all trades of the strategy and hence making a positive profit again with out any investment. As explained before both situations are arbitrage situations are not possible or will disappear immediately.

4 Derivatives

Derivatives are financial assets whose value depends on the evolution of another asset. For example forward contract can be considered a derivative because the price of the forward, F, depends on the current level of the underlying asset $S(t)$. The definition seems very wide and indeed is because you can trade virtually any kind of dependency from other assets. But the following concepts tend to be common aspects of all derivatives:

- **Premium** The amount paid for the contract initially. How to find this value is the final objective of this Chapter.
- **Underlying** The financial instrument on which the option value depends.
- **Strike** The amount for which the underlying can be bought or sold.
- **Expiration** Date on which the derivative matures.
- **Payoff** The function that describes the profit or loss of the derivative depending on the current level of the underlying.

In the above example of the forward contract, the *premium* was zero, the *underlying* was S, the strike F, the *expiration* T and *payoff function* was $f(x) = x - F$. We often use also the letter K for the strike in the following sections.

4.1 Plain Vanilla Options

While forward terms are written in such a way that the trade becomes an obligation for both parties A and B. The next type of derivatives are options which make asymmetric obligations for A and B.

: Call Option

An agreement where one party, A, has the right but not the obligation to buy an asset, S, from another party, B, at a specific time in the future, T, at some specific price, K, agreed today t. The terms of it make it a right for A but an obligation for B should A exercise the right.

: Put Option

An agreement where one party, A, has the right but not the obligation to sell an asset, S, from another party, B, at a specific time in the future, T, at some specific price, K, agreed today t. The terms of it make it a right for A but an obligation for B should A exercise the right.

On a forward contract the obligation is binding for both parties A and B and hence the fair price is 0, neither of them knows the outcome and the agreed price K or F is the expected payoff. On calls or puts one party, A, has an advantageous position because depending on the evolution of the underlying S, A might decide to exercise the option or not, while B is a passive player and in the event of an exercise is obliged to perform the sell or the buy. Therefore it is clear that the price of the option is never 0 and A will always need to pay B a premium at t.

Options are traded for the same reasons as for forward, which is hedging and speculation.

The understanding of options is helped by the visual interpretation of an option's value at expiration, i.e. the payoff function.

Let's start with a CALL option and assume you hold a CALL with underlying S and strike K, then at time $t = T$:

- $\boldsymbol{S_T > K}$: In this situation you will exercise the right to buy the asset S at a price K because you could make an instant benefit by selling it to the current market price, that is a profit of $S_T - K$.

- $\boldsymbol{S_T < K}$: You will not exercise the right to buy the asset at price K because you can buy it in the market at S_T which is cheaper, hence your benefit is 0.

To summarize it the payoff function at $t = T$ is $\max(S_T - K, 0)$

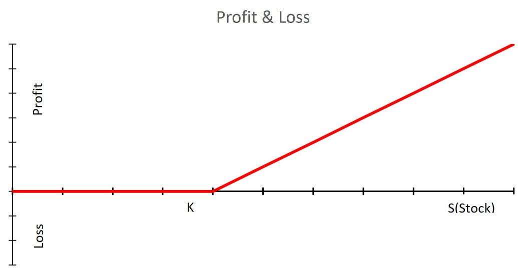


Figure 1. Payoff function for a CALL option

Obviously options give the holder the right but not the obligation and hence payoff functions need to incorporate the premium to account for the potential loss if there is no exercise. At the very least you know that there is no downside to own an option, it gives you right but no obligations.

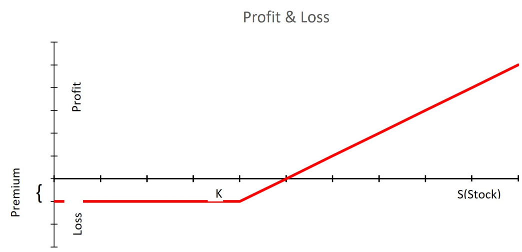


Figure 2. Payoff function for a CALL option including Premium. The intersection with the x-axis is called Breakeven point.

Note that being short a CALL option has the same payoff function which is now reflected along side the x-axis:

Figure 3. Short position on a CALL option.

In a similar way one can build up the payoff of a long position in a PUT option:

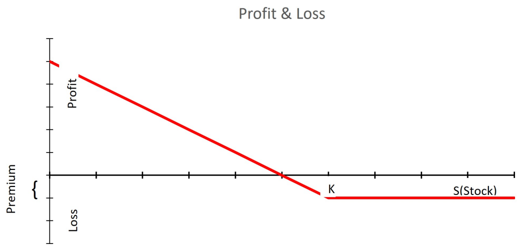


Figure 4. Payoff function for a PUT option

: Premium

The aim of this module is to obtain a formula to compute the premium of vanilla, calls and puts, options.

We have now seen how much calls and puts worth at maturity, but the central question is how much they worth before expiry at $t < T$.

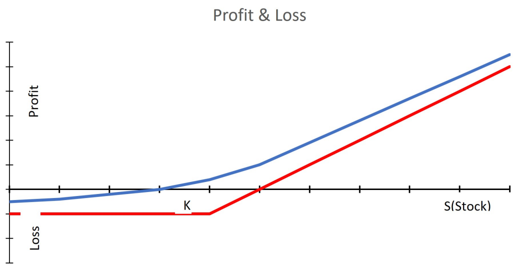


Figure 5. Red line represent Call value at $t = T$ and blue line at $t < T$.

The following variables seem to determine the value of an option before expiry:

- **Value of underlying today**: The higher or lower the value S_t the higher or lower it can be at S_T.
- **Time to maturity**: If the option has a long expiration, S may have more time to rise or fall to reach the strike of the option.
- **Volatility**: Similar to the time to expire effect, the more volatile is S the more possibilities has the value S_T to reach K.
- **Strike**: Conversely, given all the rest of the parameters constants the different strikes may change the probability of S to reach certain levels and hence the option to be exercise.
- **Interest Rates**: Not obvious up to now, but if you think of an option as the expected outcome discounted to present time, you will notice that interest rate need to play a role in option premium computation.

: Exercise

- What is worth more a PUT or a CALL option (all common parameters equal)?
- Does the increase in volatility make an option more or less expensive?
- Does a longer time to maturity make an option more or less expensive?
- Given two identical CALL options with only different strike, $K_1 < K_2$, which option is more expensive? The one with the low strike or the high strike?
- Given two identical PUT options with only different strike, $K_1 < K_2$, which option is more expensive? The one with the low strike or the high strike?

5 The Put-Call parity

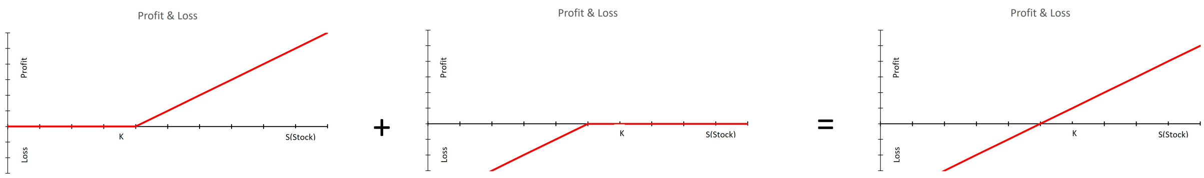
Let us denote by $C(t, T, K, S)$ the price of a CALL option at time t with strike K of S and maturity T, analogously let us denote by $P(t, T, K, S)$ the price of a PUT option. As t gets closer to T the price of a CALL or a PUT option becomes its payoff function, that is:

$$\begin{aligned} \lim_{t \to T} C(t, T, K, S) &= \max(S_T - K, 0) \\ \lim_{t \to T} P(t, T, K, S) &= \max(K - S_T, 0) \end{aligned}$$

Therefore, one can observe that being long a CALL option plus being short a PUT option produces the payoff of a Forward contract at T, indeed:

$$C(T, T, K, S) - P(T, T, K, S) = \max(S_T - K, 0) - \max(K - S_T, 0) = S_T - K$$

Or graphically



Since the right hand side of the equation has the same risk as the left hand side, we can apply the non-arbitrage principle and argue that if both part of the equation are equal at $t = T$, there must also satisfy the equation at $t < T$:

$$\begin{aligned} C(T, T, K, S) - P(T, T, K, S) &= S_T - K \\ C(t, T, K, S) - P(t, T, K, S) &= S_t - Ke^{-r(T-t)} \end{aligned}$$

Even though we do not know how to compute the price of a CALL or a PUT option, we know from the above relation that there is a equation with respect of both prices.

6 Recap

We have presented the basic foundation of financial markets and introduced that principle of no-arbitrage. Using this principle we were able to derive the forward price at which two parties may enter into a Forward. The forward price does not take into account any of the views of the parties and represents the fair price evolution of the underlying. We also introduced vanilla option, CALLS and PUTS, and the relation between them, the PUT-CALL parity. The aim of the following sections is to derive a plausible model for the evolution of S_t which will allow us to find the premiums of CALLs and PUTs.

References

[1] Paul Willmott (2013) *Introduction to Quantitative Finance*, Wiley.