A Contraction for Sovereign Debt Models

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- The Uniqueness of Markov equilibria in Eaton and Gersovitz (1981)
- Auclert and Rognlie (2016) proves it by contradiction
- Challenge: Bellman operator is not a contraction mapping (with constant $\beta < 1$)
- Contribution: 'Inverse of value function' is the fp of a contraction.

Key Idea

- Initial Problem: (not contraction)
  Given a debt level, the government maximizes its utility

$$v(b) = \max\{v_d, v_r(b)\}$$

- Dual Problem: (contraction)
  Given an at least utility level of government, what is the maximum debt value

$$b(v_r) = \max \cdots$$

Whether Default

- Given exogenous state s and a debt level b
- The value function of the government is

$$v(s, b) = \max \{\text{Default}, \text{Not Default}\}$$

- Default: can never borrow from or lend to the market
- Default $= v_d(s)$
  (solution to a pure optimal saving model)
- Not Default: repay the debt b, and decide the next period b'

Continuation Value

$$v_r(s, b) = \max_{(c,b')\in\Gamma(s,b)} \left\{ u(c) + \beta \sum_{s'} \max\{v_d(s'), v_r(s', b')\} P(s, s') \right\}$$

- today's action b' = tomorrow's debt level b'
- Q: What is $\Gamma(s, b)$

Feasible Correspondence

$$v_r(s,b) = \max_{(c,b')\in\Gamma(s,b)} \left\{ u(c) + \beta \sum_{s'} \max\{v_d(s'), v_r(s',b')\} P(s,s') \right\}$$

$$\Gamma(s,b) := \{(c,b') \in \mathbb{R}_+ \times \mathbb{R} : y(s) - b - c + q(s,b')b' \geqslant 0\}$$

- income - repayment - consumption + **'do activity in market'** $\geqslant 0$
- $b' > 0$: borrow money from market; $b' \leqslant 0$: invest in the market
- $q(s,b')$: price: present value of tomorrow's 1 dollar
- Q: What is $q(s,b')$ of interest

Markov Equilibrium

$$v_r(s,b) = \max_{(c,b')\in\Gamma(s,b)} \left\{ u(c) + \beta \sum_{s'} \max\{v_d(s'), v_r(s',b')\} P(s,s') \right\}$$

$$\Gamma(s,b) := \{(c,b') \in \mathbb{R}_+ \times \mathbb{R} : \, y(s) - b - c + q(s,b')b' \geqslant 0\}$$

$$q(s,b') = \begin{cases} \frac{1}{R}, & b' \leqslant 0 \\ \frac{1}{R}\sum_{s'} \mathbb{1}\{v_r(s',b') \geqslant v_d(s')\} P(s,s'), & b' > 0 \end{cases}$$

Issues

$$v_r(s, b) = \max_{(c,b') \in \Gamma(s,b)} \left\{ u(c) + \beta \sum_{s'} \max\{v_d(s'), v_r(s', b')\} P(s, s') \right\}$$

$$\Gamma(s, b) := \{(c, b') \in \mathbb{R}_+ \times \mathbb{R} : y(s) - b + q(s, b')b' \geqslant c\}$$

$$q(s, b') = \begin{cases} \frac{1}{R}, & b' \leqslant 0 \\ \frac{1}{R} \sum_{s'} \mathbb{1}\{v_r(s', b') \geqslant v_d(s')\} P(s, s'), & b' > 0 \end{cases}$$

- **Generally, it is not a contraction**
- (Challenge to work with ADP: T_σ depends on v)

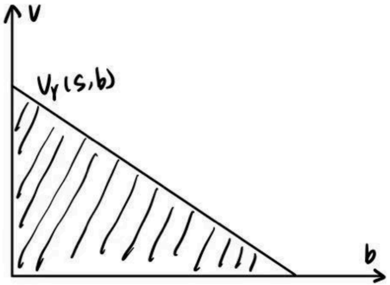
Properties

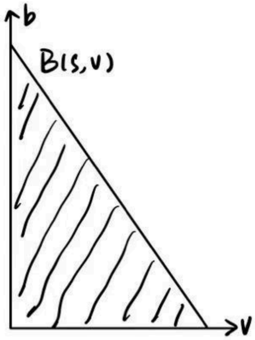
$$v_r(s, b) = \max_{(c,b')\in\Gamma(s,b)} \left\{ u(c) + \beta \sum_{s'} \max\{v_d(s'), v_r(s', b')\} P(s, s') \right\}$$

$$\Gamma(s, b) := \{(c, b') \in \mathbb{R}_+ \times \mathbb{R} : \ y(s) - b + q(s, b')b' \geqslant c\}$$

- $v_r(s, b)$ is strictly decreasing in b
- The maximizer (c, b') satisfies

$$b = y(s) - c + q(s, b')b'$$


v
$V_Y(s,b)$
b


b
B(s,v)
v

Dual Problem

- The inverse function is

$$B(s,v) = \max_{c,b'} \{y(s) - c + q(s,b')b'\}$$

subject to

$$v \leqslant u(c) + \beta \sum_{s'} \max \{v_r(s',b'), v_d(s')\} P(s,s')$$

Bellman operator

- Given $b' = B(s', w(s'))$, we have $v_r(s', b') = w(s')$

$$(Tb)(s, v) = \max_{c, b', \{w(s')\}} \{y(s) - c + q(s, b')b'\}$$

subject to

$$v \leqslant u(c) + \beta \sum_{s'} \max \{w(s'), v_d(s')\} P(s, s')$$

$$b' \leqslant b(s', w(s')) \text{ for } s' \text{ with } w(s') \geqslant v_d(s')$$

where

$$q(s, b') = R^{-1} \left[\mathbb{1}\{b' \leqslant 0\} + \mathbb{1}\{b' > 0\} \sum_{s'} \mathbb{1}\{w(s') \geqslant v_d(s')\} P(s, s') \right]$$

Contraction

- Let S be finite and $V \subset \mathbb{R}$ be compact
- Let W := $b(\mathsf{S} \times V)$
- With some assumptions, they show
  - $T\mathsf{W} \subset \mathsf{W}$
  - T is order preserving
  - $T[(b + \lambda)](s, v) \leqslant (Tb)(s, v) + R^{-1}\lambda$ for any $\lambda \geqslant 0$

AUCLERT, A. AND M. ROGNLIE (2016): "Unique equilibrium in the Eaton–Gersovitz model of sovereign debt," *Journal of Monetary Economics*, 84, 134–146.

EATON, J. AND M. GERSOVITZ (1981): "Debt with Potential Repudiation: Theoretical and Empirical Analysis," *The Review of Economic Studies*, 48, 289–309.