Weak formulation of the heat diffusion equation

$$\frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T)) \tag{1}$$

We define a space of test functions and then, multiply each term of the PDE by any arbitrary function as a member of this space:

$$\mathcal{V} = \left\{ v(\mathbf{x}) | \mathbf{x} \in \Omega, v(\mathbf{x}) \in \mathcal{H}^1(\Omega), \text{ and } v(\mathbf{x}) = 0 \text{ on } \Gamma \right\} \tag{2}$$

in which the Ω is the domain of interest, Γ is the boundary of Ω, and $\mathcal{H}^1$ denotes the Sobolev space of the domain Ω, which is a space of functions whose derivatives are square-integrable functions in Ω. The solution of the PDE belongs to a trial function space, which is similarly defined as:

$$\mathcal{S}_t = \left\{ T(\mathbf{x}, t) | \mathbf{x} \in \Omega, t > 0, T(\mathbf{x}, t) \in \mathcal{H}^1(\Omega), \text{ and } \frac{\partial T}{\partial n} = 0 \text{ on } \Gamma \right\} \tag{3}$$

We multiply Eq. 1 to an arbitrary function $v \in \mathcal{V}$:

$$\frac{\partial T}{\partial t} v = \nabla \cdot (k \nabla T) \, v \tag{4}$$

Integrating over the whole domain yields:

$$\int_\Omega \frac{\partial T}{\partial t} v d\omega = \int_\Omega \nabla \cdot (k \nabla T) v d\omega \tag{5}$$

The diffusion term can be split using the integration by parts technique:

$$\int_\Omega \nabla \cdot (k \nabla T) v d\omega = \int_\Omega \nabla \cdot [v(k \nabla T)] d\omega - \int_\Omega (\nabla v) \cdot (k \nabla T) d\omega \tag{6}$$

in which the second term can be converted to a surface integral on the domain boundary by applying the Green's divergence theory. The surface integral is zero because there is a no-flux boundary condition on the boundary of the computational domain (defined in the trial function space according to Eq. 3):

$$\int_\Omega \nabla \cdot [v(k \nabla T)] d\omega = \int_\Gamma k v \frac{\partial T}{\partial n} d\gamma = 0 \tag{7}$$

For the temporal term, we use the finite difference method and apply a first-order backward Euler scheme for discretization, which makes it possible to solve the PDE implicitly:

$$\frac{\partial T}{\partial t} = \frac{T - T^n}{\Delta t} \tag{8}$$

where T^n denotes the value of the state variable in the previous time step (or initial condition for the first time step). Inserting Eqs. 6, 7, and 8 into Eq. 5 yields:

$$\int_\Omega \frac{T - T^n}{\Delta t} v d\omega = -\int_\Omega k \nabla T \cdot \nabla v d\omega \tag{9}$$

By reordering the equation, we get the weak form of Eq. 1:

$$\int_\Omega T v d\omega + \int_\Omega \Delta t k \nabla T \cdot \nabla v d\omega = \int_\Omega T^n v d\omega \tag{10}$$