SGN – Assignment #1

Davide Iafrate, 962657

1 Impulsive guidance

Exercise 1

Let $\mathbf{x}(t) = \varphi(\mathbf{x}_0, t_0; t)$ be the flow of the geocentric two-body model. 1) Using one of Matlab's built-in integrators, implement and validate* a propagator that returns $\mathbf{x}(t)$ for given $\mathbf{x}_0$, t_0, t, and μ. 2) Given the pairs $\{\mathbf{r}_1, \mathbf{r}_2\}$ and $\{t_1, t_2\}$, develop a solver that finds $\mathbf{v}_1$ such that $\mathbf{r}(t_2) = \mathbf{r}_2$, where $(\mathbf{r}(t), \mathbf{v}(t))^\top = \varphi((\mathbf{r}_1, \mathbf{v}_1)^\top, t_1; t_2)$ (Lambert's problem). To compute the derivatives of the shooting function, use either a) finite differences or b) the state transition matrix $\Phi = \mathrm{d}\varphi/\mathrm{d}\mathbf{x}_0$. Validate the algorithms against the classic Lambert solver. 3) Using the propagator of point 1) in the heliocentric case, and reading the motion of the Earth and Mars from SPICE, solve the shooting problem

$$\min_{\mathbf{x}_1, t_1, t_2} \Delta v \quad \text{s.t.} \begin{cases} \mathbf{r}_1 = \mathbf{r}_E(t_1) \\ \mathbf{r}(t_2) = \mathbf{r}_M(t_2) \\ t_1^L \le t_1 \le t_1^U \\ t_2^L \le t_2 \le t_2^U \\ t_2 \ge t_1 \end{cases} \tag{1}$$

where $\Delta v = \Delta v_1 + \Delta v_2$, $\boldsymbol{\Delta}\mathbf{v}_1 = \mathbf{v}_1 - \mathbf{v}_E(t_1)$, $\boldsymbol{\Delta}\mathbf{v}_2 = \mathbf{v}(t_2) - \mathbf{v}_M(t_2)$. $\mathbf{x}_1 = (\mathbf{r}_1, \mathbf{v}_1)^\top$, and $(\mathbf{r}(t), \mathbf{v}(t))^\top = \varphi(\mathbf{x}_1, t_1; t_2)$. Define lower and upper bounds, and make sure to solve the problem stated in Eq. (1) for different initial guesses.

The flow of the two body problem has been developed using MATLAB's ode113 integrator, which is the most suited to orbital dynamics problems, typically characterized by smooth solutions and requiring high accuracy. The flow $\mathbf{x}(t) = \varphi(\mathbf{x}_0, t_0; t)$ simply returns the final state at time t computed by the integrator, given the initial time t_0 and initial conditions $\mathbf{x}_0$. The flow has been validated by computing the error in the state $||\mathbf{x}(t_f) - \mathbf{x}_0||$ after 1 full orbital period ($t_f = T_{orbit}$) starting from the periapsis of the orbit. The resulting error is in the order of 10^{-7} so the flow produces the correct solution.

To solve the shooting problem the `fsolve` function has been employed. Both a central difference approximation of the Jacobian (computed by fsolve) and a Jacobian specified through the STM (computed through a MATLAB function) have been used. The Jacobain for the STM case is simply the sub-matrix relating the change in final position to the change in initial velocity: given the STM

$$\begin{Bmatrix} \delta\mathbf{r}(t) \\ \delta\mathbf{v}(t) \end{Bmatrix} = \begin{bmatrix} \Phi_{rr} & \Phi_{rv} \\ \Phi_{vr} & \Phi_{vv} \end{bmatrix} \begin{Bmatrix} \delta\mathbf{r_0} \\ \delta\mathbf{v_0} \end{Bmatrix} \tag{2}$$

being $\delta\mathbf{r_0} = 0$, the defect with respect to the final position is

$$\delta\mathbf{r}_f = \Phi_{rv}\delta\mathbf{v}_0 \tag{3}$$

and thus the Jacobian of the objective function is simply

$$J = \Phi_{rv} \tag{4}$$

Both methods converge to the same solution, however the central difference gradient method requires 7 times as many function evaluations for each step. The lambert shooting solver has

*It can be done by taking $\mathbf{x}_0$ at the periapsis of elliptic orbits and t_f equal to their period; get μ from SPICE.

been validated by computing the same arc using an analytical `lambertMR` function and the two results turn out to be the same.

For the Earth to Mars transfer two formulations of the shooting problem have been studied and compared. The first has the initial state as an optimization variable:

$$\min_{\mathbf{x}_1, t_1, t_2} \Delta v \quad \text{s.t.} \begin{cases} \mathbf{r}_1 = \mathbf{r}_E(t_1) \\ \mathbf{r}(t_2) = \mathbf{r}_M(t_2) \\ t_1^L \leq t_1 \leq t_1^U \\ t_2^L \leq t_2 \leq t_2^U \\ t_2 \geq t_1 \end{cases} \tag{5}$$

with $\Delta v = \Delta v_1 + \Delta v_2$, $\mathbf{\Delta v}_1 = \mathbf{v}_1 - \mathbf{v}_E(t_1)$, $\mathbf{\Delta v}_2 = \mathbf{v}(t_2) - \mathbf{v}_M(t_2)$. $\mathbf{x}_1 = (\mathbf{r}_1, \mathbf{v}_1)^\top$, and $(\mathbf{r}(t), \mathbf{v}(t))^\top = \varphi(\mathbf{x}_1, t_1; t_2)$. This method propagates the orbit in a sun-centerd two body problem starting from the state $\mathbf{x}_1$, which is constrained in terms of position and evaluates the total Δv and the constraint violation at the end of the propagation (the difference with respect to Mars' position at final time). The problem has been solved using the `fmincon` optimization function. By changing the initial guess given it is noticed that the problem results very sensitive to initial conditions, as changing the departure epoch guess produces significant differences in the trajectory obtained: a good initial guess has to be provided in order to converge to a low Δv solution. The initial guess for the state $\mathbf{x}_1^{guess}$ is given by the Earth state at departure time with a velocity in the same direction as Earth's orbital velocity and increased by 3km/s.

The lower and upper bounds for departure and arrival times have been defined as:

Table 1: Transfer windows

	Lower bound	Upper bound
Departure	2022-01-01 00:00:00	2023-01-01 00:00:00
Arrival	2022-03-01 00:00:00	2023-12-31 00:00:00

and appropriate lower and upper bounds for the state have been enforced as linear constraints.

The initial guess time of flight has been chosen as 300 days. The optimal solution is reported in Tab.[2] and the transfer arc is shown in Fig.[1]

Table 2: Free initial state x_1 optimal solution

Departure	Arrival	ΔV
2022-09-08 23:58:50	2023-08-04 03:54:51	7.11 km/s

To obtain better convergence a second formulation that has been studied presents the shooting problem as:

$$\min_{t_1, t_2} \Delta v \quad \text{s.t.} \begin{cases} t_1^L \leq t_1 \leq t_1^U \\ t_2^L \leq t_2 \leq t_2^U \\ t_2 \geq t_1 \end{cases} \tag{6}$$

In this formulation the cost function computes the Lambert arc required to connect the points at initial and final times, using the previously developed shooting Lambert solver and the resulting Δv required to enter and exit the transfer arc. For this solution the initial guesses have been computed starting from a guess departure date and adding a time of flight of 300 days, once a month starting from March. This method is much less sensitive to the initial guess of departure date as shown in Tab.[3] where it is evident that the solver converges to the same solution most of the times. The optimal transfer arc is shown in Fig.[2].

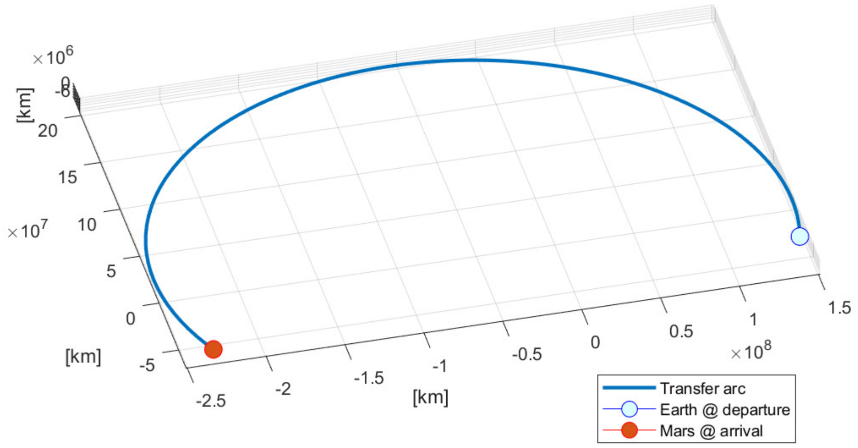


Figure 1: Free initial state x_1 transfer arc

Table 3: Lambert transfer arc optimal solutions

Departure guess	Departure	Arrival	ΔV
2022-03-20 00:00:00	2022-08-31 18:17:18	2023-08-16 17:58:33	6.48 km/s
2022-04-20 00:00:00	2022-08-31 18:17:18	2023-08-16 17:58:33	6.48 km/s
2022-05-20 00:00:00	2022-08-31 18:33:35	2023-08-16 18:56:43	6.48 km/s
2022-06-20 00:00:00	2022-08-31 18:34:31	2023-08-16 18:48:13	6.48 km/s
2022-07-20 00:00:00	2022-08-31 18:42:40	2023-08-16 18:5:593	6.48 km/s
2022-08-20 00:00:00	2022-08-31 18:34:48	2023-08-16 18:57:27	6.48 km/s
2022-09-20 00:00:00	2022-09-19 12:33:54	2023-05-11 23:54:04	7.27 km/s
2022-10-20 00:00:00	2022-09-19 12:24:24	2023-05-11 23:49:52	7.27 km/s
2022-11-20 00:00:00	2022-08-31 18:38:10	2023-08-16 18:48:04	6.48 km/s
2022-12-20 00:00:00	2022-08-31 18:38:49	2023-08-16 19:07:55	6.48 km/s

Exercise 2

A spacecraft is initially parked on a 200 km circular orbit around the Earth. The final destination is the Earth–Moon Lagrange point L_2 (EML2)†. The spacecraft is transferred by applying a sequence of three impulsive maneuvers that mimic a bi-elliptic transfer, namely: i) a first impulse injects the spacecraft into an "elliptic orbit 1" with apogee higher than 500,000 km; ii) a second impulse is carried out at the apogee of elliptic orbit 1 to achieve "elliptic orbit 2", having the perigee at EML2; iii) a third impulse is carried out at the perigee of elliptic orbit 2 to achieve the EML2 state. 1) Using multiple shooting, formalize an unambiguous statement of the problem (akin to the one of Eq. (1)) that minimizes the cost of the transfer by considering that the spacecraft is subject to the gravitational attractions of the Earth, Moon, and Sun all acting simultaneously. 2) Develop the code that solves the problem formulated in point 1) and solve it for arbitrary values of the departure epoch, transfer time, and number of segments. 3) Repeat point 2) for departure epochs spanning an entire year (two points per month of suggested discretization). Plot the the optimal transfer cost as a function of departure epoch.

†EML2 can be modeled as a point aligned with the Earth–Moon line, 60,000 km beyond the Moon, and having the velocity of the Moon.

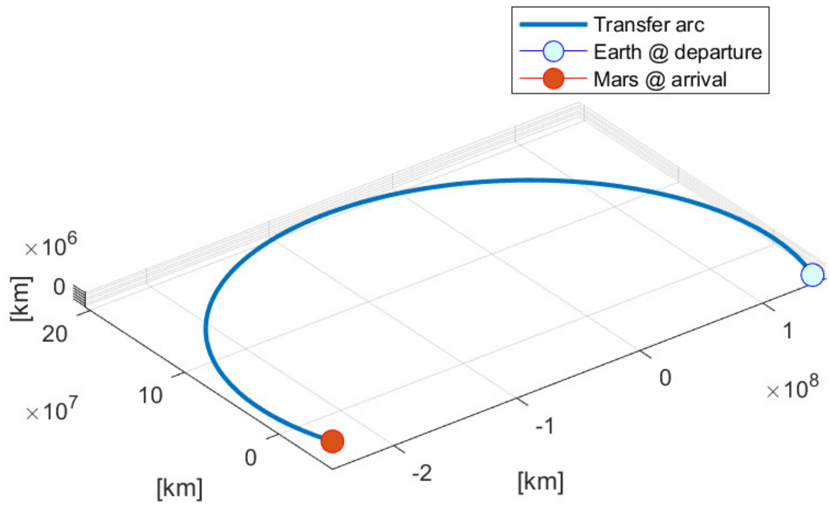


Figure 2: Lambert transfer arc

The global transfer has been split in 2 orbits: starting from an equatorial ($r_{1z} = 0$) circular orbit at an height of 200km (radius r_0) the spacecraft performs a first impulse to reach a distance r_3 and then a further one to reach the EML2 location $\mathbf{r}_{L2}$. The first velocity $\mathbf{v}_1$ has been chosen to be bounded in magnitude by the circular velocity v_0 and the escape velocity v_{esc}. The two orbits are split in two arcs for the solution of the problem, in order to reduce the error due to the long propagation times and improve convergence. State continuity of the two arcs is enforced as non-linear equality constraints, such that the flow $\varphi(t_1, \mathbf{x}_; t_2)$ from the first burn point is equal to the state $\mathbf{x}_2$ at the beginning of the second arc of the first orbit, and analogously for the two arcs of the second orbit.
The lower bound for the apogee distance $||\mathbf{r}_3||$ is given and the upper bound has been selected about the size of Earth's SOI. Linear constraints on the burn times t_1,t_3 and t_5 are enforced. The NLP vector of variables is thus formulated as: $\mathbf{z} = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, t_1, t_3, t_5\}$. The mathematical formulation of the problem is shown in Eq.[7].

$$\min_{\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4,t_1,t_3,t_5} \Delta v \quad \text{s.t.} \begin{cases} ||\mathbf{r}_1|| = r_0 \\ r_{1z} = 0 \\ \varphi(t_1, \mathbf{x}_1; t_2) = \mathbf{x}_2 \\ \mathbf{r}(t_3) = \mathbf{r}_3 \\ \varphi(t_3, \mathbf{x}_3; t_4) = \mathbf{x}_4 \\ \mathbf{r}_5 = \mathbf{r}_{L2}(t_5) \\ ||\mathbf{r}_3|| \geq 500'000\, km \\ ||\mathbf{r}_3|| \leq 1'000'000\, km \\ ||\mathbf{v}_1|| \leq \sqrt{\frac{2\mu_E}{r_1}} \\ ||\mathbf{v}_1|| \geq \sqrt{\frac{\mu_E}{r_1}} \\ t_3 \geq t_1 \\ t_5 \geq t_3 \end{cases} \tag{7}$$

The cost function is the total Δv of the transfer $\Delta v = ||\mathbf{\Delta} v_1|| + ||\mathbf{\Delta} v_2|| + ||\mathbf{\Delta} v_3||$, with

$\mathbf{\Delta v}_1 = \mathbf{v}_1 - \mathbf{v}_0$, $\mathbf{\Delta v}_2 = \mathbf{v}_3 - \mathbf{v}(t_3)$ and $\mathbf{\Delta v}_3 = \mathbf{v}_{L2} - \mathbf{v}(t_5)$ where each $\mathbf{v}(t)$ is the velocity given by the flow from the previous state. The initial guess has been generated from a bielliptical transfer in a 2 body problem approximation. The transfer time is equal to the sum of the two semi-periods of the transfer orbits. The apogee radius r_3 that results in the L2 point being at the nodal line at the end of the transfer is found iteratively through `fsolve` and the two transfer arcs are propagated, starting from a position $\mathbf{r}_1$ in the direction of the L2 position at final time, going up to the apogee of the first orbit (still on the equatorial plane) and then changing inclination at this location (which sits on the moon's nodal line). The spacecraft travels then for another half orbit up to the L2 position and performs the third burn to match the velocity of the L2 point. The initial guess shape, along with the sampling points for the initial guesses are shown in Fig[3].

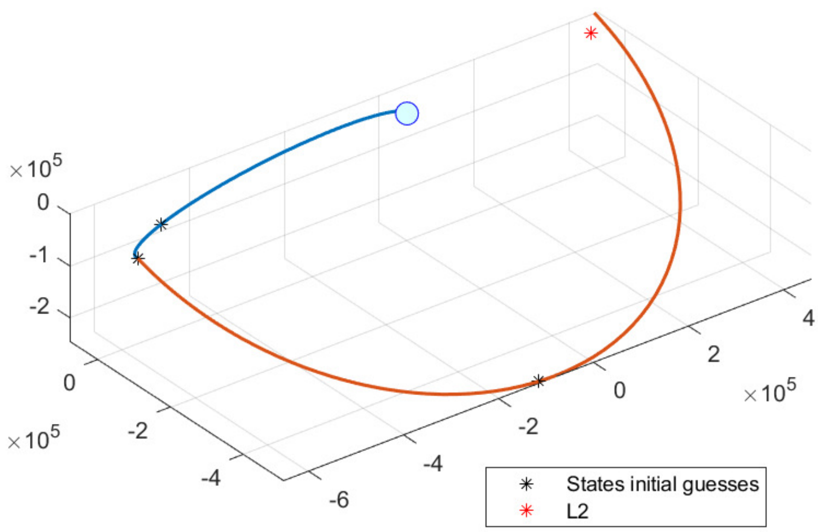


Figure 3: Initial guess for the transfer

The problem is then solved through the fmincon function, creating the non-linear constraint function and computing the Δv through the cost function `costFcn`. The orbits are propagated relative to the Earth in the *J2000* reference frame using an n-body propagator restricted to only the Earth, Moon and Sun gravitational accelerations. The `cspice` package is used to get the states of each body at the given time instant and the integrator chosen is MATLAB's ode113.

The typical transfer shape is shown in Fig.[4]

The transfer is computed for 24 dates in the span of 1 year, from 01-01-2022 to 31-12-2022. The change in total Δv and of each Δv_i is shown in Fig.[5].

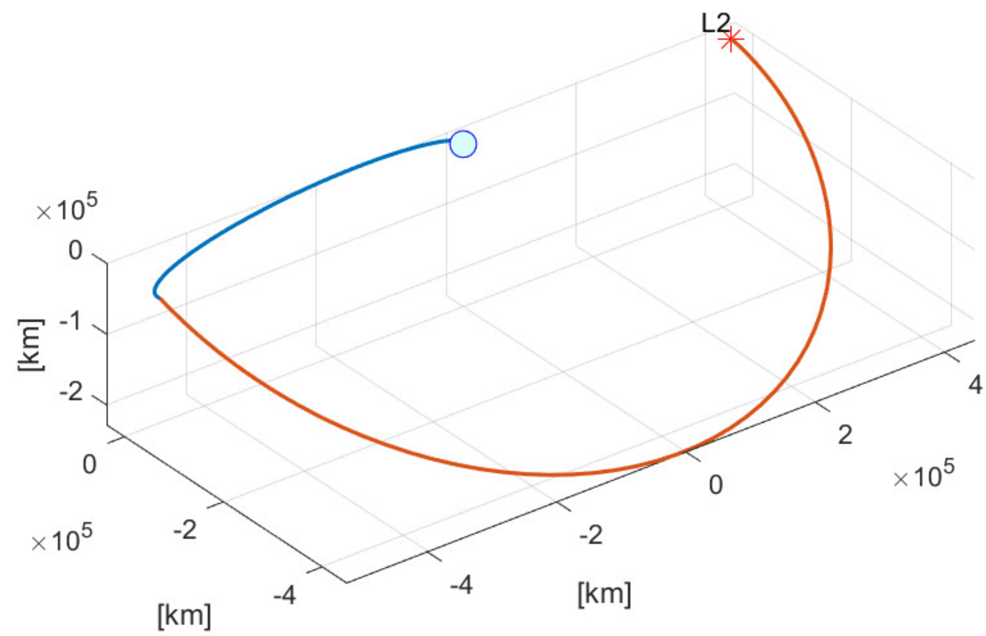


Figure 4: Optimized transfer

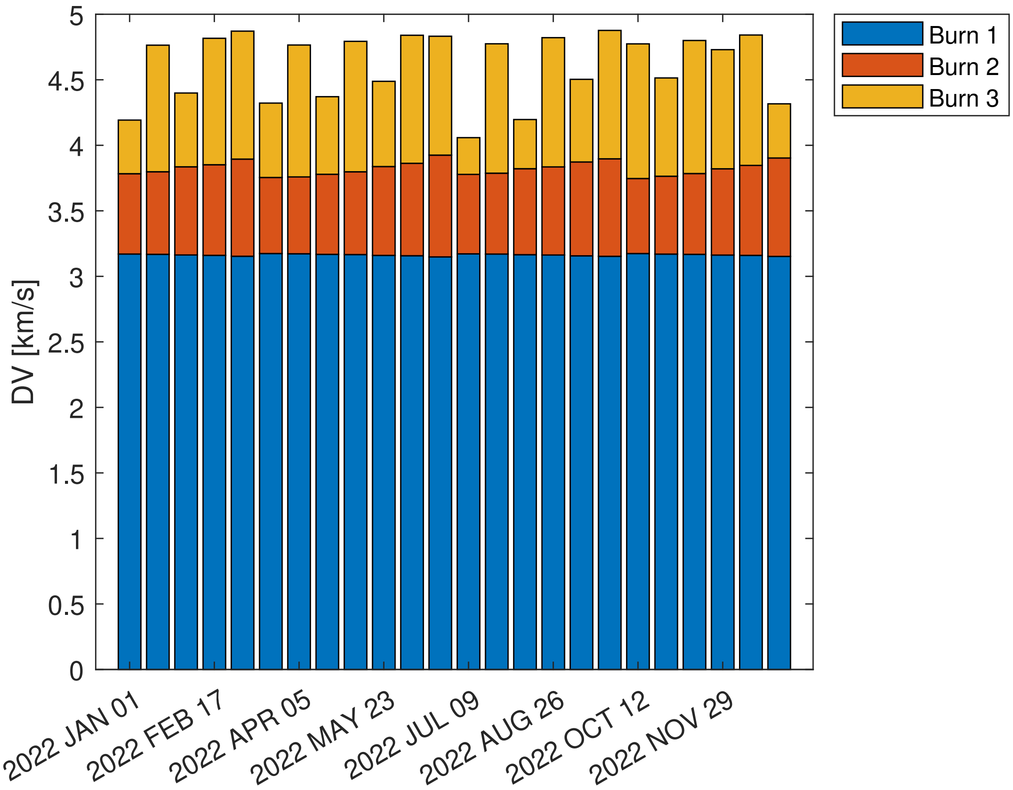


Figure 5: Transfer cost as function of departure date

2 Continuous guidance

Exercise 3

A spacecraft equipped with low-thrust propulsion moves in the geocentric two-body problem. The spacecraft has to accomplish a *time-optimal* transfer from the initial point $(\mathbf{r}_0, \mathbf{v}_0, m_0)^\top$ to the final point $(\mathbf{r}_f, \mathbf{v}_f)^\top$. 1) Write down the spacecraft equations of motion, the costate dynamics, and the zero-finding problem for the unknowns $\{\boldsymbol{\lambda}_0, t_f\}$. 2) Solve the problem with the following data: $\mathbf{r}_0 = (0, -29597.43, 0)^\top$ km, $\mathbf{v}_0 = (1.8349, 0.0002, 3.1783)^\top$ km/s, $m_0 =$ 735 kg; $\mathbf{r}_f = (0, -29617.43, 0)^\top$ km, $\mathbf{v}_f = (1.8371, 0.0002, 3.1755)^\top$ km/s; $T_{\max} = 100$ mN, $I_{sp} = 3000$ s. 3) Optional: solve the problem in 2) for several values of $T_{\max}$ and plot $t_f(T_{\max})$.

A time-optimal transfer is employed to transfer the S/C in the minimum time possible between initial and final states. In the case of a S/C with low-thrust propulsion in the geocentric two-body problem the *Variational Approach* is employed. The Hamiltonian of the system is:

$$H = 1 + \lambda_{\mathbf{r}} \cdot \mathbf{v} - \frac{\mu}{r^3}\lambda_{\mathbf{v}} \cdot \mathbf{r} + u\frac{T_{max}}{m}\lambda_{\mathbf{v}} \cdot \hat{\alpha} - \lambda_m u \frac{T_{max}}{I_{sp}g_0} \tag{8}$$

with λ_i being the costates of each state, $\mathbf{r}$ the position vector, $\mathbf{v}$ the velocity vector, m the mass, T_{max} the maximum thrust, I_{sp} the specific impulse, g_0 the standard gravity at sea level, $u \in [0, 1]$ the magnitude of the thrust relative to the maximum and $\hat{\alpha}$ a unitary thrust direction vector. The S/C equations of motion and costate dynamics are:

$$\dot{\mathbf{x}} = \frac{\partial H}{\partial \lambda} = \begin{cases} \dot{\mathbf{r}} = \mathbf{v} \\ \dot{\mathbf{v}} = -\frac{\mu}{r^3}\mathbf{r} - u\frac{T_{MAX}}{m}\hat{\alpha} \\ \dot{m} = -u\frac{T_{max}}{I_{sp}g_0} \end{cases} \tag{9}$$

$$\dot{\lambda} = \frac{\partial H}{\partial \mathbf{x}} = \begin{cases} \dot{\lambda_{\mathbf{r}}} = -\frac{3\mu}{r^5}(\mathbf{r} \cdot \lambda_{\mathbf{v}})\mathbf{r} + \frac{\mu}{r^3}\lambda_{\mathbf{v}} \\ \dot{\lambda_{\mathbf{v}}} = -\lambda_{\mathbf{r}} \\ \dot{\lambda_m} = -u\frac{\lambda_v T_{MAX}}{m^2} \end{cases} \tag{10}$$

with *boundary conditions*

$$\begin{cases} \mathbf{r}(t_0) = \mathbf{r}_0 \\ \mathbf{v}(t_0) = \mathbf{v}_0 \\ m(t_0) = m_0 \\ \mathbf{r}(t_f) = \mathbf{r}_f \\ \mathbf{v}(t_f) = \mathbf{v}_f \\ \lambda_m(t_f) = 0 \end{cases} \tag{11}$$

The control law is divided in a control law for the relative magnitude of the thrust u and a unitary direction vector $\hat{\alpha}$. To choose a control law for u and $\hat{\alpha}$ the Pontryagin Maximum Principle (PMP) is used. The control vector $\mathbf{u}$ is taken to be the argument that minimzes the Hamiltonian of the system:

$$\mathbf{u} = arg\,min\,H(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}) \tag{12}$$

The throttle u is computed based on the switching function

$$S_t = -||\lambda_v||\frac{I_{sp}g_0}{m} - \lambda_m \tag{13}$$

as

$$u := \begin{cases} 0 & if\ S(t) \geq 0 \\ 1 & if\ S(t) < 0 \end{cases} \tag{14}$$

while the direction is chosen to be opposite to the "primer vector" $\lambda_v/||\lambda_v||$:

$$\hat{\alpha} := -\frac{\boldsymbol{\lambda}_v}{||\boldsymbol{\lambda}_v||} \tag{15}$$

In the case of free final time we have to apply the transversality condition $H(t_f) = 0$. The zero finding problem becomes then

$$\text{Find } \boldsymbol{\lambda}_0, t_f \text{ such that } Z_t(\boldsymbol{\lambda}_0, t_f) = \begin{Bmatrix} \mathbf{r}(t_f) - \mathbf{r}_f \\ \mathbf{v}(t_f) - \mathbf{v}_f \\ \lambda_m(t_f) \\ H(t_f) \end{Bmatrix} = 0 \tag{16}$$

The resulting orbit is depicted in Fig.[6]. In time-optimal problem it is proven that the derivative of the mass costate $\dot{\lambda}_m$ has to always be negative and the control throttle should be always $u = 1$. Both behaviours are verified in the solution as shown in Fig.[7a] and Fig.[7b] respectively. Changing the thrust level in the range [100-1000]mN we can see that the transfer time decreases but not very significantly as shown in Fig.[8].

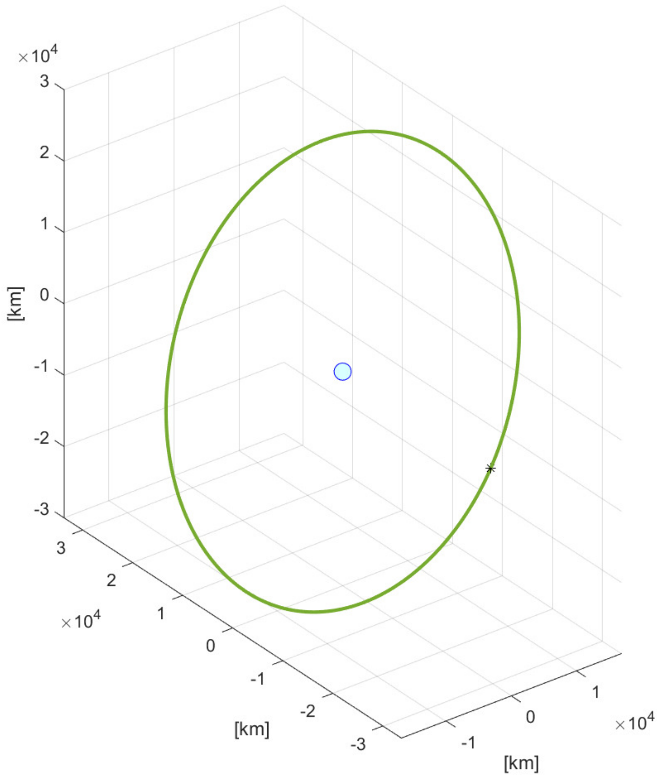
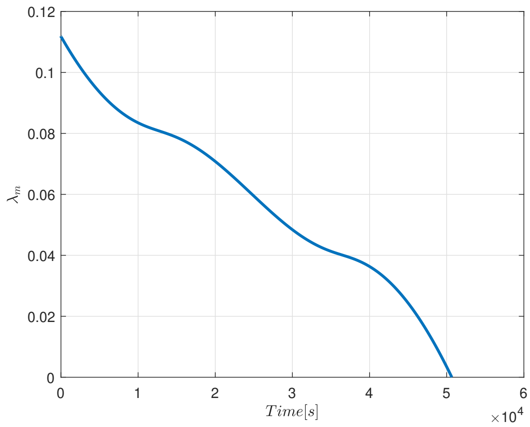
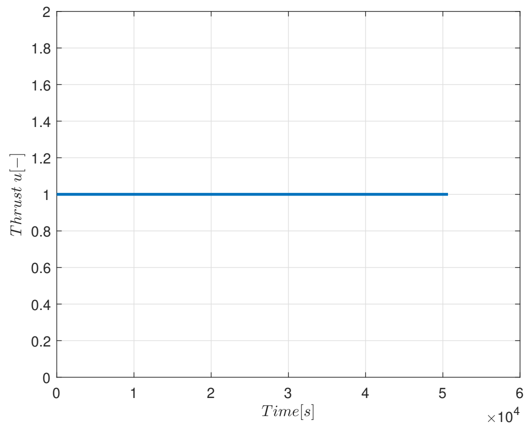


Figure 6: Controlled trajectory



(a) λ_m time evolution



(b) Control Thrust throttle

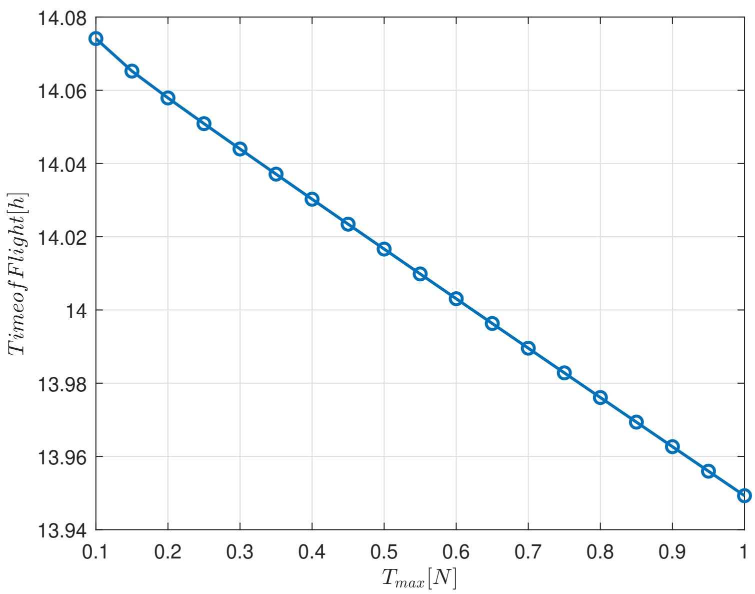


Figure 8: Time of Transfer vs Maximum Thrust level