Weak formulation of the steady diffusion equation

$$-\nabla \cdot (D\nabla u)) = f \tag{1}$$

We define a space of test functions and then, multiply each term of the PDE by any arbitrary function as a member of this space:

$$\mathcal{V} = \left\{v(\mathbf{x})|\mathbf{x} \in \Omega, v(\mathbf{x}) \in \mathcal{H}^1(\Omega),\ \text{and } v(\mathbf{x}) = 0 \text{ on } \Gamma\right\} \tag{2}$$

in which the Ω is the domain of interest, Γ is the boundary of Ω, and $\mathcal{H}^1$ denotes the Sobolev space of the domain Ω, which is a space of functions whose derivatives are square-integrable functions in Ω. The solution of the PDE belongs to a trial function space, which is similarly defined as:

$$\mathcal{S}_t = \left\{u(\mathbf{x})|\mathbf{x} \in \Omega, u(\mathbf{x}) \in \mathcal{H}^1(\Omega),\ \text{and } \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma\right\} \tag{3}$$

We multiply Eq. 1 to an arbitrary function $v \in \mathcal{V}$:

$$-\nabla \cdot (D\nabla u)\, v = fv \tag{4}$$

Integrating over the whole domain yields:

$$-\int_\Omega \nabla \cdot (D\nabla u) v d\omega = \int_\Omega f v d\omega \tag{5}$$

The diffusion term can be split using the integration by parts technique:

$$\int_\Omega \nabla \cdot (D\nabla u) v d\omega = \int_\Omega \nabla \cdot [v(D\nabla u)] d\omega - \int_\Omega (\nabla v) \cdot (D\nabla u) d\omega \tag{6}$$

in which the second term can be converted to a surface integral on the domain boundary by applying the Green's divergence theory. The surface integral is zero because there is a no-flux boundary condition on the boundary of the computational domain (defined in the trial function space according to Eq. 3):

$$\int_\Omega \nabla \cdot [v(D\nabla u)] d\omega = \int_\Gamma Dv \frac{\partial u}{\partial n} d\gamma = 0 \tag{7}$$

Inserting Eqs. 6 into 7 yields:

$$\int_{\Omega} \nabla \cdot (D\nabla u) v d\omega = -\int_{\Omega} (\nabla v) \cdot (D\nabla u) d\omega \tag{8}$$

Substituting this into Eq. 5 results in the weak form of Eq. 1:

$$\int_{\Omega} (\nabla v) \cdot (D\nabla u) d\omega = \int_{\Omega} f v d\omega \tag{9}$$

or

$$\int_{\Omega} (\nabla v) \cdot (D\nabla u) d\omega - \int_{\Omega} f v d\omega = 0 \tag{10}$$