Hierarchy of astrodynamics models

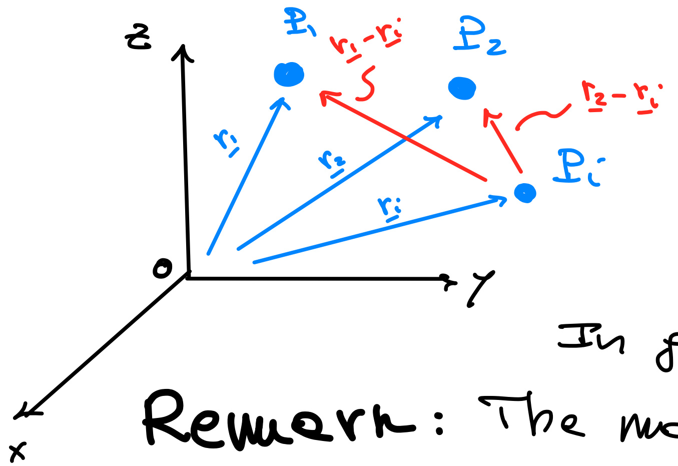
$$\underline{\dot{x}} = \underline{f}(\underline{x}, t)$$

$\underline{x}$: STATE, t: TIME, $\underline{f}$: RHS (VECTOR FIELD)

Aim: Develop models for s/c flying "well beyond" LEO (no Earth oblateness, no atmospheric drag).

General and restricted problems

We have n celestial bodies (P_i, $i=1,\dots,n$) that interact mutually by virtue of their gravitational attractions. The eqs. of motion for the i-th body (P_i)



in an inertial frame (centered in the SSB):

$$m_i \ddot{\underline{r}}_i = \frac{G m_1 m_i}{\|\underline{r}_1 - \underline{r}_i\|^3}(\underline{r}_1 - \underline{r}_i) + \frac{G m_2 m_i}{\|\underline{r}_2 - \underline{r}_i\|^3}(\underline{r}_2 - \underline{r}_i) + \dots$$

In general, we have to write this equation n times.

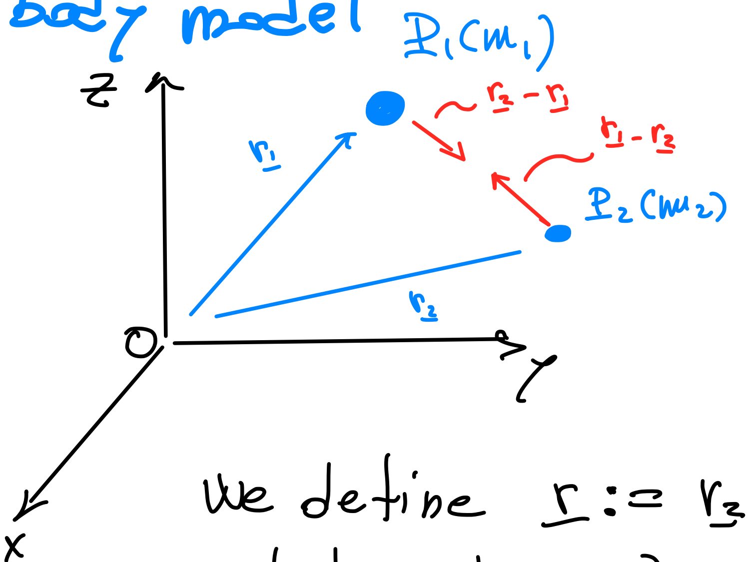
Remark: The motion of $n-1$ celestial bodies is given (by ephemeris models); we are interested in the motion of the "massless" particle (or spacecraft) that does not affect the motion of the other $n-1$ bodies.

Hierarchy of models

(One has to use the model that is most appropriate depending on the needs!)

- 2-body
- 3-body (circular or elliptic, Hill's, etc.)
- 4-body (coherent or non-coherent, concentric or bicircular)
- ...
- n-body (perturbed Kepler o 3-body, full ephemeries w/ non-gravit. pert.)

2-body model



The eqs. of motion are:

$$\begin{cases} m_1 \ddot{\underline{r}}_1 = \dfrac{G m_1 m_2}{\| \underline{r}_2 - \underline{r}_1 \|^3} (\underline{r}_2 - \underline{r}_1) & (P_1) \\ m_2 \ddot{\underline{r}}_2 = \dfrac{G m_1 m_2}{\| \underline{r}_1 - \underline{r}_2 \|^3} (\underline{r}_1 - \underline{r}_2) & (P_2) \end{cases}$$

We define $\underline{r} := \underline{r}_2 - \underline{r}_1$ (to describe the motion of P_2 relative to P_1); we also define $\mu := G(m_1 + m_2)$, we have:

$$\boxed{\ddot{\underline{r}} + \frac{\mu}{r^3} \underline{r} = 0}$$

Remarks

i) in the case of the 2-body model we can solve both the general and the restricted problem. The difference is in the definition of μ:

$\mu = G(m_1 + m_2)$ general pb

$\mu = G m_1$ restricted pb.

ii) The canonical form for astrodynamics models:

$$\underline{x} = \begin{pmatrix} \underline{r} \\ \underline{v} \end{pmatrix} \qquad \begin{cases} \dot{\underline{r}} = \underline{v} \\ \dot{\underline{v}} = \underline{g}(\underline{r}) \end{cases} \Rightarrow \dot{\underline{x}} = \underline{f}(\underline{x}) \Rightarrow \begin{pmatrix} \dot{\underline{r}} \\ \dot{\underline{v}} \end{pmatrix} = \begin{pmatrix} \underline{v} \\ -\frac{\mu}{r^3}\underline{r} \end{pmatrix}$$

iii) We want to solve the IVP: $\begin{cases} \dot{\underline{x}} = \underline{f}(\underline{x}, t) \\ \underline{x}(t_0) = \underline{x}_0 \end{cases}$

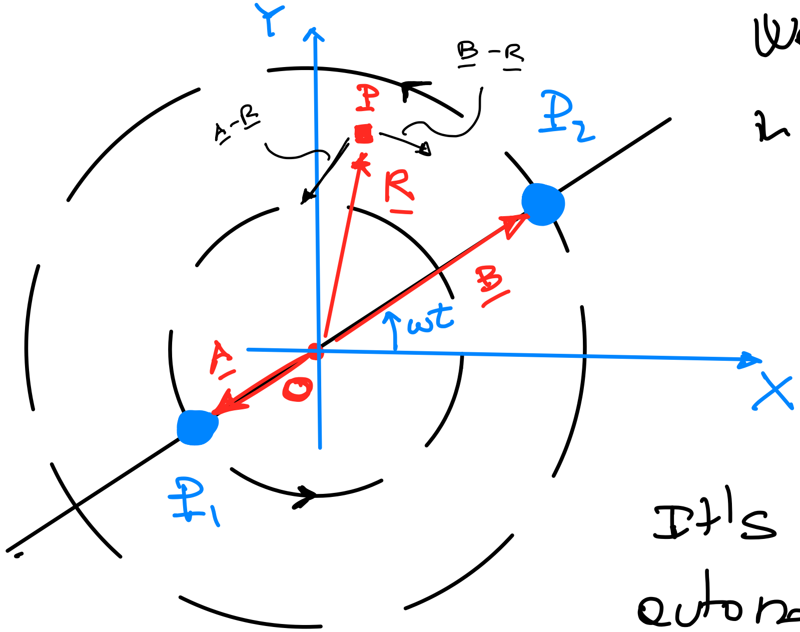
In the 2-body problem, we can solve the IVP by "pushing forward" $\underline{x}_0$

$$\begin{cases} \underline{r}(t) = F\,\underline{r}_0 + G\,\underline{v}_0 \\ \underline{v}(t) = F_t\,\underline{r}_0 + G_t\,\underline{v}_0 \end{cases} \Rightarrow \begin{pmatrix} \underline{r}(t) \\ \underline{v}(t) \end{pmatrix} = \begin{bmatrix} F & G \\ F_t & G_t \end{bmatrix} \begin{pmatrix} \underline{r}_0 \\ \underline{v}_0 \end{pmatrix}$$

F, G, F_t, G_t are Lagrange coefficients, which require the true anomaly (given by Kepler's implicit equation).

3-body model (circular)

Spacecraft (P) subject to gravitational attraction of 2 primaries (P_1, P_2), which move in circular orbits by virtue of their mutual gravitational attraction.



We can write the eqs. of motion for P in an inertial frame centered at O (CoM):

$$\ddot{\underline{R}} = \frac{G m_1}{\|\underline{A}-\underline{R}\|^3}(\underline{A}-\underline{R}) + \frac{G m_2}{\|\underline{B}-\underline{R}\|^3}(\underline{B}-\underline{R})$$

We'd like to write this as $\dot{\underline{x}} = \underline{f}(\underline{x}, t)$

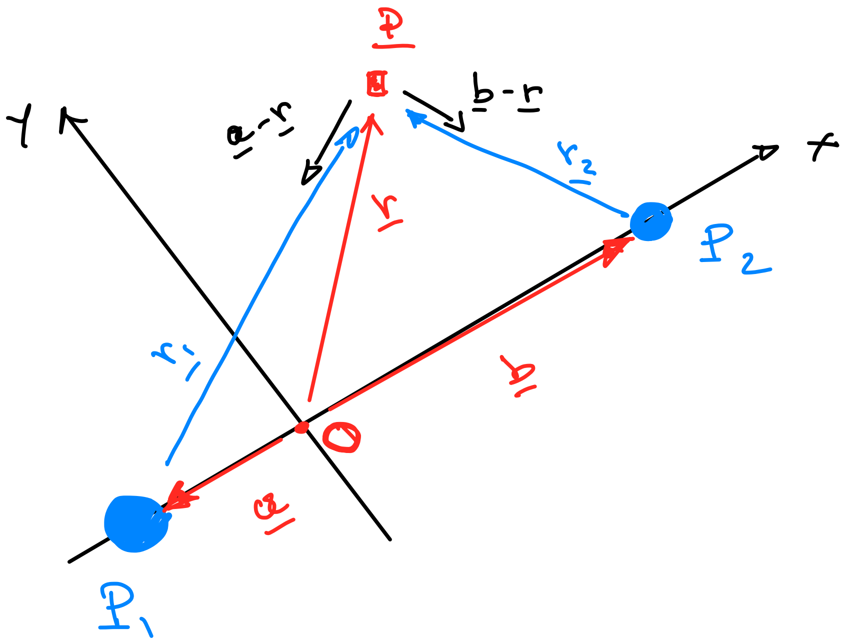
It's much more convenient to work with autonomous systems ($\dot{\underline{x}} = \underline{f}(\underline{x})$) because their features are valid for all times!

In the equation above we have

$$\begin{cases} \underline{A} = \underline{A}(t) = -A\cos\omega t\,\underline{I} - A\sin\omega t\,\underline{J} \\ \underline{B} = \underline{B}(t) = B\cos\omega t\,\underline{I} + B\sin\omega t\,\underline{J} \end{cases}$$

Hence, we have

$$\begin{cases} \dot{\underline{r}} = \underline{v} \\ \dot{\underline{v}} = \underline{g}(\underline{r}, t) \end{cases} \Rightarrow \dot{\underline{x}} = \underline{f}(\underline{x}, t)$$



$$\begin{cases} \underline{a} = -a\,\underline{i} \\ \underline{b} = b\,\underline{i} \end{cases}, \qquad \underline{\omega} = \omega\,\underline{k}$$

In rotating frames $\dfrac{d\square}{dt} = \dot{\square} + \underline{\omega} \times \square$

$$\frac{d\underline{r}}{dt} = \dot{\underline{r}} + \underline{\omega} \times \underline{r}$$

$$\frac{d^2\underline{r}}{dt^2} = \underbrace{\frac{d}{dt}(\dot{\underline{r}})} + \frac{d}{dt}(\underline{\omega} \times \underline{r}) =$$

$$= \ddot{\underline{r}} + \underline{\omega} \times \dot{\underline{r}} + \dot{\underline{\omega}} \times \underline{r} + \underline{\omega} \times (\dot{\underline{r}} + \underline{\omega} \times \underline{r}) =$$

$$= \ddot{\underline{r}} + 2\,\underline{\omega} \times \dot{\underline{r}} + \dot{\underline{\omega}} \times \underline{r} + \underline{\omega} \times (\underline{\omega} \times \underline{r})$$

Eqs. of motion for P:

$$\ddot{\underline{r}} + 2\underline{\omega} \times \underline{r} + \underbrace{\dot{\underline{\omega}}}_{0} \times \underline{r} + \underline{\omega} \times (\underline{\omega} \times \underline{r}) = \frac{G m_1}{\|\underline{a} - \underline{r}\|^3}(\underline{a} - \underline{r}) + \frac{G m_2}{\|\underline{b} - \underline{r}\|^3}(\underline{b} - \underline{r})$$

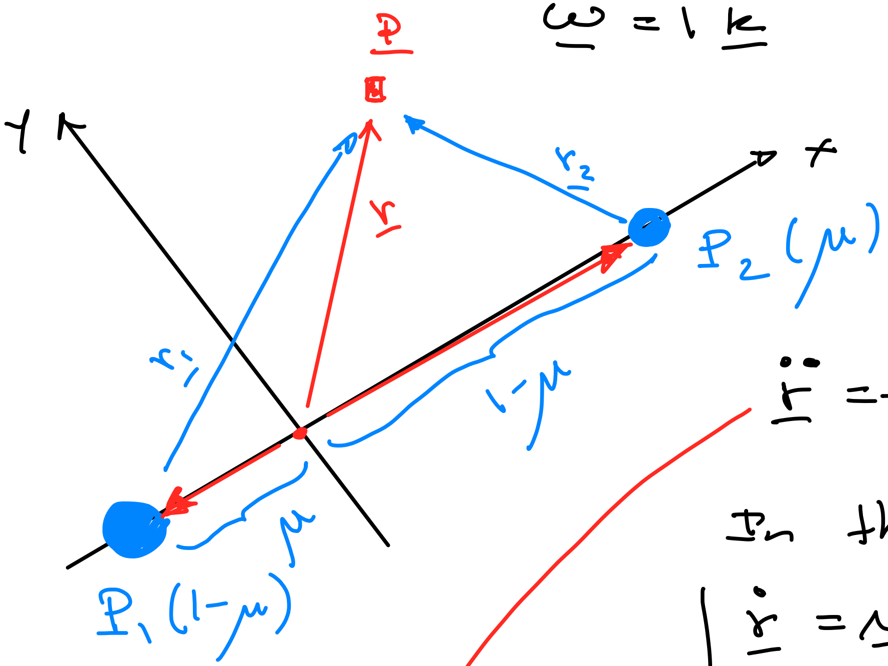
$$\underline{r} = x\,\underline{i} + y\,\underline{j}$$

$$\begin{cases} \underline{r}_1 = \underline{r} - \underline{a} = (x + a)\,\underline{i} + y\,\underline{j} \\ \underline{r}_2 = \underline{r} - \underline{b} = (x - b)\,\underline{i} + y\,\underline{j} \end{cases}$$

Remark It can be shown that the system depends on one only parameter:

$$\mu := \frac{m_2}{m_1 + m_2}$$

Assumptions: $\begin{cases} P_1 - P_2 \text{ distance} = 1 \\ P_1 - P_2 \text{ period} = 2\pi \quad (\omega = 1) \end{cases}$



$\boldsymbol{\omega} = 1\,\mathbf{k}$

$$\begin{cases} \mathbf{r}_1 = (x+\mu)\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} \\ \mathbf{r}_2 = (x+\mu-1)\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} \end{cases}$$

Eqs. of motion for P_1 (scaled):

$$\ddot{\mathbf{r}} = -2\boldsymbol{\omega}\times\dot{\mathbf{r}} - \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) - \frac{(1-\mu)}{r_1^3}\mathbf{r}_1 - \frac{\mu}{r_2^3}\mathbf{r}_2$$

In the canonical form:

$$\begin{cases} \dot{\mathbf{r}} = \mathbf{v} \\ \dot{\mathbf{v}} = \underbrace{-\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) - \frac{(1-\mu)}{r_1^3}\mathbf{r}_1 - \frac{\mu}{r_2^3}\mathbf{r}_2}_{\mathbf{g}(\mathbf{r})} \underbrace{- 2\boldsymbol{\omega}\times\mathbf{v}}_{\mathbf{h}(\mathbf{v})} \end{cases}$$

Component-wise this is:

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = \begin{pmatrix} 2\dot{y} \\ -2\dot{x} \\ 0 \end{pmatrix} + \underbrace{\begin{pmatrix} x \\ y \\ 0 \end{pmatrix} - \frac{(1-\mu)}{r_1^3}\begin{pmatrix} x+\mu \\ y \\ z \end{pmatrix} - \frac{\mu}{r_2^3}\begin{pmatrix} x+\mu-1 \\ y \\ z \end{pmatrix}}_{\text{this term is the gradient of the potential function}}$$

$$\Omega(x,y,z) = \frac{1}{2}(x^2+y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$$

$$\boxed{\begin{aligned} \ddot{x} - 2\dot{y} &= \Omega_x \\ \ddot{y} + 2\dot{x} &= \Omega_y \\ \ddot{z} &= \Omega_z \end{aligned}}$$

with $\nabla\Omega = \begin{pmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{pmatrix}$

Lagrange points

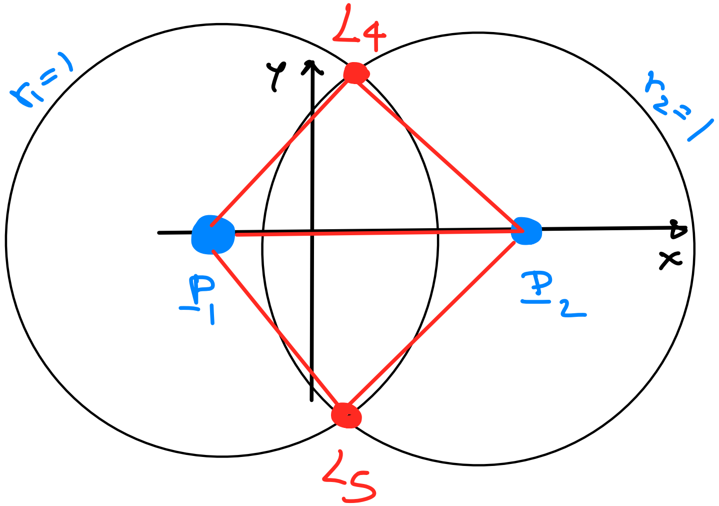
Equilibrium points of the RTBP. In general the equilibrium points of $\dot{\underline{x}} = \underline{f}(\underline{x})$ are those points $\underline{x}$ s.t. $\underline{f}(\underline{x}) = 0$.

$$\dot{\underline{x}} = \underline{f}(\underline{x}) \Rightarrow \begin{cases} \dot{x} = \cancel{v_x} \\ \dot{y} = \cancel{v_y} \\ \dot{z} = \cancel{v_z} \\ \dot{v}_x = \cancel{2v_y} + x - \dfrac{(1-\mu)}{r_1^3}(x+\mu) - \dfrac{\mu}{r_2^3}(x+\mu-1) \\ \dot{v}_y = -\cancel{2v_x} + y - \dfrac{(1-\mu)}{r_1^3}y - \dfrac{\mu}{r_2^3}y \\ \dot{v}_z = -\cancel{\dfrac{(1-\mu)}{r_1^3}z} - \cancel{\dfrac{\mu}{r_2^3}z} \end{cases}$$

In order to set the RHS to zero we need to take

- $v_x = v_y = v_z = 0$
- $z = 0$ (see last equation) Equilibrium points lie on (x,y)-plane

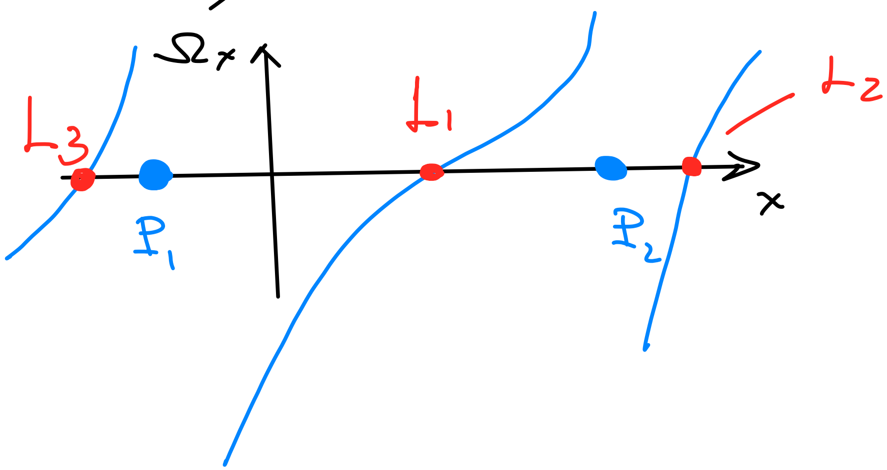
1) $r_1 = r_2 = 1$ produces $\underline{f}(\underline{x}) = 0$



$\Rightarrow \{L_4, L_5\}$: Triangular points

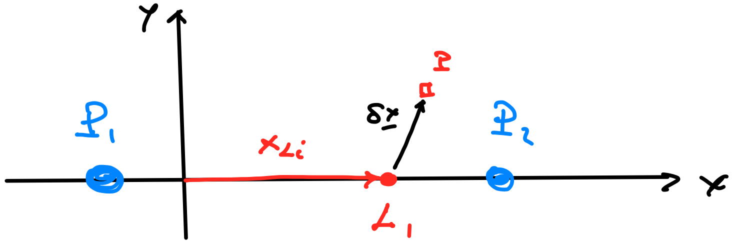
2) $y=0$ (equilibrium points on x-axis)

The function $\Omega_x(x,0,0)$ is



$\Rightarrow \{L_1, L_2, L_3\}$: collinear points

Motion about collinear points



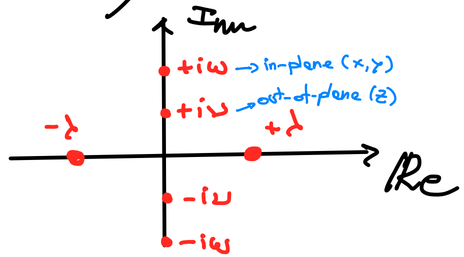
$\delta\underline{x} := \underline{x} - \underline{x}_{eq}$

$\dot{\underline{x}} = \underline{f}(\underline{x}) \Rightarrow \cancel{\dot{\underline{x}}_{eq}} + \delta\dot{\underline{x}} = \cancel{\underline{f}(\underline{x}_{eq})} + \frac{\partial \underline{f}}{\partial \underline{x}}\Big|_{\underline{x}_{eq}} \delta\underline{x}$ $\Rightarrow$ $\boxed{\delta\dot{\underline{x}} = A\,\delta\underline{x}}$ $\quad A := \frac{\partial \underline{f}}{\partial \underline{x}}\Big|_{\underline{x}_{eq}}$

For L_1 and L_2, the linearized eqs. of motion are

$$\begin{cases} \ddot{x} - 2\dot{y} - (1+2c_2) = 0 \\ \ddot{y} + 2\dot{x} + (c_2 - 1) = 0 \\ \ddot{z} + c_2 z = 0 \end{cases}$$

with $c_2(\mu) = \dfrac{\mu}{(1-\mu-x_{Li})^3} + \dfrac{1-\mu}{(\mu+x_{Li})^3}$



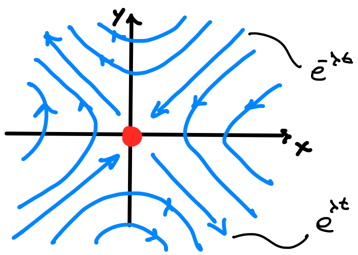
The eigenvalues of A are: $eig(A) = \{\pm\lambda, \pm i\omega, \pm i\nu\}$

The linearized solution is (in normal form)

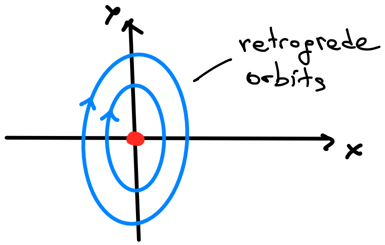
$$\begin{cases} x(t) = A_1 e^{\lambda t} + A_2 e^{-\lambda t} + A_x \cos(\omega t + \varphi) \\ y(t) = -k_1 A_1 e^{\lambda t} + k_1 A_2 e^{-\lambda t} - k_2 A_x \sin(\omega t + \varphi) \\ z(t) = A_z \cos(\nu t + \psi) \end{cases}$$

$k_1 = \dfrac{2c_2 + 1 - \lambda^2}{2\lambda}$, $k_2 = \dfrac{2c_2 + 1 + \omega^2}{2\omega}$

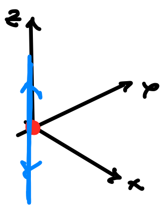
1) Hyperbolic orbits
($A_1 \neq 0, A_2 \neq 0, A_x = \varphi = A_z = \psi = 0$)



2) Planar orbits
($A_x \neq 0, \varphi \neq 0, A_1 = A_2 = A_z = \psi = 0$)



3) Vertical orbits
($A_z \neq 0, \psi \neq 0, A_1 = A_2 = A_x = \varphi = 0$)

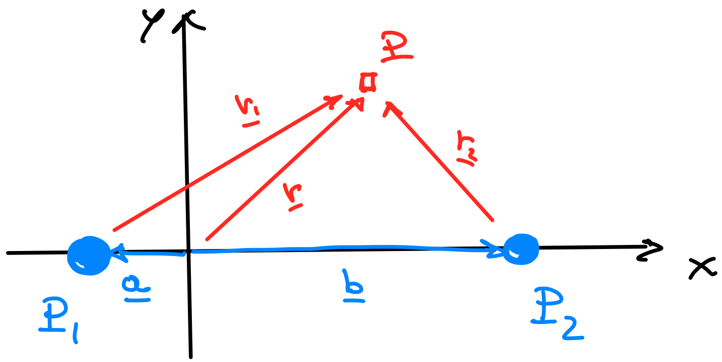


Remark

Three-dimensional "halo" orbits exist about collinear Lagrange points. They arise from a 3rd-order (or higher) analysis of the solution.

3-body model (elliptic)

Similar to the RTBP, but P_1 and P_2 move on elliptic orbits



$$a(\theta) = \frac{a_1}{1+e\cos\theta} \Rightarrow \underline{r}_1 = \underline{a} - \underline{r} \Rightarrow \underline{r}_1 = \underline{r}_1(\theta)$$

$$b(\theta) = \frac{a_2}{1+e\cos\theta} \Rightarrow \underline{r}_2 = \underline{b} - \underline{r} \Rightarrow \underline{r}_2 = \underline{r}_2(\theta)$$

The eqs. of motion for P are:

$$\ddot{\underline{r}} + 2\underline{\omega}\times\dot{\underline{r}} + \underline{\omega}\times(\underline{\omega}\times\underline{r}) + \overset{\text{NON ZERO}}{\overbrace{\dot{\underline{\omega}}}}\times\underline{r} = -\frac{G m_1}{r_1(\theta)^3}\underline{r_1(\theta)} - \frac{G m_2}{r_2(\theta)^3}\underline{r_2(\theta)}$$

Remarks

i) Despite using a rotating frame, the system is still time-dependent

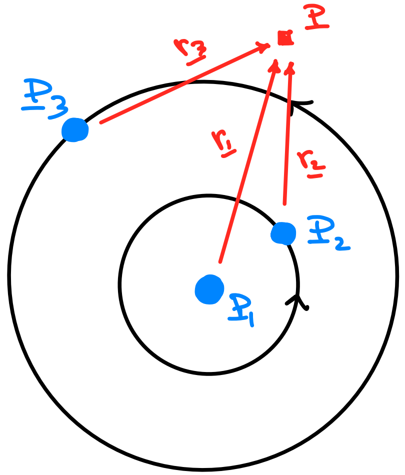
ii) It is convenient to use θ as independent variable $\Rightarrow \dot{\underline{x}} = \underline{f}(\underline{x},\theta)$

4-body models

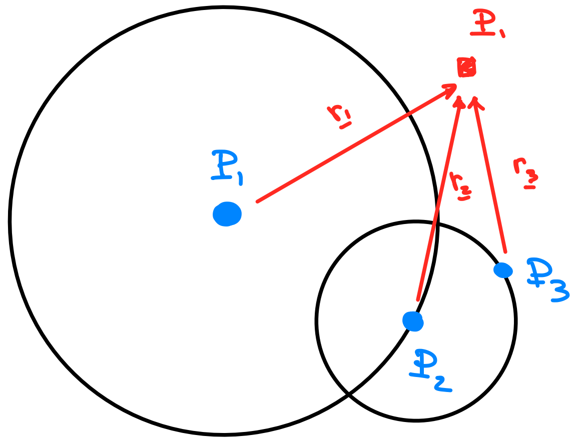
P_1, P_2, and P_3 that move under their mutual gravitational attractions

There are two different models

- concentric • (planet – moon 1 – moon 2)
- bicircular • (Sun – Earth – Moon)



$\leftarrow \left\{ \begin{array}{l} P_2, P_3 \text{ on circular orbits} \\ P_1, P_2, P_3 \text{ on same plane} \end{array} \right\} \rightarrow$

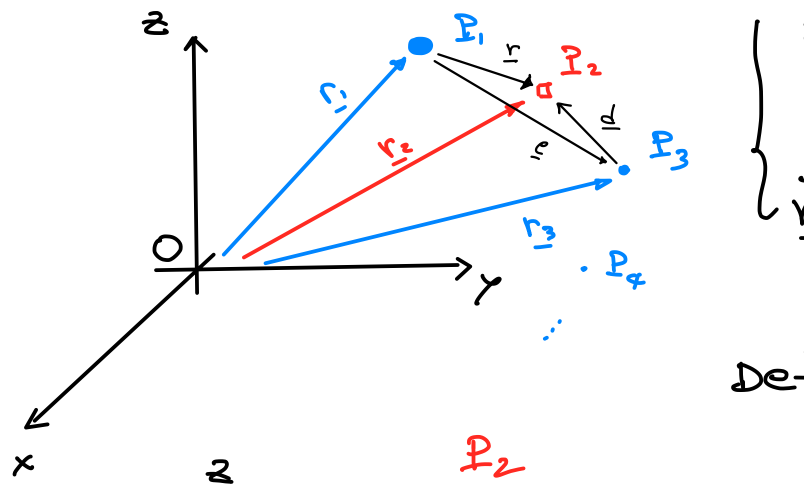


Remarks

i) All 4-body models are time-dependent

ii) These models are not coherent (they do not verify the eqs. of motion of the general $P_1 - P_2 - P_3$ problem), yet they are very close!

n-body model

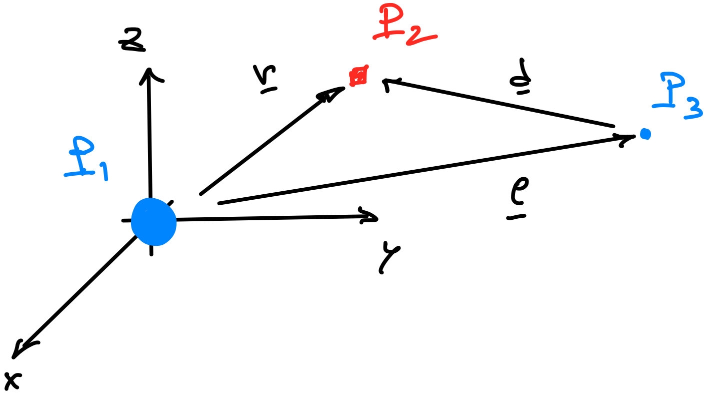


The eqs. of motion of P_1 and P_2 in an inertial frame are:

(restricted)

$$\begin{cases} \ddot{\underline{r}}_1 = \dfrac{Gm_2}{\|\underline{r}_2 - \underline{r}_1\|^3}(\underline{r}_2 - \underline{r}_1) + \dfrac{Gm_3}{\|\underline{r}_3 - \underline{r}_1\|}(\underline{r}_3 - \underline{r}_1) + \dots \\ \ddot{\underline{r}}_2 = \dfrac{Gm_1}{\|\underline{r}_1 - \underline{r}_2\|^3}(\underline{r}_1 - \underline{r}_2) + \dfrac{Gm_3}{\|\underline{r}_3 - \underline{r}_2\|^3}(\underline{r}_3 - \underline{r}_2) + \dots \end{cases}$$

Defining $\underline{r} := \underline{r}_2 - \underline{r}_1$, $\underline{d} = \underline{r}_2 - \underline{r}_3$, $\underline{e} = \underline{r}_3 - \underline{r}_1$



$$\ddot{\underline{r}} = -\frac{Gm_1}{r^3}\underline{r} - Gm_3\left(\frac{\underline{d}}{d^3} + \frac{\underline{e}}{e^3}\right) + \dots$$

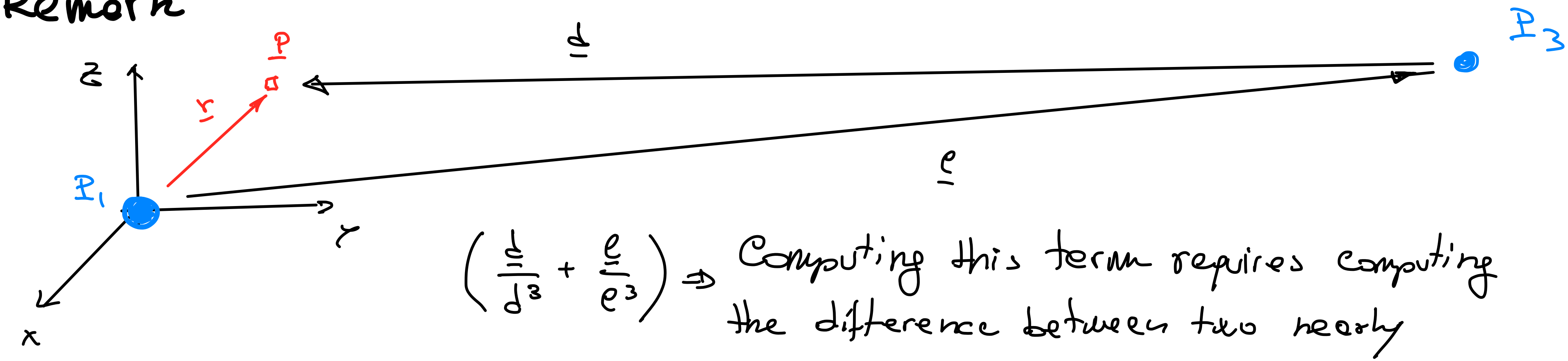
$$\underbrace{\ddot{\underline{r}} + \underbrace{\frac{\overset{Gm_1}{\mu}}{r^3}}_{\text{main attractor}}\underline{r}}_{\text{2-body}} = \underbrace{-Gm_3\left(\underbrace{\frac{\underline{d}}{d^3}}_{\text{pull by } P_3} + \underbrace{\frac{\underline{e}}{e^3}}_{\text{non-inertial frame}}\right) + \dots}_{\text{3rd-body perturbation}}$$

In general we have:

$$\boxed{\ddot{\underline{r}} + \frac{\mu}{r^3}\underline{r} = -G\sum_{j=3}^{n} m_j\left(\frac{\underline{d}_j}{d_j^{\,3}} + \frac{\underline{e}_j}{e_j^{\,3}}\right)}$$

Perturbed Kepler problem

Remark



$$\left(\frac{\underline{d}}{d^3} + \frac{\underline{e}}{e^3}\right) \Rightarrow$$ Computing this term requires computing the difference between two nearly equal numbers $\Rightarrow$ LOSS OF SIGNIFICANCE

In general (see Battin or Curtis Appendix F) the difference between two nearly equal numbers

$$\frac{\underline{p}}{p^3} - \frac{\underline{d}}{d^3}$$

can be computed as

1) $\underline{b} = \underline{p} - \underline{d}$ (known! it's $\underline{r}$ in our case)

2) $$q = \frac{\underline{b} \cdot (\underline{b} - 2\underline{p})}{\underline{p} \cdot \underline{p}}$$

3) $$\frac{\underline{p}}{p^3} - \frac{\underline{d}}{d^3} = \frac{1}{d^3}\left(q \, \frac{3+3q+q^2}{1+(1+q)^{3/2}} \, \underline{p} + \underline{b}\right)$$