Stochastic Optimal Growth with Unbounded Shock

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Introduction

Unbounded Shock

Why: $\mathbb{R}_{++}$ and distributions in mathematical statistics unbounded.

f is unbounded but $\|Pf\|$ bounded

Definitions

- A **semidynamical system** (U, T)
  1. U is a metric space
  2. T is a continuous self map on U
- (U, T) is **asymptotically stable** if it is *globally stable.*
- $A \subseteq U$ is **precompact** if every sequence in A has a convergent subsequence.
- (U, T) is **Lagrange stable** if $\{T^n x\}$ is precompact for every $x \in U$.

Fixed Point Theorem I

Theorem

Let U be a **nonempty convex closed** *subset of a normed linear space X. Let $T : X \to X$ be linear and continuous, with $TU \subseteq U$.*

$$(U, T) \text{ is Lagrange stable} \Rightarrow T \text{ has a fixed point in } U.$$

Sketch of Proof.

1. Fix x and let $\gamma(x) = \{T^n x\}$.
2. $\gamma(\hat{x})$ (precompact) is the convex hull and $\mathrm{cl}(\gamma(\hat{x}))$ (compact) is its closure.
3. $T\mathrm{cl}(\gamma(\hat{x})) \subset \mathrm{cl}(\gamma(\hat{x})) \subset U$
4. T has a fixed point in $\mathrm{cl}(\gamma(\hat{x}))$

Definitions

- T is said to be a **contraction mapping** if there exists an $\alpha < 1$ such that

$$d(Tx, Ty) \leq \alpha d(x, y), \quad \forall x, y \in U \tag{1}$$

- A semidynamical system (U, T) is called **contractive** if

$$d(Tx, Ty) \leq d(x, y), \quad \forall x, y \in U \tag{2}$$

- A semidynamical system (U, T) is called **strongly contractive** if, in addition,

$$d(Tx, Ty) < d(x, y), \quad \forall x, y \in U, \quad x \neq y. \tag{3}$$

$$\mathbf{(1)} \Rightarrow \mathbf{(3)} \Rightarrow \mathbf{(2)}$$

Fixed Point Theorem II

Lemma (Joshi and Bose 4.1.6)

Let (U, T) be a semidynamical system. If (U, T) is strongly contractive and U is compact, then (U, T) is asymptotically stable.

- Compactness might not hold

  Densities on $\mathbb{R}_{++}$ not compact, i.e. $f_n(x) = n \cdot \mathbf{1}_{[0,1/n]}(x)$

- Possible to replace it with Lagrange stable (recover some local compactness)

Fixed Point Theorem II

Theorem

Let X be a metric space, let U be a nonempty closed subset of X, and let $T : X \to X$ be a continuous function invariant on U.

$$(U, T) \text{ Lagrange stable \& strongly contractive} \Rightarrow T \text{ asymptotically stable.}$$

Sketch of Proof.

1. Fix x and let $\Gamma(x)$ be the closure of $\{T^n x\}$.
2. (U, T) Lagrange stable $\Rightarrow$ $\Gamma(x)$ compact and $T\Gamma(x) \subset \Gamma(x)$ (continuity)
3. $(\Gamma(x), T)$ strongly contractive on compact set and T has a fixed point in $\Gamma(x)$
4. Strong contractivity implies unique fixed point

Application

At each time t, the agent:

- receives income x_t
- chooses consumption $c_t \in [0, x_t]$
- invests the remainder $x_t - c_t$ in production,
- and enters the next period with income x_{t+1} as defined

$$x_{t+1} = f(x_t - c_t) \cdot \varepsilon_t,$$

Assumptions

Assumption 1. The production function $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ is zero at zero, strictly increasing, strictly concave, differentiable, and satisfies the Inada conditions

$$\lim_{x \downarrow 0} f'(x) = \infty \quad \text{and} \quad \lim_{x \uparrow \infty} f'(x) = 0.$$

Assumption 2. The utility function $u\colon \mathbb{R}_+ \to \mathbb{R}$ is strictly increasing, strictly concave, differentiable, and satisfies the interiority condition

$$\lim_{x \downarrow 0} u'(x) = \infty.$$

Assumption 3. The shocks are i.i.d.. The distribution of ε is represented by density ψ. The shock has finite mean $\mathbb{E}(\varepsilon)$. In addition, ε satisfies $\mathbb{E}(1/\varepsilon) < 1$. The shock is less than one with positive probability, i.e.,

$$\int_0^1 \psi(x)\, dx \neq 0.$$

Main Result

Theorem

Let u, f, and ψ satisfy Assumptions 1–3. The following statements are true.

1. *The economy (u, f, ψ) has at least one (nonzero) equilibrium*

   fixed point Thm I

2. *If, in addition, ψ is everywhere positive, then the equilibrium is unique and globally stable.*

   fixed point Thm II

Sketch 1 : Lagrange Stability

1. Construct semidynamical system $(D(\mathbb{R}_{++}), Q)$

   $D(\mathbb{R}_{++})$ densities and at time t, income distributed by $Q^t\psi_0$

2. Construct a dense subset $\mu \subset D(\mathbb{R}_{++})$ with elements h s.t.

$$\int_0^\infty x \; h(x)dx < \infty \quad \text{and} \quad \int_0^\infty \frac{1}{x} \; h(x)dx$$

3. Truncate any ψ by $h_k^0 = \mathbf{1}_{(\frac{1}{k},k)}\psi$ and let $\{h_k = \frac{h_k^0}{||h_k^0||}\}$, so $\{h_k\} \subset \mu$

4. $\{Q^n h\}$ weakly precompact (precompact in fixed point Thm1)

Q is both linear and continuous and $D(\mathbb{R}_{++})$ is closed and convex.

Sketch 2: Strong Contractiveness

Lemma

Given measure space (U, Σ, ν), let p be a stochastic kernel, and let P be the associated Markov operator.

If $p > 0$ on $U \times U$, then $(\mathcal{D}(U), P)$ is strongly contractive.

ψ is everywhere positive