1. Escriviu la sèrie de Fourier només en termes de sinus (i també la que és només en termes de cosinus) de la funció

$$f(x) = \begin{cases} \pi/3 & \text{si } x \in (0, \pi/3) \\ 0 & \text{si } x \in (\pi/3, 2\pi/3) \\ -\pi/3 & \text{si } x \in (2\pi/3, \pi) \end{cases}.$$

2. Sigui f una funció 2π-periòdica. Demostreu que si f és decreixent a $[0, 2\pi)$ llavors el coeficient de Fourier

$$b_n(f) = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx) dx, \quad n \geq 1$$

és no-negatiu.

3. Donada una funció 2π-periòdica, tal que $|f(x)| \leq 1$, demostreu que

$$|\widehat{f}(1) - \widehat{f}(0)| \leq \frac{4}{\pi}.$$

Trobeu un exemple que compleixi la igualtat.

Indicació: potser us pot resultar útil utilitzar alguna fórmula trigonomètrica de l'angle doble a l'hora de calcular la integral

4. Sigui $f \in \mathcal{C}^k$ una funció $2\pi-$periòdica. Demostreu que $\widehat{f}(n) = O(|n|^{-k})$ quan $|n| \to \infty$ (*i.e. existeix una constant* $C > 0$ *tal que* $|\widehat{f}(n)| \leq C|n|^{-k}$).

5. Sigui f una funció 2π-periòdica i $n \in \mathbb{N}$. Definim $p(x) = f(nx)$. Demostreu que per a $m \in \mathbb{Z}$ es compleix

$$\widehat{p}(m) = \begin{cases} \widehat{f}(\frac{m}{n}), & \text{si } n|m, \\ 0, & \text{en cas contrari.} \end{cases}$$

2. f 2π-periòdica i decreixent a $[0,2\pi) \Rightarrow \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx) \geq 0$ $\quad n \geq 1$

$$b_n(f) = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx) = \frac{1}{\pi}\cdot\sum_{k=0}^{n-1}\int_{\frac{2\pi k}{n}}^{\frac{2\pi}{n}(k+1)} f(x)\sin(nx)\,dx$$

Per $n \geq 1$ i $0 \leq k \leq n-1$, $\int_{\frac{2\pi k}{n}}^{\frac{2\pi(k+1)}{n}} \sin(nx)\,dx = 0$, per tant:

$$\int_{\frac{2\pi k}{n}}^{\frac{2\pi(k+1)}{n}} f(x)\sin(nx)\,dx = \int_{\frac{2\pi k}{n}}^{\frac{2\pi k+\pi}{n}} \left[f(x) - f\left(\frac{(2k+1)\pi}{n}\right)\right]\sin(nx)\,dx$$

$$+\int_{\frac{2\pi k+\pi}{n}}^{\frac{2\pi(k+1)}{n}} \left[f(x) - f\left(\frac{(2k+1)\pi}{n}\right)\right]\sin(nx)\,dx = A + B$$

A A estem integrant dues funcions positives, ja que

$$\begin{cases} f(x) \geq f\left(\frac{(2k+1)\pi}{n}\right) & \text{per } x \in \left[\frac{2\pi k}{n}, \frac{2\pi k+\pi}{n}\right] \text{ (f decreix)} \\ \sin(nx) \geq 0 & \Rightarrow A \geq 0 \end{cases}$$

i en B estem integrant dues funcions negatives:

$$\left.\begin{array}{l} f(x) \leq f\left(\frac{(2k+1)\pi}{n}\right) \\ \sin(nx) \leq 0 \end{array}\right\} \text{per } x \in \left[\frac{2\pi k+\pi}{n}, \frac{(2k+1)\pi}{n}\right]$$

$$\Rightarrow B \geq 0$$

$\Rightarrow A + B \geq 0$ $\square$

3. f 2π-periòdica i $|f(x)| \leq 1 \Rightarrow |\hat{f}(1) - \hat{f}(0)| \leq \frac{4}{\pi}$

Ex. on es compleixi la igualtat.

$$\boxed{|\hat{f}(1)-\hat{f}(0)|} = \frac{1}{2\pi}\left|\int_{-\pi}^{\pi}(e^{-ix}f(x) - f(x))\,dx\right| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}|e^{-ix}-1|\cdot|f(x)|\,dx$$

$$\leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{|1-e^{ix}|}{|e^{ix}|}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}|1-e^{ix}|\,dx =$$

$1-e^{ix} = (1-\cos x, -\sin x)$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sqrt{(1-\cos x)^2+(-\sin x)^2}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sqrt{2-2\cos x}\,dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sqrt{2(1-\cos x)}\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\left|\sin\left(\frac{x}{2}\right)\right|dx = \boxed{\frac{4}{\pi}}$$

$$\sqrt{\frac{1-\cos x}{2}} = \sqrt{\sin^2\left(\frac{x}{2}\right)} = \left|\sin\left(\frac{x}{2}\right)\right|$$

i $|\int| \leq \int||$

Si provem amb $f(x) = \dfrac{e^{ix}-1}{|e^{ix}-1|}$

(És únics lloc on hem fet desigualtat és $|f(x)| \leq 1$)

cal a $|f(x)|=1$)

$z \cdot \bar{z} = |z|^2$

$$|\hat{f}(1)-\hat{f}(0)| = \frac{1}{2\pi}\left|\int_{-\pi}^{\pi}(e^{-ix}-1)\cdot\frac{e^{ix}-1}{|e^{ix}-1|}\,dx\right| =$$

$$= \frac{1}{2\pi}\left|\int_{-\pi}^{\pi}\frac{|e^{ix}-1|^2}{|e^{ix}-1|}\,dx\right| = \frac{1}{2\pi}\int_{-\pi}^{\pi}|e^{ix}-1|\,dx = \frac{4}{\pi} \quad \square$$

4. $f \in \mathcal{C}^k$ 2π-periòdica $\Rightarrow$ $|\hat{f}(n)| = O(|n|^{-k})$ $|n| \to \infty$

Notem que $\widehat{f'}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(t)e^{-int}\,dt = \left(\begin{matrix} u = e^{-int} & u' = e^{-int}(-in) \\ v' = f' & v = f \end{matrix} \right.$

$= \cancel{\frac{1}{2\pi} f(t)e^{-int}\Big]_{-\pi}^{\pi}} \to 0 + \frac{in}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt = in\cdot\hat{f}(n)$ $k=1$

$\downarrow$

$= \frac{1}{2\pi}\left(f(\pi)e^{-in\pi} - f(-\pi)e^{in\pi}\right) = \frac{1}{2\pi}\left(f(\pi)\cos(n\pi) - f(-\pi)\cos(n\pi)\right)$

$= \frac{\cos(n\pi)}{2\pi}\cancel{(f(\pi) - f(-\pi))} \to 0$ Per continuïtat i 2π-periodicitat.

Per inducció obtenim $\widehat{f^{(k)}}(n) = (in)^k\cdot\hat{f}(n)$

(Càlcul tb vàlid per $n=0$)

Així: (per $n \neq 0$) $|\hat{f}(n)| = \frac{|\widehat{f^{(k)}}(n)|}{|(in)^k|} =$

$= \frac{|\widehat{f^{(k)}}(n)|}{|n|^k} \leq \frac{1}{|n|^k}\underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi}|f^{(k)}(t)|\,dt}_{\text{com que } f^{(k)} \text{ és cont. a } [-\pi,\pi] \text{ té un màx. } | \ | \leq C} \leq \frac{C}{|n|^k}$

5. $p(x) = f(nx)$

$\hat{p}(m) = \frac{1}{2\pi}\int_0^{2\pi} p(t)\cdot e^{-imt}\,dt = \frac{1}{2\pi}\sum_{k=0}^{n-1}\int_{\frac{2\pi k}{n}}^{\frac{2\pi(k+1)}{n}} p(t)\,e^{imt}\,dt$

A cada integral faig el canvi $\theta = t - \frac{2\pi k}{n}$.

Així:
$$\int_{\frac{2\pi k}{n}}^{\frac{2\pi(k+1)}{n}} p(t)\cdot e^{-imt}\,dt = \int_0^{\frac{2\pi}{n}} p\left(\theta + \frac{2\pi k}{n}\right)\cdot e^{-im\left(\theta+\frac{2\pi k}{n}\right)}\,d\theta$$
$$= \circledast$$

La funció $p(x) = f(nx)$ és $\frac{2\pi}{n}$ - periòdica (pq f era 2π-periòdica). Per tant:

$$\left[p\left(x+\frac{2\pi}{n}\right) = f(nx+2\pi) = f(nx) = p(x) \right]$$

$$\circledast = e^{-im\frac{2\pi k}{n}} \cdot \int_0^{\frac{2\pi}{n}} p(\theta)\cdot e^{-im\theta}\,d\theta$$

Substituint-ho a la primera suma tenim que:

$$\hat{p}(m) = \frac{1}{2\pi}\left[\sum_{k=0}^{n-1} e^{-im\frac{2\pi k}{n}}\right]\cdot \int_0^{\frac{2\pi}{n}} p(t)\cdot e^{-imt}\,dt \quad \left[\begin{array}{l} t = \frac{x}{n} \\ dt = \frac{dx}{n}\end{array}\right]$$

$$= \frac{1}{2\pi n}\int_0^{2\pi} p\left(\frac{x}{n}\right)\cdot e^{-i\frac{m}{n}x}\,dx \left(\sum_{k=0}^{n-1} e^{-i\frac{m2\pi k}{n}}\right)$$

Si ~~...~~ $n \mid m \Rightarrow e^{-i\frac{2m\pi k}{n}} = 1 \Rightarrow$

$$\hat{p}(m) = \frac{\cancel{n}}{2\pi \cancel{n}}\int_0^{2\pi} \underbrace{p\left(\frac{x}{n}\right)}_{=f(x)} e^{-i\frac{m}{n}x}\,dx = \hat{f}\left(\frac{m}{n}\right)$$

Si m no és múltiple de n $\Rightarrow$

$$\sum_{k=0}^{n-1} e^{-i\frac{2mk\pi}{n}} \underset{\text{geomètrica}}{=} \frac{1-e^{-i\frac{2m(n)\pi}{n}}}{1-e^{-i\frac{2m\pi}{n}}} = \frac{1-1}{\neq 0} = 0$$

$\Rightarrow \hat{p}(m) = 0$ en aquest cas. □

1. Fem primer l'extensió senar de $f \Rightarrow a_k = 0\ \forall k$

i $b_k = \frac{2}{\pi}\int_0^{\pi} f(x)\cdot \sin(kx)\,dx = \frac{2}{3k}\left[1-\cos\left(\frac{\pi}{3}k\right)+(-1)^k-\cos\left(\frac{2\pi}{3}k\right)\right]$

$\left.\begin{array}{l} (-1)^k \text{ es repeteix cada } 2 \\ \cos\left(\frac{2\pi k}{3}\right) \text{ es repeteix cada } 3 \\ \cos\left(\frac{\pi k}{3}\right) \quad " \quad " \quad \text{cada } 6 \end{array}\right]$ l'expressió la pensem (mod 6)

Com que $\cos\frac{\pi k}{3}$ $\begin{cases} = 1 & \text{per } k \equiv 0 \pmod 6 \\ = 1/2 & \text{si } k \equiv \pm 1 \pmod 6 \\ = -1/2 & \text{si } k \equiv \pm 2 \pmod 6 \\ = -1 & \text{si } k \equiv 3 \pmod 6 \end{cases}$

$\Rightarrow b_k = \begin{cases} \frac{2}{k} & \text{si } k \equiv \pm 2 \pmod 6 \\ 0 & \text{si } k \not\equiv \pm 2 \pmod 6 \end{cases}$

Per tant: $$S f_{senar} = 2 \cdot \sum_{k=0}^{\infty} \left[\frac{\sin((6k+2)x)}{6k+2} + \frac{\sin((6k+4)x)}{6k+4} \right]$$

$$= \sum_{k=0}^{\infty} \left[\frac{\sin((6k+2)x)}{3k+1} + \frac{\sin((6k+4)x)}{3k+2} \right]$$

$$= \sum_{k=0}^{\infty} \frac{\sin((6k+2)x)}{3k+1} + \sum_{m=-\infty}^{-1} \frac{\sin((-6m-2)x)}{-3m-1} =$$

$$\frac{\sin((-6m-2)x)}{-3m-1} = \frac{\sin(6m+2)}{3m+1}$$

$m = -k-1$

$\sin x = -\sin(-x)$

$$= \sum_{k=0}^{\infty} \frac{\sin((6k+2)x)}{3k+1} + \sum_{m=-\infty}^{-1} \frac{\sin((6m+2)x)}{3m+1}$$

$$= \sum_{k \in \mathbb{Z}} \frac{\sin((6k+2)x)}{3m+1}$$

Si ara fem l'extensió parell de $f \to f_P$

Només tindrem termes amb cosinus i terme constant.

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)dx = 0$$

$$a_k = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(kx)dx = \frac{2}{3} \left(\int_0^{\pi/3} \cos(kx)dx - \int_{\frac{2\pi}{3}}^{\pi} \cos(kx)dx \right) =$$

$$= \frac{2}{3k}\left[\left[\sin(kx)\right]_0^{\pi/3} - \sin(kx)\Big|_{2\pi/3}^{\pi} \right] =$$

$$= \frac{2}{3k}\left(\sin\left(\frac{\pi k}{3}\right) + \sin\left(\frac{2\pi k}{3}\right)\right)$$

Com abans, com que $\sin\left(\frac{k\pi}{3}\right) = \begin{cases} = 0 & \text{si } k \equiv 0\,(3) \\ \frac{\sqrt{3}}{2} & k \equiv 1\,(6),\ k \equiv 2\,(6) \\ -\frac{\sqrt{3}}{2} & k \equiv -1\,(6),\ k \equiv -2\,(6) \end{cases}$

Així: $$a_k = \begin{cases} \pm \frac{2}{\sqrt{3}k} & k \equiv \pm 1\ (6) \\ 0 & k \not\equiv \pm 1\ (6) \end{cases}$$

Per tant:

$$Sf_P = \frac{2}{\sqrt{3}} \sum_{k=0}^{\infty} \left[\frac{\cos((6k+1)x)}{6k+1} - \frac{\cos((6k+5)x)}{6k+5} \right]$$

$$= \frac{2}{\sqrt{3}} \sum_{k \in \mathbb{Z}} \frac{\cos((6k+1)x)}{6k+1}$$

on hem fet servir que $\cos(-x) = \cos x$ i hem canviat k per $-k-1$ en el segon sumand (com al cas de l'extensió senar).