Precautionary Saving in a Markovian earnings environment

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- Consider a prudent agent,

$$\uparrow \text{Future income risk} \quad \rightarrow \quad \text{Saving} \uparrow$$

- Miller (1976): In IID earnings process, u' is convex
- Contribution: In the Markov earnings process, under what conditions
- AR(1) process satisfies those conditions

Optimal Saving Model

- Endogenous s and exogenous y
- Timing of events

$$s_t \longrightarrow \text{receive } y_t \longrightarrow \text{choose } c_t \longrightarrow s_{t+1} = Rs_t + y_t - c_t$$

- Value function

$$v(s, y) = \max_{c \in [0, Rs+y]} \left\{ u(c) + \beta \int v(Rs + y - c, y') P(y, \mathrm{d}y') \right\}$$

Properties with respect to s

$$(Tf)(s,y) = \max_{c \in [0, Rs+y]} \left\{ u(c) + \beta \int f(Rs + y - c, y') P(y, \mathrm{d}y') \right\}$$

- Assume that
  - P has the Feller property
  - u is bounded, strictly increasing, strictly concave, continuously differentiable and that $u'(0) = \infty$
- Then
  - v is strictly increasing in s
  - v is strictly concave in s
  - If f is concave in s, then σ^f is unique and hence continuous
  - If f is continuously differentiable w.r.t. s, then $(Tf)'(s,y) = Ru'(\sigma^f(s,y))$

Key Technique

- Let V be all such f and let E be closed
- If the following conditions hold
  - There exists $f \in V$ such that $f' \in E$
  - $f' \in E$ and $Tf \in V$ can imply $(Tf)' \in E$
- Then
  - $T^n f \to v \in V$
  - $(T^n f)'(s,y) = Ru'(\sigma_n(s,y)) \to Ru'(\sigma(s,y)) = v'(s,y)$
  - Hence, $v' \in E$

Properties with respect to y

$$v(s, y) = \max_{c \in [0, Rs+y]} \left\{ u(c) + \beta \int v(Rs + y - c, y')P(y, \mathrm{d}y') \right\}$$

- $v(s, y)$ is concave whenever P is CVP

$$f \text{ is concave} \quad \Rightarrow \quad \int f(s, y')P(y, \mathrm{d}y') \text{ is concave}$$

- $v'(s, y) = \frac{\partial v}{\partial s}(s, y)$ is decreasing in y whenever P is monotone

$$f \text{ is monotone} \quad \Rightarrow \quad \int f(s, y')P(y, \mathrm{d}y') \text{ is monotone}$$

- $v'(s, y)$ is convex whenever u' is convex and P is CVP

Increase in Future Income Risk

- A Markov kernel Q is riskier than a Markov kernel P if for any $y \in \mathsf{Y}$

$$P(y, \cdot) \text{ \textbf{second order stochastically dominates} } Q(y, \cdot)$$

- i.e. for all concave functions $f \in ib\mathsf{Y}$ and $y \in \mathsf{Y}$

$$\int f(y') Q(y, \mathrm{d}y') \leqslant \int f(y') P(y, \mathrm{d}y')$$

Main Result

- If the following conditions hold

  1. u' is convex (the agent is prudent)
  2. P is monotone
  3. P is CVP
  4. Q is riskier than P (No condition on Q)

then $\sigma(w, y, Q) \leqslant \sigma(w, y, P)$ for all $(w, y) \in \mathsf{W} \times \mathsf{Y}$

Key in Proof

$$f' \in E \text{ and } Tf \in V \text{ can imply } (Tf)' \in E$$

- Let $f(s, y) \in V$
- E: $f'(s, y)$ is convex and decreasing in y and that there exists $g(s, y) \in V$ such that $f'(s, y) \leqslant g'(s, y)$
- Notes that
  - $(Tf)'(s, y, P)$ is convex and decreasing in y
  - $(Tf)(s, y, P)$ and $(Tg)(s, y, Q)$ are in V
- FOC, SOC, and Envelope Condition imply

$$(Tf)'(s, y, P) = Ru'(\sigma^f(s, y, P)) \leqslant Ru'(\sigma^g(s, y, Q)) = (Tg)'(s, y, Q)$$

Miller, B. L. (1976): "The effect on optimal consumption of increased uncertainty in labor income in the multiperiod case," *Journal of Economic Theory*, 13, 154–167.