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Orbital Mechanics
Module 2: Orbit representation

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ORBIT REPRESENTATION

Cartesian and Keplerian elements

Basic orbit representation

- The most common representations for the state of an orbiting body are the **Cartesian coordinates** and the **Keplerian elements** [1]:
  - **Cartesian coordinates** $\mathbf{s} = [\mathbf{r}, \mathbf{v}]$. Position $\mathbf{r}$ and velocity $\mathbf{v}$ at a given time t.
  - **Keplerian elements** $\boldsymbol{\alpha} = [a, e, i, \Omega, \omega, M]$. Based on a geometrical description of the conic:
    - a and e give the geometry of the planar conic.
    - i, Ω, and ω give the orientation of the conic in space (Euler angles).
    - M (or true anomaly f) gives the object position in the orbit. M and f are related through Kepler's equation.
- It is possible to convert between both [1]:
  - You will need `kep2car` and `car2kep` functions for the labs.
  - Be careful with singularities at $e = 0$ and $i = 0$ deg.
  - Test your functions converting a set of element back and forth and checking if you recover the original value: $\mathbf{s} \rightarrow \boldsymbol{\alpha} \rightarrow \tilde{\mathbf{s}} = \mathbf{s}$

[1] Curtis, H. D.. *Orbital mechanics for engineering students,* Butterworth-Heinemann , 2014

Cartesian and Keplerian elements

Osculating orbit and singularities

- Strictly, **Keplerian elements are defined for Keplerian orbits (2BP)**
  - $[a, e, i, \Omega, \omega]$ constant, M changes in time. That is, the orbit remains constant and the body moves along it.
  - In the presence of perturbations the orbit changes slowly in time. **Osculating orbit:** *Keplerian orbit that the object would have around the central body at a particular time instant if perturbations were absent.* **Osculating elements:** *Keplerian elements of the osculating orbit. They evolve in time.*
- Keplerian elements have 2 singularities:
  - **Zero inclination:** The line of nodes is not defined. Ω can take an arbitrary value, the angle of the apse line with the X axis is $\Omega + \omega$. $\rightarrow \omega = \underset{\text{RAAN}}{0} + \omega$
  - **Circular orbit:** The line of apses is not defined. ω can take an arbitrary value, the angular position of the body with the line of nodes is $\omega + f$.

$\omega = \emptyset \rightarrow$

Ephemeris

Locating objects in space

- *A table of the coordinates of celestial bodies as a function of time is called an **ephemeris*** [1].
  - Can be provided in the form of a look-up table, polynomial fittings, TLEs, etc.
  - Computed from astronomical observations and precise orbit propagation.
  - Importance of the **reference time** (epoch of the ephemeris).
- We will make an overview of one of the most common sources of ephemeris: the **Simplified General Perturbation (SGP)** models and the **Two-Line Elements (TLE)**.
- Other ephemeris will be provided in future labs when needed.

[1] Curtis, H. D.. *Orbital mechanics for engineering students,* Butterworth-Heinemann , 2014

Simplified General Perturbation models

Overview

- **Simplified General Perturbation (SGP) models:**
  - Analytical methods for propagating the orbital state of a satellite or debris, **from some ad-hoc mean elements**.
  - They include expressions for the **secular, long-periodic and short-periodic evolutions**, depending on the force models included.
- The US Air Force began development of their SGP model series in the 1960s:
  - Five different models: SGP, SGP4/SDP4, and SGP8/SDP8.
  - SGP/SGP4/SPG8 for close-Earth objects (period <225 minutes), SDP4/SDP8 for deep-space objects (period $\geq$ 225 minutes).
  - Most used one is SGP4/SDP4.
  - Mean elements distributed as **Two-Line Element (TLE) sets**.
  - SGP4/SDP4 was documented in Spacetrack Report #3, to promote compatibility in the operational community.

Simplified General Perturbation models

Two-line elements

- **NORAD (North American Aerospace Defense Command)** provides the TLEs for SGP4/SDP4.
    - TLEs for SGP/SGP8/SDP8 can be derived from them (some fields are included only for this purpose).
- TLEs are computed from observations and orbit determination procedures.
    - Exact data and models used are classified information.
    - A single TLE allows to propagate the orbital state for a **limited time range** around the TLE epoch (accuracy degradation).
- Data is stored in two different lines (hence the name):
    - Originally in 80-column punch cards.
    - Currently in 70-column ASCII files.
    - They are made publicly available through Space-Track, but not all objects can be queried (e.g. classified satellites are excluded).

Simplified General Perturbation models

Two-line elements – Line 1

```
1 20580U 90037B   19341.79883884  .00000350  00000-0  10277-4 0  9996
2 20580  28.4681 153.8207 0002675 174.2378 205.9686 15.09309002426911
```

Column	Description
01	Line number
03-07	Satellite number
08	Classification (U=Unclassified)
10-11	International Designator (last two digits of launch year)
12-14	International Designator (launch number of the year)
15-17	International Designator (piece of the launch)
19-20	Epoch year (last two digits of year)
21-32	Epoch (day of the year and fraction portion of the day)
34-43	First time derivative of the mean motion
45-52	Second Time Derivative of Mean Motion (decimal point assumed)
54-61	BSTAR drag term (decimal point assumed)
63	Ephemeris type
65-68	Element number
69	Checksum (Modulo 10)

Simplified General Perturbation models

Two-line elements – Line 2

```
1 20580U 90037B   19341.79883884  .00000350  00000-0  10277-4 0  9996
2 20580  28.4681 153.8207 0002675 174.2378 205.9686 15.09309002426911
```

Column	Description
01	Line Number of Element Data
03-07	Satellite Number
09-16	Inclination [deg]
18-25	Right Ascension of the Ascending Node [deg]
27-33	Eccentricity (decimal point assumed)
35-42	Argument of Perigee [deg]
44-51	Mean Anomaly [deg]
53-63	Mean Motion [rev/day]
64-68	Revolution number at epoch [rev]
69	Checksum (Modulo 10)

CelesTrak

https://celestrak.com/

CelesTrak

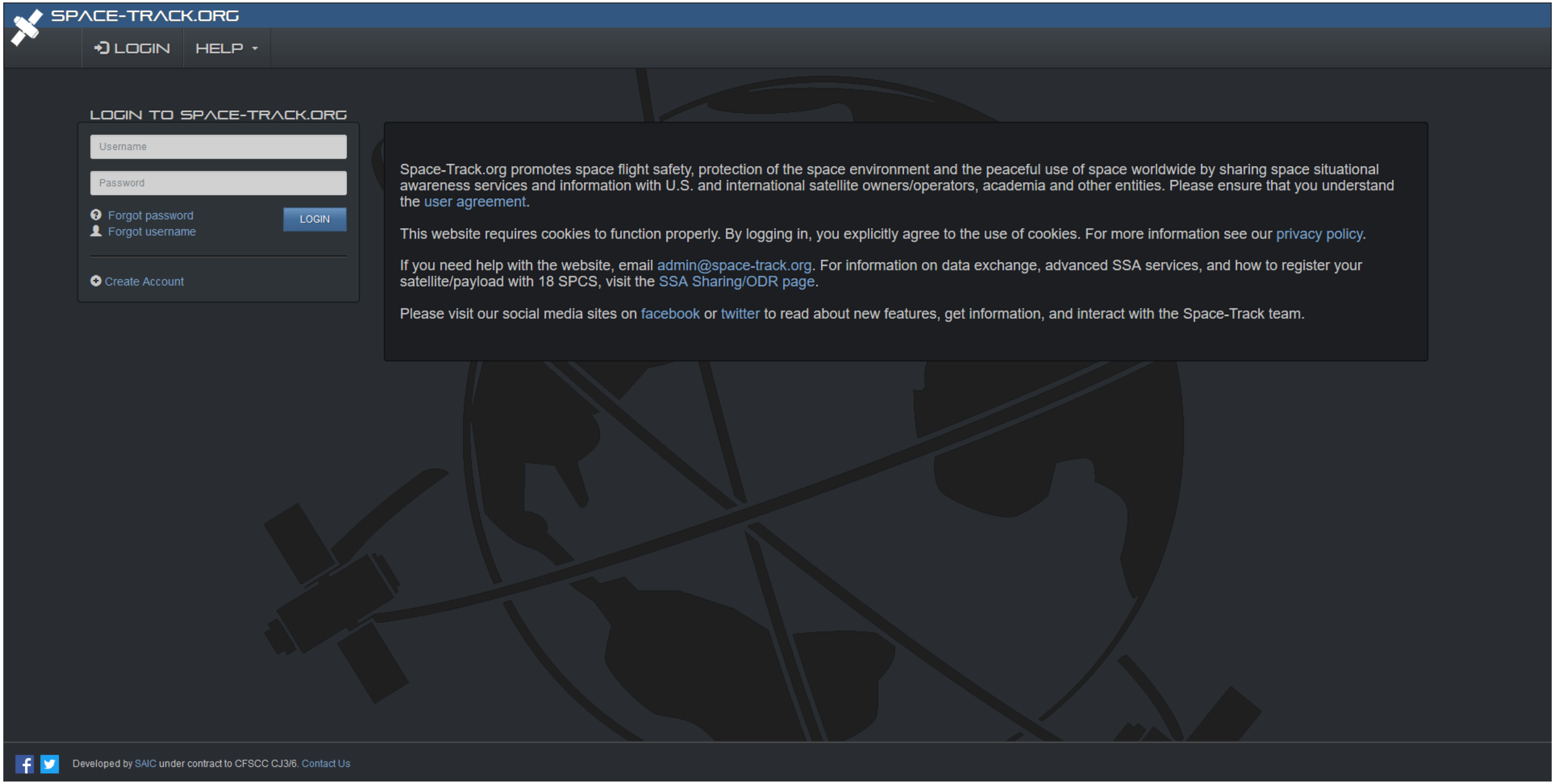
https://celestrak.com/

- Created and maintained by **Dr. T.S. Kelso**.
  - Originally introduced in 1985 to make TLEs electronically available (NASA was providing them as printed bulletins).

- Contains a lot of **useful information**:
  - Current TLEs for non-classified satellites, organized by types of satellites/missions.
  - Historical TLEs archive for selected objects.
  - Satellite catalogue (SATCAT).
  - Documentation and Q&A (including a detailed description of TLEs).
  - Software.
  - Others: GPS data, space weather data, etc.
  - Orbit visualization tool.

Space-Track

https://www.space-track.org



Space-Track

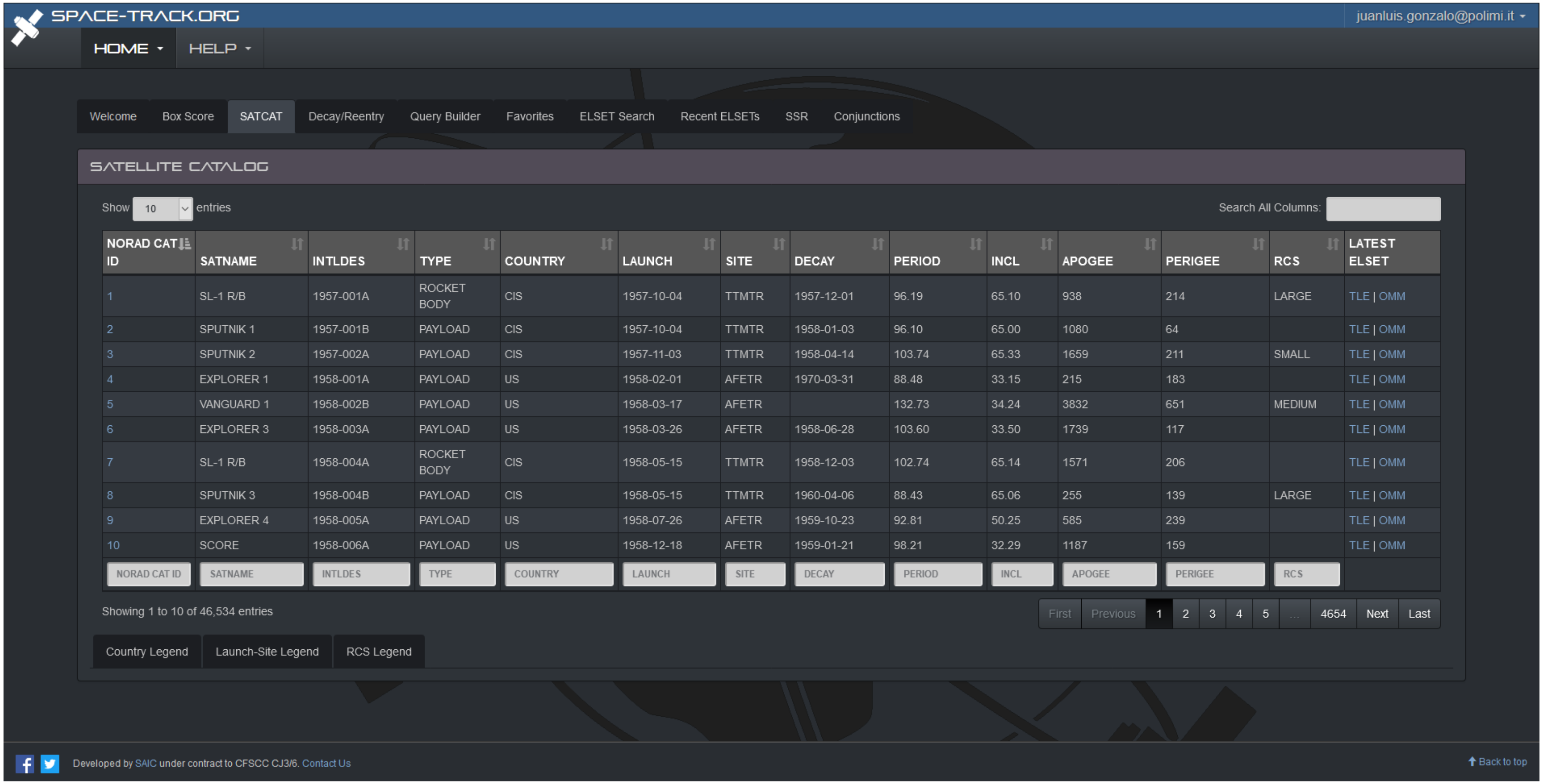
https://www.space-track.org

- **TLE information published by USSPACECOM/NORAD:**
  - Objective: *promoting a safe, stable, sustainable, and secure space environment through Space Situational Awareness information sharing*.
  - Includes a Windows/Linux implementation of SGP4.
  - API for automatic queries.
  - Information about decay/reentry of satellites.
  - Requires an account.

Space-Track

SATCAT

- Satellite catalog can be filtered, ordered and queried for different fields.

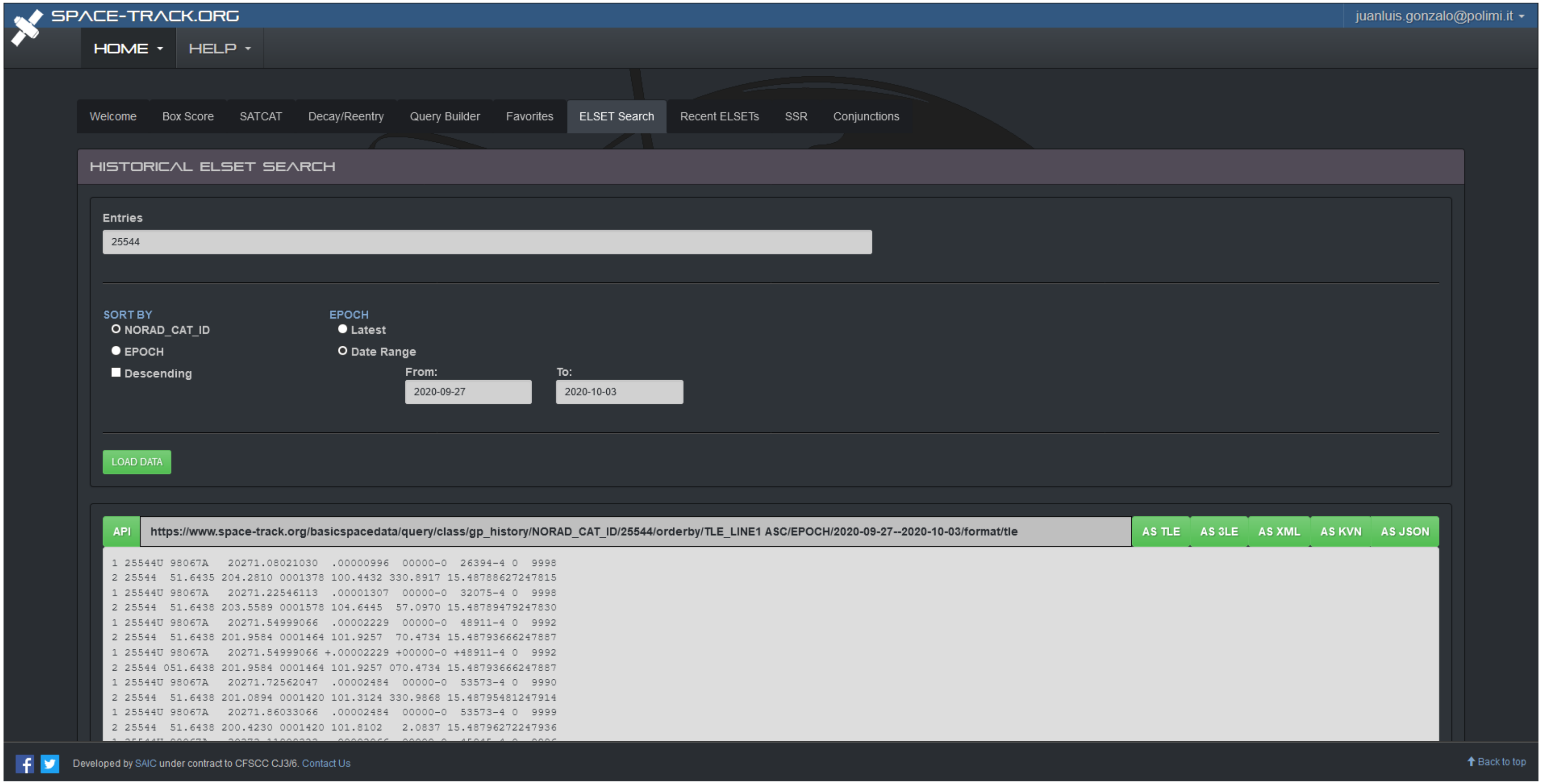


NORAD CAT ID	SATNAME	INTLDES	TYPE	COUNTRY	LAUNCH	SITE	DECAY	PERIOD	INCL	APOGEE	PERIGEE	RCS	LATEST ELSET
1	SL-1 R/B	1957-001A	ROCKET BODY	CIS	1957-10-04	TTMTR	1957-12-01	96.19	65.10	938	214	LARGE	TLE | OMM
2	SPUTNIK 1	1957-001B	PAYLOAD	CIS	1957-10-04	TTMTR	1958-01-03	96.10	65.00	1080	64		TLE | OMM
3	SPUTNIK 2	1957-002A	PAYLOAD	CIS	1957-11-03	TTMTR	1958-04-14	103.74	65.33	1659	211	SMALL	TLE | OMM
4	EXPLORER 1	1958-001A	PAYLOAD	US	1958-02-01	AFETR	1970-03-31	88.48	33.15	215	183		TLE | OMM
5	VANGUARD 1	1958-002B	PAYLOAD	US	1958-03-17	AFETR		132.73	34.24	3832	651	MEDIUM	TLE | OMM
6	EXPLORER 3	1958-003A	PAYLOAD	US	1958-03-26	AFETR	1958-06-28	103.60	33.50	1739	117		TLE | OMM
7	SL-1 R/B	1958-004A	ROCKET BODY	CIS	1958-05-15	TTMTR	1958-12-03	102.74	65.14	1571	206		TLE | OMM
8	SPUTNIK 3	1958-004B	PAYLOAD	CIS	1958-05-15	TTMTR	1960-04-06	88.43	65.06	255	139	LARGE	TLE | OMM
9	EXPLORER 4	1958-005A	PAYLOAD	US	1958-07-26	AFETR	1959-10-23	92.81	50.25	585	239		TLE | OMM
10	SCORE	1958-006A	PAYLOAD	US	1958-12-18	AFETR	1959-01-21	98.21	32.29	1187	159		TLE | OMM

Space-Track

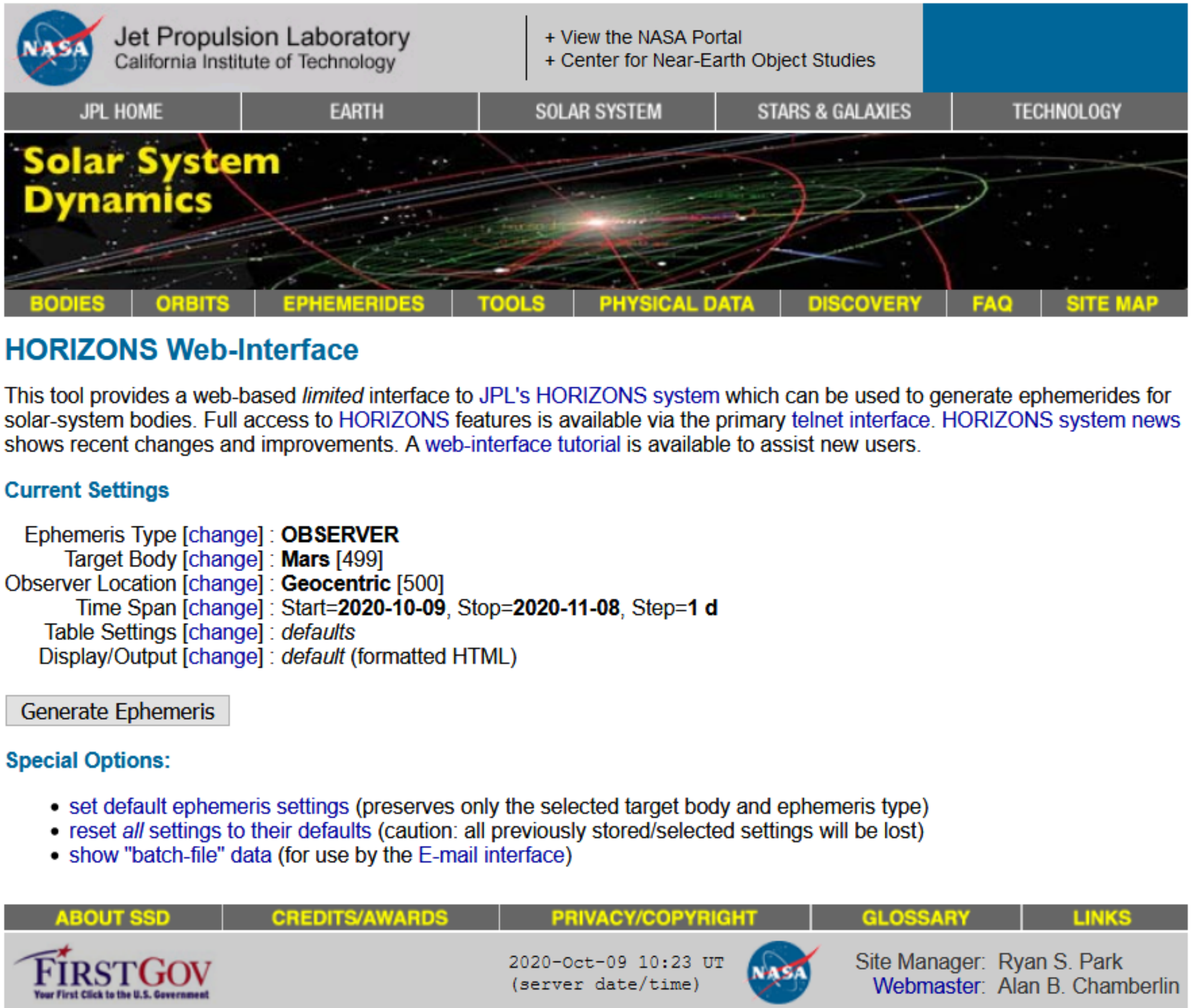
ELSET Search

- Search of historical element sets (TLEs, 3LEs, Keplerian, …) for given satellites and epoch ranges.



NASA/JPL HORIZONS

https://ssd.jpl.nasa.gov/horizons.cgi



Jet Propulsion Laboratory
California Institute of Technology

+ View the NASA Portal
+ Center for Near-Earth Object Studies

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Solar System Dynamics

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HORIZONS Web-Interface

This tool provides a web-based *limited* interface to JPL's HORIZONS system which can be used to generate ephemerides for solar-system bodies. Full access to HORIZONS features is available via the primary telnet interface. HORIZONS system news shows recent changes and improvements. A web-interface tutorial is available to assist new users.

Current Settings

Ephemeris Type [change] : **OBSERVER**
Target Body [change] : **Mars** [499]
Observer Location [change] : **Geocentric** [500]
Time Span [change] : Start=**2020-10-09**, Stop=**2020-11-08**, Step=**1 d**
Table Settings [change] : *defaults*
Display/Output [change] : *default* (formatted HTML)

Generate Ephemeris

Special Options:

- set default ephemeris settings (preserves only the selected target body and ephemeris type)
- reset *all* settings to their defaults (caution: all previously stored/selected settings will be lost)
- show "batch-file" data (for use by the E-mail interface)

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FirstGov Your First Click to the U.S. Government

2020-Oct-09 10:23 UT
(server date/time)

Site Manager: Ryan S. Park
Webmaster: Alan B. Chamberlin

NASA/JPL HORIZONS

https://ssd.jpl.nasa.gov/horizons.cgi

- NASA/JPL HORIZONS is a **solar system data and ephemeris computation service**:
  - Maintained by the Solar System Dynamics Group of the Jet Propulsion Laboratory (JPL).
  - Provides access to key solar system data.
  - Computes highly accurate ephemeris for solar system objects (856810 asteroids, 3598 comets, 210 planetary satellites, 8 planets, the Sun, L1, L2, select spacecraft, and system barycenters).
  - Few Earth-orbiting objects are included in the database, but users can provide their own set of TLEs (i.e. we can use it as a SGP4 implementation).
  - Full access through telnet connection or email (submitting batch-style input files).
  - A web interface is available (limited capabilities).

NASA/JPL HORIZONS

https://ssd.jpl.nasa.gov/horizons.cgi

- Settings for the web interface (remember to confirm all changes before proceeding to the next setting):
  - **Ephemeris type:** Orbital Elements.
  - **Target body:** Select a satellite from the list, or provide up to 750 sets of SGP4/SDP4 TLEs in standard form.
  - **Observer Location:** Geocentric (code 500) for Earth-orbiting objects.
  - **Time span:** Initial time, final time, and step size.
  - **Table settings:** Available options depend on the Ephemeris type. Remember to choose adequate units (km and km/s), reference plane (Earth mean equator), and set output to CSV format (easier to import into Matlab or other tools).
  - **Display/Output:** Select ‘download/save’.

References

- **CelesTrak** (TLEs and other resources): https://celestrak.com/
- **Space-Track** (satellite/debris catalogue and TLEs): https://www.space-track.org/
- **NASA/JPL's HORIZONS web-interface** (ephemeris generation): https://ssd.jpl.nasa.gov/horizons.cgi
- References about SGP4/SDP4 and TLEs (accessible in CelesTrak):

  - F.R. Hoots, "Spacetrack report no. 3, models for propagation of NORAD element sets", *Department of Commerce, National Technical Information Service*, 1980.
  - D. Vallado, P. Crawford, Hujsak, R., and Kelso, T.S., "Revisiting Spacetrack Report #3," *AIAA/AAS Astrodynamics Specialist Conference*, Keystone, CO, 21-24 August 2006.

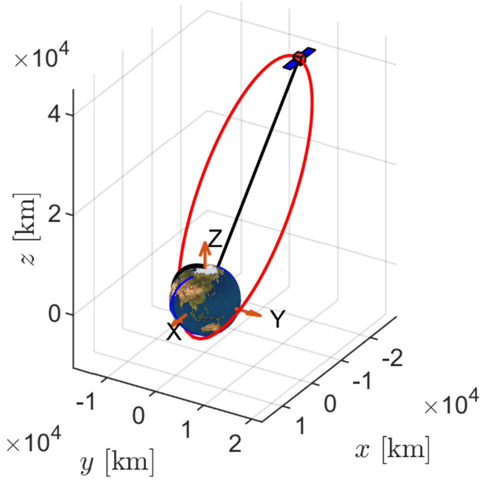
GROUND TRACK

Ground Track

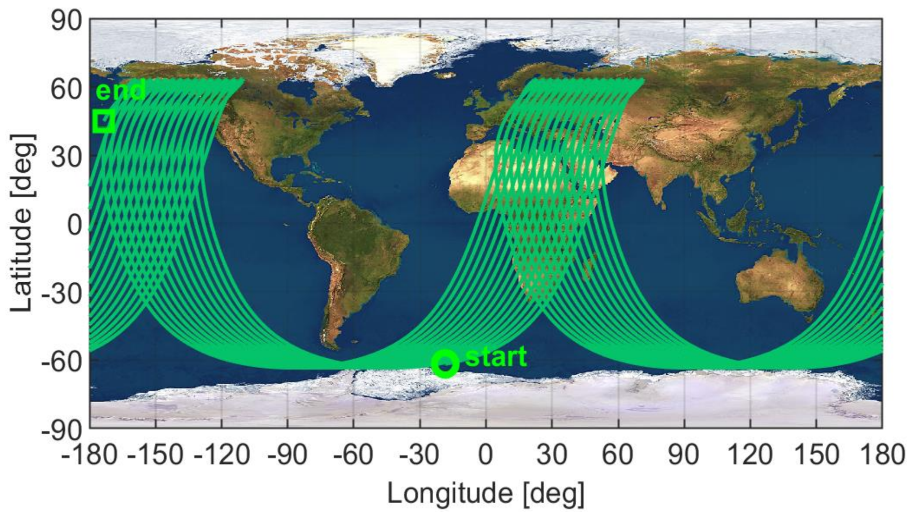
Definition

Ground track: *Projection of a satellite's orbit onto the Earth's surface [1].*

- Neglecting Earth's oblateness, it can be plotted as the trace left on the planet's surface by the line connecting the centre of the Earth and the satellite as it travels its orbit.
- At each time t, the corresponding ground track point is located by its **latitude** ϕ and **longitude** λ relative to the **rotating Earth**.



Molniya orbit



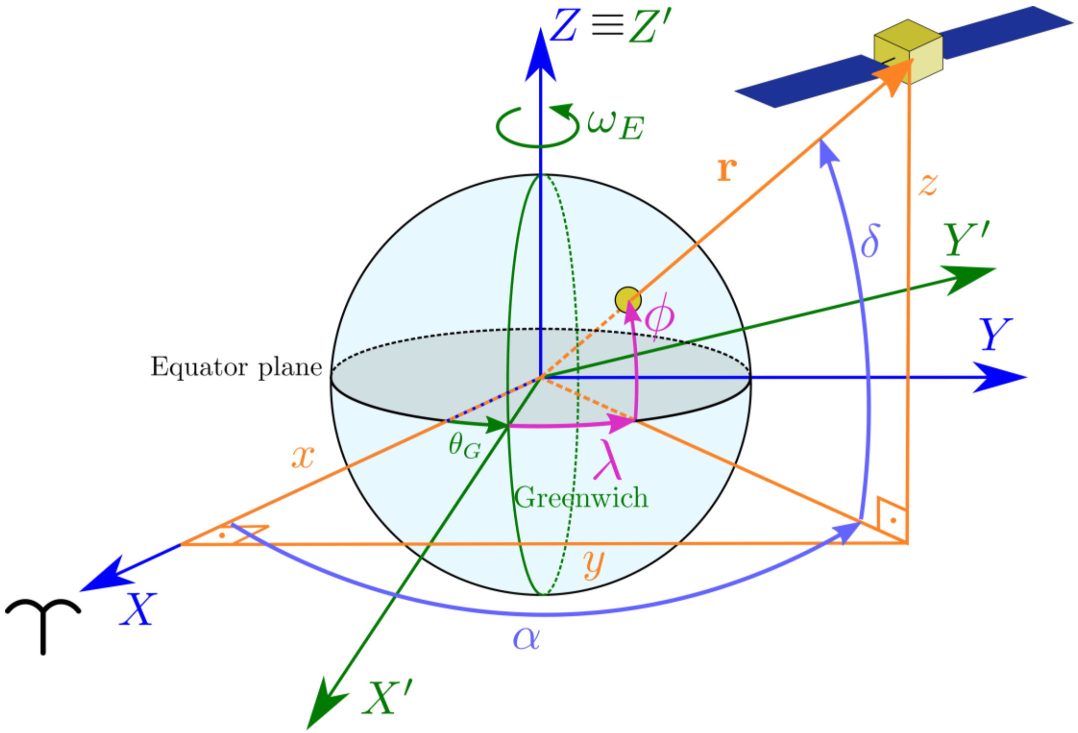
$a = 26600$ km
$e = 0.74$
$i = 63.4$ deg
$\Omega = 50$ deg
$\omega = 280$ deg
$f_0 = 0$ deg ‡
30 orbits

‡ f denotes true anomaly

[1] Curtis, H. D.. *Orbital mechanics for engineering students,* Butterworth-Heinemann , 2014

Ground Track

Angles for the ground track



$$\theta_G(t) = \theta_G(t_0) + \omega_E\,(t - t_0)$$

Declination

$$\delta = \operatorname{asin}\frac{z}{r}$$

Right ascension

$$\alpha = \begin{cases} \operatorname{acos}\dfrac{x/r}{\cos\delta} & \dfrac{y}{r} > 0 \\ 2\pi - \operatorname{acos}\dfrac{x/r}{\cos\delta} & \dfrac{y}{r} \le 0 \end{cases}$$

Longitude

$$\lambda = \alpha - \theta_G$$

Latitude

$$\phi = \delta$$

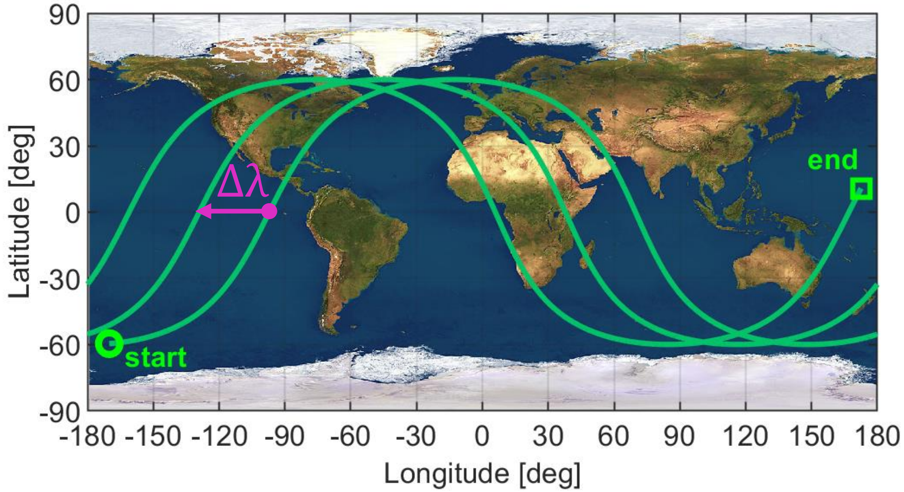
Ground Track

Geometrical properties

- For each orbital revolution, the ground track presents *2 equator crossings*, *one maximum in latitude*, and *one minimum in latitude*.
- For the unperturbed two-body problem, the orbit remains constant. However, **the ground track advances westward** by an angle $\Delta\lambda$ equal to Earth's rotation during one orbital period T of the satellite:

$$\Delta\lambda = T\,\omega_E$$

Earth's rotation velocity (eastwards): $\omega_E = 15.04\ \mathrm{deg/h}$



$a = 8350\ \mathrm{km}$

$e = 0.19760$

$i = 60\ \mathrm{deg}$

$\Omega = 270\ \mathrm{deg}$

$\omega = 45\ \mathrm{deg}$

$f_0 = 230\ \mathrm{deg}$

3.25 orbits

Ground Track

Repeating ground tracks

- A ground track will repeat itself after k revolutions of the satellite and m rotations of the planet if the total ground track drift $k\Delta\lambda$ is equal to the corresponding planet rotation $m2\pi$:

$$k\,\Delta\lambda = m\,2\pi \;\Rightarrow\; \frac{\Delta\lambda}{2\pi} = \frac{m}{k} \qquad \text{with } \; k, m \in \mathbb{N}$$

- Substituting $\Delta\lambda = T\omega_E$ and operating, we see that this is equivalent to imposing that *the ratio of the satellite's orbital period* T *and Earth's rotational period* $T_E = 2\pi/\omega_E$ *is a rational number*:

$$\frac{T}{T_E} = \frac{m}{k} \qquad \text{with } \; k, m \in \mathbb{N}$$

- Or expressed in terms of the satellite's mean motion $n = 2\pi/T$ and Earth's rotation velocity ω_E:

$$\frac{\omega_E}{n} = \frac{m}{k} \qquad \text{with } \; k, m \in \mathbb{N}$$

Ground Track

Repeating ground tracks

- Therefore, **repeating ground tracks can be obtained by choosing an orbit with a period T such that:**

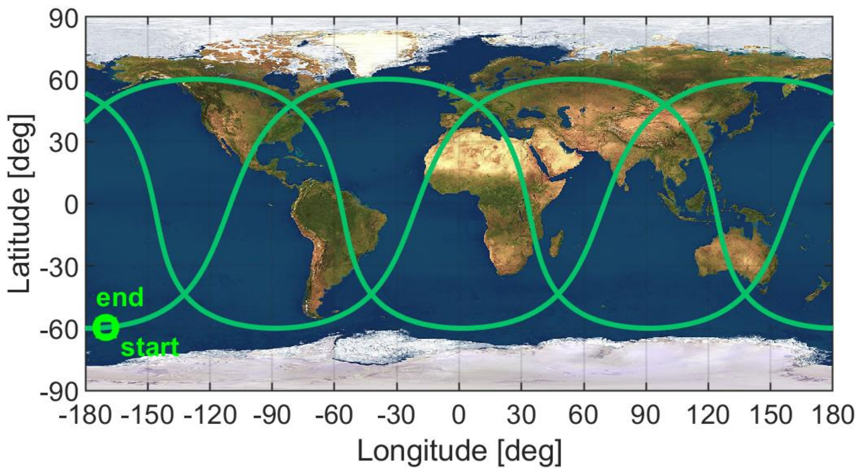
$$T = T_E \frac{m}{k} = \frac{2\pi}{\omega_E} \frac{m}{k}$$

- Or equivalently, with a mean motion n such that:

$$n = \omega_E \frac{k}{m}$$

- Keep in mind that period T and mean motion n only depend on the semimajor axis a:

$$n = \frac{2\pi}{T} = \sqrt{\frac{\mu}{a^3}}$$



$a = 16733.65$ km
$e = 0.19760$
$i = 60$ deg
$\Omega = 270$ deg
$\omega = 45$ deg
$f_0 = 230$ deg
$k = 4$
$m = 1$

Exercise 1: Computation of ground tracks

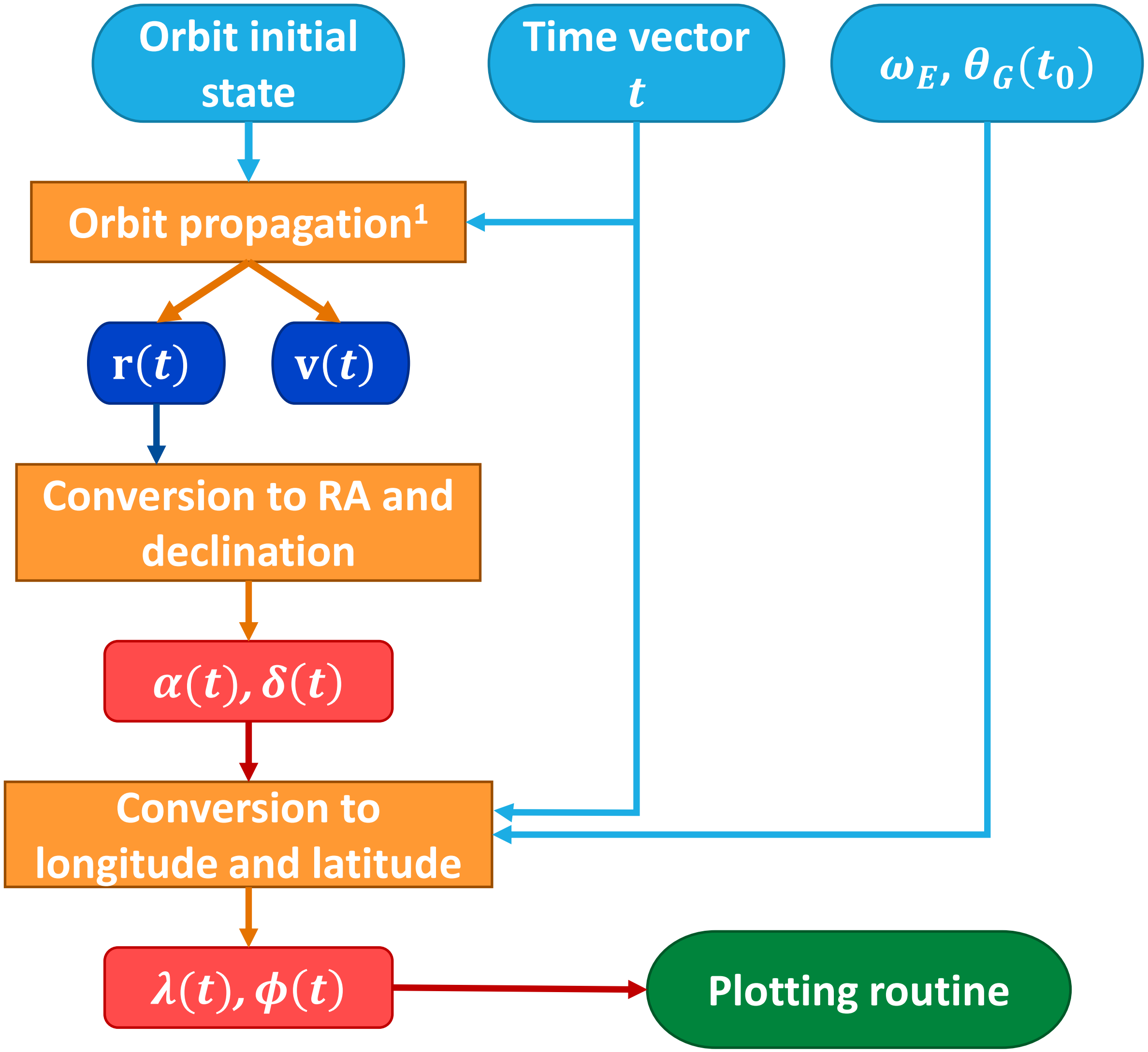
Exercise 1: Computation of ground tracks

1. Implement a function `groundTrack` that computes the ground track of an orbit
   - **Inputs:**
     - State of the orbit at the initial time (either in Cartesian or Keplerian elements)
     - Longitude of Greenwich meridian at initial time
     - Vector of times at which the ground track will be computed
     - Other inputs that you consider useful (e.g. ω_E, μ, t_0)
   - **Outputs:**
     - `alpha`: right ascension in Earth Centered Equatorial Inertial frame
     - `delta`: declination in Earth Centered Equatorial Inertial frame
     - `lon`: longitude with respect to rotating Earth (0 deg at Greenwich meridian)
     - `lat`: latitude with respect to rotating Earth

Exercise 1: Computation of ground tracks

Flow diagram

Hint: For numerical orbit propagation[1], you can reuse the code from **Module 1**.



[1] **Orbit propagation**: prediction of a body's orbital characteristics at a future date given the current orbital characteristics.

Exercise 1: Computation of ground tracks

Exercise 1: Computation of ground tracks

2. Plot the ground track for the following orbits:
   1. $a = 8350$ km, $e = 0.1976$, $i = 60$ deg, $\Omega = 270$ deg, $\omega = 45$ deg, $f_0 = 230$ deg (taken from [1], Example 4.12)
   2. A Molniya orbit with $a = 26600$ km, $e = 0.74$, $i = 63.4$ deg, $\Omega = 50$ deg, $\omega = 280$ deg, $f_0 = 0$ deg
   3. Three circular LEO orbits with altitude 800 km, $\Omega = 0$ deg, $\omega = 40$ deg, $f_0 = 0$ deg, and different inclinations: 0 deg , 98 deg , and 30 deg
   4. A geostationary orbit (GEO)

Data:

$\mu_\oplus = 398600\ \mathrm{km^3/s^2}$

$\omega_E = 15.04\ \mathrm{deg/h}$

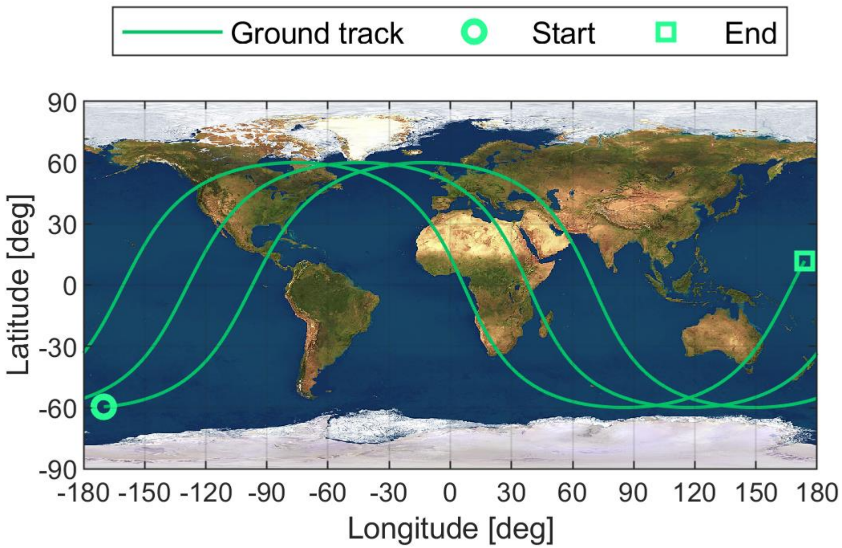
$R_\oplus = 6378.137$ km

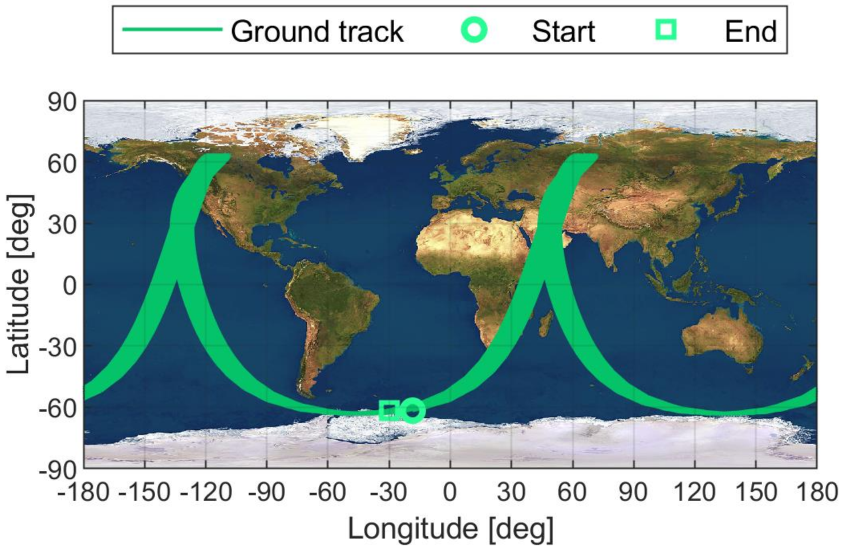
$\theta_G(t_0) = 0$ deg

[1] Curtis, H. D.. *Orbital mechanics for engineering students,* Butterworth-Heinemann , 2014

Exercise 1: Computation of ground tracks

Sample solutions





Case 1

$a = 8350$ km, $e = 0.19760$
$i = 60$ deg, $\Omega = 270$ deg
$\omega = 45$ deg, $f_0 = 230$ deg
$\theta_G(t_0) = 0$ deg
3.25 orbits

Case 2

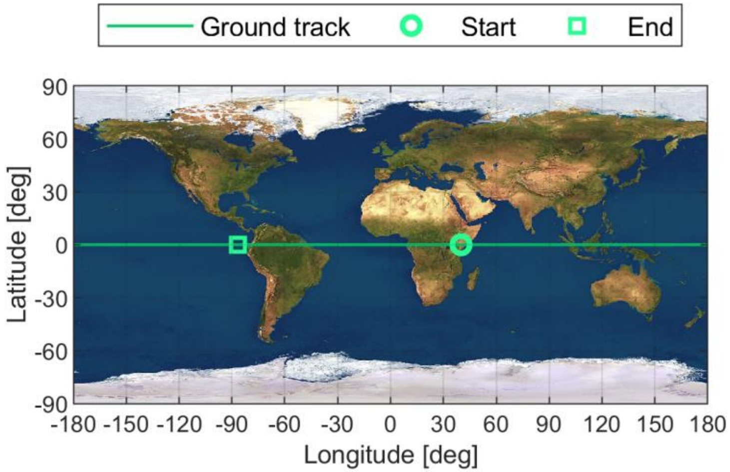
$a = 26600$ km, $e = 0.74$
$i = 63.4$ deg, $\Omega = 50$ deg
$\omega = 280$ deg, $f_0 = 0$ deg
$\theta_G(t_0) = 0$ deg
30 orbits

Exercise 1: Computation of ground tracks

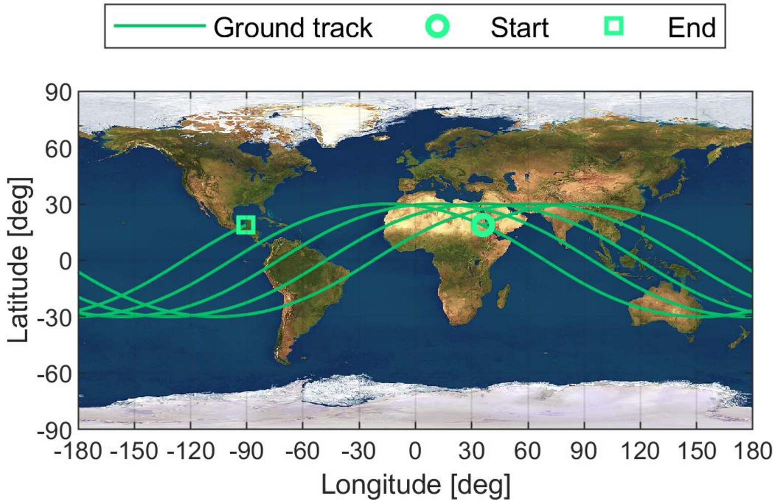
Sample solutions

Case 3

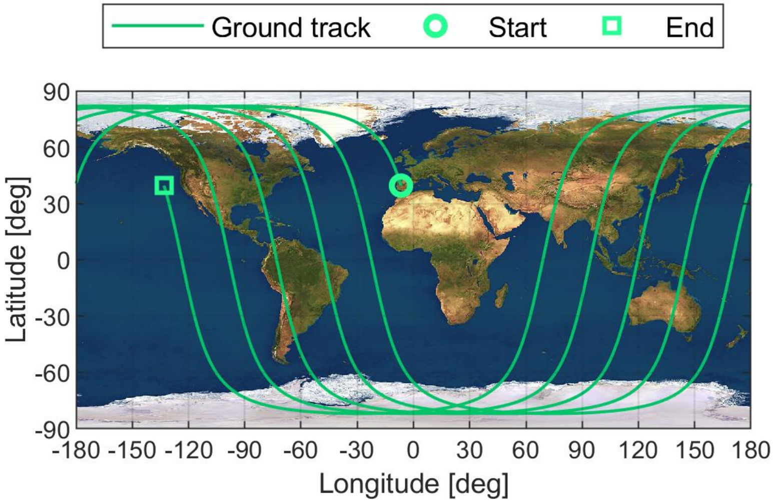
$a = 7178.137$ km, $e = 0$
$\Omega = 0$ deg, $\omega = 40$ deg,
$f_0 = 0$ deg
$\theta_G(t_0) = 0$ deg
5 orbits



$i = 0$ deg



$i = 30$ deg



$i = 98$ deg

Exercise 2: Design of repeating ground tracks

Exercise 2: Design of repeating ground tracks

1. Implement a function that computes the required a for a repeating ground track with k satellite revolutions and m Earth revolutions.
2. Modify the semimajor axis of the orbits in **Exercise 1** (cases 1 to 3) to get repeating ground tracks with:
   - **Case 1:** $k = 12, m = 1$
   - **Case 2 (Molniya orbit):** $k = 2, m = 1$
   - **Case 3 (circular LEOs):**
     - For $i = 0 \text{ deg}$: $k = 20, m = 2$
     - For $i = 30 \text{ deg}$: $k = 29, m = 2$
     - For $i = 98 \text{ deg}$: $k = 15, m = 1$
3. Plot each modified ground track together with the original one and compare them

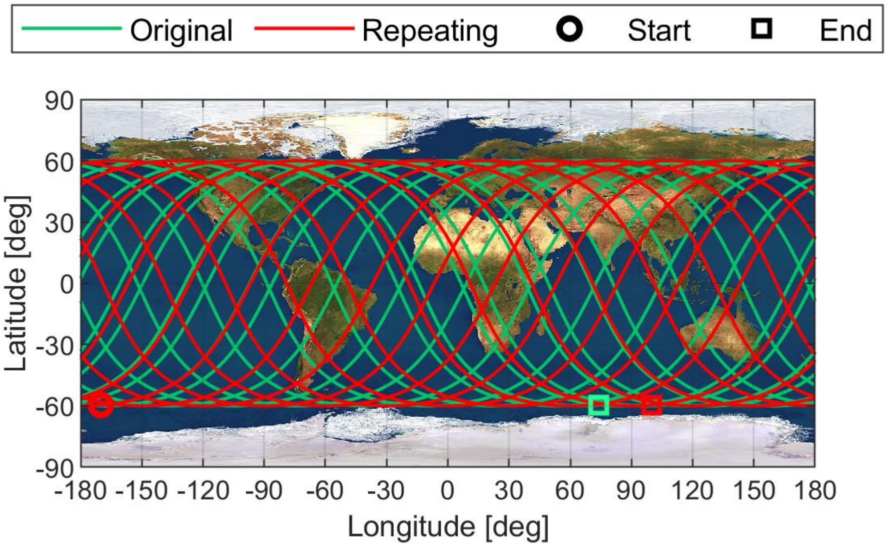
Exercise 2: Design of repeating ground tracks

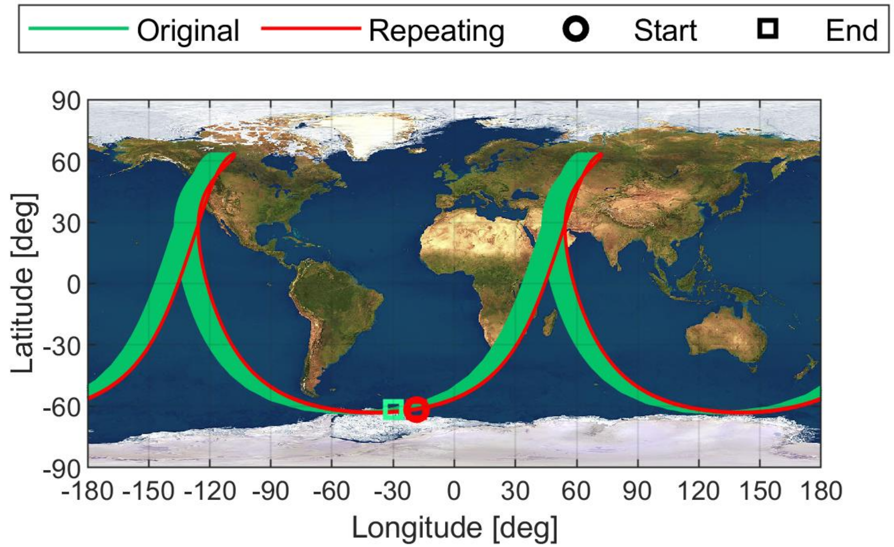
Hints

- Because **longitude λ** is defined in $[\,0, 360\,]\ \mathrm{deg}$, the ground track plot will be **discontinuous** at the boundaries of the angular range in λ:
  - If you include a `NaN` in the data arrays for a plot, the resulting line will have a discontinuity at that point.
- **Radians** are more convenient for calculations, while **degrees** are better for graphical representation.
- If you want, you can **add Earth's surface as background,** using function `imread` to load the image as a matrix of colored pixels, and function `image` to plot it.
  - Check the **documentation center** for detailed information on how to use them.
- Don't forget to *adjust the limits for the plotting regions*, to *label the axes*, and to *add a legend* if more than one line is represented.

Exercise 2: Design of repeating ground tracks

Sample solutions





Case 1

$a_0 = 8350$ km, $e = 0.19760$
$i = 60$ deg, $\Omega = 270$ deg
$\omega = 45$ deg, $f_0 = 230$ deg
$\theta_G(t_0) = 0$ deg, 15 orbits

$a_{\text{new}} = 8044.699$ km
$k = 12, m = 1$

Case 2

$a_0 = 26600$ km, $e = 0.74$
$i = 63.4$ deg, $\Omega = 50$ deg
$\omega = 280$ deg, $f_0 = 0$ deg
$\theta_G(t_0) = 0$ deg, 30 orbits

$a_{\text{new}} = 26563.011$ km
$k = 2, m = 1$

Exercise 2: Design of repeating ground tracks

Sample solutions

Case 3

$a_0 = 7178.137$ km

$e = 0$

$\Omega = 0$ deg

$\omega = 40$ deg

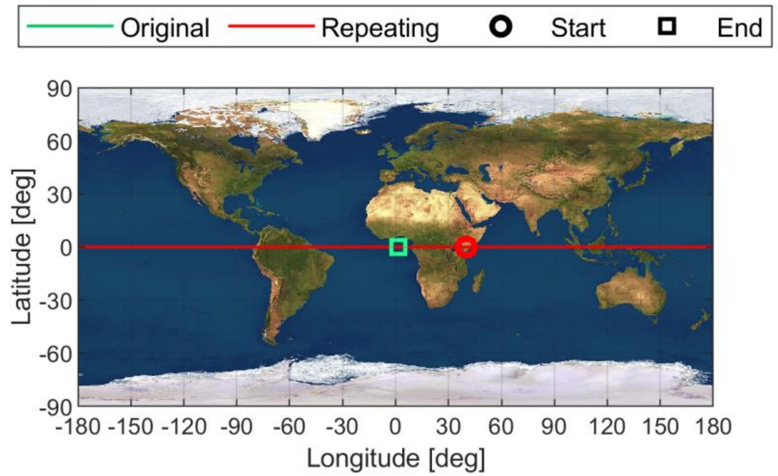
$f_0 = 0$ deg

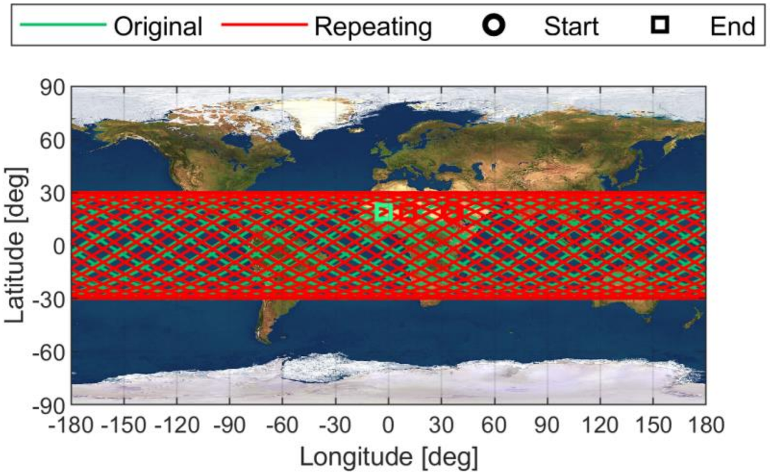
$\theta_G(t_0) = 0$ deg , 30 orbits

$i = 0$ deg

$a_{\text{new}} = 9084.422$ km

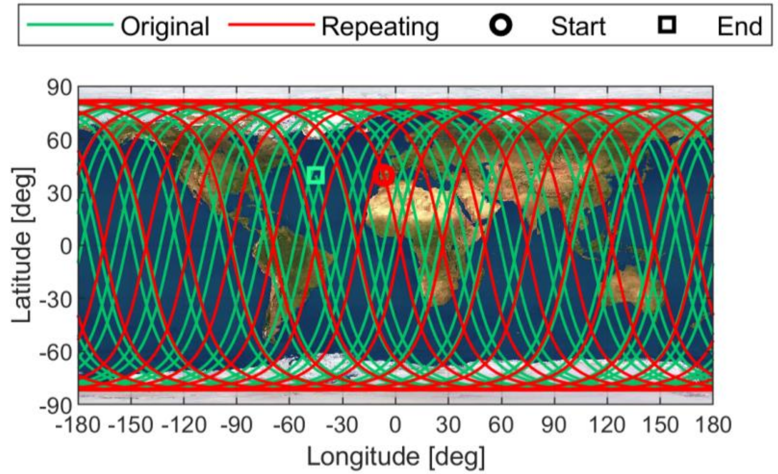
$k = 20, m = 2$





$i = 30$ deg

$a_{\text{new}} = 7091.182$ km, $k = 29, m = 2$



$i = 98$ deg

$a_{\text{new}} = 6932.711$ km, $k = 15, m = 1$

Repeating ground track with perturbations

Nodal periods

- Previous results correspond to the **unperturbed 2BP.**
- The repeating ground track condition can be generalized introducing the **nodal periods for satellite and Earth** [2]:
  - Satellite nodal period $\tilde{T}$: Time between two successive equator crossings.
  - Greenwich nodal period $\tilde{T}_E$: Earth's rotation period with respect to the ascending node.
- $\tilde{T}$ and $\tilde{T}_E$ are constant and equal to T and T_E in the 2BP, but may change under other perturbations (J_2, atmospheric drag, etc.)
- Same as before, *the condition for repeating ground track is that the ratio of the periods must be a rational number:*

as initial guess

$$\frac{\tilde{T}}{\tilde{T}_E} = \frac{m}{k} \quad \text{with } k, m \in \mathbb{N}$$

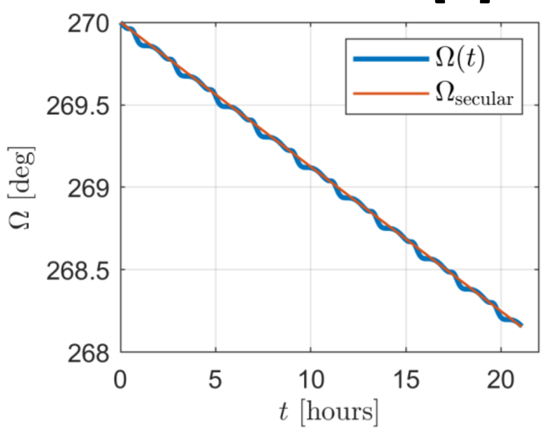
[2] Vallado, D.A. *Fundamental of Astrodynamics and Applications*, 4th Ed, Microcosm Press, 2013. Section 11.4.2.

Repeating ground track with perturbations

Effect of J_2

- The Earth bulges out at the equator due to centrifugal forces, taking the form of an **oblate spheroid**.
  - **Oblateness** is defined as: $f = \frac{R_e - R_p}{R_e}$ R_e: equatorial radius R_p: polar radius
  - **Geocentric latitude**: Oblateness modifies the latitude with respect to the spherical case (the normal to Earth's surface below the satellite is no longer parallel to $\mathbf{r}$). We will not consider this effect for the computer labs.

- J_2 introduces time variations in the Keplerian elements of the orbit [1]
  - Short-term variations in all elements.
  - **Long-term variations** only for Ω, ω, and M
    - The **secular (average long-term)** evolution for Ω, ω, and M are straight lines. Analytical approximation are available for the **average change rates**.
    - For a, e, and i long-term variations average out.



Evolution of Ω for the initial conditions of Exercise 1, case 1

[1] Curtis, H. D.. *Orbital mechanics for engineering students,* Butterworth-Heinemann , 2014

Repeating ground track with perturbations

Secular effects of J_2

- The secular evolution (average evolution for long times) of Ω, ω, and $M_0 = M(t_0)$ are straight lines with average change rates [1]:

$$\dot{\Omega} = -\left[\frac{3}{2}\frac{\sqrt{\mu}J_2R_e^2}{(1-e^2)^2a^{7/2}}\right]\cos i$$

$$\dot{\omega} = -\left[\frac{3}{2}\frac{\sqrt{\mu}J_2R_e^2}{(1-e^2)^2a^{7/2}}\right]\left(\frac{5}{2}\sin^2 i - 2\right)$$

$$\dot{M}_0 = -\left[\frac{3}{2}\frac{\sqrt{\mu}J_2R_e^2}{(1-e^2)^2a^{7/2}}\right]\left(1 - \frac{3}{2}\sin^2 i\right)$$

- These effects are known as:
    - **Regression of the node**: The line of nodes drifts westward for prograde orbits, and eastward for retrograde orbits.
    - **Advance of the perigee:** The perigee advances in the direction of the satellite motion for $0 \text{ deg} \leq i < 63.4 \text{ deg}$ or $116.6 \text{ deg} < i \leq 180 \text{ deg}$, and opposite to it for $63.4 \text{ deg} < i < 116.6 \text{ deg}$. It does not move for $i = 63.4 \text{ deg}$ or 116.6 deg.

[1] Curtis, H. D.. *Orbital mechanics for engineering students,* Butterworth-Heinemann , 2014

Repeating ground track with perturbations

Repeating ground track with J_2

- The **secular effects of J_2** modify the nodal periods of satellite and Earth:
  - Satellite nodal period $\tilde{T}$ is modified by $\dot{\omega}$ and $\dot{M}_0$

$$\tilde{T} = \frac{2\pi}{n + \dot{M}_0 + \dot{\omega}}$$

  - Greenwich nodal period $\tilde{T}_E$ is modified due to the regression of the node (because the ascending node is changing in time)

$$\tilde{T}_E = \frac{2\pi}{\omega_E - \dot{\Omega}}$$

- The **repeating ground track** with **secular effects of J_2** becomes an **implicit equation** involving a, e, and i [2]

as initial guess a_0
use the calculations
for UNPERTURBED

$$\frac{m}{k} = \frac{\tilde{T}}{\tilde{T}_E} = \frac{\omega_E - \dot{\Omega}}{n + \dot{\omega} + \dot{M}_0}$$

- Keep in mind that this solution only accounts for the **secular effects**.

[2] Vallado, D.A. *Fundamental of Astrodynamics and Applications*, 4th Ed, Microcosm Press, 2013. Section 11.4.2.

Exercise 3: Ground tracks with J_2

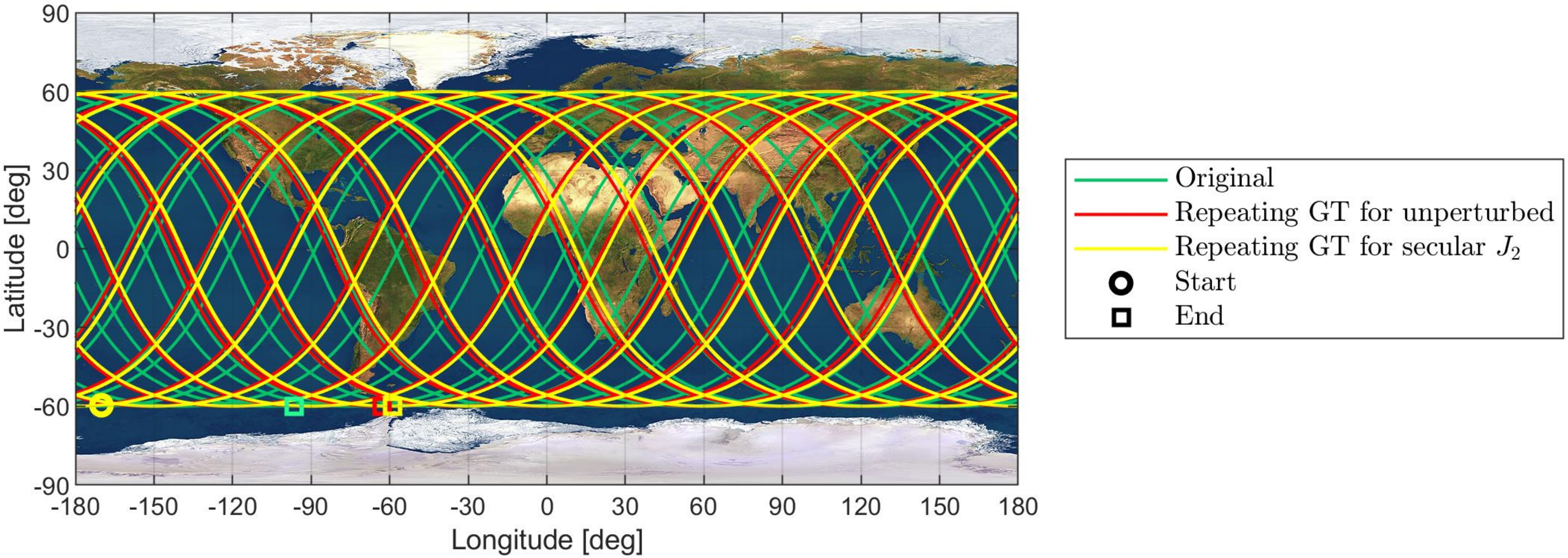
Exercise 3: Ground tracks with J_2

1. Modify function `groundTrack` from **Exercise 1** to include J_2 effects
   - Use values $R_e = 6378.137$ km and $J_2 = 0.00108263$ (taken from [3])
   - **Hint**: Recall the exercise on orbit propagation with J_2 from the previous module
2. Implement a new function to compute the required semimajor axis a for a repeating ground track with secular J_2 effects, with k satellite revolutions and m Earth revolutions
   - **Hint:** Solve the implicit equation in a using `fzero` or `fsolve`
3. Plot again the test cases in **Exercises 1 and 2**
   - Include the modified ground tracks obtained imposing the repeating GT conditions both for unperturbed orbit and with secular J_2 effects
   - Check that the repeating GT for unperturbed orbit no longer closes
   - Does the repeating GT with secular J_2 perfectly close? Why?
   - Is the GT of the GEO still a point under the effect of J_2?

[3] https://nssdc.gsfc.nasa.gov/planetary/factsheet/earthfact.html

Exercise 3: Ground tracks with J_2

Sample solutions



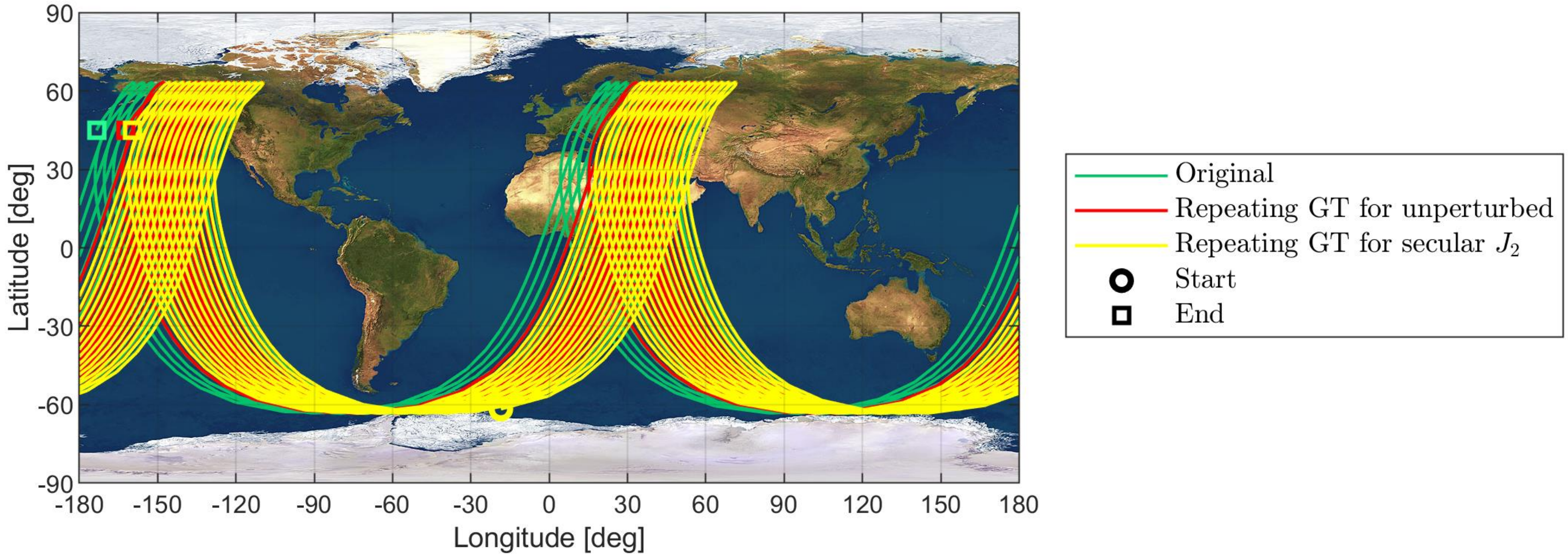
Case 1

$a_0 = 8350$ km, $e = 0.19760$, $i = 60$ deg, $\Omega = 270$ deg, $\omega = 45$ deg, $f_0 = 230$ deg
$\theta_G(t_0) = 0$ deg, 20 orbits

Repeating ground tracks for $k = 12$, $m = 1$:
$a_{\text{unpert}} = 8044.699$ km, $a_{\text{secular } J_2} = 8008.782$ km

Exercise 3: Ground tracks with J_2

Sample solutions



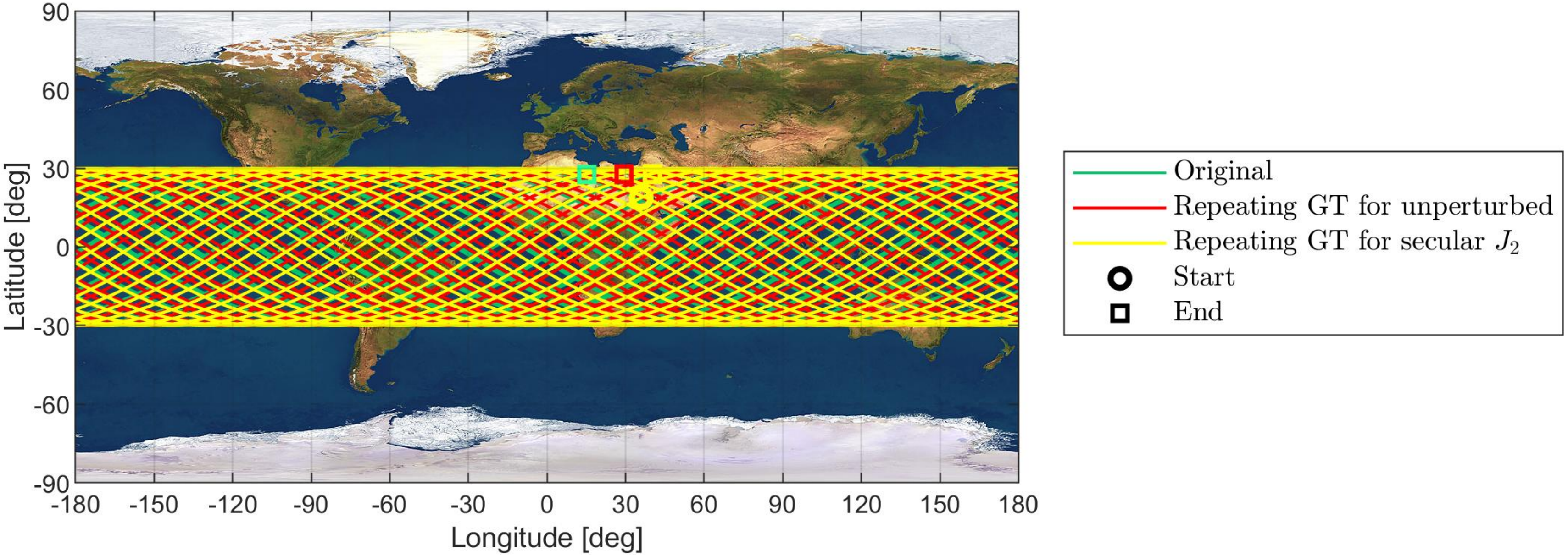
Case 2

$a_0 = 26600$ km, $e = 0.74$, $i = 63.4$ deg, $\Omega = 50$ deg, $\omega = 280$ deg, $f_0 = 0$ deg
$\theta_G(t_0) = 0$ deg, 30 orbits

Repeating ground tracks for $k = 2$, $m = 1$:
$a_{\text{unpert}} = 26563.011$ km, $a_{\text{secular } J_2} = 26554.674$ km

Exercise 3: Ground tracks with J_2

Sample solutions



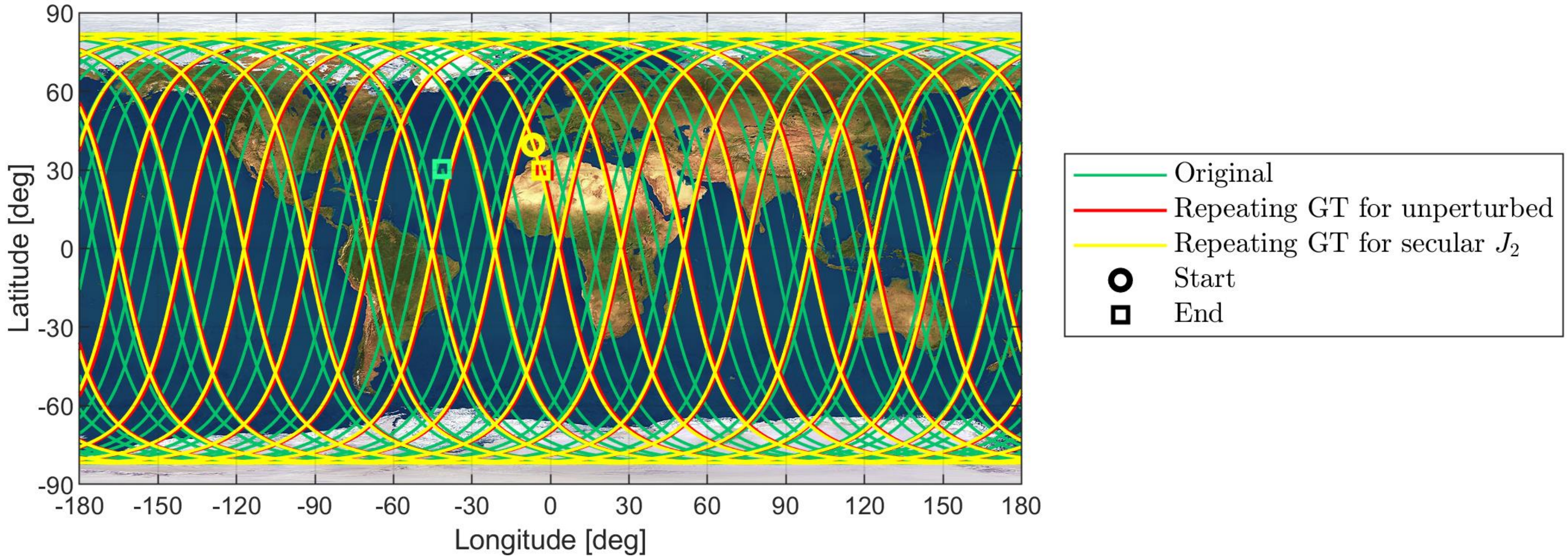
Case 3

$a_0 = 7178.137$ km, $e = 0$, $i = 30$ deg, $\Omega = 0$ deg, $\omega = 40$ deg, $f_0 = 0$ deg
$\theta_G(t_0) = 0$ deg, 30 orbits

Repeating ground tracks for $k = 29$, $m = 2$:
$a_{\text{unpert}} = 7091.182$ km, $a_{\text{secular } J_2} = 7024.197$ km

Exercise 3: Ground tracks with J_2

Sample solutions



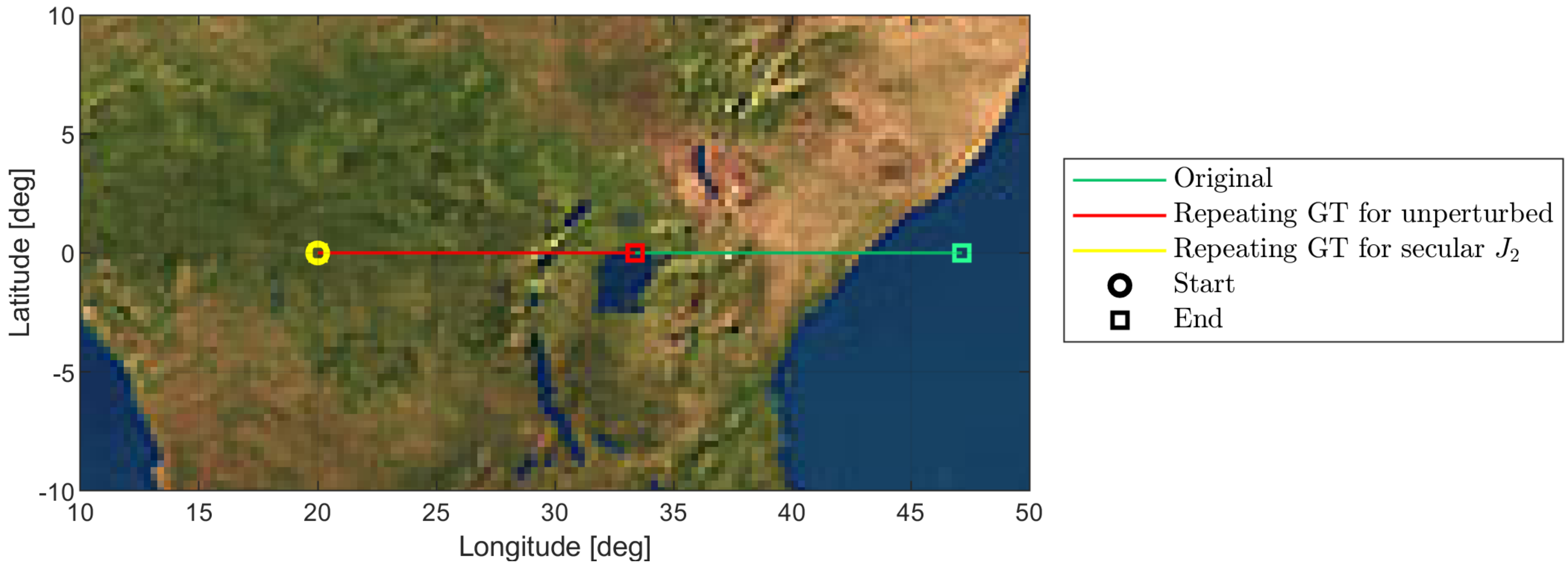
Case 3

$a_0 = 7178.137$ km, $e = 0$, $i = 98$ deg, $\Omega = 0$ deg, $\omega = 40$ deg, $f_0 = 0$ deg
$\theta_G(t_0) = 0$ deg, 30 orbits

Repeating ground tracks for $k = 15$, $m = 1$:
$a_{\text{unpert}} = 6932.711$ km, $a_{\text{secular } J_2} = 6940.095$ km

Exercise 3: Ground tracks with J_2

Sample solutions



Case 4

$a_0 = 42164$ km, $e = 0$, $i = 0$ deg, $\Omega = 0$ deg, $\omega = 0$ deg, $f_0 = 20$ deg
$\theta_G(t_0) = 0$ deg, 500 orbits

Repeating ground tracks for $k = 1$, $m = 1$:
$a_{\text{unpert}} = 42166.151$ km, $a_{\text{secular } J_2} = 42168.240$ km