Financial Econometrics: Linear Time Series

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Purpose of financial econometrics

1. Model prices with stochastic processes, either via SDEs or time series.
2. Identify particulariteies of the data(jump, fat tails , stochastic vol. etc) and include it in the model
3. Apply the result for rish management, or portfilio management.

Why we learn the linear time series

Treating an asset return (e.g., log return r_t of a stock) as a collection of random variables over time

1. Linear time series analysis provides a netural framework to study the dynamic structure of such a series
2. To better appreciate the more sophisticated models in the future

Stationarity

The foundation of time series analysis is stationarity.

Definition (Strictly Stationary)

A time series $\{r_t\}$ is said to be strictly stationary if the joint distribution of $(r_1, \cdots, r_k)$ is identical to that of $(r_{1+t}, \cdots, r_{k+t})$ for all t.

Definition (Weakly Stationary)

A time series $\{r_t\}$ is weakly stationary if both the mean of r_t and covariance between r_t and r_{t-l} are time invariant, where l is arbitary integer. More specific, if $\{r_t\}$ is weakly stationary sequence if

1. $E(r_t) = \mu$
2. $Cov(r_t, r_{t-l}) = \gamma_l$

In the following discussion, if we refer to stationary, we mean weakly stationary.

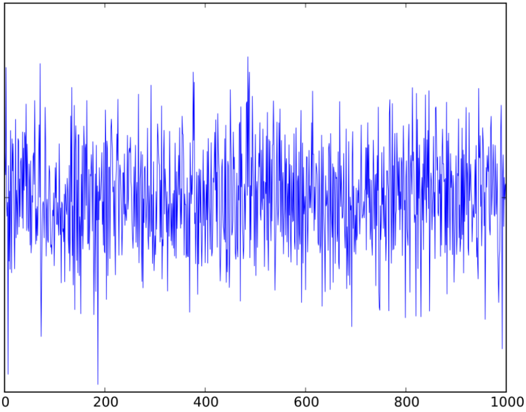
Stationary Time Series - Example

Definition (White Noise)

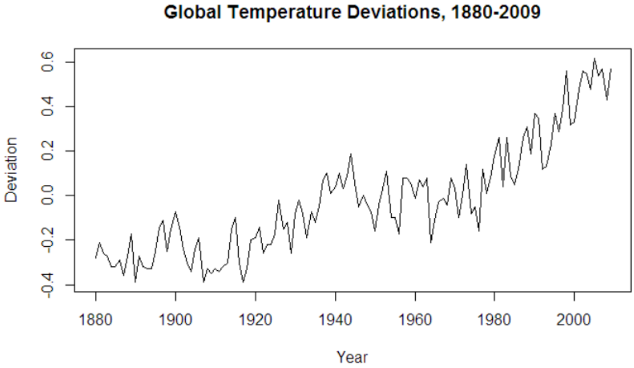
a collection of iid random variables $\{\varepsilon_t\}$ *with mean 0 and finite variance* σ_ε^2

White noise is standard example for stationary time series

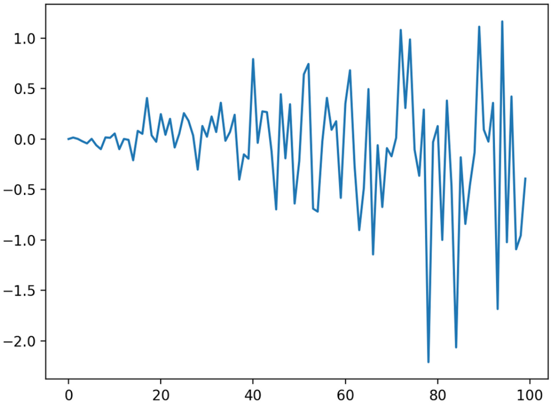
White Noise - picture



Nonstationary Time Series - Violating Constant Mean



Nonstationary Time series - Violating Constant Variance



Stationary

In the finance literature, it is common to assume that an asset return series is weakly stationary.

Definition (Autocorrelation Function(ACF))

$$\rho(l) = \frac{Cov(r_t, r_{t-l})}{\sqrt{Var(r_t)Var(r_{t-l})}} = \frac{Cov(r_t, r_{t-l})}{Var(r_t)} = \frac{\gamma_l}{\gamma_0}$$

Note that: We assume the series is stationary, the ACF only depends on l We also denote the autocovariance function of stationary process as

$$\gamma(l) = Cov(r_t, r_{t+l}) = E(r_t - \mu)(r_{t+l} - \mu)$$

Relation between two definition of stationary

Theorem

1. *Strict stationarity* $\rightarrow$ *stationarity*
2. *Stationarity* $+$ *Gaussian* $\rightarrow$ *strict stationarity*
3. *Stationarity* $\rightarrow \gamma(h) = \gamma(-h)$

Relation between two definition of stationary

Proof.

The first one is easy, the distribution invariant would imply that the moments must be the same. For the second statement, notice the first two moments parametrices the normal distribution. For third one,

$$\gamma(h) = Cov(r_t, r_{t+l}) = Cov(r_{t-l}, r_t) = \gamma(-h)$$

The second equality comes from the definition of stationarity. $\square$

More examples

To understand the dynamic of time series, the key component is to explore the correlation structure of series. We would give some famous processes.

Random Walk

Definition (Random Walk)

We say that the series $\{r_t\}$ is random walk if

$$r_t = r_{t-1} + \varepsilon_t$$

where ε_t is white noise.
We sat that the series $\{r_t\}$ is random walk with drift δ if

$$r_t = r_{t-1} + \delta + \varepsilon_t$$

Random Walk

If $s > t$, the autocovariance of Random Walk is

$$\gamma(s,t) = Cov(r_s, r_t) = Cov(r_t + \sum_{l=t+1}^{s} \varepsilon_l, r_t) = Cov(r_t, r_t)$$

$$= \sum_{l=1}^{t} Cov(\varepsilon_l, \varepsilon_l) = t\sigma_\varepsilon^2$$

Similarly, if $s < t$ the autocovariance of Random walk is $s\sigma_\varepsilon^2$, therefore

$$\gamma(s,t) = \min\{s,t\}\sigma_\varepsilon^2$$

So, it is not stationary(violating constant variance).

Moving Average

We consider the process

$$r_t = \frac{1}{2}(\varepsilon_t + \varepsilon_{t-1})$$

Therefore,

$$\gamma(s,t) = Cov(r_s, r_t) = \begin{cases} \frac{1}{2}\sigma_\varepsilon^2, & \text{for } s = t \\ \frac{1}{4}\sigma_\varepsilon^2, & \text{for } |s-t| = 1 \\ 0, & \text{for } |s-t| > 1 \end{cases}$$

Plus $\mu_t = 0$, it implies this process is stationary.

Linear process

Definition (Linear process)

We say $\{r_t\}$ is linear process if it is a linear combination of time -lagged white noise

$$r_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j \varepsilon_{t-j}$$

with coefficients

$$\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$$

Linear process

The autocovariance of linear process would be

$$\gamma(h) = Cov(r_t, r_{t+h}) = Cov(\mu + \sum_{j=-\infty}^{\infty} \psi_j \varepsilon_{t-j}, \mu + \sum_{l=-\infty}^{\infty} \psi_l \varepsilon_{t+h-l})$$

$$= \sum_{j=-\infty}^{\infty} Cov(\varepsilon_{t-j}, \varepsilon_{t-j}) \psi_j \psi_{j+h}$$

$$= \sum_{j=-\infty}^{\infty} \sigma_\varepsilon^2 \psi_j \psi_{j+h}$$

AR(p)

When we find the stochastic process (discrete time) is stationary, then we can consider a model(process) to describe it. If we observe a high dependence on time-lagged variables, the popular choice would be AR(p) process.

Definition (AR(p))

A zero - mean stationary process is an autoregressuve process if order p if it satisfies

$$r_t = \phi_1 r_{t-1} + \cdots + \phi_p r_{t-p} + \varepsilon_t$$

where ε_t *is white noise*