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NETWORKS, LINES AND FIELDS

By

JOHN D. RYDER

Dean of the School of Engineering
Michigan State University

SECOND EDITION

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PREFACE TO SECOND EDITION

The science of the electric circuit, when broadened to include currents or fields in lumped networks, distributed constant lines, wave guides, and space, is a primary concern of the modern electrical engineer or physicist. Analysis of performance and synthesis for specified operation is a major field of study beyond that usually possible in a basic alternating-current circuits course.

The author has attempted to develop the subjects of networks, resonance, and filters in such a way as to give the undergraduate student a definite familiarity with network theory, or the description of network performance by certain measured or defined parameters. In the second edition the material on matrix solution of networks has been increased, consideration of circuit duality expanded, and a treatment of Foster's reactance theorem and canonic networks added.

The student is led to discover that these same fundamental network concepts still apply when he is introduced to the circuit of distributed constants, by study of lines at both radio and power frequencies. Lines are treated throughout in terms of exponential factors and the reflection coefficient, thus reminding the student at all times of the possible presence of the incident and reflected waves, resulting in a clearer physical picture than can be obtained by the use of hyperbolic functions. The exponential forms of the line equations also have major computational advantages over the hyperbolic functions when complex angles are involved. Further emphasis has been given to the Smith chart for radio-frequency lines in the second edition.

The study of fields in transmission of energy, particularly the plane wave, skin effect, and guided waves, is considerably simplified by extensive use of the transmission-line analogy, employing the incident and reflected wave viewpoint, for which the student is well prepared by the treatment of lines through the exponential factors and the reflection coefficient. In response to a considerable demand, the second edition employs the vector analysis approach and vector notation where applicable in the chapters on time-varying fields and radiation, and a chapter on antenna theory and simple systems has been added. The vector material is developed as needed, and no preceding work in this direction is assumed.

It is suggested that Sections 4-1 to 4-5 of the chapter on filters be included in courses covering lines and circuits of distributed constants, even though the rest of the chapter on filters is omitted, in order to establish a proper groundwork for the network concepts used as an introduction to circuits of distributed constants.

The suggestions of numerous instructors have been followed where possible, and are hereby gratefully acknowledged. In particular the author wishes to thank Dean W. L. Everitt, of the University of Illinois, Prof. W. L. Cassell, of Iowa State College, and Prof. H. A. Morgan, of the Thayer School of Dartmouth College.

J. D. Ryder

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Chapter 1

NETWORK TRANSFORMATIONS

Electric circuits are the foundation on which the study of most electrical phenomena is based. Many circuits are complicated, and analysis for currents, voltages, power, or frequency performance becomes difficult unless systematic methods are developed and employed. This chapter will deal with methods that will aid in circuit analysis.

It is assumed that the student is familiar with the representation of voltages and currents, varying sinusoidally with time, by exponential functions such as $E\epsilon^{j\omega t}$. It is customary to take $t = 0$ in this usage, and to designate the complex number remaining by use of boldface italic type. The terms *complexor* or *phasor* have been proposed for this quantity, to avoid usage of the word vector, which is more properly reserved for quantities directed in space.

1-1. Network definitions

It is desirable to introduce certain definitions before going into the discussion of network solutions. The following list is not complete, and other items will be introduced as needed.

Circuit element. Any individual circuit component (inductor, resistor, capacitor, generator, etc.) with two terminals by which it may be connected to other electric components.

Branch. A group of elements, usually in series, and having two terminals.

Potential source. A hypothetical generator which maintains its value of potential independent of the output current. As an a-c source it will be indicated by a circle enclosing a wavy line.

Current source. A hypothetical generator which maintains an output current independent of the voltage across its terminals. The current source will be indicated by a circle enclosing an arrow, the arrow indicating the assumed reference current direction.

Network. An electric network is any interconnection of electric circuit elements, or branches. In Fig. 1-1(a) is shown a complicated network of elements or impedances and sources, where each branch may include R, L, C, or other types of elements. Such a network, having two distinct pairs of terminals is called a *four-terminal* network. If one of the 1,1 terminals is common to the 2,2 pair, as at (b), the circuit is a *three-terminal* network, and if the 2,2 terminals are short-circuited, it becomes a *two-terminal* network.

Lumped networks. One in which physically separate resistors, inductors, capacitors, can be represented.

Distributed network. A network in which the resistors, inductors, capacitors, cannot be electrically separated and individually

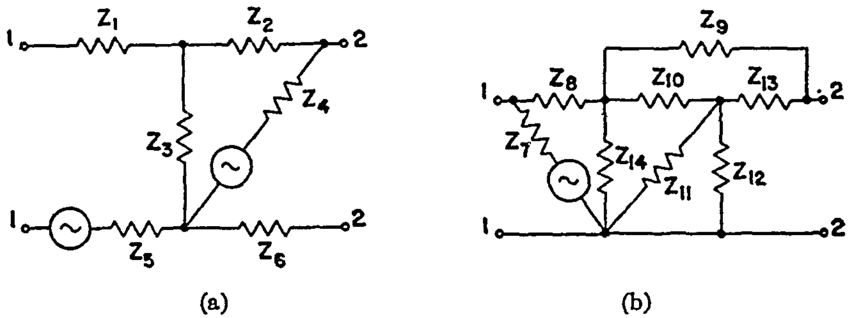


Fig. 1-1. Possible forms of networks: (a) a four-terminal network; (b) a three-terminal network.

isolated as separate elements. A transmission line is such a network.

Passive network. A network containing circuit elements without energy sources.

Active network. A network containing generators or energy sources as well as other elements.

Linear element. A circuit element is linear if the relation between current and voltage involves a constant coefficient as in

$$e = Ri, \qquad e = L\,di/dt, \qquad e = \frac{1}{C}\int i\,dt$$

Iron-cored reactors and incandescent lamps are examples of elements that are *not* linear, or in which the coefficient is a function of current.

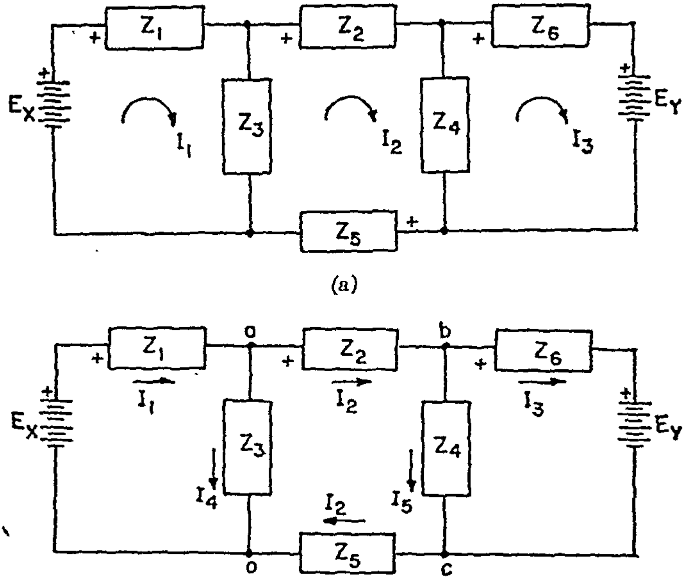
Linear networks are those in which the differential equation relating the instantaneous current and voltage is a linear equation with constant coefficients.

Mesh (*or loop*). A set of branches forming a closed path in a network, provided that if one branch is omitted, the remaining branches do not form a closed path.

Node (or junction). A terminal of any branch of a network, or a terminal common to two or more branches.

1-2. Mesh and node circuit analysis

Kirchhoff's basic circuit laws provide two methods for the solution of networks. The potential law states that the *algebraic summation*



(a)

(b)

Fig. 1-2. (a) Mesh or loop analysis; (b) nodal currents.

of potentials around a closed traverse of a circuit is zero, and this leads to the method of network solution known as *mesh or loop analysis.* The current law states that the *algebraic summation of the currents toward a junction point is zero,* and this leads to *node or junction analysis.*

The mesh or loop method is illustrated by the circuit of Fig 1-2(a) wherein three mesh or loop currents, I_1, I_2, and I_3 are assumed and given reference directions as indicated by the circuital arrows. It is then possible to write the algebraic summation of potentials around each of the closed paths as

$$I_1Z_1 + (I_1 - I_2)Z_3 -$$

$$-(I_1 - I_2)Z_3 + I_2(Z_2 + Z_5) - (I_3 - I_2)Z_4 = 0 \tag{1-2}$$

$$(I_3 - I_2)Z_4 + I_3Z_6 + E_y = 0 \tag{1-3}$$

A more orderly form of these equations would be

$$\begin{aligned}
(Z_1 + Z_3)I_1 - \qquad\qquad\qquad Z_3I_2 \qquad\qquad\qquad &= E_x \\
-Z_3I_1 + (Z_2 + Z_3 + Z_4 + Z_5)I_2 - \qquad Z_4I_3 &= 0 \\
-Z_4I_2 + (Z_4 + Z_6)I_3 &= -E_y
\end{aligned}$$

A solution for the three currents can then be obtained. However, the circuit solution is not yet complete, since potentials between junctions are usually of primary interest. By forming the two additional currents

$$I_4 = I_1 - I_2 \tag{1-4}$$

$$I_5 = I_2 - I_3 \tag{1-5}$$

which appear in the original equations above, the circuit becomes that of Fig. 1-2(b), and potentials between any pair of junctions

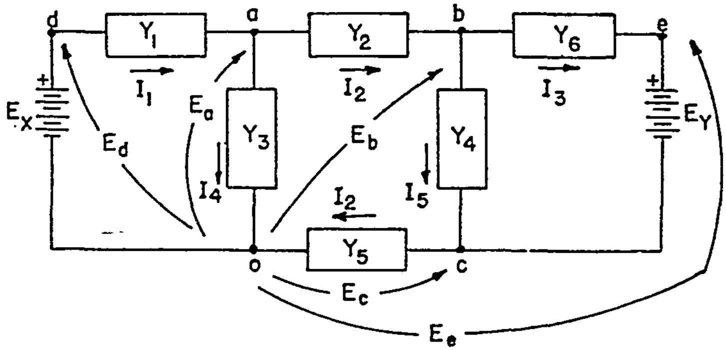


Fig. 1-3. Node or junction analysis.

in the network may be obtained and the solution is complete. Five relations were needed.

If voltages between junctions or nodes are the primary objective of the solution, and this is very often the case in practice, then it would have been as easy and in some cases easier, to have started with the assumed *branch* currents of Fig. 1-2(b), and by use of admittance instead of impedance, to have written equations involving the voltages between junctions directly. This is the node or junction method of solution.

Some convenient junction between elements is chosen as a reference junction or node, and is indicated as 0 in Fig. 1-3. In many circuits this reference is most conveniently chosen as a common terminal or as the ground connection. It may be seen that there are six nodes between elements in the network, but only four of these exist at junctions of three or more elements. With six nodes 0, a, b, c, d, e, and using 0 as reference, it is possible to write $N - 1$, or five, nodal equations or branch current summations at the nodes, in terms of the potentials E_a, E_b, E_c, E_d, E_e, and the branch admittances.

Two of the node voltage equations may be written directly as

$$E_d = E_x \tag{1-6}$$

$$E_e = E_y + E_c \tag{1-7}$$

Writing a current summation at junction a gives

$$I_1 - I_2 - I_4 = 0$$

which, in terms of potentials and admittances is

$$(E_x - E_a)Y_1 - (E_a - E_b)Y_2 - E_aY_3 = 0 \tag{1-8}$$

At junction b:

$$I_2 - I_3 - I_5 = 0$$

$$(E_a - E_b)Y_2 - (E_b - E_c - E_y)Y_6 - (E_b - E_c)Y_4 = 0 \tag{1-9}$$

At junction c:

$$I_3 - I_2 + I_5 = 0$$

$$(E_b - E_c - E_y)Y_6 + (E_b - E_c)Y_4 - E_cY_5 = 0 \tag{1-10}$$

These equations may be more systematically set down as

$$\begin{aligned}
-(Y_1 + Y_2 + Y_3)E_a \qquad & + Y_2E_b && = -Y_1E_x \\
Y_2E_a \qquad & - (Y_2 + Y_4 + Y_6)E_b + (Y_4 + Y_6)E_c && = Y_6E_y \\
& (Y_4 + Y_6)E_b - (Y_4 + Y_5 + Y_6)E_c && = -Y_6E_y
\end{aligned}$$

Solutions of Eqs. 1-8, 1-9, and 1-10, yield all potential information at once. Since at high frequencies voltmeters are more commonly used and are more accurate than ammeters, the node method of solution correlates well with practice.

In general, if N junction points and B branches exist in a circuit,

$N - 1$ current summations are required by the node-voltage method of solution. Likewise, $B - (N - 1) = B - N + 1$ voltage summations will be required by the loop or mesh-current method. For the circuit discussed, $B = 6$, $N = 4$, and $N - 1 = 3$, or $B - N + 1 = 3$, so that the number of equations required was the same in each case. However, when

$$(N - 1) < (B - N + 1)$$

the node-voltage method will yield fewer equations. Frequently other considerations will enter into the choice of method. In particular, if current sources are present, the node-voltage method may be easier; if potentials to a common reference are the desired elements of the solution, the node-voltage method may also be the most desirable.

1-3. Principle of duality

Ordinarily a voltage is considered the driving force, and a current the response of the circuit. However, except for a mental difficulty induced by habit, there is no reason why a current cannot be considered the driving force, and a voltage the response. The reversal

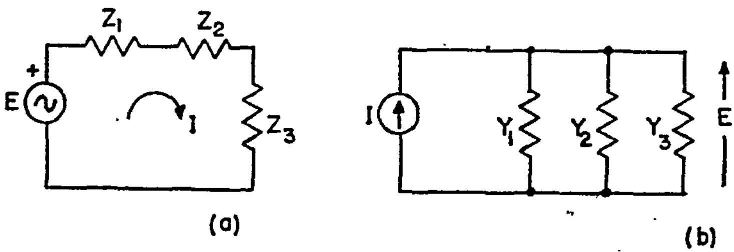


Fig. 1-4. Dual networks.

of philosophy here is merely that due to the interchange of the dependent and independent variables in the problem, and leads to the concept of *duality* in networks.

Some dual relations involving exchange of current for voltage are illustrated by the following pairs of equations:

$$e = Ri \qquad i = Ge$$

$$e = L\,di/dt \qquad i = C\,de/dt$$

$$e = \frac{1}{C}\int i\,dt \qquad i = \frac{1}{L}\int e\,dt$$

These show that resistance and conductance, inductance and capacitance, also have dual relationships. Duality also appears as a relation between networks. Referring to the simple circuit of Fig. 1-4(a), the loop equation is

$$I(Z_1 + Z_2 + Z_3) = E \tag{1-11}$$

whereas the node voltage equation for (b) is

$$E(Y_1 + Y_2 + Y_3) = I \tag{1-12}$$

Since these relations are of identical mathematical form, but with interchanged dependent and independent variables, they are *dual networks*. Because of the identical mathematical form they must

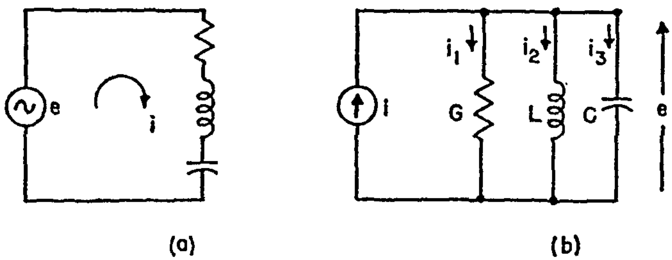


Fig. 1-5. Series and parallel networks as dual.

have identical *forms* of solutions. That is, (a) will behave with respect to element voltage as (b) does with respect to element current.

The voltage drops in one circuit are analogous to the branch currents in the other circuit. In one case network equations may be most easily written by summation of voltage drops, and in the other case the equations are most easily written by summing the currents at the junctions. Thus it can be seen that the property of duality actually comes from, and is contained in, Kirchhoff's two circuit laws.

The circuit-dual concept can be carried further by referring to the circuits of Fig. 1-5, where it can be shown that a parallel *GLC* circuit driven by a current source is the dual of a series *RLC* circuit excited by a voltage source. The mesh equation for the series case may be written

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt = e \tag{1-13}$$

and for the dual parallel circuit,

$$Ge + C\frac{de}{dt} + \frac{1}{L}\int e\,dt = i \tag{1-14}$$

illustrating the various dual relations.

It should be emphasized that duals are not equivalent in performance. Duality implies only that the equations of circuit performance are similar in mathematical form, and will behave similarly, but with interchanged variables. For instance, in (a) of Fig. 1-5, the sum of the voltages across L and C will go to zero at resonance; in (b) the sum of the currents i_2 and i_3 into the L and C branches will go to zero at parallel resonance.

Further cases of duality will be discussed as they arise.

1-4. Reduction of a complicated network

Many circuits will be found complicated and difficult to analyze. If they can be reduced or simplified it is frequently possible better to understand their performance. This reduction can be accomplished through use of certain types of simple networks that are *equivalent* in performance.

One passive network is said to be equivalent to a second network if the second can be substituted for the first without change in currents and voltages appearing at the network terminals. It must be emphasized that an equivalent network is not identical internally but has only identical values of external voltages and currents at the terminals. Thus two networks are equivalent if each can be placed in a box with terminals and if the boxes can be substituted one for the other in a circuit without change in the circuit operation. Thus it may be possible to find a simple network which may be substituted for a more complicated network..

Assume that the box of Fig. 1-6(a) contains a linear passive electric network, no matter how complicated in internal connection. Certain external terminals may be designated 1,1 and 2,2; and impedance measurements may be made at these terminals. The input impedance may be measured at either pair of terminals with any selected value of impedance connected to the other pair. Convenient impedance values which may be chosen are open circuit and short circuit. The impedance measured across the 1,1 terminals with the 2,2 terminals open-circuited may be designated Z_{1oc}.

Likewise, the measurement at the 1,1 terminals with the 2,2 terminals short-circuited might be called Z_{1k}. Similar measurements taken at the 2,2 terminals would result in $Z_{2\infty}$ and Z_{2k}, giving four such possible measurements.

Since these are measurements of terminal impedance, any other network which can be found with identical measurements should

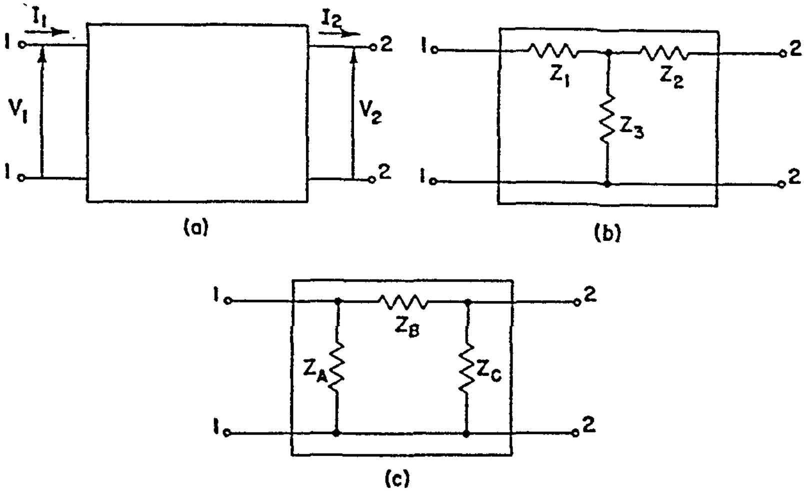


Fig. 1-6. (a) Any complicated network, with terminal voltages and currents indicated; (b) a T network which may be made equivalent to the network in the box of (a); (c) a π network equivalent to (b) and (a).

be equivalent. The simplest form for a network equivalent to the one in the box may now be considered. There are eight quantities, the magnitudes and phase angles of V_1, V_2, I_1, and I_2, which must be established by the new network, in order that the conditions for equivalence be satisfied. However, if a load be connected to the 2,2 terminals and called Z_R, then

$$V_2 = I_2 Z_R$$

so that although there are apparently four phase angles and four magnitudes to be specified, only six of these are functions of the network, one magnitude and one phase angle being determined by the load impedance Z_R. With three magnitudes and three phase angles to be determined, a minimum of three adjustable impedances is needed to set up a network which may be equivalent to that in the box. Only two arrangements of three-element circuits are possible; these are shown at (b) and (c) of Fig. 1-6 as the T (Y) and π (delta) networks.

Consider the T network of (b), Fig. 1-6, as a possible simplified circuit, that by proper selection of values of Z_1, Z_2, and Z_3 might be made to have the same external currents and voltages as the original network of (a). The T network would then be said to be equivalent to the network at (a). If the T section is equivalent, the open- and short-circuit measurements must be identical. If these measurements are made on the T circuit, the results will be

$$Z_{1oc} = Z_1 + Z_3 \tag{1-15}$$

$$Z_{1sc} = Z_1 + \frac{Z_2 Z_3}{Z_2 + Z_3} \tag{1-16}$$

$$Z_{2oc} = Z_2 + Z_3 \tag{1-17}$$

$$Z_{2sc} = Z_2 + \frac{Z_1 Z_3}{Z_1 + Z_3} \tag{1-18}$$

A T equivalent for any passive network having a given set of open and short-circuit measurements can then be found by solving the above equations for Z_1, Z_2, and Z_3. Since there are four equations and only three unknowns, one equation and one measurement are not needed.

Subtracting Eq. 1-16 from Eq. 1-15,

$$Z_{1oc} - Z_{1sc} = Z_3 - \frac{Z_2 Z_3}{Z_2 + Z_3} = \frac{Z_3^2}{Z_2 + Z_3}$$

After recognizing that the denominator is Z_{2oc}, from Eq. 1-17, there results

$$Z_3 = \pm \sqrt{Z_{2oc}(Z_{1oc} - Z_{1sc})}$$

Use of the above equation leads to

$$Z_1 = Z_{1oc} - Z_3 \tag{1-19}$$

$$Z_2 = Z_{2oc} - Z_3 \tag{1-20}$$

$$Z_3 = \pm \sqrt{Z_{2oc}(Z_{1oc} - Z_{1sc})} \tag{1-21}$$

as the values for a T network which will be equivalent in performance to the original network. Choice of the + or − sign on the radical of Z_3 will lead to two different T networks. These will be equivalent as to current and voltages under the definition of equivalence, except for an ambiguity of 180° in output phase angle. That this is inherent in the method is illustrated by the simple expedient of reversing the output leads of the network in Fig. 1-6(a). The open and short-circuit impedances will remain unchanged, but there will be a change of 180° in the output phase conditions. Thus the method cannot resolve this ambiguity without additional data as to the phase relations of the original network.

If it is desired to find an equivalent π section, the same three measurements on the π circuit of (c), Fig. 1-6 would be

$$Z_{1\infty} = \frac{Z_A(Z_B + Z_C)}{Z_A + Z_B + Z_C} \tag{1-22}$$

$$Z_{2\infty} = \frac{Z_C(Z_A + Z_B)}{Z_A + Z_B + Z_C} \tag{1-23}$$

$$Z_{1s} = \frac{Z_A Z_B}{Z_A + Z_B} \tag{1-24}$$

Multiplying Eqs. 1-23 and 1-24 gives

$$Z_{2\infty} Z_{1s} = \frac{Z_A Z_B Z_C}{Z_A + Z_B + Z_C} \tag{1-25}$$

and subtracting Eq. 1-24 from 1-22,

$$Z_{1\infty} - Z_{1s} = \frac{Z_C Z_A^2}{(Z_A + Z_B + Z_C)(Z_A + Z_B)}$$

Use of Eq. 1-23 leads to

$$Z_{2\infty}(Z_{1\infty} - Z_{1s}) = \frac{Z_A^2 Z_C^2}{(Z_A + Z_B + Z_C)^2} \tag{1-26}$$

Values may then be obtained for the three π-section branches as

$$Z_A = \frac{Z_{2\infty} Z_{1s}}{Z_{2\infty} - \sqrt{Z_{2\infty}(Z_{1\infty} - Z_{1s})}} \tag{1-27}$$

$$Z_B = \frac{Z_{2\infty} Z_{1s}}{\sqrt{Z_{2\infty}(Z_{1\infty} - Z_{1s})}} \tag{1-28}$$

$$Z_C = \frac{Z_{2\infty} Z_{1s}}{Z_{1\infty} - \sqrt{Z_{2\infty}(Z_{1\infty} - Z_{1s})}} \tag{1-29}$$

These three equations permit designing a π network equivalent to any complicated network.

It should be noted that the equivalent circuits, in terms of Z are perfectly general; but when interpreted in terms of R, L, and C values, the circuits become equivalent at only one frequency. Impossibly large or small values of inductance or capacitance, or negative resistances, may occasionally be called for by the mathematical relations. Such results do not affect the validity of the mathematical equations, but they do limit the physical realizability of some of the networks.

The results of the preceding paragraphs may now be summarized as follows:

Any linear, bilateral, passive electrical network can be represented, at a single frequency, by a T *or* π *network.*

The term *bilateral* will be defined in Section 1-7. Frequently the knowledge that it can be done is more important than the ability to make the transformation to the equivalent T or π section.

1-5. Conversions between T and π sections

It is convenient to be able to make conversions directly from T to equivalent π sections, or vice versa, without the necessity of employing the external open- and short-circuit measurements of Section 1-4. For the T and π sections of (a) and (b), Fig. 1-7, these measurements are

T *Section* | π *Section*

$$Z_{1oc} = Z_1 + Z_3 \qquad\qquad Z_{1oc} = \frac{Z_A(Z_B + Z_C)}{Z_A + Z_B + Z_C}$$

$$Z_{2oc} = Z_2 + Z_3 \qquad\qquad Z_{2oc} = \frac{Z_C(Z_A + Z_B)}{Z_A + Z_B + Z_C}$$

$$Z_{1sc} = Z_1 + \frac{Z_2 Z_3}{Z_2 + Z_3} \qquad\qquad Z_{1sc} = \frac{Z_A Z_B}{Z_A + Z_B}$$

If the two circuits are to be equivalent, the external measurements must be equivalent. If the two networks were placed in boxes and only the terminals brought out, it would then be impossible to distinguish the T from the π circuit at a given frequency. Equating

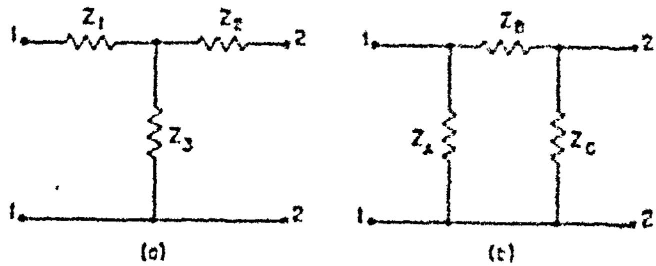


Fig. 1-7. (a) A T network; (b) a π network which may be made equivalent to the T network of (a).

the measurements leads to

$$Z_1 + Z_3 = \frac{Z_A(Z_B + Z_C)}{Z_A + Z_B + Z_C} \tag{1-30}$$

$$Z_2 + Z_3 = \frac{Z_C(Z_A + Z_B)}{Z_A + Z_B + Z_C} \tag{1-31}$$

$$Z_1 + \frac{Z_2Z_3}{Z_2 + Z_3} = \frac{Z_AZ_B}{Z_A + Z_B} \tag{1-32}$$

Subtracting Eq. 1-32 from 1-30 yields

$$Z_3 - \frac{Z_2Z_3}{Z_2 + Z_3} = \frac{Z_A(Z_B + Z_C)}{Z_A + Z_B + Z_C} - \frac{Z_AZ_B}{Z_A + Z_B}$$

$$\frac{Z_3^2}{Z_2 + Z_3} = \frac{Z_A^2Z_C}{(Z_A + Z_B + Z_C)(Z_A + Z_B)}$$

Equation 1-31 should be inserted for the denominator of the left-hand term, giving a value for Z_3. All branches of the T network follow as

$$Z_1 = \frac{Z_AZ_B}{Z_A + Z_B + Z_C} \tag{1-33}$$

$$Z_2 = \frac{Z_BZ_C}{Z_A + Z_B + Z_C} \tag{1-34}$$

$$Z_3 = \frac{Z_AZ_C}{Z_A + Z_B + Z_C} \tag{1-35}$$

If the impedances of the π circuit are known, the above equations may be used to determine the equivalent T network.

The inverse transformation from T to π can be obtained if Eq. 1-34 is substituted in Eq. 1-32, resulting in

$$Z_1Z_2 + Z_2Z_3 + Z_3Z_1 = \frac{Z_AZ_BZ_C}{Z_A + Z_B + Z_C}$$

Successive use of Eqs. 1-35, 1-33, and 1-34 in the right-hand side gives

$$Z_A = \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_2} \tag{1-36}$$

$$Z_B = \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_3} \tag{1-37}$$

$$Z_C = \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_1} \tag{1-38}$$

thus permitting a π circuit equivalent to a given T network to be designed.

The various relations for transformation between T and π networks can be summarized through Fig. 1-8. The impedance for a T arm may be obtained from the π by noting that the T impedance is calculated by using the product of the two adjacent π network impedances divided by the sum of all three π impedances. Likewise, a π-section impedance can be found by use of the sum of the double products of the T branches divided by the impedance of the T branch opposite the desired π arm.

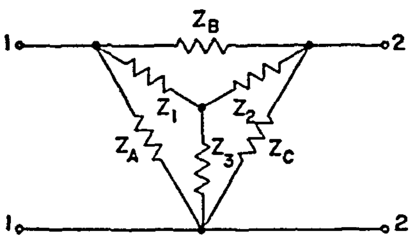


Fig. 1-8. A diagram for use in remembering the T-to-π and the π-to-T transformation relations.

If Eqs. 1-33, 1-34, 1-35 for the branches of the T are written in terms of admittances, the following are obtained:

$$Y_1 = \frac{Y_AY_B + Y_BY_C + Y_CY_A}{Y_C} \tag{1-39}$$

$$Y_2 = \frac{Y_AY_B + Y_BY_C + Y_CY_A}{Y_A} \tag{1-40}$$

$$Y_3 = \frac{Y_AY_B + Y_BY_C + Y_CY_A}{Y_B} \tag{1-41}$$

Comparison of these equations with those for the branches of the π network in Eqs. 1-36, 1-37, 1-38 is interesting, and shows that the *π and T networks are duals.* This is further reinforced if Eqs. 1-36, 1-37, and 1-38 are rewritten in terms of admittances, when dual equations for the T network are obtained.

1-6. The bridged-T network

A circuit frequently encountered in filters and frequency-response equalizing networks is the *bridged*-T circuit. This network, shown at (a), Fig. 1-9, may be analyzed as an example of the reduction of more complex networks to T sections by successive steps. The circuit may be redrawn as at (b), Fig. 1-9, to emphasize the fact

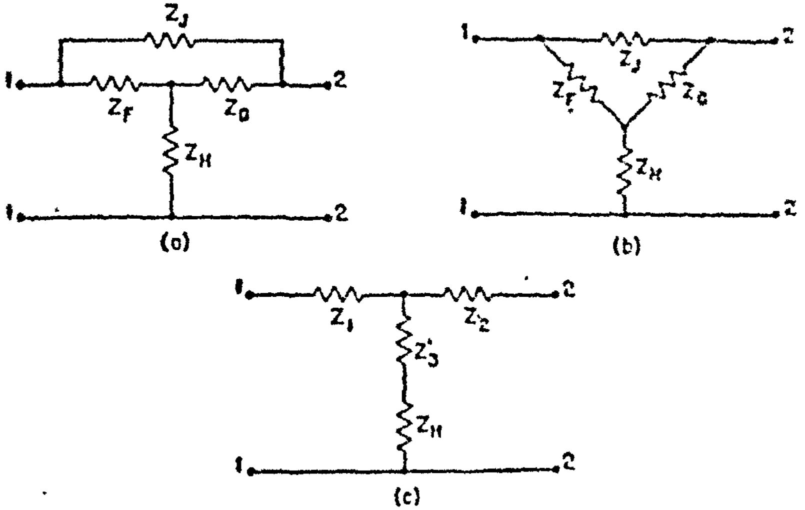


Fig. 1-9. The bridged-T network: (a) the form usually seen; (b) with a portion of the circuit redrawn as a π section; (c) the circuit reduced to an equivalent T section.

that the Z_F, Z_G, and Z_J branches form a π (delta) network. Using π to T transformation relations, the π may be changed to the equivalent T as indicated at (c), where the equivalent T branches Z_1 and Z_2 are given by

$$Z_1 = \frac{Z_F Z_J}{Z_F + Z_J + Z_G} \tag{1-42}$$

$$Z_2 = \frac{Z_J Z_G}{Z_F + Z_J + Z_G} \tag{1-43}$$

The impedance of the Z_3 shunt branch for the bridged T is the sum of Z_H and the equivalent Z_3' of the π section, or

$$Z_3 = Z_3' + Z_H = \frac{Z_F Z_G}{Z_F + Z_J + Z_G} + Z_H \tag{1-44}$$

Various other complex networks can be reduced to equivalent T circuits, at one frequency, by successive reduction of various parts of the circuit from π to T sections, or vice versa, and combining series or parallel combinations of impedances thus obtained.

1-7. The lattice network

A network frequently encountered is the *lattice*, shown at (a), Fig. 1-10, and redrawn at (b) as a bridge, in which form it is more

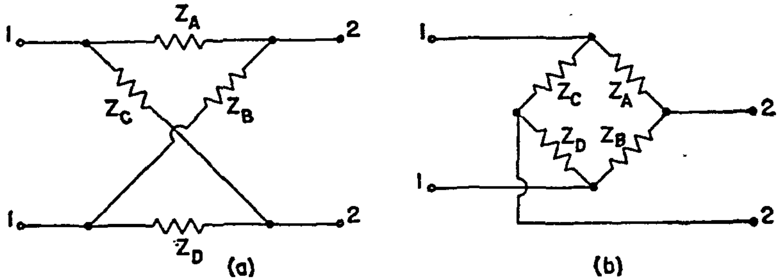


Fig. 1-10. (a) The lattice network; (b) the lattice network redrawn in the form of a bridge.

often seen. Its reduction to an equivalent T is outlined here; certain other properties are discussed in Section 1-13.

Taking measurements as before,

$$Z_{1\infty} = \frac{(Z_A + Z_B)(Z_C + Z_D)}{Z_A + Z_B + Z_C + Z_D} \tag{1-45}$$

$$Z_{2\infty} = \frac{(Z_A + Z_C)(Z_B + Z_D)}{Z_A + Z_B + Z_C + Z_D} \tag{1-46}$$

$$Z_{1sc} = \frac{Z_A Z_C}{Z_A + Z_C} + \frac{Z_B Z_D}{Z_B + Z_D} \tag{1-47}$$

Equating these to the like values obtained for a T section from Eqs. 1-15, 1-16, 1-17 gives

$$Z_1 + Z_3 = \frac{(Z_A + Z_B)(Z_C + Z_D)}{Z_A + Z_B + Z_C + Z_D} \tag{1-48}$$

$$Z_2 + Z_3 = \frac{(Z_A + Z_C)(Z_B + Z_D)}{Z_A + Z_B + Z_C + Z_D} \tag{1-49}$$

$$Z_1 + \frac{Z_2 Z_3}{Z_2 + Z_3} = \frac{Z_A Z_C}{Z_A + Z_C} + \frac{Z_B Z_D}{Z_B + Z_D} \tag{1-50}$$

Subtracting Eq. 1-50 from 1-48 and simplifying leads to

$$\frac{Z_3^2}{Z_2 + Z_3} = \frac{(Z_B Z_C - Z_A Z_D)^2}{(Z_A + Z_B + Z_C + Z_D)(Z_A + Z_C)(Z_B + Z_D)}$$

After substitution of Eq. 1-49 for the denominator of the left-hand side, and choice of the positive root,

$$Z_3 = \frac{Z_B Z_C - Z_A Z_D}{Z_A + Z_B + Z_C + Z_D} \tag{1-51}$$

Values for the other two arms of the T section are then easily obtained as

$$Z_1 = \frac{Z_A Z_C + 2Z_A Z_D + Z_B Z_D}{Z_A + Z_B + Z_C + Z_D} \tag{1-52}$$

$$Z_2 = \frac{Z_A Z_B + 2Z_A Z_D + Z_C Z_D}{Z_A + Z_B + Z_C + Z_D} \tag{1-53}$$

A T section is then established which is equivalent, at a single frequency, to the lattice network.

1-8. The superposition theorem

In any linear network containing bilateral linear impedances and energy sources, the current flowing in any element is the vector sum of the currents that are separately caused to flow in that element by each energy source.

The direct proportionality and linearity between voltage and current, as expressed by Ohm's law, is both reason for and proof of the superposition theorem. A generator of 1 volt applied to an impedance of 1 ohm produces a current of 1 ampere. Two generators in phase and in series, each of 1 volt and connected to the same 1-ohm impedance, cause a current of 2 amperes to flow. Therefore each generator is still the cause for a current of 1 ampere in the circuit. The total current is thus the sum of the currents produced by the individual generators.

Consider the case of two generators feeding into an involved network. The network may be replaced with an equivalent T section as at (a), Fig. 1-11, the currents being designated I_1 and I_2. Writing the mesh equations

$$\begin{aligned} E_1 &= I_1(Z_1 + Z_3) + I_2 Z_3 \\ E_2 &= I_1 Z_3 + I_2(Z_2 + Z_3) \end{aligned} \tag{1-54}$$

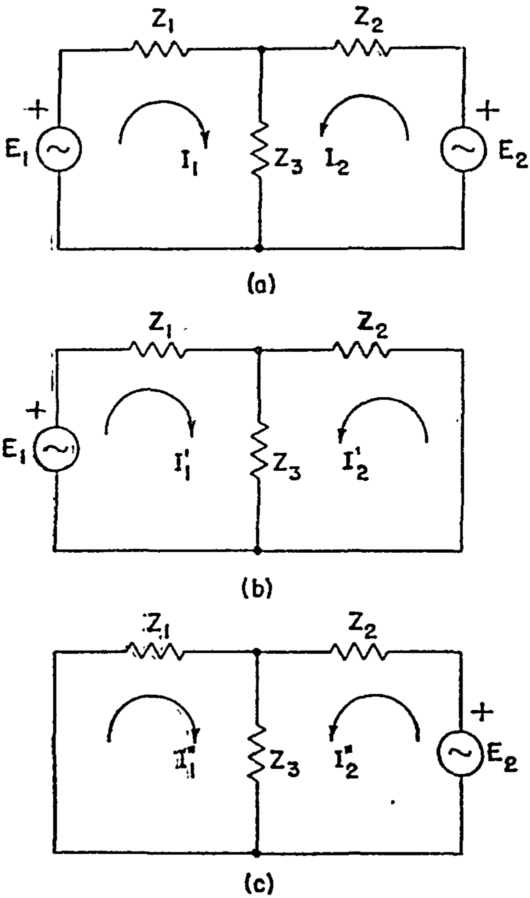


Fig. 1-11. (a) A two-mesh network; (b), (c) the two circuits, which, when superposed, result in the network of (a).

and solving for the currents leads to

$$I_1 = \frac{Z_2 + Z_3}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1} E_1 - \frac{Z_3}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1} E_2 \tag{1-55}$$

$$I_2 = -\frac{Z_3}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1} E_1 + \frac{Z_1 + Z_3}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1} E_2 \tag{1-56}$$

If now each generator is in turn removed by short-circuiting its emf (but not its impedance), the circuits of (b) and (c), Fig. 1-11 are obtained. The superposition theorem states that the sum of the currents of (b) and (c) equals the currents in (a), or

$$I_1 = I_1' + I_1''$$
$$I_2 = I_2' + I_2''$$

Solution of the circuits of (b) and (c) for their respective currents gives

$$I_1'' = -\frac{Z_3}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1} E_2 \tag{1-57}$$

$$I_1' = \frac{Z_2 + Z_3}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1} E_1 \tag{1-58}$$

$$I_2'' = \frac{Z_1 + Z_3}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1} E_2 \tag{1-59}$$

$$I_2' = -\frac{Z_3}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1} E_1 \tag{1-60}$$

Comparison term by term with Eqs. 1-55 and 1-56 for the total currents proves the theorem. The proof of the theorem may be extended to include any number of separate generators.

The theorem allows computation of the currents, in a network supplied by generators of several different frequencies, as the sum of the individual currents due to each frequency, consideration being given to changes in reactances due to the different frequencies.

The superposition theorem is a result of the linear relation between current and voltage in circuits having linear impedances. In circuits containing incandescent lamps, vacuum or gas tubes, or iron-core reactors, the theorem is not applicable or must be used with care.

1-9. The reciprocity theorem

*In any linear network containing bilateral linear impedances and energy sources, the ratio of a voltage **E** introduced in one mesh to the current **I** in any second mesh is the same as the ratio obtained if the positions of **E** and **I** are interchanged, other emfs being removed.*

This *reciprocity theorem* leads to a definition of the transfer impedance Z_T, where if E_1 is the voltage in mesh 1 and I_2 is the current in any second mesh, then

$$Z_{T12} = \frac{E_1}{I_2} \tag{1-61}$$

Proof of the theorem is supplied by the material of the preceding section. Consideration of (a), Fig. 1-11 and of Eq. 1-56, with $E_2 = 0$, gives

$$Z_{T12} = \frac{E_1}{I_2} = -\frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_3}$$

Likewise, if $E_1 = 0$, then from (a), Fig. 1-11, and Equation 1-55,

$$Z_{T21} = \frac{E_2}{I_1} = -\frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_3}$$

Therefore

$$Z_{T12} = Z_{T21} \tag{1-62}$$

and the theorem is proved.

A network in which Eq. 1-62 holds is said to be bilateral. In such a network, transmission in one direction only need be studied, since the results obtained are equally applicable to transmission in the reverse direction. Networks containing ordinary resistors, inductors, or capacitors as circuit elements are bilateral. When electron tubes or other control devices are introduced, the circuits may not be bilateral and the reciprocity theorem may not apply.

1-10. Thevenin's theorem; the voltage-source equivalent circuit

Any two-terminal linear network containing energy sources (generators) and impedances can be replaced with an equivalent circuit consisting of a voltage source E' in series with an impedance Z'. The value of E' is the open-circuit voltage between the terminals of the network, and Z' is the impedance measured between the terminals with all energy sources eliminated (but not their impedances).

The network represented by the box between generator and terminals 1,1 of (a), Fig. 1-12, may be replaced with its equivalent T section as in (b). The theorem then states that the circuit of (c) will be equivalent to that of (b) or (a), thus implying identity of voltages and currents associated with the load Z_R in all three cases.

Solution of the circuit of (b), for the load current I_R can be made from the mesh equations

$$E = I_1(Z_1 + Z_3) - I_RZ_3$$

$$0 = -I_1Z_3 + I_R(Z_2 + Z_3 + Z_R)$$

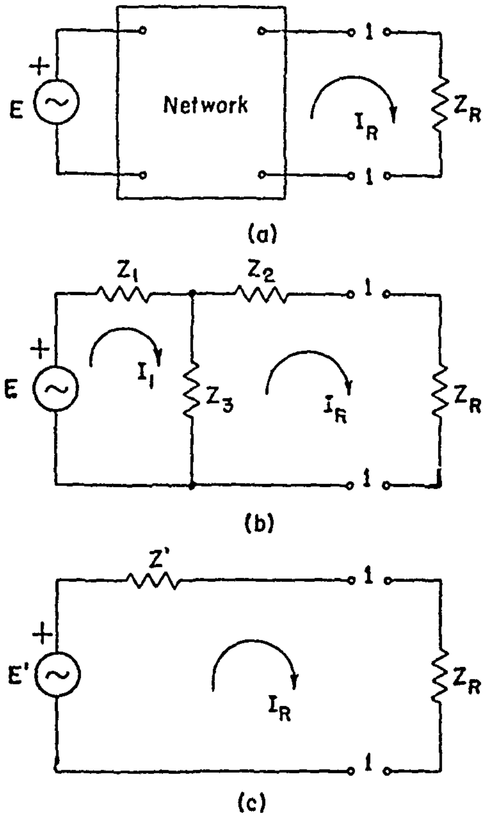


Fig. 1-12. (a) An active network connected to a load Z_R; (b) the passive portion of the network replaced with its equivalent T network; (c) the voltage-source equivalent of (b).

Then
$$I_1 = I_R \frac{Z_2 + Z_3 + Z_R}{Z_3}$$

and
$$I_R = \frac{E}{\dfrac{(Z_1 + Z_3)(Z_2 + Z_3 + Z_R)}{Z_3} - Z_3}$$

$$= \frac{EZ_3}{Z_2(Z_1 + Z_3) + Z_1Z_3 + (Z_1 + Z_3)Z_R}$$

Division by the term $(Z_1 + Z_3)$ gives

$$I_R = \frac{E\left(\dfrac{Z_3}{Z_1 + Z_3}\right)}{Z_2 + \dfrac{Z_1 Z_3}{Z_1 + Z_3} + Z_R} \tag{1-63}$$

as the value of the load current.

By inspection of (b), Fig. 1-12, the open-circuit voltage at the 1,1 terminals is

$$E' = E\left(\frac{Z_3}{Z_1 + Z_3}\right) \tag{1-64}$$

and the impedance of the network measured between the 1,1 terminals with all generator emf's short-circuited is

$$Z' = Z_2 + \frac{Z_1 Z_3}{Z_1 + Z_3} \tag{1-65}$$

It can then be seen by comparison of these equations with Eq. 1-63 that

$$I_R = \frac{E'}{Z' + Z_R} \tag{1-66}$$

which is the current equation for the network of (c), Fig. 1-12. The theorem has therefore been proved for networks containing one generator. It may be generalized to any number of generators by application of the superposition theorem, permitting each generator and associated circuit to be considered separately. Thus an equivalent circuit for an active network is obtained.

If a complicated network can be divided into several two-terminal networks, each containing a generator, equivalent circuits may be found for each of these two-terminal networks, and combination of these equivalent circuits may provide an easier solution than by other means. The circuit at (a), Fig. 1-13, may be used as an example, in which the current in Z_R is desired. The portion of the circuit to the left of terminals a,b may be reduced to a series equivalent as at (b). The emf E' becomes

$$E' = E_1\left(\frac{Z_2}{Z_1 + Z_2}\right)$$

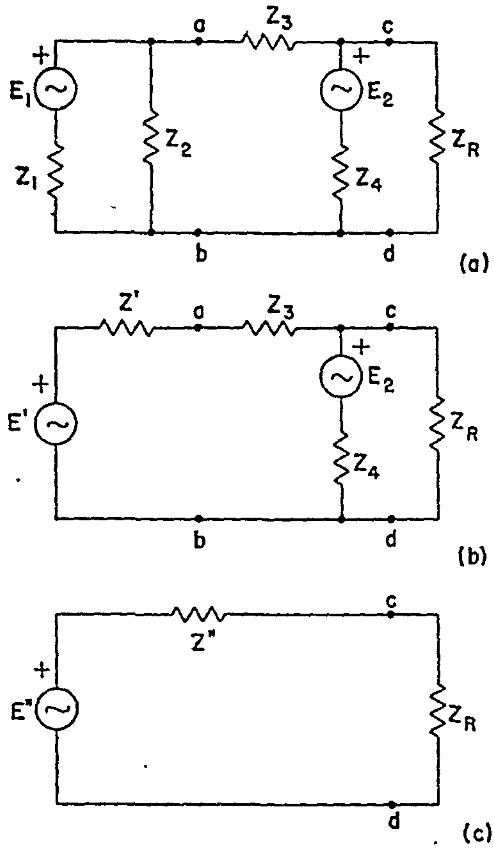


Fig. 1-13. (a) A network with two generators; (b), (c) successive steps in the use of Thevenin's theorem to reduce the network of (a) to a simple voltage-source equivalent circuit.

and the impedance Z' can be seen as

$$Z' = \frac{Z_1 Z_2}{Z_1 + Z_2}$$

By another application of Thevenin's theorem to the part of the network to the left of terminals c,d, the original circuit finally becomes the simple series circuit of (c), Fig. 1-13, from which the current in

Z_R is easily obtained. For the final step,

$$E'' = E_2 + \frac{(E' - E_2)Z_4}{Z' + Z_3 + Z_4}$$

and
$$Z'' = \frac{Z_4(Z' + Z_3)}{Z_4 + Z' + Z_3}$$

and the current in Z_R can then be written

$$I_R = \frac{E''}{Z'' + Z_R}$$

1-11. Norton's theorem; the current-source equivalent circuit

Any two-terminal linear network containing energy sources (generators) and impedances can be replaced with an equivalent circuit consisting of a current source I' in parallel with an admittance Y'. The value of I' is the short-circuit current between the terminals of the network, and Y' is the admittance measured between the terminals, with all energy sources eliminated (but not their admittances).

This theorem produces an equivalent circuit that is as useful for parallel connected active networks as the voltage-source equivalent is for series-connected active networks. The theorem may be proved by considering any network reduced by Thevenin's theorem to that of (b), Fig. 1-14. The value of the current I_R in (b) is

$$I_R = \frac{E'}{Z' + Z_R} = E'Y'\left(\frac{Y_R}{Y' + Y_R}\right) \tag{1-67}$$

Now consider the circuit at (c), Fig. 1-14, wherein a constant-current generator of value I' supplies current to a parallel network of two admittances, Y' and Y_R, these being, respectively, the reciprocals of Z' and Z_R of (b). By use of the appropriate current division factor the current I_R' in the load admittance Y_R may be written

$$I_R' = I'\left(\frac{Y_R}{Y' + Y_R}\right) \tag{1-68}$$

The load current I_R' may be made equal to that of I_R in the circuit of (b), and in Eq. 1-67, if

$$I' = E'Y' = \frac{E'}{Z'} \tag{1-69}$$

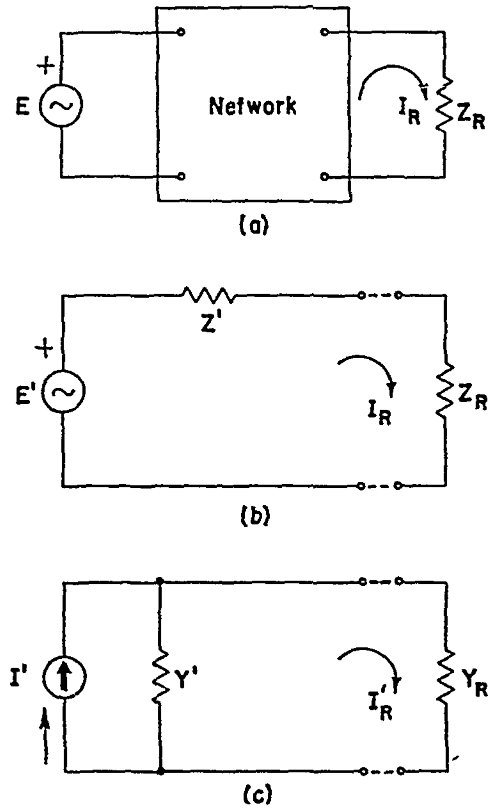


Fig. 1-14. (a) An active network connected to a load Z_R; (b) the same network converted to the voltage-source equivalent by Thevenin's theorem; (c) the current-source equivalent of (b).

and the circuits of (b) and (c) will then be equivalent. The current I' may be recognized as the short-circuit current of the Thevenin's equivalent network of (b). If then the current output of the current generator is given the value of Eq. 1-69, the current-source network of (c), is shown equivalent to the Thevenin circuit and to the original network.

Interchange of voltage and current sources by means of Thevenin's and Norton's theorems provides a powerful method of circuit analysis. In such conversion, it should be noted that a voltage source is removed from a circuit by short-circuiting its emf, whereas a current source is removed by opening its circuit. An example of this use of the theorems is given in Fig. 1-15.

Starting with (a), the portion of the circuit to the left of the terminals a, b may be transformed by Norton's theorem to (b), Z_1 and

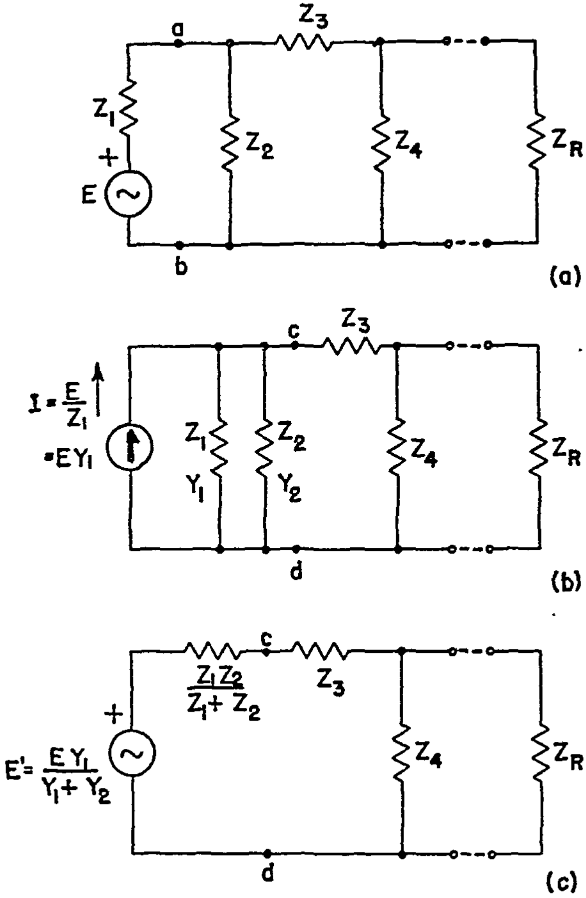


Fig. 1-15. (a) An active network; (b), (c) successive steps in reduction of the network of (a) to a simplified equivalent.

Z_2 combined in parallel as admittances and the circuit to the left of c, d changed back by Thevenin's theorem to that of (c). One more transformation by Norton's theorem would give a current-source circuit, from which the load voltage can be readily calculated.

In Fig. 1-12 (c), which was obtained through use of Thevenin's theorem, the load voltage E_R is given by

$$E_R = E'\left(\frac{Z_R}{Z' + Z_R}\right)$$

Norton's theorem was used to obtain the equivalent circuit of Fig. 1-14(c), in which the load current I_R may be written

$$I_R = I'\left(\frac{Y_R}{Y' + Y_R}\right)$$

It may be noted that these two equations are of identical mathematical form, but one relates voltages and impedances, the other currents and admittances. Thus the voltage-source and current-source circuits are duals. This particular duality is often of great importance.

1-12. The compensation theorem

Any impedance having an emf across its terminals may be replaced by a generator of zero internal impedance whose emf is at every instant equal to the emf across the impedance.

A circuit equation, such as

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int i\,dt = e$$

is simply a summation of emf's. For the purposes of the equation, it makes no difference whether an emf is produced by a flow of current through an impedance or by action of a generator. In fact, in transient studies, inductances and capacitances become energy sources or generators. The theorem is therefore apparent.

The theorem applies equally well to the effect produced on the network currents by a change ΔZ in the impedance of a network branch. The effect of this ΔZ change on the currents may be found by replacing ΔZ with a generator of emf $-I\Delta Z$, where I is the original current which flowed in the branch. Some authors restrict the compensation theorem to this form relating to ΔZ changes. The form stated above as a theorem is believed to be more useful although less general in form.

1-13. The maximum power-transfer theorem

Maximum power will be delivered by a network to an impedance Z_R if the impedance of Z_R is the conjugate of the impedance Z' of the network, measured looking back into the terminals of the network.

A corollary to the above theorem states:

If only the absolute magnitude and not the angle of Z_R may be varied, then the greatest power output will be delivered from the network if the absolute magnitude of Z_R is made equal to the absolute magnitude of Z'.

The amount of power delivered by matching magnitudes under the corollary will be somewhat less than the amount possible if both magnitude and angle are adjusted to the conjugate condition called for by the theorem. Measurement of Z' is, of course, made with all generator emf's removed, but not their impedances.

Actually, the network or generator may not be capable of supplying the maximum possible power to the conjugate impedance specified above without overheating of the generator or network. Thus it is not always physically possible to use the ideal conjugate load, a higher impedance load being required to avoid burnout or damage. Hence there are two limiting factors on the maximum possible power which may be obtained from a given source: first, the limitation fixed by reason of overheating or damage; and second, the adjustment of the load to achieve a conjugate impedance. The ordinary d-c generator is usually limited in output due to heating; in vacuum-tube amplifiers the conjugate condition may be readily achieved under certain operating conditions, whereas under other conditions heating becomes the limiting factor and a matched conjugate load cannot be used.

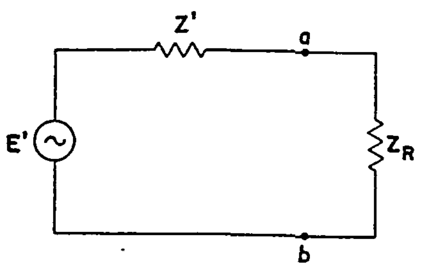


Fig. 1-16. A voltage-source equivalent of any active network connected to a load Z_R.

The theorem may be readily proved. Any linear network containing energy sources may be reduced by Thevenin's theorem to that of Fig. 1-16. If the load $Z_R = R_R + jX_R$ and the internal impedance $Z' = R' + jX'$, then the current I flowing in the circuit will be

$$I = \frac{E'}{(R_R + R') + j(X_R + X')} \tag{1-70}$$

The power then delivered to the load is

$$P = \frac{(E')^2 R_R}{(R_R + R')^2 + (X_R + X')^2} \tag{1-71}$$

Maximizing this expression with respect to X_R leads to

$$\frac{\partial P}{\partial X_R} = \frac{-2(E')^2 R_R (X_R + X')}{[(R_R + R')^2 + (X_R + X')^2]^2} = 0$$

from which $$X_R + X' = 0$$

Therefore the condition for maximum power, when reactance only is varied, is reached when

$$X_R = -X' \tag{1-72}$$

or the reactance of the load is of opposite sign to the reactance of the source network.

If the condition of Eq. 1-72 is substituted in Eq. 1-71, the expression for power in the load becomes

$$P = \frac{(E')^2 R_R}{(R_R + R')^2}$$

Maximizing this expression with respect to R_R gives

$$\frac{\partial P}{\partial R_R} = \frac{(E')^2(R_R + R')^2 - 2(E')^2 R_R (R_R + R')}{(R_R + R')^4} = 0$$

$$(E')^2(R_R + R') - 2(E')^2 R_R = 0$$

Assuming X_R to have been adjusted to equal $-X'$, then the condition for maximum power delivered, if R_R is varied, is derived from the equation above:

$$R_R = R' \tag{1-73}$$

Thus for a network of internal impedance $Z' = R' + jX'$ to deliver the greatest possible power output with internal heating neglected, the terminating impedance Z_R should be the conjugate of the internal impedance, or

$$Z_R = R' - jX' \tag{1-74}$$

Proof of the corollary to the theorem may be easily established if Eq. 1-71 for the power delivered to the load be written

$$P = \frac{(E')^2|Z_R| \cos \theta}{(R' + |Z_R| \cos \theta)^2 + (X' + |Z_R| \sin \theta)^2} \tag{1-75}$$

This expression may be maximized with respect to $|Z_R|$, after which some simplification gives

$$\begin{aligned} (R')^2 + (X')^2 - |Z_R|^2(\sin^2 \theta + \cos^2 \theta) &= 0 \\ |Z_R|^2 &= (R')^2 + (X')^2 \\ |Z_R| &= |Z'| \end{aligned} \tag{1-76}$$

Therefore, if the angle of the load impedance cannot be altered, the magnitude of the load impedance should be made equal to the magnitude of the network impedance to obtain from the network the greatest amount of power, again with internal heating given no consideration.

It should be observed that under the condition of conjugate impedances the power delivered to the load is

$$P = \left(\frac{E'}{2R_R}\right)^2 R_R = \frac{(E')^2}{4R_R} \tag{1-77}$$

Since $R_R = R'$, the power lost in the internal generator or network resistance is equal to the power delivered to the load, and the power efficiency is only 50 per cent. Where the cost of installed generating equipment is very large compared with the cost of the input energy, obtaining the greatest possible power output per unit of equipment is more important than the efficiency of energy utilization. This may occur in telephone and radio systems. In power systems operating at constant voltage it is not possible to use matched impedances, since variation of the load is the usual means for setting the power demand, and efficiency is economically important.

1-14. Driving-point impedance; transfer impedance

Any four-terminal network and connected load Z_R is specified as to performance if we know three of the four quantities I_1, I_2, E_1, and E_2. Such a network and load may be represented by an equivalent as in (b), Fig. 1-17. The input and output currents and voltages may be used to define certain terms that have been found useful in network analysis.

The ratio of E_1 to I_1, or the impedance seen by the generator at the terminals a,b, with all other generator emf's removed, is called

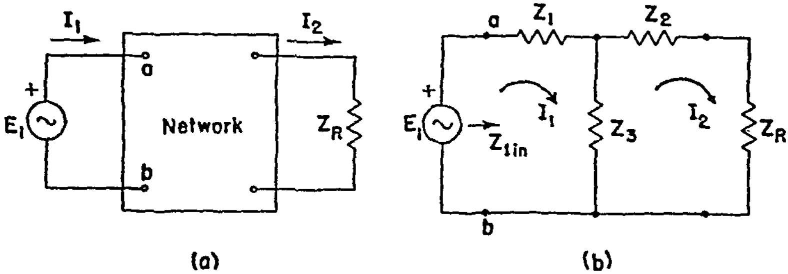


Fig. 1-17. (a) A network and load; (b) the network of (a) replaced with its equivalent T network.

the *input* or *driving-point impedance* $Z_{1\,\text{in}}$ at the terminals a,b. Its reciprocal is known as the *driving-point admittance* $Y_{1\,\text{in}}$. For (b), Fig. 1-17, the value of $Z_{1\,\text{in}}$ is

$$Z_{1\,\text{in}} = Z_1 + \frac{Z_3(Z_2 + Z_R)}{Z_2 + Z_3 + Z_R} \tag{1-78}$$

In general, the ratio of voltage to current in the jth mesh, or the impedance seen by a test generator inserted in the mesh, is called $Z_{j\,\text{in}}$.

The *transfer impedance* has been previously defined in Section 1-9 as the ratio of voltage applied in the jth mesh to the current flowing in any other mesh k with all other emf's removed, and is designated Z_{Tjk}. With bilateral impedances present, the principle of reciprocity requires that the ratio of voltage E_j to current I_k be the same as the ratio of voltage E_k to current I_j, or

$$Z_{Tjk} = Z_{Tkj} \tag{1-79}$$

The *transfer admittance* Y_{Tjk} is the reciprocal of Z_{Tjk}.

The transfer impedance for the circuit of (b), Fig. 1-17, is

$$Z_{T12} = \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1 + (Z_1 + Z_3)Z_R}{Z_3} \tag{1-80}$$

The impedance measured around any mesh, with all other meshes open-circuited, is called a *mesh impedance*. For the kth mesh it bears the symbol Z_{kk}. In the first mesh of the circuit of (b), the mesh impedance is

$$Z_{11} = Z_1 + Z_3$$

and in the second mesh,

$$Z_{22} = Z_2 + Z_3 + Z_R$$

The *mutual impedance between two meshes is defined as the ratio of the voltage introduced in one mesh by a current flowing in a second mesh, to the current in the second mesh, with all meshes except the second open-circuited.* With the mesh current directions as specified in (b), the mutual impedance between meshes 1 and 2 would be $-Z_3$, since the voltage introduced in the second mesh would be $-I_1Z_3$, and the ratio of this to I_1 would be $-Z_3$. The negative sign is determined by the oppositely assumed positive current *directions, additive current directions in the mutual impedance* resulting in a positive impedance. In general, the mutual impedance between mesh j and mesh k is given by a symbol Z_{jk}. For the example above,

$$Z_{12} = -Z_3 = Z_{21}$$

1-15. Alternating-current bridges (lattice network)

Alternating-current bridges are used for impedance measurement, for frequency measurement, and occasionally for frequency-selective circuits, as in the lattice and parallel-T networks.

A conventional bridge is shown in Fig. 1-18, and at balance

$$\frac{Z_1}{Z_2} = \frac{Z_3}{Z_x}$$

or

$$\frac{|Z_1|\angle\theta_1}{|Z_2|\angle\theta_2} = \frac{|Z_3|\angle\theta_3}{|Z_x|\angle\theta_x} \tag{1-81}$$

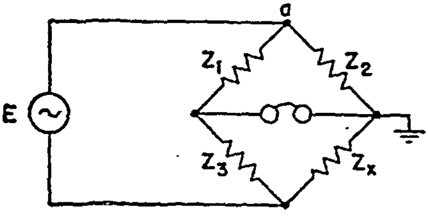


Fig. 1-18. An alternating-current bridge.

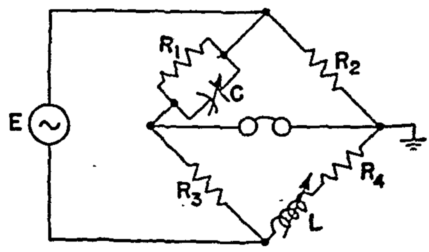


Fig. 1-19. The Maxwell bridge for inductance and capacity measurements.

It is customary to design the bridge with the Z_1 and Z_2 arms as known resistors, or

$$|Z_1|\angle\theta_1 = R_1$$
$$|Z_2|\angle\theta_2 = R_2$$

Then the balance conditions become

$$\frac{R_1}{R_2} = \frac{|Z_3|\angle\theta_3}{|Z_x|\angle\theta_x} = \frac{R_3 + jX_3}{R_x + jX_x} \tag{1-82}$$

which requires that θ_3 be made equal to θ_x so that

$$\tan^{-1}\frac{X_3}{R_3} = \tan^{-1}\frac{X_x}{R_x}, \qquad \frac{X_3}{R_3} = \frac{X_x}{R_x}$$

Thus
$$\frac{R_3}{R_x} = \frac{X_3}{X_x} \tag{1-83}$$

is a condition which must be attained before final balance according to Eq. 1-82 can be obtained. The additional adjustments required to achieve the equality of the phase angles of the unknown and standard arms make the balancing of an impedance bridge somewhat more difficult than a simple d-c resistance bridge.

Although a pair of headphones is shown as the indicator of unbalance in Fig. 1-18, it is more common to use a cathode-ray oscillograph and its associated amplifiers. Grounding of the bridge at some point is always advisable.

A specialized form of the above bridge shown in Fig. 1-19, known as the *Maxwell bridge*, is useful for measuring either inductance or capacitance. The usual balance relations are

$$\frac{Z_1}{Z_3} = \frac{Z_2}{Z_4}$$

and in this case

$$\frac{-jR_1/\omega C}{R_3(R_1 - j/\omega C)} = \frac{R_2}{R_4 + j\omega L}$$

$$\frac{-jR_1R_4}{\omega C} + \frac{R_1L}{C} = R_1R_2R_3 - \frac{jR_2R_3}{\omega C} \tag{1-84}$$

Equating the real and the imaginary terms,

$$\frac{R_1L}{C} = R_1R_2R_3, \qquad \frac{R_1R_4}{\omega C} = \frac{R_2R_3}{\omega C}$$

If L and its resistance R_4 are the unknowns, then

$$L = R_2R_3C \tag{1-85}$$

$$R_4 = \frac{R_2R_3}{R_1} \tag{1-86}$$

whereas if C and its shunting resistance R_1 are the unknowns, then

$$C = \frac{L}{R_2R_3} \tag{1-87}$$

$$R_1 = \frac{R_2R_3}{R_4} \tag{1-88}$$

The units of L, C, and R are the henry, farad, and ohm, respectively.

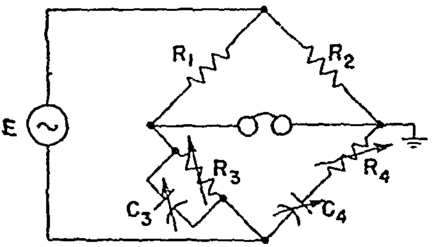


Fig. 1-20. The Wien bridge for measurement of frequency.

A form of circuit used to measure frequency is the Wien bridge of Fig. 1-20. The balance conditions for this circuit are

$$\frac{-R_1\omega C_3/jR_3}{R_3 - j/\omega C_3} = \frac{R_2}{R_4 - j/\omega C_4}$$

Then

$$R_1R_4 - \frac{jR_1}{\omega C_4} = \frac{R_2R_3}{1 + j\omega R_3C_3}$$

$$\omega C_4R_1R_4 - jR_1 + j\omega^2R_1R_3C_3R_4C_4 + \omega R_1R_3C_3 = \omega C_4R_2R_3$$

from which $$\omega^2R_3C_3R_4C_4 = 1$$

$$f = \frac{1}{2\pi}\sqrt{\frac{1}{R_3C_3R_4C_4}} \tag{1-89}$$

From the real terms,

$$\frac{C_3}{C_4} = \frac{R_2}{R_1} - \frac{R_4}{R_3} \tag{1-90}$$

can be obtained as the second condition of balance. If the bridge is designed so that

$$\frac{C_3}{C_4} = \frac{R_4}{R_3} = 1$$

and $$\frac{R_2}{R_1} = 2$$

then Eq. 1-90 is fulfilled and the equation for the frequency of balance becomes

$$f = \frac{1}{2\pi R_3 C_3} \tag{1-91}$$

By switching fixed values of C_3 and C_4 into the circuit and ganging R_3 and R_4 on a common shaft, the measurement of an unknown frequency becomes simple.

1-16. Sensitivity in bridge measurements

The design of a bridge circuit is made with consideration of the magnitude of the element or unknown to be measured, the type of detecting or unbalance-indicating element, and by determination of the sensitivity or voltage available at the detector terminals for δ change in the unknown arm or element. It is usually desired that the sensitivity be as large as possible.

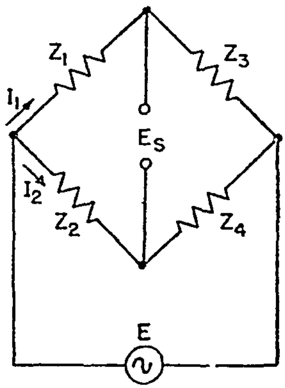


Fig. 1-21. Bridge measurements of Z_1.

If the bridge circuit of Fig. 1-21 is initially in balance, then $E_s = 0$, and

$$\frac{Z_1}{Z_2} = \frac{Z_3}{Z_4} \tag{1-92}$$

If the unknown is the Z_1 arm, and Z_1 is changed by a small variation δ to $Z_1 + \delta Z_1$, the currents are

$$I_1 = \frac{E}{Z_1 + \delta Z_1 + Z_3}$$

$$I_2 = \frac{E}{Z_2 + Z_4}$$

and with the detector terminals open,

$$E_s = I_1(Z_1 + \delta Z_1) - I_2 Z_2$$

$$\frac{E_s}{E} = \frac{Z_1 + \delta Z_1}{Z_1 + \delta Z_1 + Z_3} - \frac{Z_2}{Z_2 + Z_4}$$

which, in view of Eq. 1-92, reduces to

$$\frac{E_s}{E} = \frac{\delta Z_1 Z_4}{(Z_1 + Z_3)(Z_2 + Z_4)}$$

after neglecting δZ_1 in relation to $Z_1 + Z_3$ in the denominator.

If the arms are related to each other as

$$\left|\frac{Z_1}{Z_3}\right| = \left|\frac{Z_2}{Z_4}\right| = a$$

then $$\left|\frac{Z_2 + Z_4}{Z_4}\right| = 1 + a, \qquad \left|\frac{Z_1 + Z_3}{Z_1}\right| \approx 1 + \frac{1}{a}$$

and $$E_s = \frac{\delta E}{(1 + a)(1 + 1/a)} = \frac{\delta a}{(1 + a)^2} E \tag{1-93}$$

The sensitivity or unbalance voltage E_s is a function of the ratio of the bridge elements a, and may be maximized with respect to this ratio as

$$\frac{\partial E_s}{\partial a} = \frac{\delta[(1 + a)^2 - 2(1 + a)a]E}{(1 + a)^4} = 0$$

from which the optimum value is

$$a = 1$$

The impedance ratios $|Z_1/Z_3|$ and $|Z_2/Z_4|$ should therefore approximate unity, or an equal-arm bridge is desired, for maximum sensitivity.

The above analysis has been made under the assumption of a fixed applied voltage E. However, for a detector of infinite impedance, such as a vacuum-tube amplifier, increased sensitivity may be obtained by increasing E, at the same time increasing Z_3, so that I_1 is constant, or the power input to the unknown Z_1 is constant to prevent overheating. Since E_s is multiplied by E, the sensitivity increases due to increase of E, but decreases due to departure of the ratio $Z_1/Z_3 = a$ from unity. While infinite E and infinite Z_1/Z_3 will then give the greatest sensitivity, practical values of $Z_1/Z_3 = 0.2$ will give an increase in sensitivity of about three times that at $a = 1$.

If the detector has a finite impedance and requires power for operation, the maximum power output to the detector will occur with matched impedances. With negligible impedance in the source E, the input impedance at the detector terminals is

$$Z_{\text{in}} = \frac{Z_1 Z_3}{Z_1 + Z_3} + \frac{Z_2 Z_4}{Z_2 + Z_4} \tag{1-94}$$

and for equal arms, Z_{in} is equal to the impedance of a single arm, this then being the optimum value for the detector impedance.

1-17. The parallel-T network

Two symmetrical T networks, having Z_1 and Z_2 arms equal, as shown in (a), Fig. 1-22, may be connected in parallel; and if the

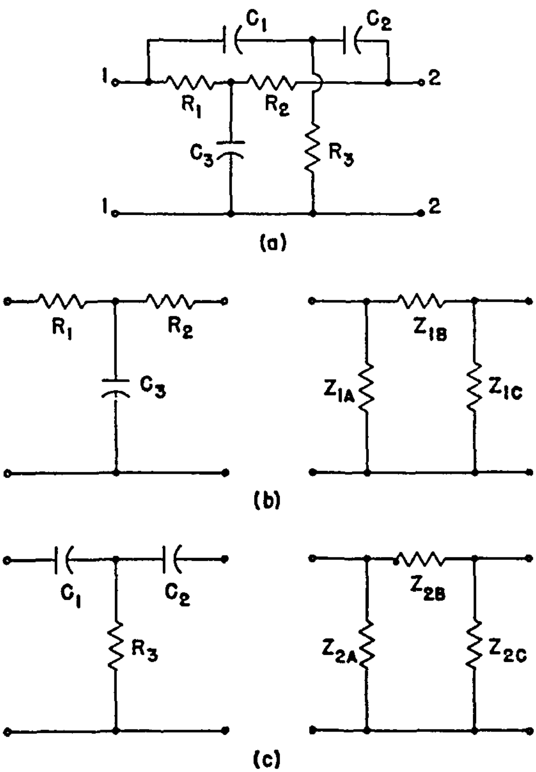


Fig. 1-22. (a) The parallel-T network; (b) one of the T networks and its equivalent π; (c) the second T network and its equivalent π.

reactances and resistances are properly chosen, these networks may be used to balance out or reject a particular frequency. The two T networks may be considered individually and changed to equivalent π circuits, as at (b) and (c). The branch impedances will then be

$$\left.\begin{aligned} Z_{1A} &= R_1 - \frac{jX_3(R_1 + R_2)}{R_2} \\ Z_{1B} &= R_1 + R_2 + \frac{jR_1R_2}{X_3} \\ Z_{1C} &= R_2 - \frac{jX_3(R_1 + R_2)}{R_1} \end{aligned}\right\} \tag{1-95}$$

$$\left.\begin{aligned} Z_{2A} &= \frac{R_3(X_1 + X_2)}{X_2} - jX_1 \\ Z_{2B} &= \frac{-X_1X_2}{R_3} - j(X_1 + X_2) \\ Z_{2C} &= \frac{R_3(X_1 + X_2)}{X_1} - jX_2 \end{aligned}\right\} \tag{1-96}$$

where the reactances (X) refer to the capacitances having the same subscripts.

The π networks may then be placed in parallel, resulting in one π section having

$$Z_A = \frac{\left[R_1 - jX_3\left(\frac{R_1 + R_2}{R_2}\right)\right]\left[R_3\left(\frac{X_1 + X_2}{X_2}\right) - jX_1\right]}{\left[R_1 + R_3\left(\frac{X_1 + X_2}{X_2}\right)\right] - j\left[X_1 + X_3\left(\frac{R_1 + R_2}{R_2}\right)\right]} \tag{1-97a}$$

$$Z_B = \frac{\left[R_1 + R_2 + j\frac{R_1R_2}{X_3}\right]\left[\frac{-X_1X_2}{R_3} - j(X_1 + X_2)\right]}{\left[R_1 + R_2 - \frac{X_1X_2}{R_3}\right] + j\left[\frac{R_1R_2}{X_3} - X_1 - X_2\right]} \tag{1-97b}$$

$$Z_C = \frac{\left[R_2 - jX_3\left(\frac{R_1 + R_2}{R_1}\right)\right]\left[R_3\left(\frac{X_1 + X_2}{X_1}\right) - jX_2\right]}{\left[R_2 + R_3\left(\frac{X_1 + X_2}{X_1}\right)\right] - j\left[X_2 + X_3\left(\frac{R_1 + R_2}{R_1}\right)\right]} \tag{1-97c}$$

Balance or zero transmission can be achieved if $Z_A = 0$, $Z_C = 0$, or $Z_B =$ infinity. Elements Z_A and Z_C can be zero only if their denominators are infinite, which is impossible with ordinary circuit elements. The element Z_B can be infinite, however, if its denominator is zero, or

$$(R_1 + R_2)R_3 = X_1X_2 \tag{1-98}$$

$$(X_1 + X_2)X_3 = R_1R_2 \tag{1-99}$$

These are the balance conditions for the parallel-T circuit. If $R_1 = R_2$ and $X_1 = X_2$, the frequency at which balance or zero transmission occurs is

$$f = \frac{1}{2\pi}\sqrt{\frac{1}{2R_1R_3C_1^2}} \tag{1-100}$$

Values for R_1, R_2, C_1, and C_2 may be assumed, after which Eqs. 1-99 and 1-100 may be used to obtain R_3 and C_3.

The search for factors that may cause an expression to go to zero or infinity at some frequency is an important part of the study of network performance.

1-18. Matrices and determinants

For a general network such as Fig. 1-23, with the mesh currents as shown, the three mesh equations may be written

$$\begin{aligned} E &= (Z_1 + Z_3)I_1 - Z_3I_2 \\ 0 &= -Z_3I_1 + (Z_2 + Z_3 + Z_4)I_2 - Z_4I_3 \\ 0 &= -Z_4I_2 + (Z_4 + Z_5)I_3 \end{aligned}$$

The coefficients of the currents in these circuit equations may be written in the form

$$[Z] = \begin{bmatrix} (Z_1 + Z_3), & -Z_3, & 0 \\ -Z_3, & (Z_2 + Z_3 + Z_4), & -Z_4 \\ 0, & -Z_4, & (Z_4 + Z_5) \end{bmatrix} \tag{1-101}$$

This form is called the *impedance matrix* of the network in Fig. 1-23.

A *matrix* is an ordered rectangular array of numbers. It is denoted by being enclosed in brackets, as [Z], may have any number m rows of real or complex numbers arranged in n columns, and is not necessarily square (or m is not necessarily equal to n). It is possible, however, to make any matrix square by inserting rows or columns of zeros. For bilateral circuits where $Z_{Tjk} = Z_{Tkj}$ the impedance matrix is symmetric about the *principal diagonal* from upper left to lower right.

Matrix 1-101 may be simplified by observation. It may be noted that the term $(Z_1 + Z_3)$ is the first mesh impedance Z_{11}, the term $(Z_2 + Z_3 + Z_4)$ is the second mesh impedance Z_{22}, and $(Z_4 + Z_5)$ is the impedance of the third mesh Z_{33}. Also, $(-Z_3)$ is the mutual impedance Z_{12} between meshes 1 and 2, and $(-Z_4)$ is the mutual impedance Z_{23} between meshes 2 and 3. Since there are no mutual impedances between meshes 1 and 3 zeros are written in the 1, 3 and 3, 1 positions of the matrix. Matrix 1-101 may then be written

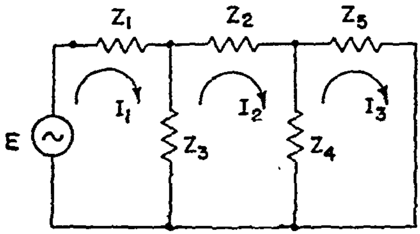


Fig. 1-23. A three-mesh network.

$$[Z] = \begin{bmatrix} Z_{11}, & Z_{12}, & 0 \\ Z_{12}, & Z_{22}, & Z_{23} \\ 0, & Z_{23}, & Z_{33} \end{bmatrix} \tag{1-102}$$

By use of this form the impedance matrix for any circuit of any number of meshes can be written directly by inspection of the circuit, without first writing the mesh equations. It may be noted that the mutual-impedance terms are positive if the assumed positive currents flow in the same direction through the mutual impedance. The mutual-impedance terms will be negative if the assumed positive currents flow in opposite directions through the impedance. To systematize writing the mutual terms, it is advisable to assume the reference current directions always the same, that is, all clockwise or all counterclockwise. It is also desirable to arrange the current paths so that only one current flows through each emf.

The original mesh equations may then be written

$$[E] = [Z][I] \quad \text{or} \quad \mathcal{E} = \mathcal{Z}\mathcal{I} \tag{1-103}$$

where $[E]$ and $[I]$ indicate column matrices

$$[E] = \begin{bmatrix} E \\ 0 \\ 0 \end{bmatrix}, \qquad [I] = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}$$

An admittance matrix may also be written from inspection of the circuit, and for Fig. 1-24 would be

$$[Y] = \begin{bmatrix} Y_1 + Y_2 + Y_3, & -Y_2 \\ -Y_2, & Y_2 + Y_4 + Y_5 \end{bmatrix} \tag{1-104}$$

since the node-voltage equations are

$$(Y_1 + Y_3)E_1 + Y_2(E_1 - E_2) = I_1$$
$$Y_2(E_1 - E_2) - Y_4E_2 - Y_5E_2 = 0$$

It can be seen that the Y_{11} position is taken by the sum of the admittances connected to node 1, the $Y_{12} = Y_{21}$ positions are

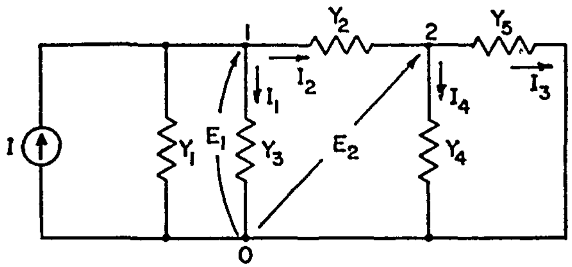


Fig. 1-24. Admittance equivalent of circuit in Fig. 1-23.

occupied by the negative of the admittance joining nodes 1 and 2, (for the indicated current directions), and the Y_{22} position is taken by the sum of the admittances connected to node 2. The matrix equation would then be

$$[Y][E] = [I] \quad \text{or} \quad \mathcal{Y}\mathcal{E} = \mathcal{I}$$

where
$$[E] = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}, \qquad [I] = \begin{bmatrix} I_1 \\ 0 \end{bmatrix}$$

The matrix impedances may be further interpreted as

$$Z_{11} = E_1/I_1 \quad \text{with } I_2 = 0, \qquad Z_{12} = E_1/I_2 \quad \text{with } I_1 = 0$$
$$Z_{22} = E_2/I_2 \quad \text{with } I_1 = 0, \qquad Z_{21} = E_2/I_1 \quad \text{with } I_2 = 0$$

the current and voltage conventions being as in Fig. 1-25. These impedances are then for the open-circuit condition.

The admittance matrix can be interpreted in a similar manner as

$$Y_{11} = I_1/E_1 \quad \text{with } E_2 = 0, \qquad Y_{12} = I_1/E_2 \quad \text{with } E_1 = 0$$
$$Y_{22} = I_2/E_2 \quad \text{with } E_1 = 0, \qquad Y_{21} = I_2/E_1 \quad \text{with } E_2 = 0$$

and these admittances are for the short-circuit condition.

A *determinant* is a number associated with a square matrix of n rows and n columns. This determinant, as a number, serves much the same purpose in matrix algebra as does the absolute value of $(a + jb)$ in complex algebra. It is unfortunate that the term "determinant" is often used to mean both an array and a number associated with that array. The determinant of a matrix $[Z]$ can be written as $d[Z]$ or Δ, where

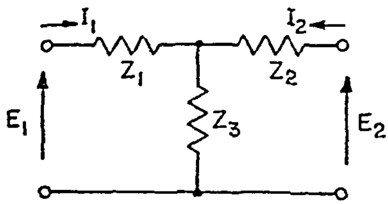


Fig. 1-25. Voltage and current conventions.

$$d[Z] = \begin{vmatrix} Z_{11}, & Z_{12} \\ Z_{21}, & Z_{22} \end{vmatrix} = \Delta \tag{1-105}$$

The true determinant is an expansion of the array, and its expanded form consists of all the terms that can be made, each containing n factors, *one factor taken from each column in order*, but no two from the same row. The assignment of signs to the terms will be explained later. Thus

$$\Delta_z = \begin{vmatrix} Z_{11}, & Z_{12} \\ Z_{21}, & Z_{22} \end{vmatrix} = Z_{11}Z_{22} - Z_{21}Z_{12} \tag{1-106}$$

and for a third-order determinant,

$$\begin{vmatrix} Z_{11}, & Z_{12}, & Z_{13} \\ Z_{21}, & Z_{22}, & Z_{23} \\ Z_{31}, & Z_{32}, & Z_{33} \end{vmatrix} = Z_{11}Z_{22}Z_{33} - Z_{11}Z_{32}Z_{23} - Z_{21}Z_{12}Z_{33} \\ + Z_{21}Z_{32}Z_{13} - Z_{31}Z_{22}Z_{13} + Z_{31}Z_{12}Z_{23} \tag{1-107}$$

For the third-order determinant there are six terms, or $3!$; and for the nth-order determinant there will be $n!$ terms.

The *normal order* of the subscripts is $1, 2, 3, \ldots, n$, each number being less than those following. For the third-order determinant, the normal order is 1, 2, 3. If the factors are chosen successively from each column as specified above, the second subscripts of each term will be in the normal order.

After checking for normal order of the second subscripts of each term, the signs of the terms may be determined by the smallest number of interchanges of the first subscripts needed to bring the first subscripts into the normal order. If the first subscripts appear as 3, 1, 2, then two interchanges are needed, namely, 3 over 1 and 3 over 2. If the subscripts appear as 1, 3, 2, one interchange is needed, 3 over 2. The sign of a term in a determinant is plus if an even number of interchanges are needed to restore the normal order, minus if an odd number of interchanges are needed. The signs of the terms in Eq. 1-107 may be checked by this rule.

Because of the systematic procedures possible when matrices are used in network analysis, further study will be given to them here. It should be emphasized, however, that the matrix method is only a tool, and not a new method of analysis. It is an orderly manner of presenting the data for a network, and separates the independent and dependent variables and the impedances or admittances, so that each may be operated on and studied separately.

1-19. Matrix manipulation

Certain rules are necessary for arithmetic manipulation of matrices. Two matrices may be *added* term by term as

$$\mathcal{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \qquad \mathcal{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

Then

$$\mathcal{A} + \mathcal{B} = \begin{bmatrix} a_{11} + b_{11}, & a_{12} + b_{12} \\ a_{21} + b_{21}, & a_{22} + b_{22} \end{bmatrix} \tag{1-108}$$

A matrix may be *subtracted* from another by giving each term a negative sign and adding.

Two matrices must be multiplied by a special rule. To form the c_{rs} term of the product matrix, the rth row of the first matrix is multiplied term by term with the sth column of the second matrix, and the terms added. That is, any term c_{rs} is given by

$$c_{rs} = \sum_{k=1}^{n} a_{rk} b_{ks}$$

where n = number of rows of $\mathcal{A}$ = number of columns of $\mathcal{B}$. As an example

$$\mathcal{A} \times \mathcal{B} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21}, & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21}, & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix} \tag{1-109}$$

Multiplication is not commutative, or

$$\mathcal{A}\mathcal{B} \neq \mathcal{B}\mathcal{A}$$

and the order of multiplication must be preserved. Multiplication is associative and distributive, or

$$\mathcal{A}(\mathcal{B} + \mathcal{C}) = \mathcal{A}\mathcal{B} + \mathcal{A}\mathcal{C}$$
$$\mathcal{A}(\mathcal{B}\mathcal{C}) = \mathcal{A}\mathcal{B}(\mathcal{C})$$

Division does not exist for matrices; however, the *inverse operation* takes its place. That is, if

$$\mathcal{E} = \mathcal{Z}\mathcal{I} \quad \text{then} \quad \mathcal{I} = \mathcal{Z}^{-1}\mathcal{E} \tag{1-110}$$

where $\mathcal{Z}^{-1}$ is the inverse of $\mathcal{Z}$ and has the dimensions of admittance. The inverse of a matrix is another matrix of the same order. If $\mathcal{C}$ is the inverse of $\mathcal{B}$, then the terms of $\mathcal{C}$ are

$$c_{rs} = \frac{\text{cofactor of } b_{sr}}{\text{determinant of } \mathcal{B}}$$

A *cofactor* is a determinant obtained from $\mathcal{B}$ by striking out a row and a column, giving a determinant of lower order by one. The sign to be used is fixed by $(-1)^{r+s}$, where r and s are the numbers of the row and column omitted. Thus the inverse of matrix $\mathcal{A}$

$$\mathcal{A}^{-1} = \begin{vmatrix} \dfrac{a_{22}}{\Delta_a} & \dfrac{-a_{21}}{\Delta_a} \\ \dfrac{-a_{12}}{\Delta_a} & \dfrac{a_{11}}{\Delta_a} \end{vmatrix} = \frac{1}{\Delta_a}\begin{vmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{vmatrix} \tag{1-111}$$

An inverse matrix exists only if Δ_a is not zero.

The *transpose* is another matrix formed from a first by interchange of rows and columns. Thus

$$\mathcal{A}^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \tag{1-112}$$

1-20. Network calculations using matrices

A single two-mesh circuit, as in Fig. 1-26, may be chosen to illustrate matrix circuit calculations. The mesh equations may be written

$$E = (Z_1 + Z_3)I_1 - Z_3I_2 \tag{1-113}$$
$$0 = -Z_3I_1 + (Z_2 + Z_3 + Z_R)I_2$$

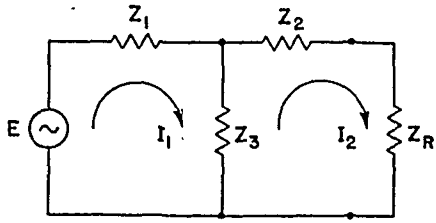


Fig. 1-26. A two-mesh network.

The impedance matrix $[Z] = \mathcal{Z}$ will then be

$$\mathcal{Z} = \begin{bmatrix} (Z_1 + Z_3), & -Z_3 \\ -Z_3, & (Z_2 + Z_3 + Z_R) \end{bmatrix}$$

so that Eq. (1-113) may be written

$$\mathcal{E} = \mathcal{Z}\mathcal{I}$$

the script symbols indicating matrices as before. The circuit may be solved by conventional substitution methods, giving for the currents:

$$I_1 = \frac{(Z_2 + Z_3 + Z_R)E}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1 + (Z_1 + Z_3)Z_R} \tag{1-114}$$

$$I_2 = \frac{Z_3E}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1 + (Z_1 + Z_3)Z_R} \tag{1-115}$$

The determinant of $\mathcal{Z}$ may be written

$$\begin{aligned} \Delta_z &= (Z_1 + Z_3)(Z_2 + Z_3 + Z_R) - Z_3^2 \\ &= Z_1Z_2 + Z_2Z_3 + Z_3Z_1 + (Z_1 + Z_3)Z_R \end{aligned} \tag{1-116}$$

and it may be noted that this determinant is the denominator in the expressions for each of the currents I_1 and I_2. After some further experimentation, it might be discovered that if the *first* column of the matrix be replaced by the left-hand side or the voltages of Eq. 1-113, then a determinant

$$\Delta_1 = \begin{vmatrix} E, & -Z_3 \\ 0, & Z_2 + Z_3 + Z_R \end{vmatrix} = (Z_2 + Z_3 + Z_R)E \tag{1-117}$$

is formed, and this determinant is the numerator of the expression for I_1 in Eq. 1-114. Likewise, if a matrix is formed by replacing the *second* column of Z with the ℰ column matrix, then a determinant

$$\Delta_2 = \begin{vmatrix} Z_1 + Z_3, & E \\ -Z_3, & 0 \end{vmatrix} = Z_3E \tag{1-118}$$

is formed, and this is the numerator for the I_2 expression.

From these observations, if I_r is to be solved for:

1. *Write the determinant* Δ_r, *formed by replacing the* rth *column with the mesh voltage column.*

2. *Divide* Δ_r *by the matrix determinant* Δ_z.

The solution may be written

$$I_r = \frac{\Delta_r}{\Delta_z} \tag{1-119}$$

for any current. Voltages in the node method follow by analogy.

It is frequently desired to find the input or driving-point impedance of a circuit. By inspection of Fig. 1-26, the input impedance $Z_{1\,\text{in}}$ of the first mesh is

$$\begin{aligned} Z_{1\,\text{in}} &= Z_1 + \frac{Z_3(Z_2 + Z_R)}{Z_2 + Z_3 + Z_R} \\ &= \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1 + (Z_1 + Z_3)Z_R}{Z_2 + Z_3 + Z_R} \\ &= \frac{\Delta_z}{Z_2 + Z_3 + Z_R} \end{aligned} \tag{1-120}$$

It may be discovered that the denominator consists of the determinant of Z with the first row and first column deleted. This is called Δ_{11} and may be recognized as the cofactor of the 1,1 term of Δ. Then

$$Z_{1\,\text{in}} = \frac{\Delta_z}{\Delta_{11}}$$

and that of the rth mesh

$$Z_{r\,\text{in}} = \frac{\Delta_z}{\Delta_{rr}} \tag{1-121}$$

Thus:

To find the driving point impedance of the rth *mesh, divide the determinant* Δ_z *by the cofactor of the* r,r *term.*

This operation saves considerable labor for networks of many meshes.

It might be suspected that a similar rule would apply for the transfer impedance Z_T. This is true, and a transfer impedance may be written

$$Z_{Trs} = \frac{\Delta_z}{\Delta_{rs}}(-1)^{r+s} \tag{1-122}$$

or

To find the transfer impedance between the r*th and* s*th meshes, divide* Δ_z *by the cofactor of the* r,s *term. The result is multiplied by* $(-1)^{r+s}$ *to provide the appropriate sign.*

Networks of various configurations are frequently cascaded, one following another, and the performance or response of the whole chain is desired. For this purpose another form of circuit matrix is useful. Referring again to the conventions of current and voltage of Fig. 1-25, equations for a T network may be written

$$E_1 = (Z_1 + Z_3)I_1 + Z_3I_2 \tag{1-123}$$

$$E_2 = Z_3I_1 + (Z_2 + Z_3)I_2 \tag{1-124}$$

Solving Eq. (1-124) for I_1 gives

$$I_1 = \frac{E_2}{Z_3} - \frac{Z_2 + Z_3}{Z_3} I_2$$

and upon substitution in Eq. (1-123) there results

$$E_1 = \frac{Z_1 + Z_3}{Z_3} E_2 - \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_3} I_2$$

Rewriting the last two equations in matrix form gives

$$\begin{bmatrix} E_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} \dfrac{Z_1 + Z_3}{Z_3}, & \dfrac{\Delta_z}{Z_3} \\ \dfrac{1}{Z_3}, & \dfrac{Z_2 + Z_3}{Z_3} \end{bmatrix} \begin{bmatrix} E_2 \\ -I_2 \end{bmatrix} \tag{1-125}$$

$$= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} E_2 \\ -I_2 \end{bmatrix}$$

the minus sign being transferred to I_2. The above form of matrix of Eq. (1-125) for a network is sometimes called an $\mathcal{A}$ matrix. In power work it is frequently referred to as an *A, B, C, D* matrix, where

$$\begin{bmatrix} E_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E_2 \\ -I_2 \end{bmatrix} \tag{1-126}$$

where the coefficients A, B, C, D can be seen to have the general values for any network, of

$$A = \frac{Z_{11}}{Z_{21}}, \qquad C = \frac{1}{Z_{21}}$$

$$B = \frac{\Delta_z}{Z_{21}}, \qquad D = \frac{Z_{22}}{Z_{21}}$$

and where $\Delta_z = Z_{11}Z_{22} - Z_{12}^2$. Equation 1-126 is useful as an expression relating input and output quantities, and further discussion will occur in Chapter 8.

Consider the three cascaded networks in Fig. 1-27(a), whose

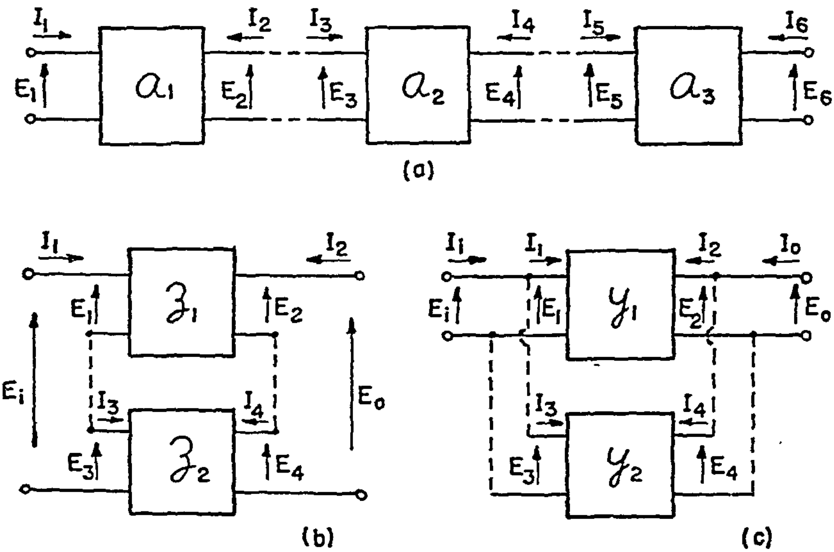


Fig. 1-27. Connection of networks.

matrices are of the form of Eq. 1-125 and are designated $\mathcal{a}_1$, $\mathcal{a}_2$, and $\mathcal{a}_3$. The individual network relations can be expressed as

$$\begin{bmatrix} E_1 \\ I_1 \end{bmatrix} = \mathcal{a}_1 \begin{bmatrix} E_2 \\ -I_2 \end{bmatrix}, \quad \begin{bmatrix} E_3 \\ I_3 \end{bmatrix} = \mathcal{a}_2 \begin{bmatrix} E_4 \\ -I_4 \end{bmatrix}, \quad \begin{bmatrix} E_5 \\ I_5 \end{bmatrix} = \mathcal{a}_3 \begin{bmatrix} E_6 \\ -I_6 \end{bmatrix}$$

It may be noted from the circuit that

$$\begin{bmatrix} E_2 \\ -I_2 \end{bmatrix} = \begin{bmatrix} E_3 \\ I_3 \end{bmatrix}, \qquad \begin{bmatrix} E_4 \\ -I_4 \end{bmatrix} = \begin{bmatrix} E_5 \\ I_5 \end{bmatrix}$$

so the combined matrix equation for the cascaded networks is simply obtained by substitution as

$$\begin{bmatrix} E_1 \\ I_1 \end{bmatrix} = \mathfrak{a}_1\mathfrak{a}_2\mathfrak{a}_3 \begin{bmatrix} E_4 \\ -I_4 \end{bmatrix} \tag{1-127}$$

Thus the $\mathfrak{a}$ matrix for the cascaded network is the product of the individual $\mathfrak{a}$ matrices. This is a considerable saver of time in network analysis.

Series connected networks as in Fig. 1-27(b) are also easily handled. Using the Z matrices for the networks, of the form of Eq. 1-102, then

$$\begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = \mathbf{Z}_1 \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} E_3 \\ E_4 \end{bmatrix} = \mathbf{Z}_2 \begin{bmatrix} I_3 \\ I_4 \end{bmatrix}$$

From the circuit

$$E_a = E_1 + E_3, \qquad I_1 = I_3$$
$$E_b = E_2 + E_4, \qquad I_2 = I_4$$

so that addition of the network matrix equations yields

$$\begin{bmatrix} E_a \\ E_b \end{bmatrix} = [\mathbf{Z}_1 + \mathbf{Z}_2] \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \tag{1-128}$$

as the matrix equations for the two networks in series.

Networks may be placed in parallel as in Fig. 1-27(c). The individual network equations, using $\mathfrak{Y}$ matrices of the form of Eq. 1-101 are

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \mathfrak{Y}_1 \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} I_3 \\ I_4 \end{bmatrix} = \mathfrak{Y}_2 \begin{bmatrix} E_3 \\ E_4 \end{bmatrix}$$

From the circuit

$$I_a = I_1 + I_3, \qquad E_1 = E_3$$
$$I_b = I_2 + I_4, \qquad E_2 = E_4$$

and addition of the two admittance matrices yields

$$\begin{bmatrix} I_a \\ I_b \end{bmatrix} = [\mathfrak{Y}_1 + \mathfrak{Y}_2] \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \tag{1-129}$$

as the method for finding the relations existing between input and output when two networks are in parallel.

Many additional methods of systematizing network analysis through use of matrix algebra exist.

PROBLEMS

1-1. (a) By use of computed values of the external measured impedances, reduce the networks of Fig. 1-28 to equivalent T networks.

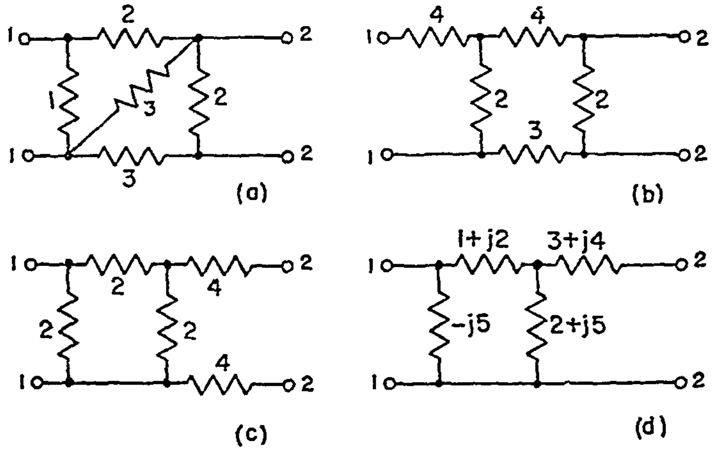


Fig. 1-28.

(b) Change the T networks of (a) to π networks.

1-2. By successive use of T-to-π transformations, or vice versa, convert the networks of Fig. 1-28 to equivalent π networks.

1-3. Convert the networks of Fig. 1-29 to T and π equivalent circuits.

1-4. (a) By use of superposition find the current output of each generator of Fig. 1-30, assuming their voltages to be in phase.

(b) Repeat if the 2-v generator lags the 1-v generator by 60°.

1-5. A certain network, including a generator, has an open-circuit voltage of 125 v, and on short circuit produces a current of 5.59 amp. When a 10-ohm resistive load is connected, the current is 4.41 amp. By Thevenin's theorem, find an equivalent voltage-source circuit for the network. How could the sign of the reactance be determined?

1-6. Solve the network of Fig. 1-30 by node voltages, to obtain the voltages across the $j20$ impedances directly.

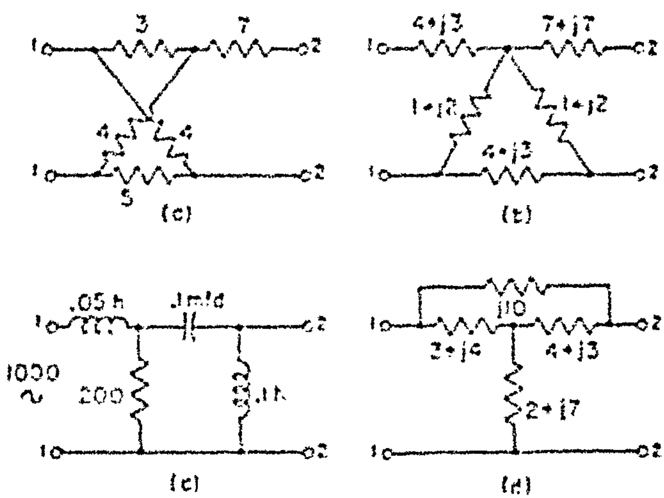


Fig. 1-29.

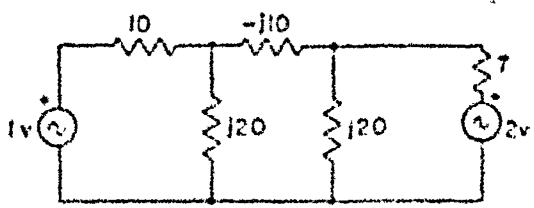


Fig. 1-30.

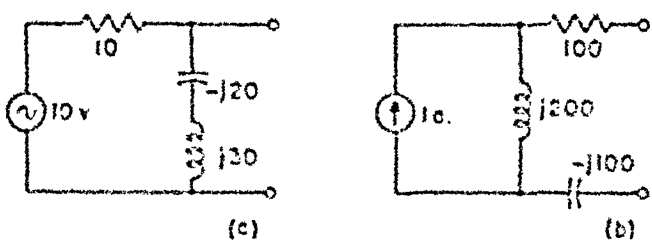


Fig. 1-31.

1-7. Draw a dual network for that of Fig. 1-30.

1-8. (a) Find the voltage-source equivalent circuit for Fig. 1-31b.

(b) Determine the value of load for maximum power.

(c) Find the power received in the matching load.

(d) Convert the circuit to a current-source equivalent and show that the same value of power is delivered to the matching load.

1-9. (a) What load is needed to obtain maximum power output of the terminals of the network of Fig. 1-31(b)?

(b) With the load determined for (a), what will be the generator current, load current, the power generated, and the power delivered to the load?

(c) Considering efficiency as the ratio of the power in the load to the power generated, what is the efficiency in (b)?

1-10. A resistance, variable between 0 and 500 ohms, is available for use as a load on the network of (b), Fig. 1-31.

(a) At what value of resistance should it be set to develop maximum possible power in the resistor?

(b) How much power will be developed?

(c) Plot a curve of power output vs. resistance load, showing that the selected value of resistance does develop maximum power.

1-11. If the measurements made on a box enclosing a network having four terminals are

$$Z_{1oc} = 40 \underline{/0°}$$
$$Z_{1sc} = 20.3 \underline{/29.8°}$$

and it is known that the network is symmetrical, or $Z_1 = Z_2$ of the equivalent T section, find the values for the equivalent T network and draw the circuit.

1-12. A box containing impedances has the following measurements made at 60 c:

$$Z_{1oc} = 60 + j0, \qquad Z_{2oc} = 40 + j20, \qquad Z_{1sc} = 50 + j20$$

Construct the equivalent π network.

1-13. By successive exchange of current-source and voltage-source equivalent circuits, simplify the circuit of Fig. 1-30 to a series circuit and compute the current flowing in the $-j10$ impedance.

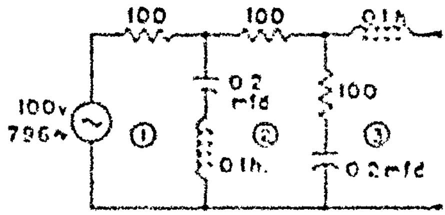
1-14. (a) If the network is short-circuited at the output terminals, compute the driving-point impedance of the generator mesh of the circuit of (b), Fig. 1-32.

(b) Find the transfer impedance between meshes 1 and 2; between meshes 1 and 3.

(c) Find the mutual impedance between each of the meshes.

1-15. If the network of Fig. 1-32(a) is short-circuited, write the impedance matrix by inspection. Solve for the current in the third mesh.

1-16. Write the admittance matrix of Fig. 1-32(b) by inspection, if the circuit is short-circuited. Find the potential across the $-j200$ ohm impedance.



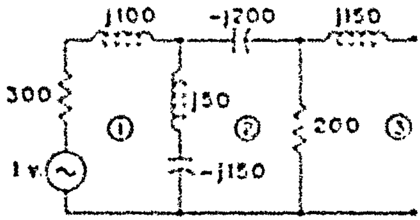


Fig. 1-32.

1-17. Draw the dual of the circuit of Fig. 1-31(a).

1-18. By use of the impedance matrix equations for Fig. 1-31(b), replace all mesh voltages by branch currents, all mesh currents by node voltages, all impedances by admittances, and draw the dual as specified, indicating branch currents and nodes.

1-19. Using a current source of internal admittance Y/θ, prove that $Y/-\theta$ will absorb maximum power.

1-20. The networks of Fig. 1-33 are connected in cascade. Write the cascaded impedance matrix and find an expression for the voltage across Z_n at open circuit.

1-21. The networks of Fig. 1-33 are connected in parallel. Write the impedance matrix for the combined circuit

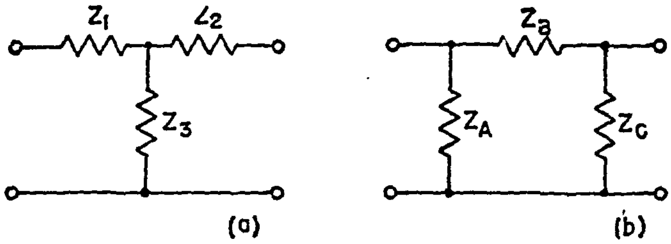


Fig. 1-33.

1-22. Assuming generators of 10-v emf and zero internal impedance connected to the 1,1 terminals and resistors of 25 ohms connected to the 2,2 terminals of the circuits of Fig. 1-28, write the impedance matrices by inspection.

REFERENCES

1. Shea, T. E., *Transmission Networks and Wave Filters*, D. Van Nostrand Company, Inc., New York, 1929.
2. Johnson, K. S., *Transmission Circuits for Telephone Communication*, D. Van Nostrand Company, Inc., New York, 1927.
3. Thevenin, M. L., "Sur un nouveau théorème d'electricité dynamique," *Compt. Rend.*, **97**, 159 (1883).
4. Scott, H. H., "A New Type of Selective Circuit and Some Applications," *Proc. I.R.E.*, **26**, 226 (1938).
5. Hague, B., *Alternating-Current Bridge Methods*, Pitman Publishing Corporation, New York, 1930.
6. Pipes, L. A., *Applied Mathematics for Engineers and Physicists*. McGraw-Hill Book Company, Inc., New York, 1946.
7. Guillemin, E. A., *Communication Networks*, Vol. I, John Wiley & Sons, Inc., New York, 1931.
8. Reed, M. B., *Alternating-Current Circuits*, Harper & Brothers, New York, 1948.
9. LePage, W. R., and Seeley, S., *General Network Analysis*, McGraw-Hill Book Company, Inc., New York, 1952.
10. Le Corbeiller, P., *Matrix Analysis of Electric Networks*, Harvard University Press, Cambridge, 1950.

Chapter 2

RESONANCE

The property of cancellation of reactance when inductive and capacitive reactances are in series, or cancellation of susceptance when in parallel, has become known as *resonance.* Such cancellation leads to operation of reactive circuits under unity power-factor conditions, or with current and voltage in phase. Two types of resonance are recognized: *series-circuit* resonance and *parallel-circuit* resonance. The latter is sometimes referred to as *antiresonance,* to afford a means of distinguishing between the two types.

The valuable properties of resonant circuits have long been appreciated and are probably more frequently employed than any other circuit property. Certain fundamental features and applications are discussed in this chapter, with other applications deferred to Chapter 3.

2-1. Definition of Q, the factor of merit

Since inductive or capacitive reactances are essentially energy-storing devices, it is convenient to discover the efficiency with which energy is stored and to compare various inductor or capacitor designs in terms of that efficiency. As a measure of such efficiency, a *figure of merit,* or Q, has been defined as

$$Q = 2\pi \times \frac{\text{maximum energy stored per cycle}}{\text{energy dissipated per cycle}}$$

The maximum value of stored energy is present in an inductor at maximum current, and is given by $LI_m^2/2$. The average power dissipated in the inductor, in terms of maximum current is $I_m^2R_s/2$, where R_s is the series resistance of the inductor, as in Fig. 2-1(a). The energy dissipated per cycle is then $I_m^2R_s/2f$. The definition for the Q of an inductor then is

$$Q = \frac{2\pi LI_m^2}{I_m^2R_s/f} = \frac{2\pi fL}{R_s} = \frac{\omega L}{R_s} \qquad (2\text{-}1)$$

where I_m is the maximum value of the current through the inductor and resistor.

A capacitor is usually assumed as represented by a parallel circuit, with its leakage resistance in shunt, as in Fig. 2-1(b). The maximum stored energy may be computed from capacitance C and the maximum applied voltage E_m, and is $CE_m^2/2$. The average power lost in the resistor is $E_m^2/2R_p$. The value of Q is defined for a capacitor as

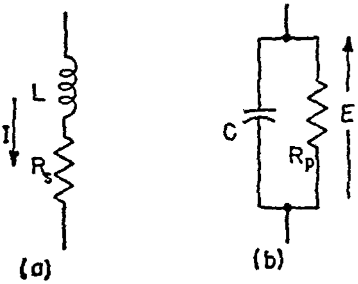


Fig. 2-1. Internal resistances of inductors and capacitors.

$$Q = \frac{2\pi C E_m^2}{E_m^2/R_p f} = 2\pi f C R_p = \omega C R_p \tag{2-2}$$

where the same voltage E_m appears across both C and R_p.*

When the factor Q is used as a figure of merit for a coil, it should be coupled with a statement of the frequency of measurement, since the resistance does not usually vary directly with frequency and Q is a frequency-dependent function. Since the losses of usual capacitors are so small, or R_p so high, Q values for capacitors are large, and in resonant circuits the Q of the inductor is usually the controlling factor. *When the Q of a resonant circuit is stated, the resonant frequency of the circuit is implied as that of the measurement.*

The resistance of a coil and the value of Q are affected by the size of wire, the diameter and length of the coil or its shape, and whether the turns are closely wound or spaced. The material of the form has a bearing on the effective resistance, since it is in the field of the coil and subject to dielectric loss if not of efficient insulating material at the frequency of operation. The dielectric loss introduces an in-phase or resistive component into the coil current. To prevent this loss many high-frequency coils are wound to be almost self-supporting. The effect of losses in the form is illustrated by curves 2 and 4 of Fig. 2-2, which are for coils of identical wire size and approximately equal inductance. The coil of 4 was wound on a low quality plastic form, whereas that of 2 was almost self-supporting. The effect of large wire is illustrated

* If R' is the series resistance of a capacitor, it may be converted to an equivalent shunt resistance R_p by use of $R_p = 1/R'(\omega C)^2$.

by curve 1, and all curves illustrate to some extent the effect of coil diameter in raising Q.

At low frequency ω is low, so that the Q of a given coil is low. As the frequency is increased, Q rises as a result of increasing ω, but at higher frequencies R increases more rapidly because of skin effect

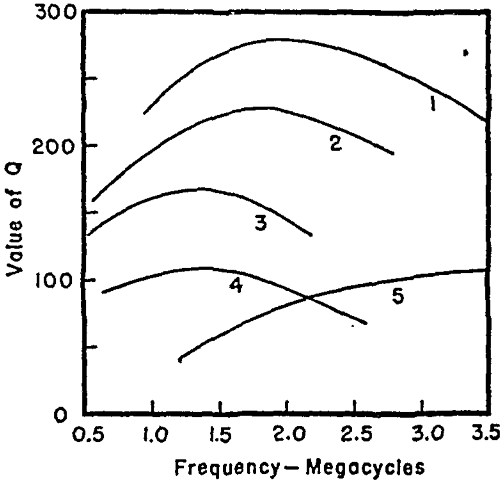


Fig. 2-2. Variation of the Q of various coils as a function of frequency.

(1) $L = 40\ \mu h$; 3 in. diameter; No. 14 D.C.C. wire on thin bakelite tube.
(2) $L = 91\ \mu h$; 3 in. diameter; No. 24 S.C.C. wire, no form.
(3) $L = 187\ \mu h$; 2½ in. diameter; No. 24 S.C.C. wire, bakelite form.
(4) $L = 110\ \mu h$; 2½ in. diameter; No. 24 S.C.C. wire, molded bakelite form.
(5) $L = 50\ \mu h$; 1½ in. diameter; No. 26 D.C.C. wire, molded bakelite form.

and dielectric loss, so that eventually Q drops. Thus the curves of Q have a definite, but quite broad maximum at some frequency.

The Q of a coil may be raised by adding a core of a magnetic material, usually in compressed powder form. By thus raising the permeability the amount of wire needed for a given inductance is reduced, and if core material of low eddy and hysteresis loss is used, the reduction of copper losses may be made greater than the increase in core loss, thus raising Q.

While the losses in capacitors are usually small, they are still affected by the type of dielectric used. Air has the lowest losses of the materials frequently used, followed by polystyrene, mica, and paper.

2-2. Series resonance

A series circuit of R, L, and C is shown in Fig. 2-3, driven by a generator E. The resistance R includes generator resistance,

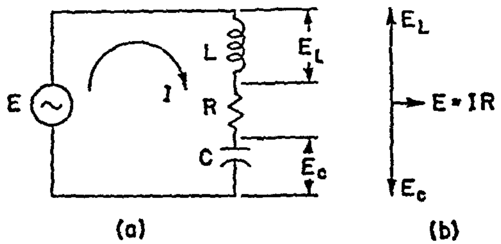


Fig. 2-3. The series R, L, C circuit; (b) phasor diagram of the circuit of (a), at resonance.

resistance of the inductor and capacitor, and any resistance introduced into the circuit as a load.

A circuit equation may be written

$$E = RI + j\omega LI - \frac{jI}{\omega C}$$

$$= I\left[R + j\left(\omega L - \frac{1}{\omega C}\right)\right] \tag{2-3}$$

Since resonance has been defined as operation of a reactive circuit at unity power factor, then at the resonant frequency, ω_r, the reactive term of Eq. 2-3 must be zero, or

$$\omega_r L = \frac{1}{\omega_r C}$$

$$\omega_r^2 LC = 1 \tag{2-4}$$

from which the frequency of resonance, f_r, may be obtained for the series circuit as

$$f_r = \frac{1}{2\pi\sqrt{LC}} \tag{2-5}$$

If the resonance condition given by Eq. 2-4 is inserted in the circuit equation, then the value of the current at resonance, I_r, is

$$I_r = \frac{E}{R} \tag{2-6}$$

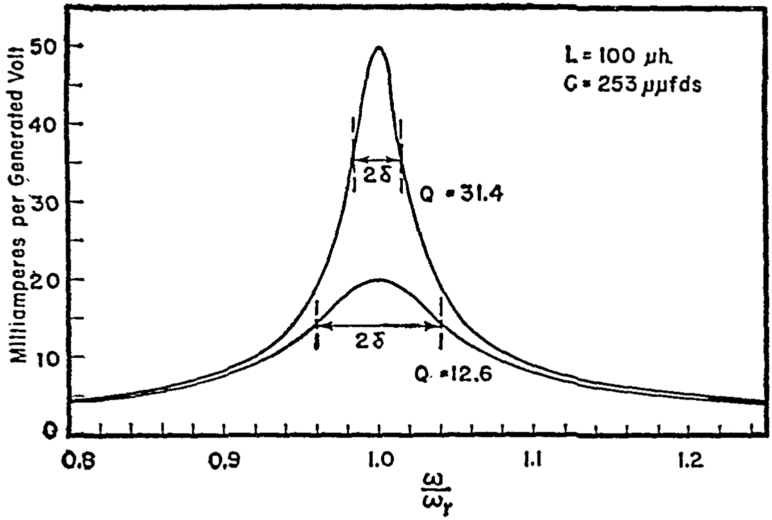


Fig. 2-4. Performance of a series R, L, C circuit as a frequency selector, shown to be a function of circuit Q. The band width is $2\delta = 1/Q$.

The current is seen to be limited solely by the circuit resistance R, as defined above. If this resistance can be made small, the current at resonance can rise to a very high value.

At resonance, the voltage developed across the inductance L is

$$E_L = \frac{j\omega_r LE}{R} = jQE \tag{2-7}$$

Likewise, the voltage across the capacitance C is

$$E_C = -\frac{jE}{\omega_r CR}$$

and since

$$\omega_r L = \frac{1}{\omega_r C}$$

$$E_C = -jQE \tag{2-8}$$

where Q is that of the circuit at resonance. The voltages E_L and E_C are seen to be equal in magnitude but opposite in sign, thus canceling, so that the voltage impressed on the circuit need be only sufficient to overcome the resistance drop, as is shown in (b), Fig. 2-3. Since Q is ordinarily a number considerably greater than unity, then the voltage developed across either L or C is larger than the source voltage E by the factor Q, or the circuit is a voltage amplifier.

At frequencies below resonance, the capacitive reactance term will exceed the inductive reactance and the circuit will appear as a capacitive reactance of value $(1/\omega C - \omega L)$ in series with R. At frequencies above resonance, the circuit will appear as an inductive reactance of value $(\omega L - 1/\omega C)$ in series with R.

The impedance Z of the series circuit is

$$Z = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

and near resonance the reactive term involves the calculation of the difference of two nearly equal numbers, so that accuracy is difficult to obtain. A transformation allows this situation to be avoided. The expression for the impedance may be written

$$Z = R + j\sqrt{\frac{L}{C}}\left(\omega\sqrt{LC} - \frac{1}{\omega\sqrt{LC}}\right) \tag{2-9}$$

$$= R + j\sqrt{\frac{L}{C}}\left(\frac{\omega}{\omega_r} - \frac{\omega_r}{\omega}\right)$$

where ω_r is defined from Eq. 2-4 as

$$\omega_r = \frac{1}{\sqrt{LC}} \tag{2-10}$$

Using this definition,

$$Q = \frac{\omega_r L}{R} = \frac{1}{R}\sqrt{\frac{L}{C}} \tag{2-11}$$

and the impedance of the series circuit becomes

$$Z = R\left[1 + j\frac{1}{R}\sqrt{\frac{L}{C}}\left(\frac{\omega}{\omega_r} - \frac{\omega_r}{\omega}\right)\right]$$

$$= R\left[1 + jQ\left(\frac{\omega}{\omega_r} - \frac{\omega_r}{\omega}\right)\right] \tag{2-12}$$

A new variable δ may be defined as

$$\delta = \frac{f - f_r}{f_r} = \frac{\omega - \omega_r}{\omega_r} \tag{2-13}$$

or

$$\frac{\omega}{\omega_r} = 1 + \delta$$

It can be seen that δ *is the fractional deviation of the actual frequency from the resonant frequency.* If δ is introduced into Eq. 2-12, then

$$Z = R\left[1 + jQ\left(1 + \delta - \frac{1}{1 + \delta}\right)\right]$$

The binomial theorem may be used to expand $1/(1 + \delta) = (1 + \delta)^{-1}$, giving

$$(1 + \delta)^{-1} \cong 1 - \delta + \delta^2$$

neglecting higher powers of δ, since δ is always small with respect to unity. Then

$$Z = R[1 + jQ\delta(2 - \delta)] \tag{2-14}$$

Equation 2-14 expresses the impedance of the series resonant circuit *for small deviations from the resonant frequency, in a form well* suited to computation.

2-3. Band width of the series-resonant circuit

Resonant circuits are often employed as frequency-selective devices in which it is desired that the circuit respond to one frequency, or a narrow band of frequencies, and have no response to all other frequencies. How well a series-resonant circuit may perform as a frequency selector is illustrated in Fig. 2-4, which shows the circuit current as a function of frequency, near the point of resonance. It may be noted that the circuit of low resistance or high Q gives a more selective curve than that for the circuit of higher resistance, although neither is ideal from the standpoint of complete rejection of all frequencies except the desired one. It is convenient to have a measure of the effectiveness with which a series-resonant circuit selects a given frequency or band of frequencies and rejects others, and this measure is supplied by the arbitrarily defined factor *band width.* The band width or frequency discrimination of a resonant circuit is defined *as the width of the resonant curve, in cycles, at the frequency at which the power in the circuit is one-half the maximum power.*

Actually the shape of resonance curve desired for many applications would be a rectangle, of desired band width across the top, and with vertical sides falling to zero, thus securing perfect rejection of all undesired frequencies. As will be shown later, special circuits may be employed to approach more nearly this ideal shape than is possible with the simple circuits discussed here. Bandwidth requirements depend on the application, and in terms of cycles vary from one or two hundred cycles in the audio frequencies for telegraph reception, to band widths of many megacycles for reception of special types of signals at very high radio frequencies.

If I_r is the current in the circuit at resonance, then at the half-maximum-power point the power, in terms of the power at resonance, is

$$\frac{P_r}{2} = \frac{I_r^2 R}{2} = \left(\frac{I_r}{\sqrt{2}}\right)^2 R$$

Since $I_r = E/R$, then the current at the half-power point is

$$\frac{I_r}{\sqrt{2}} = \frac{1}{\sqrt{2}}\frac{E}{R} = \frac{E}{\sqrt{R^2 + X^2}} = \frac{E}{Z_{1/2}}$$

where $\sqrt{R^2 + X^2}$ is the impedance of the circuit at the half-power frequency. It is apparent that

$$\sqrt{2}\,R = \sqrt{R^2 + X^2}$$

from which

$$R = X \tag{2-15}$$

and the resistance and reactance are equal at the half-power frequencies.

The reactance and resistance may be found from Eq. 2-14 and equated as

$$1 = Q\delta_{1/2}(2 - \delta_{1/2})$$

For most selective circuits the value of $\delta_{1/2}$ at the half power frequencies is small with respect to 2, so that

$$1 = 2Q\delta_{1/2}$$

The frequency deviation of each half-power point from resonance will be $\delta_{1/2}$, so that the deviation between the two half-power frequencies will be $2\delta_{1/2}$. Thus the band width B in cycles is

$$B = \Delta f = 2\delta_{1/2} f_r = \frac{f_r}{Q} \tag{2-16}$$

for the series resonant circuit, where Q is that of the complete circuit, including generator, inductor, and capacitor series resistances plus that of any connected load.

Separating the effect of generator resistance R_g, and circuit resistance $R' = R_{\text{coil}} + R_{\text{load}}$, the band width may be written

$$B = \left(\frac{R'}{\omega L} + \frac{R_g}{\omega L}\right) f_r$$

$$= \left(\frac{1}{Q_{\text{circuit}}} + \frac{R_g}{\omega L}\right) f_r \qquad (2\text{-}17)$$

It should be noted that since the impedance of a series circuit at resonance is equal to the total resistance and should be small if the circuit is to be selective (or have a small band width), series resonant circuits should be used with voltage sources of low internal resistance. This is desirable not only from the standpoint of matching impedances for obtaining maximum power delivery but also from the standpoint of obtaining good frequency selectivity, since the generator resistance enters directly into the band-width expression. If maximum power delivery is desired from the generator and the generator resistance R_g is made equal to the circuit resistance R' at resonance, then *for matched conditions*

$$\text{Band width} = \frac{2}{Q} f_r$$

Band width furnishes a means of comparing the selectivity of various circuit designs, as in Fig. 2-4. It should be noted that since the curves are plotted in terms of ω/ω_r, they are applicable for any frequency range. This observation emphasizes the fact that band width as defined is a percentage of the resonant frequency, and when measured in cycles becomes greater for increasing frequencies of resonance. It can be seen that for the curve for $Q = 31.4$, the band width as indicated bears out the relation

$$2\delta = 0.0318 \frac{\omega}{\omega_r} = \frac{1}{Q}$$

If $\omega_r = 1000$ kc, the frequency width to the half-power points is 31.8 kc, whereas if $\omega_r = 10$ mc, the frequency width to the half-power points is 318 kc.

2-4. Parallel resonance or antiresonance

The parallel or antiresonant form of reactive circuit is shown in Fig. 2-5, connected to a generator of internal resistance R_g. The capacitor C will usually have negligible resistance, so that R may

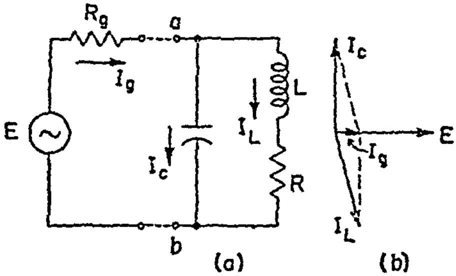


Fig. 2-5. (a) A parallel-connected R, L, C circuit; (b) phasor diagram of (a) at antiresonance.

be considered the resistance of the inductor L plus any resistance introduced as a load.

The admittance of the capacitive branch of the circuit is

$$Y_C = j\omega C$$

and that of the inductive branch is

$$Y_L = \frac{R - j\omega L}{R^2 + \omega^2 L^2} \tag{2-18}$$

The total admittance of the antiresonant circuit to the right of the terminals a,b, Fig. 2-5, is

$$Y = \frac{R}{R^2 + \omega^2 L^2} - j\left(\frac{\omega L}{R^2 + \omega^2 L^2} - \omega C\right) \tag{2-19}$$

For antiresonance, the circuit must have unity power factor, or the j term must be zero. Setting the reactive term equal to zero at ω_{ar} gives

$$\frac{\omega_{ar} L}{R^2 + \omega_{ar}^2 L^2} = \omega_{ar} C$$

or

$$R^2 + \omega_{ar}^2 L^2 = \frac{L}{C} \tag{2-20}$$

from which it is possible to solve for the frequency at which anti resonance occurs as

$$f_{ar} = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{L^2}} \tag{2-21}$$

It may be seen that Eq. 2-21 is the expression for the resonant frequency of the series circuit, modified by the usually small term R^2/L^2 under the radical. It should be noted that resonance is impossible for values of R that make $R^2/L^2 > 1/LC$. This contrasts with the series circuit, which can be resonant or have unity power factor for all values of resistance present.

Equation 2-21 may be put into a useful form by rewriting as follows:

$$f_{ar} = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}\sqrt{1 - \frac{R^2C}{L}}$$

and in view of Eq. 2-11 for Q,

$$f_{ar} = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}\sqrt{1 - \frac{1}{Q^2}} \tag{2-22}$$

where Q is that of the circuit to the right of the terminals a,b, at antiresonance. Equation 2-22 shows that the antiresonant frequency differs from that of a series resonant circuit with the same circuit elements only by the factor $\sqrt{1 - 1/Q^2}$. If Q is greater than 10, the error in neglecting the last radical of Eq. 2-22 is less than 1 per cent. At the same time the radical shows that resonance is not possible for circuits with values of Q less than unity.

From Eq. 2-22 it may be found that

$$\omega_{ar}^2LC = 1 - \frac{1}{Q^2} \tag{2-23}$$

and the reactances of inductive and capacitive branches are not quite equal at unity power factor resonance as they were for the series-resonant case. That is, at antiresonance

$$X_L = X_C\left(1 - \frac{1}{Q^2}\right) \tag{2-24}$$

With the condition of unity power factor imposed, the admittance at antiresonance at the a,b terminals of Fig. 2-5 is

$$Y_{ar} = \frac{R}{R^2 + \omega_{ar}^2 L^2}$$

from which the antiresonant impedance is obtained as

$$Z_{ar} = R_{ar} = \frac{R^2 + \omega_{ar}^2 L^2}{R} \tag{2-25}$$

$$= R + Q\omega_{ar}L = R(1 + Q^2) \tag{2-26}$$

and for circuits of high Q, the term R_{ar} reduces to RQ^2, and can be high in value. Another expression for R_{ar} is possible, by use of Eq. 2-20 in Eq. 2-25, giving

$$R_{ar} = \frac{L}{CR} \tag{2-27}$$

The latter expression indicates that the resonant resistance of the circuit is a function of the L/C ratio chosen by the circuit designer and that R_{ar} can be made quite large if inductors of low resistance are employed.

2-5. Conditions for maximum impedance

The conditions that lead to unity power-factor resonance may not necessarily result in a circuit having the maximum possible impedance. The latter condition is occasionally desirable.

The admittance of the parallel circuit may be written from Eq. 2-19 as

$$Y = \frac{R - j[\omega L - \omega C(R^2 + \omega^2 L^2)]}{R^2 + \omega^2 L^2} \tag{2-28}$$

The square of the absolute value of the admittance will be found to reduce to

$$|Y|^2 = \frac{1 - 2\omega^2 LC + \omega^2 C^2 (R^2 + \omega^2 L^2)}{R^2 + \omega^2 L^2} \tag{2-29}$$

and this expression may be minimized to find the conditions for maximum impedance magnitude.

If the frequency or ω is varied, Eq. 2-29 may be minimized with respect to ω, giving

$$\frac{d|Y|^2}{d\omega} = \frac{-4\omega LCR^2 + 2\omega C^2R^4 + 4\omega^3L^2C^2R^2 + 2\omega^5L^4C^2 - 2\omega L^2}{(R^2 + \omega^2L^2)^2} = 0$$

$$\omega^4 + 2\omega^2\frac{R^2}{L^2} - \frac{1}{L^2C^2} + \frac{2R^2}{L^3C} - \frac{R^4}{L^4} = 0$$

$$\omega = \sqrt{\frac{1}{LC}\sqrt{1 + \frac{2CR^2}{L}} - \frac{R^2}{L^2}} \qquad (2\text{-}30)$$

Thus the condition for maximum impedance with the frequency variable is not quite the unity power factor condition of antiresonance, although it will reduce to that condition for $R \ll L$, or for Q large.

If the frequency is held constant and the capacitance C is varied to produce maximum impedance, Eq. 2-29 may be differentiated with respect to C and minimized as

$$\frac{d|Y|^2}{dC} = \frac{-\omega^2L + \omega^2CR^2 + \omega^4L^2C}{(R^2 + \omega^2L^2)^2} = 0$$

from which

$$R^2 + \omega^2L^2 = \frac{L}{C} \qquad (2\text{-}31)$$

which is Eq. 2-20, and is the condition for unity power factor resonance, which will be simultaneously obtained when capacitance is varied for maximum impedance. This is the manner of tuning much radio equipment.

If the inductance is varied to produce maximum impedance, then

$$\frac{d|Y|^2}{dL} = \frac{(-2\omega^2C+2\omega^4LC^2)(R^2+\omega^2L^2)-[1-2\omega^2LC+\omega^2C(R^2+\omega^2L^2)]2\omega^2L}{(R^2+\omega^2L^2)^2} = 0$$

which leads to

$$\omega^2L^2C - L - CR^2 = 0$$

and

$$\omega^2LC = \frac{1 \pm \sqrt{1 + 4\omega^2C^2R^2}}{2} \qquad (2\text{-}32)$$

and this is not the value for usual antiresonance. Thus if L is adjusted for maximum impedance the power factor will not be unity.

If Q is high, these conditions all approach unity power factor resonance. However, all circuits do not have high Q, especially those employed in the output circuits of radio transmitters, where a resistive load is coupled into the circuit. Adjustment for maximum impedance may then not lead to unity power factor for inductance or frequency variation.

2-6. Currents in antiresonant circuits

At antiresonance, the power delivered by the generator to the circuit of Fig. 2-5 is

$$P = I_g^2 R_{ar} \tag{2-33}$$

The power dissipated in the parallel circuit, assuming negligible capacitor losses, is

$$P = I_L^2 R \tag{2-34}$$

and is equal to the power supplied by the generator, since there are no other power-dissipating elements in the circuit. Equating the input power to the power delivered gives

$$\frac{I_g^2}{I_L^2} = \frac{R}{R_{ar}} \tag{2-35}$$

By use of Eq. 2-27,

$$I_L^2 = \frac{L}{CR^2} I_g^2$$

and in view of the definition of $Q = (1/R)\sqrt{L/C}$ in Eq. 2-11, the current magnitudes are related as

$$I_L = QI_g \tag{2-36}$$

and the circuit is a current amplifier.

The voltage across the terminals a,b is equal to the voltage E_C across the capacitor, and

$$E_C = I_g R_{ar} \tag{2-37}$$

Since $I_C = \omega_{ar} C E_C$, then

$$I_g = \frac{I_C}{\omega_{ar} C R_{ar}}$$

Substitution in Eq. 2-35 leads to

$$\frac{I_C^2}{I_L^2} = \omega_{ar}^2 C^2 R R_{ar} = \omega_{ar}^2 LC$$

in view of Eq. 2-27. Use of Eq. 2-23 then gives

$$\frac{I_C}{I_L} = \sqrt{1 - \frac{1}{Q^2}} \tag{2-38}$$

This is the ratio of the magnitude of the current in the capacitive branch to that in the inductive branch, at unity power factor resonance. The two currents are not quite equal if the resistance is appreciable, approaching equality as R is decreased. This discrepancy is shown in (b), Fig. 2-5, and results in a small value of generator current. The higher the value of Q, the higher will I_C and I_L be, and the lower the generator current. At infinite Q, currents I_C and I_L will be infinite and I_g will be zero.

2-7. Impedance variation with frequency; universal resonance curves

To plot a curve of impedance or admittance of the antiresonant circuit near resonance involves the computation of the difference of two nearly equal large numbers in the reactance term (Eq. 2-19), and many significant figures must be carried if calculations are to be accurate. As for the series circuit, certain simplifications can be made. The admittance expression of Eq. 2-19 can be written

$$Y = \frac{R\left(1 - \dfrac{j\omega L}{R} + \dfrac{\omega^3 CL^2}{R} + \omega CR\right)}{R^2 + \omega^2 L^2} \tag{2-39}$$

and it may be altered in form by again introducing δ so that

$$\frac{\omega L}{R} = \frac{\omega_{ar} L}{R} \frac{\omega}{\omega_{ar}} = Q(1 + \delta) \tag{2-40}$$

so as to permit Q at the antiresonant frequency to be written for $\omega L/R$ when it appears. Also

$$\omega^2 LC = \omega_{ar}^2 LC(1 + \delta)^2 = \left(1 - \frac{1}{Q^2}\right)(1 + \delta)^2$$

by use of Eq. 2-23. Then Eq. 2-39 becomes

$$Y = \frac{1 - jQ(1+\delta)\left\{1 - \left(1 - \frac{1}{Q^2}\right)(1+\delta)^2\left[1 - \frac{1}{Q^2(1+\delta)^2}\right]\right\}}{R[1 + Q^2(1+\delta)^2]} \tag{2-41}$$

In practice, Q^2 will usually be large with respect to unity, at least if $Q > 10$, so that Eq. 2-41 simplifies to

$$Y = \frac{1 - jQ(1+\delta)[1 - (1+\delta)^2]}{RQ^2(1+\delta)^2}$$

from which the impedance may be written

$$Z = \frac{RQ^2(1+\delta)^2}{1 + jQ\delta(1+\delta)(2+\delta)} \tag{2-42}$$

which is in a form better suited to computation than is Eq. 2-39. No assumptions have been made other than $Q > 10$, and the expression is usable for any value of δ.

At antiresonance $\delta = 0$ and Eq. 2-42 reduces to

$$Z_{ar} = R(1 + Q^2) \cong RQ^2$$

so that the ratio of the impedance to that at antiresonance is

$$\frac{Z}{Z_{ar}} = \frac{(1+\delta)^2}{1 + jQ\delta(1+\delta)(2+\delta)} \tag{2-43}$$

Near antiresonance where $\delta \ll 1$, this reduces to

$$\frac{Z}{Z_{ar}} = \frac{1}{1 + j2Q\delta} = A\underline{/\theta^\circ} \tag{2-44}$$

The magnitude A and angle θ of Eq. 2-44 may be plotted in terms of the parameter $Q\delta$, as in Fig. 2-6. Thus plotted in terms of the circuit Q and the frequency variation δ, the curves are universally applicable to any antiresonant circuit, giving the impedance Z at any small variation δ, in terms of Z_{ar}, the impedance at antiresonance. Likewise, the angle θ of the impedance Z is given. For negative values of δ or for frequencies below antiresonance, the angle θ is positive or the circuit is inductive, whereas for positive δ or frequencies above antiresonance the circuit becomes capacitive.

The variation of impedance magnitude and angle for any circuit

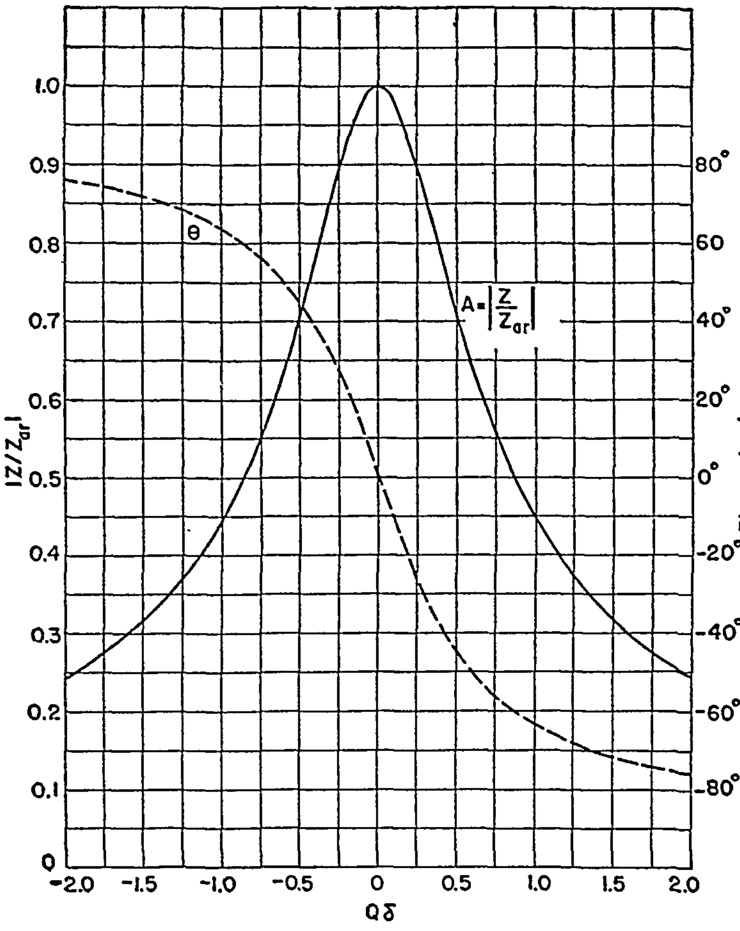


Fig. 2-6. The universal resonance curve for antiresonant circuits, plotted in terms of the function $Q\delta$. The magnitude $|Z/Z_{ar}|$, in terms of impedance magnitude at resonance, and the angle of the impedance, are shown for various $Q\delta$ products.

can be readily plotted with values chosen from the curves illustrated in Fig. 2-6.

2-8. Band width of antiresonant circuits

Figure 2-7 shows the curve of the capacitor voltage E_C plotted against frequency for the case of a generator of zero internal resist-

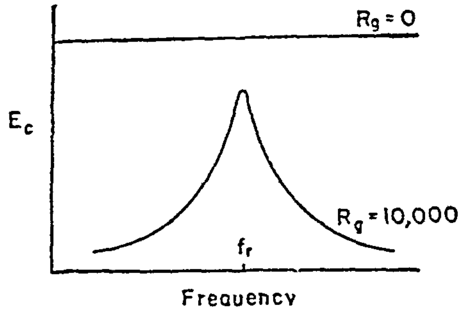


Fig. 2-7. Effect of generator resistance on frequency response of the antiresonant circuit.

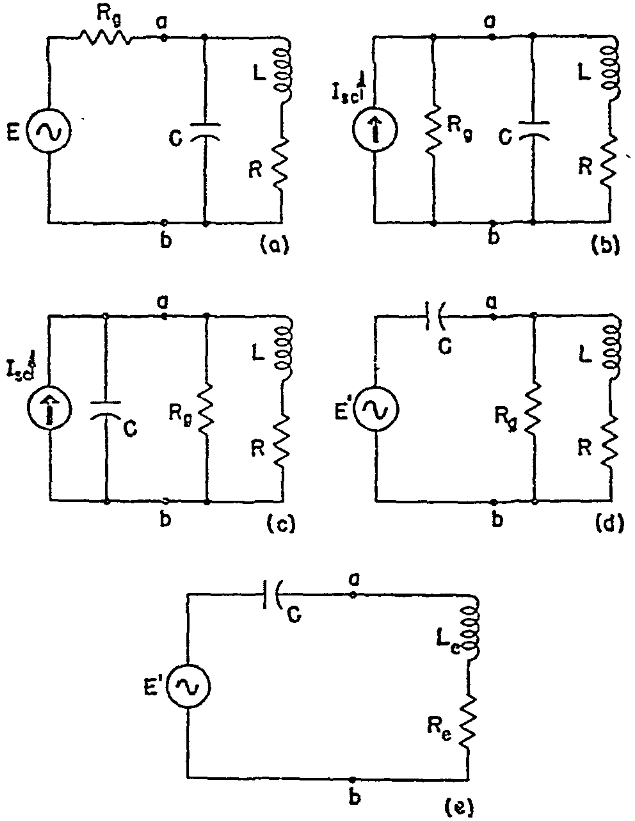


Fig. 2-8. (a) Parallel, R, L, C circuit connected to a generator of internal resistance R_g; (b), (c), (d) successive steps in reduction of (a) to its series voltage-source equivalent at (e).

ance connected to the a,b terminals of Fig. 2-5, and for the case of a generator of 10,000 ohms internal resistance. It should be noted that if frequency selectivity or discrimination is desired, a generator having a high internal resistance must be used with the parallel circuit. A high-resistance generator is also necessary if optimum matching of impedances for maximum power output is to be achieved.

From Fig. 2-7 it is obvious that the band width or selectivity of the circuit is dependent on the value of connected generator resistance R_g. An analysis for the circuit band width, including the effect of R_g, may be made from the circuit of Fig. 2-5, redrawn at (a), Fig. 2-8. By the current-source theorem, the circuit of (a) may be reduced to its equivalent at (b). Since all branches are now in parallel, no change is made in any of the branch currents if the C and R_g branches are interchanged in position as at (c). By Thevenin's theorem for the voltage-source circuit, the circuit of (d) can be drawn as equivalent to (c), considering the capacitor C as the internal impedance of a new generator E'. The new voltage source E' will have an emf of

$$E' = \frac{E}{\omega C R_g} \tag{2-45}$$

dependent on frequency.

The circuit of (d) is then equivalent to the simple series form of (e), Fig. 2-8, with circuit values as obtained below. Considering R_g and the L,R branch in parallel, the equivalent impedance Z_e is

$$Z_e = \frac{R_g(R + j\omega L)}{R_g + R + j\omega L} = \frac{R_g R + j\omega L R_g}{(R_g + R) + j\omega L}$$

After rationalizing and collecting terms,

$$Z_e = \frac{R_g^2 R + R_g R^2 + R_g \omega^2 L^2 + j\omega L R_g^2}{(R_g + R)^2 + \omega^2 L^2}$$

from which it can be seen that

$$R_e = \frac{R_g^2 R + R_g R^2 + R_g \omega^2 L^2}{(R_g + R)^2 + \omega^2 L^2} \tag{2-46}$$

$$\omega L_e = \frac{\omega L R_g^2}{(R_g + R)^2 + \omega^2 L^2} \tag{2-47}$$

In Section 2-3 it has been shown that the *band width of a series resonant circuit* is given by the reciprocal of Q of the circuit. Figure 2-8(e) *is a series circuit*, equivalent to the parallel circuit of (a), so that the band width should be proportional to the reciprocal of Q of the circuit of (e). The value of Q is

$$Q = \frac{\omega L_e}{R_e} = \frac{\omega L R_g^2}{R_g^2 R + R^2 R_g + R_g \omega^2 L^2} \tag{2-48}$$

The band width then is

$$\begin{aligned}\Delta f &= f_2 - f_1 = \frac{f_{ar}}{Q}\\ &= \left(\frac{R_g^2 R + R^2 R_g + \omega^2 L^2 R_g}{\omega L R_g^2}\right) f_{ar}\\ &= \left(\frac{R}{\omega L} + \frac{R R_{ar}}{\omega L R_g}\right) f_{ar} = \left(\frac{1}{Q} + \frac{R R_{ar}}{\omega L R_g}\right) f_{ar}\end{aligned}$$

$$\Delta f = \left(\frac{1}{Q} + \frac{1}{Q}\frac{R_{ar}}{R_g}\right) f_{ar} \tag{2-49}$$

Equation 2-49 shows that the band width is inversely proportional to the Q of the original parallel circuit, modified by a factor dependent on R_g. For a generator of infinite resistance, the band width of a given parallel circuit is that fixed by the R, L, C elements alone, but for any generator of finite resistance the band width is increased by the presence of the generator. Thus it may readily be seen that for the greatest selectivity or least band width of the antiresonant circuit, a generator of very high resistance should be used.

If it is desired to match impedances, so as to obtain the greatest possible power delivery from generator to load, *then the circuit* should be so designed that

$$R_g = R_{ar}$$

It is then seen that the *band width for matched conditions* will be

$$\Delta f = \frac{2}{Q} f_{ar} \tag{2-50}$$

a value identical with that obtained for the same R, L, C elements in a series-connected circuit when matched to a low-resistance generator.

By a slight transformation of Eq. 2-49, certain circuit design factors affecting band width may be discovered. Rewriting Eq. 2-49 by insertion of Eq. 2-27 gives

$$\Delta f = \left(\frac{1}{Q} + \frac{1}{Q}\frac{L}{CRR_g}\right) f_{ar} \tag{2-51}$$

showing that the band width will be greater, or the circuit less selective, if L is chosen large and C small, for a given frequency of resonance. For greater selectivity the inductance should be reduced and C increased, but this lowers the value of R_{ar} and may be undesirable.

2-9. The general case—resistance present in both branches

In some types of antiresonant circuits a resistance may be present in series with the capacitive branch as well as with the inductive branch, as shown in Fig. 2-9(a).

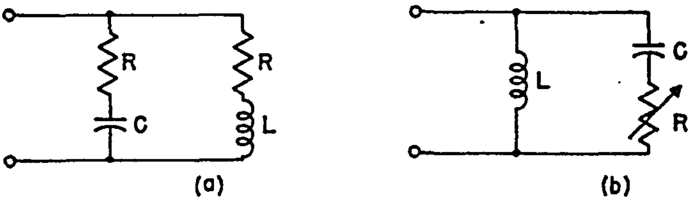


Fig. 2-9. (a) Antiresonant circuit with resistance in both branches; (b) phase shifting, constant impedance circuit.

The admittance Y_L of the inductive branch is then

$$Y_L = \frac{R_1 - j\omega L}{R_1^2 + \omega^2 L^2}$$

and that of the capacitive branch is

$$Y_C = \frac{R_2 + j/\omega C}{R_2^2 + 1/\omega^2 C^2}$$

The total admittance of the parallel combination then is

$$Y_T = \frac{R_1}{R_1^2 + \omega^2 L^2} + \frac{R_2}{R_2^2 + 1/\omega^2 C^2} - j\left(\frac{\omega L}{R_1^2 + \omega^2 L^2} - \frac{1/\omega C}{R_2^2 + 1/\omega^2 C^2}\right) \tag{2-52}$$

For antiresonance the reactive term must be zero, or

$$\omega_{ar}L\left(R_2^2 + \frac{1}{\omega^2C^2}\right) - \frac{1}{\omega_{ar}C}(R_1^2 + \omega^2L^2) = 0 \tag{2-53}$$

and

$$f_{ar} = \frac{1}{2\pi}\sqrt{\frac{1}{LC}\left(\frac{L - R_1^2C}{L - R_2^2C}\right)} \tag{2-54}$$

Equation 2-54 gives the antiresonant frequency for such a circuit with resistance in both branches. If R_2 is made zero, then Eq. 2-54 reduces to the expression (Eq. 2-21) for a circuit with resistance in series with the inductance only, as it should.

If the resistances are small in comparison with the reactances, then $I_L \cong I_C$, and R_2 may be combined with R_1 in the inductive branch, without altering circuit operation, and the analysis of the preceding sections applies.

2-10. Antiresonance at all frequencies; variable phase-angle circuit

If the two resistances of the circuit of Fig. 2-9(a) are given values

$$R_1 = R_2 = \sqrt{\frac{L}{C}} \tag{2-55}$$

it is apparent that Eq. 2-54 is indeterminate. If these resistance values are substituted in Eq. 2-53, it is found that the reactance term is zero, or the circuit is at unity power factor for all frequencies.

If the condition of Eq. 2-55 is imposed on the total admittance, Eq. 2-52, there results

$$Y_T = \frac{\sqrt{L/C}}{L/C + \omega^2L^2} + \frac{\sqrt{L/C}}{L/C + 1/\omega^2C^2} = \sqrt{C/L}$$

as the admittance of the parallel circuit. At all frequencies, the impedance of a parallel circuit with $R_1 = R_2 = \sqrt{L/C}$ is

$$Z = \sqrt{\frac{L}{C}} \tag{2-56}$$

Another interesting circuit is shown in Fig. 2-9(b), whose impedance is

$$Z = \frac{j\omega L(R - j/\omega C)}{R + j(\omega L - 1/\omega C)}$$

and if at a given frequency $\omega L = 2/\omega C$, then

$$Z = \frac{j(2/\omega C)(R - j/\omega C)}{R + j/\omega C} = 2/\omega C \underline{/90^\circ - 2\tan^{-1}(1/\omega RC)}$$

As R is varied, the impedance magnitude is constant, but the impedance phase angle goes from -90° at $R = 0$, to $+90^\circ$ at $R = \infty$.

2-11. Reactance curves

It can be seen from the definition

$$jX_L = j2\pi fL$$

that inductive reactance is a linear positive function of frequency which plots as a straight line through the origin. Likewise from the definition

$$jX_C = \frac{-j}{2\pi fC}$$

capacitive reactance appears as an inverse negative function of frequency, which plots as a negative hyperbola, asymptotic to

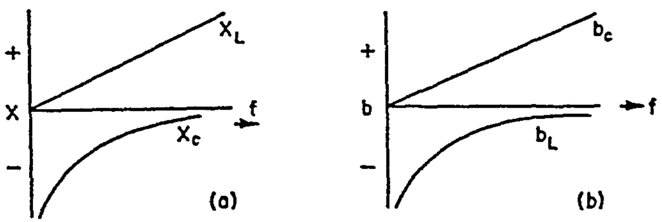


Fig. 2-10. (a) Reactance vs frequency plots of an inductor and a capacitor; (b) susceptance vs frequency plots of the same elements.

the reactance and frequency axes. These curves are shown at (a), Fig. 2-10.

The susceptance of an inductance is

$$jb_L = \frac{-j}{2\pi fL}$$

whereas that of a capacitance is

$$jb_C = j2\pi fC$$

Inductive susceptance plots as an inverse frequency function or as a negative hyperbola, and capacitive susceptance as a positive linear function of frequency. The susceptance functions are plotted at (b), Fig. 2-10.

Thus these functions plot as

Linear positive	*Hyperbolic negative*
inductive reactance	capacitive reactance
capacitive susceptance	inductive susceptance

The reciprocal of a positive linear relation is a negative hyperbola, and vice versa.

For circuits in which the resistances are small enough to be neglected, these reactance or susceptance curves can be combined, giving idealized composite curves of network performance versus frequency. In Fig. 2-11, curves for X_L and X_C are added alge-

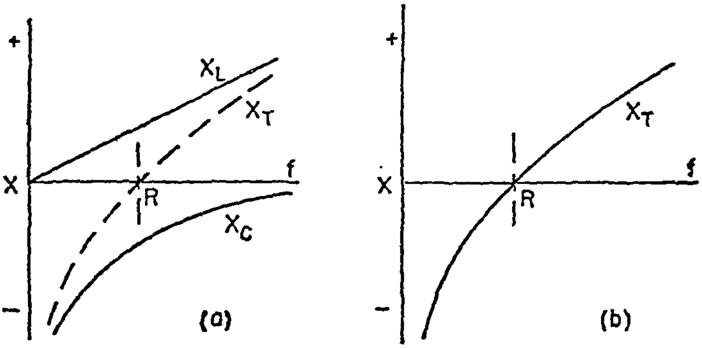


Fig. 2-11. Reactance vs frequency plot for L and C of a series L, C circuit; total reactance of the series circuit at X_T. The resonance point is designated as R.

braically, as would be the case for the series resonant circuit. The curve labeled X_T gives the circuit performance as a frequency function, showing a zero reactance or resonant point, with the circuit having capacitive reactance at frequencies below resonance, and inductive reactance at frequencies above resonance.

In (a), Fig. 2-12, susceptances are added to produce a total susceptance curve b_T, for a parallel LC circuit. The reciprocal of b_T is taken and plotted as X_T in (b), giving a curve of reactance vs. frequency for the parallel resonant circuit. The antiresonant point, at which the reactance theoretically goes to infinity, is shown at AR.

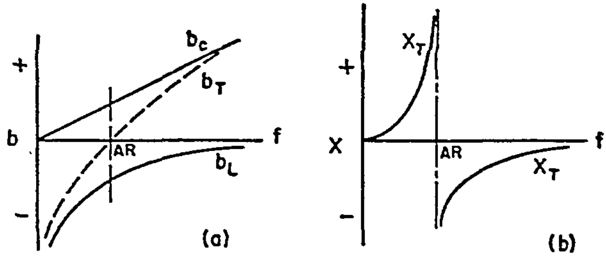


Fig. 2-12. (a) Susceptance vs frequency plots for the L and C branches of a parallel L, C circuit; total susceptance at b_T. (b) Reciprocal of b_T plotted as X_T, *showing antiresonant point designated AR.*

It also appears that the circuit is inductive below, and capacitive above, antiresonance.

More complicated circuits may be analyzed in a similar manner, as in Fig. 2-13. The circuit at (a) may be considered by parts, and the L_2,C_2 branch series reactance is shown as X_2 in (b). The reciprocal of X_2, or its susceptance b_2, is taken and plotted in (c), where it is algebraically added to b_{C1}, since the two branches are in parallel.

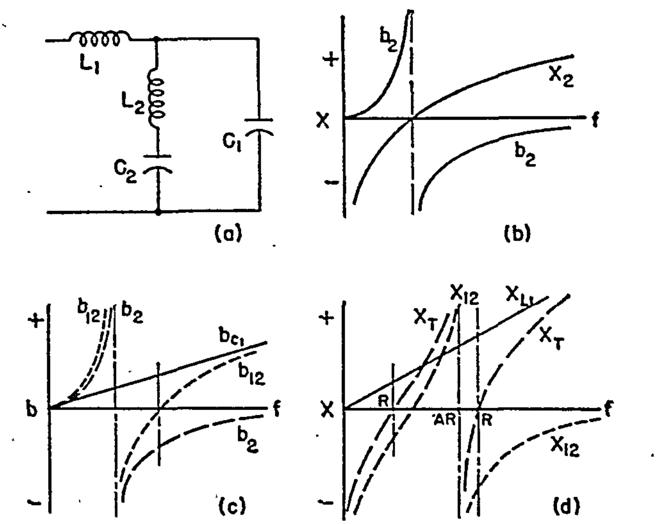


Fig. 2-13. Steps in plotting the reactance vs frequency curve for the circuit of (a). The final curve is X_T and shows two resonant and one antiresonant points.

The total susceptance b_{12} of the two branches results. The reciprocal of b_{12}, or the reactance X_{12}, is plotted in (d), to which is then added X_{L1}, as a series element, giving the total reactance curve X_T, which is the idealized performance of the circuit of (a) as a function of frequency. It can be seen that the circuit has two resonant points separated by one antiresonant point. Analysis of this nature is of considerable help in roughly predicting the frequency performance of complicated reactive networks.

It should be remembered that the curves are plotted by considering the resistances negligible, and if resistance is present the impedances obtained will go neither to infinity nor to zero. The impedance actually will reach some high value at antiresonance and some low value at resonance.

In general, a network may have a total of resonant and antiresonant points not exceeding its number of meshes plus one. The resonant and antiresonant points must alternate in occurrence, since after crossing the zero axis at a resonant point, the curve must go through infinity before again approaching the zero axis, due to the positive slope of all the curves.

2-12. Foster's reactance networks

Because of the extent to which resonance and antiresonance are employed in networks, it is desirable to study their effects in more general form. The resonances, or *zeros*, so called because the reactance there goes to zero, and the antiresonances, or *poles*, so called because the reactance there goes to infinity, characterize the performance of any reactive network. It may be shown that specification of the location and nature of the zeros and poles, plus the knowledge of the impedance at one other frequency, is sufficient information to specify the complete network.

It is usually desired that dissipation or losses be minimized, and the theory and design are based upon the behavior of networks with negligible resistance. The presence of actual resistance in the physical network then slightly modifies the predicted nondissipative performance.

The following theory, developed by R. M. Foster, determines the physical structure of a reactive network from its functional characteristics.

The input impedance Z_{in} of an arbitrary reactive network will be given by

$$Z_{in} = \frac{\Delta}{\Delta_{11}} \tag{2-57}$$

from Eq. 1-121. Each of the mesh and mutual impedance terms in the determinant Δ will be functions of ω, due to operations with $Z = j\omega L$ and $1/j\omega C$. As an example, series combination of L and C would yield

$$j\left(\omega L - \frac{1}{\omega C}\right) = -\frac{L}{j\omega}\left(\omega^2 - \frac{1}{LC}\right)$$

as a typical term of the determinant.

Both Δ and Δ_{11} will then be polynomials in ω and Eq. 2-57 may be written in expanded general form as

$$Z_{in} = S\,\frac{a_0 + a_2\omega^2 + a_4\omega^4 + a_6\omega^6 + \dots + a_{2M}\omega^{2M}}{b_0 + b_2\omega^2 + b_4\omega^4 + b_6\omega^6 + \dots + b_{2M}\omega^{2M}} \tag{2-58}$$

where the coefficient S includes $j\omega$ as

$$S = \pm j\omega H \quad \text{or} \quad \pm\frac{H}{j\omega} \tag{2-59}$$

and H is a *scale factor*.

In general, Δ will involve only even powers of ω, and the denominator only odd powers, or vice versa; the numerator and denominator differ by 1 in the degree of the variable ω, as determined by the position of ω in the coefficient S. In addition, the highest power of ω that can appear in Eq. 2-58 is $2M$, where M is the order of the network determinant Δ.

The numerator and denominator polynomials can be factored so that

$$Z_{in} = S\,\frac{(\omega^2 - \omega_1^2)(\omega^2 - \omega_3^2)(\omega^2 - \omega_5^2)\dots(\omega^2 - \omega_{2M-1}^2)}{(\omega^2 - \omega_2^2)(\omega^2 - \omega_4^2)(\omega^2 - \omega_6^2)\dots(\omega^2 - \omega_{2M}^2)} \tag{2-60}$$

It is apparent that the roots of Δ contribute zeros of the function, and the roots of Δ_{11} contribute poles. That is, at $\omega = \omega_1, \omega_3, \omega_5, \dots$, the impedance function has a zero, and these are the resonant angular frequencies of the network. Likewise at $\omega = \omega_2, \omega_4, \omega_6, \dots$, the impedance function has a pole or becomes infinite, and these are the antiresonant angular frequencies. Because of the ω

present in S, the value $\omega = 0$ may be either a zero or a pole of the function. Also, as ω becomes infinite, the impedance may approach either zero or infinity. Thus Z_{in} may occur in any one of four forms, dependent on the occurrence of a zero or pole at the origin and infinity, as dictated by which of the four forms of S is present (Eq. 2-59).

If all the zeros and poles of the function are known, all the terms of Eq. 2-60 in parentheses are known. If the value of Z_{in} is known at some point other than a pole or zero, the scale factor H is known, and Z_{in} is completely established. That is, in general *an input impedance is completely specified by the location of its poles and zeros, and by its value at a nonzero, nonpole, frequency.* This is essentially Foster's reactance theorem. If the resonant and antiresonant frequencies are specified for a network, and the value of the input impedance is known at one additional frequency, Eq. 2-60 may be written, representing a physical network.

The slope of the reactance function is always positive, as may be seen from the reactance curves of the preceding section. At a pole the function must change sign, and again pass through zero before reaching another pole, so that the zeros and poles must alternate. If the zeros and poles did not alternate, the slope would have to be negative over part of a frequency region. This alternation of poles and zeros is called the *separation property* of the zeros and poles of the reactance function.

As an example of the further development of the network, consider Eq. 2-60 written for the case in which $S = -H/j\omega$. This makes Eq. 2-60 appear as

$$Z_{in} = -\frac{H}{j\omega}\frac{(\omega^2 - \omega_1^2)(\omega^2 - \omega_3^2)(\omega^2 - \omega_5^2)\ \cdots}{(\omega^2 - \omega_2^2)(\omega^2 - \omega_4^2)(\omega^2 - \omega_6^2)\ \cdots} \tag{2-61}$$

Expansion by the partial fraction method then yields

$$Z_{in} = j\omega H\left(1 + \frac{A_0}{\omega^2} + \frac{A_2}{\omega^2 - \omega_2^2} + \frac{A_4}{\omega^2 - \omega_4^2} + \cdots + \frac{A_{2M-2}}{\omega^2 - \omega_{2M-2}^2}\right) \tag{2-62}$$

This impedance function represents a series combination of physical circuits. The unity term, when combined with $j\omega H$, may be recognized as the reactance of a series inductor. The second term is in the form of the reactance of a series capacitor, and the succeeding terms are in the form taken by the reactance of an

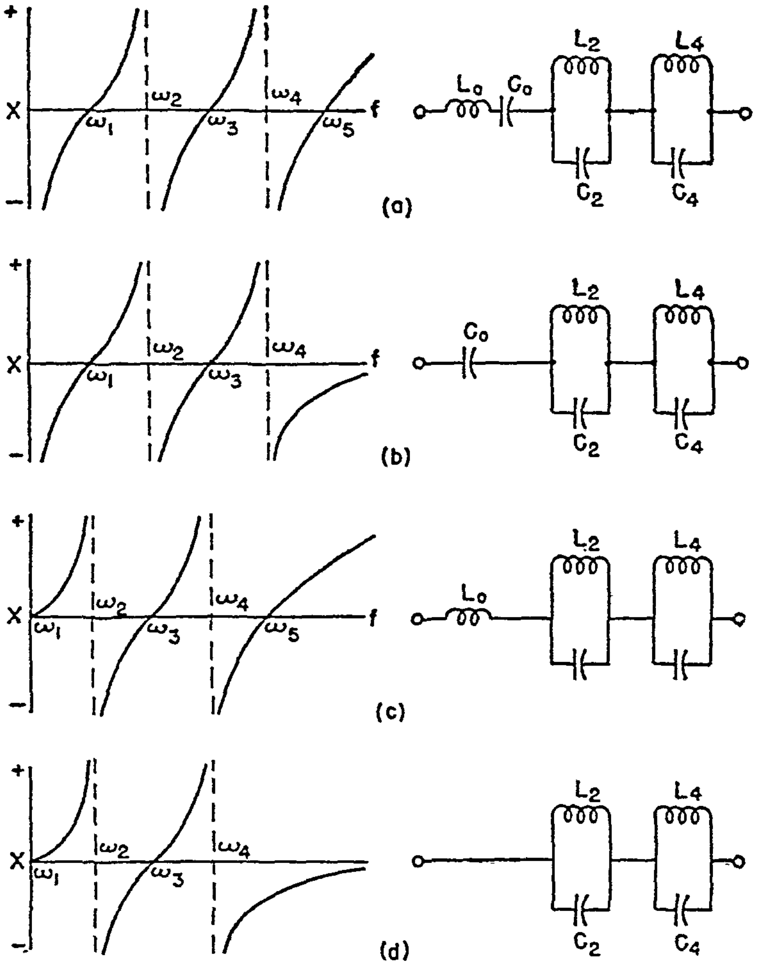


Fig. 2-14. Performance of the series canonic circuits.

inductor and capacitor in parallel, as can be demonstrated by

$$Z_k = \frac{j\omega L(-j/\omega C)}{j(\omega L - 1/\omega C)} = \frac{j\omega(-1/C)}{\omega^2 - 1/LC} = \frac{j\omega(-1/C)}{\omega^2 - \omega_k^2}$$

where $\omega_k^2 = 1/LC$. The partial fraction expansion as in Eq. 2-62 represents a physical circuit as at (a), Fig. 2-14.

Determination of the A_k coefficients may be made if Eqs. 2-61 and 2-62 are equated and multiplied by ω^2 as

$$\frac{(\omega^2 - \omega_1^2)(\omega^2 - \omega_3^2)(\omega^2 - \omega_5^2)\ \cdots}{(\omega^2 - \omega_2^2)(\omega_2 - \omega_4^2)(\omega^2 - \omega_6^2)\ \cdots} = \omega^2 + A_0 + \frac{A_2\omega^2}{\omega^2 - \omega_2^2} + \frac{A_4\omega^2}{\omega^2 - \omega_4^2} + \cdots \tag{2-63}$$

If $\omega = 0$, then

$$A_0 = \frac{(-\omega_1^2)(-\omega_3^2)(-\omega_5^2)\ \cdots}{(-\omega_2^2)(-\omega_4^2)(-\omega_6^2)\ \cdots} \tag{2-64}$$

A similar method, involving multiplication of Eq. 2-63 by $(\omega^2 - \omega_k^2)$ and making $\omega = \omega_k$, will yield the other A_k coefficients.

Choice of the other three possibilities involved in S yields three more equations and their partial fraction expansions as follows.

For $S = H/j\omega$:

$$Z_{\text{in}} = \frac{H}{j\omega}\frac{(\omega^2 - \omega_1^2)(\omega^2 - \omega_3^2)(\omega^2 - \omega_5^2)\ \cdots}{(\omega^2 - \omega_2^2)(\omega^2 - \omega_4^2)(\omega^2 - \omega_6^2)\ \cdots} \tag{2-65}$$

for which

$$Z_{\text{in}} = j\omega H\left(\frac{A_0}{\omega^2} + \frac{A_2}{\omega^2 - \omega_2^2} + \frac{A_4}{\omega^2 - \omega_4^2} + \cdots\right) \tag{2-66}$$

The circuit representative of this function appears at (b), Fig. 2-14.

For $S = j\omega H$:

$$Z_{\text{in}} = j\omega H\frac{(\omega^2 - \omega_3^2)(\omega^2 - \omega_5^2)(\omega^2 - \omega_7^2)\ \cdots}{(\omega^2 - \omega_2^2)(\omega^2 - \omega_4^2)(\omega^2 - \omega_6^2)\ \cdots} \tag{2-67}$$

for which

$$Z_{\text{in}} = j\omega H\left(1 + \frac{A_2}{\omega^2 - \omega_2^2} + \frac{A_4}{\omega^2 - \omega_4^2} + \frac{A_6}{\omega^2 - \omega_6^2} + \cdots\right) \tag{2-68}$$

and the circuit representative of this function appears at (c), Fig. 2-14.

For $S = -j\omega H$:

$$Z_{\text{in}} = -j\omega H\frac{(\omega^2 - \omega_3^2)(\omega^2 - \omega_5^2)\ \cdots}{(\omega^2 - \omega_2^2)(\omega^2 - \omega_4^2)(\omega^2 - \omega_6^2)\ \cdots} \tag{2-69}$$

for which

$$Z_{\text{in}} = j\omega H\left(\frac{A_2}{\omega^2 - \omega_2^2} + \frac{A_4}{\omega^2 - \omega_4^2} + \frac{A_6}{\omega^2 - \omega_6^2} + \cdots\right) \tag{2-70}$$

and the circuit for this function appears in (d), Fig. 2-14.

In all the above, odd subscripts on ω represent zeros, even subscripts represent poles. The four circuits in Fig. 2-14 represent the fundamental or *canonic* forms of Foster's networks. Four susceptance dual networks are also possible.

The types of networks discussed above will yield all possible types of reactive network frequency performance. Choice of a particular form by reason of its behavior at the origin and at infinite frequency, coupled with a knowledge of the H value, will allow reproduction of any desired frequency performance. These networks contain the *least* number of circuit elements needed to represent a given function. This least number of elements equals one more than the sum of the *internal* poles and zeros. The poles or zeros at the origin and at infinity are called *external* poles and zeros; thus the internal poles or zeros are those having the factors $(\omega^2 - \omega_k^2)$ present, the external poles or zeros involving the term $j\omega$.

2-13. Nondissipative network design, using Foster's methods

A discussion of the method by which Foster's circuits may be designed may aid in understanding the preceding section and is presented here. It covers only nondissipative reactive networks, but serves as a foundation for a considerable class of circuits. Other modern methods, employing knowledge of the complex variable, are available for network synthesis with consideration of dissipative elements, but are beyond the scope of this text.

The problem of design revolves around the determination of the circuit parameters in terms of the coefficients of the partial fraction expansions. Using Eq. 2-62 as an example, it is apparent from the first term inside the bracket that

$$L_0 = H \text{ henrys} \tag{2-71}$$

Determination of the C_0 capacitor value can be made from the second term of Eq. 2-62, which represents a capacitive reactance as

$$\frac{1}{j\omega C} = \frac{jHA_0}{\omega} = -\frac{HA_0}{j\omega}$$

so that

$$C_0 = -\frac{1}{HA_0} \tag{2-72}$$

and A_0 can be found by Eq. 2-64.

The other A_k coefficients may be found by study of Eqs. 2-62, 2-66, 2-68, and 2-70 at a frequency very close to ω_k. For one of these frequencies the term involving A_k becomes extremely large, and as $\omega \to \omega_k$, all other terms in the expansion become negligible in comparison. Thus it is possible to write from Eq. 2-62:

$$Z_{\text{in}} = \frac{j\omega H A_k}{\omega^2 - \omega_k^2}\bigg]_{\omega=\omega_k} \tag{2-73}$$

At $\omega = \omega_k$ the input impedance has a pole due to the parallel resonance of an L_k,C_k pair. The impedance of such a parallel pair is

$$Z_k = \frac{\omega L_k(-j/\omega C_k)}{\omega L_k - 1/\omega C_k} = \frac{j\omega(-1/\omega C_k)}{\omega^2 - \omega_k^2}$$

where $\omega_k = 1/\sqrt{L_k C_k}$. This expression may then be equated to Eq. 2-73, since the input impedance at $\omega = \omega_k$ is due to that of the L_k,C_k parallel pair. Then

$$Z_{\text{in}} = \frac{j\omega H A_k}{\omega^2 - \omega_k^2}\bigg]_{\omega=\omega_k} = \frac{j\omega(-1/C_k)}{\omega^2 - \omega_k^2}\bigg]_{\omega=\omega_k} \tag{2-74}$$

from which

$$HA_k = -\frac{1}{C_k}$$

This result may be substituted in Eq. 2-73, and solved for C_k, giving

$$C_k = \frac{-j\omega}{Z_{\text{in}}(\omega^2 - \omega_k^2)}\bigg]_{\omega=\omega_k} \text{ farads} \tag{2-75}$$

Values of the resonating capacitors are thus determined. The necessary inductor values, other than L_0, can then be obtained from the specification

$$\omega_k^2 = \frac{1}{L_k C_k} \tag{2-76}$$

and completing the design of the circuit, which is shown in (a), Fig. 2-14.

Example: A two-terminal network is required to have zeros at $\omega_1 = 5000$, $\omega_3 = 7000$, $\omega_5 = 9000$, and poles at $\omega_2 = 6000$, $\omega_4 = 8000$, and at infinity. The input impedance at $\omega = 1000$ is $-j1000$.

Because of the negative sign of the impedance at $\omega = 1000$, or below ω_1, it appears that there will be a pole at zero frequency. Since a pole is also specified at infinity, the frequency performance must be of the form of (a), Fig. 2-14, and Eq. 2-61, with $S = -H/j\omega$, applies.

The value of H may be found from Eq. 2-61 as

$$-j1000 = -\frac{H}{j1000}\frac{(1000^2 - 5000^2)(1000^2 - 7000^2)(1000^2 - 9000^2)}{(1000^2 - 6000^2)(1000^2 - 8000^2)}$$

$$H = 0.0239 \text{ henry}$$

To find C_0, use Eq. 2-64 and obtain A_0 as

$$A_0 = \frac{(-5000^2)(-7000^2)(-9000^2)}{(-6000^2)(-8000^2)}$$
$$= -43.04 \times 10^6$$

Then

$$C_0 = -\frac{1}{HA_0} = \frac{1}{0.0239 \times 43.04 \times 10^6}$$
$$= 0.097\ \mu\text{f}$$

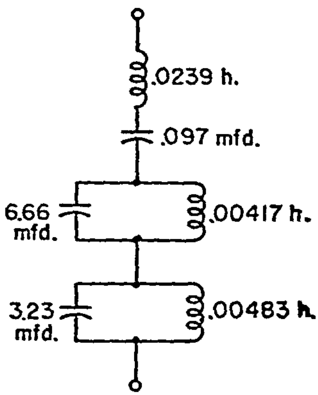


Fig. 2-15. Network as designed.

To find C_2, use Eq. 2-75 as

$$C_2 = \frac{-j\omega}{\dfrac{-H}{j\omega}\dfrac{(\omega^2 - \omega_1^2)(\omega^2 - \omega_3^2)(\omega^2 - \omega_5^2)}{\omega^2 - \omega_4^2}}$$

with $\omega = \omega_2$ and this becomes

$$C_2 = \frac{6000^2}{-0.0239\dfrac{(6000^2 - 5000^2)(6000^2 - 7000^2)(6000^2 - 9000^2)}{6000^2 - 8000^2}}$$

$$= 6.66\ \mu\text{f}$$

In similar fashion, with $\omega = \omega_4$

$$C_4 = \frac{8000^2}{-0.0239\dfrac{(8000^2 - 5000^2)(8000^2 - 7000^2)(8000^2 - 9000^2)}{8000^2 - 6000^2}}$$
$$= 3.23\ \mu\text{f}$$

By use of the relation $\omega_k^2 = 1/L_kC_k$ it is possible to calculate L_2 and L_4 as $L_2 = 0.00417$ henry, $L_4 = 0.00483$ henry. The complete network is shown in Fig. 2-15.

PROBLEMS

2-1. A series circuit, having a capacitor of negligible resistance and a 120-μh coil of 18 ohms resistance, is resonant at 1 megacycle (10^6 cycles), and is connected to a generator of $R_g = 0$, 1 volt, 1 megacycle frequency.

(a) What will be the voltage across the capacitor?

(b) What currents will flow at resonance and at 10 kc above resonance?

(c) What is the band width in cycles?

2-2. Plot a curve of current against frequency for the above circuit over the region from 0.9 to 1.1 megacycles.

2-3. A series circuit is in resonance at 8×10^6 c, and has a coil of 35 μh and 10 ohms resistance.

(a) Find the current at resonance.

(b) Repeat for a frequency of 8.01 megacycles.

(c) What capacitance is required at resonance?

(d) What impedance will this circuit present to the second harmonic of the resonant frequency?

2-4. A series resonant circuit is to act as a frequency-selective shunt, as in Fig. 2-16, to separate two signals, f_A at 1.5 megacycles, and f_B at 1.51 megacycles. The circuit is to have a band width of 5000 c for f_A, the desired signal, and the output E_0 for f_A must be 5 v.

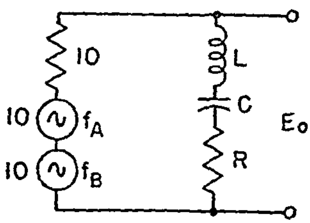


Fig. 2-16.

(a) Specify the circuit components required.

(b) If the 1.5 mc signal is desired, compute the ratio of the desired to the 1.51 mc signal.

(c) Repeat (b) at the half-power frequency lying between f_A and f_B.

2-5. (a) A series circuit with $Q = 100$ is in series resonance at 500 kc. Plot the curve of impedance vs frequency. Repeat on the same sheet if $Q = 10$.

2-6. A series circuit of $Q = 50$ is in resonance at 1 megacycle. Plot the impedance and phase angle against frequency. Repeat for a circuit of identical Q at a frequency of 10 megacycles.

2-7. A coil in series with a 400-$\mu\mu$f capacitor is resonant at 0.8 megacycle. If the power supplied to the circuit by a zero-resistance generator at 0.78 megacycle is half that supplied by the same generator at resonance, specify the values of the circuit components.

2-8. An inductor of 180 μh with a Q of 150 is used in a parallel circuit with a capacitor, and the circuit resonant at 1200 kc. The circuit is supplied by a generator of 10 v, 1 megohm resistance. Plot the magnitude of voltage across the circuit from 1100 to 1300 kc.

2-9. A parallel resonant circuit has a coil of 150 μh with a Q of 60, and is resonated at 1 megacycle.

(a) Specify the value of the required capacitance.

(b) What is the resistance of L?

(c) What is the circuit impedance at antiresonance?

(d) If the Q is reduced to 4 by adding additional series resistance, how much resistance is needed?

(e) What is the antiresonant impedance with the new Q?

(f) What is the new antiresonant frequency?

2-10. A parallel resonant circuit has fixed C and variable L. The Q of the inductor is 4 and constant. Find the values of L and C for a circuit impedance of $1000 + j0$ at $f = 2.4$ megacycles. What is the band width?

2-11. A parallel circuit has an inductor of 150 μh and 15 ohms resistance, tuned by capacitor to 1800 kc.

(a) Plot a curve of impedance vs. frequency.

(b) Plot a curve of impedance vs. frequency if 50 ohms is added in series with L.

(c) By measurements on the curves, check the equation for band width.

2-12. Since a parallel circuit is resistive at antiresonance, it may be used as a load on a generator and the resonant resistance may be adjusted for maximum power output. Such a circuit, made of a coil and a 300-$\mu\mu$f capacitor, is to match a 10-v, 0.5×10^6 c, 20,000-

ohm internal-resistance generator. Assuming no resistance in the capacitor, what values of L and R must the inductor possess to achieve an impedance match? Find the band width of the complete circuit.

2-13. An inductor of 50 μh is used in parallel with an appropriate capacitor as a load for a generator of 15 v, 10^7 c, and 15,000 ohms internal resistance.

(a) Assuming no resistance in the capacitor, what are the values of Q and R of the coil that will cause the parallel circuit to load the generator for maximum output?

(b) What will the generator current be?

(c) What is the inductor current?

(d) Show that the generator power output is equal to the power dissipated in the parallel circuit.

2-14. (a) In Fig. 2-17(a), $C = 100$ $\mu\mu$f. If $Q = 7$, find the value of L needed for antiresonance, R being the coil resistance.

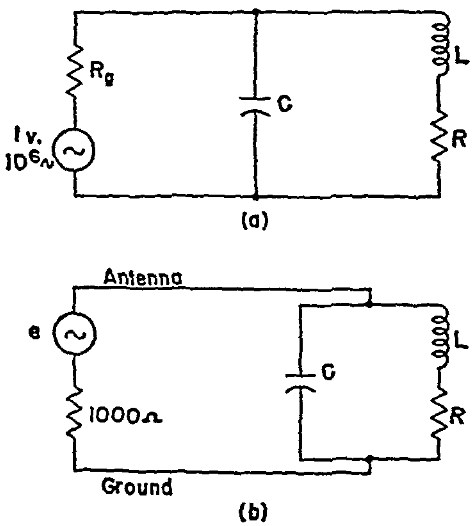


Fig. 2-17.

(b) Find the resistance R of the coil.

(c) What value should R_g have to supply maximum power to the circuit?

(d) What is the maximum amount of power that may be obtained from the generator?

(e) Find the band width of the circuit.

(f) Find the generator current at a frequency of 1.05×10^6 c.

2-15. A voltage induced in the antenna of a radio receiver, Fig. 2-17(b), may be expressed as

$$e = 0.025 \sin (2\pi \times 1.5 \times 10^6 t) + 0.010 \sin (2\pi \times 1.510 \times 10^6 t) + 0.012 \sin (2\pi \times 1.550 \times 10^6 t)$$

This emf is applied to a parallel circuit having a Q of 10, $L = 100\ \mu$h, and C to resonate the circuit at 1.5×10^6 c.

Find the voltage produced across C for each frequency present.

2-16. Two inductors are used in separate parallel circuits, each antiresonant at 7 megacycles. Inductor A has $L = 25\ \mu$h, $Q = 50$; and inductor B has $L = 120\ \mu$h, $Q = 50$. Plot curves of circuit impedance vs frequency. Find the band width when each circuit is supplied by a generator of 50,000 ohms internal resistance.

2-17. Plot reactance curves for the circuits of Fig. 2-18, designating all resonant and antiresonant points.

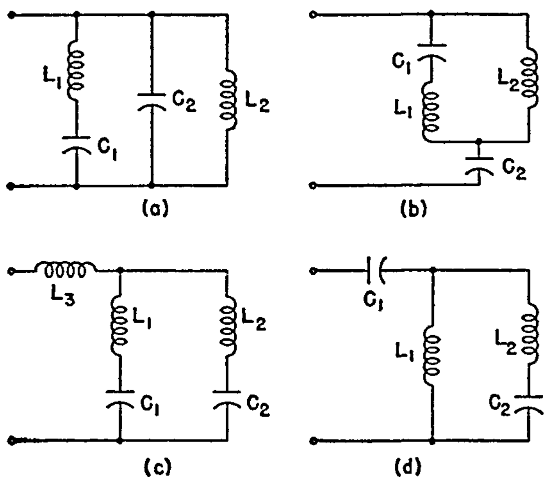


Fig. 2-18.

2-18. Develop Eq. 2-62 from Eq. 2-61.

2-19. Develop Eq. 2-68 from Eq. 2-67.

2-20. Design a network to have zeros at 0, 159, and 318 c, with poles at 74.5 and 233 c. At 250 c the input impedance must be $-j1200$ ohms.

2-21. Find the Foster network which will respond as follows: zero at $\omega = 1000$, no external poles, reactance at $\omega = 5000$ is $+j250$ ohms.

2-22. Find the Foster network which will respond as follows: zeros at $\omega = 2000$, $\omega = 6000$, poles at $\omega = 4000$, $\omega = 8000$, and reactance at $\omega = 1000$ to be $-j50$ ohms.

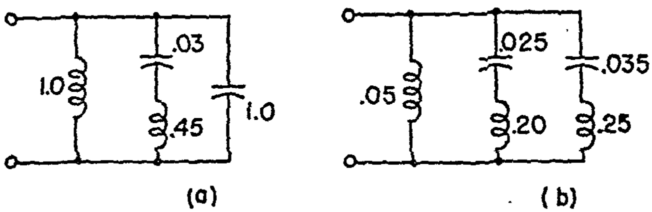


Fig. 2-19. Units are henrys and microfarads.

2-23. Draw reactance plots for each of the circuits in Fig. 2-19, and write the input impedance expression, including the value of H.

REFERENCES

1. Terman, F. E., *Radio Engineering*, 3d ed., McGraw-Hill Book Company, Inc., New York, 1947.
2. Everitt, W. L., *Communication Engineering*, 2d ed., McGraw-Hill Book Company, Inc., New York, 1937.
3. LePage, W. R., and Seely, S., *General Network Analysis*, McGraw-Hill Book Company, Inc., New York, 1952.
4. Reed, M. B., *Alternating-Current Circuits*, Harper & Brothers, New York, 1948.
5. Foster, R. M., "A Reactance Theorem," *Bell Syst. Tech. J.*, April, 1924, p. 259.
6. Shea, T. E., *Transmission Networks and Wave Filters*, D. Van Nostrand Company, Inc., New York, 1929.

Chapter 3

IMPEDANCE TRANSFORMATION AND COUPLED CIRCUITS

In Chapters 1 and 2 it has been assumed that a desirable load for any given generator, with the load value fixed either by heating limitations of the generator or by maximum-power-transfer conditions, may readily be found. In practice it is hardly to be expected that a specified load will, in general, be suited to the generator which is to be connected to it, since many other factors enter into the determination of the impedance of these devices. For example, the impedance of the usual loudspeaker is of the order of 8 to 10 ohms, whereas a desirable load for the vacuum-tube generator supplying power to the loudspeaker is usually several thousand ohms. Likewise, a radio transmitter, limited by internal heat losses to a load of 4000 ohms, may be required to supply power efficiently to an antenna having a resistance of possibly 70 ohms. Since it is highly desirable that the maximum possible or permissible amounts of power be obtained from the generators used in communications service, methods of transforming impedance values to the desired values have been developed. Hence it is possible to make the 8 ohms of the loudspeaker appear as several thousand ohms to the vacuum tube, or to transform the 70 ohms of the antenna so that to the radio transmitter the load appears as the desired value of 4000 ohms.

In this chapter, methods of transforming impedances are developed whereby load and generator impedances may be matched for maximum power transfer, or adjusted to the values fixed by internal loss limitations.

3-1. Transformation of impedances with tapped resonant circuits

A parallel LC circuit represents a resistance at antiresonance and may therefore be used as a power-absorbing load on a generator of large internal resistance. The value of antiresonant resistance R_{ar}, however, is dependent on the L/C ratio chosen for the circuit.

and may be higher or lower than that needed for the desired power transfer. In such a case the resistive impedance into which the generator supplies power may be reduced by tapping the external generator connections across only a portion of the inductance, as in Fig. 3-1(a).

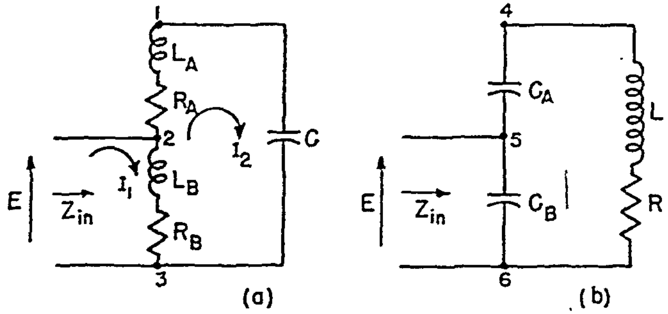


Fig. 3-1. Forms of tapped antiresonant circuits for impedance transformation.

Assuming that the mutual inductance between L_A and L_B is such that the total inductance is

$$L = L_A + L_B + 2M \tag{3-1}$$

then antiresonance will be found to occur between terminals 1 and 3 if

$$\omega(L_A + L_B + 2M) = \frac{1}{\omega C} \tag{3-2}$$

Considering the terminals 2,3 of (a), antiresonance will occur if

$$\omega(L_B + M) = \frac{1}{\omega C} - \omega(L_A + M) \tag{3-3}$$

since part of the inductor is now in series with C, but this branch must still have a net capacitive reactance.

It is apparent that Eq. 3-3 is identical with Eq. 3-2, and that if a parallel LC circuit is antiresonant at a given frequency, antiresonance will occur simultaneously between any two other points in the circuit. Accordingly, tapping a circuit as at (a), Fig. 3-1, does not alter the antiresonance conditions.

The circuit equations for (a) may be written in the form

$$\begin{bmatrix} E \\ \\ 0 \end{bmatrix} = \begin{bmatrix} R_B+jX_{LBb}; & -(R_B+jX_{LB}+jX_M) \\ -(R_B+jX_{LB}+jX_M); & (R_A+R_B+jX_{LA} \\ & +jX_{LB}+j2X_M-jX_C) \end{bmatrix} \begin{bmatrix} I_1 \\ \\ I_2 \end{bmatrix}$$

and since $Z_{in} = Z_{2,3} = \Delta/\Delta_{11}$, then

$$Z_{2,3} = \frac{(R_B + jX_{LB})(R_A + R_B + jX_{LA} + jX_{LB} + j2X_M - jX_C) - (R_B + jX_{LB} + jX_M)^2}{R_A + R_B + jX_{LA} + jX_{LB} + j2X_M - jX_C}$$

In view of Eq. 3-1, and letting $R = R_A + R_B$,

$$Z_{2,3} = R_B + jX_{LB} - \frac{(R_B + jX_{LB} + jX_M)^2}{R + j(X_L - X_C)} \tag{3-4}$$

If the circuit is antiresonant and of reasonable Q, then $X_L = X_C$; also $X_{LB} \gg R_B$, and R_B may be dropped, giving

$$Z_{2,3} = jX_{LB} + \frac{(X_{LB} + X_M)^2}{R} \tag{3-5}$$

The magnitude of the first term will be small with respect to the second, for reasonable Q values, and the reactive term may be dropped. If the value for impedance $Z_{2,3}$ is then compared with the resonant impedance across the 1,3 terminals, where

$$Z_{1,3} = \frac{R^2 + \omega^2L^2}{R} = \frac{R^2 + X_L^2}{R}$$

then
$$\frac{Z_{2,3}}{Z_{1,3}} = \frac{(X_{LB} + X_M)^2}{R^2 + X_L^2} \tag{3-6}$$

Again, $R \ll X_L$, for reasonable Q, and

$$\frac{Z_{2,3}}{Z_{1,3}} = \frac{(X_{LB} + X_M)^2}{X_L^2} = \frac{(L_B + M)^2}{L^2} \tag{3-7}$$

For values of L_B considerably greater than M, the effect of tapping down on the inductance varies as the square of the fraction of the inductance across which the generator is connected. This would make the impedance vary approximately as the fraction of the total turns.

If the circuit capacitance is split into two capacitors in series, equivalent in capacitance to the single capacitor C, and if the external generator is tapped between the two capacitors, and since $\omega L = 1/\omega C$,

$$Z_{5,6} = \frac{(X_{C2})^2}{R}$$

and
$$Z_{4,6} = \frac{(X_{C1} + X_{C2})^2}{R}$$

The effect of tapping down on the capacitive side of the circuit is then given by

$$\frac{Z_{5,6}}{Z_{4,6}} = \frac{(X_{C2})^2}{(X_{C1} + X_{C2})^2} = \frac{C_1^2}{(C_1 + C_2)^2} \tag{3-8}$$

also showing a reduction of impedance. These methods become very convenient at high frequencies.

3-2. Reactance L sections for impedance transformation

Two reactances of opposite sign may be arranged as in Fig. 3-2 to transform at one frequency a load resistance R to provide a desired load R_{in} for the generator, where $R < R_{in}$. Such a reactance circuit as that between the terminals a,b and c,d of Fig. 3-2 is called an

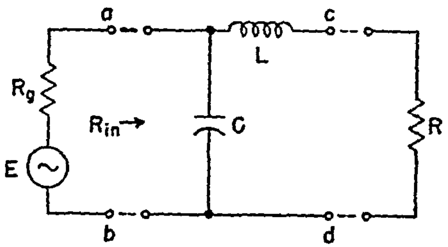


Fig. 3-2. A reactance L section for impedance transformation ($R < R_g$).

L section because of its appearance when drawn in the circuit diagram. Operation of the circuit may be readily understood if it is noted that the circuit to the right of terminals a,b constitutes a parallel circuit that at antiresonance appears as a resistance load on the generator. The value of this resistance load is a function of the L/C ratio chosen for the reactance matching section. Therefore the circuit designer can select values resulting in an antiresonant load R_{in}, for matching to R_g, or for any other desired value of load.

Simultaneous conditions to be realized are that the circuit to the right of a,b be in antiresonance and have an antiresonant impedance equal to R_{in}. These conditions may be expressed by Eqs. 2-21 and 2-27 as

$$\omega = \sqrt{\frac{1}{LC} - \frac{R^2}{L^2}} \tag{3-9}$$

$$R_{ar} = R_{in} = \frac{L}{CR} \tag{3-10}$$

From Eq. 3-10, $$L = R_{in}RC$$

which if inserted in Eq. 3-9 leads to

$$\omega^2 = \frac{1}{R_{in}RC^2} - \frac{1}{R_{in}^2C^2}$$

$$\omega C = \sqrt{\frac{R_{in} - R}{R_{in}^2R}}$$

which gives for the value of capacitance C needed for the L section:

$$C = \frac{1}{\omega R_{in}}\sqrt{\frac{R_{in}}{R} - 1} \tag{3-11}$$

Likewise, from Eq. 3-10,

$$C = \frac{L}{R_{in}R}$$

which if inserted in Eq. 3-9 leads to

$$\omega^2 = \frac{R_{in}R}{L^2} - \frac{R^2}{L^2}, \qquad \omega L = \sqrt{R_{in}R - R^2}$$

$$L = \frac{R}{\omega}\sqrt{\frac{R_{in}}{R} - 1} \tag{3-12}$$

where L is the value of inductance needed for the L section to ensure the desired value of load R_{in}, where $R < R_{in}$.

For the case in which $R > R_g$, the L section may be reversed, as in Fig. 3-3. It can then be recognized that the equations just

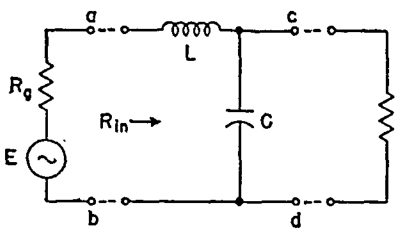


Fig. 3-3. The reversed L section for impedance transformation ($R > R_g$).

developed will apply if R_{in} is substituted for R and R for R_{in}. If $R > R_g$, the transforming L section should contain the following components:

$$C = \frac{1}{\omega R}\sqrt{\frac{R}{R_{\text{in}}} - 1} \tag{3-13}$$

$$L = \frac{R_{\text{in}}}{\omega}\sqrt{\frac{R}{R_{\text{in}}} - 1} \tag{3-14}$$

the value of R_{in} normally being equal to R_g, for matched conditions.

It should be emphasized that because of the resonant properties of the circuits developed, the desired resistive load is obtained only at the one frequency selected.

3-3. Image impedances; reactance matching

Consider a T section of impedances interposed between a generator having internal impedance Z_{1i} and a load of impedance Z_{2i}, as in Fig. 3-4. It is desired that the impedance at the 1,1 terminals, into which the generator supplies power, be equal to the generator

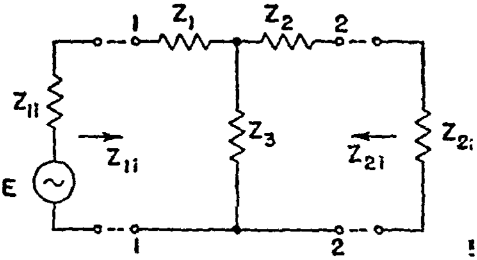


Fig. 3-4. Equivalent network showing image impedances.

impedance, and that the impedance looking into the 2,2 terminals be equal to the load Z_{2i}. Under these conditions the impedance at 1,1 looking in one direction is the image of the impedance looking in the other direction, and Z_{1i} is called an *image impedance* of the network. Likewise, at 2,2 the impedance looking in one direction is the same as that looking in the other, so that Z_{2i} is also an image impedance at the 2,2 terminals. The network is then said to be *matched on an image basis*.

The values of the image impedances of the T section may be computed. The impedance $Z_{1\,\text{in}}$ at the 1,1 terminals is required to be Z_{1i} and is

$$Z_{1\,\text{in}} = Z_{1i} = Z_1 + \frac{Z_3(Z_2 + Z_{2i})}{Z_2 + Z_3 + Z_{2i}}$$

Likewise, the impedance looking into the 2,2 terminals is required to be Z_{2i} and is

$$Z_{2i} = Z_2 + \frac{Z_3(Z_1 + Z_{1i})}{Z_1 + Z_3 + Z_{1i}}$$

Upon solving for Z_{1i} and Z_{2i},

$$Z_{1i} = \sqrt{\frac{(Z_1 + Z_3)(Z_1Z_2 + Z_2Z_3 + Z_3Z_1)}{Z_2 + Z_3}} \tag{3-15}$$

$$Z_{2i} = \sqrt{\frac{(Z_2 + Z_3)(Z_1Z_2 + Z_2Z_3 + Z_3Z_1)}{Z_1 + Z_3}} \tag{3-16}$$

In Chapter 1, networks were analyzed in terms of open- and short-circuit measurements. Making an impedance measurement on the T section at the 1,1 terminals with the 2,2 terminals open gives

$$Z_{1oc} = Z_1 + Z_3$$

A similar measurement at the 1,1 terminals with the 2,2 terminals short-circuited gives

$$Z_{1sc} = Z_1 + \frac{Z_2Z_3}{Z_2 + Z_3}$$

$$= \frac{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}{Z_2 + Z_3}$$

Observation of the equations for the image impedances then shows that

$$Z_{1i} = \sqrt{Z_{1oc}Z_{1sc}} \tag{3-17}$$

Similar measurements made at the 2,2 terminals would lead to

$$Z_{2i} = \sqrt{Z_{2oc}Z_{2sc}} \tag{3-18}$$

These equations show that Z_{1i} and Z_{2i} are readily obtained by simple open- and short-circuit measurements on any network.

Thus a properly designed T network may have the property of transformation of an impedance to produce matching of a load and a source. Since the T network is used only to transfer power from the source to the load, it will operate most efficiently if it be lossless or be made of pure reactances. By proper choice of magnitude and sign of the reactance arms of the T section, Z_{1i} can be made to have any desired value to match the generator at the 1,1 terminals. Maximum power will then be transferred from the generator to the reactance T section. Since the T section consists of pure reactances, no power can be dissipated therein; consequently, the power delivered by the generator must all be transferred to the load. If the maximum possible power output of the generator is delivered to the load, then the load must be considered as matched to the generator. By this reasoning it is possible to develop a theorem, originally due to Everitt, as follows:

If, in a network of pure reactances, terminated in a dissipative load and supplied from a generator having an internal resistance, a conjugate impedance match occurs at one pair of terminals, the impedances will have a conjugate match at every other pair of terminals.

It will frequently be desirable to consider loads and generators as having wholly resistive impedances. A great many of the most common generators are resistive (the carbon-button microphone, or the vacuum tube, for instance), and loads must have resistive components if they are to absorb power. That such treatment does not lack generality is easily seen if it is remembered that any reactance in either generator or load can be canceled by introduction of a properly chosen series-reactive element, tuned to resonance. Such cancellation restricts operation to one frequency; but since reactance T sections can be designed for proper matching at only one frequency, no loss of generality occurs.

3-4. Reactance T networks for impedance transformation

The idea of matching load and source by use of a connecting reactance T network, as proposed in the preceding section, may be further developed in terms of the load and source resistances which are to be matched for maximum power transfer. In Fig. 3-5

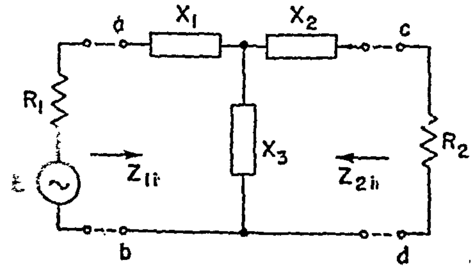


Fig. 3-5. Reactance network for impedance transformation.

is shown a generator of internal resistance R_1 connected to a load R_2 through a T network of pure reactances. If the generator is to transfer maximum power to the load it is necessary only that the image impedance Z_{1i} at terminals a,b be equal to R_1, by the theorem of the preceding section. In effect then, the load resistance R_2 is transformed by the T network to a value, at the a,b terminals, equal to R_1.

With load R_2 connected, the image impedance Z_{1i} must be

$$Z_{1i} = R_1 = jX_1 + \frac{jX_3(R_2 + jX_2)}{R_2 + jX_2 + jX_3} \tag{3-19}$$

where the X may have either $+$ or $-$ signs, or may be either capacitive or inductive. Then

$$R_1R_2 + jR_1(X_2 + X_3) = -(X_1X_2 + X_2X_3 + X_3X_1) + jR_2(X_1 + X_3)$$

By equating the real terms,

$$R_1R_2 = -(X_1X_2 + X_2X_3 + X_3X_1) \tag{3-20}$$

The negative sign on the right appears there as a result of placing all positive signs on the j terms in the original Eq. 3-19. However, both R_1 and R_2 are real and positive, and the bracket on the right must be positive. Thus one or more of the terms on the right must be positive, and the only manner in which the product of two reactances can be positive is for one of the reactances to be opposite in sign to the other, that is: $jX_a(-jX_b) = +X_aX_b$. To force this condition, and thus a positive sign on the bracket on the right side of Eq. 3-20, requires that *one reactive arm of the* T *network be opposite in sign to the sign of the other two arms.* That is, the T network must actually be composed of one capacitance and two inductances, or vice versa. This fixes one design condition of the network.

By equating the imaginary terms

$$R_1(X_2 + X_3) = R_2(X_1 + X_3)$$

$$X_2 + X_3 = \frac{R_2}{R_1}(X_1 + X_3) \tag{3-21}$$

Equation 3-20 may be rewritten

$$R_1R_2 = -[(X_1 + X_3)(X_2 + X_3) - X_3^2] \tag{3-22}$$

and substitution of Eq. 3-21 leads to

$$R_1R_2 = -\left[(X_1 + X_3)^2 \frac{R_2}{R_1} - X_3^2\right]$$

$$X_1 + X_3 = \pm \sqrt{\frac{R_1}{R_2}(X_3^2 - R_1R_2)} \tag{3-23}$$

from which the value for one of the reactance arms is

$$X_1 = -X_3 \pm \sqrt{\frac{R_1}{R_2}(X_3^2 - R_1R_2)} \tag{3-24}$$

Substitution of Eq. 3-24 into 3-22 gives

$$X_2 + X_3 = \pm \frac{R_2}{R_1}\sqrt{\frac{R_1}{R_2}(X_3^2 - R_1R_2)}$$

$$X_2 = -X_3 \pm \sqrt{\frac{R_2}{R_1}(X_3^2 - R_1R_2)} \tag{3-25}$$

for the second reactance arm.

Equations 3-24 and 3-25 supply values for the X_1 and X_2 arms of the T, in terms of the third arm X_3. Since the $\pm$ sign of Eq. 3-24 leads directly to the $\pm$ sign of Eq. 3-25, the choice of $+$ or $-$ in the expression for X_1 requires a similar choice in the expression for X_2, and only *two* matching T networks exist. Since there are three unknowns and only two equations, it is necessary to assume a value for one of the reactance arms, after which the other two are readily determined.

A convenient method is to choose X_3. Another choice is to assume X_1 or X_2 equal to zero, thereby determining X_3. The latter choice is also an economical assumption, since it permits the T to be built with two reactances instead of three. If X_3 is then chosen capacitive, the circuit reduces to the L section of Section 3-2, and it will be found that Eqs. 3-24 and 3-25 reduce to the equations of that section. If X_3 is chosen inductive, the results compare with Section 2-9 with $R_1 = 0$, the load being in series with a capacitive X_2.

If reactances have been placed in series with the generator or load to cancel internal reactance, arms X_1 or X_2 may be chosen as of equal magnitude and opposite sign to that of the reactance with which they are in series. In such a case, although electrically X_1 or X_2 is present, yet physically they are absent, since the net arm reactance would be zero.

As a special case, X_3 may be so chosen that

$$X_3 = \pm \sqrt{R_1 R_2} \tag{3-26}$$

which leads to the condition that

$$|X_1| = |X_2| = |X_3| \tag{3-27}$$

the sign of the mutual reactance being opposite to that of the other two. This is the condition of *critical coupling*. Under this condition there is only one possible value of X_1 and X_2, and only one matching T section may be built. When $X_3^2 > R_1R_2$, two T sections are possible, as mentioned above.

If $X_3^2 < R_1R_2$, the radicals in Eqs. 3-24 and 3-25 become imaginary. If a value of X_3 less than that specified by Eq. 3-26 is chosen, power will be transferred from generator to load; but it will not be the maximum possible value of power, or the impedances will not be matched, and the coupling is said to be *insufficient*.

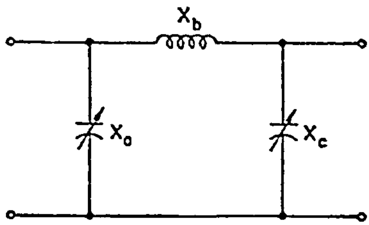


Fig. 3-6. The π-section impedance-transforming section as used at radio frequencies.

By T to π transformations, the T networks designed by the equations of this section may readily be transformed to π sections. If given the form of Fig. 3-6 by choice of X_b as inductance and X_a and X_c as capacitances, the circuit becomes useful for coupling a transmitter to its load at radio frequencies. The ease of tuning contributed by the two adjustable capacitors and the possibility of grounding the rotors of the variable capacitors are helpful. The circuit also has valuable properties in suppression of certain frequencies, as will be considered in Chapter 4.

3-5. Coupled circuits

A two-mesh network may be represented as in Fig. 3-7 and the mesh equations written as

$$E = I_1(Z_1 + Z_3) - I_2Z_3$$
$$0 = -I_1Z_3 + I_2(Z_2 + Z_3)$$

which, if the definitions of mesh and mutual impedance are introduced as

$$Z_{11} = Z_1 + Z_3, \qquad Z_{22} = Z_2 + Z_3, \qquad Z_{12} = -Z_3$$

results in the impedance matrix for the circuit

$$\begin{bmatrix} Z_{11}, & Z_{12} \\ Z_{12}, & Z_{22} \end{bmatrix}$$

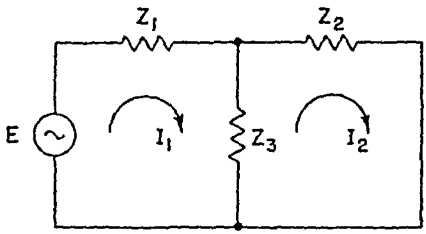


Fig. 3-7. A two-mesh coupled circuit.

The determinant of the matrix is

$$\Delta = Z_{11}Z_{22} - Z_{12}^2 \tag{3-28}$$

and after application of the rules of Section 1-20, the driving point or input impedance of mesh 1 may be determined as

$$\begin{aligned} Z_{1\,\text{in}} &= \frac{Z_{11}Z_{22} - Z_{12}^2}{Z_{22}} = \frac{\Delta}{\Delta_{11}} \\ &= Z_{11} - \frac{Z_{12}^2}{Z_{22}} \end{aligned} \tag{3-29}$$

Equation 3-29 shows that the input impedance of the first mesh is modified by the presence of the second mesh because of the term $-Z_{12}^2/Z_{22}$.

The transfer impedance may also be readily determined as

$$\begin{aligned} Z_{T12} &= \frac{Z_{11}Z_{22} - Z_{12}^2}{-Z_{12}} = \frac{\Delta}{\Delta_{12}} \\ &= Z_{12} - \frac{Z_{11}Z_{22}}{Z_{12}} \end{aligned} \tag{3-30}$$

The two meshes are said to be *coupled* by the presence of the impedance Z_3 common to both meshes. This impedance might take any form, such as resistance, self-inductance, capacitance, or the mutual inductance common to two coils. Equations 3-29 and 3-30 are perfectly general and apply to all such cases.

3-6. Mutual inductance

If the current through an inductance varies with time, the flux through the coil likewise varies with time and an emf is induced in the coil. The direction of this *self-induced* emf is such as to oppose the circuit emf if the current is increasing or to add to the circuit emf if the current is decreasing. Experimentally, it has been found that the emf of self-induction is proportional to the rate of change of the coil current, or

$$e = -L\frac{di}{dt} \quad \text{volts} \tag{3-31}$$

where L is the constant of proportionality and is called the self-inductance of the coil. The minus sign expresses mathematically the statement in the second sentence of this paragraph.

A form of Faraday's law is

$$e = -N\frac{d\phi}{dt} \tag{3-32}$$

which is one of the basic laws of electromagnetics. Equations 3-31 and 3-32 may be combined, giving

$$N\frac{d\phi}{dt} = L\frac{di}{dt}$$

$$L = N\frac{d\phi}{di} \quad \text{henrys} \tag{3-33}$$

as a definition of inductance in terms of the rate of change of flux with respect to current, N being constant.

In Fig. 3-8, if current i_1 varies with time, then flux ϕ_1 varies with time. Flux ϕ_{12}, the portion of ϕ_1 that passes through the second

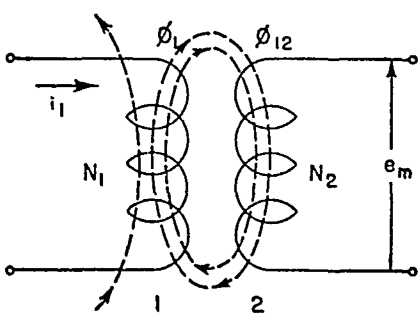


Fig. 3-8. Two coils, magnetically coupled. Flux ϕ_1 is the total flux in N_1; flux ϕ_{12} is the total flux in N_2.

coil of N_2 turns, likewise will vary with time. Flux ϕ_1 will produce a self-induced voltage e_1 in N_1 and flux ϕ_{12} will induce an emf in N_2. The emf e_{12} induced in N_2 is called an emf of *mutual induction*, since it is an emf produced in a second circuit by a change of current in a first circuit. The mutual emf is proportional to the rate of change of current i_1 in the inducing circuit, or

$$e_{12} = -M_{12}\frac{di_1}{dt} \quad \text{volts} \tag{3-34}$$

The constant of proportionality M_{12} is called the *mutual inductance* between coils N_1 and N_2.

As before, if the rate of change of flux ϕ_{12} is considered,

$$e_{12} = -N_2\frac{d\phi_{12}}{dt} \quad \text{volts}$$

from which

$$M_{12} = N_2\frac{d\phi_{12}}{di_1} \quad \text{henrys} \tag{3-35}$$

which shows the mutual inductance proportional to the rate of change of flux in N_2 with respect to the change in current i_1.

The use of mutual inductance in the analysis of a circuit is illustrated in Fig. 3-9. Two coils are shown coupled by a magnetic flux

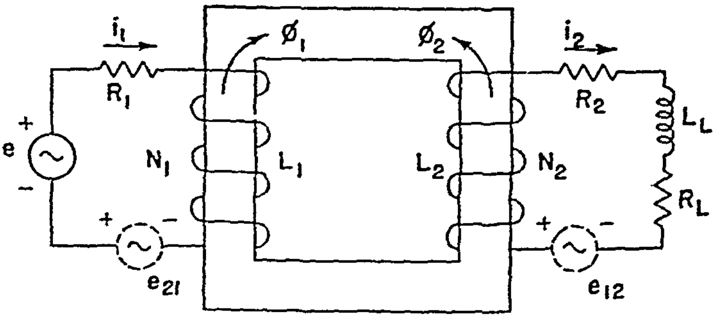


Fig. 3-9. Two coils, magnetically coupled. The core is indicated only for case of analysis; it is not necessary.

path, with currents i_1 and i_2 *assumed* to have the reference directions indicated. The emf's e_{12} and e_{21} are fictitious generators to be discussed below.

A circuit equation for the first mesh may be written in terms of instantaneous currents and voltages as

$$e = R_1 i_1 + L_1 \frac{di_1}{dt} \pm M_{12} \frac{di_2}{dt} \tag{3-36}$$

where the last term represents an emf induced in the first mesh due to a changing current in the second mesh.

The sign of the mutual voltage term may be determined readily. According to the right-hand rule, magnetic flux ϕ_2 is produced in the direction indicated by the assumed current i_2. Also, flux ϕ_1 is produced in the indicated direction by the assumed current i_1. If ϕ_{21} is considered the portion of flux ϕ_2 which couples with or passes down through coil N_1, then ϕ_{21} must induce an emf e_{21} in N_1 that will cause a component of current to flow in N_1 which produces an opposite or upward flux, by Lenz's law. This induced emf e_{21} may be considered as that of a fictitious generator in series in the first mesh as shown, where the magnitude of e_{21} is

$$e_{21} = M_{21} \frac{di_2}{dt}$$

By the right-hand rule the current produced by the fictitious generator e_{21} must flow downward in coil N_1 to produce an upward flux. The polarity of the generator must then be as indicated to produce this current, and e_{21} will appear as a rise in voltage, or as a negative drop. To write it as a drop the sign of the mutual voltage term must be negative in Eq. 3-36, as

$$e = R_1 i_1 + L_1 \frac{di_1}{dt} - M_{21} \frac{di_2}{dt} \tag{3-37}$$

In similar manner, the portion of magnetic flux ϕ_1 which couples with or passes through N_2 may be called ϕ_{12}. Then flux ϕ_{12} in passing down through coil N_2 must induce an emf e_{12} which will cause an upward current and an upward flux, opposing ϕ_{12}. This current is produced by an emf such as the fictitious generator e_{12} with polarity as indicated. The emf equation for the second mesh can then be written as

$$0 = (R_2 + R_L) i_2 + (L_2 + L_L) \frac{di_2}{dt} - M_{12} \frac{di_1}{dt} \tag{3-38}$$

the mutual emf again being written as a potential drop. The emf

$-M_{12}\, di_1/dt$ may also be considered the source voltage for the current flowing in the second mesh.

From the reasoning given, if the reference direction assumed for i_2 were to be reversed, the right-hand rule would require reversal of ϕ_2 and ϕ_{21}, and the polarity of generator e_{21} would reverse, resulting in a positive sign for the mutual voltage terms $M_{21}\, di_2/dt$ and $M_{12}\, di_1/dt$. If the direction of winding of either coil were reversed, the direction of flux would change and call for a readjustment of the sign of the mutual voltage terms. Therefore the sign attached to these terms, and by convention to M, is a function of the *assumed* directions of current and of the manner in which the coils are wound.

It is possible to formulate a convention for ready determination of this sign as follows:

After assumption of the positive or reference direction of current in a magnetically coupled circuit, *the mutual voltage term is given a positive sign if the positive currents produce fluxes in the same direction through the coil; the mutual voltage is given a negative sign if the positive currents produce fluxes in opposite directions through the coil.*

That is, if the fluxes oppose, e_{21} and e_{12} will appear as rises in voltage: if the fluxes add, e_{21} and e_{12} will appear as drops in voltage.

If the assumed currents are sinusoidal, Eqs. 3-37 and 3-38 may be written in terms of effective current as

$$\begin{bmatrix} E \\ 0 \end{bmatrix} = \begin{bmatrix} R_1 + j\omega L_1; & -j\omega M \\ -j\omega M; & R_2 + R_L + j\omega(L_2 + L_L) \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \tag{3-39}$$

If μ is constant, $M_{12} = M_{21}$ and may be represented by M; then it appears that

$$\begin{aligned} Z_{11} &= R_1 + j\omega L_1 \\ Z_{22} &= R_2 + R_L + j\omega(L_2 + L_L) \\ Z_{12} &= -j\omega M \end{aligned}$$

from which the matrix equation becomes

$$\begin{bmatrix} E \\ 0 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \tag{3-40}$$

Solution for the two currents follows since $\Delta = Z_{11}Z_{22} - Z_{12}^2$.

Then

$$I_1 = \frac{\Delta_1}{\Delta} = \left(\frac{Z_{22}}{Z_{11}Z_{22} - Z_{12}^2}\right) E \tag{3-41}$$

$$I_2 = \frac{\Delta_2}{\Delta} = \left(\frac{-Z_{12}}{Z_{11}Z_{22} - Z_{12}^2}\right) E \tag{3-42}$$

It should be noted that the terms in parentheses are the reciprocals of $Z_{1\,\text{in}}$ and Z_{T12} of Eqs. 3-29 and 3-30, respectively, so that the currents could have been obtained directly by the use of the input and transfer impedances of Section 3-5. A magnetically coupled circuit is then seen to fall into the general class of coupled circuits considered in Section 3-5, with ωM as the coupling or mutual reactance.

3-7. Coefficient of coupling

From the preceding section

$$\frac{M_{12}}{L_1} = \frac{N_2}{N_1}\frac{d\phi_{12}/di_1}{d\phi_1/di_1} = \frac{N_2}{N_1}\frac{d\phi_{12}}{d\phi_1} \tag{3-43}$$

If all the flux due to i_1 in coil N_1 also links all the turns of coil N_2, or there is no *leakage* flux, then

$$\phi_1 = \phi_{12} \quad \text{and} \quad d\phi_1 = d\phi_{12}$$

so that

$$\frac{M_{12}}{L_1} = \frac{N_2}{N_1} = \frac{1}{a} \tag{3-44}$$

where a is known as the *turns ratio.*

If, however, only a fraction k_1 of the flux ϕ_1 links with the turns of N_2, then $\phi_{12} = k_1\phi_1$ and

$$\frac{M_{12}}{L_1} = k_1\frac{N_2}{N_1} \tag{3-45}$$

Expressions similar to Eq. 3-43 may be written for the second coil N_2 as

$$\frac{M_{21}}{L_2} = \frac{N_1}{N_2}\frac{d\phi_{21}/di_2}{d\phi_2/di_2} = \frac{N_1}{N_2}\frac{d\phi_{21}}{d\phi_2} \tag{3-46}$$

If there is no leakage flux and $\phi_{21} = \phi_2$,

$$\frac{M_{21}}{L_2} = \frac{N_1}{N_2} = a$$

but if only a fraction k_2 of the flux ϕ_2 links with the turns of N_1, then $\phi_{21} = k_2\phi_2$ and

$$\frac{M_{21}}{L_2} = k_2 \frac{N_1}{N_2} \tag{3-47}$$

Equations 3-45 and 3-47 may be combined as

$$k_1k_2 = \frac{M_{12}M_{21}}{L_1L_2}$$

As before, if the circuit is linear or μ is constant, then $M_{12} = M_{21} = M$. Then by defining a *coefficient of coupling* as $k = \sqrt{k_1k_2}$,

$$k = \frac{M}{\sqrt{L_1L_2}} \tag{3-48}$$

The coefficient of coupling is usually expressed as a decimal or a percentage, having a maximum value of unity or 100 per cent. It can be seen that k is a function of the excellence of the magnetic path between N_1 and N_2, being usually small if this path is air, large if it is of good magnetic steel and the leakage flux is small.

Under conditions of zero leakage flux and $M_{12} = M_{21}$, Eqs. 3-45 and 3-47 may be combined to give

$$\frac{L_1}{L_2} = \frac{N_1^2}{N_2^2} = a^2 \tag{3-49}$$

which is very nearly correct for iron-cored transformers of good design.

If two magnetically coupled coils are connected in series, the total inductance between terminals may be measured as

$$L = L_1 \pm M + L_2 \pm M$$

where the signs on the M terms will be alike, as discussed above, and will depend on the relative winding directions. The value of M may be determined by subtraction of the two possible values, L_A and L_B, which may be measured, so that

$$M = \frac{L_A - L_B}{4} \tag{3-50}$$

The leakage inductance L_{11} is due to the portion of the flux ϕ_1

that does not link the turns of N_2. This leakage flux for the primary is

$$\phi_{11} = (1 - k)\phi_1$$

and likewise, the secondary leakage flux is

$$\phi_{22} = (1 - k)\phi_2$$

The primary leakage inductance L_{11} is then

$$L_{11} = N_1 \frac{d[(1 - k)\phi_1]}{di_1}$$

and since $L_1 = N_1\, d\phi_1/di_1$,

$$L_{11} = (1 - k)L_1 = L_1 - \sqrt{\frac{L_1}{L_2}}\, M \tag{3-51}$$

Similarly, the secondary leakage inductance is

$$L_{22} = (1 - k)L_2 = L_2 - \sqrt{\frac{L_2}{L_1}}\, M \tag{3-52}$$

relating the leakage inductances to the mutual inductance.

3-8. Equivalent T network for the magnetically coupled circuit

For convenience in circuit analysis it is desirable to be able to replace a magnetically coupled circuit, such as (a), Fig. 3-10, with an equivalent T network. Neglecting losses in any iron present, and with μ constant, the mesh equations for a short-circuited secondary appear as

$$\begin{bmatrix} E \\ 0 \end{bmatrix} = \begin{bmatrix} R_1 + j\omega L_1; & -j\omega M \\ -j\omega M; & R_2 + j\omega L_2 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \tag{3-53}$$

Similarly the T section at (b) will have mesh equations as

$$\begin{bmatrix} E \\ 0 \end{bmatrix} = \begin{bmatrix} Z_1 + Z_3; & -Z_3 \\ -Z_3; & Z_2 + Z_3 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \tag{3-54}$$

For equivalence of circuits (a) and (b):

$$\begin{aligned} Z_1 + Z_3 &= R_1 + j\omega L_1 \\ Z_2 + Z_3 &= R_2 + j\omega L_2 \\ Z_3 &= j\omega M \end{aligned}$$

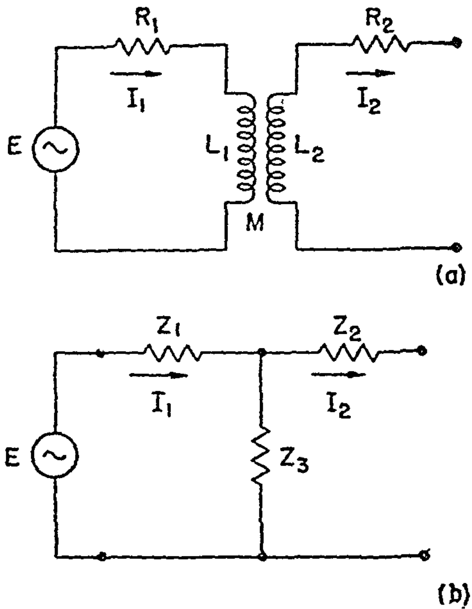


Fig. 3-10. (a) A magnetically coupled circuit; (b) a T section equivalent to (a).

It can then be found that the equivalent T network must have

$$\left.\begin{aligned} Z_1 &= R_1 + j\omega(L_1 - M) \\ Z_2 &= R_2 + j\omega(L_2 - M) \\ Z_3 &= j\omega M \end{aligned}\right\} \tag{3-55}$$

as values for its arms. Consequently the T network will have the appearance of Fig. 3-11, and will be equivalent to the magnetically coupled circuit at a given frequency. The equivalent is of considerable aid in analyzing circuits including transformers.

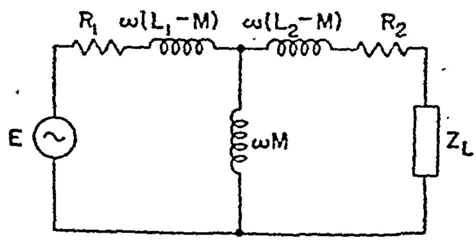


Fig. 3-11. A T section equivalent for the magnetically coupled circuit of (a), Fig. 3-9.

The circuit can be recognized as a form of the impedance-transforming T network, although it will meet the required conditions fully only if k is relatively large. In that case, since M will be nearly the geometric mean of L_1 and L_2, M will be larger than one of the inductances, $(L_1 \neq L_2)$; therefore either $L_1 - M$ or $L_2 - M$ will result in a reactance of sign opposite to that of the other two arms, a condition of the impedance-transforming T network.

Before using this equivalent circuit for an iron-cored transformer an investigation of the magnitude of the core losses should be made. In many audio transformers these losses are not negligible and must be considered as a resistance added in shunt to M.

3-9. Iron-core transformers; the ideal transformer

At audio frequencies it is customary to use an iron-core transformer for transformation of impedances. It is possible to design these transformers with k large, usually 0.97 or higher. In that case

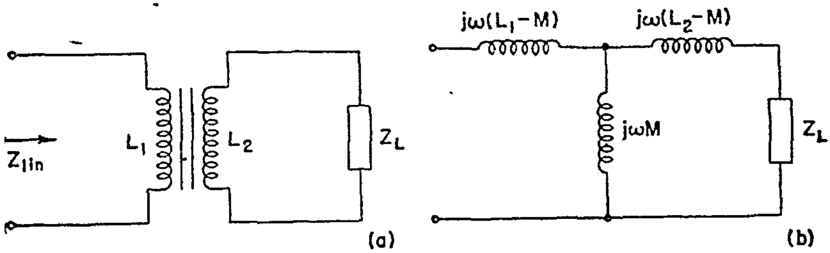


Fig. 3-12. (a) The ideal transformer and load; (b) the equivalent T network for (a).

the argument in the next to the last paragraph of Section 3-8 holds, and a T section is obtained that will approximate the requirements of Eqs. 3-24 and 3-25 without the use of capacitors. Since all arms of the equivalent T network of Fig. 3-12 change in like fashion with frequency, an approximate impedance match with the iron-core transformer is possible over rather wide frequency ranges. The frequency range is limited by eddy current and hysteresis losses in the iron at the higher frequencies, and also by the effect of various winding capacitances.

The analysis of the iron-core transformer for impedance transformation use is customarily carried out by the use of the concept of the *ideal transformer*. An ideal transformer is assumed to have:

(1) Zero losses.

(2) Primary and secondary winding reactances very large with respect to any connected impedance.

(3) No leakage flux, or $k = 1$.

In view of the ease with which transformers closely approaching these conditions can be built, the assumptions are reasonable. With values of k above 0.97, condition (3) is easily met; and with efficiencies of 90 per cent or better in many transformers, (1) is satisfied. Item (2) is entirely within the range of choice of the designer.

Given an ideal transformer and load as in (a), Fig. 3-12, with the T network equivalent drawn at (b), the winding resistances being considered zero. The mesh impedances may be written

$$Z_{11} = 0 + j\omega L_1 \tag{3-56}$$

$$Z_{22} = R_L + j\omega(L_2 + L_L) \tag{3-57}$$

$$Z_{12} = j\omega M \tag{3-58}$$

The driving-point impedance of a coupled circuit is

$$Z_{1\,\text{in}} = Z_{11} - \frac{Z_{12}^2}{Z_{22}}$$

Since $k = 1$ for the ideal transformer,

$$M^2 = L_1L_2$$

and substitution of this condition and the above equations into the expression for driving-point impedance gives

$$Z_{1\,\text{in}} = j\omega L_1 + \frac{\omega^2 L_1 L_2}{R_L + j\omega(L_2 + L_L)} \tag{3-59}$$

Upon rationalization, this yields

$$Z_{1\,\text{in}} = \frac{\omega^2 L_1 L_2 R_L}{R_L^2 + (\omega L_2 + \omega L_L)^2} + j\left[\omega L_1 - \frac{\omega^2 L_1 L_2(\omega L_2 + \omega L_L)}{R_L^2 + (\omega L_2 + \omega L_L)^2}\right]$$

Neglecting R_L^2 in comparison to $(\omega L_2 + \omega L_L)^2$, since the requirement is for ωL_2 to be large,

$$Z_{1\,\text{in}} = \frac{\omega^2 L_1 L_2 R_L + j\omega^2 L_1 L_L(\omega L_2 + \omega L_L)}{(\omega L_2 + \omega L_L)^2}$$

However, it is also required that $\omega L_2 \gg \omega L_L$, so that

$$Z_{1\,\text{in}} = (R_L + j\omega L_L)\frac{L_1}{L_2} \tag{3-60}$$

Since $k = 1$, by Eq. 3-49

$$\frac{L_1}{L_2} = \left(\frac{N_1}{N_2}\right)^2 = a^2$$

giving
$$Z_{1\,\text{in}} = a^2 Z_L \tag{3-61}$$

This series of operations shows that the ideal transformer *changes the magnitude, but not the phase angle, of an impedance.* The transformation is proportional to the square of the turns ratio of the transformer.

Since the transformer is assumed to have no losses, the power input equals the power output, or

$$I_1^2 Z_{1\,\text{in}} \cos\theta = I_2^2 Z_L \cos\theta$$

from which the ratio of currents is obtained as

$$\left|\frac{I_1}{I_2}\right| = \frac{1}{a} \tag{3-62}$$

Then since
$$\frac{I_1 Z_{1\,\text{in}}}{I_2 Z_L} = \frac{V_1}{V_2}$$

the voltage ratio is
$$\left|\frac{V_1}{V_2}\right| = a \tag{3-63}$$

Iron-core transformers are used extensively for impedance transformation in audio-frequency apparatus.

3-10. Singly tuned air-core transformers

In radio-circuit design the air-core transformer network of Fig. 3-13 is frequently employed. Because of the air path for the magnetic circuit, the leakage flux is large, making the value of k small. In radio practice this circuit may be employed either for obtaining the greatest transfer of power to the load R_2 or for developing the greatest possible value of secondary voltage E_2.

For analysis, the circuit can be replaced by its equivalent T network. Since, for the T network to achieve an impedance match, both magnitude and phase angle of the load must be transformed, two adjustable T-section parameters are necessary. Equations 3-24 and 3-25 also demonstrate that two adjustable elements are necessary for an impedance match.

Usually, for simplicity of operation, X_m is fixed and only one adjustment, X_{C_2}, is available. Adjustment of phase is then possible

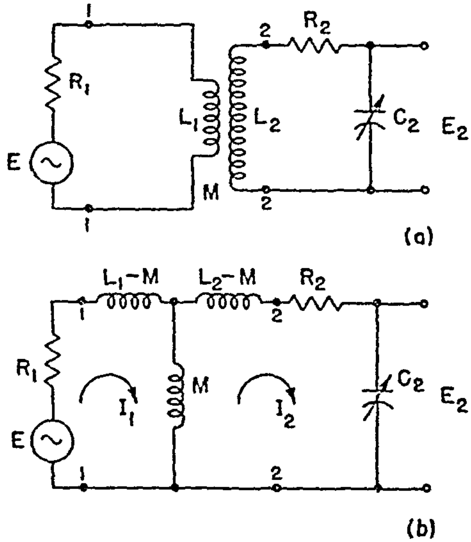


Fig. 3-13. (a) Singly-tuned air-core transformer with parallel tuning; (b) equivalent network for (a).

(making the circuit appear resistive to a resistive generator), but a simultaneous magnitude transformation is impossible. The condition is then referred to as *partial resonance.*

The secondary-tuned circuit of Fig. 3-13(a) may be analyzed for greatest power-transfer conditions under the assumption that C_2 is the only available adjustment (although M may be chosen during design). By Eq. 3-29, the input impedance at terminals 2,2 in (b), and looking to the left is

$$Z_{2\,\text{in}} = Z_{22} - \frac{Z_{12}^2}{Z_{11}}$$

if Z_{22} is taken as applying to the I_2 mesh and Z_{11} as applying to the I_1 mesh. The impedance looking to the left at 2,2 is then

$$Z_{2\,\text{in}} = \frac{\omega^2 M^2 R_1}{R_1^2 + \omega^2 L_1^2} + j\left(\omega L_2 - \frac{\omega^2 M^2}{R_1^2 + \omega^2 L_1^2}\,\omega L_1\right)$$

To give I_2 its greatest value, and thus to transfer the greatest power to R_2, the reactive term may be made zero by adjustment of C_2. To cancel the circuit reactance, the reactance of C_2 should be made equal to

$$\frac{1}{\omega C_2} = \omega L_2 - \frac{\omega^2 M^2}{R_1^2 + \omega^2 L_1^2}\omega L_1 \tag{3-64}$$

A much more common usage of the circuit of Fig. 3-13 is as a coupling between vacuum-tube amplifiers in radio receiver circuits. In such applications no appreciable power need be transferred, so that maximum power conditions need not be considered. Instead,

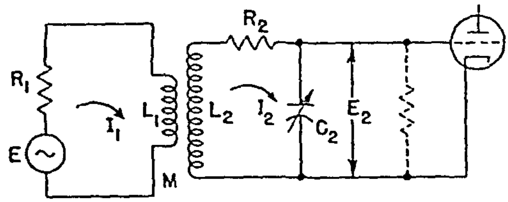


Fig. 3-14. Singly-tuned circuit as applied to radio-frequency coupling in a vacuum tube circuit.

it is desired that the maximum possible secondary voltage E_2 be developed. In this application, resistance R_1 may be considered as the plate resistance of the preceding vacuum-tube amplifier. The resistance of coil L_1 may be added to R_1 or neglected, since the plate resistance R_1 is usually large. The resistance of the secondary winding constitutes the resistance R_2, although at high frequencies the input admittance of the grid-cathode circuit of the vacuum tube should be placed in shunt with the circuit, as indicated by the dashed lines of Fig. 3-14.

The vacuum-tube grid voltage is

$$E_2 = \frac{-jI_2}{\omega C_2} \tag{3-65}$$

The secondary current I_2 can be obtained as

$$I_2 = \frac{E}{Z_{T12}} = \frac{-j\omega ME}{\omega^2 M^2 + (R_1 + j\omega L_1)\left[R_2 + j\left(\omega L_2 - \dfrac{1}{\omega C_2}\right)\right]}$$

$$= \frac{-j\omega ME}{\omega^2 M^2 + R_1 R_2 - \omega^2 L_1 L_2 + \dfrac{L_1}{C_2} + j\left[\omega L_1 R_2 + R_1\left(\omega L_2 - \dfrac{1}{\omega C_2}\right)\right]} \tag{3-66}$$

The secondary current, and therefore the voltage E_2, will have its largest value if C_2 is so adjusted as to neutralize the reactance term in the denominator. Then

$$\frac{1}{\omega C_2} = \omega\left(L_2 + \frac{R_2}{R_1}L_1\right) \tag{3-67}$$

The term $(R_2/R_1)L_1$ may be considered as the effect of the primary circuit in altering the secondary circuit inductance. Since in vacuum-tube amplifiers $R_1 \gg R_2$, Eq. 3-67 states that the secondary circuit is nearly in resonance. This condition is often assumed in such circuits and will be so assumed here, thus:

$$\omega L_2 = \frac{1}{\omega C_2}$$

Equation 3-66, under the assumptions and the condition of Eq. 3-67, then becomes

$$I_2 = \frac{-j\omega ME}{\omega^2 M^2 + R_1 R_2} \tag{3-68}$$

which lags the source E by 90°. The secondary voltage is then

$$E_2 = \frac{-ME}{C_2(\omega^2 M^2 + R_1 R_2)} = \frac{-\omega^2 M L_2 E}{\omega^2 M^2 + R_1 R_2} \tag{3-69}$$

By maximizing the expression with respect to ωM, the largest value of E_2 is found to occur when

$$\omega M = \sqrt{R_1 R_2} \tag{3-70}$$

which is the condition for *critical coupling.*

Since the coefficient of coupling k may be written

$$k = \frac{\omega M}{\omega\sqrt{L_1 L_2}}$$

the coefficient for critical coupling is

$$k_c = \frac{\sqrt{R_1 R_2}}{\omega\sqrt{L_1 L_2}} \tag{3-71}$$

The secondary voltage for critical coupling is then

$$E_2 = \frac{-\omega L_2 E}{2\sqrt{R_1 R_2}} = \frac{-E\sqrt{L_2/L_1}}{2k_c} \tag{3-72}$$

The designer may maximize this voltage through choice of the value of M. In addition the voltage is a function of the ratio L_2/L_1 and this ratio may be varied within limits.

Thus Eq. 3-69 represents the situation when only one adjustment C_2 is available in the equivalent matching T network; that is, the phase angle has been made resistive, but a magnitude match between generator and load has not been achieved. Equation 3-72 shows the secondary voltage obtainable if a second adjustment, that of X_m, has been made to achieve a magnitude match as well.

3-11. Doubly-tuned air-core transformer

If both primary and secondary circuits have adjustable capacitive reactance, as in Fig. 3-15(a), then it is possible to design the circuit

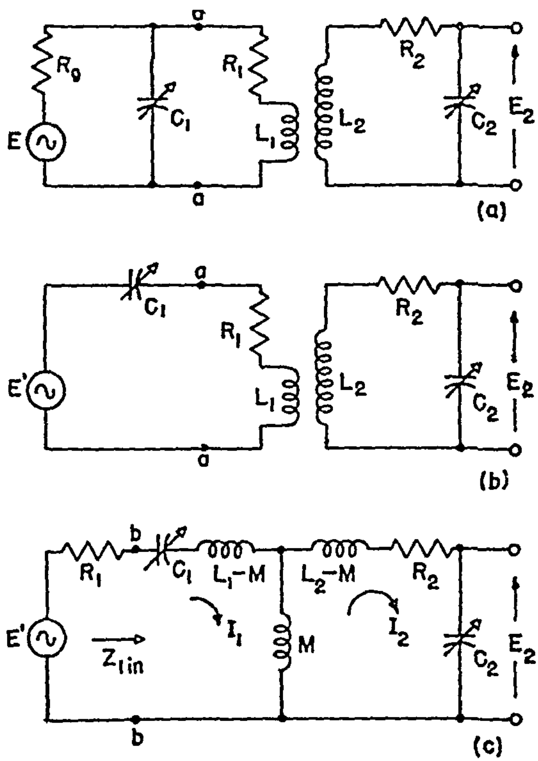


Fig. 3-15. The doubly-tuned transformer.

so that an impedance match is obtained if the coupling is sufficient, or critical or above. The capacitive reactances may be used to adjust the phase angle, making the impedance facing the generator resistive, and the mutual inductance adjusted to the critical value to achieve a magnitude match. The maximum power transfer conditions of Section 3-4 are then fulfilled, and it is said that *optimum resonance* occurs.

Resistance R_1 is that of the primary winding L_1, R_2 is that of the secondary inductor L_2, or R_2 may include resistance introduced as a load. The circuit of (a) may be simplified by using Thevenin's theorem on the portion to the left of terminals a,a. The circuit reduces to that of (b), where the equivalent generator potential E' is

$$E' = \frac{-j}{\omega C_1 R_g - 1} E$$

However, usual values in electronic circuits make $R_g \gg 1/\omega C_1$ so that to a very good approximation

$$E' = \frac{1}{j\omega C_1 R_g} E \tag{3-73}$$

The internal impedance of the equivalent generator would be

$$Z' = \frac{R_g(1/j\omega C_1)}{R_g + (1/j\omega C_1)} = \frac{R_g}{1 + j\omega C_1 R_g}$$

and under the same inequality this becomes

$$Z' = \frac{-j}{\omega C_1} \tag{3-74}$$

thus making the generator internal impedance that of C_1. The series equivalent circuit so obtained appears in (b), Fig. 3-15, and at (c) with the equivalent T network replacing the transformer.

Maximum secondary current I_2 will give both maximum power transfer to the secondary load R_2, and also maximum secondary voltage E_2. Use of the transfer impedance Z_{T12} allows the current I_2 to be obtained as

$$I_2 = \frac{E'}{Z_{T12}} = -\frac{Z_{12}E'}{Z_{11}Z_{22} - Z_{12}^2} \tag{3-75}$$

where
$$Z_{11} = R_1 + j\left(\omega_1 L - \frac{1}{\omega C_1}\right)$$
$$Z_{12} = j\omega M$$
$$Z_{22} = R_2 + j\left(\omega L_2 - \frac{1}{\omega C_2}\right)$$

for the circuit of (c), Fig. 3-15. Current I_2 will be greatest if $|Z_{11}|$ and $|Z_{22}|$ are reduced by resonating primary and secondary such that $\omega_r L_1 = 1/\omega_r C_1$ and $\omega_r L_2 = 1/\omega_r C_2$. Then

$$I_2 = \frac{-j\omega_r M E'}{R_1 R_2 + \omega_r^2 M^2} \tag{3-76}$$

which for critical coupling or $\omega M = \sqrt{R_1 R_2}$, becomes

$$I_2 = \frac{-jE'}{2\sqrt{R_1 R_2}} \tag{3-77}$$

as the value of I_2 giving greatest power transfer.

The first mesh current, at resonance, is

$$I_1 = \frac{E'}{Z_{1\,\text{in}}} = \frac{E'}{Z_{11} - Z_{12}^2/Z_{22}} \tag{3-78}$$

$$= \frac{R_2 E'}{R_1 R_2 + \omega_r^2 M^2} \tag{3-79}$$

which, for critical coupling, reduces to

$$I_1 = \frac{E'}{2R_1} \tag{3-80}$$

It is apparent from Eqs. 3-77 and 3-80 that at resonance current I_2 lags I_1 by 90°. This fact has led to several applications in radio circuits.

If considerations of secondary output voltage are dominant, it may be more convenient to write the general result of Eq. 3-76 in terms of the secondary voltage $E_2 = -jI_2/\omega_r C_2$ as

$$E_2 = \frac{-E'M/C_2}{\omega_r^2 M^2 + R_1 R_2}$$

Since $\omega_r L_2 = 1/\omega_r C_2$,

$$E_2 = \frac{-\omega_r^2 M L_2 E'}{\omega_r^2 M^2 + R_1 R_2} = \frac{-\omega_r^2 M L_2 E'/R_1 R_2}{\omega_r^2 M^2/R_1 R_2 + 1} \tag{3-81}$$

If Q_1 for the primary circuit at resonance (which is equal to the Q of the primary coil since R_g was neglected as large) is written as

$$Q_1 = \frac{\omega_r L_1}{R_1}$$

and Q_2 for the secondary circuit at resonance as

$$Q_2 = \frac{\omega_r L_2}{R_2}$$

(which is the Q of the secondary coil if R_2 is the resistance of that coil). Then since $k = M/\sqrt{L_1 L_2}$,

$$k^2 L_1 L_2 = M^2$$

and

$$k^2 Q_1 Q_2 = \frac{\omega_r^2 M^2}{R_1 R_2}$$

which allows Eq. 3-81 to be written in terms of the Q of the circuits as

$$E_2 = \frac{-kQ_1Q_2\sqrt{L_2/L_1}\,E'}{k^2Q_1Q_2 + 1} \tag{3-82}$$

Maximizing this expression with respect to k shows that maximum E_2 will occur for

$$k = k_c = \frac{1}{\sqrt{Q_1Q_2}} \tag{3-83}$$

Use of the definitions for k and Q reduces this to $\omega M = \sqrt{R_1R_2}$, showing that maximum E_2 will occur for k equal to k_c, the value of critical coupling. Hence maximum secondary voltage and maximum power both occur at critical coupling. Using Eq. 3-83 in 3-82 gives

$$E_2 = -\frac{Q_1Q_2}{2}\frac{L_2}{L_1}E' \tag{3-84}$$

as a simplified expression for the maximum value of E_2. High values of Q_1, Q_2, and L_2 are then seen to be important in contributing to a high secondary voltage, with coupling equal to critical.

3-12. Band width with insufficient coupling

If the maximum secondary voltage or power is desired, the coupling should be critical or overcritical, in the latter case being termed *sufficient.* However, selectivity considerations frequently lead to use of insufficient coupling to reduce the band width. The band width may be determined by considering that the circuit of Fig. 3-15(c), to the right of the b,b terminals, constitutes a resonant resistive load on the generator of internal resistance R_1. This resonant load is

$$Z_{1\,\text{in}} = R_{ar} = j(\omega L_1 - 1/\omega C_1) + \frac{\omega^2 M^2}{R_2 + j(\omega L_2 - 1/\omega C_2)}$$

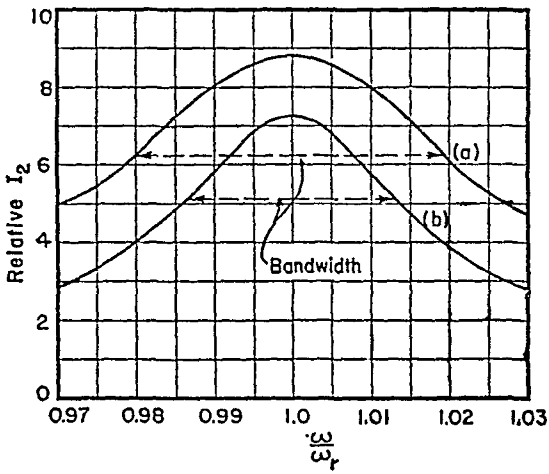


Fig. 3-16. Effect of M on band width of insufficiently coupled doubly-tuned circuit of Fig. 3-15. $L_2 = 200$ μh; $Q_2 = 100$; $R_1 =$ 10,000 ohms; (a) $M = 56.6$ μh (critical); (b) $M = 28$ μh.

Under the conditions of primary and secondary resonance this becomes

$$R_{ar} = \frac{\omega^2 M^2}{R_2} \tag{3-85}$$

Using Eq. 2-49, the band width of the circuit of Fig. 3-14(c) with insufficient coupling may be expressed as

$$\frac{\Delta f}{f_r} = \frac{1}{Q_2} + \frac{1}{Q_2}\frac{\omega^2 M^2}{R_1 R_2} \tag{3-86}$$

The band width may also be written

$$\frac{\Delta f}{f_r} = \frac{1}{Q_2} + \frac{\omega M^2}{L_2 R_1} \tag{3-87}$$

Reduction of M will reduce the band width, although a more effective reduction may be made by raising the Q and L of the secondary circuit. The effect of M on selectivity is illustrated in Fig. 3-16.

The currents and secondary voltage to be expected in this circuit are those of Eqs. 3-76, 3-79, and 3-81.

3-13. Effects of overcoupling; selectivity curves

The coupling circuits that have been discussed in this chapter are very frequently employed to achieve certain frequency-selectivity characteristics in radio receivers. In such service the design is

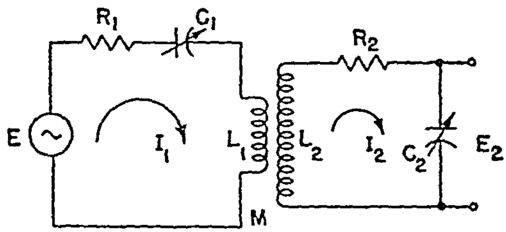


Fig. 3-17. Circuit for discussion of Section 3-13; equivalent to Fig. 3-15.

chosen not on the basis of power transfer but to obtain the greatest possible secondary voltage in a certain frequency band with rejection of all other frequencies. Coupling may be selected either over or under the critical value, so that a general analysis is necessary.

The secondary voltage E_2 has been determined as

$$E_2 = \frac{-jI_2}{\omega C_2}$$

where I_2 is

$$I_2 = \frac{E'}{Z_{T12}} = \frac{E'}{Z_{12} - Z_{11}Z_{22}/Z_{12}} \tag{3-88}$$

It is convenient to compute the transfer impedance term by term and to employ certain methods developed in Chapter 2. The term Z_{11} is

$$Z_{11} = R_1 + j\left(\omega L_1 - \frac{1}{\omega C_1}\right) = R_1 + j\left(\frac{\omega^2 L_1 C_1 - 1}{\omega C_1}\right) \tag{3-89}$$

Bringing in the term δ from Chapter 2 as

$$\delta = \frac{\omega - \omega_r}{\omega_r}$$

$$\omega = \omega_r(1 + \delta) \tag{3-90}$$

where ω_r is the resonant angular velocity at which

$$\omega_r L_1 = \frac{1}{\omega_r C_1} \qquad \omega_r L_2 = \frac{1}{\omega_r C_2}$$

it becomes possible to write Eq. 3-89, by use of the definition of Q_1, as

$$\begin{aligned} Z_{11} &= \frac{\omega_r L_1}{Q_1} + j\omega_r L_1 \left[\frac{(1+\delta)^2 - 1}{1 + \delta}\right] \\ &= \frac{\omega_r L_1}{Q_1}\left[1 + jQ_1\delta\left(\frac{2+\delta}{1+\delta}\right)\right] \end{aligned} \tag{3-91}$$

Employing the approximation

$$\frac{1}{1+\delta} \cong 1 - \delta$$

since $\delta \ll 1$, Eq. 3-91 reaches the form

$$Z_{11} = \frac{\omega_r L_1}{Q_1}[1 + jQ_1\delta(2 - \delta)] \tag{3-92}$$

The second mesh impedance Z_{22} may then be written by inspection as

$$Z_{22} = \frac{\omega_r L_2}{Q_2}[1 + jQ_2\delta(2 - \delta)] \tag{3-93}$$

The mutual impedance Z_{12} can be written as

$$Z_{12} = j\omega M = j\omega k\sqrt{L_1L_2} = j\omega_r(1+\delta)k\sqrt{L_1L_2} \tag{3-94}$$

If Eqs. 3-92, 3-93, and 3-94 are combined, the transfer impedance is obtained as

$$\begin{aligned} Z_{T12} &= j\omega_r(1+\delta)k\sqrt{L_1L_2} \\ &\quad + \frac{j\omega_r\sqrt{L_1L_2}}{Q_1Q_2k(1+\delta)}[1 + jQ_1\delta(2-\delta)][1 + jQ_2\delta(2-\delta)] \end{aligned} \tag{3-95}$$

Since the frequencies of interest are grouped closely about ω_r, then

δ is always small and may be dropped with respect to 1 or 2, so that

$$Z_{T12} = \frac{j\omega_r\sqrt{L_1L_2}}{Q_1Q_2k}\,[Q_1Q_2k^2 + (1 + j2Q_1\delta)(1 + j2Q_2\delta)] \qquad (3\text{-}96)$$

The current I_2 will be a maximum when Z_{T12} is a minimum, and vice versa. Since phase angles are not now of interest, the square of the absolute value of Z_{T12} may be differentiated with respect to δ and maximized:

$$|Z_{T12}|^2 = [4(Q_1 + Q_2)^2\delta^2 + (Q_1Q_2k^2)^2 + 16(Q_1Q_2\delta^2)^2 + 1 + 2Q_1Q_2k^2 - 8Q_1Q_2\delta^2 - 8Q_1{}^2Q_2{}^2\delta^2k^2]\left(\frac{\omega_r\sqrt{L_1L_2}}{Q_1Q_2k}\right)^2$$

$$\frac{\partial|Z_{T12}|^2}{\partial\delta} = 8\delta(Q_1 + Q_2)^2 + 64Q_1{}^2Q_2{}^2\delta^3 - 16Q_1Q_2\delta - 16Q_1{}^2Q_2{}^2k^2\delta = 0$$

Obviously, a maximum or minimum occurs at $\delta = 0$. Upon further reduction,

$$\delta^2 = \frac{1}{4Q_1Q_2} + \frac{k^2}{4} - \frac{(Q_1 + Q_2)^2}{8Q_1{}^2Q_2{}^2}$$

$$\delta = \pm\frac{1}{2}\sqrt{k^2 + \frac{1}{Q_1Q_2} - \frac{(Q_1 + Q_2)^2}{2Q_1{}^2Q_2{}^2}} \qquad (3\text{-}97)$$

Frequently the circuits are designed so that $Q_1 = Q_2$. If this equality does not exist but Q_1 does not differ too greatly from Q_2, then it is possible to assume that

$$\sqrt{Q_1Q_2} \cong \frac{Q_1 + Q_2}{2} \qquad (3\text{-}98)$$

That is, the geometric mean is approximately equal to the arithmetic mean. Equation 3-97 then becomes

$$\delta = \pm\frac{1}{2}\sqrt{k^2 - \frac{1}{Q_1Q_2}}$$

This equation shows two possible values of δ at which $|Z_{T12}|^2$ is a maximum or a minimum. It may be discovered that the above two values are minima and the $\delta = 0$ point is a maximum. Thus two maxima or peaks of current I_2 occur with a minimum value between. These peaks are spaced an equal distance each side of the center resonant frequency f_r, or the distance between peaks is given by

$$f_2 - f_1 = f_r \sqrt{k^2 - \frac{1}{Q_1Q_2}} \tag{3-99}$$

The frequencies of the individual peaks are

$$f_2 = f_r\left(1 + \frac{1}{2}\sqrt{k^2 - \frac{1}{Q_1Q_2}}\right) \tag{3-100}$$

$$f_1 = f_r\left(1 - \frac{1}{2}\sqrt{k^2 - \frac{1}{Q_1Q_2}}\right) \tag{3-101}$$

The general shape of such a resonant curve is shown in Fig. 3-18.

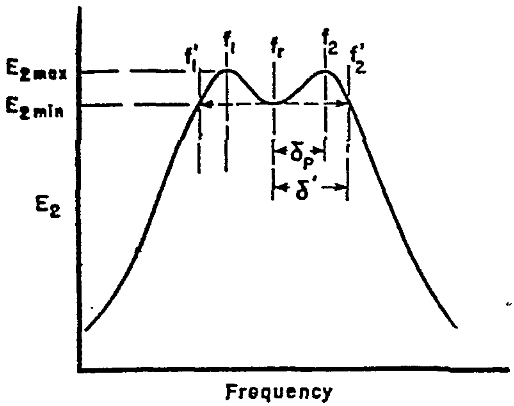


Fig. 3-18. Resonance curve for an overcoupled circuit of the form of Fig. 3-17, indicating various reference frequencies and voltages.

If k is varied, the frequency difference between peaks changes and at a value of k such that

$$k = k_c = \frac{1}{\sqrt{Q_1Q_2}}$$

the radical becomes zero and only one peak will be found. By reference to Eq. 3-83 it is seen that this is the value of critical coupling k_c. Equation 3-99 may then be simplified to

$$f_2 - f_1 = f_r k_c \sqrt{\frac{k^2}{k_c^2} - 1} \tag{3-102}$$

For values of k less than critical, only a single peak will exist, with the amplitude of the current at the peak decreasing with lower k values.

Since

$$E_2 = \frac{-jI_2}{\omega C_2} = \frac{-jI_2}{\omega_r C_2(1+\delta)} = \frac{-j\omega_r L_2 I_2}{1+\delta} \tag{3-103}$$

after neglecting δ with respect to unity it is possible, by use of Eq. 3-96, to write E_2 as

$$E_2 = \frac{-j\sqrt{\frac{L_2}{L_1}}\,Q_1Q_2kE'}{-2(Q_1+Q_2)\delta + j(Q_1Q_2k^2 - 4Q_1Q_2\delta^2 + 1)}$$

In view of the approximation of Eq. 3-98 and the value for k_c, the above equation simplifies to

$$E_2 = -j\sqrt{\frac{L_2}{L_1}}\,E'\left[\frac{k}{-4\delta k_c + j(k^2 - 4\delta^2 + k_c^2)}\right]$$

This equation may be better understood if it is rearranged as a function of δ/k_c. Thus

$$E_2 = -j\sqrt{\frac{L_2}{L_1}}\,\frac{E}{k_c}\left[\frac{\frac{k}{k_c}}{-4\frac{\delta}{k_c} + j\left(\frac{k^2}{k_c^2} - 4\frac{\delta^2}{k_c^2} + 1\right)}\right] \tag{3-104}$$

The absolute value of the term in brackets is plotted as a function of δ/k_c in Fig. 3-19, for various values of the parameter k/k_c, or the ratio of actual coupling to critical-coupling coefficient. These curves then serve as a family of universal response curves for the circuit.

The two peak frequencies, f_1 and f_2, are readily available from Eqs. 3-100 and 3-101. The secondary voltage $E_{2\,\max}$ at those points is obtainable by substitution of

$$\delta_p = \pm\frac{k_c}{2}\sqrt{\frac{k^2}{k_c^2} - 1} \tag{3-105}$$

from Eq. 3-102 into Eq. 3-104. The value of $E_{2\,\max}$ at either peak then is

$$E_{2\,\max} = -j\sqrt{\frac{L_2}{L_1}}\,\frac{E}{k_c}\left[\frac{\frac{k}{k_c}}{-2\sqrt{\frac{k^2}{k_c^2} - 1} + j2}\right] \tag{3-106}$$

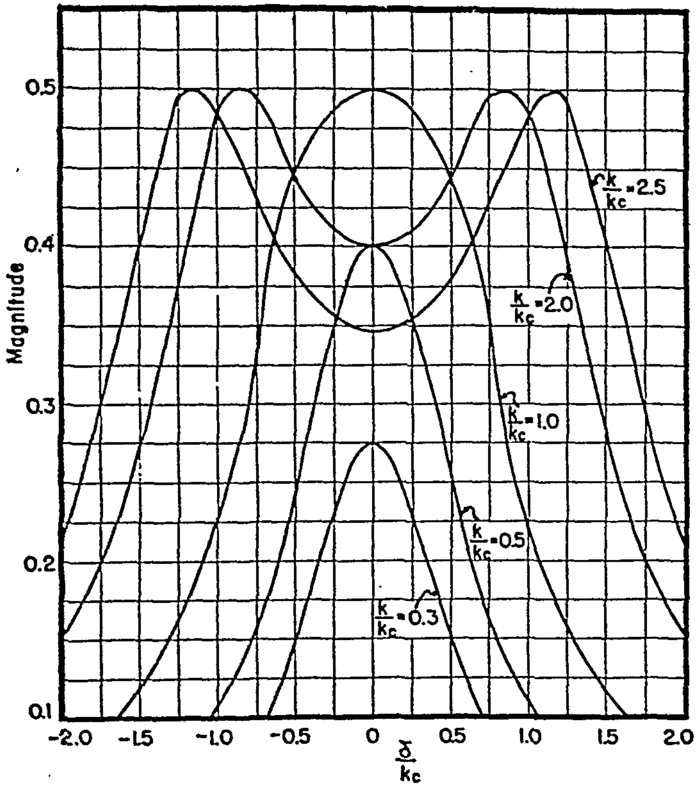


Fig. 3-19. Universal response curves for the circuit of Fig. 3-17 in terms of the parameter k/k_c. The curves are plotted in terms of the magnitude of the bracketed term of Eq. 3-104. The parameter k/k_c shows the effect of change in coupling.

The secondary voltage $E_{2\,\min}$ at the dip or frequency f_r can be readily found by substitution of $\delta = 0$ in Equation 3-104, giving

$$E_{2\,\min} = -j\sqrt{\frac{L_2}{L_1}}\frac{E}{k_c}\left[\frac{-j\,\dfrac{k}{k_c}}{\dfrac{k^2}{k_c^2}+1}\right] \tag{3-107}$$

Two additional points on the resonance curve can also be found. These are the frequencies f_1' and f_2', down each slope of the curve, at which the voltage E_2 has fallen to $E_{2\,\min}$. Equating the value of

$E_{2\,\min}$ from Eq. 3-107 to the general value of E_2 from Eq. 3-104 gives

$$\left|\frac{-\dfrac{k}{k_c}}{\dfrac{k^2}{k_c^2}+1}\right| = \left|\frac{\dfrac{k}{k_c}}{-4\dfrac{\delta}{k_c}+j\left(\dfrac{k^2}{k_c^2}-4\dfrac{\delta^2}{k_c^2}+1\right)}\right|$$

This may be solved for $\delta = \delta'$, the value of δ at frequencies f_1' and f_2', leading to

$$2\left(\frac{\delta'}{k_c}\right)^2 - \frac{k^2}{k_c^2} + 1 = 0$$

$$\delta' = \pm \frac{k_c}{\sqrt{2}}\sqrt{\frac{k^2}{k_c^2}-1} \tag{3-108}$$

It has already been determined that at the frequencies f_1 and f_2,

$$\delta_p = \pm \frac{k_c}{2}\sqrt{\frac{k^2}{k_c^2}-1}$$

so it can be seen that

$$\delta' = \sqrt{2}\,\delta_p \tag{3-109}$$

Thus frequency and voltage data for five points near the top of a resonance curve are readily obtainable.

It may be observed from Fig. 3-19 that for values of k/k_c less than unity, a response curve with only one peak is obtained, and a true impedance match is not achieved. The condition of critical coupling or $k/k_c = 1$ does lead to an impedance match and maximum power transfer with but one frequency of response. The selectivity of this curve is not so good as for $k/k_c = 0.5$ or other values less than unity, which confirms the statements of Section 3-12. For values of k/k_c greater than unity, the circuit is overcoupled. The same value of maximum voltage is reached as for critical coupling, but response occurs at two frequencies with a dip between.

If δ had not been neglected with respect to unity in the denominator of Eq. 3-103, it would be found that the voltage E_2 given by Eq. 3-106 at the low-frequency peak $(-\delta_p)$ would be higher than that developed at the high-frequency peak $(+\delta_p)$. In circuits ordinarily used, this difference would be very slight because of the smallness of δ_p. It is also found in such cases that the gain at critical coupling as given by Eq. 3-84 is approximately the mean between the gains at the low- and high-frequency peaks. Since these differences are so small, it is customary to assume the voltages at both peaks equal, as was done here, and equal to the voltage developed at critical coupling.

3-14. Design of overcoupled circuits

If the magnitude of the dip in the top of the response curve could be held to any desirable minimum, it might not affect circuit operation adversely and the overcoupled circuit could then be used where response over a wide frequency band is desired, as in intermediate-frequency amplifiers of radio receivers. The ideal response curve would be a rectangle with vertical sides and flat top, to which the overcoupled-circuit curve furnishes a reasonable approximation.

A means of controlling the magnitude of the voltage dip in the response curve may be readily found. Computing the ratio of $E_{2\,max}$ to $E_{2\,min}$ from Eqs. 3-106 and 3-107, this ratio appears as

$$\frac{\dfrac{1}{-2\sqrt{\dfrac{k^2}{k_c^2} - 1} + j2}}{\dfrac{-j}{\dfrac{k^2}{k_c^2} + 1}} \tag{3-110}$$

Taking the absolute value and calling the ratio λ,

$$\lambda = \frac{k^2/k_c^2 + 1}{2k/k_c}$$

from which $$\frac{k^2}{k_c^2} - 2\lambda \frac{k}{k_c} + 1 = 0$$

Solving for k/k_c, $$\frac{k}{k_c} = \lambda \pm \sqrt{\lambda^2 - 1}$$

The plus sign should be chosen, since for overcoupled circuits k/k_c is always greater than unity. Then

$$\frac{k}{k_c} = \lambda + \sqrt{\lambda^2 - 1} \tag{3-111}$$

Thus for any desired ratio of voltages at the peaks and at the dip, the ratio k/k_c may be determined.

The frequency band in which the voltage deviation remains less

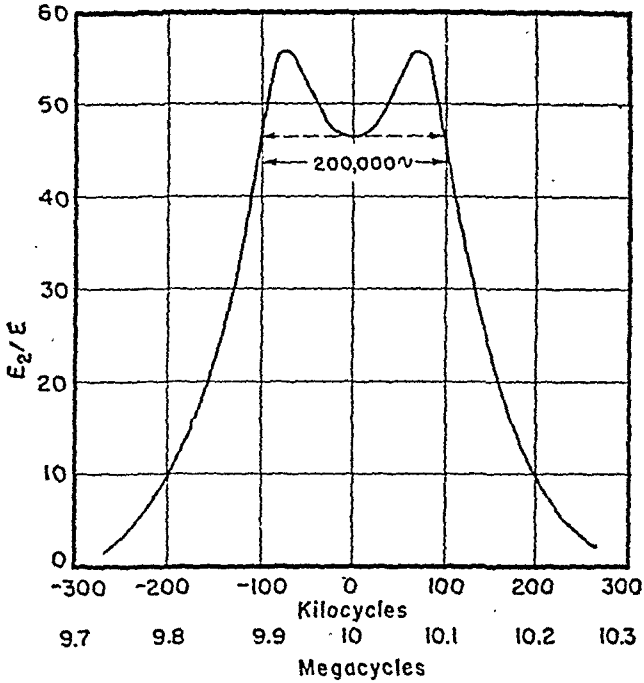


Fig. 3-20. Response curve as plotted for the example of Section 3-14. Center frequency 10 megacycles.

than λ is, from Fig. 3-18, equal to

$$\Delta f = f_2' - f_1'$$

But
$$\frac{\Delta f}{f_r} = 2\delta' = 2\sqrt{2}\,\delta_p$$

$$\Delta f = \sqrt{2}\,f_r k_c \sqrt{\frac{k^2}{k_c^2} - 1} \tag{3-112}$$

The frequency band Δf is ordinarily fixed, so that by use of Eqs. 3-111 and 3-112, values may be found for k, k_c, and $\sqrt{Q_1Q_2}$, of the circuit. This circuit will then produce the desired shape of response curve.

As an example, consider a transformer to have a value of Δf

equal to 200,000 cycles, with f_r equal to 10 megacycles. The allowable variation in voltage over the Δf band is taken as $\lambda = 1.2$.

Solving for k/k_c from Eq. 3-111,

$$\frac{k}{k_c} = 1.2 + \sqrt{1.44 - 1} = 1.865$$

From Eq. 3-112,

$$\Delta f = 200{,}000 = \sqrt{2} \times 10 \times 10^6 k_c \sqrt{(1.865)^2 - 1}$$

$$k_c = \frac{0.0141}{1.573} = 0.00899$$

Thus $\quad k = 0.01675$

Since $\quad k_c = 1/\sqrt{Q_1 Q_2}, \quad$ then $\quad \sqrt{Q_1 Q_2} = 111$

If $Q_1 = Q_2$, then the respective values of Q are 111.

Thus by choosing the Q value of the primary and secondary circuits and setting the coupling to a definite value, any desired value of frequency band and ratio between $E_{2\,\max}$ and $E_{2\,\min}$ may be obtained. It should be noted that if the top of the curve is to be held flat, of if λ is small and Δf is to be large, then the voltage E_2 will be reduced.

The response curve for the example above is shown in Fig. 3-20.

PROBLEMS

3-1. A parallel circuit antiresonant at 3×10^6 c, with $Q = 20$ and $L = 100\ \mu$h, is available as a load for a generator of 20,000 ohms internal resistance. If the circuit capacitance is changed to two capacitances in series, find the capacitances required and specify across which one the generator should be connected for an impedance match.

3-2. A parallel circuit antiresonant at 1.5×10^6 c has a Q of 9 and matches a generator of 10,000 ohms resistance, generated voltage 1 v, 1.5×10^6 c. Find

(a) The values of L and C required.

(b) The generator current and power output.

(c) The currents in L and C and the power delivered to the circuit.

3-3. You have a generator of 1 v, 0.7×10^6 c, 1000 ohms internal resistance.

(a) Design an L-section network to couple this generator to a 100-ohm load with maximum power transfer.

(b) Repeat for a load of 10,000 ohms.

3-4. A given generator of 1 v, 1.5×10^6 c, has an internal impedance of $500 + j500$ ohms.

(a) Design an L-section network to couple this to 8000 ohms resistance for an impedance match.

(b) Find the current and power delivered to the load.

3-5. A parallel circuit, antiresonant at 24×10^6 c, has a Q of 5 and is used as a load on a generator of 12,000 ohms resistance, generated voltage 1 v, 2.4×10^6 c. If maximum power is to be delivered, find

(a) Values of L and C required in the circuit.

(b) Generator current.

(c) Capacitor current.

(d) Power transferred to the resonant load.

3-6. An inductor is to be used with a 500-$\mu\mu$f capacitor as a load to match a generator of 10 v, 0.7×10^6 c, 20,000 ohms internal resistance. Assuming no resistance in capacitor or leads, find L, R, and Q of the coil.

3-7. Design a reactance-matching section to match a 10,000-ohm generator to a 250-ohm load for critical coupling at a frequency of 7.0 megacycles. State the values of the elements in terms of microhenrys and micromicrofarads.

3-8. Referring to (a), Fig. 3-21, with $|M| = 10$ μh, $L_1 = 30$ μh, $L_2 = 200$ μh, and $\omega L_2 = 1/\omega C_2$ at 2 mc, find the voltage developed across C_2 in both magnitude and angle.

3-9. A radio transmitter having an internal resistance of 4000 ohms is to be coupled to an antenna having a resistance of 72 ohms at 10 megacycles. The mutual impedance is to be a capacitance and the circuit is to respond at only one setting of the variables. Design the required π network to transfer maximum power.

3-10. In the circuit of Fig. 3-21(b), $R_1 = 10{,}000$ ohms, $L_1 = 20$ μh, $L_2 = 50$ μh, $R_2 = 100$ ohms with a frequency of 3.5×10^6 c. Find the values of M, C_1, and C_2 needed for an impedance match, with response at one frequency only.

3-11. Using an air-core transformer with $L_1 = 50$ μh, $R_1 = R_2 = 10$ ohms, $L_2 = 200$ μh, $M = 40$ μh, and any values of capacitance

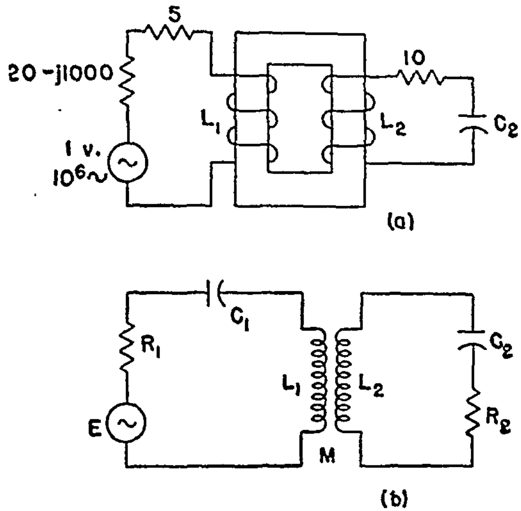


Fig. 3-21.

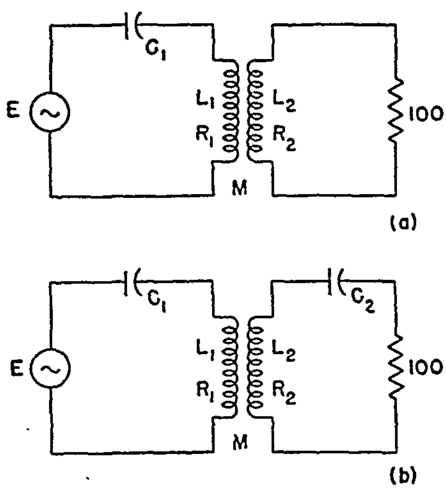


Fig. 3-22.

needed, design a circuit in terms of reactances, to match a generator of 10^6 c, $1990 + j1000$ ohms internal impedance to a load of $10 + j20$ ohms.

3-12. In the circuit of Fig. 3-22(b), $L_1 = 200$ μh, $L_2 = 50$ μh, $-M = 10$ μh, $R_1 = R_2 = 5$ ohms. If $\omega L_1 = 1/\omega C_1$, $\omega L_2 = 1/\omega C_2$

at 1 megacycle, find the effective Q of the primary or generator mesh, as changed by coupling of the second mesh.

3-13. With $L_1 = 100\ \mu h$, $R_1 = 10$ ohms, $L_2 = 50\ \mu h$, $R_2 = 5$ ohms, $M = 20\ \mu h$, what value of C_1 will be required to produce unity-power-factor resonance in the generator circuit at 1.2×10^6 c in Fig. 3-22(a)?

3-14. The primary winding of a radio-frequency transformer is connected in series with an antenna. The primary has an inductance of 30 μh and resistance of 5 ohms.

The effective series values for the antenna are $L_a = 20\ \mu h$, $C_a = 200\ \mu\mu f$, $R_a = 20$ ohms.

The secondary of the transformer has $L_2 = 200\ \mu h$, $R_2 = 10$ ohms, with $M = 10\ \mu h$. Across the secondary is a variable capacitor C_2, and at 10^6 c, $\omega L_2 = 1/\omega C_2$.

(a) A signal of 1 mv, 10^6 c, is induced in the antenna. What will be the voltage developed across C_2?

(b) Without changing the value of C_2 in the circuit above, a signal of 1 mv, 1.2×10^6 c is induced in the antenna. What is the ratio of E_2 for the signal at 10^6 c to E_2 for the 1.2- $\times$ 10^6-c signal?

3-15. You have one variable inductance and several variable capacitors. Design a network to match a 50-v, 20,000-ohm generator at 3.5×10^6 c to a 1200-ohm load. Coupling is to be 20 per cent above the critical value.

3-16. A certain antenna coupling device matches a 600-ohm resistance to a 72-ohm antenna load. This device is built to have a mutual impedance of 208 ohms. Why was this value selected?

3-17. If the mutual impedance must be 0.1-μf capacitor, design a T network to match a generator of 4000 ohms, 10 v, 796 c, to a load of $1000 + j1200$ ohms. What is the power delivered to the load?

3-18. An air-core transformer has $L_1 = 30\ \mu h$, $L_2 = 200\ \mu h$, $M = 10\ \mu h$, $R_1 = 5$ ohms, $R_2 = 20$ ohms. It is supplied by a generator of 5 v, 10^6 c, 10,000 ohms.

(a) Compute k.

(b) Find I_1,I_2, and voltage across the load for loads of $0 + j750$, $0 - j1256$, $0 - j2000$, $1256 + j0$ ohms.

(c) What is the band width of the circuit as seen from the generator if the load is C_2 and $\omega L_2 \doteq 1/\omega C_2$ at 10^6 c?

3-19. The transformer of Fig. 3-21(b) has $L_1 = L_2 = 100\ \mu$h, $R_1 = 1000$ ohms, $R_2 = 25$ ohms, with $E = 1$ v, 2×10^6 c. Here M is 150 per cent of critical value.

(a) Find values of C_1 and C_2 required for an impedance match.

(b) Find the maximum secondary voltage, E_2.

(c) Plot the frequency-response curve.

3-20. A vacuum-tube generator with output voltage of 100 v at 1000 c, and with internal resistance of 7000 ohms, is to deliver power to a 16-ohm load.

(a) Using an ideal transformer for matching, what is the required turns ratio?

(b) What are the tube current, 16-ohm load current, and power delivered to the load?

3-21. An ideal transformer is used to couple a generator of 10 v, 1200 c, 7000-ohm internal resistance to a load of $200 - j300$.

(a) For best power transfer, what turns ratio should be used?

(b) What amount of power will be delivered to the load?

3-22. Determine the proper design for the windings of an intermediate-frequency transformer having $L_1 = L_2$ and each winding tuned to resonance by a capacitor of 100 $\mu\mu$f. The secondary voltage is not to fall below 0.88 of the peak value inside a band of 8 kc, with the resonant frequency at 465 kc. Find k, L_1, L_2, Q_1 and Q_2, and the secondary voltage developed with 1 v, 465 kc applied to the primary.

REFERENCES

1. Everitt, W. L., *Communication Engineering*, 2d ed., McGraw-Hill Book Company, Inc., New York, 1937.

2. Aiken, C. B., "Two-Mesh-Tuned Coupled-Circuit Filters," *Proc. I.R.E.*, 25, 230 (1937).

3. Glasgow, R. S., *Principles of Radio Engineering*, McGraw-Hill Book Company, Inc., New York, 1936.

Chapter 4

FILTERS

Resonant circuits that will select relatively narrow bands of frequencies and reject others have already been discussed. Certain other reactive networks are available that will freely pass desired bands of frequencies while almost totally suppressing other bands of frequencies. Such reactive networks, called *filters*, were first investigated by G. A. Campbell and O. J. Zobel of the Bell Telephone Laboratories.

An ideal filter would pass all frequencies in a given band without reduction in magnitude, and totally suppress all other frequencies. Such ideal performance is not possible but can be approached with complex designs, if the need warrants. Filter circuits are widely used and vary in complexity from the relatively simple power-supply filter of the a-c operated radio receiver to complex filter sets used to separate the various voice channels in carrier-frequency telephone circuits. Whenever alternating currents occupying different frequency bands are to be separated, filter circuits have an application.

Here analysis of filter circuits is carried out on the basis of certain definitions from the general field of electric network theory, *under the assumption of symmetrical network sections.*

4-1. The neper; the decibel

In filter circuits and other electric networks it is frequently convenient to appraise the performance of a circuit in terms of the ratio of input-current to output-current magnitude. If the input and output image impedances, or the ratios of voltage to current at input and output of the network, are equal, then the ratios of the input to output currents, or input to output voltages, may equally well be written

$$\left|\frac{I_1}{I_2}\right| = \left|\frac{V_1}{V_2}\right| \tag{4-1}$$

If several networks are used in succession as in Fig. 4-1, the over-all performance may be appraised as

$$\left|\frac{V_1}{V_2}\right| \times \left|\frac{V_2}{V_3}\right| \times \left|\frac{V_3}{V_4}\right| \times \cdots \left|\frac{V_{n-1}}{V_n}\right| = \left|\frac{V_1}{V_n}\right| \tag{4-2}$$

which may also be stated as

$$A_1\underline{/\alpha} \times A_2\underline{/\beta} \times A_3\underline{/\gamma} \times A_4\underline{/\delta} = A_1A_2A_3A_4\underline{/\alpha + \beta + \gamma + \delta}$$

both processes employing multiplication of magnitudes. In general, the process of addition or subtraction may be carried out with

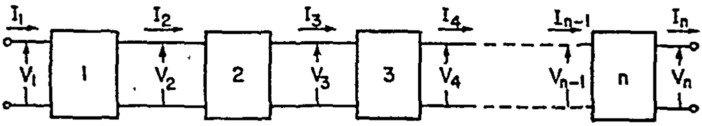


Fig. 4-1. A succession of n networks in cascade.

greater ease than the process of multiplication or division. It is therefore of interest to note that

$$\epsilon^a \times \epsilon^b \times \epsilon^c \times \cdots \epsilon^n = \epsilon^{a+b+c+\cdots n}$$

is an application in which addition is substituted for multiplication. If the voltage ratios of Eq. 4-2 are defined as

$$\left|\frac{V_1}{V_2}\right| = \epsilon^a; \quad \left|\frac{V_2}{V_3}\right| = \epsilon^b; \quad \left|\frac{V_3}{V_4}\right| = \epsilon^c; \quad \text{etc.}$$

then Eq. 4-2 becomes

$$\left|\frac{V_1}{V_n}\right| = \epsilon^{a+b+c+\cdots n}$$

and if the natural logarithm (ln)* of both sides is taken, then

$$\ln\left|\frac{V_1}{V_n}\right| = a + b + c + \cdots n \tag{4-3}$$

Consequently, if the ratio of each individual network is given as ϵ to an exponent, the logarithm of the current or voltage ratio for all the networks in series is very easily obtained as the simple sum of the various exponents. It has become common, for this reason, to

* In this text the abbreviation "ln" will be used to indicate the natural logarithm, whereas "log" will indicate the logarithm to the base 10.

define

$$\left|\frac{V_1}{V_2}\right| = \left|\frac{I_1}{I_2}\right| = \epsilon^N \tag{4-4}$$

under conditions of equal impedance associated with input and output circuits. The unit of "N" has been given the name *neper* and defined as

$$\text{N nepers} = \ln\left|\frac{V_1}{V_2}\right| = \ln\left|\frac{I_1}{I_2}\right| \tag{4-5}$$

Two voltages, or currents, differ by one neper when one of them is ϵ times as large as the other.

Obviously, ratios of input to output power may also be expressed in this fashion. That is,

$$\frac{P_1}{P_2} = \epsilon^{2N}$$

The number of nepers represents a convenient measure of the power loss or gain of a given network. Losses or gains of successive networks then may be introduced by addition or subtraction of their appropriate N values.

The telephone industry proposed and has popularized a similar unit based on logarithms to the base 10, naming the unit the *bel* for Alexander Graham Bell. The bel is defined as the logarithm of a power ratio,

$$\text{number of bels} = \log\frac{P_1}{P_2}$$

It has been found that a unit one-tenth as large is more convenient, and the smaller unit is called the *decibel*, abbreviated "db," defined as

$$\text{db} = 10\log\frac{P_1}{P_2} \tag{4-6}$$

For the case of equal impedances in input and output circuits,

$$\text{db} = 20\log\frac{I_1}{I_2} = 20\log\frac{V_1}{V_2} \tag{4-7}$$

Equating the values for the power ratios,

$$\epsilon^{2N} = 10^{\text{db}/10}$$

and taking the logarithm of both sides,

$$8.686\ \mathrm{N} = \mathrm{db}$$

or

$$1 \text{ neper} = 8.686 \text{ db}$$

is obtained as the relation between nepers and decibels.

Most of the energy transmitted through electric networks, lines, and filters is ultimately converted to acoustic energy and heard by the human ear as sound. Further substantiation of the logic of a logarithmic unit for measurement of energy to be ultimately delivered to the ear is furnished by an important property of the human ear. This property may be stated as "the ear hears logarithmically." More elaborately, the ear is observed to obey the Weber-Fechner law, which states, "The change in stimulus necessary to produce a perceptible change in response is proportional to the stimulus already existing." The ear hears sound intensities on a proportional, or logarithmic, scale and not on a linear one. The loudness L may be expressed as

$$L = K \log_b S$$

where K is a constant of proportionality, b is the base of the logarithms, and S is the sound power.

A sound of original power S_1 and loudness L_1 may be caused to increase in power to S_2 units, with loudness L_2. The change in loudness is then

$$\Delta L = L_2 - L_1 = K \log_b S_2 - K \log_b S_1$$

$$\Delta L = K \log_b \frac{S_2}{S_1} \tag{4-8}$$

Upon assigning values to K and b, the change in power is expressed in

nepers, if $K = 0.5$ and the base $b = \epsilon$

bels, if $K = 1.0$ and the base $b = 10$

decibels, if $K = 10$ and the base $b = 10$

The logic of the logarithmic units, the neper and the decibel, is then apparent.

The use of these logarithmic units may be illustrated by a few examples. For instance, under one condition the output of a network is 2 watts. Under a changed condition the output is 3.2 watts. The output is then said to have changed by

$$10 \log \frac{3.2}{2} = 2.04 \text{ db}$$

If this were acoustic power reaching the ear, the change would appear as a noticeable increase, since 1 decibel represents approximately the minimum perceptible audible change.

If instead of increasing the power to 3.2 watts it had been reduced to 0.5 watt, then

$$10 \log \frac{0.5}{2.0} = -10 \log \frac{2.0}{0.5} = -6.02 \text{ db}$$

the minus sign indicating that a reduction in power has taken place.

Although the decibel as discussed is a *power ratio*, it can be used for absolute measurements if a certain reference, or zero level, for P_1 is adopted beforehand and known or stated. Various reference levels which have been used in the telephone and broadcasting industries are 1, 6, 10, and 12.5 milliwatts. Until a uniform agreement on a single standard for *zero level* has been reached, it will always be necessary to state specifically which level is meant as reference.

Using the 6-milliwatt level as reference, the original 2-watt output of the network above is

$$10 \log \frac{2}{0.006} = 10 \times 2.523 = 25.23 \text{ db above reference}$$

After the change to 3.2 watts the output is 27.27 db above zero, the reduction to 0.5 watt output changing the level to 19.21 db above the reference. These figures would again indicate that a loss of -6.02 db had taken place.

When referred to a reference, negative decibel values are powers smaller than the reference level.

When the concept of logarithmic power ratios is carried into other fields, various standard references are adopted. As an example, in acoustic measurements sound levels are measured in decibels with reference or zero power level of 10^{-16} watt per square centimeter.

4-2. Characteristic impedance of symmetrical networks

When $Z_1 = Z_2$ or the two series arms of a T network are equal, or $Z_a = Z_c$ and the shunt arms of a π network are equal, the networks are said to be *symmetrical.*

Filter networks are ordinarily set up as symmetrical sections, basically of the T or π type, such as shown at (b) and (d), Fig. 4-2.

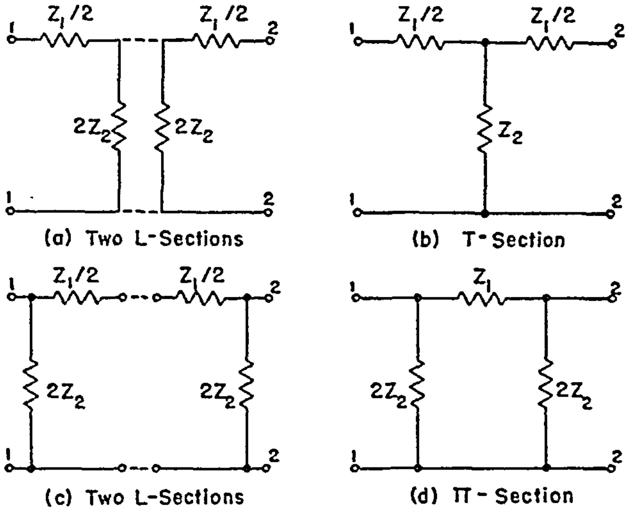


Fig. 4-2. The T and π sections as derived from unsymmetrical L sections, showing notation used in symmetrical network analysis.

Attention is called to the peculiarities of notation employed on the various arms. This peculiarity is largely dictated by custom, arising from the fact that both T and π networks can be considered as built of unsymmetrical L half sections, connected together in one fashion for the T network, and oppositely for the π network as at (a) and (c), Fig. 4-2. A series connection of several T or π networks leads to so-called "ladder networks," which are indistinguishable one from the other except for the end or terminating L half sections, as can be seen in Fig. 4-3.

For a symmetrical network the image impedances Z_{1i} and Z_{2i}, of Eqs. 3-15 and 3-16, are equal to each other, and the image impedance is then called the *characteristic impedance* or the *iterative impedance,* Z_0. That is, if a symmetrical T network is terminated in Z_0, its input impedance will also be Z_0, or its impedance transformation ratio is unity. The term iterative impedance is apparent if the terminating impedance Z_0 is considered as the input impedance

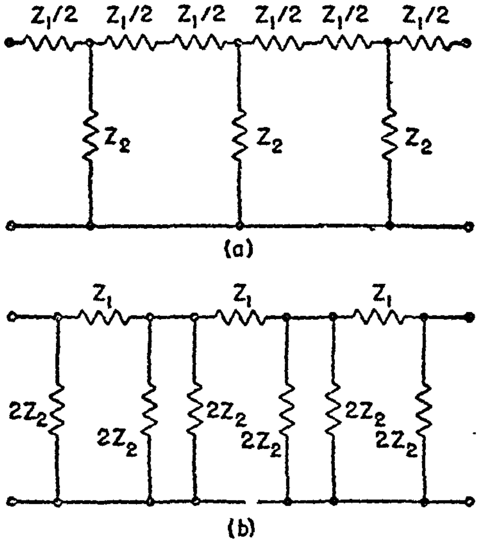


Fig. 4-3. (a) Ladder network made from T sections; (b) ladder network built from π sections. The parallel shunt arms will be combined.

of a chain of similar networks, in which case Z_0 is iterated at the input to each network.

The value of Z_0 for a symmetrical network can be easily determined. For the T network of Fig. 4-4(a), terminated in an impedance Z_0, the input impedance is

$$Z_{1\,\text{in}} = \frac{Z_1}{2} + \frac{Z_2(Z_1/2 + Z_0)}{Z_1/2 + Z_2 + Z_0} \tag{4-9}$$

It can be assumed that if Z_0 is properly chosen in terms of the network arms, it should be possible to make $Z_{1\,\text{in}}$ equal to Z_0. Requiring this equality gives

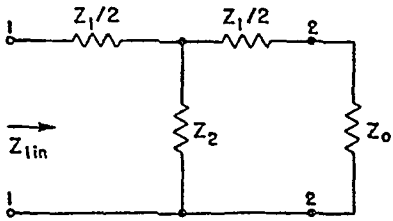
$$Z_0 = \frac{Z_1^2/4 + Z_1Z_2 + Z_2Z_0 + Z_1Z_0/2}{Z_1/2 + Z_2 + Z_0}$$

$$Z_0^2 = \frac{Z_1^2}{4} + Z_1Z_2$$

For the symmetrical T section, then,

$$Z_{0T} = \sqrt{\frac{Z_1^2}{4} + Z_1Z_2} = \sqrt{Z_1Z_2\left(1 + \frac{Z_1}{4Z_2}\right)} \tag{4-10}$$

becomes the characteristic impedance. This result could also have been immediately obtained from Eqs. 3-15 and 3-16 for the image



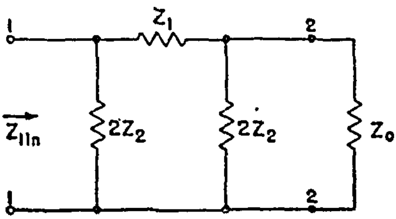


Fig. 4-4. Determination of Z_0: (a) for a T section; (b) for a π section.

impedance of a T section, by using the values of the arms of Fig. 4-4.

Similarly, for the π section of Fig. 4-4 (b) the input impedance is

$$Z_{1\,in} = \frac{\left[Z_1 + \left(\frac{2Z_2Z_0}{2Z_2 + Z_0}\right)\right] 2Z_2}{Z_1 + \frac{2Z_2Z_0}{2Z_2 + Z_0} + 2Z_2}$$

Requiring that $Z_{1\,in} = Z_0$ leads to

$$Z_{0\pi} = \sqrt{\frac{Z_1Z_2}{1 + Z_1/4Z_2}} \tag{4-11}$$

which is the characteristic impedance of the symmetrical π section.

In Chapter 1, certain information concerning networks was developed from measurements of Z_∞ and Z_{sc}. If these measurements are made on the T section of (a), Fig. 4-4, exclusive of the load Z_0, then

$$Z_{1\infty} = Z_{\infty} = \frac{Z_1}{2} + Z_2$$

$$Z_{1sc} = Z_{sc} = \frac{Z_1}{2} + \frac{Z_1Z_2/2}{Z_1/2 + Z_2}$$

$$Z_{\infty}Z_{sc} = \frac{Z_1^2}{4} + Z_1Z_2 = Z_{0T}^2 \tag{4-12}$$

Similar work for the π section leads to

$$Z_{\infty}Z_{sc} = \frac{4Z_2^2Z_1}{Z_1 + 4Z_2} = Z_{0\pi}^2$$

Therefore, for a symmetrical network,

$$Z_0 = \sqrt{Z_{\infty}Z_{sc}} \tag{4-13}$$

This result could have been directly obtained from the image impedance relations of Section 3-3. It is a valuable relationship, since it supplies an easy experimental means of determining the Z_0 of any symmetrical network.

4-3. Current and voltage ratios as exponentials; the propagation constant

In Section 4-1, under the assumption of equal input and output impedances, which may now be interpreted as a Z_0 termination on the network, the absolute value of the ratio of input current to output current of a given *symmetrical network* was defined as an exponential function,* for the purpose of simplifying network calculations. Obviously, the magnitude ratio does not express the

* In the general case of unsymmetrical 4-terminal networks, terminated on an image basis, it is customary to define a *transfer constant* θ, by

$$\epsilon^{2\theta} = \frac{E_1I_1}{E_2I_2} = \frac{\text{input volt-amperes}}{\text{output volt-amperes}}$$

or

$$\theta = \frac{1}{2} \ln \frac{E_1I_1}{E_2I_2}$$

where θ is in general a complex number. For symmetrical networks $Z_{1i} = Z_{2i} = Z_0$, and with a termination of Z_0, the above discussion follows, with γ customarily replacing θ and implying symmetry and Z_0 termination.

complete network performance, the phase angle between the currents being needed as well. The use of the exponential can be extended to include the phasor current ratio *if it be defined that, under the condition of Z_0 termination,*

$$\frac{I_1}{I_2} = \epsilon^{\gamma} \tag{4-14}$$

where γ is a complex number defined as

$$\gamma = \alpha + j\beta \tag{4-15}$$

Hence
$$\frac{I_1}{I_2} = \epsilon^{\gamma} = \epsilon^{\alpha + j\beta}$$

To illustrate further, if $I_1/I_2 = A\underline{/\beta}$, then

$$A = |I_1/I_2| = \epsilon^{\alpha}$$

$$\underline{/\beta} = \epsilon^{j\beta}$$

Since the input and output impedances are equal under the Z_0 termination, it is also true that

$$\frac{V_1}{V_2} = \epsilon^{\gamma}$$

The term γ has been given the name *propagation constant*. The exponent α is known as the *attenuation constant*, since it determines the magnitude ratio between input and output quantities, or the attenuation produced in passing through the network. The units of α are nepers. The exponent β is the *phase constant* as it determines the phase angle between input and output quantities, or the shift in phase introduced by the network. The units of β are radians.

If a number of sections all having a common Z_0 value are cascaded, the ratio of currents is

$$\frac{I_1}{I_2} \times \frac{I_2}{I_3} \times \frac{I_3}{I_4} \times \cdots = \frac{I_1}{I_n}$$

from which

$$\epsilon^{\gamma_1} \times \epsilon^{\gamma_2} \times \epsilon^{\gamma_3} \times \cdots = \epsilon^{\gamma_n}$$

and taking the natural logarithm,

$$\gamma_1 + \gamma_2 + \gamma_3 + \cdots = \gamma_n \tag{4-16}$$

Thus the over-all propagation constant is equal to the sum of the *individual propagation constants.*

4-4. Hyperbolic trigonometry

It is assumed that the student is familiar with some of the properties of hyperbolic functions, at least for real angles. Hyperbolic angles also have geometric meaning, being related to a hyperbola in the same way that trigonometric functions are related to a circle. This is illustrated in Fig. 4-5, wherein the hyperbola is the locus for the radius r, and $\sinh u = a/r$, $\cosh u = b/r$, $\tanh u = a/b$.

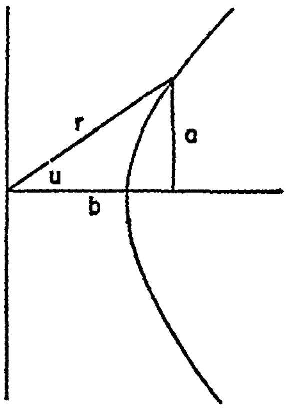


Fig. 4-5. Meaning of a hyperbolic angle.

As they will be used here, hyperbolic functions simplify the writing of certain exponential relations, and knowledge of their limits is particularly useful. A few properties are here summarized and extended to the case of complex angles:

$$\sinh u = \frac{\epsilon^u - \epsilon^{-u}}{2} \tag{4-17}$$

$$\cosh u = \frac{\epsilon^u + \epsilon^{-u}}{2} \tag{4-18}$$

$$\tanh u = \frac{\sinh u}{\cosh u} = \frac{1}{\coth u} \tag{4-19}$$

$$\cosh^2 u - \sinh^2 u = 1 \tag{4-20}$$

The values of the functions at the limits $u = 0$, and $u = \infty$ are

	$u = 0$	$u = \infty$
$\sinh u$	0	∞
$\cosh u$	1	∞
$\tanh u$	0	1

For u large, $\sinh u = \cosh u$. If u is imaginary or $u = jw$, then

$$\sinh jw = \frac{\epsilon^{jw} - \epsilon^{-jw}}{2} = j \sin w \tag{4-21}$$

$$\cosh jw = \frac{\epsilon^{jw} + \epsilon^{-jw}}{2} = \cos w \tag{4-22}$$

Expressions for complex angles, where $u = a + jb$, can be obtained by expansion. That is

$$\begin{aligned}\sinh(a + jb) &= \sinh a \cosh jb + \cosh a \sinh jb \\ &= \sinh a \cos b + j \cosh a \sin b \qquad (4\text{-}23)\\ \cosh(a + jb) &= \cosh a \cosh jb + \sinh a \sinh jb \\ &= \cosh a \cos b + j \sinh a \sin b \qquad (4\text{-}24)\end{aligned}$$

A few useful half-angle identities, which can be proved from the above are:

$$\sinh \frac{u}{2} = \sqrt{\frac{1}{2}(\cosh u - 1)} \qquad (4\text{-}25)$$

$$\cosh \frac{u}{2} = \sqrt{\frac{1}{2}(\cosh u + 1)} \qquad (4\text{-}26)$$

$$\sinh u = 2 \sinh \frac{u}{2} \cosh \frac{u}{2} \qquad (4\text{-}27)$$

A considerable number of hyperbolic functions will prove useful in the sections to follow.

4-5. Properties of symmetrical networks

Use of the definition of γ, and the introduction of ϵ^γ as the current ratio for a Z_0 terminated network, leads to further useful results.

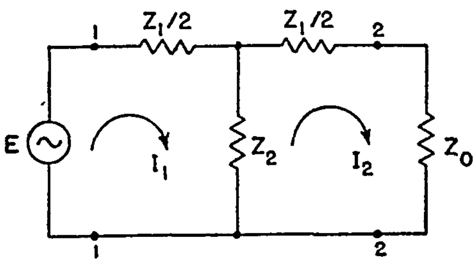


Fig. 4-6. Symmetrical network with generator and load.

In Fig. 4-6, the T network is considered equivalent to any connected symmetrical network, and is terminated in a load Z_0. The mesh equations are

$$E = I_1\left(\frac{Z_1}{2} + Z_2\right) - I_2 Z_3 \qquad (4\text{-}28)$$

$$0 = -I_1 Z_2 + I_2\left(\frac{Z_1}{2} + Z_2 + Z_0\right) \qquad (4\text{-}29)$$

The current ratio for the two meshes, which is equal to ϵ^γ by definition, can be obtained from the second equation as

$$\frac{I_1}{I_2} = \frac{Z_1/2 + Z_2 + Z_0}{Z_2} = \epsilon^\gamma \tag{4-30}$$

After thus introducing ϵ^γ, the above may be written

$$Z_0 = Z_2(\epsilon^\gamma - 1) - \frac{Z_1}{2} \tag{4-31}$$

From Eq. 4-10 for the characteristic impedance,

$$Z_0^2 = \frac{Z_1^2}{4} + Z_1Z_2 \tag{4-32}$$

If Z_0 is eliminated by use of Eq. 4-32 in Eq. 4-31, there results

$$Z_2(\epsilon^\gamma - 1)^2 - Z_1\epsilon^\gamma = 0$$

$$\epsilon^{2\gamma} - 2\epsilon^\gamma + 1 = \frac{Z_1}{Z_2}\epsilon^\gamma$$

$$\frac{\epsilon^\gamma + \epsilon^{-\gamma}}{2} = 1 + \frac{Z_1}{2Z_2}$$

$$\cosh\gamma = 1 + \frac{Z_1}{2Z_2} \tag{4-33}$$

Equation 4-33 and its other derived forms will be of considerable value in the study of filters.

By use of the identity, Eq. 4-20, that

$$\cosh^2\gamma - \sinh^2\gamma = 1$$

it is possible to write

$$\sinh\gamma = \frac{Z_0}{Z_2} \tag{4-34}$$

Combining Eqs. 4-33 and 4-34 leads to

$$\tanh\gamma = \frac{Z_0}{Z_1/2 + Z_2} \tag{4-35}$$

By use of Eq. 4-25 it is possible to write

$$\sinh\frac{\gamma}{2} = \sqrt{\frac{1}{2}\left(1 + \frac{Z_1}{2Z_2} - 1\right)}$$
$$= \sqrt{\frac{Z_1}{4Z_2}} \tag{4-36}$$

an expression which will serve to predict filter performance.

The propagation constant γ can be related to the network parameters by use of Eq. 4-10, for Z_{0T}, in Eq. 4-30 as

$$\epsilon^{\gamma} = 1 + \frac{Z_1}{2Z_2} + \sqrt{\left(\frac{Z_1}{2Z_2}\right)^2 + \frac{Z_1}{Z_2}} \tag{4-37}$$

Taking the natural logarithm

$$\gamma = \ln\left[1 + \frac{Z_1}{2Z_2} + \sqrt{\left(\frac{Z_1}{2Z_2}\right)^2 + \frac{Z_1}{Z_2}}\right] \tag{4-38}$$

For a network of pure reactances this is not difficult to compute. For an impedance it may be noted that the logarithm of a complex quantity $B\underline{/\alpha} = \ln B + j\alpha$.

The input impedance of any T network, terminated in any impedance Z_R, may also be written in terms of hyperbolic functions of γ. Writing

$$Z_{in} = Z_{11} - \frac{Z_{12}^2}{Z_{22}}$$

and substituting the required mesh relations from Fig. 4-6, with Z_0 replaced by Z_R, then

$$Z_{in} = \frac{Z_1}{2} + Z_2 - \frac{Z_2^2}{Z_1/2 + Z_2 + Z_R}$$
$$= \frac{Z_1^2/4 + Z_1Z_2 + (Z_1/2 + Z_2)Z_R}{Z_1/2 + Z_2 + Z_R}$$

Use of Eqs. 4-32 and 4-35 leads to

$$Z_{in} = \frac{Z_0^2 + Z_RZ_0/\tanh\gamma}{Z_0/\tanh\gamma + Z_R}$$
$$= Z_0\left(\frac{Z_R\cosh\gamma + Z_0\sinh\gamma}{Z_0\cosh\gamma + Z_R\sinh\gamma}\right) \tag{4-39}$$

This is the input impedance of a symmetrical T network terminated in a load Z_R, in terms of the propagation constant and Z_0 of the network.

For a short-circuited network $Z_R = 0$. The input impedance is then Z_{sc} where, from the above equation,

$$Z_{sc} = Z_0 \tanh \gamma \tag{4-40}$$

For an open circuit $Z_R = \infty$ in the limit, and Z_{oc} is then

$$\underset{\lim Z_R \to \infty}{Z_{oc}} = \frac{Z_0}{\tanh \gamma} \tag{4-41}$$

From these two equations it can be seen that

$$\tanh \gamma = \sqrt{\frac{Z_{sc}}{Z_{oc}}} \tag{4-42}$$

and

$$Z_0 = \sqrt{Z_{oc}Z_{sc}}$$

which has already been proved from the properties of the characteristic impedance.

In Chapter 1, open-circuit and short-circuit measurements were used to describe the performance of a network. In this chapter, two new parameters, the characteristic impedance Z_0, and the propagation constant γ, have been introduced, and the properties of the network have been developed in terms of these new parameters. The last few equations are relations between the two sets of parameters.

4-6. Filter fundamentals; pass and stop bands

Ideally it is desired that a filter network transmit or *pass* a desired frequency band without loss, whereas it should *stop* or completely *attenuate* all undesired frequencies. The propagation constant $\gamma = \alpha + j\beta$, being a function of frequency by Eq. 4-38, can supply information on the ability of the filter to perform as desired. If $\alpha = 0$ or $I_1 = I_2$, then there is no attenuation, only a phase shift, in transmitting a signal through the filter, and operation is in a *pass band* of frequencies. When α has a positive value, then I_2 is smaller in magnitude than I_1, attenuation has occurred and operation is in an attenuation or *stop band* of frequencies.

The propagation constant γ may be conveniently studied by use of Eq. 4-36:

$$\sinh \frac{\gamma}{2} = \sqrt{\frac{Z_1}{4Z_2}} \tag{4-43}$$

It will first be assumed that the network contains only pure reactances, and thus $Z_1/4Z_2$ will be real, and either positive or negative, depending on the type of reactance used for Z_1 and Z_2. Expanding gives

$$\sinh \frac{\gamma}{2} = \sinh \left(\frac{\alpha}{2} + \frac{j\beta}{2}\right)$$

$$= \sinh \frac{\alpha}{2} \cos \frac{\beta}{2} + j \cosh \frac{\alpha}{2} \sin \frac{\beta}{2} \tag{4-44}$$

as an equation containing much information.

If Z_1 and Z_2 *are the same type of reactance* then $|Z_1/4Z_2| > 0$, or the ratio is positive and real. This requires that $\sinh \gamma/2$ be real, which means that the imaginary term in Eq. 4-44 must equal zero and that

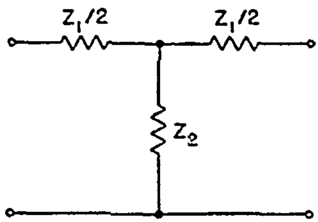


Fig. 4-7. Symmetrical T network.

(a) $\cosh \frac{\alpha}{2} \sin \frac{\beta}{2} = 0$

(b) $\sinh \frac{\alpha}{2} \cos \frac{\beta}{2} = \sqrt{\frac{Z_1}{4Z_2}}$

are simultaneously satisfied.

From (a),

$$\sin \frac{\beta}{2} = 0; \quad \beta = n\pi \qquad \text{where } n = 0, 2, 4, \ldots$$

From (b), since $\cos \beta/2 = 1$, then

$$\sinh \frac{\alpha}{2} = \sqrt{\frac{Z_1}{4Z_2}}$$

and the attenuation will be given by

$$\alpha = 2 \sinh^{-1} \sqrt{\frac{Z_1}{4Z_2}} \tag{4-45}$$

Thus the *condition that* $|Z_1/4Z_2| > 0$ *implies a stop or attenuation band of frequencies.*

If Z_1 *and* Z_2 *are opposite types of reactance* then $Z_1/4Z_2$ is negative, $|Z_1/4Z_2| < 0$, and the radical of Eq. 4-43 is imaginary. The real term in Eq. 4-44 must then be zero, so that

(c) $$\sinh\frac{\alpha}{2}\cos\frac{\beta}{2} = 0$$

(d) $$\cosh\frac{\alpha}{2}\sin\frac{\beta}{2} = \sqrt{\frac{Z_1}{4Z_2}}$$

must be satisfied.

Two conditions are possible from the above:

I. $\sinh\frac{\alpha}{2} = 0$; therefore

$$\alpha = 0; \qquad \beta \neq 0; \qquad \sin\frac{\beta}{2} = \sqrt{\frac{Z_1}{4Z_2}}$$

II. $\cos\frac{\beta}{2} = 0$; therefore $\sin\frac{\beta}{2} = \pm 1$ and

$$\alpha \neq 0; \qquad \beta = (2n-1)\pi; \qquad \cosh\frac{\alpha}{2} = \sqrt{\frac{Z_1}{4Z_2}}$$

Condition I leads to a pass band, or region of zero attenuation, which is limited by the upper limit on the sine, or by $\sin\frac{\beta}{2} = 1$, or it is required that

$$-1 < \frac{Z_1}{4Z_2} < 0$$

The phase angle in this pass band will be given by

$$\beta = 2\sin^{-1}\sqrt{\frac{Z_1}{4Z_2}} \tag{4-46}$$

Condition II leads to a stop or attenuation band since $\alpha \neq 0$. The phase angle is π, and the attenuation is given by

$$\alpha = 2\cosh^{-1}\sqrt{\frac{Z_1}{4Z_2}} \tag{4-47}$$

Because the hyperbolic cosine has no value below unity, it appears that the region in which condition II applies is a stop band where

$$\frac{Z_1}{4Z_2} < -1$$

Values of $Z_1/4Z_2$ can then be classified into three regions, with corresponding values of α and β, these regions being bounded by $Z_1/4Z_2$ values of $+\infty$, 0, -1 and $-\infty$, as given below:

$Z_1/4Z_2 =$	$+\infty$ to 0	0 to -1	-1 to $-\infty$
Reactance type:	Same	Opposite	Opposite
Band:	Stop	Pass	Stop
α:	$2\sinh^{-1}\sqrt{\dfrac{Z_1}{4Z_2}}$	0	$2\cosh^{-1}\sqrt{\dfrac{Z_1}{4Z_2}}$
β:	π	$2\sin^{-1}\sqrt{\dfrac{Z_1}{4Z_2}}$	π

The frequencies at which the network changes from a pass network to a stop network, or vice versa, are called *cutoff frequencies.* These frequencies occur when

$$\left.\begin{aligned} \frac{Z_1}{4Z_2} &= 0 \quad \text{or} \quad Z_1 = 0 \\ \frac{Z_1}{4Z_2} &= -1 \quad \text{or} \quad Z_1 = -4Z_2 \end{aligned}\right\} \tag{4-48}$$

where Z_1 and Z_2 are opposite types of reactance.

Since Z_1 and Z_2 may have a number of configurations, as L and C elements, or as parallel and series combinations, a variety of types of performance are possible.

The elements considered above were assumed pure reactances, and design is ordinarily carried out on this basis. Measurements of actual performance are then made and adjustments introduced into the design to compensate for deviation of the results from the ideal. In addition to minimizing the losses of physical elements it is also necessary to reduce stray electric and magnetic couplings between elements to obtain more nearly the predicted performance.

4-7. Behavior of the characteristic impedance

It has been shown that for a symmetrical T network

$$Z_{0T} = \sqrt{Z_1Z_2\left(1 + \frac{Z_1}{4Z_2}\right)}$$

In a network made up entirely of pure reactances this expression for the characteristic impedance becomes

$$Z_{0T} = \sqrt{-X_1X_2\left(1 + \frac{X_1}{4X_2}\right)} \tag{4-49}$$

where the X terms will carry their own signs, the minus sign under the radical being due to j^2.

In Section 4-6 it was shown that a stop band exists where X_1 and X_2 are the same type of reactance. The ratio $X_1/4X_2$ will be real and positive, and the characteristic impedance will be a pure reactance in this attenuation region.

A pass band was shown to exist where X_1 and X_2 were of opposite reactance types and $-1 < X_1/4X_2 < 0$. Placing these conditions in Eq. 4-49 results in the product X_1X_2 being negative, with the bracketed term positive. The over-all sign under the radical will be positive and Z_0 will be real, and thus able to absorb power from a source.

A stop band exists with X_1 and X_2 of opposite types, but with $X_1/4X_2 < -1$. This implies that the product X_1X_2 is a negative term, and that the bracketed term is negative. When combined with the negative sign present in Eq. 4-49, the over-all sign under the radical will be negative and Z_0 will be a pure reactance in this stop region.

It has been shown that in a pass band Z_0 is real and positive. If the reactive network is terminated with a resistive $Z_0 = R_0$, then the input impedance is R_0, and the network can accept power and will transmit it to the resistive load without loss or attenuation. If the network is supplied by a source having R_0 as its internal impedance, the system will be matched at each set of terminals, and maximum power will be delivered from generator to load.

In a stop band Z_0 has been shown to be reactive. If the network is terminated in its reactive Z_0, it will appear as a totally reactive circuit and as such cannot accept or transmit power, since there is no resistive element in which the power may be dissipated. The network may transmit voltage or current, but with a 90° phase angle between the two and with considerable attenuation.

Similar reasoning may be applied to the Z_0 for a π network if it is noted that

$$Z_{0\pi} = \frac{Z_1 Z_2}{Z_{0T}} \tag{4-50}$$

and Z_1Z_2 is always real for Z_1 and Z_2 as pure reactances. Thus it is seen that the conditions developed for pass and stop bands for T sections likewise apply for π sections.

4-8. The constant-k low-pass filter

If Z_1 and Z_2 of a reactance network are unlike reactance arms, then

$$Z_1 Z_2 = k^2$$

where k is a constant independent of frequency. Networks or filter sections for which this relation holds are called *constant-k* filters.

As a special case, let $Z_1 = j\omega L$ and $Z_2 = -j/\omega C$, then the product

$$Z_1 Z_2 = \frac{L}{C} = R_k^2 \tag{4-51}$$

The term R_k is used since k must be real if Z_1 and Z_2 are of opposite type. A T section so designed would appear as at (a), Fig. 4-8.

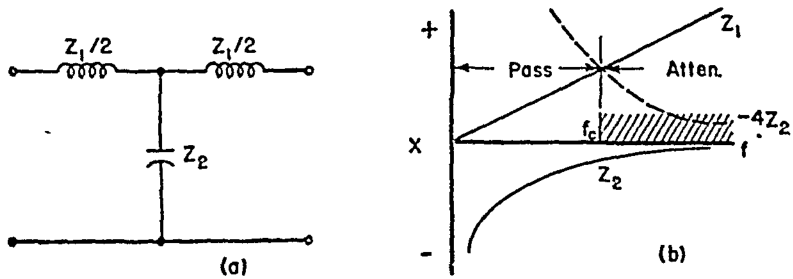


Fig. 4-8. (a) Low-pass filter section; (b) reactance curves demonstrating that (a) is a low-pass section or has a pass band between $Z_1 = 0$ and $Z_1 = -4Z_2$.

The reactances of Z_1 and $4Z_2$ will vary with frequency as sketched at (b), Fig. 4-7. The curve representing $-4Z_2$ may be drawn and compared with the curve for Z_1. It has been shown by Eq. 4-48 that a pass band starts at the frequency at which $Z_1 = 0$ and runs to the frequency at which $Z_1 = -4Z_2$. Thus the reactance curves show that a pass band starts at $f = 0$ and continues to some higher frequency f_c. All frequencies above f_c lie in a stop, or attenuation, band. Thus the network is called a *low-pass* filter.

The cutoff frequency f_c may be readily determined, since at that point

$$Z_1 = -4Z_2, \qquad j\omega_c L = \frac{4j}{\omega_c C}$$

$$f_c = \frac{1}{\pi \sqrt{LC}} \tag{4-52}$$

This expression may be used to develop certain relations applicable to the low-pass network. Then $\sinh \gamma/2$ may be evaluated as

$$\sinh \frac{\gamma}{2} = \sqrt{\frac{Z_1}{4Z_2}} = \sqrt{\frac{-\omega^2 LC}{4}} = \frac{j\omega \sqrt{LC}}{2}$$

and in view of Eq. 4-52 this is

$$\sinh \frac{\gamma}{2} = j\frac{f}{f_c} \tag{4-53}$$

Then if the frequency f is in the pass band or $f/f_c < 1$, so that $-1 < Z_1/4Z_2 < 0$, then

$$\frac{f}{f_c} < 1, \qquad \alpha = 0, \qquad \beta = 2 \sin^{-1}\left(\frac{f}{f_c}\right)$$

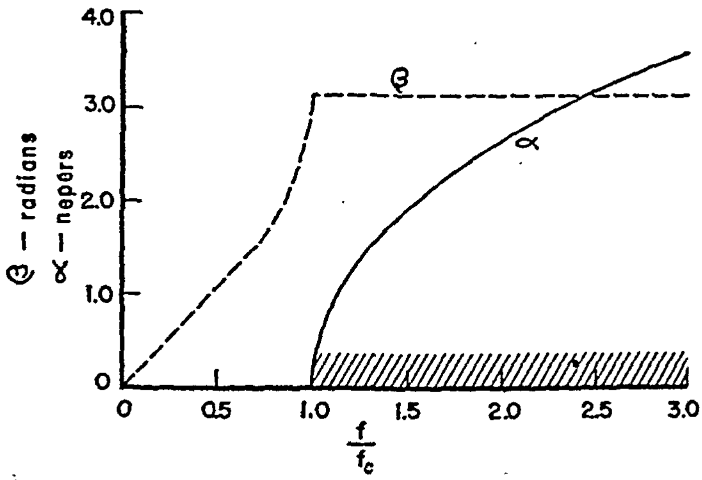


Fig. 4-9. Variation of α and β with frequency for the low-pass section.

whereas if frequency f is in the attenuation band or $f/f_c > 1$, so that $Z_1/4Z_2 < -1$, then

$$\frac{f}{f_c} > 1, \qquad \alpha = 2\cosh^{-1}\left(\frac{f}{f_c}\right), \qquad \beta = \pi$$

thereby allowing determination of α and β. The variation of α and β is plotted in Fig. 4-9 as a function of f/f_c. This method shows that the attenuation α is zero throughout the pass band but rises gradually from the cutoff frequency at $f/f_c = 1.0$ to a value of ∞ at infinite frequency. The phase shift β is zero at zero frequency and increases gradually through the pass band, reaching π at f_c and remaining at π for all higher frequencies.

The characteristic impedance of a T section was obtained as

$$Z_{0T} = \sqrt{Z_1Z_2\left(1 + \frac{Z_1}{4Z_2}\right)}$$

which becomes

$$Z_{0T} = \sqrt{\frac{L}{C}\left(1 - \frac{\omega^2LC}{4}\right)} \tag{4-54}$$

for the low-pass constant-k section under discussion. By use of Eq. 4-52 the characteristic impedance of a low-pass filter may be stated as

$$Z_{0T} = \sqrt{\frac{L}{C}\left[1 - \left(\frac{f}{f_c}\right)^2\right]} \tag{4-55}$$

$$= R_k\sqrt{1 - \left(\frac{f}{f_c}\right)^2} \tag{4-56}$$

in accordance with the definition of R_k in Eq. 4-51. Values of Z_{0T}/R_k are plotted against f/f_c in Fig. 4-10. It may be seen that Z_{0T} varies throughout the pass band, reaching a value of zero at cutoff, then becomes imaginary in the attenuation band, rising to infinite reactance at infinite frequency.

A low-pass filter may be designed from a knowledge of the cutoff frequency desired and the load resistance to be supplied. It is desirable that the Z_0 in the pass band match the load; but because of the nature of the Z_0 curve in Fig. 4-10, this result can occur at only one frequency. This match may be arranged to occur at any frequency which it is desired to favor by an impedance match.

For reasons which will appear in Section 4-13, the load is chosen as $R = R_k = \sqrt{L/C}$, which will favor zero frequency for a low-pass filter.

The design of a low-pass filter may be readily carried out. From

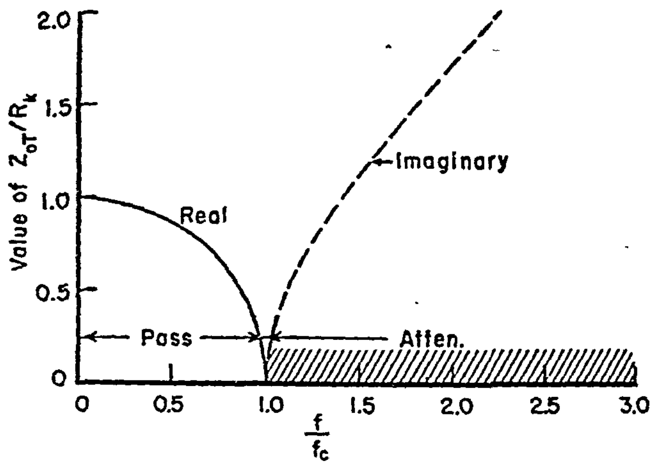


Fig. 4-10. Variation of Z_{0T}/R_k with frequency for the low-pass section.

the relation that at cutoff

$$Z_1 = -4Z_2$$

it is seen that

$$\omega_c L = \frac{4}{\omega_c C}$$

Using the cutoff frequency equation changes this to

$$\pi^2 f_c^2 LC = 1$$

and use of the relation $R = \sqrt{L/C}$ gives for the value of the shunt capacitance arm

$$C = \frac{1}{\pi f_c R} \tag{4-57}$$

By similar methods the inductance for Z_1 is obtained as

$$L = \frac{R}{\pi f_c} \tag{4-58}$$

Since the design is based on an impedance match at zero frequency only, power transfer to a matched load will drop at higher pass-band frequencies. This condition may be undesirable in certain applications, and a remedy will be discussed in Section 4-13.

A network such as is described here is called a *prototype section.* It may be employed when a sharp cutoff is not required, although cutoff may be sharpened by using a number of such networks in cascade. This is not usually an economic use of circuit elements, and introduces excessive losses over other available methods of raising the attenuation near the cutoff frequency.

4-9. The constant-k high-pass filter

If the positions of inductance and capacitance are interchanged to make $Z_1 = -j/\omega C$ and $Z_2 = j\omega L$, then Z_1Z_2 will still be given by

$$Z_1Z_2 = k^2$$

and the filter design obtained will be of the constant-k type. The T section will then appear as in (a), Fig. 4-11. The reactances of Z_1 and Z_2 are sketched as functions of frequency in (b), and Z_1

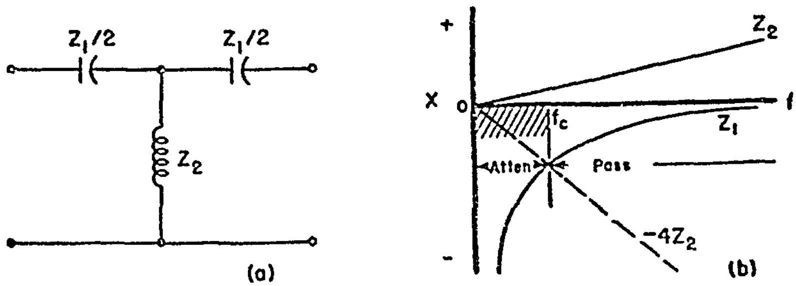


Fig. 4-11. (a) High-pass filter section; (b) reactance curves demonstrating that (a) is a high-pass section or has a pass band between $Z_1 = 0$ and $Z_1 = -4Z_2$.

is compared with $-4Z_2$, showing a cutoff frequency at the point at which Z_1 equals $-4Z_2$, with a pass band from that frequency to infinity where $Z_1 = 0$. The network is thus a *high-pass* filter. All frequencies below f_c lie in an attenuation, or stop, band.

The cutoff frequency is determined as the frequency at which $Z_1 = -4Z_2$, or

$$\frac{-j}{\omega_c C} = -j4\omega_c L, \qquad 4\omega_c^2 LC = 1$$

$$f_c = \frac{1}{4\pi\sqrt{LC}} \tag{4-59}$$

Using the above expression

$$\sinh\frac{\gamma}{2} = \sqrt{\frac{Z_1}{4Z_2}} = \sqrt{-\frac{1}{4\omega^2LC}} = \frac{-j}{2\omega\sqrt{LC}} = -j\frac{f_c}{f} \qquad (4\text{-}60)$$

The region in which $f_c/f < 1$ is a pass band, so that the variation of γ inside and outside the pass band will be identical with the values for the low-pass filter, and the curves of Fig. 4-9 will apply if the abscissa be considered as calibrated in terms of f_c/f, except that the phase angle β will be negative, changing from 0 at infinite frequency or $f_c/f = 0$, to $-\pi$ at cutoff or $f_c/f = 1$.

The high-pass filter may be designed by again choosing a resistive load R equal to R_k such that

$$R = R_k = \sqrt{\frac{L}{C}} \qquad (4\text{-}61)$$

From the relation that at cutoff $Z_1 = -4Z_2$ it was shown that

$$4\omega_c^2LC = 1$$

and again $L/C = R^2$, so that the value of the capacitance for Z_1, the series element, is

$$C = \frac{1}{4\pi f_c R} \qquad (4\text{-}62)$$

It should be noted that since $Z_1/2$ is the value of each series arm, the capacity used in each series $Z_1/2$ element should be $2C$. By similar methods the value for the inductance for Z_2, the shunt arm, is

$$L = \frac{R}{4\pi f_c} \qquad (4\text{-}63)$$

The characteristic impedance for the *high-pass filter* may be transformed to

$$Z_{0T} = R_k\sqrt{1 - \left(\frac{f_c}{f}\right)^2} \qquad (4\text{-}64)$$

4-10. The m-derived T section

The constant-k prototype filter section, though simple, has two major disadvantages. The attenuation does not rise very rapidly at cutoff, so that frequencies just outside the pass band are not appreciably attenuated with respect to frequencies just inside the pass band. Also, the characteristic impedance varies widely over the pass band, so that a satisfactory impedance match is not possible. In cases where an impedance match is not important, the attenuation may be built up near cutoff by cascading or connecting a number of constant-k sections in series.

It is more economical to attempt to raise the attenuation near cutoff by other means. Consider first the circuit of (a), Fig. 4-12. The reactance curves sketched at (b) show that this circuit is a low-pass filter. However, it can be seen that the shunt arm is a series

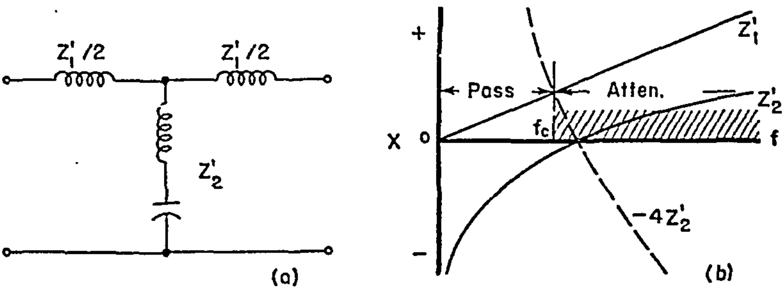


Fig. 4-12. (a) Derivation of a low-pass section having a sharp cutoff action; (b) reactance curves for (a).

circuit resonant at a frequency above f_c. At this resonant frequency. the shunt arm appears as a short circuit on the network, or the attenuation becomes infinite. This frequency of infinite or high attenuation is called f_∞; and by reason of the requirement that below f_c the shunt circuit appear as a capacitance, the frequency of resonance, f_∞, will always be higher in value than f_c. If, then, f_∞ can be chosen arbitrarily close to f_c, the attenuation near cutoff may be made high.

The attenuation above f_∞ will fall to low values, so that if high attenuation is desired over the whole attenuation band, it is necessary to use a section such as in Fig. 4-12 for high attenuation near cutoff, in series with a prototype section to provide high attenuation at frequencies well removed from cutoff. For satisfactory matching of several such types of filters in series, it is necessary that the Z_0 of all be identical at all points in the pass band. They will consequently also all have the same pass band.

The network of Fig. 4-12 may be derived by assuming that

$$Z_1' = mZ_1 \tag{4-65}$$

the primes indicating the *derived section*. It is then necessary to find the value for Z_2' such that $Z_0' = Z_0$. Setting the characteristic impedances equal,

$$Z_0' = Z_0$$

$$\frac{(mZ_1)^2}{4} + mZ_1Z_2' = \frac{Z_1^2}{4} + Z_1Z_2$$

$$Z_2' = \frac{Z_2}{m} + \frac{1 - m^2}{4m} Z_1 \tag{4-66}$$

It then appears that the shunt arm Z_2' consists of two impedances in series, as shown in Fig. 4-13. As required, the characteristic impedance and f_c remain equal to those of the T section prototype containing Z_1 and Z_2 values.

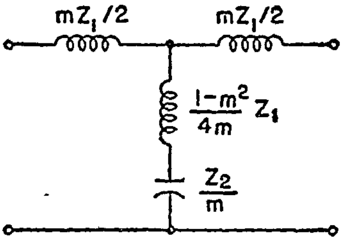


Fig. 4-13. The m-derived low-pass filter.

Since m is arbitrary, it is possible to design an infinite variety of filter networks meeting the required conditions on Z_0 and f_c. However, Z_2 will be opposite in sign to Z_1, and it is desired that this relation continue in the two series impedances given by Eq. 4-66 for the Z_2' arm. Equation 4-66 then indicates that $(1 - m^2)/4m$ must be positive, forcing the terms $1 - m^2$ and m always to be positive. Thus m must always be chosen so that

$$0 < m < 1$$

Filter sections obtained in this manner are called *m-derived sections.*

The shunt arm is to be chosen so that it is resonant at some frequency f_∞ above f_c. This means that at the resonant frequency

$$\left|\frac{Z_2}{m}\right| = \left|\frac{1 - m^2}{4m} Z_1\right| \tag{4-67}$$

and for the *low-pass filter*

$$\frac{1}{2\pi f_\infty mC} = \frac{1 - m^2}{4m} 2\pi f_\infty L$$

$$f_\infty = \frac{1}{\pi \sqrt{(1 - m^2)LC}}$$

Since the cutoff frequency for the low-pass filter is

$$f_c = \frac{1}{\pi \sqrt{LC}}$$

the frequency of infinite attenuation will be

$$f_\infty = \frac{f_c}{\sqrt{1 - m^2}} \tag{4-68}$$

from which

$$m = \sqrt{1 - (f_c/f_\infty)^2} \tag{4-69}$$

This equation determines the m to be used for a particular f_∞.

Similar relations for the *high-pass filter* can be derived as

$$f_\infty = f_c \sqrt{1 - m^2} \tag{4-70}$$

and

$$m = \sqrt{1 - (f_\infty/f_c)^2} \tag{4-71}$$

The m-derived section is designed following the design of the prototype T section. The use of a prototype and one or more m-derived sections in series results in a *composite filter*. If a sharp cutoff is desired, an m-derived section may be used with f_∞ near f_c, followed by as many m-derived sections as desired to place frequencies of high attenuation where needed to suppress various signal components or to produce a high attenuation over the entire attenuation band.

The variation of attenuation over the attenuation band for a *low-pass m-derived section* in the stop band is dependent on the sign of the reactances or

$$\alpha = 2 \cosh^{-1} \sqrt{\left|\frac{Z_1}{4Z_2}\right|} \quad \text{or} \quad \alpha = 2 \sinh^{-1} \sqrt{\left|\frac{Z_1}{4Z_2}\right|}$$

$$f_c < f < f_\infty \qquad\qquad f_\infty < f$$

For $Z_1 = j\omega L$ and $Z_2 = -j/\omega C$ for the prototype, then

$$\left|\frac{Z_1}{4Z_2}\right| = \frac{m\omega L}{4[1/m\omega C - \omega L(1 - m^2)/4m]}$$

so that for $f_c < f < f_\infty$

$$\alpha = 2 \cosh^{-1} \frac{mf/f_c}{\sqrt{1 - f^2/f_\infty^2}} \tag{4-72}$$

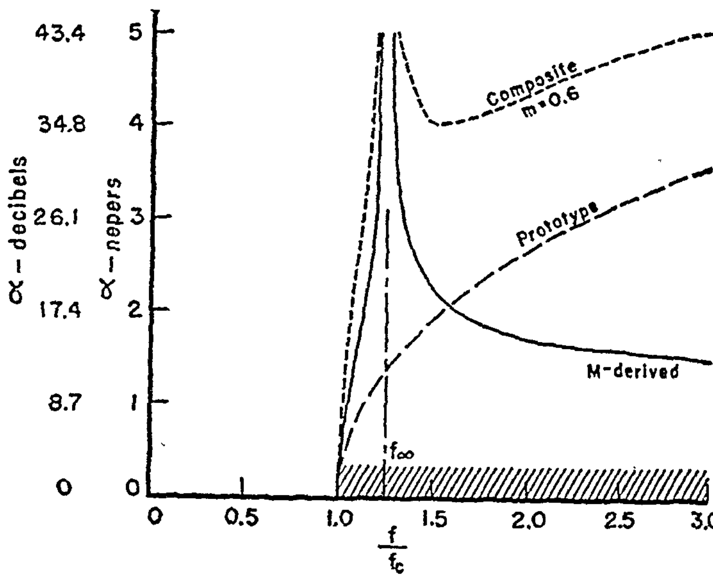


Fig. 4-14. Variation of attenuation for the prototype and m-derived sections, and the composite result of the two in series.

and for $f_\infty < f$

$$\alpha = 2 \sinh^{-1} \frac{mf/f_c}{\sqrt{f^2/f_\infty^2 - 1}} \tag{4-73}$$

The value of α may be determined from this expression. Figure 4–14 is a plot of α against f/f_c for $m = 0.6$, which gives a value of f_∞ equal to 1.25 times the cutoff frequency f_c. The great increase in sharpness of cutoff for the m-derived section over the prototype is apparent. The higher attenuation over the whole attenuation band obtained by use of a prototype section and an m-derived section in series as a composite filter is also readily seen.

Again following the procedure of Section 4–8, the phase shift constant β may be determined, in the pass band, from

$$\beta = 2 \sin^{-1} \sqrt{\left|\frac{Z_1}{4Z_2}\right|}$$

$$= 2 \sin^{-1} \frac{mf/f_c}{\sqrt{1 - (f^2/f_c^2)(1 - m^2)}} \tag{4-74}$$

In the attenuation band, up to f_∞, β has the value π. Above f_∞ the value of β drops to zero, because the shunt arm becomes inductive above resonance. The phase shift of the m-derived section is plotted as a function of f/f_c in Fig. 4-15.

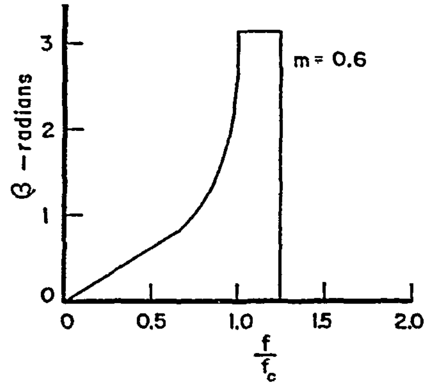


Fig. 4-15. Variation of phase shift β, for the m-derived filter.

This material demonstrates the ability of the m-derived section to overcome the lack of a sharp cutoff in the simple prototype filter. Although it may be noted that the sharpness of cutoff increases for small values of m, the attenuation beyond the point of peak attenuation becomes smaller for small m. This emphasizes the necessity of supplementing the m-derived section with a prototype section in series to raise the attenuation for frequencies well removed from cutoff.

4-11. The m-derived π section

An m-derived π section may also be obtained. The characteristic impedance of the π section is

$$Z_{0\pi} = \frac{Z_1 Z_2}{\sqrt{Z_1 Z_2 (1 + Z_1/4Z_2)}}$$

The characteristic impedances of the prototype and m-derived sections are to be equal so that they may be joined without mismatch. By use of the transformation for the shunt arm,

$$Z_2' = \frac{Z_2}{m} \tag{4-75}$$

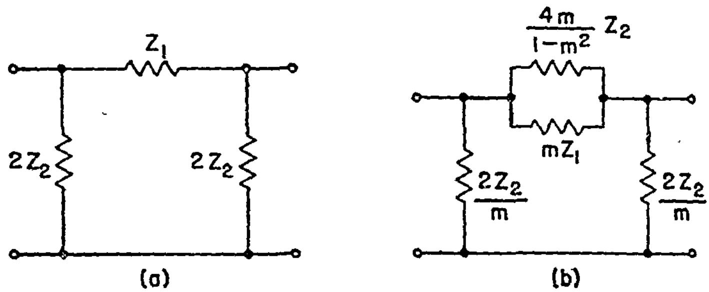


Fig. 4-16. (a) Usual symmetrical π section; (b) the m-derived π filter.

it is possible to equate the characteristic impedances as

$$\frac{Z_1'Z_2/m}{\sqrt{Z_1(Z_2/m)(1 + Z_1'm/4Z_2)}} = \frac{Z_1Z_2}{\sqrt{Z_1Z_2(1 + Z_1/4Z_2)}}$$

from which

$$Z_1' = \frac{1}{\dfrac{1}{mZ_1} + \dfrac{1}{\dfrac{4m}{1-m^2}Z_2}} \tag{4-76}$$

It is apparent that the series arm Z_1' is represented by two impedances in parallel, one being mZ_1, the other being $4m/(1 - m^2)Z_2$ in value.

Equations 4-75 and 4-76 thus give the values to be used in designing the m-derived π section. The circuit is drawn in Fig. 4-16.

4-12. Variation of characteristic impedance over the pass band

It has been shown in Section 4-8 that for a low-pass T section

$$Z_{0T} = R_k\sqrt{1 - \left(\frac{f}{f_c}\right)^2} \tag{4-77}$$

The characteristic impedance for a π section is

$$Z_{0\pi} = \frac{Z_1 Z_2}{\sqrt{Z_1 Z_2 (1 + Z_1/4Z_2)}}$$

Since $Z_1 = j\omega L$ and $Z_2 = -j/\omega C$ for the *low-pass filter*, use of the cutoff frequency expression permits the characteristic impedance of

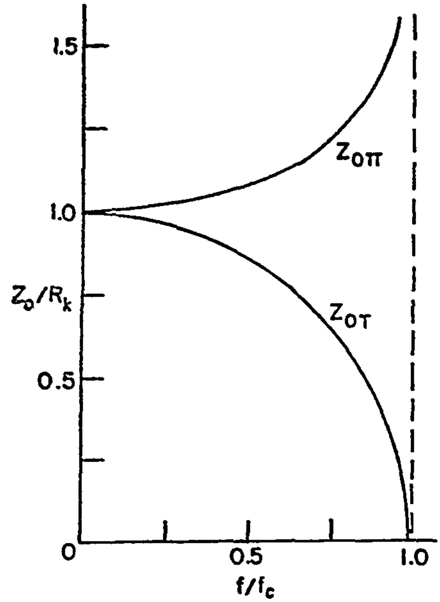


Fig. 4-17. Manner of variation of Z_0 over the pass band for the T and π networks.

the low-pass π section to be expressed as

$$Z_{0\pi} = \frac{L/C}{\sqrt{L/C[1 - (f/f_c)^2]}}$$

$$= \frac{R_k}{\sqrt{1 - (f/f_c)^2}} \tag{4-78}$$

The π-section characteristic impedance is plotted over the pass band in Fig. 4-17 as a function of f/f_c and is compared with the curve for the T section, reproduced from Fig. 4-10.

The curves show that the characteristic impedance of neither section is sufficiently constant over the pass band that a load equal to R_k will give a satisfactory impedance match.

4-13. Termination with m-derived half sections

In Chapter 3, reactance L sections were designed that would transform a given resistance to a more desired value. The problem

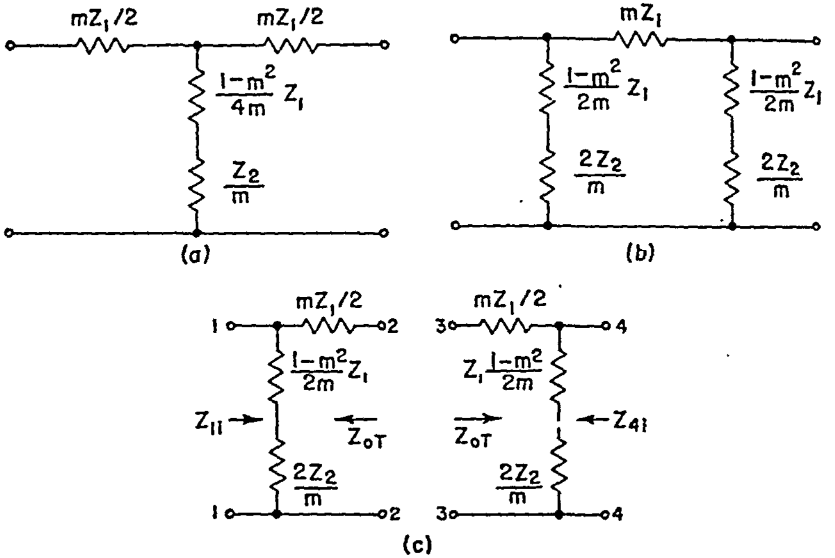


Fig. 4-18. (a) m-derived T section; (b) π section formed by rearranging constants of (a); (c) circuit of (b) split into two L sections.

of satisfactorily terminating or matching a T or π filter to a given load could be performed by an L-section network at one frequency. The real problem, however, is that of causing an L section to change its characteristics with frequency in such a way that the filter is approximately matched to its load at all frequencies over the pass band. Zobel discovered that an m-derived half section or L section could be made to have the desired properties over most of the pass band.

Consider the π section in (b), Fig. 4-18, formed by use of the elements of the m-derived T section of (a). This π section can be split into two half sections, with values as at (c). The image impedance of the left half section at the 1,1 terminals is given by Eq. 3-17 as

$$Z_{1i} = \sqrt{Z_{1\infty} Z_{1sc}}$$

so that

$$Z_{1i} = \sqrt{\frac{[(1-m^2)Z_1/2m + 2Z_2/m]^2(mZ_1/2)}{(1-m^2)Z_1/2m + 2Z_2/m + mZ_1/2}}$$

$$= \left(\frac{1-m^2}{2m} Z_1 + \frac{2Z_2}{m}\right)\sqrt{\frac{(mZ_1/2)^2}{(1-m^2)Z_1^2/4 + Z_1Z_2 + m^2Z_1^2/4}}$$

$$= \left[1 + (1-m^2)\frac{Z_1}{4Z_2}\right]\sqrt{\frac{Z_1Z_2}{1 + Z_1/4Z_2}} \qquad (4\text{-}80)$$

By Eq. 4-11, the above is recognizable as

$$Z_{1i} = \left[1 + (1-m^2)\frac{Z_1}{4Z_2}\right]Z_{0\tau} \qquad (4\text{-}81)$$

Equation 4-81 shows the 1,1 image impedance to be a function of $Z_{0\tau}$ modified by the factor $1 + (1-m^2)\, Z_1/4Z_2$ and thus to have possibilities of variation of image impedance with values of m.

For the *low-pass* filter, Eq. 4-81 may be written as

$$Z_{1i} = \frac{R_k[1 - (1-m^2)f^2/f_c^2]}{\sqrt{1 - f^2/f_c^2}} \qquad (4\text{-}82)$$

The variation of Z_{1i}, or the image impedance at the 1,1 terminals of Fig. 4-18, is plotted over the pass band for several values of m in Fig. 4-19. It can be seen that, by use of the value $m = 0.6$ for the half section, a nearly constant value of image impedance equal to R_k is obtained at the 1,1 terminals over 85 per cent of the pass band. A source impedance equal to R_k then could be matched satisfactorily on an image basis at the 1,1 terminals over most of the pass band.

A similar variation with m can be developed for the *high-pass* filter.

The image impedance of the left half section of Fig. 4-18(c) at the 2,2 terminals may be written

$$Z_{2i} = \sqrt{Z_{2oc}Z_{2sc}}$$

$$= \sqrt{\left(\frac{mZ_1}{2} + \frac{1-m^2}{2m}Z_1 + \frac{2Z_2}{m}\right)\frac{mZ_1}{2}}$$

$$= \sqrt{Z_1Z_2\left(1 + \frac{Z_1}{4Z_2}\right)} \qquad (4\text{-}83)$$

$$Z_{2i} = Z_{0T} \qquad (4\text{-}84)$$

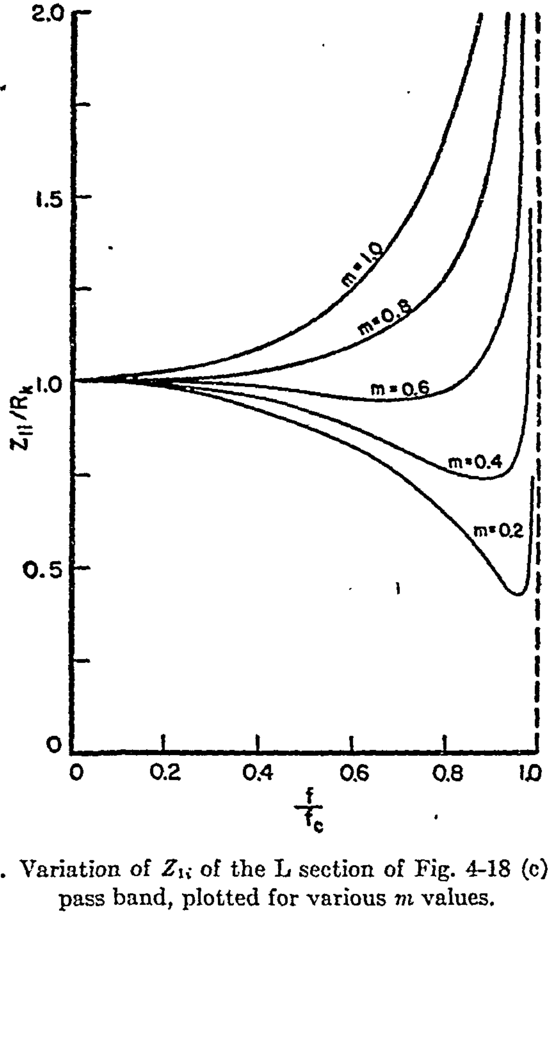


Fig. 4-19. Variation of Z_{1i} of the L section of Fig. 4-18 (c) over the pass band, plotted for various m values.

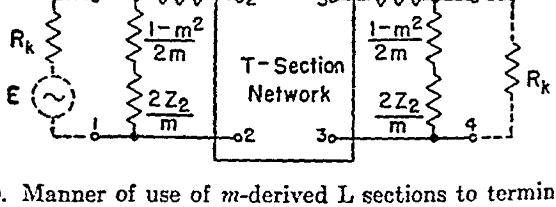


Fig. 4-20. Manner of use of m-derived L sections to terminate a T-section filter. The value of m should be 0.6.

Thus the image impedance looking to the left at the 2,2 terminals, Fig. 4-18(c), is that of a T section. By the same reasoning, the image impedance looking to the right at the 3,3 terminals is also Z_{0T}; and that looking to the left at the 4,4 terminals is the modified value of $Z_{0\pi}$ given by Eq. 4-81.

Thus a generator of internal impedance R_k may be connected to the 1,1 terminals of such a half section and a satisfactory image impedance match obtained over the pass band except close to cutoff. Likewise, a load of value R_k may be connected to the 4,4 terminals with a satisfactory match. Between the 2,2 and 3,3 terminals may be inserted prototype and m-derived T sections designed for a value

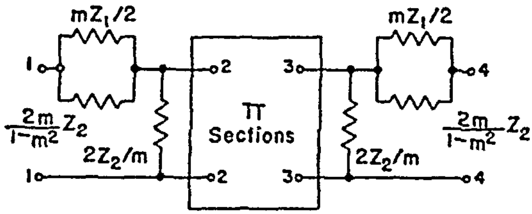


Fig. 4-21. Use of L sections to terminate a π-section filter, with $m = 0.6$.

of $R = R_k$. These sections will be working into terminals of the half sections that have image impedances equal to the Z_0 of the T sections, and thus will be matched. The over-all characteristic impedance will be very nearly constant except near cutoff, at a value equal to R_k, and all sections will be closely matched for maximum power transfer over at least 85 per cent of the pass band. The general arrangement will be as in Fig. 4-20.

The two half m-derived sections, known as *terminating half-sections*, are normally added to the design of any filter to provide uniform termination and matching characteristics. Moreover, they provide a point of high attenuation at a frequency 1.25 that of f_c, thus improving the attenuation characteristics of the filter. If additional attenuation is needed in the stop band or if the cutoff must be made sharper, then additional full m-derived sections with different m values may be added in series with the prototype T sections. Derived sections may be used alone, with nothing between, if attenuation at $f = \infty$ is not important.

The mismatch at frequencies near cutoff tends to slightly decrease the sharpness of the change in attenuation at the cutoff point, producing a small amount of attenuation inside the pass band near cutoff.

Prototype or m-derived sections of the π type may be analyzed in a similar manner. It is then found that if an m-derived π section is rearranged as a T section and split into half sections, these half sections, with $m = 0.6$, will give similarly satisfactory matching of impedances. These half sections are shown in Fig. 4-21.

4-14. Band-pass filters

Occasionally it is desirable to pass a band of frequencies and to attenuate frequencies on both sides of the pass band. The action

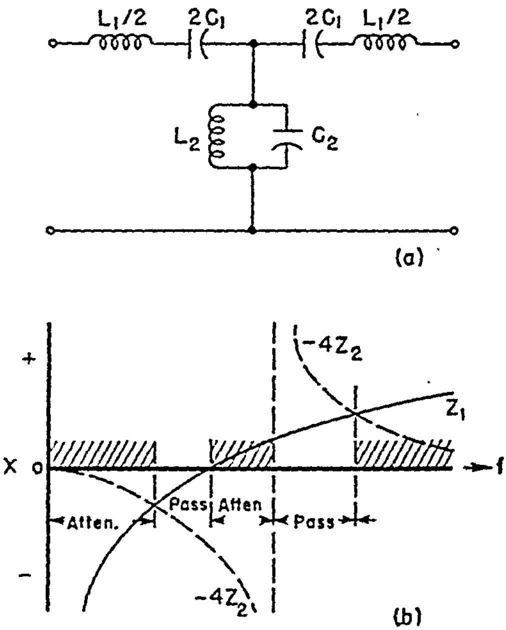


Fig. 4-22. (a) Band-pass filter network; (b) reactance curves showing possibility of two bands.

might be thought of as that of low-pass and high-pass filters in series, in which the cutoff frequency of the low-pass filter is above the cutoff frequency of the high-pass filter, the overlap thus allowing only a band of frequencies to pass. Although such a design would function, it is more economical to combine the low- and high-pass functions into a single filter section.

Consider the circuit of (a), Fig. 4-22, with a series-resonant series arm and an antiresonant shunt arm. In general, the reactance curves show that two pass bands might exist. If, however, the antiresonant frequency of the shunt arm is made to correspond to the resonant frequency of the series arm, the reactance curves become as shown in Fig. 4-23 and only one pass band appears. For

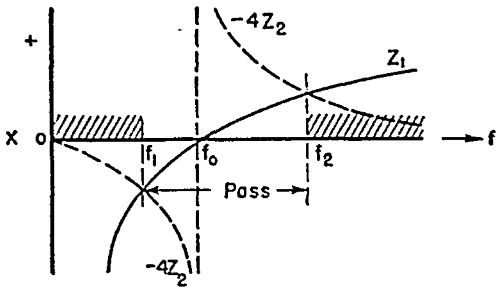


Fig. 4-23. Reactance curves for the band-pass network when resonant and antiresonant frequencies are properly adjusted.

this condition of equal resonant frequencies,

$$\omega_0^2 L_1 C_1 = 1 = \omega_0^2 L_2 C_2 \tag{4-85}$$

or

$$L_1 C_1 = L_2 C_2$$

The impedances of the arms are

$$Z_1 = j\left(\omega L_1 - \frac{1}{\omega C_1}\right) = j\,\frac{(\omega^2 L_1 C_1 - 1)}{\omega C_1} \tag{4-86}$$

$$Z_2 = \frac{j\omega L_2(-j/\omega C_2)}{j(\omega L_2 - 1/\omega C_2)} = \frac{j\omega L_2}{1 - \omega^2 L_2 C_2} \tag{4-87}$$

That a network such as (a), Fig. 4-23 is still a constant-k filter is easily shown as

$$Z_1 Z_2 = -\frac{L_2(\omega^2 L_2 C_1 - 1)}{C_1(1 - \omega^2 L_2 C_2)}$$

and if $L_1C_1 = L_2C_2$, then

$$Z_1 Z_2 = \frac{L_2}{C_1} = \frac{L_1}{C_2} = R_k^2 \tag{4-88}$$

Thus the previously developed theory still applies.

At the cutoff frequencies,

$$Z_1 = -4Z_2$$

Multiplying by Z_1 gives

$$Z_1^2 = -4Z_1Z_2 = -4R_k^2$$

from which the value of Z_1 at the cutoff frequencies is obtained as

$$Z_1 = \pm j2R_k \tag{4-89}$$

so that

$$Z_1 \text{ at lower cutoff } f_1 = -Z_1 \text{ at upper cutoff } f_2$$

The reactance of the series arm at the cutoff frequencies then can be written by use of the above as

$$\frac{1}{\omega_1 C_1} - \omega_1 L_1 = \omega_2 L_1 - \frac{1}{\omega_2 C_1}$$

$$1 - \omega_1^2 L_1 C_1 = \frac{\omega_1}{\omega_2}(\omega_2^2 L_1 C_1 - 1) \tag{4-90}$$

Now from Eq. 4-85,

$$L_1 C_1 = \frac{1}{\omega_0^2} \tag{4-91}$$

so that Eq. 4-90 may be written as

$$1 - \frac{f_1^2}{f_0^2} = \frac{f_1}{f_2}\left(\frac{f_2^2}{f_0^2} - 1\right)$$

$$f_0^2(f_1 + f_2) = f_1 f_2 (f_1 + f_2)$$

$$f_0 = \sqrt{f_1 f_2} \tag{4-92}$$

or the frequency of resonance of the individual arms should be the geometric mean of the two frequencies of cutoff.

If the filter is terminated in a load $R = R_k$, as is customary, then the values of the circuit components can be determined in terms of R and the cutoff frequencies f_1 and f_2. At the lower cutoff frequency,

$$\frac{1}{\omega_1 C_1} - \omega_1 L_1 = 2R$$

$$1 - \frac{f_1^2}{f_0^2} = 4\pi R f_1 C_1$$

In view of Eq. 4-92, the expression for C_1 becomes

$$C_1 = \frac{f_2 - f_1}{4\pi R f_1 f_2} \tag{4-93}$$

It follows, then, from Eqs. 4-91 and 4-92, that

$$L_1 = \frac{R}{\pi(f_2 - f_1)} \tag{4-94}$$

From Eq. 4-88, it is possible to obtain the values for the shunt arm as

$$L_2 = C_1 R^2 = \frac{R(f_2 - f_1)}{4\pi f_1 f_2} \tag{4-95}$$

$$C_2 = \frac{L_1}{R^2} = \frac{1}{\pi R(f_2 - f_1)} \tag{4-96}$$

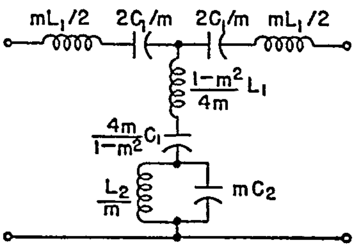


Fig. 4-24. m-derived band-pass section.

This completes the design of the prototype band-pass filter.

An m-derived band-pass section is also possible. Use of the transformation relations developed in Section 4-10 leads to a network of the form of Fig. 4-24. The shunt arm then consists of series-resonant and antiresonant circuits in series. Plotting reactance curves for these two circuits and adding to obtain the reactance variation of the shunt arm, Z_2 of the filter, gives the dashed curve of Fig. 4-25. The antiresonant frequency of the arm as a whole must, by previous reasoning, be f_0 of the filter. The reactance curve for Z_2 then shows that the shunt arm becomes resonant at a frequency below f_0 and again at a frequency above f_0. At these

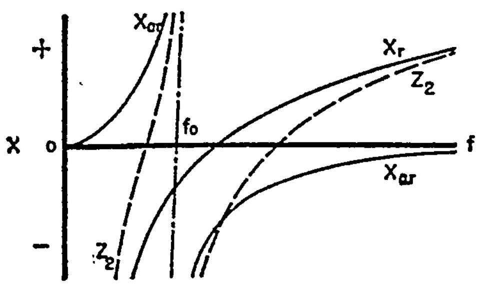


Fig. 4-25. Reactance curves for the shunt arm of the m-derived band-pass section.

frequencies the network is short-circuited, and thus they are frequencies of high attenuation, f_∞. These frequencies of high attenuation are placed on each side of the pass band, and the m-derived section may be used to increase the attenuation near cutoff, as for the high- or low-pass cases.

At one f_∞, the reactances X_r and X_{ar} are equal and opposite, so that

$$j\omega_\infty\left(\frac{1-m^2}{4m}\right)L_1 - \frac{j}{\omega_\infty[4m/(1-m^2)]C_1} = \frac{(j\omega_\infty L_2/m)(-j/\omega_\infty mC_2)}{j(1/\omega_\infty mC_2 - \omega_\infty L_2/m)}$$

$$\frac{1-m^2}{4}(\omega_\infty{}^2L_1C_1 - 1) = \frac{\omega_\infty{}^2L_2C_1}{\omega_\infty{}^2L_2C_2 - 1} \tag{4-97}$$

In view of the fact that

$$L_1C_1 = L_2C_2 = \frac{1}{\omega_0{}^2}$$

Eq. 4-97 becomes

$$\frac{1-m^2}{4}\left(\frac{f_\infty{}^2}{f_0{}^2} - 1\right)^2 = 4\pi^2 f_\infty{}^2 L_2C_1 \tag{4-98}$$

The term L_2C_1 can be evaluated as a function of f_1 and f_2 from Eqs. 4-93 and 4-95:

$$L_2C_1 = \frac{f_2 - f_1}{4\pi R f_1 f_2}\left[\frac{(f_2 - f_1)R}{4\pi f_1 f_2}\right] = \frac{(f_2 - f_1)^2}{16\pi^2 f_0{}^4}$$

Equation 4-98 then reduces to

$$(1 - m^2)(f_\infty{}^2 - f_1f_2)^2 = f_\infty{}^2(f_2 - f_1)^2 \tag{4-99}$$

$$f_\infty{}^2 - \frac{(f_2 - f_1)f_\infty}{\sqrt{1-m^2}} - f_1f_2 = 0$$

Solving for the values of the frequencies of peak attenuation,

$$f_\infty = \frac{f_2 - f_1}{2\sqrt{1-m^2}} \pm \sqrt{\frac{(f_2 - f_1)^2}{4(1-m^2)} + f_1f_2} \tag{4-100}$$

It is apparent that the radical is larger than $(f_2 - f_1)/2\sqrt{1-m^2}$, and thus one root would appear as a negative frequency that has no physical significance here. Thus the expression for f_∞ should be reversed so that the two frequencies of peak attenuation are

$$f_{\infty 1} = \sqrt{\frac{(f_2 - f_1)^2}{4(1 - m^2)} + f_1 f_2} - \frac{f_2 - f_1}{2\sqrt{1 - m^2}} \tag{4-101}$$

$$f_{\infty 2} = \sqrt{\frac{(f_2 - f_1)^2}{4(1 - m^2)} + f_1 f_2} + \frac{f_2 - f_1}{2\sqrt{1 - m^2}} \tag{4-102}$$

Equation 4-99 may be solved to determine the value of m, giving

$$m = \sqrt{1 - \left[\frac{f_\infty (f_2 - f_1)}{f_\infty^2 - f_1 f_2}\right]^2}$$

$$= \frac{\sqrt{(f_\infty^2 - f_1^2)(f_\infty^2 - f_2^2)}}{f_\infty^2 - f_1 f_2} \tag{4-103}$$

The value of m may be chosen to place either one of the two frequencies of peak attenuation at a desired point, the other frequency of peak attenuation then being definitely fixed. That this is true may be chosen by forming the product for $f_{\infty 1} f_{\infty 2}$ from Eqs. 4-101 and 4-102:

$$f_{\infty 1} f_{\infty 2} = \frac{f_2^2 + 2f_1 f_2 + f_1^2 - 4m^2 f_1 f_2 - f_2^2 + 2f_1 f_2 - f_1^2}{4(1 - m^2)} = f_1 f_2 \tag{4-104}$$

$$\sqrt{f_{\infty 1} f_{\infty 2}} = f_0 \tag{4-105}$$

Thus f_0 is the geometric mean of the frequencies of peak attenuation and, by Eq. 4-92, of the cutoff frequencies as well. If m is selected to place $f_{\infty 1}$ at a desired point, then by Eq. 4-105, $f_{\infty 2}$ is automatically fixed, and vice versa. It is possible to fix both $f_{\infty 1}$ and $f_{\infty 2}$ by the use of two m's or an mm'-derived filter, as shown by Zobel (Reference 2).

An m-derived T section, rearranged as a π, may be split into two half sections and used as terminating half sections. If m is given the value 0.6, then satisfactory impedance matching conditions are maintained over the pass band. This usage follows the previously developed theory for low- or high-pass sections.

4-15. Band-elimination filters

If the series- and parallel-tuned arms of the band-pass filter are interchanged, the result is the band-elimination filter of (a), Fig.

4-26. That this circuit does eliminate or attenuate a given frequency band is shown by the reactance curves for Z_1 and $-4Z_2$ at (b). The action may be thought of as that of a low-pass filter in parallel with a high-pass section, in which the cut-off frequency of the low-pass filter is below that of the high-pass filter.

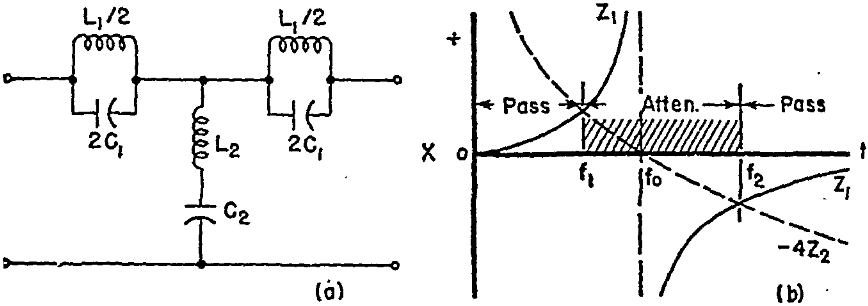


Fig. 4-26. (a) Band-elimination filter; (b) reactance curves showing action of band-elimination section.

As for the band-pass filter, the series and shunt arms are made antiresonant and resonant at the same frequency f_0. Again, it is possible to show that

$$R_k^2 = \frac{L_1}{C_2} = \frac{L_2}{C_1} \tag{4-106}$$

and that

$$f_0 = \sqrt{f_1 f_2} \tag{4-107}$$

At the cutoff frequencies,

$$Z_1 = -4Z_2, \qquad Z_1 Z_2 = -4Z_2^2 = R_k^2$$

$$Z_2 = \pm \frac{jR_k}{2} \tag{4-108}$$

If the filter is terminated in a load $R = R_k$, then at the lower cutoff frequency,

$$Z_2 = j\left(\frac{1}{\omega_1 C_2} - \omega_1 L_2\right) = \frac{jR}{2}$$

Since $L_2 C_2 = 1/\omega_0^2$,

$$1 - \frac{f_1^2}{f_0^2} = \pi f_1 C_2 R$$

$$C_2 = \frac{1}{\pi R}\left(\frac{f_2 - f_1}{f_1 f_2}\right) \tag{4-109}$$

In view of the fact that

$$f_0 = \sqrt{f_1 f_2} = \frac{1}{2\pi\sqrt{L_2 C_2}}$$

then
$$L_2 = \frac{R}{4\pi(f_2 - f_1)} \tag{4-110}$$

By use of Eq. 4-106, the values for the series arm are obtained as

$$L_1 = \frac{R(f_2 - f_1)}{\pi f_1 f_2} \tag{4-111}$$

$$C_1 = \frac{1}{4\pi R(f_2 - f_1)} \tag{4-112}$$

Sections of the m-derived form may also be obtained.

4-16. Filter-circuit design

As an example of the manner in which a complete filter may be designed, it will be convenient to carry out the calculations on a typical filter to meet the following specifications:

A composite low-pass filter is to be terminated in 500 ohms resistance. It must have a cutoff frequency of 1000 cycles, with very high attenuation at 1065, 1250, and ∞ cycles. The prototype is designed first as

$$L = \frac{R}{\pi f_c} = \frac{500}{\pi \times 1000} = 0.159 \text{ henry}$$

$$C = \frac{1}{\pi f_c R} = \frac{1}{\pi \times 1000 \times 500} = 0.636 \text{ microfarad}$$

This prototype section meets the specification for high attenuation at infinity. The assembly of the section is illustrated in the circuit of Fig. 4-27, with inductance of $L/2$ in each series arm.

The m-derived section to provide high attenuation at $f_\infty = 1065$ cycles may then be designed:

$$m = \sqrt{1 - \left(\frac{f_c}{f_\infty}\right)^2} = \sqrt{1 - \left(\frac{1000}{1065}\right)^2}$$

$$= \sqrt{1 - 0.882} = 0.343$$

$$\frac{mL}{2} = \frac{0.343 \times 0.159}{2} = 0.0273 \text{ h}$$

$$\frac{1 - m^2}{4m} L = \frac{1 - 0.118}{1.372} \times 0.159 = 0.102 \text{ h}$$

$$mC = 0.343 \times 0.636 = 0.218 \text{ microfarad}$$

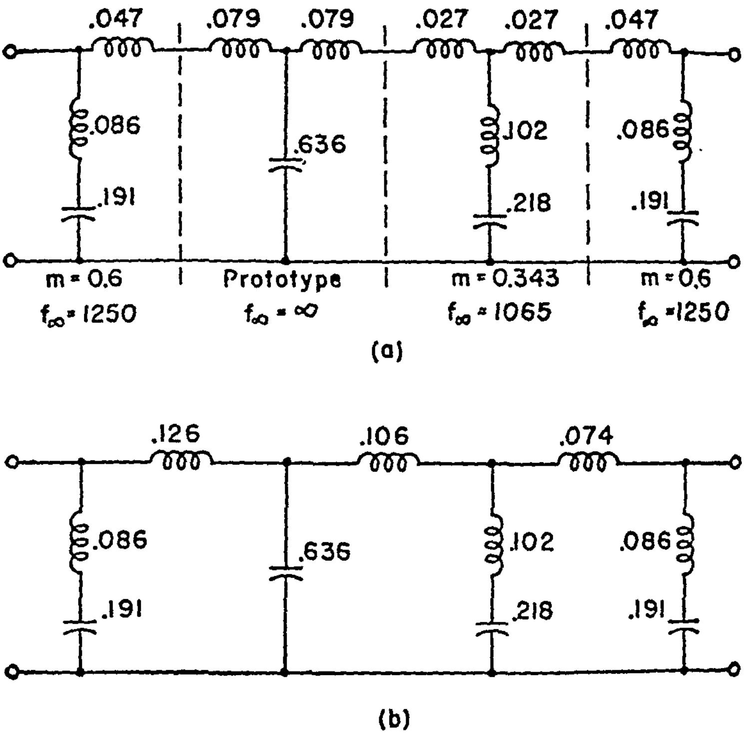


Fig. 4-27. (a) Filter as designed; (b) as it would be combined.

These values are then assembled as in Fig. 4-27.

The section providing high attenuation at 1250 cycles will have a value of m given by

$$m = \sqrt{1 - \left(\frac{1000}{1250}\right)^2} = \sqrt{1 - 0.640} = 0.6$$

and this will be appropriate for use in the terminal half sections.

Then the circuit elements should be

$$\frac{mL}{2} = \frac{0.6 \times 0.159}{2} = 0.0474 \text{ henry}$$

$$\frac{1 - m^2}{2m} L = \frac{1 - 0.36}{1.2} \times 0.159 = 0.086 \text{ henry}$$

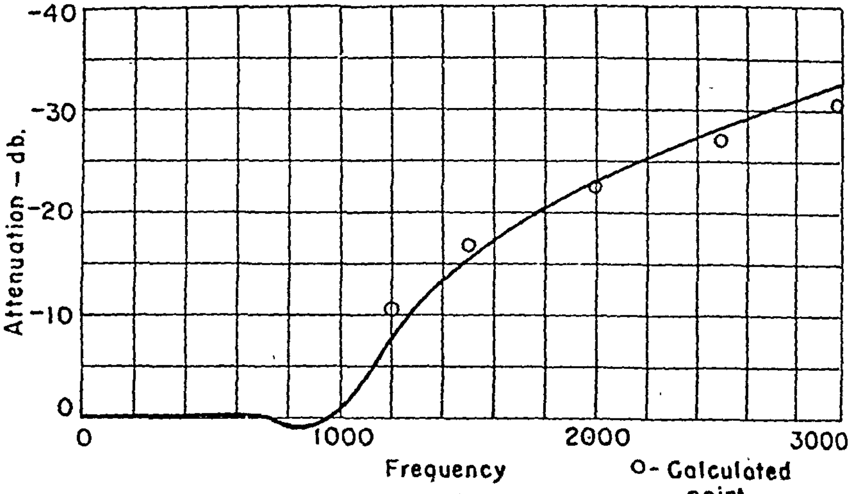
$$\frac{mC}{2} = \frac{0.6 \times 0.636}{2} = 0.191 \text{ microfarad}$$

These sections are all shown individually in Fig. 4-27(a). For economy, it is customary to combine elements wherever possible. The series inductors may then be added, resulting in the final design of (b). In making a mechanical assembly it is necessary to avoid electric and magnetic couplings. To aid in this and to improve the magnetic circuit, the inductors are ordinarily wound as toroids on ring cores. Magnetic materials of very high permeability are used, these usually being high-nickel alloys such as permalloy, or powdered iron or permalloy materials in compressed form. The values of Q obtained should be as high as possible so that the filter performance will closely approximate the performance calculated for pure reactances.

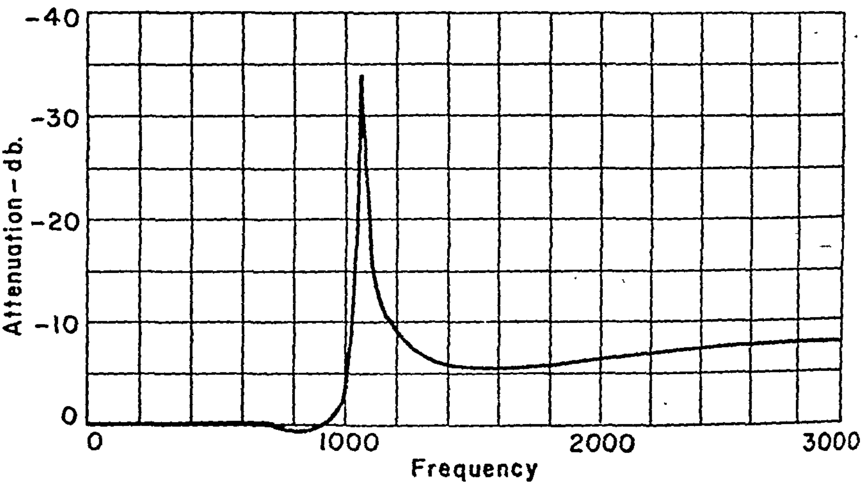
4-17. Filter performance

To illustrate the sort of approach to the theoretical ideal which is possible in filter design, laboratory filters were assembled in accordance with the designs of Fig. 4-27. The inductors used were toroids on compressed molybdenum-permalloy dust cores, and had Q values of approximately 40. These inductors would not be considered as having very high Q by commercial standards, as values of 100 to 300 are available. The filters were designed for 500 ohms resistance termination, and were so used for each measurement.

Attenuation measurements were made over the pass and stop bands on each of the filter sections. The attenuation of the prototype section, terminated in 500 ohms, is shown in (a), Fig. 4-28. The presence of resistance and the insertion loss of the section causes a rounding of the attenuation curve near cutoff, but otherwise the shape of the curve reasonably fits the theoretical curve of Fig. 4-9. Calculation of attenuation, based on pure reactances and the



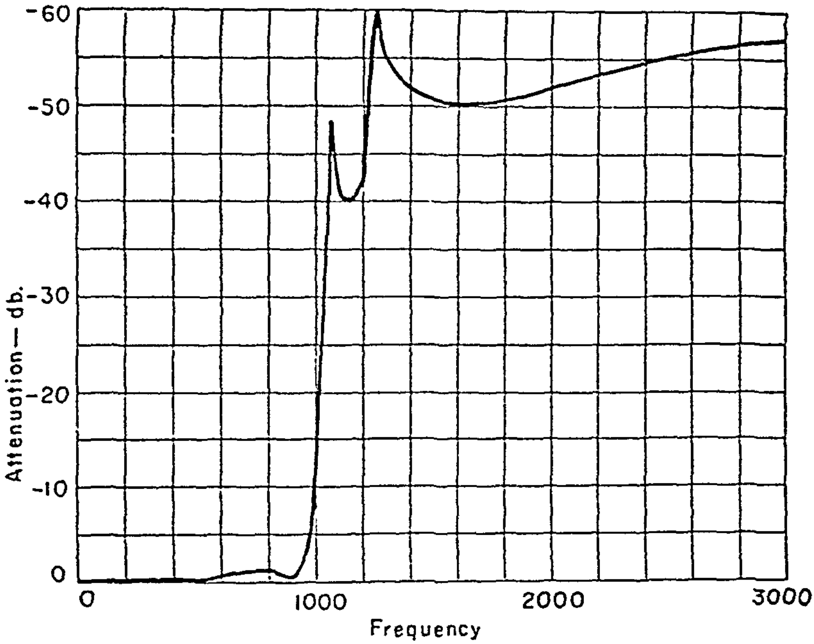
(a)



(b)

Fig. 4-28. (a) Attenuation of the prototype: $f_c = 1000$ cycles, $R_k = 500$ ohms. (b) Attenuation of the m-derived section: $m = 0.346$, $f_c = 1000$, $f_\infty = 1065$, $R_k = 500$ ohms. (c) Action of composite filter of Fig. 4-27(b).

theoretical equations of Section 4-8, gives 16.7 db attenuation at 1500 cycles or $f/f_c = 1.5$. The measured curve shows 15.5 db, which is a reasonable check. Calculated values of α for a pure



(c)

Fig. 4-28. *(cont.)*.

reactance network are shown as marked points for a few frequencies in Fig. 4-28(a), for comparison with the measurements.

Figure 4-28(b) illustrates the measured attenuation for the m-derived section alone, with $m = 0.343$. This section was designed to have $f_\infty = 1065$ cycles and a cutoff of 1000 cycles, and a peak of attenuation of -34 db is obtained at the frequency specified. The usually undesirable low α values above f_∞ are also found, the slight rise at 3000 cycles probably being due to increased losses in the coils used.

Figure 4-28(c) shows the over-all attenuation of the complete filter of Fig. 4-27, when the prototype and m-derived sectons of (a) and (b) are combined with terminal half sections having $m = 0.6$ and terminated in 500 ohms. The terminal half sections give an additional value of high attenuation at $f_\infty = 1250$ cycles.

Due to probable slight mismatches and losses, there is a small irregularity in the pass band, but reasonably sharp cutoff characteristics are obtained.

4-18. Crystal filters

The lattice structure can also be shown to have filter properties. Considering the network of Fig. 4-29,

$$Z_{oc} = \frac{(Z_1/2 + 2Z_2)^2}{2(Z_1/2 + 2Z_2)}$$

$$= \frac{Z_1}{4} + Z_2 \tag{4-113}$$

$$Z_{sc} = \frac{Z_1Z_2}{Z_1/2 + 2Z_2} + \frac{Z_1Z_2}{Z_1/2 + 2Z_2}$$

$$= \frac{Z_1Z_2}{Z_1/4 + Z_2} \tag{4-114}$$

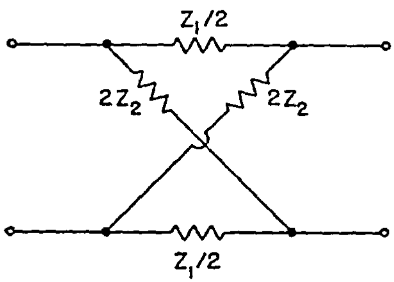


Fig. 4-29. Lattice filter section.

The characteristic impedance of the lattice section then is

$$Z_{0L} = \sqrt{Z_{oc}Z_{sc}} = \sqrt{Z_1Z_2} \tag{4-115}$$

Thus if the section elements are reactive, Z_{0L} is real, or a pass band exists for frequencies for which Z_1 and Z_2 are of opposite sign. Over ranges where Z_1 and Z_2 have the same sign, an attenuation band exists.

Propagation can be investigated further by noting that

$$\tanh \gamma = \sqrt{\frac{Z_{sc}}{Z_{oc}}} = \sqrt{\frac{Z_1}{Z_2}}\left[\frac{1}{1 + Z_1/4Z_2}\right] \tag{4-116}$$

It may be noted that Z_{0L} depends on the product of Z_1 and Z_2, whereas γ depends on the ratio of Z_1 to Z_2. This feature permits somewhat greater versatility in design of the lattice section over the T or π section, especially for filters in which certain of the elements are constructed of piezoelectric crystals. These crystals have a resonant frequency of mechanical vibration dependent on certain of their dimensions; and because of the very high equivalent Q of the crystals, it is possible to make very narrow band filters and filters in which the attenuation rises very rapidly at cutoff.

The equivalent electric circuit of a quartz mechanical-filter crystal is shown in Fig. 4-30(a), which shows a possibility of both resonance and antiresonance occurring. The inductance L_x is very large, being in henrys for crystals resonating near 500 kc, so that while R_x may approximate a few hundred or few thousand ohms, the effective Q may be in the range of 10,000 to 30,000. Considering the properties of resonant circuits, such as Q would provide a band width of 20 to 50 cycles at 500 kc.

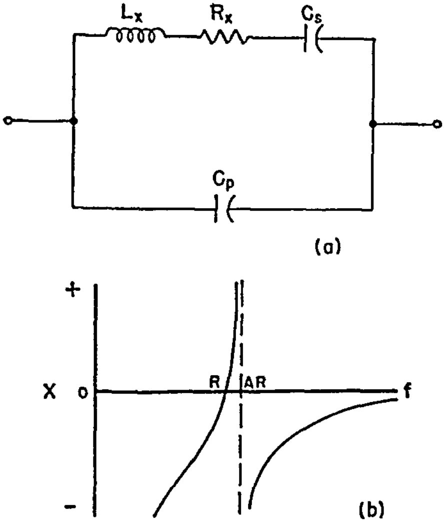


Fig. 4-30. (a) Equivalent electrical circuit for a piezoelectric crystal; (b) reactance curves for the circuit of (a).

The resistance of the crystal is due largely to mechanical damping introduced by the electrodes and by the surrounding atmosphere. By placing a crystal in an evacuated container, the value of Q can be notably increased. The electrodes are normally electroplated onto the crystal faces and need not introduce much damping.

Capacitance C_s is the equivalent series capacitance of the crystal forming a resonant circuit with L_x. Capacitance C_p is the parallel capacitance introduced by the crystal electrodes. The values of C_s and C_p are such that $C_p \gg C_s$, so that the resonant and antiresonant frequencies of the circuit lie very close together, differing by a fraction of 1 per cent of the resonant frequency. The reactance-curve sketch of Fig. 4-30(b) shows the resonant frequency below the antiresonant one. By placing adjustable capacitors in parallel with the crystal, C_p can be increased, resulting in the antiresonant frequency being moved closer to the resonant point.

Since the crystal represents either a resonant or antiresonant circuit, it may be used to replace the normal elements of the band-pass or band-elimination filter. As previously shown for band-pass action, the resonant frequency of one arm must equal the anti-resonant frequency of the other arm. The pass band with crystal

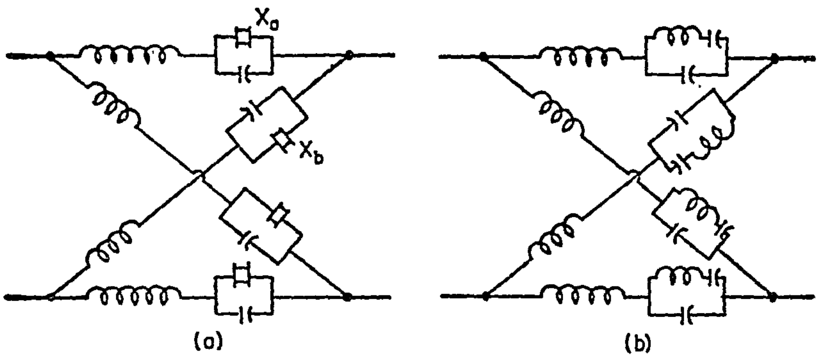


Fig. 4-31. (a) Circuit of a lattice crystal filter with series inductors and parallel capacitors; (b) the electrical equivalent of (a).

elements will then be found to extend from the lowest crystal resonant frequency to the highest crystal antiresonant frequency, or a width of pass band equal to twice the separation of the resonant and antiresonant frequencies of one crystal. This range will result in a pass band a fraction of 1 per cent wide. The band width can be reduced by putting adjustable capacitors in parallel with the crystal, furnishing a means of adjustment of the width of the pass band.

By the addition of coils in series with the crystals the pass bands may be widened. Since the added coils have Q values very much below those of the crystals, there will be some loss in sharpness at cutoff. A circuit including series coils is shown in Fig. 4-31(a), with its equivalent drawn at (b). The reactance curves for the A and B portions of this circuit are drawn in Fig. 4-32(a), which shows how the resonances and antiresonances are arranged. The presence of the series coil adds an additional resonance, and the pass band exists from the lowest resonance of one crystal to the highest resonance of the other. If f_1 and f_2 are the frequencies of resonance of one of the circuits and f_R is that of the antiresonance, then

$$f_{1,2} = f_R \sqrt{1 \mp \frac{C_s}{C_p}}$$

The separation of f_1 and f_2 represents two-thirds of the pass band and is seen to depend on the $\sqrt{C_s/C_p}$ ratio. Since C_s/C_p may be of the order of 0.01, it can be seen that the separation of f_1 and f_2 may be of

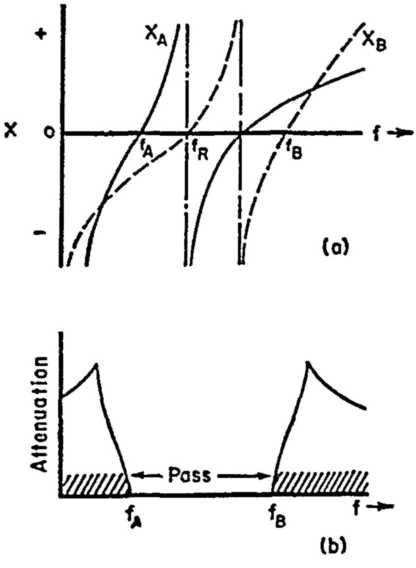


Fig. 4-32. (a) Reactance curves for the circuit of Fig. 4-31(a); (b) attenuation curves for that circuit.

the order of 0.10 f_R, or 10 per cent of the resonant frequency. By placing coils in series with the crystals, it has been possible to widen the pass band considerably. By adjustment of C_p it is then possible to narrow the band to any desired amount.

Thus the use of coils permits the bands to be widened to pass speech frequencies, and crystal filters are quite generally used to separate the various channels in carrier telephone circuits, in the range above 50 kilocycles.

PROBLEMS

4-1. (a) A generator with output of 0.0025 w supplies power to an amplifier which in turn supplies a 600-ohm load. If the power level in the load is to be +16 db above 0.001 w reference, what power gain in decibels is required of the amplifier?

(b) If a transformer of 65 per cent efficiency is placed between amplifier and load, what amplifier power gain will be needed, in decibels, to maintain the same load power?

4-2. A radio receiver has an input impedance that is 200 ohms resistive. The signal picked up and applied to this input is 400 μv. The electric output to the loudspeaker is to be at +32 db level (6 mw reference). Find

(a) Input power level in decibels.
(b) Decibels gain in the receiver.
(c) Output power in watts.

4-3. The output of a certain vacuum tube is 5.2 w.

(a) Find the decibel level (0.001 w reference).

(b) How many decibels will be added to the output level if the power is made 4 times greater?

(c) A transformer of 70 per cent efficiency is used between the output and the load. What is the decibel loss in the transformer?

4-4. The output of a certain amplifier is at a level of +37 db (0.006 w reference). It is applied to a telephone line of 60 per cent efficiency, followed by a transformer with losses measured at 2.4 w.

(a) Find the power delivered by the transformer, in watts and decibels.

(b) Find the line loss in decibels and nepers.
(c) Find the decibels transformer loss.
(d) Find the current in the load if it is 20 ohms, resistive.

4-5. A T section having $Z_1/2 = j125$ ohms, $Z_2 = -j600$ ohms, is used between a 10-v generator of Z_0 impedance and a Z_0 load.

(a) Find the power transferred to the load.
(b) Find the values of γ, α, and β.
(c) If Z_2 is changed to $+j600$ ohms, repeat (a) and (b).

4-6. A π section has the series arm made up of a 100-mh coil, each shunt arm consisting of a 0.15-μf, capacitor. Plot the magnitude and angle of Z_0, from zero frequency to 2500 c.

4-7. If the inductor in Prob. 4-6 has a resistance of 50 ohms, and neglecting the resistance of the capacitors, calculate by means of open- and short-circuit impedances, the Z_0 at 0, 500, 1000, 1500, and 2000 c.

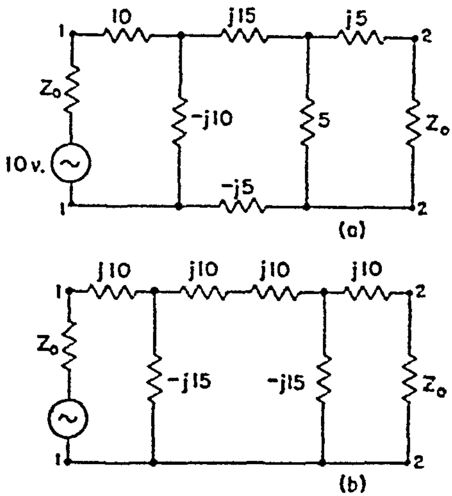


Fig. 4-33.

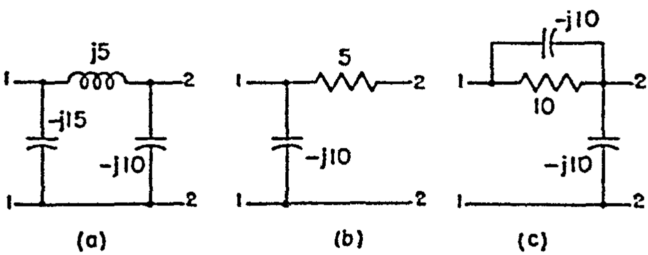


Fig. 4-34.

4-8. In Fig. 4-33(a), find the values of α and β, the power delivered to the Z_0 load, and the nepers and decibels loss in the network, between 1,1 and 2,2 terminals.

4-9. Repeat Prob. 4-8, using Fig. 4-33(b). $E = 10$ v.

4-10. Design a low-pass T section with cutoff at 2500 c to work in an audio amplifier between a vacuum tube of 10,000 ohms plate resistance and a 10,000-ohm load.

(a) Compute the attenuation in nepers and decibels at 3000 c, and at the second harmonic of the cutoff, 5000 c.

(b) Find the phase shift at 500, 1000, 1500, and 2000 c.

4-11. Find the values of circuit elements needed for a high-pass T section filter with cutoff of 1500 c, to work into a 1000-ohm load. Draw the circuit diagram.

4-12. For the networks of Fig. 4-34, determine Z_{1i} and Z_{2i}.

4-13. Compute the elements of a constant-k low-pass network having $f_c = 2000$ c and load of 5000 ohms. Compute α and β for the network over the range of 0 to 6000 c, and plot the results.

4-14. The series arm Z_1 of a filter consists of a 0.5-μf capacitor in series with an inductor of 0.35 h. If $R_k = 500$ ohms, determine the elements for the shunt arm and their manner of connection.

4-15. A filter on the input to a telephone line is to attenuate all frequencies above 1500 c, with particularly large attenuation at 2000 c. The input resistance of the telephone line is 550 ohms. Design and draw the resultant circuit diagram, assuming a reasonably constant Z_0 is desired. Plot the attenuation characteristic in the stop band.

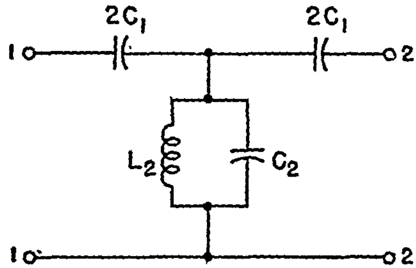


Fig. 4-35.

4-16. Determine the propagation characteristics of the circuit shown in Fig. 4-35 by use of reactance curves.

4-17. A power-supply filter for a radio receiver consists of a π section with a 20-h series arm and 20-μf capacitors as shunt arms.

(a) Calculate the cutoff frequency.

(b) Find the decibels attenuation at 60 and 120 c.

(c) If a voltage

$$e = 400 + 320 \sin 2\pi 60t + 160 \sin 2\pi 120t$$

is applied to this filter, find the current of each frequency flowing in a 4000-ohm resistive load.

(d) Find the ratio of 60-c power to d-c power in the load, in decibels.

4-18. A composite low-pass filter has two T prototype sections and an m-derived section with $m = 0.4$. For the filter $R_k = 600$ ohms, and $f_c = 796$ c, plot α and β for the range from 0 to 5000 c.

4-19. Design a low-pass filter to work into 1250 ohms, with cutoff at 600 c and with high attenuation at 900 and 1200 c. Terminate it properly and draw the complete circuit. Plot the attenuation characteristic over the attenuation band, in decibels.

4-20. A filter to pass the band between 1250 and 2000 c, with high attenuation at 2500 c, is desired to work into a 4000-ohm load. Design the circuit and find the second frequency of high attenuation. Use proper terminating sections.

4-21. An antenna filter in a 3-megacycle radio transmitter is needed to suppress the second harmonic of the output frequency. Choose and design a section to be used, keeping economy of equipment in mind. The load may be considered as 100 ohms.

4-22. Set up the basic circuit for the T type m-derived band-elimination filter.

REFERENCES

1. Campbell, G. A., "Physical Theory of the Electric Wave-Filter," *Bell System Tech. J.*, **1,** 1 (1922).
2. Zobel, O. J., "Theory and Design of Uniform and Composite Electric Wave-Filters," *Bell System Tech. J.*, **2,** 1 (1923).
3. ———, "Transmission Characteristics of Electric Wave-Filters," *Bell System Tech. J.*, **3,** 567 (1924).
4. Shea, T. E., *Transmission Networks and Wave Filters*, D. Van Nostrand Company, Inc., New York, 1929.
5. Guillemin, E. A., *Communication Networks*, Vol. II, John Wiley & Sons, Inc., New York, 1935.
6. Mason, W. P., "Electric Wave Filters Employing Quartz Crystals as Elements," *Bell System Tech. J.*, **13,** 405 (1934).
7. ———, *Electromechanical Transducers and Wave Filters*, D. Van Nostrand Company, Inc., New York, 1942.
8. Scowen, F., *An Introduction to the Theory and Design of Electric Wave Filters*, Chapman & Hall, Ltd., London, 1945.

9. Cauer, W., "New Theory and Design of Wave Filters," *Physics*, **2**, 247 (1932).

10. Bode, H. W., "A General Theory of Electric Wave Filters," *J. Math. Phys.*, **13**, 275 (1934).

11. Lane, C. E., "Crystal Channel Filters for the Cable Carrier System," *Bell System Tech. J.*, **17**, 125 (1938).

12. Martin, W. H., "Decibel—the Name for the Transmission Unit," *Bell System Tech. J.*, **8**, 1 (1929).

Chapter 5

TRANSMISSION-LINE PARAMETERS

The networks which have so far been discussed are called circuits of *lumped parameters*, wherein the resistance, inductance, and capacitance are individually concentrated or lumped at discrete points in the circuit, and can be identified definitely as representing a particular parameter. The electric line used for transmission of telephone messages or for the transmission of power is a common example of an electric circuit with *distributed parameters*. This term implies that the resistance, inductance, and capacitance are distributed along the circuit, each elemental length of the circuit having its own values, and concentration of the individual parameters is not possible.

The first few sections of this chapter will treat the inductance and capacitance of two special forms of line, namely, the open-wire and the coaxial line. The later sections will develop methods for calculation of the parameters of more general forms of multiconductor lines.

5-1. Line parameters

A common form of transmission line is known as the *open-wire line* because of its construction. The ordinary telephone line, strung on cross arms on poles, or the power transmission line on towers, are examples of the open line. The conductors of such lines may be considered parallel and separated by air dielectric.

Another form of line construction is the *cable*. For telephone use this consists of hundreds of individually paper-insulated conductors, twisted in pairs, and combined inside a protective lead or plastic sheath. Power transmission cables will employ only two or three large conductors, insulated with oil impregnated paper or other solid dielectric, inside the protective sheath. In either case the conductors may be considered again as parallel, with a solid dielectric.

A different form of construction is employed with the *coaxial* *line*, in which one conductor is a hollow tube, the second conductor being located inside and coaxial with the tube. The dielectric may be solid or gaseous, but if the cable is gas filled, occasional solid dielectric disks are employed to maintain accurate spacing and location of the central wire.

Knowledge of the values of electric circuit parameters associated with these forms of line is necessary to understanding and design. These parameters include *resistance*, which is uniformly distributed along the length of the conductors. Since current will be present, the conductors will be surrounded and linked by magnetic flux, and this phenomena will demonstrate its effect in distributed *inductance* along the line. The conductors are separated by insulating dielectric, so that *capacitance* will be distributed along the conductor length. This dielectric, or the insulators of the open-wire line, may not be perfect, and a leakage current will flow and *leakage conductance* will exist between the conductors.

These four parameters, all distributed along the line, are known by the symbols R, L, C, and G, where usually quantities *per unit length of line* are meant. Thus for resistance, both wires are included in the value of R for a unit of line.

Methods of calculating resistance will be discussed later. Calculations for inductance and capacitance will be presented here, starting from fundamental relations.

5-2. Inductance of a line of two parallel round conductors

Self-inductance was defined in Chapter 3 as

$$L = N\frac{d\phi}{di} \tag{5-1}$$

Permeability μ is defined as

$$\mu = \mu_r\mu_v \tag{5-2}$$

where μ_v is the magnetic permeability of space and has the value $4\pi \times 10^{-7}$ in MKS units, and μ_r is the relative permeability of the particular material.

In a material or region where μ_r is independent of flux density, as in air and most dielectrics,

$$\frac{d\phi}{di} = \frac{\phi}{i}$$

and
$$L = \frac{N\phi}{i} = \frac{\lambda}{i} \text{ henrys} \tag{5-3}$$

where $\lambda = N\phi$, or is the *flux linkage* of the current.

The open-wire telephone line, or the cable pair, may be considered as two parallel, round conductors immersed in air or a solid dielectric, and if the flux linkages $N\phi = \lambda$ can be calculated, Eq. 5-3 will compute the inductance. For this purpose a unit flux linkage is defined as a surrounding or linking of one weber of flux with the

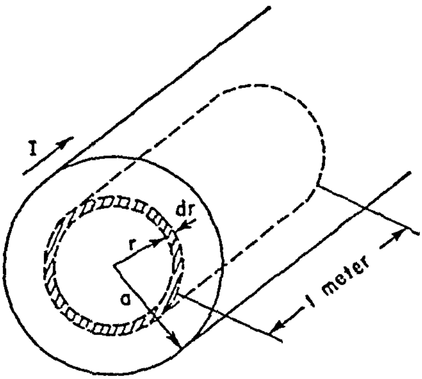


Fig. 5-1. A section of a long round wire with a hypothetical cylinder of length 1 meter and wall thickness dr shown.

entire conductor current. Thus one weber of flux linking one-half the current in the conductor produces one-half of a flux linkage, whereas one weber of flux entirely linking the same current in n turns of a conductor gives rise to n flux linkages. Inductance of a conductor in a region of constant permeability is then stated as flux linkages per ampere of current flowing in the circuit. This will also apply for conductors of magnetic material if μ is assumed constant or independent of flux density.

The flux paths surrounding a current in a long, straight, round wire are concentric circles, as in Fig. 5-1. The magnetic field intensity at any distance r from the center of the conductor is, by definition

$$H_r = \frac{NI_r}{l} \tag{5-4}$$

where I_r is the current enclosed by the flux path around which H is measured. In the figure, the length of the path at any distance r is the circumference of the dotted circle, or $l = 2\pi r$. The magnetic field intensity at any point inside or outside the conductor is

$$H_r = \frac{I_r}{2\pi r} \tag{5-5}$$

since only one turn or current path is linked.

For a flux path *inside* the conductor, as shown by the dashed circle at radius r in Fig. 5-1, the current enclosed is

$$I_r = \frac{\pi r^2}{\pi a^2} I \tag{5-6}$$

where I is the total conductor current, assumed to flow uniformly distributed over the whole cross section of the conductor. Then

$$H_r = \frac{rI}{2\pi a^2}$$

and since $B = \mu H$,

$$B_r = \frac{\mu r I}{2\pi a^2} \tag{5-7}$$

This is the flux density *at any point inside the conductor*. The conductor may be magnetic, that is μ_r may have a value other than unity, the only requirement on later use of this expression being that μ_r is independent of flux density.

The flux present in the wall of the hollow cylinder of elemental thickness dr, and 1 meter in length as shown in the figure, is

$$d\phi = B_r\, dA = \frac{\mu r I}{2\pi a^2}\, dr \tag{5-8}$$

This flux in the elemental cylinder wall links the fraction r^2/a^2 of the total current I, so that each weber of flux produces the fraction r^2/a^2 of a full flux linkage. Thus the linkages $d\lambda$ due to the flux passing in the wall of the cylinder of radius r, and elemental thickness dr, in length 1 meter, is

$$d\lambda = \frac{\mu r I}{2\pi a^2}\frac{r^2}{a^2}\, dr \tag{5-9}$$

The total internal linkages λ_{int} due to all the flux inside the conductor are obtained by integrating from the center of the conductor to the radius a:

$$\lambda_{\text{int}} = \frac{\mu I}{2\pi a^4} \int_0^a r^3 \, dr = \frac{\mu I}{8\pi} \text{ linkages/m} \tag{5-10}$$

where μ is that of the conductor material.

It is then necessary to determine the external flux linkages.

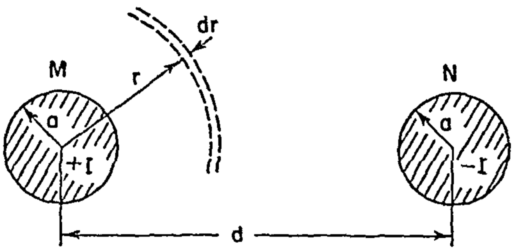


Fig. 5-2. Cross section of a parallel-wire line.

Referring to Fig. 5-2, the flux density in the *space outside* the conductor is

$$B_r = \mu_v H_r = \frac{\mu_v I}{2\pi r} \text{ webers/m}^2 \tag{5-11}$$

The flux present in the dashed hollow cylinder, drawn at radius r about conductor M with thickness dr and length 1 meter, is

$$d\phi = \frac{\mu_v I}{2\pi} \frac{dr}{r}$$

Flux passing in the space between the two conductors, and around the current in M, or having a radius between a and $d - a$, therefore contributes full linkages of the current in M, and these linkages are given by

$$\lambda_1 = \frac{\mu_v I}{2\pi} \int_a^{d-a} \frac{dr}{r} \tag{5-12}$$

Flux due to the current in M, that passes through wire N, or has a radius between $d - a$ and $d + a$, links values of current varying between $+I$ for a flux element having a radius just greater than $d - a$, and zero current for a flux element having a radius of $d + a$. An average value may be obtained by integrating to the center of wire N, thus counting as full linkages that flux passing through the left half of N, and as zero linkages the flux passing through the right half of N. The contribution by the flux cutting wire N is

$$\lambda_2 = \frac{\mu_v I}{2\pi} \int_{d-a}^{d} \frac{dr}{r} \tag{5-13}$$

Flux due to the current in M that has a radius greater than $d + a$, or that surrounds conductor N, links both $+I$ and $-I$ currents or zero net current, and contributes zero flux linkages.

The total *external* flux linkages about wire M are then

$$\lambda_{ext} = \lambda_1 + \lambda_2 = \frac{\mu_v I}{2\pi} \left[\int_a^{b-a} \frac{dr}{r} + \int_{d-a}^{d} \frac{dr}{r} \right]$$

$$\lambda_{ext} = \frac{\mu_v I}{2\pi} \int_a^d \frac{dr}{r} = \frac{\mu_v I}{2\pi} \ln \frac{d}{a} \text{ linkages/m}$$

The total inductance of both wires or of the circuit is then obtained from the definition:

$$L = \frac{\lambda}{I} = 2\left(\frac{\mu}{8\pi} + \frac{\mu_v}{2\pi} \ln \frac{d}{a}\right) \tag{5-14}$$

Rearranging and using the value of μ_v:

$$L = 10^{-7} \left(\frac{\mu}{\mu_v} + 4 \ln \frac{d}{a}\right) \text{ henrys/m} \tag{5-15}$$

$$= 10^{-7} \left(\mu_r + 9.210 \log_{10} \frac{d}{a}\right) \text{ henrys/m} \tag{5-16}$$

$$= 0.1609\mu_r + 1.482 \log_{10} \frac{d}{a} \text{ mh/mile} \tag{5-17}$$

The first term on the right of each of these four expressions is called the *internal inductance* of the line, since it is due to the internal flux linkages in the conductors. The second term on the right is the *external inductance* due to linkages with flux external to the wires. It should be emphasized that this relation was derived under the assumption of uniform current distribution over the conductors. This can be true only if $d \gg a$, or the wires are widely spaced with respect to the radius of a wire; otherwise the magnetic fields force a nonuniform distribution known as *proximity effect*. With alternating current, the current also crowds toward the surface, and at radio frequencies the current flows so near the surface that the internal inductance becomes negligible.

5-3. Inductance of the coaxial line

Figure 5-3 represents a cross section of a coaxial line, in which there is a central conductor enclosed by an outer conducting sheath, the two separated by a dielectric. It is assumed that the current is uniformly distributed over the conductor cross section, or that the frequency is low.

A current of $+I$ amperes will be carried on the inner conductor and a current of $-I$ amperes on the outer conductor. A flux path surrounding the outer conductor encloses zero net current, and therefore there is no flux external to the outer conductor. The flux linkages of the inner and outer conductors may be added to obtain the total inductance.

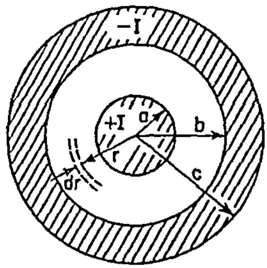


Fig. 5-3. Cross section of a coaxial line with flux paths at radius r.

The flux linkages internal to the central conductor are obtained from Eq. 5-10 as

$$\lambda_1 = \frac{\mu_1 I}{8\pi} \text{ linkages/m} \tag{5-18}$$

The linkages of the central conductor current $+I$, due to flux in the dielectric between conductors, is given by Eq. 5-11, as

$$\lambda_2 = \frac{\mu_v I}{2\pi} \int_a^b \frac{dr}{r} = \frac{\mu_v I}{2\pi} \ln \frac{b}{a} \text{ linkages/m} \tag{5-19}$$

Flux surrounding the central conductor and passing within the outer conductor material is produced by a net current that varies from $+I$ at the inner surface to zero at the outer surface of the outer conductor. The current enclosed by flux at radius r, where $b < r < c$, is

$$I + \frac{\pi r^2 - \pi b^2}{\pi c^2 - \pi b^2}(-I) = I\frac{(c^2 - r^2)}{(c^2 - b^2)}$$

and the flux density B_r at radius r within the outer conductor material, due to the above current, is

$$B_r = \frac{\mu_3 I}{2\pi r}\frac{(c^2 - r^2)}{(c^2 - b^2)} \tag{5-20}$$

A flux line at the inner surface of the outer conductor at radius b makes a complete linkage of current I, whereas a flux line at the radius c links zero current. The fractional part of a linkage contributed by a flux line at radius r between b and c is

$$1 - \frac{\pi r^2 - \pi b^2}{\pi c^2 - \pi b^2} = \frac{c^2 - r^2}{c^2 - b^2}$$

The linkages due to the flux passing through the outer conductor, between b and c, are

$$\lambda_3 = \frac{\mu_3}{2\pi} \int_b^c \frac{(c^2 - r^2)}{(c^2 - b^2)} \frac{(c^2 - r^2)}{(c^2 - b^2)} I \frac{dr}{r}$$

$$= \frac{\mu_3 I}{2\pi(c^2 - b^2)^2} \left[c^4 \ln \frac{c}{b} - c^2(c^2 - b^2) + \frac{1}{4}(c^4 - b^4) \right] \text{linkages/m} \tag{5-21}$$

The inductance of the line may be found as the sum of λ_1, λ_2, and λ_3, divided by the current I. Ordinarily the materials of a line are nonmagnetic, so that $\mu_1 = \mu_3 = \mu_v$. The inductance of the coaxial line can then be found as

$$L = \frac{\lambda_1 + \lambda_2 + \lambda_3}{I}$$

$$L = \frac{\mu_v}{8\pi} \left\{ 1 + 4 \ln \frac{b}{a} + \frac{4}{(c^2 - b^2)^2} \left[c^4 \ln \frac{c}{b} - c^2(c^2 - b^2) + \frac{1}{4}(c^4 - b^4) \right] \right\}$$

Inserting the value of μ_v and rearranging gives

$$L = 10^{-7} \left[2 \ln \frac{b}{a} + \frac{2c^4 \ln \frac{c}{b}}{(c^2 - b^2)^2} - \frac{c^2}{c^2 - b^2} \right] \text{henrys/m} \tag{5-22}$$

as the value of the inductance of the coaxial line.

At high radio frequencies, the current crowds to the outer surface of the inner conductor and the inner surface of the outer conductor, as explained qualitatively in the next section. There is then no flux inside the inner or outer conductors, eliminating λ_1 and λ_3 from the above. All flux then exists in the dielectric between radii a and b, and only the linkages λ_2 contribute to the inductance, so that

$$L = \frac{\mu_r}{2\pi} \ln \frac{b}{a} = 2 \times 10^{-7} \ln \frac{b}{a} \text{ henrys/m} \tag{5-23}$$

$$= 4.60 \times 10^{-7} \log_{10} \frac{b}{a} \qquad \text{henrys/m} \tag{5-24}$$

$$= 0.741 \log_{10} \frac{b}{a} \qquad \text{mh/mile} \tag{5-25}$$

In order to produce a flexible coaxial cable the outer conductor is frequently braided from small wires. The assumption of uniform current flow in the outer conductor is then upset, and because of the discontinuities contributed by the braiding, some flux will leak out and appear external to the cable. This may be undesirable in some applications where extreme shielding is desired. Doubly shielded cables are available to reduce this leakage of flux, if solid outer conductors cannot be used.

5-4. Qualitative discussion of skin effect

Consider a conductor made up of a large number of fine strands of wire, all strands carrying the same current. A strand at the center is linked by all the internal flux in the conductor, whereas a strand on the surface is not linked by the internal flux. The inductance and reactance of the strand at the center is greater than that of the strand at the surface. If the current is permitted to vary with time and all strands still carry the same current, then the voltage drop along the center strand will be greater than that along an outside strand. This effect, however, is a direct violation of Kirchhoff's law. Therefore the currents carried by the strands cannot be equal in magnitude, since the impedances are unequal. The interior strand thus carries less current than the outer so as to produce equal impedance drops along the strands. This phenomenon is known as *skin effect*. Though causing some deviation from uniform current distribution even at 60 cycles, it is so effective at radio frequencies that essentially the whole current flows in a layer a few thousandths of an inch thick on the conductor surface.

The decrease in effective conductor cross section at the higher frequencies increases the conductor resistance. The increase in resistance can be shown to be proportional to the square root of the frequency.

A more fundamental treatment of the subject is given in Chapter 8.

5-5. Capacitance of two parallel round conductors

In order to arrive at a means of calculation of the capacitance between the two conductors of a transmission line, it is desirable to start with consideration of the very long (infinite) round conductor of Fig. 5-4. This conductor is given an electric charge of q coulombs

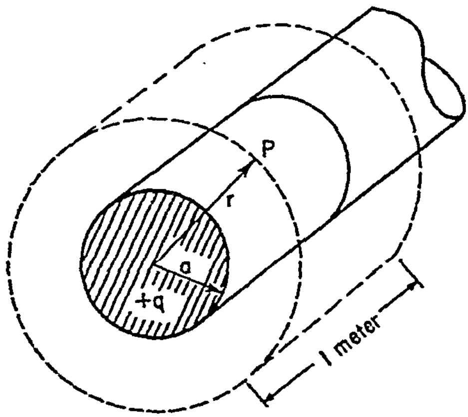


Fig. 5-4. A section of a long round conductor of radius a. Inscribed about a 1-meter length is a hypothetical cylinder of radius r.

per meter of length. According to Gauss's law, the electric flux passing outward from the wire is equal to q lines per meter of wire length. By reason of the specification that the wire be long, no *net* flux emerges from the two drumhead surfaces of any section of the wire. If a section of wire one meter in length be considered, the area of an enclosing cylinder of radius r meters is

$$A = 2\pi r \cdot 1 \quad (\text{m}^2)$$

The electric flux density D at P, a point on the enclosing cylinder, is

$$D = \frac{q}{2\pi r} \text{ coulombs/m}^2$$

TABLE 1

PROPERTIES OF DIELECTRICS

Material	Dielectric constant	Dielectric strength v/meter	Volume resistivity, ohm-meters
Vacuum	1.00000	....	....
Air	1.00059	3×10^6	....
Bakelite, pure resin	4.5	20×10^6	2×10^{14}
Celluloid	7	12×10^6	2×10^8
Ceresin wax	2.2	....	5×10^{16}
Fiber, vulcanized	2.5	7×10^6	2×10^8
Glass, Pyrex	4.8	13×10^6	1×10^{12}
Mica	7	90×10^6	2×10^{15}
Mycalex	8.0	14×10^6	5×10^{13}
Paper, kraft	3.5	3×10^6	....
Polyethylene	2.3	30×10^6	....
Polystyrene	2.5	20×10^6	....
Quartz, fused	4.2	80×10^6	5×10^{16}
Rubber, hard	3	17×10^6	1×10^{16}
Steatite	5.9	....	2.5×10^{12}
Sulphur	4	....	1×10^{15}

from which the electric field intensity at radius r is

$$\mathcal{E} = \frac{D}{\epsilon} = \frac{q}{2\pi r\epsilon} \text{ (v/m)} \tag{5-26}$$

The *permittivity* ϵ is defined as

$$\epsilon = \epsilon_r \epsilon_v \tag{5-27}$$

where ϵ_r is the *dielectric constant* or *relative permittivity*, and ϵ_v is the permittivity of space, which in MKS units is given the value $10^{-9}/36\pi = 8.85 \times 10^{-12}$. Properties of some common dielectrics are summarized in Table 1.

The potential at point P, in Fig. 5-4, with respect to the conductor, is then obtained as the negative of the integral of the field from a to r:

$$V_r = -\int_a^r \frac{q}{2\pi r\epsilon}\, dr = \frac{-q}{2\pi\epsilon} \ln \frac{r}{a} \text{ volts} \tag{5-28}$$

A parallel-wire transmission line is ordinarily constructed so that the spacing d between wires, Fig. 5-5, is large with respect to the radius a of a conductor. With that condition established it is permissible to consider the charge as uniformly distributed around

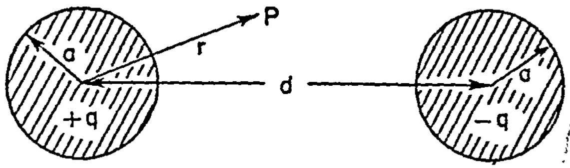


Fig. 5-5. Section of parallel-wire transmission line.

the periphery of each conductor; and then Eq. 5-28 obtained from the single isolated wire of Fig. 5-4, becomes valid for the parallel-wire case.

The potential difference between the two conductors of Fig. 5-5, when the left wire has a charge of $+q$ and the right wire a charge of $-q$ coulombs per unit length, and with the negatively charged wire as reference, is

$$V = \frac{q}{2\pi\epsilon} \ln \frac{d-a}{a} - \frac{(-q)}{2\pi\epsilon} \ln \frac{d-a}{a}$$

$$= \frac{q}{2\pi\epsilon} \ln \frac{d-a}{a} \text{ volts} \tag{5-29}$$

The capacitance C is defined as the charge which may be held on the electrodes per volt of potential difference, so

$$C = \frac{q}{V} = \frac{q}{\dfrac{q}{\pi\epsilon} \ln \dfrac{d-a}{a}} = \frac{\pi\epsilon}{\ln \dfrac{d-a}{a}} \text{ farads/m}$$

It has already been assumed that d is large with respect to a so that the term $d - a$ in the logarithm may be reduced to d, giving

$$C = \frac{\pi\epsilon}{\ln \dfrac{d}{a}} \quad \text{farads/m} \tag{5-30}$$

$$= \frac{12.07\epsilon_r}{\log_{10} \dfrac{d}{a}} \text{ farads/m} \tag{5-31}$$

For the telephone line, capacitance per mile is more useful, and since 1 mile = 1609.4 meters,

$$C = \frac{1.943 \times 10^4 \epsilon_r}{\log_{10} \dfrac{d}{a}} \ \mu\mu\text{f/mile} \tag{5-32}$$

If the wires are located such that d is not large with respect to a, then the charge is not uniformly distributed around the periphery of the wire, because of the attraction of unlike charges. A more detailed analysis in which this effect is considered gives, for the capacity of two round wires,

$$C = \frac{\pi\epsilon}{\cosh^{-1} \dfrac{d}{2a}} \ \text{farads/m} \tag{5-33}$$

Equations 5-30 and 5-33 give equivalent results within better than 2 per cent for values of d/a above 5.

5-6. Capacitance of the coaxial line

A cross section of a coaxial line is shown in Fig. 5-6. If the inner conductor is given a charge of $+q$ coulombs per meter of length, and the outer conductor an equal negative charge, the charges will distribute uniformly on the outer surface of the inner conductor and on the inner surface of the outer conductor. The field at point P will then be identical with that of the isolated round conductor of Fig. 5-4, because of symmetry and because no field contribution

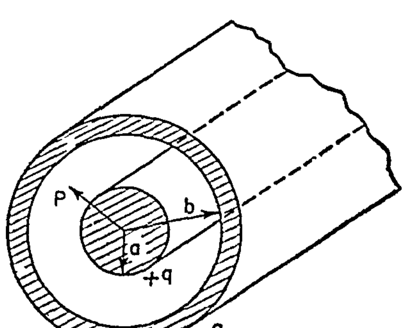


Fig. 5-6. Cross section of a coaxial transmission line.

comes from the outer charge. The field intensity at P is, from Eq. 5-26,

$$\mathcal{E} = \frac{q}{2\pi r\epsilon} \text{ (v/m)}$$

and the potential difference from outer to inner conductor, with the negative outer conductor as reference, is

$$V = -\int_b^a \frac{q}{2\pi r\epsilon}\,dr = \frac{q}{2\pi\epsilon}\ln\frac{b}{a} \text{ volts} \tag{5-34}$$

The capacitance of the coaxial line then is

$$C = \frac{q}{V} = \frac{2\pi\epsilon}{\ln\dfrac{b}{a}} \text{ farads/m} \tag{5-35}$$

$$= \frac{24.14\,\epsilon_r}{\log_{10}\dfrac{b}{a}}\ \mu\mu\text{f/m} \tag{5-36}$$

$$= \frac{3.886 \times 10^4\epsilon_r}{\log_{10}\dfrac{b}{a}}\ \mu\mu\text{f/mile} \tag{5-37}$$

The dielectric of the coaxial line may be a solid or a gas. In the use of air or nitrogen it is necessary to maintain the position of the central conductor with frequent solid dielectric spacers. These spacers raise the average value of the dielectric constant or permittivity above that of space. The effective value of the permittivity can be computed as

$$\epsilon_r' = 1 + (\epsilon_r - 1)\frac{t}{S} \tag{5-38}$$

where ϵ_r is the relative permittivity of the dielectric spacers, t the thickness, and S the center-to-center distance of the spacers. This expression applies where S is small with respect to a wavelength.

5-7. Flux linkages in a system of multiple parallel conductors

For systems involving more than two conductors, a more general method than that employed in Section 5-2 is required for computation of flux linkages and inductance.

Assume that Fig. 5-7 shows a cross section of a group of N round conductors which constitute a complete circuit, that is, the net current perpendicular to the page is zero, or

$$I_A + I_B + I_C + \ . \ . \ . + I_N = 0 \tag{5-39}$$

The conductors of the system will be labeled from A to N, and these letters as subscripts will designate conductor radii and currents.

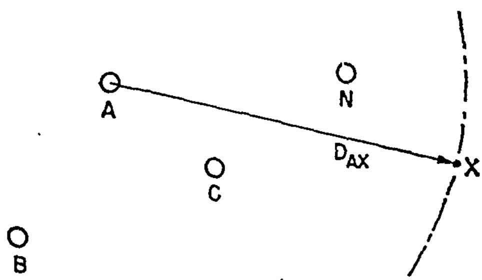


Fig. 5-7. Arbitrary arrangement of conductors A, . . . N, showing point X.

The letter D with appropriate subscripts will indicate center-to-center distances for the various elements of the system.

Let wire A be the conductor under immediate consideration, and whose flux linkages are to be computed. Select a point in the plane of the page of Fig. 5-7, much more distant from A than any other wire of the group of conductors, and label this point X as in the figure. By use of Eqs. 5-10 and 5-12, with appropriate limits, the total linkages about wire A due to its own current I_A, produced by flux surrounding A and having a radius less than the distance D_{AX} is

$$\lambda = I_A \left(\frac{\mu}{8\pi} + \frac{\mu_v}{2\pi} \int_r^{D_{AX}} \frac{dr}{r} \right)$$

and after using the value of μ_v,

$$\lambda = 10^{-7} I_A \left(\frac{\mu}{2\mu_v} + 2 \ln \frac{D_{AX}}{r_A} \right) \text{ linkages/m} \tag{5-40}$$

Now consider the flux due to a current I_J in any other conductor J. Flux produced by this current and which links A, but does not have a radius greater than D_{JX}, (X chosen so $D_{JX} \gg D_{JA}$), is given by

$$\lambda = \frac{\mu_v}{2\pi} \int_{D_{JA}}^{D_{JX}} \frac{dr}{r} = 2 \times 10^{-7} \ln \frac{D_{JX}}{D_{JA}}$$

All the other conductors likewise produce similar linkage terms, so that the total linkage around conductor A, due to flux not having a radius greater than D_{AX} is

$$\lambda_A = 10^{-7} \left(\frac{I_A}{2} \frac{\mu}{\mu_v} + 2I_A \ln \frac{D_{AX}}{r_A} + 2I_B \ln \frac{D_{BX}}{D_{AB}} + 2I_C \ln \frac{D_{CX}}{D_{AC}} + \cdots + 2I_N \ln \frac{D_{NX}}{D_{AN}} \right) \text{ linkages/m} \quad (5\text{-}41)$$

Because of Eq. 5-39 it is possible to write

$$I_N = -I_A - I_B - I_C - \cdots - I_{N-1}$$

This expression may be used to replace the last term of Eq. 5-41, giving a number of negative logarithmic terms. These may be combined with their companion positive terms as ratios, so that

$$\lambda_A = 10^{-7} \left(\frac{I_A}{2} \frac{\mu}{\mu_v} + 2I_A \ln \frac{D_{AX}}{r_A} \frac{D_{AN}}{D_{NX}} + 2I_B \ln \frac{D_{BX}}{D_{AB}} \frac{D_{AN}}{D_{NX}} + 2I_C \ln \frac{D_{CX}}{D_{AC}} \frac{D_{AN}}{D_{NX}} + \cdots + 2I_{N-1} \ln \frac{D_{(N-1)X}}{D_{A(N-1)}} \frac{D_{AN}}{D_{NX}} \right) \quad (5\text{-}42)$$

If the distances from all the wires to point X be made very large by allowing X to approach infinity, all the ratios such as D_{AX}/D_{NX}, involving distances to X, will approach unity as a limit. In this manner all flux from each wire to infinity is considered in calculating the linkages. Because of the way point X was chosen, and since the result is finite and independent of any distance to the point X, it is proved that flux which links all the conductors, or zero net current, does not contribute to the linkages or to the inductance.

Then rewriting Eq. 5-42 as

$$\lambda_A = 10^{-7} \left(\frac{I_A}{2} \frac{\mu}{\mu_v} + 2I_A \ln D_{AN} - 2I_A \ln r_A + 2I_B \ln D_{AN} - 2I_B \ln D_{AB} + \cdots + 2I_{N-1} \ln D_{AN} - 2I_{N-1} \ln D_{A(N-1)} \right)$$

The terms involving $\ln D_{AN}$ may be collected and Eq. 5-39 may again be applied, finally leading to a single term replacement as $-2I_N \ln D_{AN}$.

If *nonmagnetic conductors are assumed,* then $\mu/\mu_v = 1$, and the linkages about wire A can be written

$$\lambda_A = 2 \times 10^{-7} \left(\frac{I_A}{4} + I_A \ln \frac{1}{r_A} + I_B \ln \frac{1}{D_{AB}} + I_C \ln \frac{1}{D_{AC}} + \cdots + I_N \ln \frac{1}{D_{AN}} \right) \quad (5\text{-}43)$$

The first two terms inside the parentheses involving I_A, may be combined as

$$I_A \left(\frac{1}{4} + \ln \frac{1}{r_A} \right) = I_A \left(\ln \frac{\epsilon^{1/4}}{r_A} \right) = I_A \ln \frac{1}{r_A \epsilon^{-1/4}}$$

so that

$$\lambda_A = 2 \times 10^{-7} \left(I_A \ln \frac{1}{r_A \epsilon^{-1/4}} + I_B \ln \frac{1}{D_{AB}} + I_C \ln \frac{1}{D_{AC}} + \cdots + I_N \ln \frac{1}{D_{AN}} \right) \quad (5\text{-}44)$$

The linkages about each of the other wires of the group could then be written by inspection, through following the same pattern with respect to subscripts. The inductance of each of the conductors of the group could then be calculated by use of the known currents and the relation $L = \lambda/I$.

Now consider the two groups of conductors $a \ldots n$, and $1 \ldots N$, of Fig. 5-8. It will be assumed that these conductors in

Fig. 5-8. Cross section of parallel conductors.

each group share the current equally, each conductor of the group $a \ldots n$ carrying $+I/n$ amperes, and each conductor of the group $1 \ldots N$ carrying $-I/N$ amperes, where n does not necessarily equal N.

Following the pattern of Eq. 5-44 it is possible to write the flux linkages about any conductor, a for example, as:

$$\begin{aligned}\lambda_a = {} & 2 \times 10^{-7} \frac{I}{n}\left(\ln \frac{1}{r_a\epsilon^{-1/4}} + \ln \frac{1}{D_{ab}} + \ln \frac{1}{D_{ac}} + \cdots + \ln \frac{1}{D_{an}}\right) \\ & -2 \times 10^{-7} \frac{I}{N}\left(\ln \frac{1}{D_{a1}} + \ln \frac{1}{D_{a2}} + \ln \frac{1}{D_{a3}} + \cdots + \ln \frac{1}{D_{aN}}\right)\end{aligned} \tag{5-45}$$

This is the linkage about one conductor carrying $1/n$ of the current $+I$, and therefore each such linkage represents $1/n$ of a linkage about the total current I, or about the group $a \ldots n$. Similar expressions may be written for the linkages about any conductor of the group $a \ldots n$. The sum of these linkages, when divided by n to weight properly the fractional linkages, will give the *total* conductor linkages about current $+I$, from which the inductance of the $a \ldots n$ system may be determined. This inductance, when multiplied by dI/dt, will give the inductive voltage drop in the $a \ldots n$ group of conductors.

A similar method would lead to the inductance of the $1 \ldots N$ return group of conductors. In the case of polyphase circuits each of the other phase conductors may be considered as return circuits, with appropriate current values.

The total linkages about the conductors carrying the total current $+I$ amperes is

$$\begin{aligned}\lambda = {} & 2 \times 10^{-7} \frac{I}{n^2}\left(\ln \frac{1}{r_a\epsilon^{-1/4}} + \ln \frac{1}{D_{ab}} + \ln \frac{1}{D_{ac}} + \cdots + \ln \frac{1}{D_{an}}\right. \\ & + \ln \frac{1}{D_{ba}} + \ln \frac{1}{r_b\epsilon^{-1/4}} + \ln \frac{1}{D_{bc}} + \cdots + \ln \frac{1}{D_{bn}} \\ & + \ln \frac{1}{D_{ca}} + \ln \frac{1}{D_{cb}} + \ln \frac{1}{r_c\epsilon^{-1/4}} + \ln \frac{1}{D_{cd}} + \cdots + \ln \frac{1}{D_{cn}} \\ & \left. + \cdots + \ln \frac{1}{D_{na}} + \ln \frac{1}{D_{nb}} + \cdots + \ln \frac{1}{r_n\epsilon^{-1/4}}\right) \\ & - 2 \times 10^{-7} \frac{I}{nN}\left(\ln \frac{1}{D_{a1}} + \ln \frac{1}{D_{a2}} + \ln \frac{1}{D_{a3}} + \cdots + \ln \frac{1}{D_{aN}}\right.\end{aligned}$$

$$+ \ln \frac{1}{D_{b1}} + \ln \frac{1}{D_{b2}} + \ln \frac{1}{D_{b3}} + \cdots + \ln \frac{1}{D_{bN}}$$
$$+ \ln \frac{1}{D_{c1}} + \ln \frac{1}{D_{c2}} + \ln \frac{1}{D_{c3}} + \cdots + \ln \frac{1}{D_{cN}}$$
$$+ \cdots + \ln \frac{1}{D_{n1}} + \ln \frac{1}{D_{n2}} + \ln \frac{1}{D_{n3}} + \cdots + \ln \frac{1}{D_{nN}}\Bigg) \tag{5-46}$$

It should be noted that the first parenthesis involves distances from conductors in the $a \ldots n$ group to other conductors in the same group, and these might be referred to as *self* distances. The second parenthesis involves distances from all the conductors in the $a \ldots n$ group to all the conductors in the $1 \ldots N$ group.

By combining the various logarithms, it is possible to write

$$\lambda = 2 \times 10^{-7} I \left[\ln \frac{\sqrt[nN]{D_{a1}D_{a2} \cdots D_{aN}D_{b1}D_{b2} \cdots D_{bN} \cdots D_{n1}D_{n2} \cdots D_{nN}}}{\sqrt[n^2]{r_a\epsilon^{-1/4}D_{ab}D_{ac} \cdots D_{an}D_{ba}r_b\epsilon^{-1/4}D_{bc} \cdots D_{bn}D_{ca}D_{cb}r_c\epsilon^{-1/4} \cdots D_{cn} \cdots r_n\epsilon^{-1/4}}} \right] \tag{5-47}$$

There are nN terms under the numerator radical, for which the nNth root is to be taken, so that a geometric mean is involved. This is the geometric mean of all the distances between the conductors or elements of the system carrying the total current $+I$, and all the return current conductors. The numerator is called the *geometric mean distance* of one conductor set to all other conductors, and is abbreviated GMD and given the symbol D_M. This is a mathematically derived concept, which has no relation to the electrical problem at hand except as a useful tool.

It is then possible to write

$$\lambda = 2 \times 10^{-7} I \ln \frac{D_M}{\sqrt[n^2]{r_a\epsilon^{-1/4} \cdots r_n\epsilon^{-1/4}D_{ab} \cdots D_{an}D_{ba} \cdots D_{bn}D_{ca} \cdots D_{cn} \cdots D_{na} \cdots D_{n(n-1)}}} \tag{5-48}$$

There are n^2 terms under the denominator radical, for which the n^2 root is to be taken, so that this is again a geometric mean.

This may be further explained. Mathematically, the geometric mean of the distances between all possible pairs of points in an area is called the *geometric mean radius* of the area, abbreviated GMR. For a circular area of radius r the geometric mean radius can be shown equal to $r\epsilon^{-1/4}$. Thus it is recognized that the radical above contains terms having the GMR of each of the $a \ldots n$ conductors, times the $n(n-1)$ distances of each wire of the group to every other wire of the group. The denominator radical may be considered as the *self-geometric mean distance* among the elements of the conductor group carrying current $+I$, and is symbolized by D_S.

A simplified expression for the inductance of a conductor group carrying a total current I, with the return current $-I$ in one or more other conductors, may be written as

$$L = 2 \times 10^{-7} \ln \frac{D_M}{D_S} \text{ henrys/m} \tag{5-49}$$

where D_M and D_S have the meanings given in Eqs. 5-47 and 5-48.

It should be noted that the denominator of Eq. 5-49 relates to a conductor and to its size and shape, using the word conductor in a general sense as a group of wires carrying equal portions of a common current. The numerator is determined solely by the distances between the various wires of the several conductors, again using the latter term in a general sense.

The method may be extended to lines having individual conductors of irregular cross section as in Fig. 5-9. Each conductor

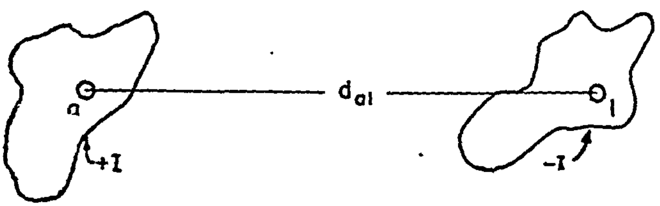


Fig. 5-9. Parallel conductors of irregular cross section.

may be divided into many elementary areas, each area carrying the same fractional current $+I/n$ or $-I/N$. To compute the inductance of the left conductor of the figure, its elementary filaments may be considered as identical to the group of wires $a \ldots n$ of Fig. 5-8. Likewise the elementary filaments of the right conductor are identical to the conductor group $1 \ldots N$ of Fig. 5-8. An expression for the flux linkages about the left conductor of Fig. 5-9 can be written in accordance with Eq. 5-47. The number of elementary conducting areas or filaments in each conductor may then be allowed to increase without limit. The radical of the numerator then approaches the GMD of the two conductor areas as a limit, and the denominator radical approaches the GMR of the conductor whose inductance is being calculated. Equation 5-49 will then apply to the case of the irregular conductors.

As an example, the two wire line of Fig. 5-2 may be used. The GMD of the two conductors is obviously the center-to-center distance $d = D_M$. The GMR of a round conductor of radius a is $a\epsilon^{-1/4}$. Therefore for *both* wires

$$L = 2 \times 2 \times 10^{-7} \ln \frac{D_M}{D_S}$$

$$= 4 \times 10^{-7} \ln \frac{d}{a\epsilon^{-1/4}} \text{ henrys/m} \qquad (5\text{-}50)$$

However, Eq. 5-15 gave

$$L = 10^{-7}\left(\frac{\mu}{\mu_v} + 4 \ln \frac{d}{a}\right) \text{ henrys/m}$$

This may be rewritten for nonmagnetic conductors as

$$L = 4 \times 10^{-7}\left(\frac{1}{4} + \ln \frac{d}{a}\right)$$

and by combining the constant $\frac{1}{4}$ with the logarithm as $\ln \epsilon^{1/4}$ this becomes

$$L = 4 \times 10^{-7} \ln \frac{d}{a\epsilon^{-1/4}} \text{ henrys/m} \qquad (5\text{-}51)$$

which is identical with the result by *GMD* methods in Eq. 5-50. Further examples for multiconductor circuits will follow.

5-8. GMR and GMD of various conductor arrangements

It has been stated that the GMR of a circular *area* of radius r is given by $r\epsilon^{-1/4} = 0.7788r$. It may be reasoned that the GMR of a circular *line* is equal to the radius of the circle. Another useful theorem, derived purely on a mathematical basis, is that the GMD between two nonoverlapping circular areas of any size is equal to the distance between their centers, this applying to two wires.

For computation with rectangular bus bars, the GMR of a rectangular area of sides a and b is closely equal to

$$D_S = 0.2235(a + b) \tag{5-52}$$

Other theorems are available in Reference 1.

Stranded cable is most frequently used in the large conductor sizes for reasons of flexibility, and the GMR value is not quite the same as that of a solid conductor of equivalent radius. Three-wire cables are used, but other common types start with a central strand surrounded by a spiraled layer made of six more strands as in Fig.

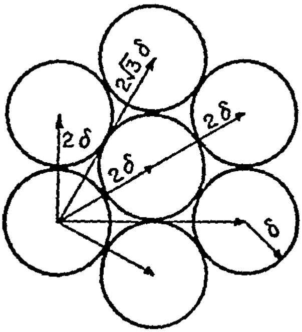


Fig. 5-10. Seven-strand cable.

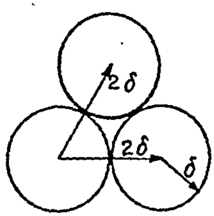


Fig. 5-11. Three-strand cable.

5-10. The next larger size is made by adding another oppositely spiraled layer which will require 12 more strands, the next layer would require 18 to fill the annular space, and so on increasing in each layer by multiples of six. The total number of strands in such cables is then 7, 19, 37, 61, 91, and so on.

Successive layers are spiraled oppositely to prevent untwisting, and also to prevent one layer from settling into the interstices of the layer below. It is desirable to illustrate the GMR calculations for several cables:

1. *Three-strand cable* of Fig. 5-11. If the radius of an individual strand is δ, then the area of each strand is $4\delta^2$, and the cable cross section $A = 3 \cdot 4\delta^2 = 12\delta^2$ *in circular measure*. The GMR value may be found by application of Eq. 5-48. First write the GMR value for a strand as $\delta\epsilon^{-1/4}$, and repeat this three times, once for each strand, giving $(\delta\epsilon^{-1/4})^3$. The distance from the center of one strand to another is 2δ, and this is repeated six times, since there are six such distances, giving $(2\delta)^6$. Collecting these nine terms

gives

$$D_S = \sqrt[9]{(\delta\epsilon^{-1/4})^3(2\delta)^6} = \delta\sqrt[9]{(\epsilon^{-1/4})^3 2^6}$$
$$= 1.46\delta$$

Since the radius R_c of the cable is equal to 2.15δ, then

$$D_S = 0.677R_c$$

2. *Seven-strand cable* of Fig. 5-10. The area A of such a cable is $A = 7 \cdot 4\delta^2 = 28\delta^2$ in circular measure. To find D_S, first write the GMR value for each strand as $\delta\epsilon^{-1/4}$, and repeat this seven times for the seven strands, giving $(\delta\epsilon^{-1/4})^7$. The distance from the center strand to the six outer strands is 2δ, and this will be repeated six times, or $(2\delta)^6$. The distances from one outside strand to the six other strands are shown in the figure, and their product is

$$2\delta \cdot 2\delta \cdot 2\delta \cdot 2\sqrt{3}\,\delta \cdot 2\sqrt{3}\,\delta \cdot 4\delta = 2^7 \cdot 3 \cdot \delta^6$$

and this quantity will be repeated six times, giving $(2^7 \cdot 3 \cdot \delta^6)^6$.

Collecting these 49 distances gives

$$D_S = \sqrt[49]{(\delta\epsilon^{-1/4})^7(2\delta)^6(2^7 \cdot 3 \cdot \delta^6)^6}$$
$$= \delta\sqrt[49]{(\epsilon^{-1/4})^7 2^{48} 3^6} = 2.18\delta$$

in terms of the radius of one strand. Since the radius R_c of the cable is 3δ, then

$$D_S = 0.726R_c$$

for the seven-strand cable.

Similar methods can be followed for cables with larger numbers of strands, leading to Table 2.

TABLE 2

GMR VALUES FOR STRANDED CABLE

Strands	D_S	Strands	D_S
3	$0.677R_c$	61	$0.772R_c$
7	$0.726R_c$	91	$0.774R_c$
19	$0.758R_c$	solid	$0.778R_c$
37	$0.768R_c$		

Some conductors are essentially tubular in shape, the long line from Hoover Dam to Los Angeles being of this form. This is desirable to provide a large outside diameter to reduce the external electric field and corona loss at the high voltage of operation. The calculation of the D_S value for a tube is complicated, but Table 3 supplies values for various ratios of outside radius c and inside radius b.

TABLE 3

GMR VALUES FOR TUBULAR CONDUCTORS

Ratio b/c	D_S	Ratio b/c	D_S
0.0 (Solid wire)	0.7788	0.6	0.879
0.1	0.781	0.7	0.908
0.2	0.791	0.8	0.938
0.3	0.806	0.9	0.967
0.4	0.826	1.0	1.0
0.5	0.850		

Aluminum is frequently employed in transmission lines as conductor material. When so used, it is customary for the central strand, or the central strand and one or more of the first layers, to be made of steel for increased mechanical strength. Such a cable is known as *aluminum cable steel reinforced* or ACSR. Because of the disparity in conductivities between aluminum and steel, a reasonable assumption is to consider that the aluminum strands carry all the current, and to compute D_S as if the steel were absent. Accurate calculations which include the effect of the steel are difficult because of the necessity for knowledge of the permeability of the steel strands. For accurate work it is best to consult tables of inductance values prepared from experimental data by the cable manufacturers.

5-9. Inductance of a symmetrical three-phase line

The simplest three-phase transmission line arrangement is the equilateral spacing of Fig. 5-12. The conductors are indicated as a, b, c, and have a center-to-center spacing D. If the currents are designated I_a, I_b, I_c, then by Eq. 5-45 the linkages about phase a are

$$\lambda_a = 2 \times 10^{-7} \left(I_a \ln \frac{1}{D_S} + I_b \ln \frac{1}{D} + I_c \ln \frac{1}{D} \right) \tag{5-53}$$

since $D_{a1} = D_{a2} = D$.

Now in a balanced three-phase system the currents are related as

$$I_b = \left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right) I_a \qquad I_c = \left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right) I_a$$

so that Eq. 5-53 becomes

$$\lambda_a = 2 \times 10^{-7} I_a \left(\ln \frac{1}{D_S} - \ln \frac{1}{D} \right)$$

$$= 2 \times 10^{-7} I_a \ln \frac{D}{D_S} \text{ linkages/m} \tag{5-54}$$

Dividing by I_a gives the inductance of phase a as

$$L_a = 2 \times 10^{-7} \ln \frac{D}{D_S} \text{ henrys/m} \tag{5-55}$$

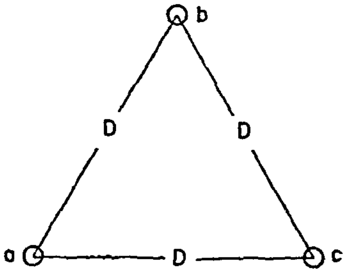


Fig. 5-12. Equilateral spacing of a three-phase line.

where D_S is the GMR of conductor a. It can be noted that the GMD between a and conductors b and c might have been written as $\sqrt[4]{D^4} = D$, and thus use of Eq. 5-49 would have lead directly to the above result.

When used in a series reactance term as ωL_a, and multiplied by the effective line current, the series voltage drop in the phase would be obtained. When this voltage drop is subtracted from the sending end line-to-neutral voltage the result is the receiving end line-to-neutral voltage.

Because of the symmetry of the conductor arrangement of Fig. 5-12, the same result would be obtained for all phases, and balanced operation would result.

5-10. The unsymmetrical three-phase line; transposition

Unsymmetrical conductor arrangements are most commonly employed for three-phase transmission, since they allow cheaper and more convenient tower and line construction. Several arrangements appear in Fig. 5-13.

Use of the linkage relations of Eq. 5-45 for either arrangement of the figure, leads to a linkage for phase a of

$$\lambda_a = 2 \times 10^{-7} \left(I_a \ln \frac{1}{D_S} + I_b \ln \frac{1}{D_{ab}} + I_c \ln \frac{1}{D_{ac}} \right)$$

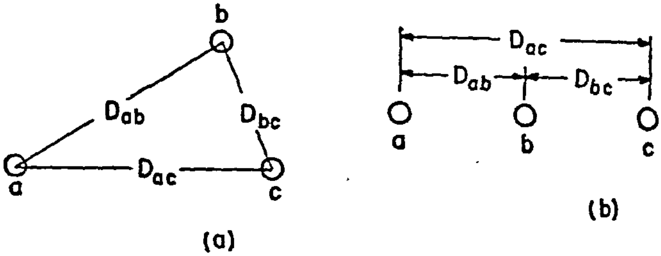


Fig. 5-13. Two unsymmetrical three-phase line arrangements.

and under *assumed balanced current* conditions

$$\lambda_a = 2 \times 10^{-7} I_a \left(\ln \frac{1}{D_S} - \frac{1}{2} \ln \frac{1}{D_{ab}} - \frac{1}{2} \ln \frac{1}{D_{ac}} \right)$$

$$= 2 \times 10^{-7} I_a \ln \frac{\sqrt{D_{ab} D_{ac}}}{D_S} \text{ linkages/m} \qquad (5\text{-}56)$$

A similar expression could be obtained for phase b as

$$\lambda_b = 2 \times 10^{-7} I_b \left(\ln \frac{1}{D_S} - \frac{1}{2} \ln \frac{1}{D_{ba}} - \frac{1}{2} \ln \frac{1}{D_{bc}} \right)$$

$$= 2 \times 10^{-7} I_b \ln \frac{\sqrt{D_{ba} D_{bc}}}{D_S} \text{ linkages/m} \qquad (5\text{-}57)$$

and for phase c as

$$\lambda_c = 2 \times 10^{-7} I_c \ln \frac{\sqrt{D_{ca} D_{cb}}}{D_S} \text{ linkages/m} \qquad (5\text{-}58)$$

It is apparent that these three linkages will differ, due to the various conductor distances involved. But if the linkages differ, the inductances will be unequal, unequal voltage drops will appear in the three lines, and this implies unbalanced operation with probable unbalanced currents, contrary to the assumption above.

While there have been constructed some high-voltage lines in which this condition is permitted because of the cost of eliminating it, it is more customary to correct the unbalance by *transposing* the line wires, so that over the full length of a line each phase will occupy each conductor position for an equal distance. A complete transposition cycle is illustrated in Fig. 5-14. The three tower or cross arm positions are indicated by 1, 2, 3, and the three conductors by a, b, c. Each wire or phase occupies each position for one-third of the cycle length, and the conductors rotate in position at the one-third and two-third points in the cycle. Such a transposition

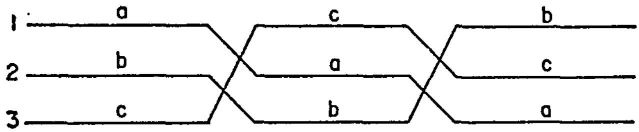


Fig. 5-14. Transposition diagram for a three-phase line.

is dictated not only to achieve balance of the phases, but also to reduce inductive interference with nearby telephone lines. This latter point even dictates that symmetrical three-phase lines should be transposed, as well.

The linkages per meter about conductor a may be computed for each of the three sections or positions, totaled, and divided by 3 to get the average value over the transposition cycle. Then

$$\lambda_a = 2 \times 10^{-7} \frac{I_a}{3} \left(\ln \frac{\sqrt{D_{ab}D_{ac}}}{D_S} + \ln \frac{\sqrt{D_{ba}D_{bc}}}{D_S} + \ln \frac{\sqrt{D_{ca}D_{cb}}}{D_S} \right)$$

$$= 2 \times 10^{-7} I_a \ln \frac{\sqrt[3]{D_{ab}D_{bc}D_{ac}}}{D_S} \text{ linkages/m} \tag{5-59}$$

Computations of average λ_b and λ_c would contain the same logarithm, proving the balance obtained by the method of transposition.

The average inductance per meter of any phase, with transpositions, will then be

$$L = 2 \times 10^{-7} \ln \frac{\sqrt[3]{D_{ab}D_{bc}D_{ac}}}{D_S} \text{ henrys/m} \tag{5-60}$$

The numerator $\sqrt[3]{D_{ab}D_{bc}D_{ac}}$ is the GMD among the three conductors. The factor D_S is, of course, computed for the particular cable used by the methods of Section 5-8. Thus the GMD method of Section 5-7 may be applied and will lead more quickly to the correct result. It is, as was stated in its derivation, a general method of solution, and will be further demonstrated in the next section.

5-11. Multicircuit lines

Occasionally single-phase or polyphase lines are built with multiple wires, connected in parallel. The four-wire line is an example at radio frequencies. The calculation of inductance may be readily carried out by the general methods of Section 5-7.

Consideration of Fig. 5-15, for the four-wire or double-circuit single-phase case, shows it to be similar to the common two-wire line, but each *conductor* of the simple case is now composed of two *wires*, a and a', b and b'. It is assumed that all four conductors are of the same radius r, and material. The symmetry of the arrangement indicates that transpositions are not required for balance, however in many applications they are used in any case, to reduce induced voltages in nearby telephone or other circuits, or to achieve balance of the circuits to ground or other conductors which happen to run nearby.

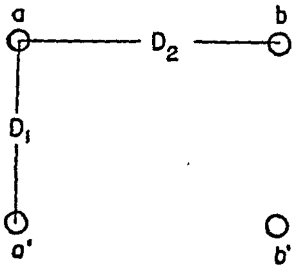


Fig. 5-15. Double-circuit or four-wire single-phase line; a and a' in parallel, b and b' in parallel.

Using the general GMD relation of Eq. 5-49 as

$$L = 2 \times 10^{-7} \ln \frac{D_m}{D_s} \text{ henrys/m}$$

will give the inductance of a conductor (the pair of wires a and a', or b and b') which carries the total current of one line $I = I_a + I_{a'}$. Computing D_S for the pair of wires a and a',

$$\begin{aligned} D_S &= \sqrt[4]{r\epsilon^{-1/4} r\epsilon^{-1/4} D_1 D_1} \\ &= \sqrt[2]{0.7788 r D_1} \end{aligned} \tag{5-61}$$

showing the use of the self-GMR of the two round conductors and the distances D_1 from a to a' and from a' to a.

Computing the GMD of the one conductor (a,a') to the other (b,b') by use of all the four distances gives

$$D_M = \sqrt[8]{(D_2 \sqrt{D_2^2 + D_1^2})^4} = \sqrt{D_2 \sqrt{D_2^2 + D_1^2}} \tag{5-62}$$

The complete expression for the inductance of *both conductors*, or of the line of Fig. 5-15 is then

$$L = 4 \times 10^{-7} \ln \frac{\sqrt{D_2 \sqrt{D_2^2 + D_1^2}}}{\sqrt{0.7788 r D_1}} \tag{5-63}$$

where r is the radius of a wire.

Frequently one polyphase line with a vertical arrangement of

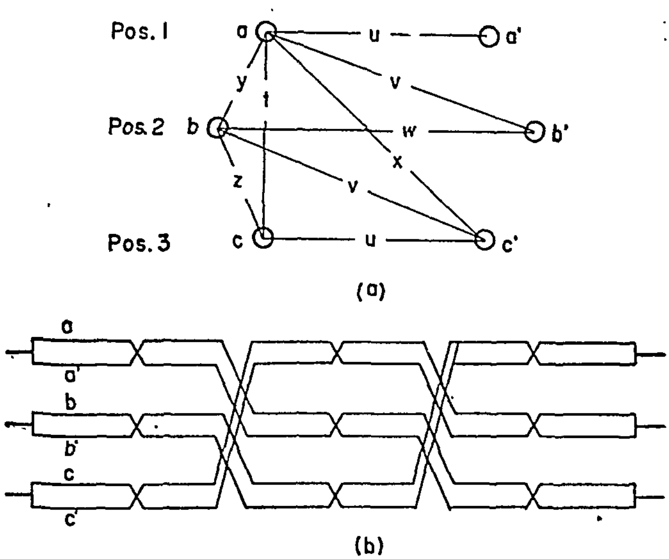


Fig. 5-16. (a) Double-circuit three-phase line; (b) transposition cycle for (a).

conductors will be built on one side of a tower, and at a later date when the electric load has increased, a second three-phase circuit will be added on the other side of the tower, giving a common form of double-circuit three-phase line shown in Fig. 5-16.

The conductors are assumed to be alike, with spacings as given in the figure, and transposed. Then using the GMD relations of Eq. 5-49, the value of D_S for a phase (of two wires) in any position over the transposition cycle, is

$$\begin{aligned} D_S &= \sqrt[12]{(r\epsilon^{-1/4} r\epsilon^{-1/4} u^2)(r\epsilon^{-1/4} r\epsilon^{-1/4} w^2)(r\epsilon^{-1/4} r\epsilon^{-1/4} u^2)} \\ &= \sqrt[6]{(r\epsilon^{-1/4})^3 u^2 w} = \sqrt[6]{(0.7788r)^3 u^2 w} \end{aligned} \tag{5-64}$$

The value of D_M for phase a (wires a,a') when in position 1 involves the distance from wires a and a' to all the other wires, as

$$D_{M1} = \sqrt[8]{(vxyt)^2}$$

When phase a is in position 2:

$$D_{M2} = \sqrt[8]{(v^2yz)^2}$$

and when in position 3:

$$D_{M3} = \sqrt[8]{(vxzt)^2}$$

The value of D_M, taken as a mean over all three portions of the transposition cycle, is then

$$D_M = \sqrt[24]{(vxyt)^2(v^2yz)^2(vxzt)^2} = \sqrt[6]{v^2xyzt} \tag{5-65}$$

The inductance, over a transposition cycle, is

$$L = 2 \times 10^{-7} \ln \frac{D_M}{D_S} = 2 \times 10^{-7} \ln \frac{\sqrt[6]{v^2xyzt}}{\sqrt[6]{(0.7788r)^3u^2w}} \tag{5-66}$$

henrys per meter for a phase of a double-circuit three-phase line of the construction of Fig. 5-16.

It is desirable to minimize the inductance of a given line. To do this would require that D_S be large and D_M small. Thus the dimensions u and w or the spacing between circuits should be large, and y and z or the spacing between phases should be as small as electrical clearances under icing and wind conditions permit.

5-12. Potentials in a system of conductors

The potential between any point x and a further point y, in the electric field of a charged wire a in space, carrying a charge of q_a coulombs per meter, can be written by the method of Eq. 5-28, as

$$V = -\int_x^y \frac{q_a}{2\pi r\epsilon}\,dr = \frac{q_a}{2\pi\epsilon} \ln \frac{y}{x} \text{ volts} \tag{5-67}$$

Now consider an arbitrary grouping of isolated conductors as was shown in Fig. 5-7. If arbitrary charges q_A, q_B, q_C, . . . , q_N are placed on these isolated conductors, each will assume a potential due to its own charge and to the fields from the other conductors.

The potential difference between two conductors will be due to the charge on the first conductor, plus that due to the charge on the second conductor, plus that due to any other charged conductors in the region. If the charges are known, it is possible to find the potential and the capacitance of each conductor since $C = q/V$.

The potential difference between any two conductors, A and B for example, is then obtained by applying Eq. 5-67, so that

$$E_{AB} = \frac{1}{2\pi\epsilon}\left(q_A \ln \frac{D_{AB}}{r_A} + q_B \ln \frac{r_B}{D_{BA}} + q_C \ln \frac{D_{CB}}{D_{CA}} + \cdots + q_N \ln \frac{D_{NB}}{D_{NA}}\right)$$

and similar expressions could be written for the potentials of each of the other conductors, all with A as reference, as

$$E_{AC} = \frac{1}{2\pi\epsilon}\left(q_A \ln \frac{D_{AC}}{r_A} + q_B \ln \frac{D_{BC}}{D_{BA}} + q_C \ln \frac{r_C}{D_{CA}} + \cdots + q_N \ln \frac{D_{NC}}{D_{NA}}\right)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$E_{AN} = \frac{1}{2\pi\epsilon}\left(q_A \ln \frac{D_{AN}}{r_A} + q_B \ln \frac{D_{BN}}{D_{BA}} + \cdots + q_N \ln \frac{r_N}{D_{NA}}\right) \tag{5-68}$$

This method assumes the charges will be known, whereas in the operation of electric systems it is the voltages which are known and not the charges. Considering the q as unknowns in the equations above, it can be seen that there are N unknowns, but only $N - 1$ equations are available for solution. To overcome this difficulty it may be assumed that the sum of all the charges is zero. That is

$$q_A + q_B + q_C + \cdots + q_N = 0 \tag{5-69}$$

and this is true for any ordinary line under normal operating conditions. Solution for the charges with known potentials is then possible, from which the capacitance value will follow.

Several special cases will be considered.

5-13. Capacitance of the symmetrical three-phase line

Assuming a balanced set of voltages applied to the equilateral three-phase line of Fig. 5-17, with each wire of radius r, equations of the form of the set of Eq. 5-68 may be written for the charges. If

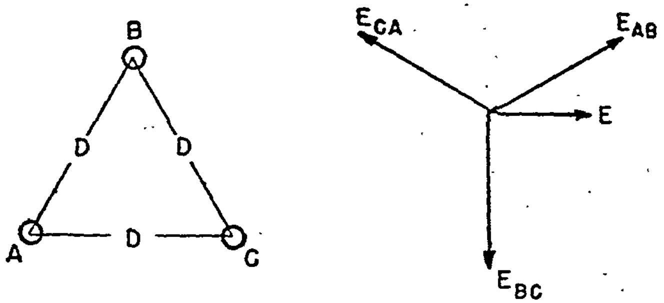


Fig. 5-17. Symmetrical three-phase line.

the line to neutral voltage is $E\underline{/0°}$, then

$$E_{AB} = E\left(\frac{3}{2} + j\frac{\sqrt{3}}{2}\right)$$

$$= \frac{1}{2\pi\epsilon}\left(q_A \ln\frac{D}{r} + q_B \ln\frac{r}{D} + q_C \ln\frac{D}{D}\right) \tag{5-70}$$

$$E_{AC} = -E_{CA} = E\left(\frac{3}{2} - j\frac{\sqrt{3}}{2}\right)$$

$$= \frac{1}{2\pi\epsilon}\left(q_A \ln\frac{D}{r} + q_B \ln\frac{D}{D} + q_C \ln\frac{r}{D}\right) \tag{5-71}$$

and also

$$q_A + q_B + q_C + \cdots + q_N = 0$$

From these equations it is possible to solve for q_A as

$$q_A = \frac{2\pi\epsilon E}{\ln\dfrac{D}{r}}$$

and since E is a line-to-neutral potential, the line-to-neutral capacitance is given by $C = q_A/E$ or

$$C = \frac{2\pi\epsilon}{\ln\dfrac{D}{r}} \quad \text{farads/m} \tag{5-72}$$

This is equivalent to

$$C = \frac{1}{18 \times 10^9 \ln\dfrac{D}{r}} \quad \text{farads/m to neutral} \tag{5-73}$$

after use of the space value of ϵ. Writing the expression on a per mile basis gives

$$C = \frac{0.0388}{\log_{10} \dfrac{D}{r}} \quad \mu\text{f/mile to neutral} \tag{5-74}$$

5-14. Capacitance of an unsymmetrical three-phase line

The unsymmetrical line requires transposition to achieve balanced charges and capacitances. For one particular portion of the transposition cycle the conductors of such a line have the arrangement of Fig. 5-18. Given a radius r for all the wires, the potentials may be written as before:

$$E_{AB} = E\left(\frac{3}{2} + j\frac{\sqrt{3}}{2}\right)$$

$$= \frac{1}{2\pi\epsilon}\left(q_A \ln \frac{D_{AB}}{r} + q_B \ln \frac{r}{D_{BA}} + q_C \ln \frac{D_{CB}}{D_{CA}}\right)$$

$$E_{AC} = E\left(\frac{3}{2} - j\frac{\sqrt{3}}{2}\right)$$

$$= \frac{1}{2\pi\epsilon}\left(q_A \ln \frac{D_{AC}}{r} + q_B \ln \frac{D_{BC}}{D_{BA}} + q_C \ln \frac{r}{D_{CA}}\right)$$

It is also required that

$$q_A + q_B + q_C = 0$$

Solution of these equations involves very considerable labor. It is necessary to solve for the charges in the first third of the transposition cycle from the above. Then in the second part of the cycle, A occupies the place of B in the first portion, so that q_{A2} equals q_{B1} rotated through 120°. In the third portion of the cycle, A is in the position of C in the first part, and q_{A3} equals q_{C1} rotated through −120°. Having thus obtained q_{A1}, q_{A2}, q_{A3}, an average value may be taken and the average value of capacitance per meter over the transposition cycle obtained. This will be a somewhat complicated expression.

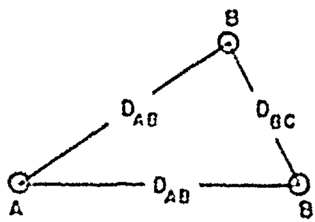


Fig. 5-18. Unsymmetrical three-phase line.

However, it will be found that an expression based on the GMD value for the given conductor arrangement will give very good agreement with the precise relation, while being simple. Such a GMD relation for the capacitance of the line of Fig. 5-18, over the transposition cycle, is

$$C = \frac{2\pi\epsilon}{\ln \dfrac{\sqrt[3]{D_{AB}D_{BC}D_{CA}}}{r}}$$

$$= \frac{2\pi\epsilon}{\ln \dfrac{D_M}{r}} \quad \text{farads/m to neutral} \tag{5-75}$$

It should be noted that while the numerator of the logarithm is the D_M value computed as in the preceding sections, the denominator of the logarithm is the radius of the conductor. The D_S value used for computation of inductance appeared because of the internal flux linkages of the conductor. However, in electrostatics the charge resides on the conductor surface and there are no internal effects; thus the appearance of the radius term is logical.

The expression above for an unsymmetrical conductor arrangement will agree with the precise relation, obtainable by use of the method outlined, within 1 per cent. For cases with symmetry the result will be exact, and this is illustrated by Eq. 5-72, in which the GMD of the three conductors was D, so that use of Eq. 5-75 would have lead directly to Eq. 5-72.

Thus for any arrangement of conductors, over a transposition cycle, Eq. 5-75 may be written with $\epsilon = \epsilon_v$ as

$$C = \frac{1}{18 \times 10^9 \ln \dfrac{D_M}{r}} \quad \text{farads/m to neutral} \tag{5-76}$$

$$= \frac{0.0388}{\log_{10} \dfrac{D_M}{r}} \quad \mu\text{f/mile to neutral} \tag{5-77}$$

5-15. Effect of ground

The presence of the conducting ground near an overhead transmission line causes a slight change in the capacitance value. The effect may be studied by the *method of images*, illustrated in Fig. 5-19.

If two parallel wires have equal and opposite charges, there will be a zero-potential plane halfway between the wires. This field configuration will be identical to that between a wire and an equipotential ground plane, so that the wire over a conducting ground problem may be studied by use of the two-wire configuration.

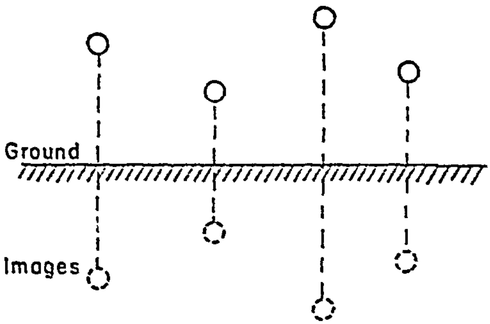


Fig. 5-19. Illustrating the method of images.

The charges may then be computed as due to the potentials between lines and images, but in terms of potentials from lines to ground.

The effect on the capacitance is of the order of 1 or 2 per cent, and is of the same order as other uncertainties in the capacitance value due to the presence of towers and other objects near the line. Because of these uncertainties in actual capacitance value, the effect of ground is usually neglected.

5-16. Relation between L and C values

It is interesting to note that if the capacitance for a two-wire line *in space*, from Eq. 5-30,

$$C = \frac{\pi\epsilon_r}{\ln \dfrac{d}{a}}$$

is multiplied by the *external inductance* of the same two-wire line, from Eq. 5-14, as

$$L_e = \frac{\mu_r}{\pi} \ln \frac{d}{a}$$

there results a constant

$$L_e C = \mu_r \epsilon_r$$

Since μ_v and ϵ_v have space values of $4\pi \times 10^{-7}$ and $10^{-9}/36\pi$ in the rationalized MKS system, this product is

$$L_e C = \mu_v \epsilon_v = \frac{1}{9 \times 10^{16}} = \frac{1}{c^2} \tag{5-78}$$

Actually this is universally true, regardless of the arrangement or number of line conductors. Thus if the value of the *external inductance* of any line be computed by the GMD methods given in detail in previous sections, the accompanying value of capacity may be immediately found by Eq. 5-78.

It should be noted that the external inductance is found from any of the GMD relations by use of the D_M term only in the logarithm, since D_S is concerned with the internal inductance. If the line is surrounded with a dielectric, appropriate values and change of constant in Eq. 5-78 will still yield useful results.

PROBLEMS

5-1. A line is made up of two No. 8 (0.1285 in. diameter) copper wires with a center-to-center spacing of 6 in. in air. Find the inductance, capacitance, and d-c resistance per mile of line. Specify separately the inductances due to internal and external linkages.

5-2. A telephone cable pair of No. 16 (0.051 in. diameter) copper wire is insulated with 0.006 in. of paper on each wire and twisted. If the dielectric is entirely of paper, find the capacitance and inductance per mile of line. If the frequency is 796 c, what shunt susceptance and series reactance are represented by the line?

5-3. A coaxial cable is made with a central conductor of No. 8 (0.1285 in. diameter) copper wire with the outer conductor of copper tubing 0.435 in. outside diameter and 0.032 in. wall thickness. The dielectric is polyethylene. Compute the shunt susceptance and series reactance per meter for a frequency of 1000 mc.

5-4. What is the inductance per loop mile of a line spaced 5 ft on centers, and having one conductor with a radius of 0.075 in. and the other of 0.15 in?

5-5. Find the D_S value for a copper cable of 6 strands of radius a, surrounding a central nonconducting core. Give the answer in terms of the radius of the cable.

5-6. A double-circuit single-phase line has the four conductors at the corners of a square 12 in. on a side. The conductors are No. 8, B & S copper wire. Find the voltage drop per thousand feet, when the line is carrying 30 kva at 2300 v. Paralleled conductors are at adjacent corners of the square, and the frequency is 60 cps.

5-7. If the paralleled conductors are on diagonally opposite corners of the square, repeat Prob. 5-6. Which arrangement gives the more desirable form of connection?

5-8. By approximating the rectangular area of Fig. 5-20 with inscribed circles, show that as the number of circles increases, the D_S value of the rectangle approaches the theoretical value of Eq. 5-52. Try at least 2 and 8 circles.

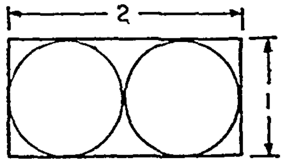


Fig. 5-20.

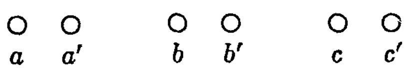
5-9. Find the D_s value for a stranded cable of four strands in terms of the cable area.

5-10. A rectangular conductor has a cross section of 1.5 million cir mils, and length to width ratio of 3:1. What will be the radius of a circular conductor having the same internal inductance per unit length? What will be the ratio of d-c resistances per unit length?

5-11. A horizontally spaced three-phase line has conductor spacing of 25 ft, each conductor being 37-strand, 500,000 cir mil copper cable. Find the inductance and capacitance per mile, assuming proper transpositions.

5-12. The above line is operated at 220 kv, for a length of 100 miles, handling 150,000 kva. If equal line currents are assumed, find the reactive line drop per mile if no transpositions are used.

5-13. A double-circuit three-phase line is hung in the arrangement



Conductors a and a', b and b', c and c' are paralleled. Spacing between wires of a phase is 15 in., spacing between a' and b, and b and c is 15 ft each. Each wire is a copper 7-strand No. 3/0 cable of 167,800 cir mils. Assuming transposition, compute the inductance and capacitance per phase per mile.

5-14. The Hoover Dam-Los Angeles line has hollow copper conductors made of interlocking segments, giving essentially a tube of 1.40 in. outside diameter and 512,000 cir mil area. The line is spaced horizontally, with 32.5 ft between conductors. Assuming transposition, find the reactance per mile per phase at 60 c.

5-15. The line of Prob. 5-14 operates at 287 kv phase voltage. Compute the capacitive charging current and kva per phase for the 275-mile line.

5-16. A double-circuit three-phase line is arranged on the points of a hexagon. If wires on opposite points are paralleled and the system is transposed, find the inductive reactance per mile per phase at 60 c. Each wire is 37-strand 500,000 cir mil copper cable.

5-17. If either adjacent wires or opposite wires of the hexagonal arrangement of Prob. 5-16 may be paralleled, specify the arrangement giving the greatest capacitance, phase to neutral.

REFERENCES

1. Maxwell, J. C., *A Treatise on Electricity and Magnetism*, Oxford University Press, New York, 1873.
2. *Electrical Transmission and Distribution Reference Book*, 3d ed., Westinghouse Electric Corp., E. Pittsburgh, Pa., 1944.
3. Woodruff, L. F., *Principles of Electric Power Transmission*, 2d ed., John Wiley & Sons, Inc., New York, 1938.
4. Ramo, S., and Whinnery, J. R., *Fields and Waves in Modern Radio*, 2d ed., John Wiley & Sons, Inc., New York, 1952.

Chapter 6

TRANSMISSION-LINE THEORY

Because of the distributed nature of the line constants, special methods must be developed for the analysis of long lines or circuits with distributed constants. This chapter will consider the transmission of electric energy over a line, with attention to the physical phenomena and the terminology associated with the distributed constant circuit viewpoint. In Chapters 9 and 10 similar phenomena will be discussed from the electromagnetic field viewpoint, leading to similar results.

6-1. A line of cascaded T sections

It is convenient to approach the study of the line through use of some of the network theory developed in Chapter 4. Therefore consider a number of identical and symmetrical T networks connected in series as in Fig. 6-1. If the final section is terminated in

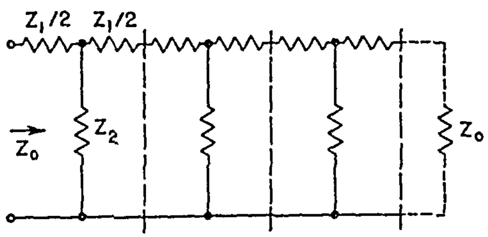


Fig. 6-1. A line of cascaded symmetrical sections of impedance.

its characteristic impedance, the input impedance at the first section is Z_0. Each section is terminated by the input impedance of the following section; and since the last section has a Z_0 termination, all sections are so terminated. The value of the characteristic impedance has been derived for a T section as

$$Z_{0T} = \sqrt{Z_1 Z_2 \left(1 + \frac{Z_1}{4Z_2}\right)} \tag{6-1}$$

If there are n such terminated sections and if the input and output currents are I_s and I_r, respectively, then

$$\frac{I_s}{I_r} = \epsilon^{n\gamma} \tag{6-2}$$

where γ is the propagation constant for one T section. As discussed in Chapter 4, γ is in general complex and is equal to $\alpha + j\beta$. From Eq. 4-37, ϵ^γ can be evaluated as

$$\epsilon^\gamma = \epsilon^{\alpha+j\beta} = 1 + \frac{Z_1}{2Z_2} + \sqrt{\frac{Z_1}{Z_2}\left(1 + \frac{Z_1}{4Z_2}\right)} \tag{6-3}$$

The above reasoning is not invalidated if the number of T sections, n, between source and load, is allowed to increase without limit. A uniform transmission line can be considered as made up of an infinity of T sections, each of infinitesimal size, each element including its proportionate share of the distributed inductance, capacitance, resistance, and leakance per unit of line length. Thus certain methods of network analysis, developed for lumped networks in Chapter 4, are fundamental to distributed networks as well.

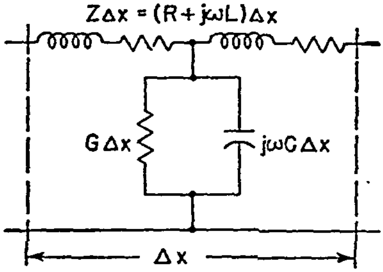


Fig. 6-2. The constants for an incremental length of transmission line.

The constants of an incremental length Δx of a line are indicated in Fig. 6-2. The series constants $Z = R + j\omega L$ are stated in ohms per unit length of line, and $Y = G + j\omega C$ are in mhos per unit length of line. Thus one T section, representing an incremental length Δx of the line, has a series impedance $Z\,\Delta x$ ohms and a shunt admittance $Y\,\Delta x$ mhos. The characteristic impedances of all the incremental sections are alike, since the sections are alike; and thus the characteristic impedance of any small section is that of the line as a whole. The characteristic impedance of a line of distributed constants can then be obtained from Eq. 6-1 for one section as

$$Z_0 = \sqrt{\frac{Z\,\Delta x}{Y\,\Delta x}\left(1 + \frac{Z\,\Delta x\,Y\,\Delta x}{4}\right)}$$

$$= \sqrt{\frac{Z}{Y}\left(1 + \frac{ZY\,\overline{\Delta x^2}}{4}\right)} \tag{6-4}$$

Allowing Δx to approach zero in the limit the value of Z_0 for the line of distributed constants is obtained as

$$Z_0 = \sqrt{\frac{Z}{Y}} \text{ ohms} \tag{6-5}$$

It should be noted that since Z and Y are defined in terms of unit length of line, the ratio Z/Y is independent of the length units chosen.

The radical in Eq. 6-3 may be expanded by the binomial theorem as

$$\sqrt{\frac{Z_1}{Z_2}\left(1 + \frac{Z_1}{4Z_2}\right)} = \sqrt{\frac{Z_1}{Z_2}}\left[1 + \frac{1}{2}\left(\frac{Z_1}{4Z_2}\right) - \frac{1}{8}\left(\frac{Z_1}{4Z_2}\right)^2 + \cdots\right]$$

so that Eq. 6-3 may be written

$$\epsilon^{\gamma} = 1 + \sqrt{\frac{Z_1}{Z_2}} + \frac{1}{2}\left(\sqrt{\frac{Z_1}{Z_2}}\right)^2 + \frac{1}{8}\left(\sqrt{\frac{Z_1}{Z_2}}\right)^3 - \frac{1}{128}\left(\sqrt{\frac{Z_1}{Z_2}}\right)^5 + \cdots$$

When applied to the incremental length of line Δx, then $Z_1 = Z\,\Delta x$, $Z_2 = 1/Y\,\Delta x$, and the propagation constant applying is $\gamma\,\Delta x$. Writing the above equation for $\epsilon^{\gamma \Delta x}$ gives

$$\epsilon^{\gamma\Delta x} = 1 + \sqrt{ZY}\,\Delta x + \frac{1}{2}(\sqrt{ZY})^2\,\overline{\Delta x}^2 + \frac{1}{8}(\sqrt{ZY})^3\,\overline{\Delta x}^3 - \frac{1}{128}(\sqrt{ZY})^5\,\overline{\Delta x}^5 + \cdots \tag{6-6}$$

If the series expansion is used for an exponential, $\epsilon^{\gamma\Delta x}$ can also be stated as

$$\epsilon^{\gamma\Delta x} = 1 + \gamma\,\Delta x + \frac{\gamma^2\,\overline{\Delta x}^2}{2!} + \frac{\gamma^3\,\overline{\Delta x}^3}{3!} + \cdots \tag{6-7}$$

Equating the two values for $\epsilon^{\gamma\Delta x}$ and canceling the unity terms,

$$\gamma\,\Delta x + \frac{\gamma^2\,\overline{\Delta x}^2}{2} + \frac{\gamma^3\,\overline{\Delta x}^3}{6} + \cdots = \sqrt{ZY}\,\Delta x + \frac{(\sqrt{ZY})^2\,\overline{\Delta x}^2}{2} + \frac{(\sqrt{ZY})^3\,\overline{\Delta x}^3}{8} + \cdots$$

Division by Δx leaves

$$\gamma + \frac{\gamma^2\,\Delta x}{2} + \frac{\gamma^3\,\overline{\Delta x}^2}{6} + \cdots = \sqrt{ZY} + \frac{(\sqrt{ZY})^2\,\Delta x}{2} + \frac{(\sqrt{ZY})^3\,\overline{\Delta x}^2}{8} + \cdots$$

Allowing Δx to approach zero in the limit it is seen that all terms but two vanish so that

$$\gamma = \sqrt{ZY} \tag{6-8}$$

This is the value for the line of distributed constants, since all elemental lengths are alike. Since Z and Y are in terms of unit length, γ is a value per unit length of line.

6-2. The transmission line—general solution

A circuit with distributed parameters requires a method of analysis somewhat different from that employed in circuits of lumped constants. Since a voltage drop occurs across each series increment of line, the voltage applied to each increment of shunt admittance is a variable and thus the shunted current is a variable along the line. Hence the line current around the loop is not a constant, as is assumed in lumped constant circuits, but varies from point to point along the line. Differential circuit equations that describe this action will be written for the steady state, from which general circuit equations can be obtained.

The notation used will be defined as follows:

R = series resistance, ohms per unit length of line (includes both wires)
L = series inductance, henrys per unit length of line
C = capacitance between conductors, farads per unit length of line
G = shunt leakage conductance between conductors, mhos per unit length of line
ωL = series reactance, ohms per unit length of line
$Z = R + j\omega L$ = series impedance, ohms per unit length of line
ωC = shunt susceptance, mhos per unit length of line
$Y = G + j\omega C$ = shunt admittance, mhos per unit length of line
s = distance to the point of observation, measured from the receiving end of the line
I = current in the line at any point
E = voltage between conductors at any point
l = length of line

Figure 6-3 illustrates a line that in the limit may be considered as made up of cascaded infinitesimal T sections, one of which is shown.

This elemental section is of length ds and carries a current I. The series line impedance being Z ohms per unit, the series impedance of the element is Zds ohms, and the voltage drop in the length ds is

$$dE = IZds$$

or

$$\frac{dE}{ds} = IZ \tag{6-9}$$

The shunt admittance per unit of length of line is Y mhos, so that

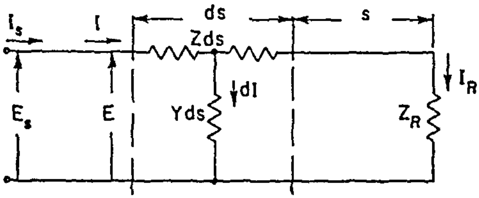


Fig. 6-3. A long line, with the elements of one of the infinitesimal sections shown.

the admittance of the element of line is Yds mhos. The current dI that flows across the line or from one conductor to the other is

$$dI = EYds$$

or

$$\frac{dI}{ds} = EY \tag{6-10}$$

Equations 6-9 and 6-10 may be differentiated with respect to s:

$$\frac{d^2E}{ds^2} = Z\frac{dI}{ds}, \qquad \frac{d^2I}{ds^2} = Y\frac{dE}{ds}$$

Then

$$\frac{d^2E}{ds^2} = ZYE \tag{6-11}$$

$$\frac{d^2I}{ds^2} = ZYI \tag{6-12}$$

These are the differential equations of the transmission line, fundamental to circuits of distributed constants. If $E = E_0\epsilon^{j\omega t}$ and $I = I_0\epsilon^{j\omega t}$ they may also be shown to be forms of the wave equation which will be discussed in Chapter 9.

Solution by conventional methods follows directly. In terms of the operator m, Eq. 6-11 becomes

$$(m^2 - ZY)E = 0$$

$$m = \pm\sqrt{ZY} \tag{6-13}$$

This result indicates two solutions, one for the plus sign and the other for the minus sign before the radical. The solutions of the differential equations then are

$$E = A\epsilon^{\sqrt{ZY}s} + B\epsilon^{-\sqrt{ZY}s} \tag{6-14}$$

$$I = C\epsilon^{\sqrt{ZY}s} + D\epsilon^{-\sqrt{ZY}s} \tag{6-15}$$

where A, B, C, and D are arbitrary constants of integration.

Since distance is measured from the receiving end of the line, it is possible to assign conditions such that at

$$s = 0, \qquad I = I_R, \qquad E = E_R$$

Then Eqs. 6-14 and 6-15 become

$$\left.\begin{aligned} E_R &= A + B \\ I_R &= C + D \end{aligned}\right\} \tag{6-16}$$

A second set of boundary conditions is not available, but the same set may be used over again if a new set of equations are formed by differentiation of Eqs. 6-14 and 6-15. Thus

$$\frac{dE}{ds} = A\sqrt{ZY}\,\epsilon^{\sqrt{ZY}s} - B\sqrt{ZY}\,\epsilon^{-\sqrt{ZY}s}$$

From Eq. 6-9, this becomes

$$IZ = A\sqrt{ZY}\,\epsilon^{\sqrt{ZY}s} - B\sqrt{ZY}\,\epsilon^{-\sqrt{ZY}s}$$

$$I = A\sqrt{\frac{Y}{Z}}\,\epsilon^{\sqrt{ZY}s} - B\sqrt{\frac{Y}{Z}}\,\epsilon^{-\sqrt{ZY}s} \tag{6-17}$$

In a similar manner,

$$\frac{dI}{ds} = C\sqrt{ZY}\,\epsilon^{\sqrt{ZY}s} - D\sqrt{ZY}\,\epsilon^{-\sqrt{ZY}s}$$

$$E = C\sqrt{\frac{Z}{Y}}\,\epsilon^{\sqrt{ZY}s} - D\sqrt{\frac{Z}{Y}}\,\epsilon^{-\sqrt{ZY}s} \tag{6-18}$$

At $s = 0$, Eqs. 6-17 and 6-18 become

$$I_R = A\sqrt{\frac{Y}{Z}} - B\sqrt{\frac{Y}{Z}} \tag{6-19}$$

$$E_R = C\sqrt{\frac{Z}{Y}} - D\sqrt{\frac{Z}{Y}} \tag{6-20}$$

Simultaneous solution of Eqs. 6-16 with Eqs. 6-19 and 6-20, along with the fact that $E_R = I_R Z_R$ and that $\sqrt{Z/Y}$ has been identified as the Z_0 of the line, leads to solutions for the constants of the above equations as

$$A = \frac{E_R}{2} + \frac{I_R}{2}\sqrt{\frac{Z}{Y}} = \frac{E_R}{2}\left(1 + \frac{Z_0}{Z_R}\right)$$

$$B = \frac{E_R}{2} - \frac{I_R}{2}\sqrt{\frac{Z}{Y}} = \frac{E_R}{2}\left(1 - \frac{Z_0}{Z_R}\right)$$

$$C = \frac{I_R}{2} + \frac{E_R}{2}\sqrt{\frac{Y}{Z}} = \frac{I_R}{2}\left(1 + \frac{Z_R}{Z_0}\right)$$

$$D = \frac{I_R}{2} - \frac{E_R}{2}\sqrt{\frac{Y}{Z}} = \frac{I_R}{2}\left(1 - \frac{Z_R}{Z_0}\right)$$

The solution of the differential equations of the transmission line may then be written

$$E = \frac{E_R}{2}\left[\left(1 + \frac{Z_0}{Z_R}\right)\epsilon^{\sqrt{ZY}s} + \left(1 - \frac{Z_0}{Z_R}\right)\epsilon^{-\sqrt{ZY}s}\right] \qquad (6\text{-}21)$$

$$I = \frac{I_R}{2}\left[\left(1 + \frac{Z_R}{Z_0}\right)\epsilon^{\sqrt{ZY}s} + \left(1 - \frac{Z_R}{Z_0}\right)\epsilon^{-\sqrt{ZY}s}\right] \qquad (6\text{-}22)$$

Then

$$E = \frac{E_R(Z_R + Z_0)}{2Z_R}\left[\epsilon^{\sqrt{ZY}s} + \left(\frac{Z_R - Z_0}{Z_R + Z_0}\right)\epsilon^{-\sqrt{ZY}s}\right] \qquad (6\text{-}23)$$

$$I = \frac{I_R(Z_R + Z_0)}{2Z_0}\left[\epsilon^{\sqrt{ZY}s} - \left(\frac{Z_R - Z_0}{Z_R + Z_0}\right)\epsilon^{-\sqrt{ZY}s}\right] \qquad (6\text{-}24)$$

These are a final and very useful form of the equations for voltage and current at any point on a transmission line, and are solutions to the wave equation of Chapter 9.

Equations 6-21 and 6-22 may also be arranged as

$$E = E_R\left(\frac{\epsilon^{\sqrt{ZY}s} + \epsilon^{-\sqrt{ZY}s}}{2}\right) + I_R Z_0\left(\frac{\epsilon^{\sqrt{ZY}s} - \epsilon^{-\sqrt{ZY}s}}{2}\right)$$

$$I = I_R\left(\frac{\epsilon^{\sqrt{ZY}s} + \epsilon^{-\sqrt{ZY}s}}{2}\right) + \frac{E_R}{Z_0}\left(\frac{\epsilon^{\sqrt{ZY}s} - \epsilon^{-\sqrt{ZY}s}}{2}\right)$$

and these equations can be recognized as

$$E = E_R \cosh \sqrt{ZY}\, s + I_R Z_0 \sinh \sqrt{ZY}\, s \tag{6-25}$$

$$I = I_R \cosh \sqrt{ZY}\, s + \frac{E_R}{Z_0} \sinh \sqrt{ZY}\, s \tag{6-26}$$

Equations 6-25 and 6-26 constitute a second very useful form for the voltage and current values at any point on a transmission line. Although these equations are extensively used, it is believed that the exponential forms of Eqs. 6-23 and 6-24 lead to a clearer physical picture of the phenomena occuring on a line. They also lead to easier calculation with complex values of $\sqrt{ZY}$.

6-3. Physical significance of the equations; the infinite line

Equation 6-26 may be written for the sending-end current I_S of a line of length l as

$$I_S = I_R \left(\cosh \sqrt{ZY}\, l + \frac{Z_R}{Z_0} \sinh \sqrt{ZY}\, l \right)$$

If the line is terminated in $Z_R = Z_0$, then

$$I_S = I_R (\cosh \sqrt{ZY}\, l + \sinh \sqrt{ZY}\, l)$$

from which

$$\frac{I_S}{I_R} = \epsilon^{\sqrt{ZY}\, l} = \epsilon^{\gamma l} \tag{6-27}$$

by reason of the determination that $\sqrt{ZY} = \gamma$. The result of Eq. 6-27 is simply a restatement, for the line, of the basic relation assumed between input and output currents in Chapter 4. The *propagation constant* γ is defined *per unit length* of line. As before, it is complex from the nature of its equivalence to $\sqrt{ZY}$, or $\gamma = \alpha + j\beta$.

Division of Eq. 6-25 by 6-26 leads to an expression for the input impedance of the line of length l as

$$Z_S = \frac{E_S}{I_S} = Z_0 \left(\frac{Z_R \cosh \gamma l + Z_0 \sinh \gamma l}{Z_0 \cosh \gamma l + Z_R \sinh \gamma l} \right) \tag{6-28}$$

This equation may be recognized as identical with Eq. 4-39 derived from network theory for a lumped constant circuit. In different form, this result may be obtained by division of Eq. 6-23 by 6-24 and by use of the fact that $E_R/I_R = Z_R$. Then

$$Z_S = \frac{E_S}{I_S} = Z_0 \left[\frac{\epsilon^{\gamma l} + \left(\frac{Z_R - Z_0}{Z_R + Z_0}\right)\epsilon^{-\gamma l}}{\epsilon^{\gamma l} - \left(\frac{Z_R - Z_0}{Z_R + Z_0}\right)\epsilon^{-\gamma l}} \right] \tag{6-29}$$

As a particular case, it is of interest to find the value of Z_S, the sending-end input impedance, when the line is terminated in its characteristic impedance; that is, when $Z_R = Z_0$. Equation 6-29 then reduces to

$$Z_S = Z_0 \tag{6-30}$$

which conforms to the definition of characteristic impedance for lumped networks and establishes the validity of the operations in Section 6-1, in which Z_0 was identified as $\sqrt{Z/Y}$ for a T section of elemental magnitude in a distributed constant circuit.

Unity has been established between the lumped-constant and distributed-constant circuits in that the description of circuit performance through Z_0 and γ has been found to hold for both cases and thus is a truly general method of description of circuit performance.

Even though a line of infinite length seems hypothetical, actually much may be learned from a study of such a line. The input impedance of an infinite line may be found by letting l approach infinity in Eq. 6-28 or 6-29. The result is

$$Z_S = Z_0 \tag{6-31}$$

Thus, by comparison of the results shown in Eqs. 6-30 and 6-31, *a line of finite length, terminated in a load equivalent to its characteristic impedance, appears to the sending-end generator as an infinite line.* A finite line terminated in Z_0 and an infinite line are indistinguishable by measurements at the source. Since the use of

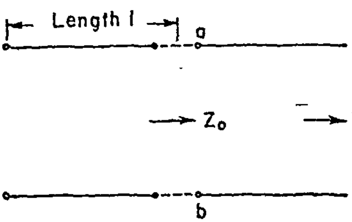


Fig. 6-4. A length l taken from an infinite line.

terminations equal to Z_0 is quite common, the Z_0-terminated finite line may be studied by analysis of the infinite line.

This result may be reasoned qualitatively by consideration of the line of infinite length of Fig. 6-4. This line may have terminals a,b placed at a finite distance l from the sending end. The remainder of the line is still infinite in length, so that the input impedance at terminals a,b is Z_0. Thus the length l of the line represents a finite length of line terminated in its characteristic impedance.

Restating Eqs. 6-23 and 6-24,

$$E = \frac{E_R(Z_R + Z_0)}{2Z_R}\left[\epsilon^{\gamma s} + \left(\frac{Z_R - Z_0}{Z_R + Z_0}\right)\epsilon^{-\gamma s}\right]$$

$$I = \frac{I_R(Z_R + Z_0)}{2Z_0}\left[\epsilon^{\gamma s} - \left(\frac{Z_R - Z_0}{Z_R + Z_0}\right)\epsilon^{-\gamma s}\right].$$

For a portion of an infinite line, or a *finite line terminated in its characteristic impedance,*

$$Z_R = Z_0$$

so that these equations reduce to

$$E = E_R\epsilon^{\gamma s} \tag{6-32}$$

$$I = I_R\epsilon^{\gamma s} \tag{6-33}$$

It is apparent that the voltage and current values change with distance s from the receiving end because of the factor $\epsilon^{\gamma s}$.

These expressions may be readily written for distance s measured from the sending end, with E_s and I_s the voltage and current values at that point, giving

$$E_R = E_s\epsilon^{-\gamma s}, \qquad I_R = I_s\epsilon^{-\gamma s}$$

If the terminals a,b be considered as any point on the infinite line, then E and I at any point are expressed in terms of E_s and I_s as

$$E = E_s\epsilon^{-\gamma s} \tag{6-34}$$

$$I = I_s\epsilon^{-\gamma s} \tag{6-35}$$

Since $\gamma = \alpha + j\beta$, then

$$E = E_s\epsilon^{-\alpha s}\epsilon^{-j\beta s} \tag{6-36}$$

$$I = I_s\epsilon^{-\alpha s}\epsilon^{-j\beta s} \tag{6-37}$$

Since the sending end values will be functions of time as $E_s = E_{s0}\epsilon^{j\omega t}$ and $I_s = I_{s0}\epsilon^{j\omega t}$, then it is seen that Eqs. 6-36 and 6-37 are functions of both distance and time. This is a property of any solution to the wave equation.

It is to be seen that as one measures along this line from the

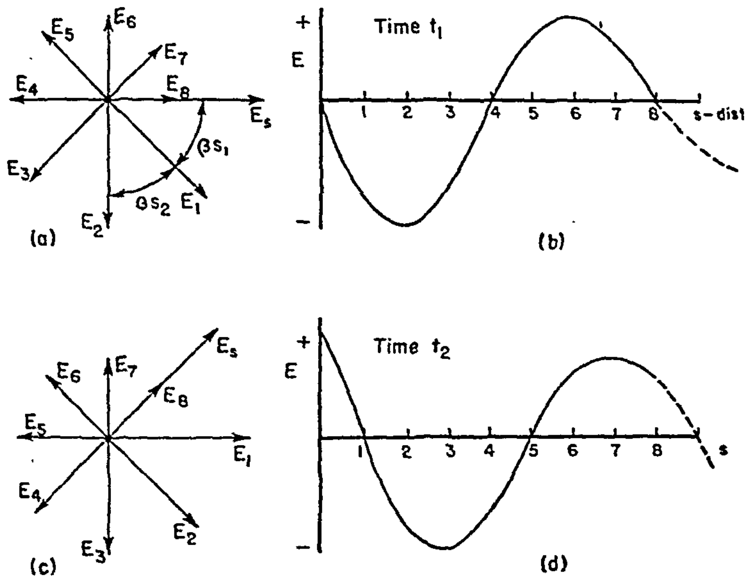


Fig. 6-5. (a) Voltage phasors at $\frac{1}{8}$-wavelength points along a line at a particular time instant; (b) a plot of the phasors of (a), showing voltage on the line as a function of distance; (c) voltage phasors one-eighth of a cycle later than (a); (d) plot of voltage along the line for the time instant of (c) showing the movement of the wave along the line.

sending end, the voltage and current become progressively smaller in magnitude because of the factor $\epsilon^{-\alpha s}$. The logic of calling α the *attenuation constant* is apparent. Also, the current and voltage phases lag progressively more and more as s increases, because of the increasing angle inherent in $\epsilon^{-j\beta s} = \angle -\beta s$.

It is essential that the student understand the use of $-s$ or s in measuring in the direction of, or counter to the direction of, the incident energy. Free use will be made of both forms as needed.

If uniformly spaced points are selected on the line or if uniformly increasing s values, such as s_1, s_2, s_3, . . . , are chosen, then voltage phasors may be drawn for each point as in (a), Fig. 6-5. These phasors may be considered as representing maximum instantaneous voltage values, and are seen to vary in magnitude as $\epsilon^{-\alpha s}$ and in phase by uniform angle increments. It may then be assumed that all the phasors are rotating in the counterclockwise direction with an angular velocity ω equal to that of the generating source at the sending end. At a given time t_1, the rotation may be stopped and the instantaneous values of voltage plotted as a function of distance, as in (b), Fig. 6-5. This plot shows a portion of an infinite line, with the instantaneous voltage conditions existing at each point along the line. Such an oscillating and attenuating voltage condition exists over the whole length of the line.

At (c), Fig. 6-5, the rotating phasors have been stopped at a time t_2, *which is one-eighth of a cycle later than time* t_1. The instantaneous values of voltage are again plotted as a function of distance in (d). Comparison of sketches (d) and (b) shows that the wave of (b) has moved to the right and become that of (d). This movement discloses the *existence of a voltage wave traveling down the line from the generator.* The student may verify this action by plotting the waves on the line for intermediate or later instants of time.

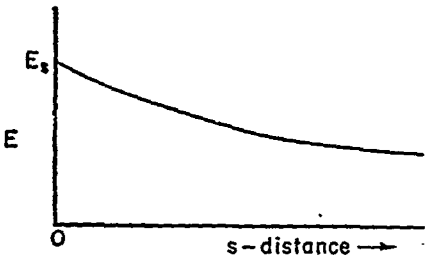


Fig. 6-6. Voltage along an infinite line as measured by an effective-reading meter.

Thus propagation takes place along the line in a wave motion, the amplitude of voltage or current being constantly reduced due to attenuation $\epsilon^{-\alpha s}$, the phase of the voltage or current constantly changing because of the phase factor $\epsilon^{-j\beta s}$. The phasors may be thought of as rotating in space, in which case the locus of the ends of the phasors appears as a tapering corkscrew. The projection of this corkscrew on a plane then constitutes the wave pattern at any instant.

This discussion has been in terms of instantaneous values. If instruments reading effective values are connected along the line, they will not show the phase angles, and thus their readings along the line would correspond to the curves for

$$E = E_s\epsilon^{-\alpha s}, \qquad I = I_s\epsilon^{-\alpha s}$$

as shown in Fig. 6-6.

Although the analysis has been given here in terms of current and voltage, it should be remembered that actually energy is being propagated along the line, its transfer taking place in the electric and magnetic fields in the region surrounding the line. The waves of current and voltage are merely convenient means of observation of the fields present. A discussion of the line from the field viewpoint is given in Chapter 12.

6-4. Wavelength; velocity of propagation

The distance the wave travels along the line while the phase angle is changing through 2π radians is called a *wavelength*. In Fig. 6-5(b), the distance from the sending end to point 8 is thus one wavelength. From the definition above, if the wavelength is represented by the symbol λ, then at a distance such that $s = \lambda$,

$$\beta\lambda = 2\pi$$

and

$$\lambda = \frac{2\pi}{\beta} \tag{6-38}$$

Since the change of 2π in phase angle represents one cycle in time and occurs in a distance of one wavelength, then

$$x = vt$$

and

$$\lambda = \frac{v}{f} \tag{6-39}$$

From this and Eq. 6-38, the velocity can be expressed in terms of the line constants as

$$v = \lambda f = \frac{2\pi f}{\beta}$$

$$v = \frac{\omega}{\beta} \tag{6-40}$$

This is the *velocity of propagation* along the line, based on observations of the change in phase along the line. It is measured in miles per second if β is in radians per mile, or in meters per second if β is in radians per meter.

If it be remembered that

$$Z = R + j\omega L$$

$$Y = G + j\omega C$$

then

$$\gamma = \alpha + j\beta = \sqrt{ZY}$$

$$= \sqrt{RG - \omega^2 LC + j\omega(LG + CR)} \tag{6-41}$$

Squaring both sides,

$$\alpha^2 + j2\alpha\beta - \beta^2 = RG - \omega^2 LC + j\omega(LG + CR)$$

Equating the reals and solving for α^2 gives

$$\alpha^2 = \beta^2 + RG - \omega^2 LC \tag{6-42}$$

Equating the imaginaries and squaring yields

$$4\alpha^2\beta^2 = \omega^2(LG + CR)^2$$

after which substitution of Eq. 6-42 gives

$$\beta^4 + \beta^2(RG - \omega^2 LC) - \frac{\omega^2}{4}(LG + CR)^2 = 0$$

A solution for β, neglecting the negative values, follows as

$$\beta = \sqrt{\frac{\omega^2 LC - RG + \sqrt{(RG - \omega^2 LC)^2 + \omega^2(LG + CR)^2}}{2}} \tag{6-43}$$

and use of Eq. 6-42 leads to a value for

$$\alpha = \sqrt{\frac{RG - \omega^2 LC + \sqrt{(RG - \omega^2 LC)^2 + \omega^2(LG + CR)^2}}{2}} \tag{6-44}$$

In a perfect line $R = 0$ and $G = 0$. Equation 6-43 then would be

$$\beta = \omega\sqrt{LC} \tag{6-45}$$

and the velocity of propagation for such an ideal line is given by

$$v = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} \text{ m/sec} \tag{6-46}$$

thus showing that the line parameter values fix the velocity of propagation. Using Eqs. 5-14 and 5-30 for L and C values for an open-wire line, the LC product becomes

$$LC = \frac{\pi\epsilon}{\ln (d/a)}\left(\frac{\mu}{4\pi} + \frac{\mu_v}{\pi}\ln\frac{d}{a}\right)$$

For a line of nonmagnetic material with air spacing

$$LC = \frac{\mu_r\epsilon_r}{4\ln (d/a)} + \mu_r\epsilon_r \tag{6-47}$$

Then

$$v = \frac{1}{\sqrt{\mu_r\epsilon_r\left[\dfrac{1}{4\ln (d/a)} + 1\right]}}\ \text{m/sec} \tag{6-48}$$

It should be noted that the first term on the right of Eq. 6-47 and the first term in the bracket under the radical of Eq. 6-48 are present because of the internal inductance of the conductors. If this internal inductance is reduced, as by skin effect, the velocity increases and reaches the limiting condition of

$$v = \frac{1}{\sqrt{\mu_r\epsilon_r}}\ \text{m/sec}$$

which in space becomes

$$3 \times 10^8 = c\ \text{m/sec} \tag{6-49}$$

which is identified as the velocity of light in space. The velocity of propagation in an actual line is slowed below this value by the effect of internal inductance, resistance, and leakage of the line.

The above substantiates the method of calculation of C presented in Section 5-16, and shows the constant in Eq. 5-78 to be equal to c^2.

6-5. An example

The relative magnitudes of quantities to be encountered in line calculations are indicated in the following example:

A generator of 1.0 volt, 1000 cycles, supplies power to a 100-mile open-wire line terminated in Z_0 and having the following parameters:

$$R = 10.4 \text{ ohms per mile}$$
$$L = 0.00367 \text{ henry per mile}$$
$$G = 0.8 \times 10^{-6} \text{ mho per mile}$$
$$C = 0.00835\ \mu\text{f per mile}$$

The line constants then are

$$Z = R + j\omega L = 10.4 + j23.0 = 25.2\underline{/66°}$$

$$Y = G + j\omega C = (0.8 + j52.5)10^{-6} = 52.6 \times 10^{-6}\underline{/90°}$$

$$Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{25.2\underline{/66°}}{52.6 \times 10^{-6}\underline{/90°}}} = 692\underline{/-12°} \text{ ohms}$$

$$\gamma = \sqrt{ZY} = \sqrt{25.2\underline{/66°} \times 52.6 \times 10^{-6}\underline{/90°}} = 0.0363\underline{/78°}$$

$$\alpha = 0.0363 \cos 78° = 0.00755 \text{ neper per mile}$$

$$\beta = 0.0363 \sin 78° = 0.0355 \text{ radian per mile}$$

$$v = \frac{\omega}{\beta} = \frac{6280}{0.0355} = 177{,}000 \text{ miles per second}$$

$$\lambda = \frac{2\pi}{\beta} = 177 \text{ miles}$$

Since the line is terminated in Z_0, then $Z_s = Z_0$, so that

$$I_s = \frac{E_s}{Z_0} = \frac{1.0}{692\underline{/-12°}} = 0.00145\underline{/12°} \text{ amp}$$

$$\frac{I_R}{I_s} = \epsilon^{-\gamma l} = \epsilon^{-\alpha l}\epsilon^{-j\beta l} = \epsilon^{-0.755}\epsilon^{-j3.55}$$

But $\epsilon^{-j3.55}$ is equivalent to an angle of -3.55 radians, or -203.8 deg from the tables in the rear of the book. Then

$$\begin{aligned} I_R &= I_s\,\epsilon^{-0.755}\underline{/-203.8°} \\ &= 0.00145\underline{/12°} \times 0.472\underline{/-203.8°} \\ &= 0.000685\underline{/-191.8°} \text{ amp } (E_s \text{ reference}) \end{aligned}$$

The received voltage is

$$\begin{aligned} E_R &= I_R Z_0 \\ &= 0.000685\underline{/-191.8} \times 692\underline{/-12°} \\ &= 0.474\underline{/-203.8°} \text{ volts } (E_s \text{ reference}) \end{aligned}$$

The received power is

$$P_R = E_R I_R \cos\theta = 318 \times 10^{-6} \text{ watts}$$

6-6. Wave-form distortion

The value of the attenuation constant α has been determined in Section 6-4 as

$$\alpha = \sqrt{\frac{RG - \omega^2 LC + \sqrt{(RG - \omega^2 LC)^2 + \omega^2(LG + CR)^2}}{2}}$$

In general, α is a function of frequency. All frequencies transmitted on a line will then not be attenuated equally. A complex applied voltage, such as a voice voltage containing many frequencies, will not have all frequencies transmitted with equal attenuation, and the received wave form will not be identical with the input wave form at the sending end. This variation is known as *frequency distortion.*

The phase constant β was shown in Section 6-4 to be

$$\beta = \sqrt{\frac{\omega^2 LC - RG + \sqrt{(RG - \omega^2 LC)^2 + \omega^2(LG + CR)^2}}{2}}$$

and can be seen to be a complicated function of frequency, in general. Since the velocity of propagation has been stated as

$$v = \frac{\omega}{\beta}$$

it is apparent that ω and β do not both involve frequency in the same manner and that the velocity of propagation will in general be some function of frequency. All frequencies applied to a transmission line will not have the same time of transmission, some frequencies being delayed more than others. For an applied voice-voltage wave the received wave form will not be identical with the input wave form at the sending end, since some components will be delayed more than those of other frequencies. This phenomenon is known as *delay* or *phase distortion.*

Frequency distortion is reduced in the transmission of high-quality radio broadcast programs over wire lines by use of *equalizers* at the line terminals. These circuits are networks whose frequency and phase characteristics are adjusted to be inverse to those of the lines, resulting in an over-all uniform frequency response over the desired frequency band.

Delay distortion is of relatively minor importance to voice and music transmission because of the characteristics of the ear. It can be very serious in circuits intended for picture transmission, and applications of the coaxial cable have been made to overcome the difficulty. In such cables the internal inductance is low at high frequencies because of skin effect, the resistance is small because of the large conductors, and capacitance and leakance are small because of the use of air dielectric with a minimum of spacers. The velocity of propagation is raised and made more nearly equal for all frequencies.

6-7. The distortionless line

If a line is to have neither frequency nor delay distortion, then α and the velocity of propagation cannot be functions of frequency. In view of the fact that

$$v = \frac{\omega}{\beta}$$

then β must be a direct function of frequency.

Consideration of Eq. 6-43,

$$\beta = \sqrt{\frac{\omega^2 LC - RG + \sqrt{(RG - \omega^2 LC)^2 + \omega^2 (LG + CR)^2}}{2}}$$

shows that if the term under the second radical be reduced to equal

$$(RG + \omega^2 LC)^2$$

then the required condition on β is obtained. Expanding the terms under the internal radical and forcing the equality gives

$$R^2G^2 - 2\omega^2 LCRG + \omega^4 L^2C^2 + \omega^2 L^2G^2 + 2\omega^2 LCRG + \omega^2 C^2R^2 = (RG + \omega^2 LC)^2$$

This reduces to

$$\omega^2 L^2G^2 - 2\omega^2 LCRG + \omega^2 C^2R^2 = 0$$

$$(LG - CR)^2 = 0$$

Therefore the condition that will make β a direct function of frequency is

$$LG = CR \tag{6-50}$$

A hypothetical line might be built to fulfill this condition. The line would then have a value of β obtained by use of Eq. 6-50 in

6-43 as

$$\beta = \omega \sqrt{LC}$$

The velocity of propagation is then

$$v = \frac{1}{\sqrt{LC}}$$

which is the same for all frequencies, thus eliminating delay distortion.

Equation 6-44 for

$$\alpha = \sqrt{\frac{RG - \omega^2LC + \sqrt{(RG - \omega^2LC)^2 + \omega^2(LG + CR)^2}}{2}}$$

may be made independent of frequency if the term under the internal radical is forced to reduce to

$$(RG + \omega^2LC)^2$$

Analysis shows that the condition of Eq. 6-50, $LG = CR$, will produce the desired result, so that it is possible to make α and the velocity independent of frequency simultaneously. Applying the condition of Eq. 6-50 to the expression for α gives

$$\alpha = \sqrt{RG}$$

which is independent of frequency, thus eliminating frequency distortion on the line.

Unfortunately, such a hypothetical line is not practical with distributed parameters, but the analysis points the way to the solution of Section 6-9. To achieve the condition

$$LG = CR$$

or

$$\frac{L}{C} = \frac{R}{G} \tag{6-51}$$

requires a very large value of L, since G is small. If G is intentionally increased, α and the attenuation are increased, resulting in poor line efficiency. To reduce R raises the size and cost of the conductors above economic limits, so that the hypothetical results cannot be achieved.

6-8. The telephone cable

In the ordinary telephone cable the wires are insulated with paper and twisted in pairs. This construction results in negligible values of inductance and conductance so that reasonable assumptions in the audio range of frequencies are that

$$Z = R \tag{6-52}$$

$$Y = j\omega C \tag{6-53}$$

Equation 6-41 stated that

$$\gamma = \sqrt{RG - \omega^2 LC + j\omega(LG + CR)}$$

With $L = G = 0$, this equation becomes

$$\gamma = \sqrt{j\omega CR} = \sqrt{\frac{j2\omega CR}{2}}$$

$$\gamma = \alpha + j\beta = (1 + j1)\sqrt{\frac{\omega CR}{2}}$$

Therefore

$$\alpha = \sqrt{\frac{\omega CR}{2}} \tag{6-54}$$

$$\beta = \sqrt{\frac{\omega CR}{2}} \tag{6-55}$$

Hence the velocity of propagation is

$$v = \frac{\omega}{\beta} = \sqrt{\frac{2\omega}{CR}} \tag{6-56}$$

It should be observed that both α and the velocity are functions of frequency, such that the higher frequencies are attenuated more and travel faster than the lower frequencies. Very considerable frequency and delay distortion is the result on telephone cable.

6-9. Inductance loading of telephone cables

The analysis of Section 6-7 concerning a distortionless line with distributed parameters suggests a remedy for the severe frequency and delay distortion experienced on long cables. It was indicated that it was necessary to increase the L/C ratio to achieve distortionless conditions. Heaviside suggested that the inductance be increased, and Pupin developed the theory that made possible this increase in the inductance by *lumped* inductors spaced at intervals along the line. This use of inductance is called *loading* the line. In some submarine cables, distributed or uniform loading is obtained by winding the cable with a high-permeability steel tape such as permalloy. This method is employed because of the practical difficulties of designing lumped loading coils for such underwater circuits.

For simplicity, consider first a uniformly loaded cable circuit for which it may be assumed that $G = 0$ and for which L has been increased so that ωL is large with respect to R. Then

$$Z = R + j\omega L$$
$$Y = j\omega C$$

Since,

$$Z = \sqrt{R^2 + \omega^2 L^2} \,\underline{/ \frac{\pi}{2} - \tan^{-1}\frac{R}{\omega L}} \tag{6-57}$$

then $\gamma = \sqrt{ZY}$

$$= \sqrt{\sqrt{R^2 + \omega^2 L^2} \,\underline{/ \frac{\pi}{2} - \tan^{-1}\frac{R}{\omega L}} \times \omega C \,\underline{/ \frac{\pi}{2}}}$$

$$= \omega \sqrt{LC} \sqrt[4]{1 + \frac{R^2}{\omega^2 L^2}} \,\underline{/ \frac{\pi}{2} - \frac{1}{2}\tan^{-1}\frac{R}{\omega L}} \tag{6-58}$$

In view of the fact that R is small with respect to ωL, the term $R^2/\omega^2 L^2$ may be dropped, and γ becomes

$$\gamma = \omega \sqrt{LC} \,\underline{/ \frac{\pi}{2} - \frac{1}{2}\tan^{-1}\frac{R}{\omega L}} \tag{6-59}$$

If $\theta = \frac{\pi}{2} - \frac{1}{2}\tan^{-1}\frac{R}{\omega L}$,

$$\cos\theta = \cos\left(\frac{\pi}{2} - \frac{1}{2}\tan^{-1}\frac{R}{\omega L}\right) = \sin\left(\frac{1}{2}\tan^{-1}\frac{R}{\omega L}\right)$$

For a small angle,

$$\sin\theta = \tan\theta = \theta, \qquad \text{so that} \qquad \cos\theta = \frac{R}{2\omega L}$$

Likewise, $$\sin\theta = \sin\left(\frac{\pi}{2} - \frac{1}{2}\tan^{-1}\frac{R}{\omega L}\right) \doteq 1$$

Equation 6-59 may then be written

$$\gamma = \omega\sqrt{LC}\,(\cos\theta + j\sin\theta) = \omega\sqrt{LC}\left(\frac{R}{2\omega L} + j\right)$$

Therefore, for the uniformly loaded cable,

$$\alpha = \frac{R}{2}\sqrt{\frac{C}{L}} \tag{6-60}$$

$$\beta = \omega\sqrt{LC} \tag{6-61}$$

and $$v = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}$$

It is readily observed that, under the assumptions of $G = 0$ and ωL large with respect to R, the attenuation and velocity are both independent of frequency and the loaded cable will be distortionless. The expression for α shows that the attenuation may be reduced by increasing L, provided that R is not also increased too greatly.

Continuous or uniform loading is expensive and achieves only a small increase in L per unit length. Lumped loading is ordinarily preferred as a means of transmission improvement for cables. The improvement obtainable on open-wire lines is usually not sufficient to justify the extra cost of the loading inductors.

An analysis for the performance of a line loaded at uniform intervals can be obtained by considering a symmetrical section of line from the center of one loading coil to the center of the next, where the loading coil impedance is Z_c. The section of line may be replaced with an equivalent T section (see Section 6-17) having symmetrical series arms. Adopting the notation of filter circuits, one of these series arms is called $Z_1/2$ and is

$$\frac{Z_1}{2} = Z_0 \tanh\frac{N\gamma}{2}$$

where N is the number of miles between loading coils and γ is the propagation constant per mile. Upon including half a loading coil, the equivalent series arm of the loaded section becomes

$$\frac{Z_1'}{2} = \frac{Z_c}{2} + Z_0\tanh\frac{N\gamma}{2}$$

The shunt Z_2 arm of the equivalent T section is

$$Z_2 = \frac{Z_0}{\sinh N\gamma}$$

An equation relating γ and the circuit elements of a T section was derived as Eq. 4-33, which may be applied to the loaded T section as

$$\begin{aligned}\cosh N\gamma' &= 1 + \frac{Z_1'}{2Z_2} \\ &= 1 + \frac{Z_c/2 + Z_0 \tanh (N\gamma/2)}{Z_0/\sinh N\gamma} \qquad (6\text{-}62)\end{aligned}$$

By use of exponentials it can be shown that

$$\tanh \frac{N\gamma}{2} = \frac{\cosh N\gamma - 1}{\sinh N\gamma}$$

so that Eq. 6-62 reduces to

$$\cosh N\gamma' = \frac{Z_c}{2Z_0} \sinh N\gamma + \cosh N\gamma \qquad (6\text{-}63)$$

This expression is known as *Campbell's equation* and permits the determination of a value for γ' of a line section consisting partially of lumped and partially of distributed elements. Campbell's equation makes possible the calculation of the effects of loading coils in reducing attenuation and distortion on lines.

For a cable, Z_2 of Fig. 6-7 is essentially capacitive and the cable capacitance plus lumped inductances appear similar to the circuit of the low-pass filter. It is found that for frequencies below cutoff,

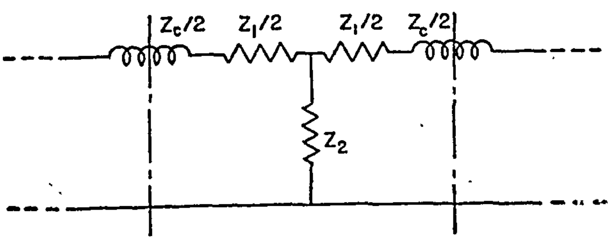


Fig. 6-7. Equivalent T section for part of a line between two lumped loading coils of impedance, Z_c.

given by

$$f_c = \frac{1}{\pi \sqrt{LC}}$$

the attenuation is reduced as expected, but above cutoff the attenuation rises as a result of filter action. This cutoff frequency forms a definite upper limit to successful transmission over cables. It can be raised by reducing L but this expedient allows the attenuation to rise. The cutoff frequency can also be raised by spacing the coils closer together, thus reducing C and more closely approximating the distributed-constant line, but the cost increases rapidly.

In practice, a truly distortionless line is not obtained by loading, because R and L are to some extent functions of frequency. Eddy-current losses in the loading inductors aggravate this condition. However, a major improvement is obtained in the loaded cable over the unloaded cable for a reasonable frequency range.

6-10. Reflection on a line not terminated in Z_0

Returning to Eqs. 6-23 and 6-24 for the voltages and currents on the line, with *s measured as positive from the receiving end,*

$$E = \frac{E_R(Z_R + Z_0)}{2Z_R}\left[\epsilon^{\gamma s} + \left(\frac{Z_R - Z_0}{Z_R + Z_0}\right)\epsilon^{-\gamma s}\right] \tag{6-23}$$

$$I = \frac{I_R(Z_R + Z_0)}{2Z_0}\left[\epsilon^{\gamma s} - \left(\frac{Z_R - Z_0}{Z_R + Z_0}\right)\epsilon^{-\gamma s}\right] \tag{6-24}$$

it may be observed that for the most general case in which Z_R is not equal to Z_0, each equation consists of two terms, one of which varies exponentially with $+s$, the other with $-s$. It has been shown in Section 6-3 that a wave travels from the sending end to the receiving end of the line, decreasing in amplitude as it approaches the receiving end. In a direction back along this wave from the receiving end, the amplitude of the wave would increase as s increases, so that the wave that travels from the sending end to the receiving end can be identified as the component varying with $\epsilon^{\gamma s}$. This wave of voltage or current is known as the *incident wave.*

Hence the second term, varying with $\epsilon^{-\gamma s}$, must represent a wave of voltage or current progressing from the receiving end toward the sending end, and decreasing in amplitude with increased distance from the load. Such a wave of voltage or current is called the *reflected wave.*

The situation may be more clearly seen in Fig. 6-8. The incident voltage values, with Z_R chosen as an open circuit for convenience,

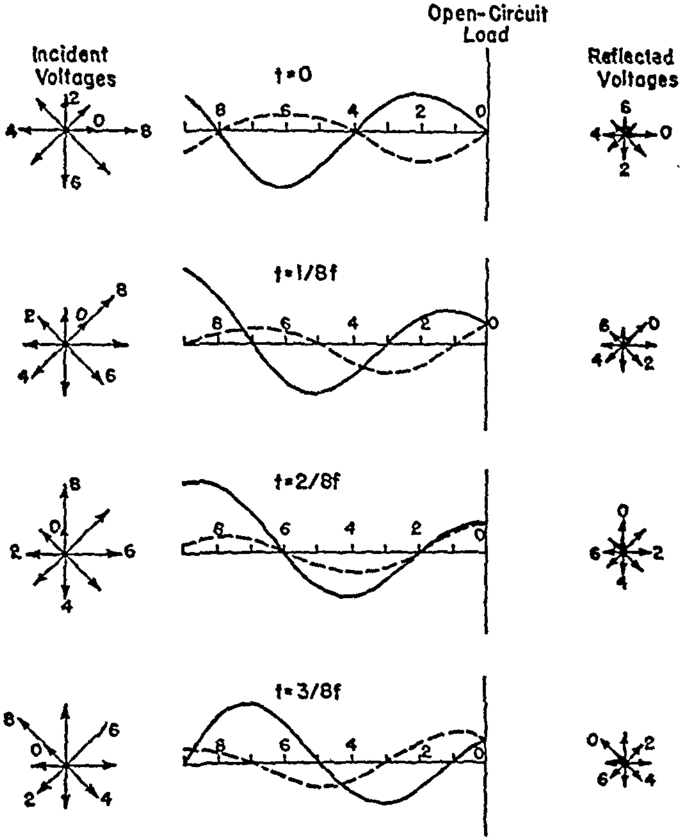


Fig. 6-8. Rotating voltage-phasor systems for incident and reflected waves for an open-circuit termination. The incident wave (solid curve) and reflected wave (dashed curve) are shown for four successive time instants.

are plotted as solid curves, as derived from the rotating maximum value phasors at the left. The waves are plotted over the last wavelength of line adjacent to the receiving end and are for four instants of time, each separated by one-eighth of a cycle. It can be seen that the voltage component varying as $\epsilon^{\gamma s}$ (s measured from the load or receiving end) is really a wave progressing from sending end to receiving end of the line. The dashed wave is derived from the rotating vectors at the right, varying in amplitude as $\epsilon^{-\gamma s}$, and it can be seen that this component is truly represented by a wave progressing from the receiving end toward the source, with initial value equal to the incident voltage at the load (for open circuit). This is the reflected wave.

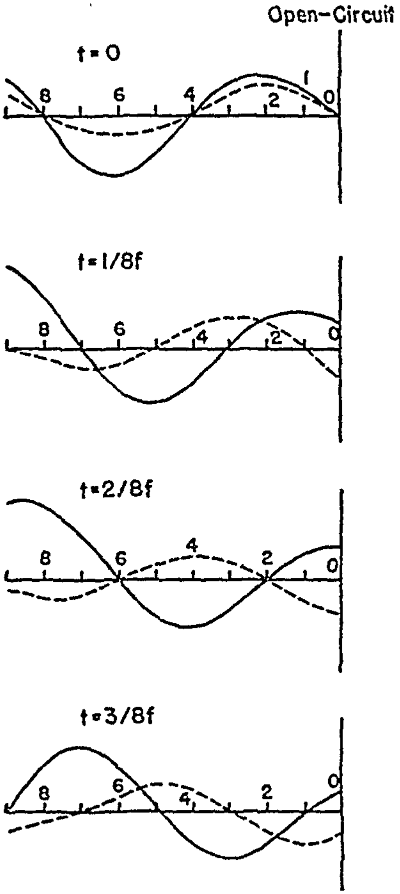


Fig. 6-9. Curves of incident (solid curve) and reflected (dashed curve) current waves for an open-circuited line, at successive time instants.

Therefore the total instantaneous voltage at any point on the line is the vector sum of the voltage of the incident and reflected waves.

From Eq. 6-23 it may be seen that the only difference between the curves for voltage and current (with Z_R infinite or as an open circuit) is in the reversed phase of the reflected current wave. The waves of instantaneous current are plotted in Fig. 6-9. It may be seen that the two current waves are equal and of opposite phase at the open-circuited receiving end, the total instantaneous current at that point always being zero as required by the open circuit.

The relative phase angles of the incident and reflected waves at the load are determined by the term $(Z_R - Z_0)/(Z_R + Z_0)$. Therefore both angle and magnitude of Z_R and Z_0 enter into the determination of the phase angle between the two waves.

In the case of an infinite line ($s = \infty$), or for $Z_R = Z_0$, the second term in the brackets of Eqs. 6-23 and 6-24 becomes zero and the reflected wave is absent. This effect seems reasonable, since in the case of the infinite line the traveling waves of energy continue in one direction indefinitely. Along such a uniform line of infinite length there is no source of energy or discontinuity to send back a reflected wave along the line. The line terminated in Z_0 behaves in exactly similar fashion, the waves traveling smoothly down the line and the energy being absorbed in the Z_0 load without setting up a reflected wave. In fact, the characteristic impedance may be looked upon as a measure of the natural rate of absorption of energy. Such a line is frequently called a *smooth* line.

It has been mentioned that the quantity actually being transmitted is energy. This energy is conveyed by the electric and magnetic fields traveling or guided along the line. A line terminated in Z_0 shows this value of impedance at all points along the line. For an ideal line with $R = G = 0$ and so terminated, the ratio of voltage to current is a constant given by

$$Z_0 = \frac{E}{I}$$

and the distribution of energy between the electric and magnetic fields is fixed. The energy conveyed in the electric field is

$$W_e = \frac{CE^2}{2} \text{ joules/m}^3$$

and that conveyed in the magnetic field is

$$W_m = \frac{LI^2}{2} \text{ joules/m}^3$$

It will be shown later that for such an ideal line $Z_0 = L/C$. Using this condition for the Z_0-terminated line where $E/I = Z_0$ everywhere, it appears that

$$W_e = W_m$$

(or the electric field energy equals the magnetic field energy) is the physical relation existing everywhere along the ideal line terminated in Z_0.

At a load where $Z_R \neq Z_0$, a different ratio of E_R to I_R

$$Z_R = \frac{E_R}{I_R}$$

is required and a redistribution of energy between the two fields is called for. This redistribution of energy acts as a source to send a reflected wave back along the line.

When the field waves strike an open circuit, for example, the magnetic field must become zero because the current is zero. The energy that was conveyed by the magnetic field cannot be dissipated, so it will appear as additional energy in the electric field, causing an increased voltage to appear. This increased voltage then sets up a returning current wave down the line. At a short circuit it is the electric field that is forced to zero by the condition of zero voltage. The transferred energy causes an increase in the magnetic field, which in turn induces a returning voltage wave down the line.

Any discontinuity in line parameters, such as the junction of an open-wire line to a cable of different Z_0, requires a redistribution of energy between the fields and thus sets up a reflected wave. A long line has an impedance of Z_0 at all points, which then determines the energy distribution in the fields. If the load is also Z_0, the voltage to current ratio is the same in the line and in the load, no energy interchange is required between the fields, and there is no opportunity for reflection to occur.

Reflection is ordinarily considered as undesirable on a transmission line. If the attenuation is not large, the returning wave appears as an echo at the sending end. Also, if reflection is present there is a reduction in efficiency and output because a portion of the received energy is rejected by the load. In passing back down the line as a reflected wave, additional energy is lost because of the R and G of the line. If the impedance of the generator is not Z_0, the reflected wave is reflected again at the sending end, becoming a new incident wave. Energy is thus transmitted back and forth on the line until dissipated in the line losses. Hence a termination in Z_0 with no reflection is desirable.

6-11. Reflection coefficient

The ratio of amplitudes of the reflected and incident *voltage waves at the receiving end of the line* is frequently called the *reflection coefficient.* From Eq. 6-23 with $s = 0$, this ratio is

$$K = \frac{\text{reflected voltage at load}}{\text{incident voltage at load}} = \frac{Z_R - Z_0}{Z_R + Z_0} \tag{6-64}$$

Equations 6-23 and 6-24 then may be written

$$E = \frac{E_R(Z_R + Z_0)}{2Z_R}\,(\epsilon^{\gamma s} + K\epsilon^{-\gamma s}) \tag{6-65}$$

$$I = \frac{I_R(Z_R + Z_0)}{2Z_0}\,(\epsilon^{\gamma s} - K\epsilon^{-\gamma s}) \tag{6-66}$$

The sign of K, and hence the polarity of the reflected wave, is dependent on the angles and magnitudes of Z_R and Z_0. For a termination of Z_0 the reflection coefficient is zero.

The reflection coefficient will be found to be extremely useful.

6-12. Line calculation

The example of Section 6-5 may be chosen as an illustration of calculation using the transmission line equations with a load other than Z_0. The problem may be stated as follows:

A generator of 1.0 volt, 1000 cycles, supplies power to a 100-mile open-wire line terminated in 200 ohms resistance. The line parameters are

$$R = 10.4 \text{ ohms per mile}$$

$$L = 0.00367 \text{ henry per mile}$$

$$G = 0.8 \times 10^{-6} \text{ mho per mile}$$

$$C = 0.00835\ \mu\text{f per mile}$$

The line constants as computed in Section 6-5 are

$$Z = 25.2\underline{/66^\circ} \text{ ohms per mile}$$

$$Y = 52.6 \times 10^{-6}\underline{/90^\circ} \text{ mho per mile}$$

$$Z_0 = 692\underline{/-12^\circ} \text{ ohms}$$

$$\gamma = 0.0363\underline{/78^\circ}$$

$$\alpha = 0.00755 \text{ neper per mile}$$

$$\beta = 0.0355 \text{ radian per mile}$$

$$\alpha l = 0.755 \text{ neper}$$

$$\beta l = 3.55 \text{ radians} = 203.8^\circ$$

The reflection coefficient K is

$$K = \frac{Z_R - Z_0}{Z_R + Z_0} = \frac{200 - 692\underline{/-12^\circ}}{200 + 692\underline{/-12^\circ}} = 0.558\underline{/172.8^\circ}$$

From Eqs. 6-29 and 6-64,

$$Z_s = Z_0\left(\frac{\epsilon^{\gamma l} + K\epsilon^{-\gamma l}}{\epsilon^{\gamma l} - K\epsilon^{-\gamma l}}\right)$$

so that values of $\epsilon^{\gamma l}$ and $\epsilon^{-\gamma l}$ are needed.
These values may be readily obtained as

$$\epsilon^{\gamma l} = \epsilon^{\alpha l}\epsilon^{j\beta l} = \epsilon^{0.755}\underline{/203.8^\circ}$$
$$= 2.12\underline{/203.8^\circ}$$
$$\epsilon^{-\gamma l} = \epsilon^{-\alpha l}\epsilon^{-j\beta l} = \epsilon^{-0.755}\underline{/-203.8^\circ}$$
$$= 0.472\underline{/-203.8^\circ}$$

Then

$$Z_s = 692\underline{/-12^\circ}\left(\frac{2.12\underline{/203.8^\circ} + 0.558\underline{/172.8^\circ} \times 0.472\underline{/-203.8^\circ}}{2.12\underline{/203.8^\circ} - 0.558\underline{/172.8^\circ} \times 0.472\underline{/-203.8^\circ}}\right)$$
$$= 692\underline{/-12^\circ}\left(\frac{1.975\underline{/210^\circ}}{2.285\underline{/198.5^\circ}}\right)$$
$$= 692\underline{/-12^\circ} \times 0.864\underline{/11.5^\circ} = 597\underline{/-0.5^\circ}$$

The input current then is

$$I_s = \frac{E_s}{Z_s} = \frac{1.0}{597\underline{/-0.5^\circ}} = 0.00167\underline{/0.5^\circ} \text{ amp, } E_s \text{ reference}$$

The received current I_R then is obtainable from

$$I_s = \frac{I_R(Z_R + Z_0)}{2Z_0}(\epsilon^{\gamma l} - K\epsilon^{-\gamma l})$$

Most of the terms in the above equation have already been computed, so that

$$0.00167\underline{/0.5^\circ} = \frac{I_R(888\underline{/-9.5^\circ})}{1384\underline{/-12^\circ}}(2.285\underline{/198.5^\circ})$$
$$I_R = \frac{2.31\underline{/-11.5^\circ}}{2030\underline{/189^\circ}}$$
$$= 0.00113\underline{/-200.5^\circ} \text{ amp, } E_s \text{ reference}$$

The load voltage E_R then is

$$E_R = I_R Z_R = 0.00113\underline{/-200.5^\circ} \times 200$$
$$= 0.226\underline{/-200.5^\circ} \text{ v, } E_s \text{ reference}$$

The power delivered to the load is

$$P_R = I_R^2 R = 0.00113^2 \times 200 = 0.000255 \text{ watt}$$

The power input to the line is

$$P_s = E_s I_s \cos\theta = 1.0 \times 0.00167 \cos 0.5^\circ = 0.00167 \text{ watt}$$

If the efficiency of transmission is of interest; it is

$$\eta = \frac{0.000255}{0.00167} \times 100\% = 15.2\%$$

6-13. Input and transfer impedance

The input impedance of a transmission line has already been obtained as

$$Z_s = Z_0 \left(\frac{Z_R \cosh \gamma l + Z_0 \sinh \gamma l}{Z_0 \cosh \gamma l + Z_R \sinh \gamma l} \right) \tag{6-67}$$

In terms of exponentials, this is

$$Z_s = Z_0 \left(\frac{\epsilon^{\gamma l} + K\epsilon^{-\gamma l}}{\epsilon^{\gamma l} - K\epsilon^{-\gamma l}} \right) \tag{6-68}$$

If the voltage at the sending-end terminals is known, it is convenient to have the transfer impedance so that the received current can be computed directly. The sending-end voltage E_s is

$$E_s = \frac{E_R(Z_R + Z_0)}{2Z_R} (\epsilon^{\gamma l} + K\epsilon^{-\gamma l})$$
$$= \frac{I_R(Z_R + Z_0)}{2} (\epsilon^{\gamma l} + K\epsilon^{-\gamma l}) \tag{6-69}$$

for which the transfer impedance is

$$Z_T = \frac{E_s}{I_R} = \frac{(Z_R + Z_0)}{2} (\epsilon^{\gamma l} + K\epsilon^{-\gamma l}) \tag{6-70}$$

By substituting for K, Eq. 6-70 becomes

$$Z_T = Z_R \left(\frac{\epsilon^{\gamma l} + \epsilon^{-\gamma l}}{2} \right) + Z_0 \left(\frac{\epsilon^{\gamma l} - \epsilon^{-\gamma l}}{2} \right)$$

which is recognizable as

$$Z_T = Z_R \cosh \gamma l + Z_0 \sinh \gamma l \tag{6-71}$$

if the expression is desired in terms of the hyperbolic functions.

6-14. Open- and short-circuited lines

As limiting cases it is convenient to consider lines terminated in open circuit or short circuit, that is, with $Z_R = \infty$ or $Z_R = 0$. The input impedance of a line of length l is

$$Z_s = Z_0 \left(\frac{Z_R \cosh \gamma l + Z_0 \sinh \gamma l}{Z_0 \cosh \gamma l + Z_R \sinh \gamma l} \right)$$

and for the short-circuit case $Z_R = 0$, so that

$$Z_{sc} = Z_0 \tanh \gamma l \tag{6-72}$$

Before the open-circuit case is considered, the input impedance should be written

$$Z_s = Z_0 \left[\frac{\cosh \gamma l + (Z_0/Z_R) \sinh \gamma l}{(Z_0/Z_R) \cosh \gamma l + \sinh \gamma l} \right]$$

The input impedance of the open-circuited line of length l, with $Z_R = \infty$, is

$$Z_{oc} = Z_o \coth \gamma l \tag{6-73}$$

By multiplying Eqs. 6-72 and 6-73 it can be seen that

$$Z_o = \sqrt{Z_{oc} Z_{sc}} \tag{6-74}$$

This is the same result as was obtained for a lumped network. Equation 6-74 supplies a very valuable means of experimentally determining the value of Z_0 of a line.

Also, from the same two equations,

$$\tanh \gamma l = \sqrt{\frac{Z_{sc}}{Z_{oc}}} \tag{6-75}$$

or

$$\gamma l = \tanh^{-1} \sqrt{\frac{Z_{sc}}{Z_{oc}}}$$

Use of this equation in experimental work requires the determination of the hyperbolic tangent of a complex angle. If

$$\tanh \gamma l = \tanh (\alpha + j\beta) l = U + jV$$

then it can be shown that

$$\tanh 2\alpha l = \frac{2U}{1 + U^2 + V^2} \tag{6-76}$$

and

$$\tan 2\beta l = \frac{2V}{1 - U^2 - V^2} \tag{6-77}$$

The value of β is uncertain as to quadrant. Its proper value may be selected if the approximate velocity of propagation is known.

6-15. Reflection factor and reflection loss

If Z_2 is not equal to Z_1, Fig. 6-10, the impedance mismatch causes a change in the ratio of voltage to current, or of energy transmitted

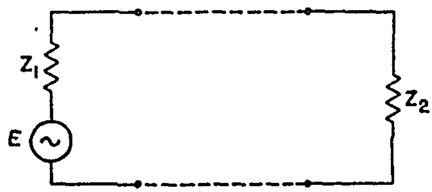


Fig. 6-10. Generator of impedance Z_1 connected to load Z_2.

by the electric field to that transmitted by the magnetic field, and thus a portion of the energy is reflected by the load. The energy delivered to the load may be less than would be delivered if impedances were matched; it is said that a *reflection loss* has occurred.

The magnitude of the reflection loss may be determined by computing the ratio of the current actually flowing in the load to that which would flow if the impedances were matched at the terminals in question. Image matching between a given generator and load might be obtained by insertion of an ideal transformer and a lossless phase shifter between source and load. According to the theory of the ideal transformer from Section 3-9,

$$\frac{I_1}{I_2} = \sqrt{\frac{Z_2}{Z_1}} \tag{6-78}$$

For image matching the magnitude of Z_2 may be adjusted to that of Z_1 by choosing the proper transformer ratio, and the phase angle of Z_2 may be adjusted to that of Z_1 by operation of the phase shifter. Under these theoretical conditions Z_2 is image matched to Z_1, and the current which would then flow through the generator would be

$$I_1 = \frac{E}{2Z_1} \tag{6-79}$$

Then the current I_2' that would flow in the load, or secondary of the transformer, *under image matching*, would be given by use of Eqs. 6-78 and 6-79 as

$$I_2' = \frac{E}{2Z_1}\sqrt{\frac{Z_1}{Z_2}} = \frac{E}{2\sqrt{Z_1 Z_2}} \tag{6-80}$$

whereas *without image matching* this current would have been

$$|I_2| = \frac{|E|}{|Z_1 + Z_2|} \tag{6-81}$$

Hence the ratio of the current actually flowing in the load to that which might flow under image matched conditions is

$$\left|\frac{I_2}{I_2'}\right| = \frac{\dfrac{|E|}{|Z_1 + Z_2|}}{\dfrac{|E|}{|2\sqrt{Z_1 Z_2}|}} = \frac{|2\sqrt{Z_1 Z_2}|}{|Z_1 + Z_2|} \tag{6-82}$$

This ratio indicates the change in current in the load due to reflection at the mismatched junction and is called the *reflection factor*, given the symbol k, where

$$k = \left|\frac{2\sqrt{Z_1 Z_2}}{Z_1 + Z_2}\right| \tag{6-83}$$

It should be noted that Z_1 and Z_2 are the impedances seen looking both ways at any junction.

The reflection loss is defined as the number of nepers or decibels by which the current in the load under image matched conditions would exceed the current actually flowing in the load. Thus the reflection loss involves the reciprocal of k, or

$$\text{reflection loss, nepers} = \ln\left|\frac{Z_1 + Z_2}{2\sqrt{Z_1 Z_2}}\right| \tag{6-84}$$

$$\text{reflection loss, db} = 20\log\left|\frac{Z_1 + Z_2}{2\sqrt{Z_1 Z_2}}\right| \tag{6-85}$$

The use of the word *reflection* here is unfortunate, and *mismatching loss* would be a better choice, since the factor k measures the relative gain or loss of actual terminations with respect to image matching.

The reflection loss may be plotted in terms of the impedance ratio

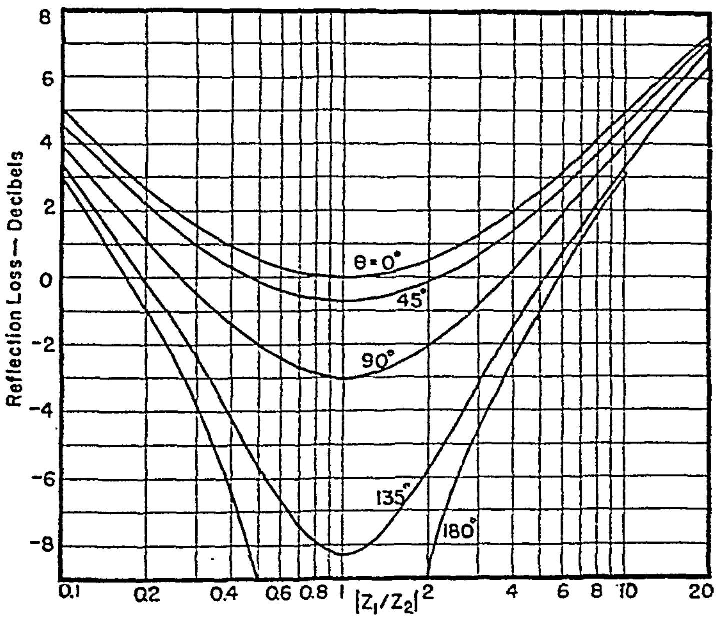


Fig. 6-11. Reflection loss due to mismatch between source and load impedances. The angle θ is the angle of Z_1/Z_2. Note that curves are symmetrical about unity on the abscissa.

$|Z_1/Z_2|$ or $|Z_2/Z_1|$ as in Fig. 6-11. It should be noted that on the logarithmic scale used, the curves are symmetric about the axis $|Z_1/Z_2| = 1$. The angle θ represents the angle of Z_1/Z_2. It can be seen that for certain conditions a negative reflection loss or a reflection gain is obtained.

6-16. Insertion loss

The insertion of a four-terminal network or a line between a generator and a load may improve or diminish the impedance match between source and load, and may also introduce dissipative elements. The net effect may be to improve or reduce the impedance match and thus to increase or decrease the power delivered to the load, resulting in a positive or negative *insertion loss*, due to insertion of the network. *The insertion loss of a line or network is defined as the number of nepers or decibels by which the current in the load is changed by the insertion.* Occasionally the insertion of a network causes an increase in load current, and this would be expected from the matching networks of Chapter 3, if made of low dissipation elements. Such an increase in load current represents a negative loss, or an *insertion gain*.

Therefore the insertion loss is the resultant of several individual losses. In Fig. 6-12, if Z_s is not equal to Z_g at the 1,1 terminals, then

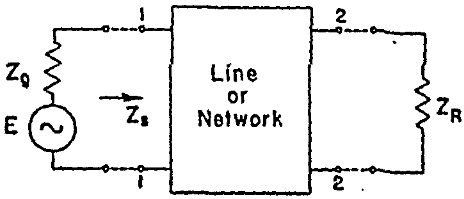


Fig. 6-12. Illustrating insertion loss.

a reflection loss occurs at that point. The line or network also may introduce an attenuation loss. If the impedances are not matched at the 2,2 terminals, then a second reflection loss occurs there. The over-all insertion loss due to insertion of a line or network between a generator and a load can be calculated from the ratio of the load current that would flow if the load were directly connected to the generator to the current actually flowing in the load.

The insertion loss due to a *line* introduced between source and load in Fig. 6-12 can be calculated from the line equations as an example. The sending-end current can be written

$$I_s = \frac{I_R(Z_R + Z_0)}{2Z_0}(\epsilon^{\gamma l} - K\epsilon^{-\gamma l}) \tag{6-86}$$

or

$$I_s = \frac{E}{Z_g + Z_s}$$

The input impedance Z_s has been obtained as

$$Z_s = Z_0\left(\frac{\epsilon^{\gamma l} + K\epsilon^{-\gamma l}}{\epsilon^{\gamma l} - K\epsilon^{-\gamma l}}\right)$$

so that $$I_s = \frac{E(\epsilon^{\gamma l} - K\epsilon^{-\gamma l})}{Z_g(\epsilon^{\gamma l} - K\epsilon^{-\gamma l}) + Z_0(\epsilon^{\gamma l} + K\epsilon^{-\gamma l})} \tag{6-87}$$

Substitution of Eq. 6-87 in 6-86 and solution for I_R gives

$$I_R = \frac{2Z_0E}{(Z_R + Z_0)[Z_g(\epsilon^{\gamma l} - K\epsilon^{-\gamma l}) + Z_0(\epsilon^{\gamma l} + K\epsilon^{-\gamma l})]}$$

Introducing the value for K, the reflection coefficient, permits this equation to be written as

$$I_R = \frac{2Z_0E}{(Z_R + Z_0)(Z_g + Z_0)\epsilon^{\gamma l} + (Z_0 - Z_g)(Z_R - Z_0)\epsilon^{-\gamma l}} \tag{6-88}$$

This is the value of current actually flowing in the load Z_R.

Without the line present, the current I_R' flowing in the load would be

$$I_R' = \frac{E}{Z_g + Z_R} \tag{6-89}$$

The ratio of the current that would flow in the load, if generator and load were directly connected, to that flowing with the line inserted is

$$\frac{I_R'}{I_R} = \frac{\dfrac{E}{Z_g + Z_R}}{\dfrac{2Z_0E}{(Z_R + Z_0)(Z_g + Z_0)\epsilon^{\gamma l} + (Z_0 - Z_g)(Z_R - Z_0)\epsilon^{-\gamma l}}}$$

$$= \frac{(Z_R + Z_0)(Z_g + Z_0)\epsilon^{\alpha l}\epsilon^{j\beta l} + (Z_0 - Z_g)(Z_R - Z_0)\epsilon^{-\alpha l}\epsilon^{-j\beta l}}{2Z_0(Z_g + Z_R)} \tag{6-90}$$

If α is large or the line is sufficiently long, the second term in the numerator may be neglected with respect to the first, leaving

$$\frac{I_R'}{I_R} = \frac{(Z_R + Z_0)(Z_g + Z_0)\epsilon^{\alpha l}\epsilon^{j\beta l}}{2Z_0(Z_g + Z_R)}$$

Greater physical meaning may be given to the expression if both numerator and denominator are multiplied by $2\sqrt{Z_gZ_R}$, giving

$$\frac{I_R'}{I_R} = \frac{2\sqrt{Z_gZ_R}\,(Z_R + Z_0)(Z_g + Z_0)\epsilon^{\alpha l}\epsilon^{j\beta l}}{4\sqrt{Z_gZ_RZ_0^2}\,(Z_g + Z_R)} \tag{6-91}$$

The insertion loss is to be calculated as a function of current magnitudes only, so that after taking absolute values and rearranging, the above expression becomes

$$\left|\frac{I_R'}{I_R}\right| = \frac{|Z_g + Z_0|}{2\sqrt{Z_g Z_0}} \cdot \frac{|Z_R + Z_0|}{2\sqrt{Z_R Z_0}} \cdot \frac{2\sqrt{Z_g Z_R}}{|Z_g + Z_R|} \cdot \epsilon^{\alpha l} \tag{6-92}$$

The coefficients on the right side may be recognized as reflection factors, where

$$\frac{2\sqrt{Z_g Z_0}}{|Z_g + Z_0|} = k_s$$

and may be considered as the reflection factor at the 1,1 terminals where the generator may be mismatched at its junction with the line. The line has been assumed long, or α large; thus its input impedance appears to be Z_0. The second term is another reflection factor

$$\frac{2\sqrt{Z_R Z_0}}{|Z_R + Z_0|} = k_R$$

and can be seen as the reflection factor at the 2,2 terminals, or the junction between line and load. The third term is

$$\frac{2\sqrt{Z_g Z_R}}{|Z_g + Z_R|} = k_{sR}$$

and may be considered as a reflection factor occurring if the generator were directly connected to the load. The fourth term is $\epsilon^{\alpha l}$ the loss in the line.

The current ratio then is

$$\left|\frac{I_R'}{I_R}\right| = \frac{k_{sR}}{k_s k_R}\,\epsilon^{\alpha l} \tag{6-93}$$

The insertion loss may then be obtained by taking the logarithm of the current ratio:

$$\text{insertion loss, nepers} = \ln\frac{1}{k_s} + \ln\frac{1}{k_R} - \ln\frac{1}{k_{sR}} + \alpha l \tag{6-94}$$

$$\text{insertion loss, db} = 20\left(\log\frac{1}{k_s} + \log\frac{1}{k_R} - \log\frac{1}{k_{sR}} + 0.4343\alpha l\right) \tag{6-95}$$

The presence of the term $\ln 1/k_{sR}$ may be perplexing. This loss has been identified as the reflection loss that would occur if the generator were connected directly to the load. As such, it was eliminated by definition, since it is not due to *insertion* of the line, and accordingly the equation shows this loss as subtracted.

If the line conditions do not justify the assumptions concerning α or l, the insertion loss may be determined by direct computation of the current in the load, with and without the line or network present.

The expression above may also apply directly to a single section network, by use of $l = 1$, and the proper value of α applying to the network.

6-17. T and π sections equivalent to lines

In Chapter 1, relations were developed permitting the design of an equivalent T section from measurements on a network. These relations were

$$Z_1 = Z_{1oc} - \sqrt{Z_{2oc}(Z_{1oc} - Z_{1sc})}$$
$$Z_2 = Z_{2oc} - \sqrt{Z_{2oc}(Z_{1oc} - Z_{1sc})}$$
$$Z_3 = \sqrt{Z_{2oc}(Z_{1oc} - Z_{1sc})}$$

The input impedances of open- and short-circuited lines were developed in Section 6-14 as

$$Z_{1oc} = \frac{Z_0}{\tanh \gamma l} = Z_0 \left(\frac{\epsilon^{\gamma l} + \epsilon^{-\gamma l}}{\epsilon^{\gamma l} - \epsilon^{-\gamma l}}\right)$$
$$Z_{1sc} = Z_0 \tanh \gamma l = Z_0 \left(\frac{\epsilon^{\gamma l} - \epsilon^{-\gamma l}}{\epsilon^{\gamma l} + \epsilon^{-\gamma l}}\right)$$

Since a line is a symmetrical network,

$$Z_{1oc} = Z_{2oc}$$

The Z_3 or shunt element of a T section that will be equivalent, in so far as external voltages and currents are concerned, to the long line can then be readily obtained as

$$Z_3 = \sqrt{\frac{Z_0}{\tanh \gamma l}\left(\frac{Z_0}{\tanh \gamma l} - Z_0 \tanh \gamma l\right)}$$
$$= \frac{Z_0}{\sinh \gamma l} \tag{6-96}$$

The series elements for the equivalent section then are

$$Z_1 = Z_2 = Z_{1oc} - Z_3 = Z_0\left(\frac{\epsilon^{\gamma l} + \epsilon^{-\gamma l}}{\epsilon^{\gamma l} - \epsilon^{-\gamma l}} - \frac{2}{\epsilon^{\gamma l} - \epsilon^{-\gamma l}}\right)$$

$$= Z_0\left[\frac{(\epsilon^{\gamma l/2} - \epsilon^{\gamma l/2})^2}{(\epsilon^{\gamma l/2} - \epsilon^{-\gamma l/2})(\epsilon^{\gamma l/2} + \epsilon^{-\gamma l/2})}\right] = Z_0\left(\frac{\epsilon^{\gamma l/2} - \epsilon^{-\gamma l/2}}{\epsilon^{\gamma l/2} + \epsilon^{-\gamma l/2}}\right)$$

$$Z_1 = Z_2 = Z_0 \tanh\frac{\gamma l}{2} \tag{6-97}$$

The T-section equivalent for the long line, made up of these elements, is shown in Fig. 6-13(a). It is useful in certain types of line calculations.

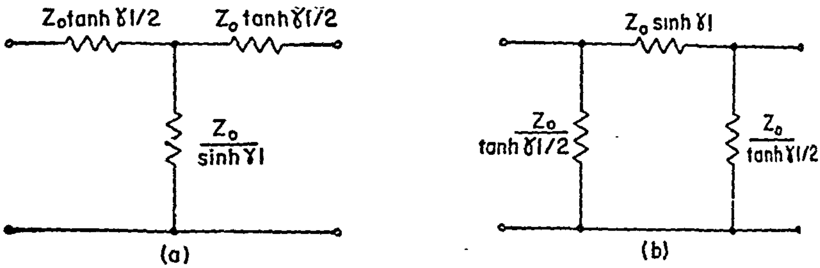


Fig. 6-13. (a) T-section equivalent circuit for a transmission line; (b) same as a π section.

A π-section equivalent for the line may likewise be determined from the terminal measurements as in Chapter 1. Because of symmetry,

$$Z_A = Z_C = \frac{Z_{2oc}Z_{1sc}}{Z_{2oc} - \sqrt{Z_{2oc}(Z_{1oc} - Z_{1sc})}}$$

$$= \frac{Z_0^2}{Z_0\left(\dfrac{\epsilon^{\gamma l} + \epsilon^{-\gamma l}}{\epsilon^{\gamma l} - \epsilon^{-\gamma l}}\right) - \dfrac{2Z_0}{\epsilon^{\gamma l} - \epsilon^{-\gamma l}}}$$

$$= \frac{Z_0(\epsilon^{\gamma l/2} - \epsilon^{-\gamma l/2})(\epsilon^{\gamma l/2} + \epsilon^{-\gamma l/2})}{(\epsilon^{\gamma l/2} - \epsilon^{-\gamma l/2})^2}$$

$$Z_A = Z_C = \frac{Z_0}{\tanh(\gamma l/2)} \tag{6-98}$$

The Z_B arm can be easily obtained as

$$Z_B = \frac{Z_{2oc}Z_{1sc}}{\sqrt{Z_{2oc}(Z_{1oc} - Z_{1sc})}}$$
$$= \frac{Z_0^2}{Z_0/\sinh \gamma l} = Z_0 \sinh \gamma l \tag{6-99}$$

The equivalent π section for a line is shown at (b), Fig. 6-13.

6-18. Distance to a line fault

The input impedance of a line is

$$Z_s = Z_0 \left(\frac{\epsilon^{\gamma l} + K\epsilon^{-\gamma l}}{\epsilon^{\gamma l} - K\epsilon^{-\gamma l}}\right)$$
$$= Z_0 \left(\frac{\epsilon^{\alpha l}\epsilon^{j\beta l} + K\epsilon^{-\alpha l}\epsilon^{-j\beta l}}{\epsilon^{\alpha l}\epsilon^{j\beta l} - K\epsilon^{-\alpha l}\epsilon^{-j\beta l}}\right) \tag{6-100}$$

Upon substituting for $\epsilon^{j\beta l}$ and $\epsilon^{-j\beta l}$,

$$Z_s = Z_0 \left[\frac{(\epsilon^{\alpha l} + K\epsilon^{-\alpha l}) \cos \beta l + j(\epsilon^{\alpha l} - K\epsilon^{-\alpha l}) \sin \beta l}{(\epsilon^{\alpha l} - K\epsilon^{-\alpha l}) \cos \beta l + j(\epsilon^{\alpha l} + K\epsilon^{-\alpha l}) \sin \beta l}\right]$$
$$= Z_0 \left[\frac{(\epsilon^{\alpha l} + K\epsilon^{-\alpha l}) + j(\epsilon^{\alpha l} - K\epsilon^{-\alpha l}) \tan \beta l}{(\epsilon^{\alpha l} - K\epsilon^{-\alpha l}) + j(\epsilon^{\alpha l} + K\epsilon^{-\alpha l}) \tan \beta l}\right] \tag{6-101}$$

This expression shows that the input impedance magnitude of the line is periodic and of period $\beta l = \pi$. Thus the input impedance oscillates between maximum and minimum values as the length of line is increased. Measurements of input impedance magnitude against length of line appear as in Fig. 6-14.

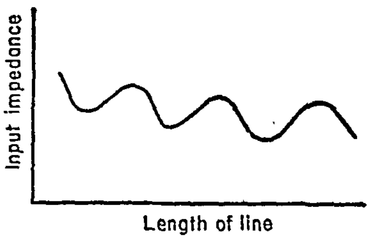


Fig. 6-14. Variation of input impedance of a line not terminated in Z_0, as the electrical length of line is increased.

Since it is βl or the electrical length that is important, the value of βl may also be changed by increasing the frequency applied to the line. The variation of impedance will appear as in Fig. 6-14, with length replaced by frequency. By the above reasoning, the difference between two maximum impedance points represents a change in electrical length of one-half wavelength. For the lower of the two frequencies of maximum impedance, the distance s to the reflecting point in terms of wavelengths is

$$s = p\lambda_1 \tag{6-102}$$

For the next higher frequency at which maximum input impedance occurs, the distance s is the same, but electrically the line is one-half wavelength longer, so that

$$s = \left(p + \frac{1}{2}\right)\lambda_2 \tag{6-103}$$

Using Eq. 6-102,

$$s = \left(\frac{s}{\lambda_1} + \frac{1}{2}\right)\lambda_2 = \frac{\lambda_1\lambda_2}{2(\lambda_1 - \lambda_2)}$$

Since $v = \lambda f$,

$$s = \frac{v_1 v_2}{2(v_1 f_2 - v_2 f_1)} \tag{6-104}$$

The velocities may be determined from $v = \omega/\beta$ by use of the known line parameters. Equation 6-104 then allows determination of the distance to a line fault or point of reflection by measurements made from one end of the line.

In many cases the velocity does not change appreciably for the frequency range between f_1 and f_2, so that Eq. 6-104 may be simplified by assuming $v_1 = v_2$, giving

$$s = \frac{v}{2(f_2 - f_1)} \tag{6-105}$$

as the distance to the reflecting point. This reflecting point may be a line fault whose location is desired.

PROBLEMS

6-1. A simulated line is composed of T sections of pure resistance, $Z_1 = Z_2 = 50$ ohms, $Z_3 = 4000$ ohms.

(a) Find α and Z_0.

(b) A line composed of 50 such sections in series is terminated in its characteristic impedance. A generator of 1 v, 400 ohms internal resistance is at the sending end. Find I_s and I_R.

(c) What is the decibel loss in the line?

6-2. A 60-mile length of 0.104 in. diameter open-wire line (see Table 4) is terminated in Z_0. A generator of 600 ohms internal resistance and 1 v, 800 c, is connected to the sending end. Find I_s, I_R and power output of generator, and power delivered to the load. Also determine wavelength and velocity of propagation.

TABLE 4

CHARACTERISTICS OF CERTAIN TELEPHONE LINES AND CABLES
(per loop mile)

Type	R ohms	L henrys	C μf	G μmhos	Wire spacing, in.
0.165-in. diameter open wire.	4.11	0.00311	0.00996	0.14	8
0.128-in. diameter open wire.	6.74	0.00353	0.00871	0.29	12
0.104-in. diameter open wire.	10.15	0.00393	0.00797	0.29	18
19-gauge cable.............	85.8	0.001	0.062	1.5	..
16-gauge cable.............	42.1	0.001	0.062	1.5	..
19-gauge cable, loaded*....	92.2	0.078	0.062	1.5	..

* Coil spacing, 6000 feet; inductance 88 mh; resistance 7.3 ohms.

6-3. Over a range of 100 to 10,000 c, plot the values of α, β, and velocity for a mile of 19-gauge cable.

6-4. A line of 0.165-in. open wire is 100 miles long and terminated in Z_0. Find Z_0, α, β, γ, v, and λ for a 500-c signal.

6-5. How much voltage must be applied to the sending end of a 30-mile line having $R = 21.4$ ohms per mile, $L = 0.001$ h per mile, $C = 0.062$ μf per mile, $G = 0.868$ μmho per mile, and terminated in Z_0, if the received power is to be at -20-db level (reference $= 0.006$ watt, $\omega = 5000$)?

6-6. A line is one wavelength long and is short-circuited. If it is of 16-gauge cable and 1 v, 800 c, is applied to it, compute values of incident, reflected, and total voltage at $\lambda/8$ intervals along the line.

6-7. A line of 16-gauge cable is 60 miles long and is terminated in a load of $400 + j300$ ohms. The received power is to be at a level of -10 db (0.001 watt reference). If the frequency is 796 cycles, find

(a) Sending end voltage, current, and power.
(b) Power loss in the line.
(c) Wavelength and velocity of propagation.

6-8. How much inductive loading (per mile) is required to make a 16-gauge cable (Table 4) distortionless? Assume no increase in R.

6-9. To what value must the shunt conductance per mile have to be changed to make the 0.128-in. open-wire line of Table 4 distortionless? How much attenuation, in decibels per mile, is then added?

6-10. If the line of Prob. 6-7 is short-circuited at the receiving end, find the sending-end current for a generator of 1 v, 500 ohms internal resistance, with $\omega = 4000$. Find the received current also.

6-11. A 16-gauge cable, 50 miles long, is inserted between a generator of 2 v, 600 ohms, $\omega = 5000$, and a load of $400 + j400$ ohms. Find the decibel loss in power in the load due to the presence of the cable.

6-12. A 19-gauge cable 32 miles long is supplied by a generator of 2 v, 400 ohms, 1200 c, and terminated in Z_0. Find the insertion loss of the line, in decibels.

6-13. Thirty miles of 0.165-in. open wire is supplied by a generator of 2 v, 600 ohms resistance, and loaded with $300 + j400$ ohms. The frequency is 1000 c. Find

(a) Values of I_R, I_s, E_R, and received power.

(b) Value of the incident and reflected voltages at $s = 0$, $s = 10$, $s = 20$, $s = 30$ miles.

6-14. A 50-mile line has the following measurements made at 1200 c:

$$Z_{1oc} = 200\underline{/-42^\circ} \text{ ohms}; \; Z_{1sc} = 1890\underline{/22^\circ} \text{ ohms}$$

Find the value of Z_0, α, β, and v for this line. The approximate velocity is 20,000 miles per second.

6-15. Plot a curve of input impedance magnitude vs length of line by 20-mile steps up to 200 miles for the 0.104-in. open-wire line at a frequency of 1000 c. The line is open-circuited.

6-16. A line of 16-gauge cable is to be loaded to improve performance with $\omega = 5000$.

(a) Find Z_0, α, β, v, λ unloaded.

(b) The line is loaded with $L = 0.246$ h and $R = 7.3$ ohms at intervals of 7.88 miles. Assume this loading to be distributed. Recalculate Z_0, α, β, v, λ.

(c) Discuss effects of differences noted in line values calculated in (b).

6-17. Calculate the decibel attenuation in 100 miles of the cable of Prob. 6-16, with and without loading.

6-18. Determine the power delivered to a load of 250 ohms resistance by 30 miles of the line of Prob. 6-16 for both the loaded and unloaded case. The line is supplied by a generator of 1 v, $\omega = 5000$, zero internal resistance.

6-19. Design an equivalent T section for 40 miles of 0.104-in. open-wire line at 1000 c.

REFERENCES

1. Shea, T. E., *Transmission Networks and Wave Filters*, D. Van Nostrand Company, Inc., New York, 1929.
2. Everitt, W. L., *Communication Engineering*, 2d ed., McGraw-Hill Book Company, Inc., New York, 1937.
3. Pupin, M. L., "Wave Transmission over Non-uniform Cables & Long Distance Air Lines," *Trans. A.I.E.E.*, **17**, 445 (1900).
4. Campbell, G. A., "Loaded Lines in Telephonic Transmission," *Phil. Mag.*, March 1903, p. 313.
5. Johnson, K. S., *Transmission Circuits for Telephonic Communication*, D. Van Nostrand Company, Inc., New York, 1927.
6. Dwight, H. B., *Tables of Integrals and Other Mathematical Data*, The Macmillan Company, New York, 1934.
7. Guillemin, E. A., *Communication Networks*, Vol. II, John Wiley & Sons, Inc., New York, 1935.

Chapter 7

THE LINE AT RADIO FREQUENCIES

When a line, either open-wire or coaxial, is used at frequencies of a megacycle or more, it is found that certain approximations may be employed leading to simplified analysis of line performance. The assumptions usually made are:

1. Very considerable skin effect, so that currents may be assumed as flowing on conductor surfaces, internal inductance then being zero.
2. That $\omega L \gg R$ when computing Z. This assumption is justifiable because it is found that the resistance increases because of skin effect with $\sqrt{f}$ while the line reactance increases directly with f.
3. The lines are well enough constructed that G may be considered zero.

The analysis is made in either of two ways, depending on whether R is merely small with respect to ωL or R is considered completely negligible compared with ωL. If R is small, the line is considered one of *small dissipation*, and this concept is useful when lines are employed as circuit elements or where resonance properties are involved. If losses were neglected then infinite currents or voltages would appear in calculations, and physical reality would not be achieved. In applications where losses may be neglected, as in transmission of power at high efficiency, R may be considered as negligible, and the line as one of *zero dissipation*. These methods will be studied separately.

7-1. Parameters of the open-wire line at high frequencies

As mentioned above, at frequencies of a few megacycles or more, the skin effect is assumed to be nearly complete, or the current is considered as flowing essentially on the surface of the conductor in a *skin* of very small depth. The internal flux and internal inductance are then reduced nearly to zero, so that Eqs. 5-15 and 5-16 for the inductance of an open-wire line become

$$L = \frac{\mu_0}{2\pi} \ln \frac{d}{a} = 4 \times 10^{-7} \ln \frac{d}{a} \text{ henrys/m} \tag{7-1}$$

$$= 9.21 \times 10^{-7} \log \frac{d}{a} \text{ henrys/m} \tag{7-2}$$

The value of capacitance of a line is not affected by skin effect or frequency, so that Eqs. 5-30 and 5-31 still apply for the *open-wire line* with air dielectric as

$$C = \frac{\pi\epsilon_v}{\ln \frac{d}{a}} \quad \text{farads/m}$$

$$= \frac{27.7}{\ln \frac{d}{a}} \quad \mu\mu\text{f/m} \tag{7-3}$$

$$= \frac{12.07}{\ln \frac{d}{a}} \quad \mu\mu\text{f/m} \tag{7-4}$$

In the case of appreciable skin effect, the current flows over the surface of the conductor in a thin layer, with a resultant reduction in effective cross section or an increase in resistance of the conductor. As will be shown in Chapter 9, the effective thickness of the surface layer of current may be considered as

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} \text{ meters} \tag{7-5}$$

where μ is the conductor permeability, and σ the conductivity of the conductor material in mhos per meter. For copper, μ is that of space or $\mu = \mu_0 = 4\pi \times 10^{-7}$, and σ has the value 5.75×10^7 mhos per meter at 20°C. Equation 7-5 then becomes, for copper,

$$\delta = \frac{0.0664}{\sqrt{f}}$$

The resistance of a round conductor of radius a meters to direct current is inversely proportional to the area as

$$R_{dc} = \frac{K}{\pi a^2} \tag{7-6}$$

while that of a round conductor with alternating current flowing in a skin of thickness δ is

$$R_{ac} = \frac{K}{2\pi a\delta} \tag{7-7}$$

Therefore the ratio of resistance to alternating current to resistance to direct current is

$$\frac{R_{ac}}{R_{dc}} = \frac{a\sqrt{\pi f\mu\sigma}}{2} \tag{7-8}$$

which becomes, for copper,

$$\frac{R_{ac}}{R_{dc}} = 7.53a\sqrt{f} \tag{7-9}$$

where a is in meters and f in cycles per second. This equation shows that the increase in resistance with increasing frequency is greater for large-radius than for small-radius conductors.

The resistance of an open-wire line of copper, with spacing greater than $20a$, can be computed as, for copper,

$$R_{ac} = \frac{8.33 \times 10^{-8}\sqrt{f}}{a} \text{ ohms/meter of line} \tag{7-10}$$

For spacings closer than $20a$, the effect of the proximity of the two currents in crowding to the sides of the conductors further increases the resistance.

7-2. Parameters of the coaxial line at high frequencies

Just as for the parameters of the open-wire line in the previous section, the parameters of the coaxial line are also modified by the presence of high-frequency currents on the line. Because of skin effect, the current flows on the outer surface of the inner conductor and the inner surface of the outer conductor. This phenomenon eliminates flux linkages due to internal conductor flux, and the inductance of the coaxial line is given by Eqs. 5-23 and 5-24 as

$$L = \frac{\mu_v}{2\pi}\ln\frac{b}{a} = 2 \times 10^{-7}\ln\frac{b}{a} \text{ henrys/m} \tag{7-11}$$

$$= 4.60 \times 10^{-7}\log\frac{b}{a} \text{ henrys/m} \tag{7-12}$$

The capacitance of the coaxial line is not affected by frequency (except as frequency may alter the relative permittivity of the dielectric), so that

$$C = \frac{2\pi\epsilon}{\ln \frac{b}{a}} \quad \text{farads/m}$$

$$= \frac{55.5\epsilon_r}{\ln \frac{b}{a}} \quad \mu\mu\text{f/m} \tag{7-13}$$

$$= \frac{24.14\epsilon_r}{\log \frac{b}{a}} \quad \mu\mu\text{f/m} \tag{7-14}$$

By use of the reasoning developed in the preceding section, the resistance of a coaxial line with appreciable skin effect may be considered as due to current in two thin-walled tubes and an expression for the resistance of a copper coaxial line obtained,

$$R_{ac} = 4.16 \times 10^{-8} \sqrt{f} \left(\frac{1}{b} + \frac{1}{a}\right) \text{ ohms/m of line} \tag{7-15}$$

where a and b are outer radius of the inner conductor and inner radius of the outer conductor in meters, respectively.

The shunt losses of air dielectric lines are zero, but many coaxial lines employ solid dielectric materials, and the conductance losses may have to be considered in some applications, especially at very high frequencies. The quality of the insulating material may be measured in terms of the *power factor* of the material, when employed as the dielectric in a capacitor C. The shunt susceptance is

$$y = g + j\omega C$$

and the power factor is then expressable from the susceptance triangle of Fig. 7-1 as the cosine of θ:

$$\text{pf} = \frac{g}{\sqrt{g^2 + \omega^2 C^2}}$$

The conductance of usual good insulating materials is very small, so that $g \ll \omega C$, and

$$\text{pf} = \frac{g}{\omega C}$$

$$g = \omega C \times \text{pf} \tag{7-16}$$

The quality of the dielectric may also be expressed in terms of the *dissipation factor*, which is the ratio of energy dissipated to energy stored in the dielectric per cycle, and is proportional to the tangent of angle φ of Fig. 7-1. For good dielectrics with small power-factor angles, where the approximation $g \ll \omega C$ holds, the dissipation factor and power factor are equal in magnitude.

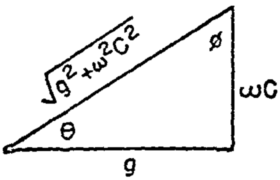


Fig. 7-1. The loss triangle for dielectrics.

Power factor or dissipation factor values for a few insulating materials are given in Table 5.

TABLE 5

DIELECTRIC LOSSES

	ϵ_r	Power factor at frequency of (cycles)			
		60	10^3	10^6	3×10^9
Quartz	3.78	0.0009	0.00075	0.0002	0.00006
Steatite	5.77	...	0.003	0.0007	0.00089
Polyethylene	2.26	<0.0002	<0.0002	<0.0002	0.00031
Polystyrene	2.56	<0.00005	<0.00005	0.00007	0.00033
Teflon	2.1	<0.0005	<0.0003	<0.0002	0.00016

7-3. Constants for the line of zero dissipation

For transmission of energy at high frequencies, where the power efficiency is high, it is possible to assume negligible losses or zero dissipation in the analysis of performance of transmission lines. This assumption of a perfect line is justified by the fact that ω is large, making ωL large. A line of a few wavelengths may be physically short, possibly only a few centimeters, but it is electrically long. The resistance of such a short line is very small compared with the reactance; and with G assumed zero because of the small number of insulators, the assumption of completely negligible losses may be made. The chief advantage of the assumption of zero dissipation is in the easy analysis and physical interpretation of line performance made possible by the method. Actually, such an assumption of a perfect line, though close to fact, may at times lead to absurd or impossible results, in which case an analysis must be made according to the methods at the end of this chapter, using the proper small value of R.

The line parameters for the line of zero dissipation are

$$Z = j\omega L$$
$$Y = j\omega C$$

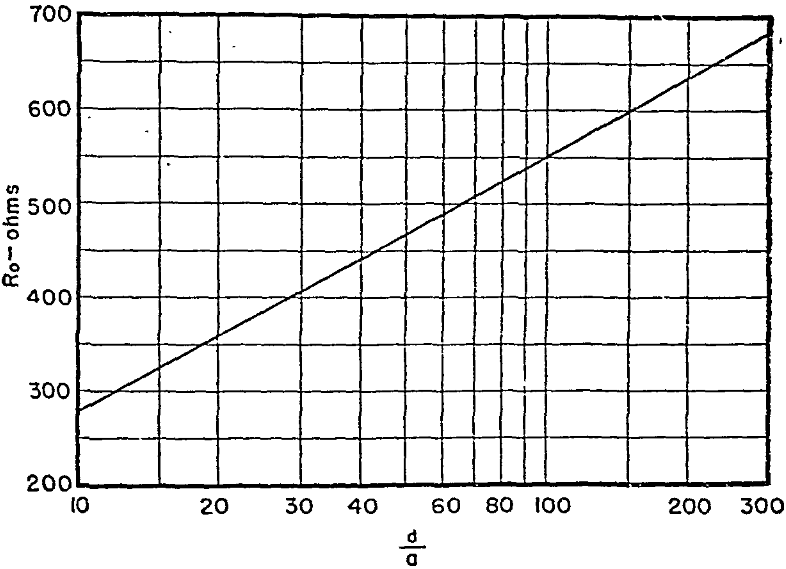


Fig. 7-2. Variation of R_0 with d/a ratio for an open-wire line.

so that the characteristic impedance, Z_0, may be written

$$Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{j\omega L}{j\omega C}}$$

$$= \sqrt{\frac{L}{C}} \text{ ohms} \tag{7-17}$$

This value is wholly resistive and may be given the symbol R_0 where

$$Z_0 = R_0 = \sqrt{\frac{L}{C}} \tag{7-18}$$

Using Eq. 7-1 for the inductance and Eq. 7-3 for the capacitance of the open-wire line at high frequency, the value of the characteristic impedance of the *open-wire line* can be found directly from line dimensions as

$$R_0 = 120 \ln \frac{d}{a} \quad \text{(ohms)} \tag{7-19}$$

$$R_0 = 276 \log \frac{d}{a} \quad \text{(ohms)} \tag{7-20}$$

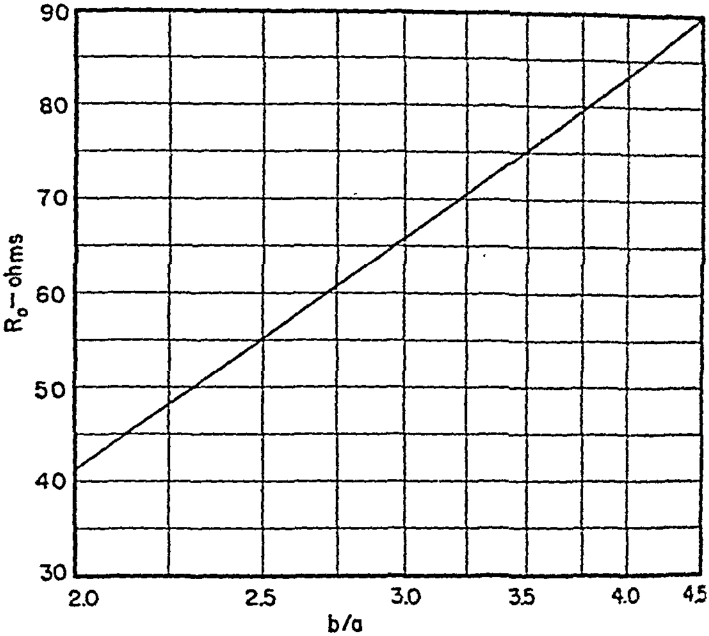


Fig. 7-3. Variation of R_0 with b/a ratio for a coaxial line.

Proximity effect has been neglected here, so that these expressions become less exact for d/a less than 10.

The characteristic impedance of the *coaxial line* can likewise be computed through use of the line dimensions as

$$R_0 = \frac{60}{\sqrt{\epsilon_r}} \ln \frac{b}{a} \quad \text{ohms} \tag{7-21}$$

$$R_0 = \frac{138}{\sqrt{\epsilon_r}} \log \frac{b}{a} \quad \text{ohms} \tag{7-22}$$

the value of ϵ_r being 1 for air-spaced lines.

The values of characteristic impedance are plotted against line dimensions in Figs. 7-2 and 7-3.

The propagation constant γ is

$$\gamma = \sqrt{ZY} = \sqrt{-\omega^2 LC}$$
$$= \alpha + j\beta = j\omega\sqrt{LC}$$

from which

$$\alpha = 0, \qquad \beta = \omega\sqrt{LC} \text{ radians/m} \tag{7-23}$$

The velocity of propagation can then be calculated as

$$v = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} \text{ m/sec} \tag{7-24}$$

With the values of L and C for an open-wire line from Section 7-1, the velocity becomes

$$v = 3 \times 10^8 \text{ m/sec}$$

Thus the velocity of propagation for the air-spaced open-wire dissipationless line is the same as the velocity of light in space. For the coaxial line

$$v = \frac{3 \times 10^8}{\sqrt{\epsilon_r}} \text{ m/sec} \tag{7-25}$$

in which case the velocity may be reduced due to the presence of a dielectric other than air between the conductors.

7-4. Voltages and currents on the dissipationless line

The voltage at any point distant s units from the receiving end of a transmission line is

$$E = \frac{E_R(Z_R + Z_0)}{2Z_R}(\epsilon^{\gamma s} + K\epsilon^{-\gamma s})$$

For the line of zero dissipation, the attenuation constant α is zero and $Z_0 = R_0$, so that

$$E = \frac{E_R(Z_R + R_0)}{2Z_R}(\epsilon^{j\beta s} + K\epsilon^{-j\beta s}) \tag{7-26}$$

The term varying with $\epsilon^{j\beta s}$ has previously been identified as a wave progressing from the source toward the load, and the term involving $\epsilon^{-j\beta s}$ as the reflected wave moving from load back toward the source.

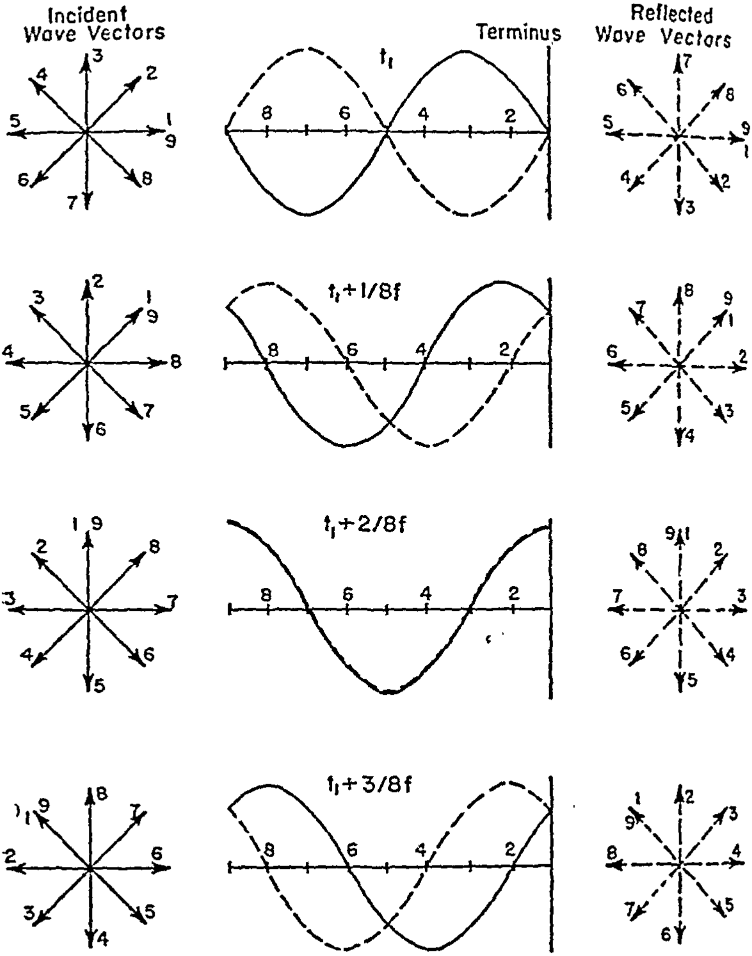


Fig. 7-4. Incident and reflected voltage-wave phasors and values along the dissipationless line for successive instants of time, for an open-circuited line.

The magnitude of the reflected wave is dependent on the value of K, the reflection coefficient. The significant difference between this analysis and that of Chapter 6 is in the absence of attentuation, the rotating vectors that may be considered as generating either the incident or reflected waves remaining constant in magnitude at all points on the line.

The successive positions of the incident and reflected waves on an open-circuited line are shown in Fig. 7-4 for values of time differing by one-eighth of a cycle. The progression toward or away from the load is readily seen. The actual voltage at any point on the transmission line is the sum of the incident and reflected wave voltages at that point. This sum voltage is plotted in Fig. 7-5(a) for the open-circuited line conditions of Fig. 7-4. It can be seen that the resultant total voltage wave appears to stand still on the line, oscillating in magnitude with time but having fixed positions of maxima and minima. Such a wave is known as a *standing wave*. If the line voltages are measured with an effective-reading voltmeter, the magnitudes will appear as in Fig. 7-5(b), since the voltmeter does not distinguish between positive and negative values.

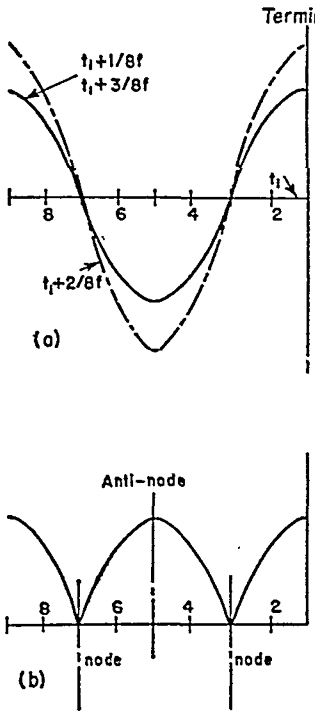


Fig. 7-5. (a) Waves of Fig. 7-4 superposed; (b) voltage values as read on line of (a) by an effective-reading voltmeter.

The standing-wave condition may be better understood if Eq. 7-26 is reduced to

$$E = \frac{E_R}{Z_R}\left[Z_R\frac{\epsilon^{j\beta s} + \epsilon^{-j\beta s})}{2} + jR_0\frac{(\epsilon^{j\beta s} - \epsilon^{-j\beta s})}{2j}\right]$$

$$= E_R \cos\beta s + jI_R R_0 \sin\beta s \tag{7-27}$$

A similar derivation may be used for the current on the line, starting with

$$I = \frac{I_R(Z_R + Z_0)}{2Z_0}\left(\epsilon^{j\beta s} - K\epsilon^{-j\beta s}\right) \tag{7-28}$$

and resulting in the current at any point on a dissipationless line as

$$I = I_R \cos\beta s + j\frac{E_R}{R_c}\sin\beta s \tag{7-29}$$

From the definition for velocity of propagation,

$$\beta = \frac{2\pi f}{v}$$

it is seen that

$$\beta = \frac{2\pi}{\lambda}$$

after which the current and voltage expressions may be written

$$E = E_R \cos \frac{2\pi s}{\lambda} + jI_R R_0 \sin \frac{2\pi s}{\lambda} \tag{7-30}$$

$$I = I_R \cos \frac{2\pi s}{\lambda} + j \frac{E_R}{R_0} \sin \frac{2\pi s}{\lambda} \tag{7-31}$$

The voltage or current distribution is then seen as the sum of cosine and quadrature sine distributions. If the line is open-circuited, I_R equals zero, and

$$E_{oc} = E_R \cos \frac{2\pi s}{\lambda} \tag{7-32}$$

$$I_{oc} = \frac{jE_R}{R_0} \sin \frac{2\pi s}{\lambda} \tag{7-33}$$

The voltage and current *magnitude* distributions for an open-circuited line 3/2 wavelengths long are plotted in Fig. 7-5(a). The current and voltage are in quadrature everywhere; thus no power is transmitted along the line.

If the line is short-circuited, E_R equals zero, and Eqs. 7-29 and 7-30 become

$$E_{sc} = jI_R R_0 \sin \frac{2\pi s}{\lambda} \tag{7-34}$$

$$I_{sc} = I_R \cos \frac{2\pi s}{\lambda} \tag{7-35}$$

and these *magnitude* distributions are plotted in Fig. 7-6(e) for a short-circuited line 3/2 wavelengths long. Again the current and voltage are in quadrature, but the current and voltage waves have shifted $\lambda/4$ from the positions for the open-circuit case. It should be noted that although the curves show points of zero voltage or current along the line, because of the losses present on even the best

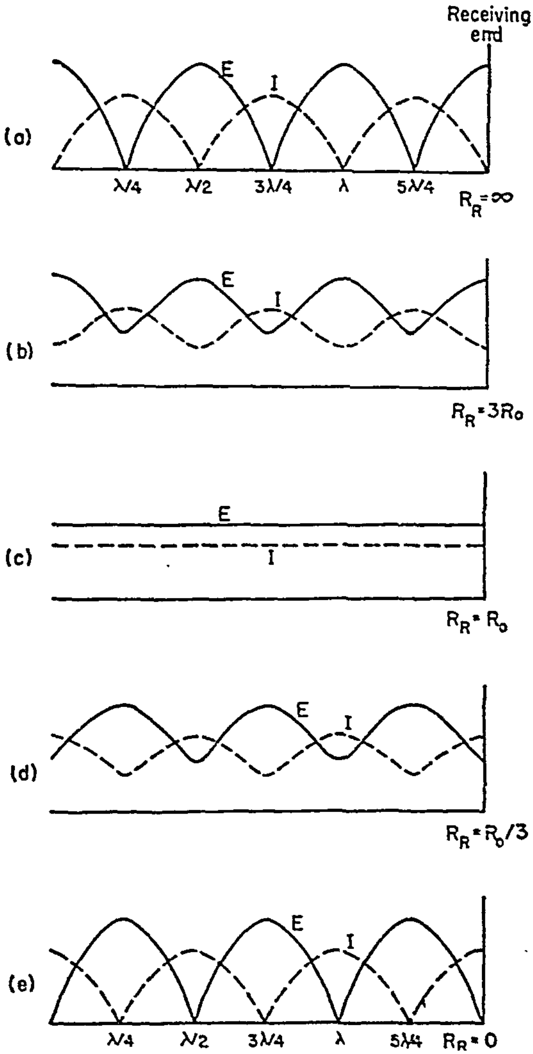


Fig. 7-6. (a) Voltages and currents on an open-circuited dissipationless line; (b) same if load $R_R = 3\ R_0$; (c) same if $R_R = R_0$; (d) same if $R_R = R_0/3$; (e) same if $R_R = 0$, or a short circuit.

lines, the voltages and currents do not reach zero but have small minimum values at the usual zero points.

Returning to Eq. 7-26 for the incident and reflected voltage waves, if the line is terminated in $Z_R = R_0$, the reflection coefficient and reflected wave become zero, and the voltage on the line is expressed by

$$E = E_R \epsilon^{j\beta s} \tag{7-36}$$

which represents a constant voltage magnitude (no attenuation) with continuously varying phase angle along the line. A similar expression may be obtained for the current as

$$I = I_R \epsilon^{j\beta s} \tag{7-37}$$

Such current or voltage distributions are represented in magnitude by the straight horizontal lines in Fig. 7-6(c).

If the line is terminated in a resistance R_R greater than R_0, the reflection coefficient K will be positive and the voltage and current conditions on the line will be intermediate to the open-circuit and R_0-terminated conditions. If, for example, $R_R = 3R_0$, the value of K is $\frac{1}{2}$ and the incident wave has an amplitude twice that of the reflected wave. The resultant voltage and current *magnitudes* are plotted in (b), indicating that there is a finite value of voltage or current at all points on the line, the zeros of (a) being replaced with minima in (b). Since both voltage and current have values other than zero at the load, some power is being transmitted.

If the load condition is made such that $R_R = R_0/3$, the value of K is $-\frac{1}{2}$ and the phase of the reflected wave is reversed. The resultant voltage and current magnitudes are plotted in (d) and appear as in (b) except for the reversed maxima and minima points. Power is again being transmitted.

In general, for resistive loads greater than R_0, the current and voltage distributions somewhat resemble those of an open-circuited line. For resistive loads less than R_0, the distributions take on some of the properties of the short-circuited line.

For loads other than resistance, the reflection coefficient K has an angle, and the reflected wave is shifted in phase with respect to the incident wave. Points of maximum or minimum voltage will not then fall at the end of the line but will be moved back up the line by an amount equal to half the angle of K.

7-5. Standing waves; nodes; standing-wave ratio

If voltage magnitudes are measured along the length of a line terminated in a load other than R_0, the plotted values will appear as in Fig. 7-7. Figure 7-7(a) is drawn for a resistive load of value not equal to R_0, and (b) is the case for either open or short circuit.

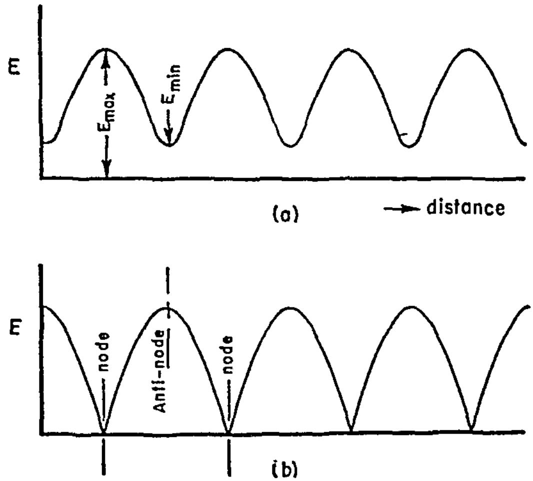


Fig. 7-7. (a) Standing waves on a dissipationless line terminated in a load not equal to R_0; (b) standing waves on a line having open- or short-circuit termination.

Current magnitudes might be plotted and would be similar except for a $\lambda/4$ shift in position of maxima and minima. Maximum and minimum values on a line are labeled as in (a), whereas the nodes and antinodes are indicated in (b). *Nodes* are points of zero voltage or current in the standing wave systems, *antinodes* or loops are points of maximum voltage or current. A line terminated in R_0 has no standing waves, and thus no nodes or loops, and is called a *smooth* line.

For *open circuit*, Fig. 7-6(a) shows that voltage nodes occur at distances $\lambda/4$, $3\lambda/4$, $5\lambda/4$, and so on, from the open end of the line. Under the same conditions, current nodes occur at distances 0, $\lambda/2$, λ, $3\lambda/2$, and so on, from the open termination. For *short circuit*, these nodal points shift by a distance of $\lambda/4$, and voltage nodes occur at 0, $\lambda/2$, λ, and so on, with current nodes at $\lambda/4$, $3\lambda/4$, $5\lambda/4$, and so on. For resistive loads greater than R_0, the voltage and current minima occur at the voltage and current nodal points for an open-circuited line. For resistive loads less than R_0, the voltage and current minima occur at the voltage and current nodal points for a short-circuited line. For pure reactive terminations the standing-wave patterns are similar to those discussed above but are shifted along the line by an angle determined by the angle of the reflection coefficient divided by 2.

The ratio of the maximum to minimum magnitudes of current or voltage on a line having standing waves is called the *standing-wave ratio*, S. That is,

$$S = \left|\frac{E_{\max}}{E_{\min}}\right| = \left|\frac{I_{\max}}{I_{\min}}\right| \tag{7-38}$$

The maxima of voltage along the line occur at points at which the incident and reflected waves are in phase and add directly. From the voltage equation,

$$E = \frac{E_R(Z_R + Z_0)}{2Z_R}(\epsilon^{j\beta s} + K\epsilon^{-j\beta s})$$

at the points where the incident and reflected waves are in phase,

$$E_{\max} = \frac{E_R(Z_R + Z_0)}{2Z_R}(1 + |K|) \tag{7-39}$$

Likewise, the voltage minima occur at points at which the incident and reflected waves are out of phase; thus

$$E_{\min} = \frac{E_R(Z_R + Z_0)}{2Z_R}(1 - |K|) \tag{7-40}$$

The standing-wave ratio then may be defined in terms of the reflection coefficient as

$$S = \frac{1 + |K|}{1 - |K|} \tag{7-41}$$

This relation may be rearranged as

$$|K| = \frac{S - 1}{S + 1} \tag{7-42}$$

$$= \frac{|E_{\max}| - |E_{\min}|}{|E_{\max}| + |E_{\min}|} \tag{7-43}$$

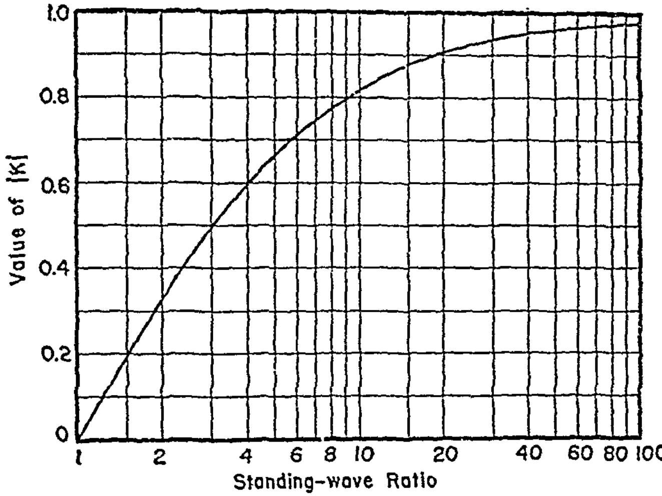


Fig. 7-8. Relation between the standing-wave ratio S and the magnitude of the reflection coefficient.

From Eqs. 7-41 and 7-43 it is possible to calculate values of $|K|$ and S from measurements of maximum and minimum voltages on the line. Exactly similar relations could be derived in terms of the maximum and minimum current values. Figure 7-8 is a plot of Eq. 7-42 that permits obtaining the magnitude of K from a knowledge of the standing-wave ratio.

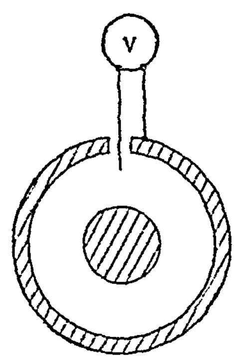


Fig. 7-9. Diagrammatic of a slotted-line section and a probe voltmeter for coaxial line measurements.

Standing-wave-ratio measurements are readily made on open-wire lines. For coaxial lines it is necessary to use a length of line in which a longitudinal slot, one-half wavelength or more long, has been cut. A wire probe is inserted into the air dielectric of the line as a pickup device, a vacuum-tube voltmeter or other detector being connected between probe and sheath as an indicator. If the meter provides linear indications, S is readily determined. If the indicator is nonlinear, corrections must be applied to the readings obtained. A probe voltmeter is indicated diagrammatically in Fig. 7-9.

The same equipment and techniques may also be used to measure the wavelength on the line, the distance between successive voltage or current maxima or minima being equal to a half wavelength. Such measurements when made on an open line are called *Lecher* measurements, after the man who performed many early high-frequency-line experiments.

For the special case of a resistive load, Eq. 7-41 becomes

$$S = \frac{1+|K|}{1-|K|} = \frac{1+\left(\dfrac{R_R - R_0}{R_R + R_0}\right)}{1-\left(\dfrac{R_R - R_0}{R_R + R_0}\right)}$$

$$S = \frac{R_R}{R_0} \quad (\text{for } R_R > R_0) \tag{7-44}$$

$$S = \frac{R_0}{R_R} \quad (\text{for } R_R < R_0) \tag{7-45}$$

7-6. Directional coupler

For direct indication and measurement of standing waves a device known as a *directional coupler* is available. Shown diagrammatically in Fig. 7-10, it consists of a section of coaxial transmission line,

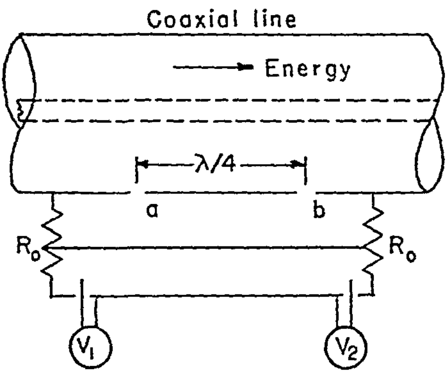


Fig. 7-10. A directional coupler on a coaxial line.

having two small holes in the outer sheath spaced by $\frac{1}{4}$ wavelength. Clamped over these holes is a small section of line, terminated in its R_0 value at both ends to prevent reflections.

Some energy will leak through the holes, and will set up a wave traveling to both left and right in the second line. If the main line is transmitting energy to the right, then a wave entering the secondary line through hole a and traveling to the right will be in phase with and reinforce a wave entering at hole b, setting up a wave traveling to the right in the secondary line.

Energy entering hole a and traveling to the left will be out of phase and will cancel the wave which entered at b and traveled to the left, since the wave reaching a from hole b will have traveled an additional distance of $\lambda/2$.

Thus a wave in the main line traveling to the right will produce a wave traveling to the right in the secondary line, and give an indication on V_2, but will not produce a wave to the left, nor will it give an indication on V_1. In a similar manner, a wave in the main line traveling to the left will give an indication on V_1, but not on V_2.

In the main line, the wave to the right might be an incident wave, and that to the left a reflected wave. The ratio of the indications of V_1 and V_2 will therefore be the ratio of incident to reflected waves in the main line. For a flat main line, with energy travel to the right, V_1 would read zero. The device then gives a continuous indication of the flatness or degree of reflection present in a line and due to any connected load.

7-7. Input impedance of the dissipationless line

The input impedance of a dissipationless line, a useful quantity, may be written from Eqs. 7-27 and 7-29 as

$$Z_s = \frac{E_s}{I_s} = \frac{E_R \cos \beta s + jI_R R_0 \sin \beta s}{I_R \cos \beta s + j\dfrac{E_R}{R_0} \sin \beta s}$$

and since $I_R Z_R = E_R$,

$$Z_s = R_0 \left(\frac{Z_R + jR_0 \tan \beta s}{R_0 + jZ_R \tan \beta s} \right) \tag{7-46}$$

It can be seen that the impedance is complex in general and is periodic with variation of βs, the period being π or $s = \lambda/2$.

Another convenient form for the input impedance may be obtained by use of Eqs. 7-26 and 7-28,

$$Z_s = \frac{E_s}{I_s} = \frac{\dfrac{I_R(Z_R + R_0)}{2} (\epsilon^{j\beta s} + K\epsilon^{-j\beta s})}{\dfrac{I_R(Z_R + R_0)}{2R_0} (\epsilon^{j\beta s} - K\epsilon^{-j\beta s})}$$

$$= R_0 \left(\frac{1 \underline{/\beta s} + |K| \underline{/\phi - \beta s}}{1 \underline{/\beta s} - |K| \underline{/\phi - \beta s}} \right) \tag{7-47}$$

where ϕ is the angle of the reflection coefficient K. This may be further simplified by dividing both numerator and denominator by $1/\underline{\beta s}$, giving

$$Z_s = R_0 \left(\frac{1 + |K|/\underline{\phi - 2\beta s}}{1 - |K|/\underline{\phi - 2\beta s}} \right) \tag{7-48}$$

In effect, the above operation has merely shifted the reference by an angle $-\beta s$.

The expression above in parentheses is shown as a phasor diagram in Fig. 7-11, the numerator and denominator being shown as two

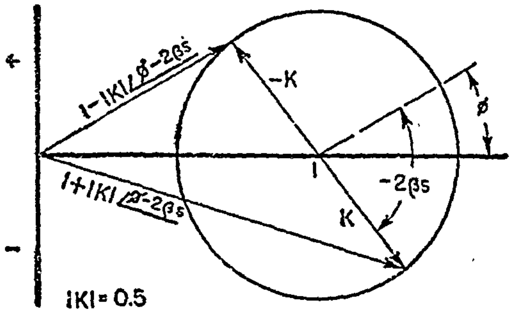


Fig. 7-11. Phasor diagram illustrating Eq. 7-47 with $|K| = 0.5$.

separate phasors, the results of adding unity to plus or minus K. The diagram is arbitrarily drawn for $|K| = 0.5$ or $S = 3$. Several interesting conclusions may be drawn. At values of $s = \phi/2\beta + n\lambda/4$, the numerator and denominator terms of Eq. 7-48 are in phase ($n = 0, 1, 2, 3, \ldots$). At these points the input of the line is purely resistive, with maximum and minimum values occurring every quarter wavelength. In view of the measurement of the angle of the K vector in terms of $2\beta s$, half a revolution represents $\lambda/4$ in terms of wavelength.

The value of the maximum input impedance (resistive), occurring at $s = \phi/2\beta + n\lambda/2$, or with phasors coincident, can be seen to be

$$\begin{aligned} R_{max} &= R_0 \left(\frac{1 + |K|}{1 - |K|} \right) \\ &= SR_0 \end{aligned} \tag{7-49}$$

The minimum input impedance, also resistive, occurring at $s = \phi/2\beta + (2n - 1)\lambda/4$, with phasors again coincident, is

$$R_{\min} = R_0\left(\frac{1 - |K|}{1 + |K|}\right)$$

$$= \frac{R_0}{S} \tag{7-50}$$

7-8. Input impedance of open- and short-circuited lines

The input impedance of a dissipationless line has been obtained in the section above as

$$Z_s = R_0\left(\frac{Z_R + jR_0 \tan \beta s}{R_0 + jZ_R \tan \beta s}\right) \tag{7-51}$$

For a short-circuited line, $Z_R = 0$, so that

$$Z_{sc} = jR_0 \tan \beta s$$

Since $\beta = 2\pi/\lambda$, this equation becomes

$$Z_{sc} = jR_0 \tan \frac{2\pi s}{\lambda} \tag{7-52}$$

The variation of $Z_{sc}/R_0 = X/R_0$ with length of line s may be plotted as in (a), Fig. 7-12.

Before the input impedance of an open-circuited line is determined, Eq. 7-51 should be rearranged as

$$Z_s = R_o\left(\frac{1 + j\dfrac{R_0}{Z_R}\tan \beta s}{\dfrac{R_0}{Z_R} + j\tan \beta s}\right)$$

For an open-circuited line, $Z_R = \infty$, so that

$$Z_{oc} = \frac{-jR_0}{\tan \beta s} = -jR_0 \cot \frac{2\pi s}{\lambda} \tag{7-53}$$

Figure 7-12(b) is a plot of $Z_{oc}/R_0 = X/R_0$ as a function of the length of line s.

It can be seen that the input impedance of either an open- or short-circuited line is a pure reactance. The value of reactance is a repetitive function of length with a period of $s = \lambda/2$ as a result of

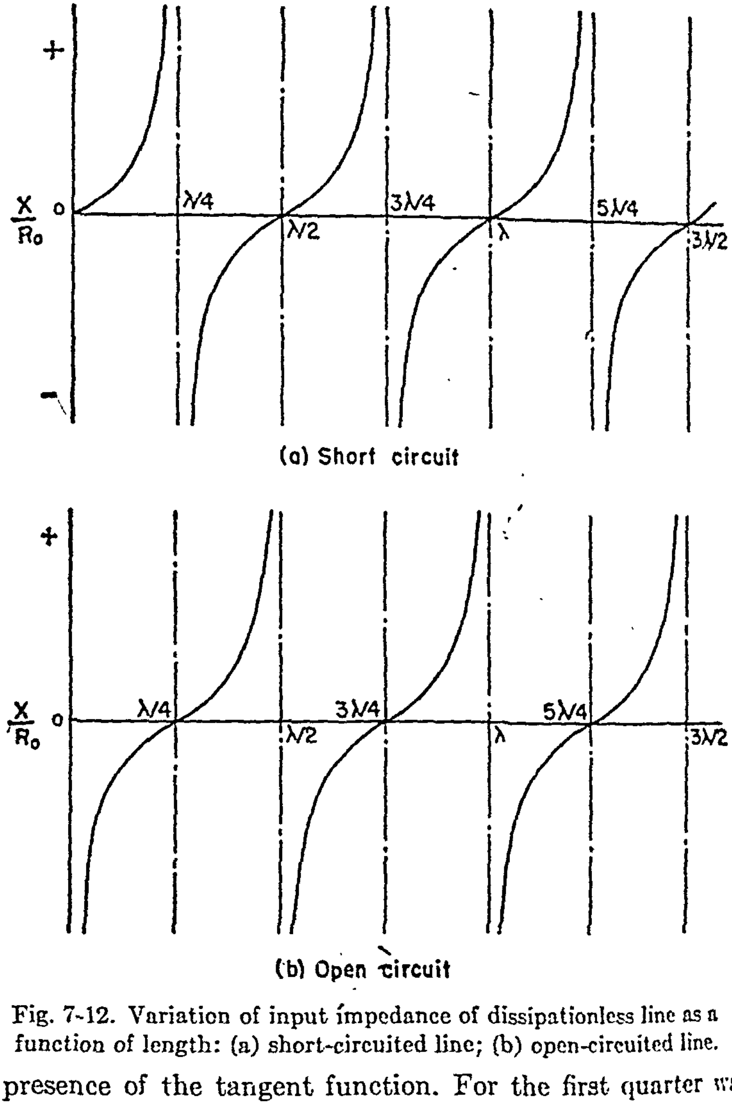


Fig. 7-12. Variation of input impedance of dissipationless line as a function of length: (a) short-circuited line; (b) open-circuited line.

the presence of the tangent function. For the first quarter wavelength, a short-circuited line acts as an inductance, whereas an open-circuited line appears as a capacitance. These reactances reverse each quarter wavelength.

The similarity of performance of open- or short-circuited lines to that of series-resonant or antiresonant circuits may be readily noted by comparison of the curves of Fig. 7-12 with the reactance curves of the resonant circuits of Chapter 2. This similarity suggests the use of lines as reactive circuit elements or as tuned circuits. As can be seen, a line one-quarter wavelength long or less offers all possible values of reactance, either inductive or capacitive. The input of the quarter-wave short-circuited line appears similar to that of a parallel resonant circuit, and the input of the quarter-wave open line as that of a series resonant circuit. It should also be observed that the input impedance of a $\lambda/4$ short-circuited line appears as an infinite reactance, whereas the input impedance of a $\lambda/4$ open-circuited line appears as a zero reactance or a short circuit. However, the curves of Fig. 7-12 are for the ideal dissipationless line. In a practical line there will always be a small resistance component of the input impedance, indicating some power loss; and zero or infinite impedances are never achieved, the actual values tending to minima and maxima. Further quantitative consideration is given to the subject later in this chapter.

7-9. Power and impedance measurement on lines

The methods of Section 7-7 may be used to rewrite the expressions for the voltage and current on the dissipationless line as

$$E = \frac{I_R(Z_R + R_0)}{2} (1 + |K| \underline{/\phi - 2\beta s}) \tag{7-54}$$

$$I = \frac{I_R(Z_R + R_0)}{2R_0} (1 - |K| \underline{/\phi - 2\beta s}) \tag{7-55}$$

It can then be seen that Fig. 7-11 may be redrawn for the voltage and current phasors as in Fig. 7-13, phasors A and B being proportional to E and I, respectively.

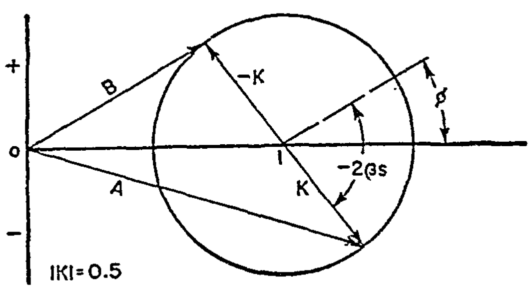


Fig. 7-13. Diagram illustrating Eqs. 7-54 and 7-55.

It has already been reasoned that at a voltage maximum the incident and reflected voltage waves are in phase. This conclusion is confirmed if in Fig. 7-13 it is noted that the line $1\underline{/0°}$ is proportional to the incident voltage and $|K|\underline{/\phi - 2\beta s}$ is proportional to the reflected voltage. Obviously, the in-phase condition is required for E_{max} in Eq. 7-54, so that

$$E_{max} = \frac{I_R|Z_R + R_0|}{2}(1 + |K|) \tag{7-56}$$

Similar reasoning shows that at a current maximum the incident and reflected current waves must also be in phase, so that

$$I_{max} = \frac{I_R|Z_R + R_0|}{2R_0}(1 + |K|) \tag{7-57}$$

Hence it can be seen that

$$\frac{E_{max}}{I_{max}} = R_0 \tag{7-58}$$

Since a change to the values at voltage and current minima requires only the reversal of phase of the reflected waves or a minus sign in front of $|K|$, it can be seen that a similar value for R_0 can be derived as

$$\frac{E_{min}}{I_{min}} = R_0 \tag{7-59}$$

These relations furnish an easy method of measuring R_0 if it cannot be readily computed.

From Eqs. 7-54 and 7-55, as from Fig. 7-13, it may be seen that a voltage maximum and a current minimum occur at the same point on the line. This phenomenon has also been noted in the discussion of standing-wave phenomena. Figure 7-13 also shows that at such a point the current and voltage are in phase; or if the impedance of the line is measured looking toward the load, the impedance seen will be resistive. By writing I_{min} as

$$I_{min} = \frac{I_R|Z_R + R_0|}{2R_0}(1 - |K|)$$

the resistive impedance seen at a voltage loop is

$$\frac{E_{max}}{I_{min}} = R_0\left(\frac{1 + |K|}{1 - |K|}\right) \tag{7-60}$$

$$= SR_0$$

But this is identifiable as R_{max}, by Eq. 7-49. Then

$$\frac{E_{max}}{I_{min}} = R_{max} \tag{7-61}$$

Since the voltage and current are again in phase at a current loop, the resistive impedance seen there may be identified as R_{min}, or

$$\frac{E_{min}}{I_{max}} = \frac{R_0}{S} = R_{min} \tag{7-62}$$

The power passing a voltage loop is the power effectively flowing into a resistance R_{max} at voltage E_{max}, so that

$$P = \frac{E^2_{max}}{R_{max}}$$

The same value of power must also pass the current loop, effectively flowing into a resistance R_{min} at voltage E_{min}, since there is no line dissipation, so that

$$P = \frac{E^2_{min}}{R_{min}}$$

Then
$$P^2 = \frac{E^2_{max}E^2_{min}}{R_{max}R_{min}}$$

and by Eqs. 7-60, 7-61, and 7-62, the expression for power passing along the line becomes

$$P = \frac{|E_{max}| \cdot |E_{min}|}{R_0} \tag{7-63}$$

The power may also be expressed as

$$P = (|I_{max}| \cdot |I_{min}|)R_0 \tag{7-64}$$

These expressions permit easy measurements of power flow on a line of negligible losses.

In many lines, especially those of coaxial construction, the dielectric strength or the voltage breakdown of the line dielectric limits the voltage on the line and thus fixes the maximum power that may be transmitted. For a given maximum line voltage, Eq. 7-63 shows that the greatest amount of power will be transmitted if $|E_{max}| = |E_{min}|$, or the line is operated as a smooth line without standing waves and with an R_0 termination. As will be shown in the next section, it is also advisable for the greatest transfer of power from line to load to operate the line with an R_0 termination, provided the line is sufficiently long that the output image impedance is substantially R_0.

The unknown value of a load impedance Z_R connected to a transmission line may be determined by standing-wave measurements on the open-wire or slotted line. A line may thus be used in lieu of some form of bridge circuit for measuring an unknown impedance. The characteristic impedance R_0 of the line must be known or calculated, and measurements must be made of the standing-wave ratio S and the distance s' from the load to the nearest point of voltage minimum.

At the point of voltage minimum it has been shown that

$$Z_s = R_{\min} = \frac{R_0}{S}$$

At any point on a line,

$$Z_s = R_0 \left[\frac{Z_R + jR_0 \tan (2\pi s/\lambda)}{R_0 + jZ_R \tan (2\pi s/\lambda)} \right]$$

so that at the point of voltage minimum, distant s' from the load,

$$\frac{R_0}{S} = R_0 \left[\frac{Z_R + jR_0 \tan (2\pi s'/\lambda)}{R_0 + jZ_R \tan (2\pi s'/\lambda)} \right]$$

Solution for Z_R gives

$$Z_R = R_0 \left[\frac{1 - jS \tan (2\pi s'/\lambda)}{S - j \tan (2\pi s'/\lambda)} \right] \tag{7-65}$$

as the value of the connected load impedance.

The point of a voltage minimum is measured rather than a voltage maximum because it is usually possible to determine the exact point of minimum voltage with greater accuracy.

7-10. Reflection losses on the unmatched line

As has been discussed in Section 6-15, if a line is not matched to its load, the energy delivered by the line to the load is less than if the impedances are properly adjusted. This effect is considered as due to reflection at the junction and makes its presence known by establishment of a reflected wave and a standing-wave system. The voltage at a maximum voltage point is due to the in-phase sum of the incident and reflected waves, so that from Eq. 7-56:

$$|E_{\max}| = |E_i| + |E_r| = \frac{|I_R(Z_R + R_0)|}{2}(1 + |K|)$$

The minimum voltage is due to the difference of the incident and reflected waves and is

$$|E_{\min}| = |E_i| - |E_r| = \frac{|I_R(Z_R + R_0)|}{2}(1 - |K|)$$

Hence the standing wave ratio is

$$S = \frac{|E_{\max}|}{|E_{\min}|} = \frac{|E_i| + |E_r|}{|E_i| - |E_r|} \tag{7-66}$$

Use of these results and Eq. 7-63 gives for the total power transmitted along the line and delivered to the load

$$\begin{aligned} P &= \frac{|E_{\max}| \cdot |E_{\min}|}{R_0} \\ &= \frac{(|E_i| + |E_r|)(|E_i| - |E_r|)}{R_0} = \frac{|E_i|^2 - |E_r|^2}{R_0} \end{aligned} \tag{7-67}$$

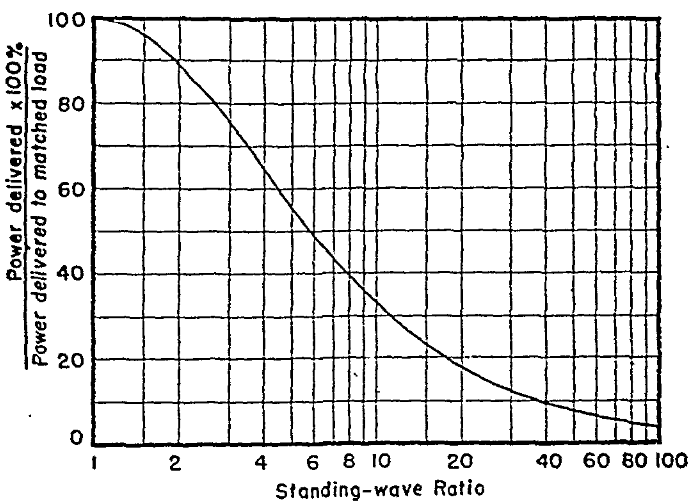


Fig. 7-14. Reflection losses as a function of standing-wave ratio.

But this expression is easily recognized as the difference of two power flows, one being the power P_i transmitted in the incident wave, the other being the power P_r traveling back in the reflected wave.

The ratio of the power P delivered to the load to the power transmitted by the incident wave is

$$\frac{P}{P_i} = \frac{P_i - P_r}{P_i} = \frac{|E_i|^2 - |E_r|^2}{|E_i|^2} = 1 - \frac{|E_r|^2}{|E_i|^2} = 1 - |K|^2$$

$$= 1 - \left(\frac{S-1}{S+1}\right)^2 = \frac{4S}{(S+1)^2} \tag{7-68}$$

in view of Eq. 7-42. The ratio of power absorbed by the load to the power transmitted is plotted as a function of S in Fig. 7-14, further illustrating the desirability of operating a line with a Z_0 termination to eliminate reflection, since with proper termination a line has its greatest power capacity.

7-11. The eighth-wave line

The input impedance of a line of length $s = \lambda/8$ is

$$Z_s = R_0 \left[\frac{Z_R + jR_0 \tan(\pi/4)}{R_0 + jZ_R \tan(\pi/4)}\right]$$

$$= R_0 \left(\frac{Z_R + jR_0}{R_0 + jZ_R}\right) \tag{7-69}$$

If such a line is terminated in a pure resistance R_R,

$$Z_s = R_0 \left(\frac{R_R + jR_0}{R_0 + jR_R}\right)$$

and the numerator and denominator have identical *magnitudes*, so that

$$|Z_s| = R_0 \tag{7-70}$$

Thus an eighth-wave line may be used to transform any resistance to an impedance with a magnitude equal to R_0 of the line, or to obtain a magnitude match between a resistance of any value and a source of R_0 internal resistance.

7-12. The quarter-wave line; impedance matching

The expression for the input impedance of a dissipationless line may be rearranged as

$$Z_s = R_0 \left[\frac{\dfrac{Z_R}{\tan(2\pi s/\lambda)} + jR_0}{\dfrac{R_0}{\tan(2\pi s/\lambda)} + jZ_R} \right]$$

If the line is made a quarter-wave long, or $s = \lambda/4$,

$$Z_s = \frac{R_0^2}{Z_R} \tag{7-71}$$

That is, the input impedance of the line is equal to the square of R_0 of the line divided by the load impedance. A quarter-wave section of line may be thought of as a transformer to match a load of Z_R ohms to a source of Z_s ohms. Such a match can be obtained if the characteristic impedance R_0' of the matching quarter-wave section of line is properly chosen according to

$$R_0' = |\sqrt{Z_s Z_R}| \tag{7-72}$$

The R_0' of the matching section should thus be equal to the geometric mean of the source and load impedances.

A quarter-wave line may be considered as an impedance inverter in that it can transform a low impedance into a high impedance and vice versa. This effect is illustrated by the action of the $\lambda/4$ short-circuited line in transforming the zero impedance short-circuit termination to an apparent open circuit, and of the open-circuited $\lambda/4$ line in transforming the open circuit termination to a low value or an apparent short circuit.

An important application of the quarter-wave matching section is to couple a transmission line to a resistive load such as an antenna. The quarter-wave matching section then must be designed to have a characteristic impedance R_0' so chosen that the antenna resistance R_A is transformed to a value equal to the characteristic impedance R_0 of the transmission line. The line then is terminated in its R_0 and is operated under conditions of no reflection. The characteristic impedance R_0' of the matching section then should be

$$R_0' = \sqrt{R_A R_0} \tag{7-73}$$

It is interesting to note that the value of R_0', the characteristic impedance of the matching section, is just the value required to achieve critical coupling and maximum power transfer from the transmission line to the load, as indicated by the coupled-circuit theory of Chapter 3.

In cases where physical spacing is greater than can be reached with a line a quarter wave in length the same transformation can be produced by a line three quarter waves long or a line any odd number of quarter waves in length. Greater lengths reduce the efficiency slightly due to increased losses in a line of practical construction.

As a practical matter, the range of values of R_A and R_0 that can be matched satisfactorily is limited roughly to about 10 to 1. The

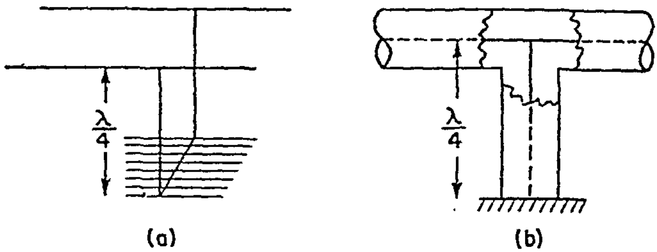


Fig. 7-15. Quarter-wave lines as insulators.

transformer is also a single-frequency or narrow-band device. The band width may be increased by using two or more quarter-wave sections in series, each accomplishing part of the total transformation.

A quarter-wave transformer may also be used if the load is not pure resistance. It should then be connected between points corresponding to I_{max} or E_{min}, at which places the transmission line has resistive impedances given by R_0/S or SR_0. For step down in impedance from the line value of R_0 the matching transformer characteristic impedance should be

$$R_0' = \sqrt{R_0 \frac{R_0}{S}} = R_0 \sqrt{\frac{1}{S}} \tag{7-74}$$

Another application of the short-circuited quarter-wave line is as an insulator to support an open-wire line or the center conductor of a coaxial line. This application, illustrated in Fig. 7-15, makes use of the fact that the input impedance of a quarter-wave shorted line is very high. Such lines are sometimes referred to as *copper insulators*. They will be given further study later in this chapter.

7-13. The half-wave line

When a length of line having $s = \lambda/2$ is used, the input impedance is

$$Z_s = R_0 \left(\frac{Z_R + jR_0 \tan \pi}{R_0 + jZ_R \tan \pi} \right)$$

$$Z_s = Z_R \tag{7-75}$$

This result is obvious, since the conditions on the line have a period of π, or one-half wavelength.

A half wavelength of line may then be considered as a one-to-one transformer. It has its greatest utility in connecting a load to a source in cases where the load and source cannot be made adjacent. A group of capacitors may be placed in parallel by connecting them with sections of line n half waves in length. As a result, insulators on a high-frequency line should not be spaced at half-wave intervals, since their effect would then be cumulative, lowering the insulation resistance of the line.

7-14. The exponential line for impedance transformation

The characteristic impedance of a line is a function of the spacing and size of the conductors among other factors. If the spacing of a dissipationless line is made to vary in a uniform manner along the length of a line, R_0 will likewise vary along the line. Using the image-impedance concept, such a tapered transmission line would not be symmetrical, the image impedance at the sending end differing from that at the load end, for reasonable electrical line lengths. This variation indicates the possibility of lines of tapered spacing being used as matching sections between a line and a load. If such a dissipationless line were matched on an image basis at both ends, it could serve as a magnitude matching transformer between some impedance Z_1 and some other load impedance Z_R.

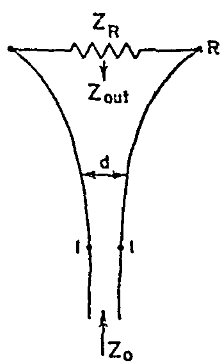


Fig. 7-16. The exponential-line impedance transformer.

While any arbitrary taper may be employed, an exponential variation of parameters may be most readily handled mathematically. Assume that the line parameters are made to vary in such a way that

$$L = L_1\epsilon^{2\theta s} \tag{7-76}$$

where L is the inductance per meter of line at any distance s from the source at point 1, Fig. 7-16, L_1 is the inductance per meter at the sending end, and θ is a *transformation function* to be defined. From the properties of lines it is apparent that

$$C = C_1\epsilon^{-2\theta s} \tag{7-77}$$

where C and C_1 are line capacitances per meter at distance s and the sending end, respectively. The line spacing is small at the sending end and large at the load for positive taper or positive θ.

It is desired that this line operate without internal reflections. It was mentioned in Section 6-10 that, for a dissipationless line operating without reflection, the energy in each of the electric and magnetic fields was maintained constant, but that if a variation of line constants (Z_0) caused an exchange of energy between the fields, a reflection was set up. It is obvious in the present case that a variation of line constants does exist. If, however, the line parameters are varied in such a way that the energy in the fields can be forced to remain constant (dissipationless line) and a wave propagated, no reflection will be produced.

If the energy in the magnetic field at any point is equated to that at the sending end as a requirement of zero reflection,

$$\frac{LI^2}{2} = \frac{L_1I_1^2}{2}$$

Likewise, for electric field energy,

$$\frac{CE^2}{2} = \frac{C_1E_1^2}{2}$$

Use of these statements and Eqs. 7-76 and 7-77 gives the following as the required manner of current and voltage variation on the exponentially tapered line:

$$I = I_1\epsilon^{-\theta s}, \qquad E = E_1\epsilon^{\theta s}$$

Introducing a phase constant β for a moving wave gives

$$E = E_1\epsilon^{\theta s}\epsilon^{j\beta s} = E_1\epsilon^{(\theta+j\beta)s} \tag{7-78}$$

$$I = I_1^{-\theta s}\epsilon^{j\beta s} = I_1\epsilon^{-(\theta-j\beta)s} \tag{7-79}$$

These voltage and current relations indicate inverse variation of current and voltage with length of line s. For positive θ the voltage increases and the current decreases along the line, transformed in an exponential manner from element to element.

The variation of voltage and current along the line with the above distributions is

$$\frac{dE}{ds} = (\theta + j\beta)E_1\epsilon^{(\theta+j\beta)s} \tag{7-80}$$

$$\frac{dI}{ds} = -(\theta - j\beta)I_1\epsilon^{-(\theta-j\beta)s} \tag{7-81}$$

If a propagating wave exists on the lossless tapered line, the line differential equations must be satisfied or

$$\frac{dE}{ds} = -ZI = -j\omega LI$$

$$\frac{dI}{ds} = -YE = -j\omega CE$$

s being measured from the sending end, and causing the negative signs. Forcing the line current and voltage to satisfy these traveling wave relations, thereby ensuring the presence of the desired propagating wave, gives

$$(\theta + j\beta)E_1\epsilon^{(\theta+j\beta)s} = -j\omega LI = -j\omega L_1\epsilon^{2\theta s}I_1\epsilon^{-(\theta-j\beta)s}$$

$$-(\theta - j\beta)I_1\epsilon^{-(\theta-j\beta)s} = -j\omega CE = -j\omega C_1\epsilon^{-2\theta s}E_1\epsilon^{(\theta+j\beta)s}$$

and cancellation of the exponentials leads to

$$(\theta + j\beta)E_1 = -j\omega L_1I_1 \tag{7-82}$$

$$-(\theta - j\beta)I_1 = -j\omega C_1E_1 \tag{7-83}$$

Eliminating E_1 and I_1 from the above gives

$$\theta^2 + \beta^2 = \omega^2L_1C_1$$

from which it is found that the phase constant β for the propagating wave must have the value

$$\beta = \pm\sqrt{\omega^2L_1C_1 - \theta^2} \tag{7-84}$$

The phase constant β may then be either real or imaginary. For large values of ω or high frequencies, β is real, and the voltage and current propagate without attenuation although they may be transformed in magnitude because of the presence of θ, the transformation function. The positive sign for the incident wave will be chosen, the negative sign merely indicating the possibility of a reflected or reverse-propagating wave.

The above expression can be recognized as the usual form for propagation on a lossless line, modified by the function θ^2. The value of β reduces to that of the lossless line if θ be made small or the rate of taper very gradual.

For large rates of taper, where $\theta^2 > \omega^2 L_1 C_1$, the phase constant β becomes imaginary or

$$\beta = j\sqrt{\theta^2 - \omega^2 L_1 C_1}$$

and the voltage and current expressions are

$$E = E_1 \epsilon^{\theta s} \epsilon^{-\beta s}, \qquad I = I_1 \epsilon^{-\theta s} \epsilon^{-\beta s}$$

indicating attenuation of both voltage and current due to the $\epsilon^{-\beta s}$ factors. The exponentially tapered line then has properties similar to those of a high-pass filter, with the cutoff frequency f_c occurring for β equal to zero or

$$f_c = \frac{\theta}{2\pi\sqrt{L_1 C_1}} \tag{7-85}$$

This indicates that there is a limiting rate of taper for a given frequency, beyond which the tapered line will not propagate a wave, or acts as a discontinuity. Physically another limit is imposed by the fact that the greatest rate of taper is reached when the two wires are directed oppositely.

The impedance at any point along the line is the ratio of E to I at that point or

$$Z = \frac{E}{I} = \frac{E_1 \epsilon^{\theta s}}{I_1 \epsilon^{-\theta s}} = Z_1 \epsilon^{2\theta s} \tag{7-86}$$

If Z_R is the load impedance to be matched, then

$$\left|\frac{Z_R}{Z_1}\right| = \epsilon^{2\theta s}$$

from which it is possible to develop a definition for θ as

$$\theta = \frac{1}{2s} \ln \left|\frac{Z_R}{Z_1}\right| \tag{7-87}$$

where Z_R is the load and Z_1 the source or uniform line impedance to which the load is to be matched. Thus the desired rate of taper, for a given length of line, may be determined.

It is also possible to show that

$$\left|\frac{E_R}{E_1}\right| = \sqrt{\left|\frac{Z_R}{Z_1}\right|} \tag{7-88}$$

which indicates that the exponential line is analogous to the ideal transformer.

The impedance at any point on the line may be found from Eq. 7-86 and the fact that

$$\sqrt{\frac{L}{C}} = \sqrt{\frac{L_1}{C_1}}\,\epsilon^{2\theta_s}$$

so

$$Z = \sqrt{\frac{L}{C}}\left(\sqrt{1 - \frac{f_c^2}{f^2}} + \frac{jf_c}{f}\right) \tag{7-89}$$

where L and C are per meter values of a small section of the line at the point under consideration. For pass-band frequency values which make $f \gg f_c$, this reduces to $Z = \sqrt{L/C} = Z_0$ of a dissipationless line, so that best operation will occur at frequencies well above cutoff value.

It is preferable that the tapered section be one-half to one wavelength long at the operating frequency and it is found that with this length a linear rate of taper may be employed without appreciable reflection. This simplifies the construction problem. A linear taper implies that Eq. 7-87 has been written by use of the logarithmic series and the first term taken. For small ratios of impedance transformation this is a good approximation.

The exponential line may be designed by use of Eq. 7-76 and the relation for parallel wire lines, as

$$\frac{L}{L_1} = \frac{\ln (d/a)}{\ln (d_1/a_1)} = \epsilon^{2\theta_s} = \left|\frac{Z_R}{Z_1}\right|$$

$$\frac{d}{a} = \left(\frac{d_1}{a_1}\right)^{\left|\frac{Z_R}{Z_1}\right|} \tag{7-90}$$

for the d/a ratio at the load end of the tapered line.

In construction of open-wire or coaxial lines it is necessary to avoid discontinuities or sudden changes in mechanical dimensions or construction, otherwise distortions will occur in the electric and magnetic fields present, and these will set up reflections. The tapered section provides a method of making transitions between line sizes and configurations without serious reflections.

7-15. Single-stub impedance matching on a line

For greatest efficiency and delivered power, a high-frequency transmission line should be operated as a smooth line or with an R_0 termination. However, the usual loads, such as antennas, do not in

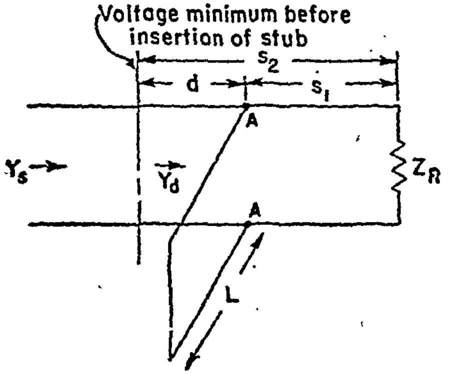


Fig. 7-17. Location of the single stub for impedance matching.

general have resistances of value equal to R_0, so that in many cases it is necessary to introduce some form of an impedance-transforming section between line and load to make the load appear to the line as a resistance of value R_0. The quarter-wave line or transformer and the tapered line are such impedance-matching devices. Another means of accomplishing the desired result is the use of an open or closed stub line of suitable length as a reactance shunted across the transmission line at a designated distance from the load, to tune the length of line and the load to resonance with an antiresonant resistance equal to R_0. The arrangement is as in Fig. 7-17.

The theory of the method may be easily stated in general terms. Since the input conductance of a line is $1/SR_0$ at a voltage maximum and S/R_0 at a voltage minimum, then at some intermediate point A the real part of the input admittance may have an intermediate value of $1/R_0$, or the input admittance at A has a value

$$Y_s = \frac{1}{R_0} \pm jB \tag{7-91}$$

The susceptance B is the shunt value at the point in question. After the point having a conductance equal to $1/R_0$ is located, a short stub line having input susceptance of $\mp B$ may be connected across the transmission line. The input admittance at this point then is

$$Y_s = \frac{1}{R_0} \pm jB \mp jB = \frac{1}{R_0}$$

or the input impedance of the transmission line at point A looking toward the load is

$$Z_s = R_0$$

The line from the source to A is then terminated in R_0 and is a smooth line. From A to the load, reflection and standing waves occur; but since this distance can always be made less than a wavelength, the losses are not severe.

Since both the location and length of the stub must be determined, two independent measurements must be made on the original line and load to secure sufficient data. The most easily obtained measurements are the standing-wave ratio S and the position of a voltage minimum, usually the minimum nearest to the load. A voltage minimum is chosen rather than a maximum, since its position usually can be determined more accurately. If the location of the stub is fixed with respect to an original voltage minimum, no knowledge of the load impedance is needed.

Because of the paralleling of elements, it is most convenient to work with admittances. From Eq. 7-48 the input admittance Y_s, looking toward the load from any point on the line, may be written as

$$Y_s = \frac{1}{R_0}\left(\frac{1 - |K|\underline{/\phi - 2\beta s}}{1 + |K|\underline{/\phi - 2\beta s}}\right) \tag{7-92}$$

Writing $G_0 = 1/R_0$ and changing to rectangular coordinates gives

$$Y_s = G_0\left[\frac{1 - |K|\cos(\phi - 2\beta s) - j|K|\sin(\phi - 2\beta s)}{1 + |K|\cos(\phi - 2\beta s) + j|K|\sin(\phi - 2\beta s)}\right]$$

and upon rationalizing,

$$Y_s = G_0\left[\frac{1 - |K|^2 - 2j|K|\sin(\phi - 2\beta s)}{1 + |K|^2 + 2|K|\cos(\phi - 2\beta s)}\right]$$

Expressing the shunt conductance as a dimensionless ratio G_s/G_0, or on a *per unit* basis,

$$\frac{G_s}{G_0} = \left[\frac{1 - |K|^2}{1 + |K|^2 + 2|K| \cos (\phi - 2\beta s)}\right] \tag{7-93}$$

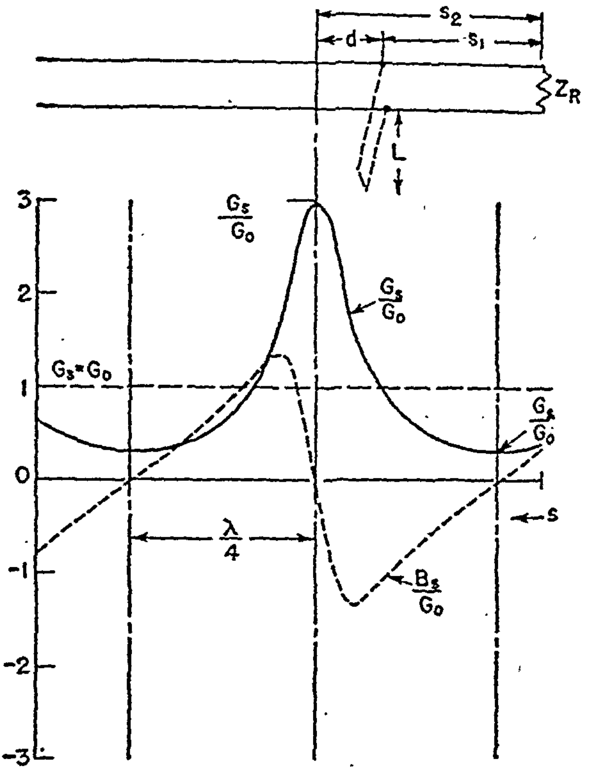


Fig. 7-18. Admittance conditions on a line indicating the proper location of the stub for $|K| = 0.5$.

and the shunt susceptance on a *per unit* basis is

$$\frac{B_s}{G_0} = \left[\frac{-2|K| \sin (\phi - 2\beta s)}{1 + |K|^2 + 2|K| \cos (\phi - 2\beta s)}\right] \tag{7-94}$$

Equations 7-93 and 7-94 are plotted in Fig. 7-18 for a value of $|K|$ arbitrarily chosen as 0.5. The value of G_s/G_0 can be seen as having a maximum. Inspection of Eq. 7-93 shows that this maximum occurs for the value s_2 at which the cosine term is -1, or

$$\phi - 2\beta s_2 = -\pi$$

$$s_2 = \frac{\phi + \pi}{2\beta} \tag{7-95}$$

The distance s_2 is identified in Fig. 7-18. At this distance from the load,

$$\begin{aligned} \frac{G_s}{G_0} &= \frac{1 - |K|^2}{1 + |K|^2 - 2|K|} \\ &= \frac{1 + |K|}{1 - |K|} = S \end{aligned} \tag{7-96}$$

Since this equation states that $R_s = R_0/S$, the point of maximum G_s/G_0 is recognized as a point of *minimum voltage*, at a distance s_2 from the load.

At a distance s_1 from the load it can be seen in Fig. 7-18 that $G_s = G_0$. This is the point at which the stub is to be connected. The value of G_s/G_0 is unity there, so that from Eq. 7-93,

$$1 = \frac{1 - |K|^2}{1 + |K|^2 + 2|K| \cos(\phi - 2\beta s_1)}$$

from which $\quad \cos(\phi - 2\beta s_1) = -|K|$

Since $\quad \cos^{-1}(-|K|) = -\pi + \cos^{-1}|K|$

$$s_1 = \frac{\phi + \pi - \cos^{-1}|K|}{2\beta} \tag{7-97}$$

Hence the distance d from the voltage minimum to the point of stub connection is

$$d = s_2 - s_1$$

which, from Eqs. 7-95 and 7-97, is

$$\begin{aligned} d &= \frac{\cos^{-1}|K|}{2\beta} \\ &= \frac{\cos^{-1}\left(\dfrac{S - 1}{S + 1}\right)\dfrac{\lambda}{4}}{\pi} \end{aligned} \tag{7-98}$$

The stub should then be located at this distance d measured in either direction from a voltage minimum. Ordinarily the stub is placed on the load side of that minimum which is nearest to the load.

The input susceptance of the line at the stub location nearest the load can be obtained from Eqs. 7-94 and 7-97 as

$$B_s = G_0\left[\frac{-2|K| \sin (\pi + \cos^{-1}|K|)}{1 + |K|^2 + 2|K| \cos (\pi + \cos^{-1}|K|)}\right]$$

For an angle whose cosine is $|K|$, the sine is $\sqrt{1 - |K|^2}$, so that

$$B_s = G_0\left(\frac{2|K|\sqrt{1 - |K|^2}}{1 - |K|^2}\right) \tag{7-99}$$

The susceptance of the stub required to cancel the line susceptance must be the negative of B_s. The susceptance of a short-circuited stub is

$$B_{sc} = -G_0 \cot \beta L$$

where L is the length of the short-circuited stub. If stub and line have equal G_0, then

$$\frac{G_0}{\tan \beta L} = G_0\left(\frac{2|K|\sqrt{1 - |K|^2}}{1 - |K|^2}\right)$$

Then
$$L = \frac{\lambda}{2\pi} \tan^{-1} \frac{\sqrt{1 - |K|^2}}{2|K|} \tag{7-100}$$

By use of the standing-wave ratio existing *before connection of the stub*, this equation may be conveniently expressed as

$$L = \frac{\lambda}{2\pi} \tan^{-1} \frac{\sqrt{S}}{S - 1} \tag{7-101}$$

This is the length of the short-circuited stub to be placed d meters toward the load from a point at which a voltage minimum existed before attachment of the stub. The susceptance of the line at d is then canceled, and the line appears to be terminated in a resistance of value R_0 at that point; it will be a smooth line between the generator and the point of connection of the stub.

It is also possible to place the stub d meters toward the source from the voltage minimum. The sign of the reactance is reversed on that side with respect to the sign for the location nearer the load, as shown by Fig. 7-18. The stub length L' then should be

$$L' = \frac{\lambda}{2} - L \tag{7-102}$$

for a short-circuited stub.

A short-circuited stub is ordinarily preferred to an open-circuited stub because of greater ease in construction and because of the inability to maintain high enough insulation resistance at the open-circuit point, to ensure that the stub is really open-circuited. A shorted stub also has a lower loss of energy due to radiation, since the short circuit can be definitely established with a large metal plate, effectively stopping all field propagation. This is especially true for a coaxial line where the short-circuiting plate can be made to seal the line completely.

7-16. The circle diagram for the dissipationless line

The diagram of Fig. 7-11 has certain inherent properties valuable in a qualitative discussion of line phenomena, but it does not present a quantitative impedance answer. A somewhat similar circle diagram may be obtained, however, that solves the impedance equation and simplifies the design of dissipationless lines considerably. The input-impedance equation for a dissipationless line may be written on a per unit basis as

$$\frac{Z_s}{R_0} = \frac{1 + |K|\underline{/\phi - 2\beta s}}{1 - |K|\underline{/\phi - 2\beta s}} \tag{7-103}$$

In terms of Z_s/R_0 the equation is applicable to all lines, regardless of their characteristic impedance values. Since Z_s/R_0 is complex, it is possible to write

$$\frac{Z_s}{R_0} = r_a + jx_a \tag{7-104}$$

where r_a and x_a are values of resistance or reactance *per unit* of R_0. Then

$$r_a + jx_a = \frac{1 + |K|\underline{/\phi - 2\beta s}}{1 - |K|\underline{/\phi - 2\beta s}} \tag{7-105}$$

The most easily measured line quantity is the standing-wave ratio S, so that it is convenient to replace $|K|$ by its equivalent in terms of S, or

$$(r_a + 1 + jx_a)\left(\frac{S-1}{S+1}\right)\underline{/\phi - 2\beta s} = r_a - 1 + jx_a \tag{7-106}$$

Equating the squares of the magnitudes and after clearing of fractions, this becomes

$$r_a^2 - r_a \frac{(S^2 + 1)}{S} + x_a^2 = -1$$

By adding a term to complete the square, there results

$$r_a^2 - \frac{2r_a(S^2 + 1)}{2S} + \left(\frac{S^2 + 1}{2S}\right)^2 + x_a^2 = \left(\frac{S^2 + 1}{2S}\right)^2 - 1$$

from which can be obtained

$$\left[r_a - \left(\frac{S^2 + 1}{2S}\right)\right]^2 + x_a^2 = \left(\frac{S^2 - 1}{2S}\right)^2 \qquad (7\text{-}107)$$

This is an equation of the form

$$(x - c)^2 + y^2 = r^2$$

which is recognizable as that of a family of circles of radius r and with centers shifted c units from the origin on the positive x axis.

An actual circle will then have a radius

$$r = \frac{S^2 - 1}{2S} = \frac{S - \frac{1}{S}}{2} \qquad (7\text{-}108)$$

and a shift of the center of the circle on the positive r_a axis (abscissa)

$$c = \frac{S^2 + 1}{2S} = \frac{S + \frac{1}{S}}{2} \qquad (7\text{-}109)$$

A family of circles may be drawn for successive values of S as in Fig. 7-19. In drawing particular circles it is interesting to note that for any circle the intercept near the origin is at $1/S$, and that far removed from the origin is at S units on the r_a axis.

Since the minimum value for S is unity, Eq. 7-109 shows that all S circles must surround the 1,0 point. In fact, the circle for $S = 1$ is represented by the 1,0 point. The maximum value of S is infinity, for the case of open-circuit or short-circuit line termination. As S increases, the radius of the S circle increases, and the center moves to the right; for the limiting case of S = infinity, the circle becomes the x_a axis.

Hence a given constant-S circle represents all possible values of

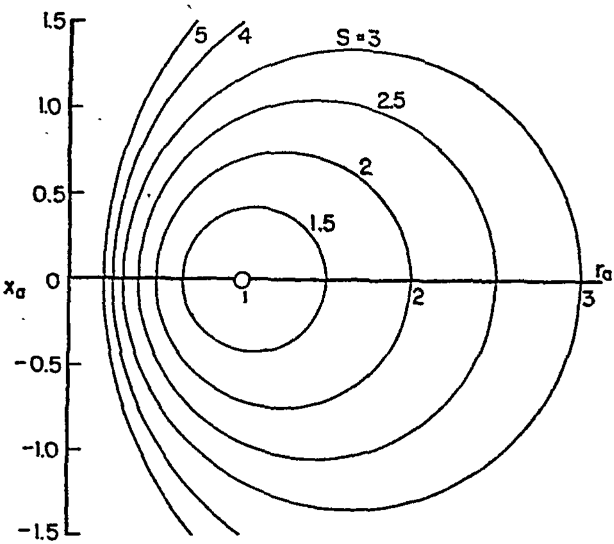


Fig. 7-19. A family of constant-S circles.

r_a and x_a for a given value of Z/R_0. The line from the origin to a given point on the circle represents Z_s/R_0 in both magnitude and angle, with real and reactive components r_a and jx_a, respectively. When Z_s/R_0 lies on the abscissa with magnitude S, the line impedance has a maximum value, and

$$\frac{Z_s}{R_0} = S = \frac{1 + |K|}{1 - |K|} \tag{7-110}$$

so that by comparison with Eq. 7-103 it is seen that

$$\phi - 2\beta s = 0$$

Thus the point $r_a = S$, $x_a = 0$ on the S circle represents a resistive line impedance at a voltage maximum. This point is chosen as the $\beta s = 0$ condition by convention.

When Z_s/R_0 terminates at the circle intercept $1/S$, the line impedance has a minimum value, and

$$\frac{Z_s}{R_0} = \frac{1}{S} = \frac{1 - |K|}{1 + |K|} \tag{7-111}$$

from which it can be reasoned that for a resistive load ($\phi = 0$),

$$\beta s = -\frac{\pi}{2}$$

Thus, moving through $\beta s = \pi/2$ radians back along the line has caused the tip of the impedance vector to travel over a *distance* on the circle of $-\pi$ radians. Hence it is seen advisable to place a βs scale on the S circles. The βs scale increases *clockwise*, or in the direction of increasing negative angles.

Rewriting Eq. 7-106 as

$$\left(\frac{S-1}{S+1}\right) \underline{/\phi - 2\beta s} = \frac{r_a - 1 + jx_a}{r_a + 1 + jx_a}$$

and rationalizing the right side gives

$$\left(\frac{S-1}{S+1}\right) \underline{/\phi - 2\beta s} = \frac{r_a^2 - 1 + x_a^2 + j2x_a}{(r_a + 1)^2 + x_a^2} \qquad (7\text{-}112)$$

The angle ϕ may be made zero in order that the βs scale may start at 0° on the abscissa. Then, equating the tangents of the angles in Eq. 7-112,

$$\tan(-2\beta s) = \frac{2x_a}{r_a^2 - 1 + x_a^2} \qquad (7\text{-}113)$$

$$r_a^2 - 1 + x_a^2 + \frac{2x_a}{\tan 2\beta s} = 0$$

After the square has been completed, this may be written

$$r_a^2 + \left(x_a + \frac{1}{\tan 2\beta s}\right)^2 = 1 + \frac{1}{\tan^2 2\beta s}$$

$$= \frac{1}{\sin^2 2\beta s} \qquad (7\text{-}114)$$

Lines of equal βs value are then seen to be circles of radius

$$= \frac{1}{\sin 2\beta s} \qquad (7\text{-}115)$$

with a shift of center downward on the x_a axis (ordinate)

$$= \frac{1}{\tan 2\beta s} \qquad (7\text{-}116)$$

A family of such circles is drawn in Fig. 7-20. All the βs circles pass through the point $r_a = 1$, $x_a = 0$.

Superposition of the βs circles on the S circles provides a scale of βs angles and results in the circle diagram of Fig. 7-21. Although the constant-βs circles are computed from Eqs. 7-116 and 7-115 in

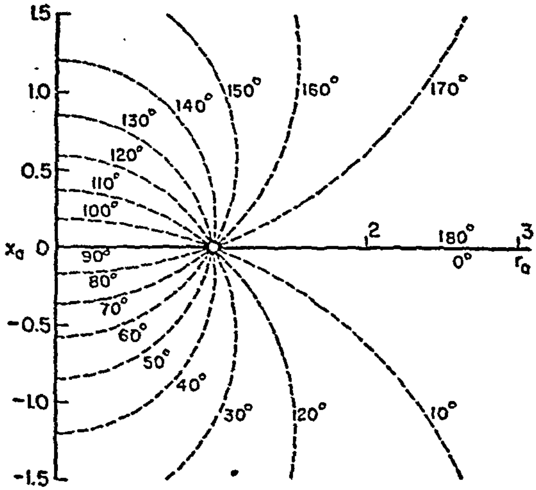


Fig. 7-20. A family of constant-βs circles.

terms of $2\beta s$, they are labeled in terms of βs for ease of use of the diagram.

Per unit admittance may be written

$$\begin{aligned} \frac{Y_s}{G_0} &= \frac{1}{r_a + jx_a} = \frac{r_a}{r_a^2 + x_a^2} - \frac{jx_a}{r_a^2 + x_a^2} \\ &= g_a - jb_a \end{aligned} \tag{7-117}$$

Thus a positive inductive reactance becomes a negative susceptance. From Eq. 7-103, it is possible to write

$$\frac{Y_s}{G_0} = g_a - jb_a = \frac{1 - |K|\underline{/\phi - 2\beta s}}{1 + |K|\underline{/\phi - 2\beta s}} \tag{7-118}$$

It can be seen that this appears as if formed from Eq. 7-103 by replacing r_a by g_a, x_a by $-b_a$, and $+|K|$ by $-|K|$. Making the same

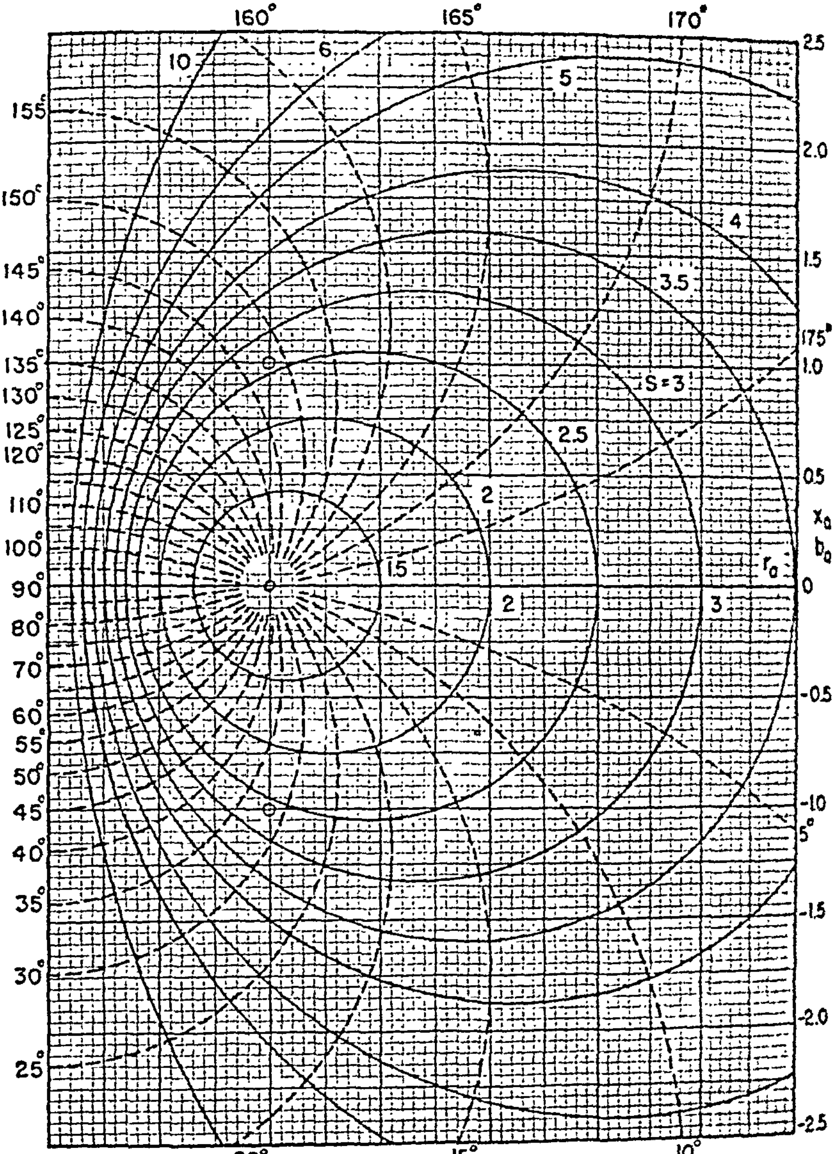


Fig. 7-21. The transmission-line circle diagram.

substitutions in Eq. 7-107 gives

$$\left[g_a - \left(\frac{S^2+1}{2S}\right)\right]^2 + b_a^2 = \left(\frac{S^2-1}{2S}\right)^2 \qquad (7\text{-}119)$$

which is an equation of exactly similar form to Eq. 7-107. The circle diagram obtained for impedance, resistance, and reactance can be used for admittance, conductance, and susceptance merely by changing the r_a scale to g_a. The substitution of $-b_a$ for x_a need not be made, since Eq. 7-119 is independent of the sign of the b_a term. The important point to be observed in the use of the chart for susceptance is that inductive reactance is negative susceptance and thus plots downward, whereas positive (capacitive) susceptance plots upward from the axis of reals, with the same scale used as for x_a.

7-17. Application of the circle diagram

The circle diagram may be used to find the input impedance of a line of any chosen length. Compute first the *per unit* value of the load impedance as

$$\frac{Z_R}{R_0} = r_a + jx_a$$

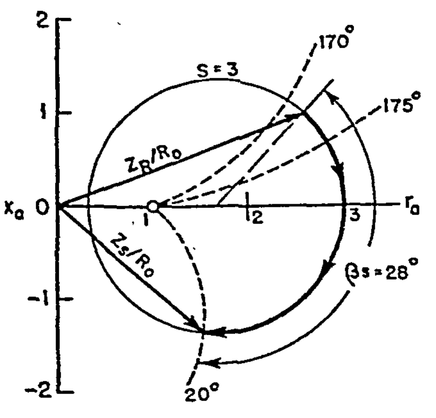


Fig. 7-22. Application of the circle diagram to obtain the input impedance of a 28 deg line terminated in $Z_R/R_0 = 2.6 + j1.0$.

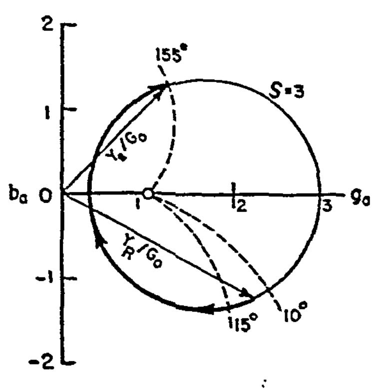


Fig. 7-23. Illustration of the application of the circle diagram to a problem employing admittances.

Locate the point on the chart having coordinates $r_a + jx_a$. This point is the end of a line from the origin that represents the per unit input impedance of the transmission line of zero length (the load only). Follow the constant-S circle passing through the point so located, in a clockwise or negative angle direction through a total number of degrees corresponding to the value of βs or the electrical length of the line, and read the coordinates r_a' and jx_a' of the point reached. It may be necessary to interpolate between βs or S circles. The input impedance of the length of line is then

$$Z_s = R_0(r_a' + jx_a') \tag{7-120}$$

The method is illustrated in Fig. 7-22. For the example chosen, the load is inductive, of value $Z_R/R_0 = 2.6 + j1$; the line is 28° long; the standing-wave ratio is found to be 3; and the per unit input impedance of the line is determined as $Z_s/R_0 = 1.58 - j1.35$.

The input admittance of a length of line may be found by a similar method. An example is given in Fig. 7-23 for a line having per unit load admittance $Y_R/G_0 = 2.25 - j1.20$, representing an inductive load component. This value is plotted below the axis of reals. The standing-wave ratio is found to be 3; and with the line 143° long, the per unit input admittance is $Y_s/G_0 = 1.21 + j1.28$, showing a capacitive component.

An open-circuited line has $S = \infty$, the corresponding S circle appearing as the vertical axis. The input impedance is then pure reactance, with the value for various electrical lengths determined by the intersections of the corresponding βs circles with the vertical axis.

A short-circuited line may be solved by determining its admittance. The S circle is again the vertical axis, and susceptance values may be read off at appropriate intersections of the βs circles with the vertical axis.

7-18. The Smith circle diagram

A modified form of circle diagram for the dissipationless line has been developed by P. H. Smith (Reference 5). The Smith diagram is obtained from a transformation of Eq. 7-112

$$\left(\frac{S-1}{S+1}\right) \underline{/\phi - 2\beta s} = |K| \underline{/\phi - 2\beta s} = \frac{r_a^2 - 1 + x_a^2 + j2x_a}{(r_a + 1)^2 + x_a^2}$$

by introducing new variables, $U + jV$. Thus

$$U + jV = \frac{r_a^2 - 1 + x_a^2}{(r_a + 1)^2 + x_a^2} + \frac{j2x_a}{(r_a + 1)^2 + x_a^2}$$

Equating reals and imaginaries,

$$U = \frac{r_a^2 - 1 + x_a^2}{(r_a + 1)^2 + x_a^2} \tag{7-121}$$

$$V = \frac{2x_a}{(r_a + 1)^2 + x_a^2} \tag{7-122}$$

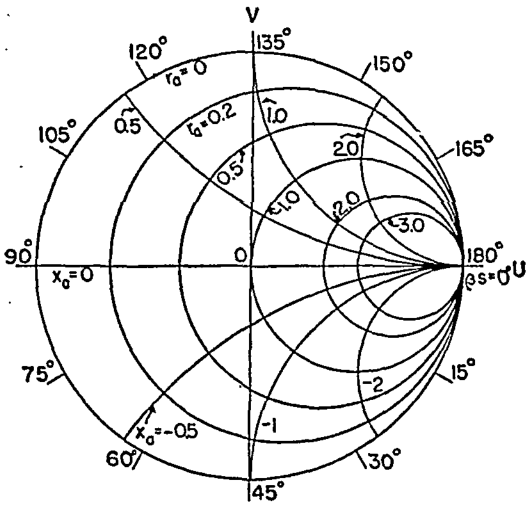


Fig. 7-24. Basis of the Smith circle diagram.

Elimination first of x_a and then of r_a from Eqs. 7-121 and 7-122 results in two equations:

$$\left[U - \left(\frac{r_a}{r_a + 1}\right)\right]^2 + V^2 = \frac{1}{(r_a + 1)^2} \tag{7-123}$$

$$(U - 1)^2 + \left(V - \frac{1}{x_a}\right)^2 = \frac{1}{x_a^2} \tag{7-124}$$

The first of these equations represents a family of constant-r_a circles having centers on the U axis at $r_a/(r_a + 1)$ and radii of $1/(r_a + 1)$. The second equation is that of a family of constant-x_a circles with

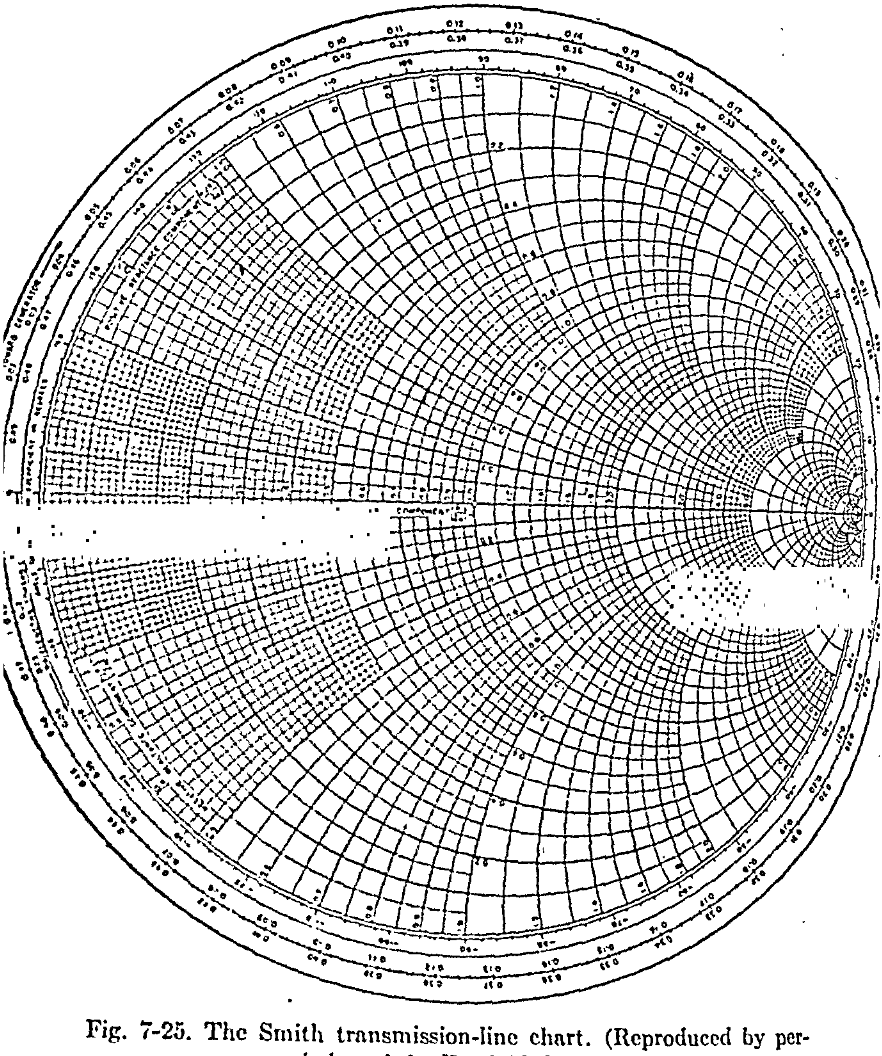


Fig. 7-25. The Smith transmission-line chart. (Reproduced by permission of the Emeloid Co., Inc.)

centers at $1 + j/x_a$ and radii equal to $1/x_a$. The two circle families are drawn in Fig. 7-24.

The diagram was obtained under the assumption that

$$|K|\underline{/\phi - 2\beta s} = U + jV$$

Hence the maximum magnitude of $U + jV$ is fixed at unity by the maximum value of K. Thus *all possible values of impedance are* *contained inside the outer circle of unit radius.* The same relation between polar and rectangular coordinates fixes the βs angles or electrical-line-length coordinates at equal increments around the $K = 1$ circle ($S = \infty$) as indicated in the figure.

The circle diagram of Section 7-16 was drawn in terms of S with values from zero to infinity, all lying outside the unit circle, which appeared as a point at 1,0. The Smith diagram is essentially an inversion of the other diagram, mapping all points or S values inside a unit circle. This diagram has achieved considerable popularity.

In a commercially available form of the Smith chart, the βs increments are indicated around the outer edge of the chart in terms of wavelengths, as in Fig. 7-25. A transparent straightedge is pivoted at the center to serve as a distance coordinate to any point on the chart. This straightedge is marked in terms of K or S, giving the effect of adding constant-S circles to the chart without complicating the figure with additional lines.

The impedance of a line may be read at any point on the appropriate standing-wave ratio or S circle. The point at the center of the chart represents the impedance of a line terminated in its characteristic impedance, where $Z/R_0 = 1$ for all distances. The point at the extreme left of the resistance or r_a axis represents zero impedance or short circuit, the point at the extreme right represents infinite impedance or open circuit, and the outer circle represents $S = \infty$.

The chart may be used for admittance as well as for impedance, the r_a and x_a axes simply becoming g_a and b_a axes, with the usual implication that capacitive susceptance is positive or above, and inductive susceptance below, the U or real axis. The point at the left of the conductance or g_a axis then represents zero conductance or an open circuit, while the point at the extreme right represents infinite conductance or a short circuit.

7-19. Application of the Smith chart

The use of the Smith chart is identical with that of the previously discussed circle diagram, and a number of applications will be illustrated.

Consider a line with load of $Z_R/R_0 = 2.6 + j1$, which is 28° long, as was discussed in Section 7-17. The input impedance may be found by entering the chart of Fig. 7-26 at point A, having coordinates $2.6 + j1$, the resistance component scale being along the

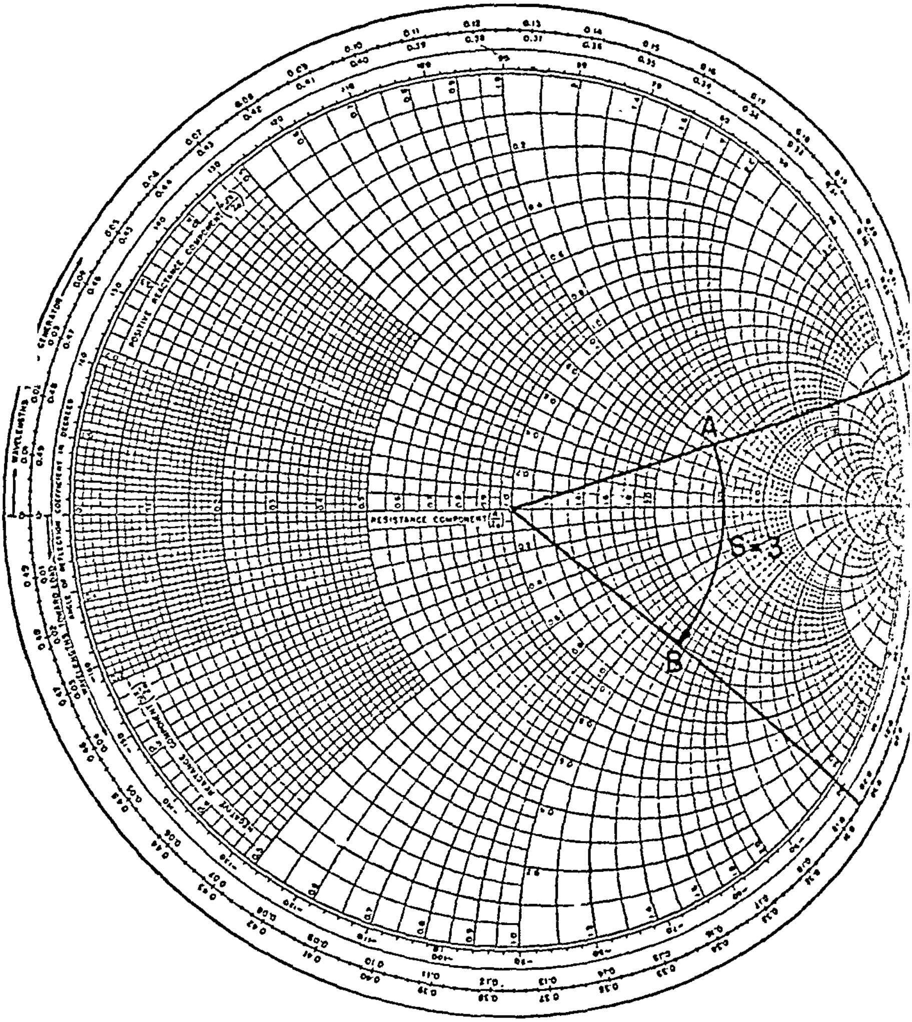


Fig. 7-26. Input impedance of a line.

horizontal axis, the reactive component having a scale at the inner edge of the bounding circle. A constant βs line through A intersects the outer distance scale at 0.227λ.

Then draw a constant-S circle as through point A, using the chart center at the point 1,0 as center. This circle is found to intersect the resistance axis at 3.0, indicating a standing-wave-ratio of 3.0. The line length is $28°/360° = 0.078\lambda$, so that the $S = 3$ circle is followed in a clockwise direction (toward the generator) to a βs

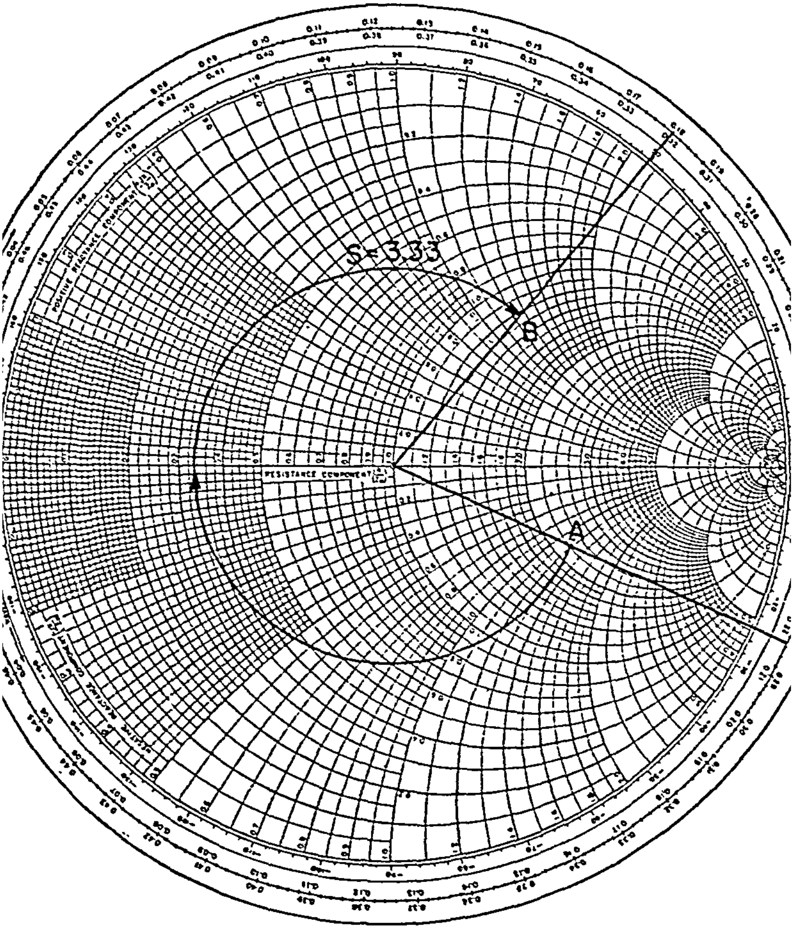


Fig. 7-27. Sending-end input admittance.

line drawn through $(0.227 + 0.078)\lambda = 0.305\lambda$. Point B at the source is thus found. The input impedance can then be read as $Z_s/R_0 = 1.58 - j1.35$, indicating a capacitive reactance for the line input, a result identical to that obtained in Section 7-17.

A similar method yields the input admittance of a loaded line. Assume a line with a load of admittance $Y_R/G_0 = 2.25 - j1.20$, representing an inductive load component. The line length is 143° $= 0.396\lambda$. Using Fig. 7-27, the load admittance is plotted at A, the

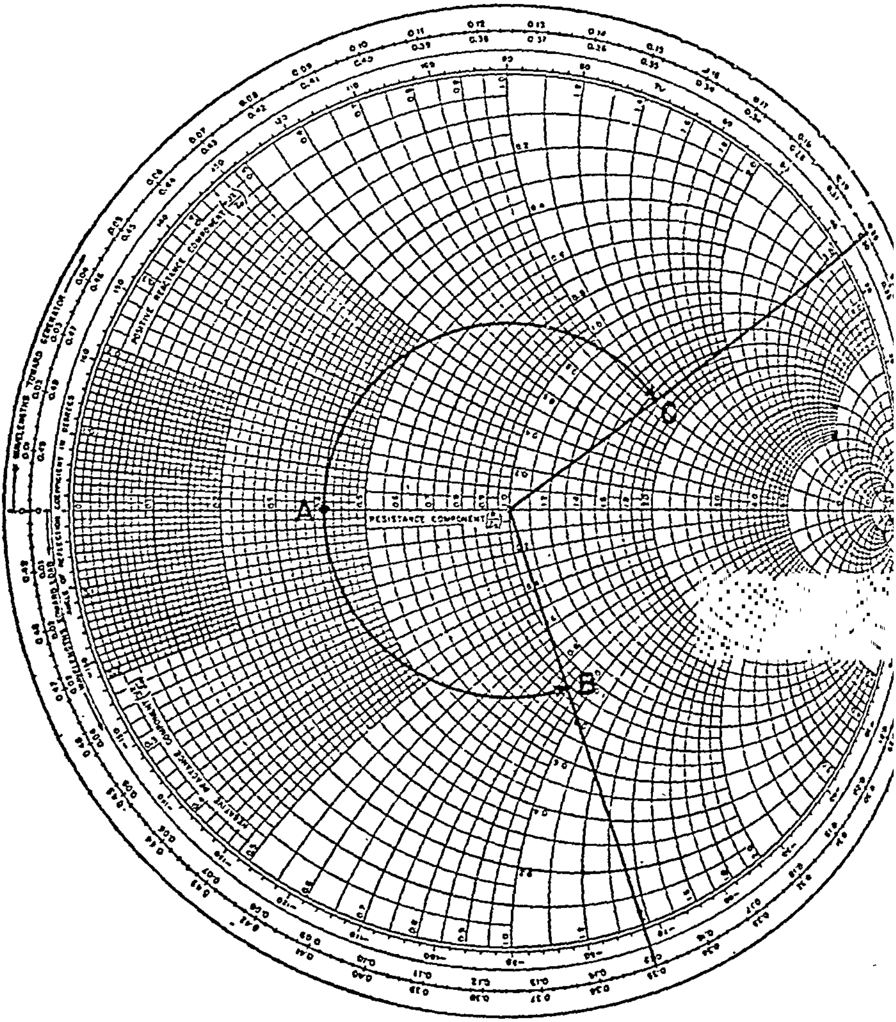


Fig. 7-28. Load and input impedances.

βs line being located at 0.284λ. Using 1,0 as a center the constant-S circle is drawn, showing a standing-wave ratio of 3.0. The βs line for the generator end of the transmission line is drawn at $(0.284 + 0.396)\lambda = 0.680\lambda$, and point B is found. The value of Y_s/G_0 is then read as $1.21 + j1.23$, which checks the value from the previous chart. Since a positive admittance is capacitive, this line will have a capacitive input admittance.

As another example, assume that a line has $S = 2.5$, and it is found that a voltage minimum exists 0.15λ from the load. Find the load and input impedances for a line of 0.35λ length. A voltage minimum occurs where the S circle crosses the left half of the resistance axis; for $S = 2.5$ this will be at point A at 0.4 in Fig. 7-28. Drawing the circle for $S = 2.5$ and measuring 0.15λ counterclockwise, in the direction away from the generator, will locate point B as the load impedance. The inner wavelength scale might also have been used in terms of distance toward the load. The load impedance at B is read as $Z_R/R_0 = 0.89 - j0.89$.

Measurement on the S circle in the clockwise direction for a distance of 0.35λ from B carries to point C which is the location of the source or generator. The input impedance of the line can here be read as $Z_s/R_0 = 1.68 + j1.03$.

The reflection factor K can be calculated from $(S - 1)/(S + 1) = 0.428$. The angle of K can be read from the βs line at the load, giving $-72°$ from the innermost scale of the chart.

It may be noted that the open-circuited line has $K = 1\underline{/0°}$ or $S = \infty$, and the appropriate S circle is the outer bound of the chart. For this circle $r_a = 0$, and the input impedance of an open line will be pure reactance, as determined by the electrical length of the line.

A short-circuited line may be considered by use of its admittance, its S circle is then the outer bound of the chart, and the input susceptance is determined by the length or βs angle around the chart.

The Smith chart may also be used for lossy lines, and the locus of points on a line then follows a spiral path toward the chart center, due to attenuation.

7-20. Single-stub matching with the Smith chart

The solution of the stub-matching problem of Section 7-15 may be easily carried out on the Smith chart, and an example is shown in Fig. 7-29. Working with admittances, since the tuning stub is to be connected in parallel with the line, enter the chart at A, Fig. 7-29, for a load having a capacitive component and given as $Y_R/G_0 = 2.75 + j1.75$. Drawing the constant-S circle shows the standing-wave ratio before use of the stub to be 4.0, by reason of the intersection of the S circle with the right half of the real axis.

Note that the chart circle for $Y/G_0 = 1$ is the locus of all points for which the real part of the line conductance is unity. This is the

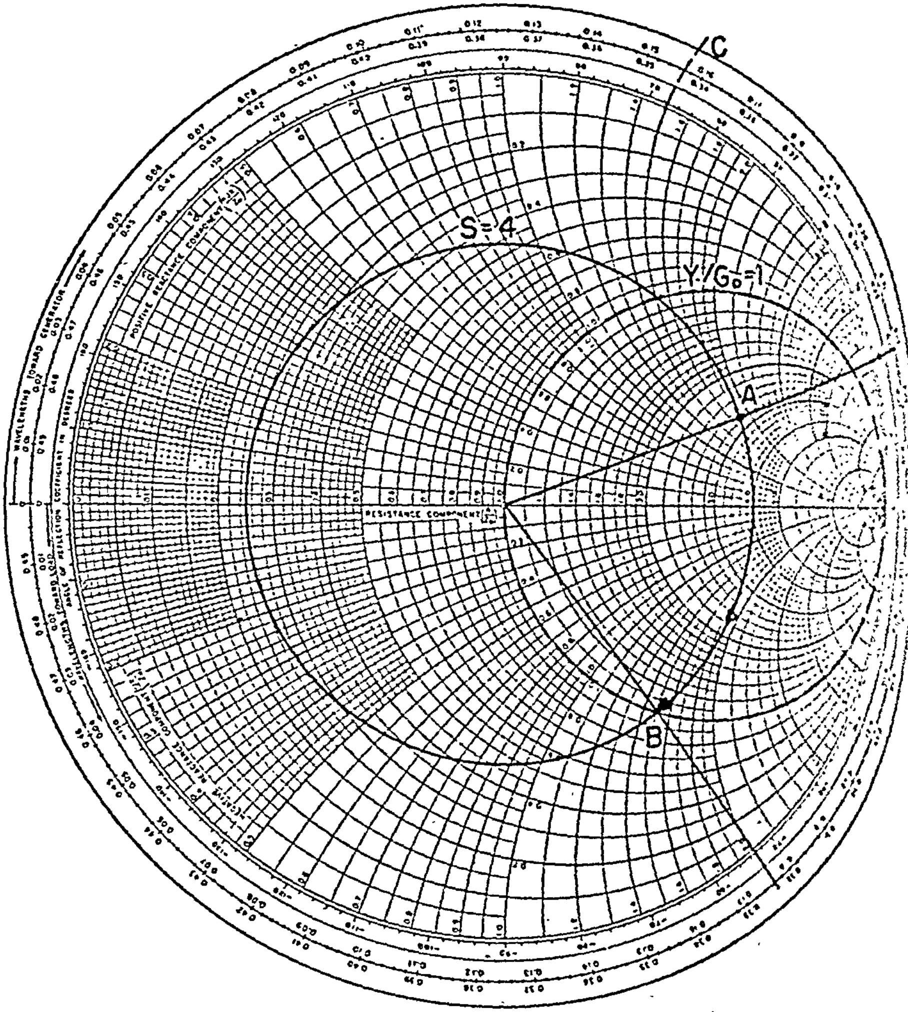


Fig. 7-29. Single stub matching.

desired condition at the point of stub connection, so that the intersection of the S circle and the chart circle for $Y/G_0 = 1$, as indicated at B, will locate the stub. Use of the βs lines through A and B shows that the stub should be located $0.102\lambda = 37°$ from the load.

The b_a value at B represents the susceptance of the line at the stub connection, and this is read as -1.5, indicating inductive susceptance. This value of line susceptance must be resonated by a stub line having an input susceptance of $+1.5$. The electrical length for a capacitive (short-circuited) stub may be computed from

$$\beta s = \cot^{-1}(-b_a)$$

or may be readily found on the chart.

The input admittance of a short-circuited stub line having capacitive susceptance of $b_a/G_0 = +1.5$ would plot at the intersection of the $+1.5$ susceptance circle and the $K = 1$ ($S = \infty$ circle) or outer bound of the chart at point C. This intersection occurs at 0.156λ on the βs scale or 0.406λ from a short circuit at the right end of the real axis or infinite admittance point (measuring toward the load or counterclockwise). A short-circuited stub line 0.406λ or $146°$ in length would then have the required capacitive susceptance.

It is also apparent that an open stub line 0.156λ or $56°$ long might be used, the length of this line in wavelengths being determined from $\beta s \doteq 0.156\lambda$ counterclockwise to zero, or on the $K = 1$ circle to the left end of the real axis which represents an open circuit or zero admittance load.

In Section 7-15 similar information was developed; there only standing-wave ratio before stub connection and the location of the voltage minimum nearest the load were the available data. This is the usual information, and the Smith chart can be readily applied. From the standing-wave ratio the proper S circle is immediately determined by reason of its intercept on the real or U axis. This intersection on the g_a axis at 4.0 for the above example represents maximum conductance where $Y/G_0 = S$, and this is also a point of minimum voltage. Following the S circle to its intersection with the $Y/G_0 = 1$ circle shows that the stub should be connected $(0.324 - 0.25)\lambda = 0.074\lambda = 27°$ toward the source as measured from the voltage minimum nearest the load. The length of stub would be as previously determined. Thus no knowledge of the load admittance is required.

7-21. Double-stub impedance matching on a line

Single-stub impedance matching requires that the stub be located at a definite point on the line. This requirement frequently calls for placement of the stub at an undesirable place from a mechanical viewpoint. For a coaxial line, it is not possible to determine the location of a voltage minimum without a slotted line section, so that placement of a stub at the exact required point is difficult. In the case of the single stub it was mentioned that two adjustments were required, these being location and length of the stub. Another possible method is to use two stubs in which the locations of the stubs are arbitrary, the two stub lengths furnishing the required adjustments. The spacing is frequently made $\lambda/4$. Half-wave spacing should be avoided because it places the two stubs in parallel, resulting in only one *effective adjustment* being available. The same difficulty arises if the stubs are closely spaced. An arrangement of two short-circuited stubs is illustrated for an open-wire line in Fig. 7-30, $\lambda/4$ spacing being indicated.

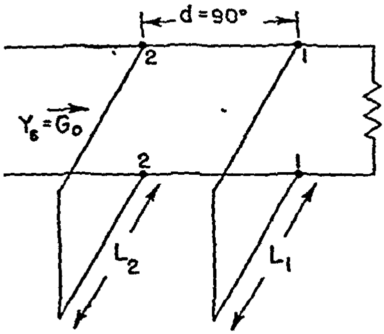


Fig. 7-30. Double-stub impedance matching.

For smooth line operation, the input admittance of the line looking toward the load at the 2,2 location of Fig. 7-30 should be

$$Y_s = G_0$$

or the line should appear terminated in its characteristic impedance at that point. Thus the 2,2 point should be at a location on the line having a per unit admittance of

$$\frac{Y_s}{G_0} = 1 \pm jb_a$$

The Y/G_0 circle passing through $g_a = 1$ will be the locus of all such admittances, and this locus is shown as circle A in short dashes in Fig. 7-31. All points on this circle can be resonated by a stub of susceptance $\mp b_a$ to give the desired per unit admittance of $Y/G_0 = 1$ at 2,2 on the line.

The transformer formed by the quarter-wave of line between 2,2 and 1,1 will transform all admittances on the locus circle A into admittances which will lie on a second locus circle B, found by displacing each point of circle A counterclockwise by a quarter wavelength on the chart (180° rotation on the chart). This second locus circle B is indicated in the figure.

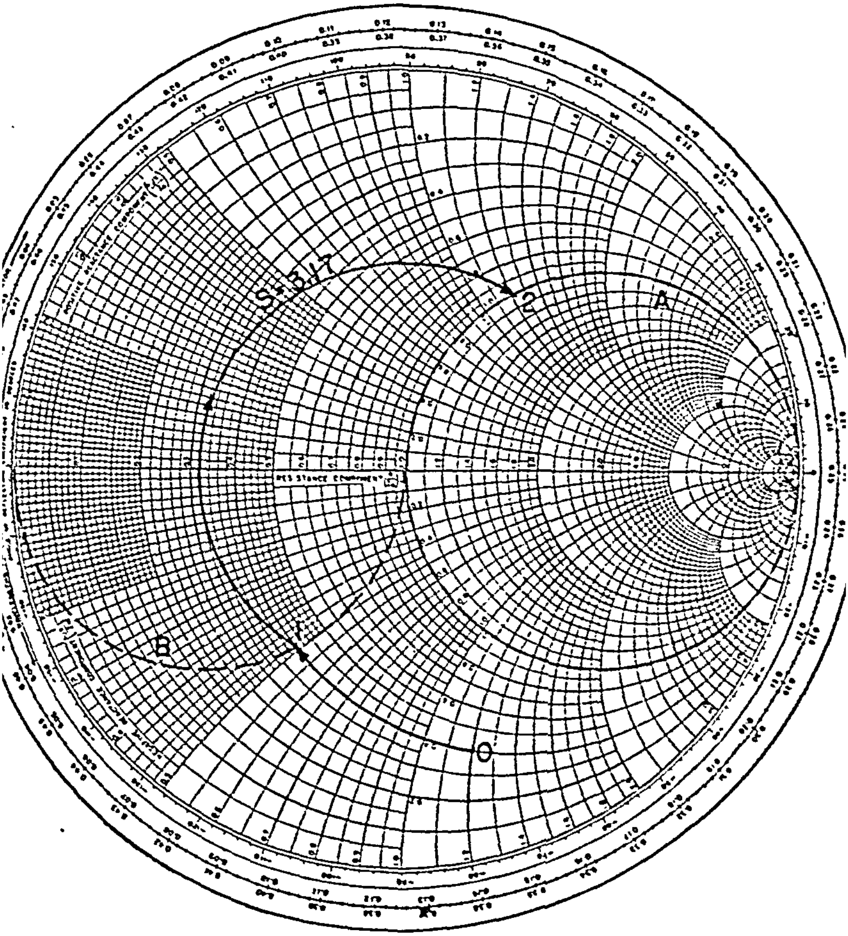


Fig. 7-31. Solution of the double-stub problem.

Therefore if stub 1 succeeds in transforming the input admittance of the line and load to the right of 1,1 into an admittance which will plot on the circle B locus, the quarter-wavelength line will further transform the admittance into a value at 2,2 which will plot on the locus circle A and will have $Y/G_0 = 1 + jb_a$. Stub 2 can then resonate the admittance at 2,2 to the desired value $Y/G_0 = 1 + j0$ for properly terminating the main line to the left of 2,2.

It is desirable that stub 1 be connected at or near the load. It is not always possible to make such a connection at the load, since the largest conductance component at 1,1 that can be transformed is $g_a = 1$, or the input admittance to the left of 1,1 when plotted on the chart cannot lie inside the $g_a = 1$ circle. If this happens, a point on the line toward the source having per unit conductance less than 1.0 must be selected for connection of the stub. Such a point will be available, since the per unit conductance varies between S and $1/S$ in a $\lambda/4$ distance along the line.

An example is illustrated by Fig. 7-31. Before connection of the stubs, the per unit admittance of the line (or of line and load) at the desired location for connection of stub 1 is plotted at 0 on the chart as

$$\frac{Y}{G_0} = 0.4 - j1.2$$

Stub 1 adds a susceptance in parallel, and this must change Y/G_0 to a value Y_1/G_0 which lies on locus circle B. Stub 1 cannot alter the conductance, so following a *constant-conductance circle* from point 0 locates point 1 on circle B, where

$$\frac{Y_1}{G_0} = 0.4 - j0.5$$

Thus stub 1 must contribute a susceptance of $b_a = +0.7$ at 1,1 on the line.

If Y_s/G_0 had a conductance greater than 1.0, a constant-conductance circle from the Y_s/G_0 plot would miss circle B and point 1 could not be established, confirming the limitation on conductance given above.

The section of line between points 1,1 and 2,2 changes the admittance at 1,1 to that of 2,2 along a constant-S circle, and S along this circle is found to have a value of 3.15. The admittance at 2,2 on the line, without stub 2 connected, can be read from the chart at point 2 on locus circle A, giving

$$\frac{Y_2}{G_0} = 1.0 + j1.2$$

Stub 2 should then be adjusted to provide an inductive susceptance of $b_a = -1.2$, after which the admittance at 2,2 will be $Y_2/G_0 = 1 + j0$, and the main line to the left of 2,2 will be properly terminated.

The required stub lengths may be readily found from the chart. Since short-circuited stubs are most desirable, because of ease of construction and adjustment, their lengths will be

$$L_1 = 0.348\lambda = 125°, \qquad L_2 = 0.11\lambda = 40°$$

If $\lambda/4$ spacing is not suitable, the design method may still be applied by drawing a different locus circle B. The desired circle B should be drawn by rotating circle A through the wavelength spacing chosen for the two stubs. That is, if $\frac{3}{8}\lambda$ spacing is desired, circle B will have its center lying on an axis rotated 270 space degrees counterclockwise from the axis of circle A, or the diameter of B will project vertically downward from the chart center.

7-22. Constants for the line of "small" dissipation

At very high frequencies, of the order of 100 megacycles or more, open- or short-circuited lines are frequently employed as reactances or as resonant circuits. The assumption of zero dissipation would indicate that all such elements have infinite Q values, an obvious fallacy. Consideration of the small losses present in all such elements is necessary to give a true quantitative understanding of the situation. At the frequencies employed, the value of ω is so large that R is always small with respect to ωL, and G is assumed zero. Under such an assumption an analysis for the line constants may readily be made. The line parameters are

$$Z = R + j\omega L, \qquad Y = j\omega C$$

so that the characteristic impedance Z_0 is

$$Z_0 = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{R + j\omega L}{j\omega C}} = \sqrt{\frac{L}{C}\left(1 - j\frac{R}{\omega L}\right)}$$

Expansion by the binomial theorem gives

$$Z_0 = \sqrt{\frac{L}{C}}\left(1 - j\frac{R}{2\omega L}\right) \tag{7-125}$$

after neglecting high order terms in $R/\omega L$. This equation shows the impedance Z_0 to have an angle

$$\theta = \tan^{-1}\left(-\frac{R}{2\omega L}\right)$$

and since $R/\omega L$ is small with respect to unity, the characteristic impedance may be written as

$$Z_0 = R_0 \cong \sqrt{\frac{L}{C}} \tag{7-126}$$

and is essentially resistive, in which case it is again given the symbol R_0.

The propagation constant γ for the line of small dissipation is

$$\gamma = \sqrt{ZY} = \sqrt{(R + j\omega L)j\omega C} \tag{7-127}$$
$$= (-\omega^2 LC + j\omega CR)^{1/2}$$

Since $R \ll \omega L$, a more convenient value for γ can be obtained by expansion by the binomial theorem as

$$\gamma = (-\omega^2 LC)^{1/2} + \frac{1}{2}(-\omega^2 LC)^{-1/2} j\omega CR + \frac{1}{8}(-\omega^2 LC)^{-3/2}\omega^2 C^2 R^2$$
$$- \frac{1}{16}(-\omega^2 LC)^{-5/2} j\omega^3 C^3 R^3 + \cdots$$
$$= j\omega\sqrt{LC} + \frac{R}{2}\sqrt{\frac{C}{L}} + j\frac{R^2}{8\omega L}\sqrt{\frac{C}{L}} - \frac{1}{16}\frac{R^3}{\omega^2 L^2}\sqrt{\frac{C}{L}} + \cdots$$
$$= \frac{R}{2}\sqrt{\frac{C}{L}}\left(1 - \frac{R^2}{8\omega^2 L^2}\right) + j\omega\sqrt{LC}\left(1 + \frac{R^2}{8\omega^2 L^2}\right) + \cdots \tag{7-128}$$

The values of α and β are then seen to be approximately

$$\alpha = \frac{R}{2}\sqrt{\frac{C}{L}}\left(1 - \frac{R^2}{8\omega^2 L^2}\right) \tag{7-129}$$

$$\beta = \omega\sqrt{LC}\left(1 + \frac{R^2}{8\omega^2 L^2}\right) \tag{7-130}$$

Usually the value of $R/\omega L$ will be such that the expressions for α and β may be reduced further to

$$\alpha \cong \frac{R}{2}\sqrt{\frac{C}{L}} = \frac{R}{2R_0} \tag{7-131}$$

$$\beta \cong \omega\sqrt{LC} \tag{7-132}$$

where all values of parameters are usually per meter.

The velocity of propagation may be found by use of Equation

7-130 as

$$v = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}\,(1 + R^2/8\omega^2L^2)} \tag{7-133}$$

In Section 6-4 it was found that for the ideal conditions of zero losses and zero internal inductance of a line, $v = 1/\sqrt{LC}$, and the velocity of propagation is equal to that of light in the medium surrounding the conductors. Equation 7-133 shows that when dissipation is considered, the velocity is less than that of light by reason of a factor dependent on line resistance. In Section 6-4 it was shown that the internal inductance of the line is responsible for slowing of propagation, and here it is seen that the resistance of the line is also a factor in reducing the velocity below the theoretical value, that of light in the medium. Thus it has been found that internal inductance, line resistance, and line insulation with a value of ϵ_r other than unity all contribute to slowing the velocity of propagation below that of light in free space. In actual lines, since $R/\omega L$ is small, the reduction in velocity due to resistance is usually only a fraction of 1 per cent.

The wavelength on the line is

$$\lambda = \frac{2\pi}{\beta} = \frac{1}{f\sqrt{LC}\,(1 + R^2/8\omega^2L^2)} \tag{7-134}$$

The wavelength in free space is

$$\lambda = \frac{1}{f\sqrt{LC}}$$

so that the wavelength on the line is reduced slightly below the free-space value by reason of the line resistance.

7-23. Voltages and currents on the line of small dissipation

Voltage and current expressions on the long line were originally developed as

$$E = \frac{E_R(Z_R + Z_0)}{2Z_R}\,(\epsilon^{\gamma s} + K\epsilon^{-\gamma s})$$

$$I = \frac{I_R(Z_R + Z_0)}{2Z_0}\,(\epsilon^{\gamma s} - K\epsilon^{-\gamma s})$$

Since it has been found that

$$\alpha = \frac{R}{2R_0}$$

where R is the line resistance per meter of length, then if R is small with respect to $2R_0$ and the line is short, the $\epsilon^{\gamma s}$ terms may be written

$$\epsilon^{\gamma s} = \epsilon^{\alpha s}\epsilon^{j\beta s} = (1 + \alpha s)(\cos \beta s + j \sin \beta s)$$

$$\epsilon^{-\gamma s} = \epsilon^{-\alpha s}\epsilon^{-j\beta s} = (1 - \alpha s)(\cos \beta s - j \sin \beta s)$$

the higher-order terms of the exponential series for $\epsilon^{\alpha s}$ being neglected as small. After insertion of the value of K, the voltage and current equations reduce to

$$E = E_R \left[\left(1 + \alpha s \frac{Z_0}{Z_R}\right) \cos \beta s + j\left(\alpha s + \frac{Z_0}{Z_R}\right) \sin \beta s\right] \quad (7\text{-}135)$$

$$I = \frac{E_R}{Z_0}\left[\left(\alpha s + \frac{Z_0}{Z_R}\right) \cos \beta s + j\left(1 + \alpha s \frac{Z_0}{Z_R}\right) \sin \beta s\right] \quad (7\text{-}136)$$

If it is desired to start with the conventional hyperbolic relations

$$E = E_R \cosh (\alpha + j\beta)s + I_R Z_0 \sinh (\alpha + j\beta)s$$

$$I = I_R \cosh (\alpha + j\beta)s + \frac{E_R}{Z_0} \sinh (\alpha + j\beta)s$$

the cosh and sinh may be expanded as the sum of two angles, and trigonometric functions substituted. If αs is sufficiently small, cosh αs may be replaced by unity and sinh αs by αs, giving the same results as obtained above from the exponential expressions.

7-24. Open- and short-circuit impedances when considering dissipation

The input impedance of a line of length $s = l$ in which the dissipation is small may be written from Eqs. 7-135 and 7-136 as

$$Z_s = \frac{E_s}{I_s} = Z_0 \left[\frac{(Z_R + \alpha l Z_0) \cos \beta l + j(Z_R \alpha l + Z_0) \sin \beta l}{(Z_R \alpha l + Z_0) \cos \beta l + j(Z_R + \alpha l Z_0) \sin \beta l}\right] \quad (7\text{-}137)$$

For a line terminated in a short circuit, $Z_R = 0$, and the input impedance becomes

$$Z_{sc} = Z_0 \left(\frac{\alpha l \cos \beta l + j \sin \beta l}{\cos \beta l + j\alpha l \sin \beta l}\right) \quad (7\text{-}138)$$

which, after rationalizing and writing $Z_0 = R_0$, becomes

$$Z_{sc} = R_0 \left[\frac{\alpha l + j(1 - \alpha^2 l^2) \sin \beta l \cos \beta l}{1 - (1 - \alpha^2 l^2) \sin^2 \beta l} \right]$$

It will be found more convenient if β is eliminated by writing

$$\beta l = \frac{2\pi l}{\lambda}$$

so that

$$Z_{sc} = R_0 \left[\frac{\alpha l + j(1 - \alpha^2 l^2) \sin (2\pi l/\lambda) \cos (2\pi l/\lambda)}{1 - (1 - \alpha^2 l^2) \sin^2 (2\pi l/\lambda)} \right] \tag{7-139}$$

Such a short-circuited line may be used as a circuit element having components as follows:

$$R_{sc} = \frac{R_0 \alpha l}{1 - (1 - \alpha^2 l^2) \sin^2 (2\pi l/\lambda)} \tag{7-140}$$

$$X_{sc} = \frac{R_0 (1 - \alpha^2 l^2) \sin (2\pi l/\lambda) \cos (2\pi l/\lambda)}{1 - (1 - \alpha^2 l^2) \sin^2 (2\pi l/\lambda)} \tag{7-141}$$

For values of l between 0 and $\lambda/4$ this reactance is inductive; for values of l between $\lambda/4$ and $\lambda/2$ it is capacitive and is seen to oscillate in sign every quarter wavelength. This performance is similar to that of the dissipationless line studied in Section 7-8 except that the above expression for reactance reaches a maximum but does not become infinite.

For certain applications it is convenient to have the admittance of the short-circuited line of small dissipation. It can be obtained by rationalizing the reciprocal of Eq. 7-138, giving

$$Y_{sc} = G_0 \left[\frac{\alpha l - j(1 - \alpha^2 l^2) \sin (2\pi l/\lambda) \cos (2\pi l/\lambda)}{1 - (1 - \alpha^2 l^2) \cos^2 (2\pi l/\lambda)} \right] \tag{7-142}$$

After the numerator and denominator of Eq. 7-137 are divided by Z_R and after Z_R has been set equal to ∞, an expression for the input impedance of an open-circuited line is obtained as

$$Z_{oc} = Z_0 \left(\frac{\cos \beta l + j\alpha l \sin \beta l}{\alpha l \cos \beta l + j \sin \beta l} \right) \tag{7-143}$$

Upon rationalizing and substituting $R_0 = Z_0$ and $\beta = 2\pi/\lambda$, this becomes

$$Z_{oc} = R_0 \left[\frac{\alpha l - j(1 - \alpha^2 l^2) \sin (2\pi l/\lambda) \cos (2\pi l/\lambda)}{1 - (1 - \alpha^2 l^2) \cos^2 (2\pi l/\lambda)} \right] \tag{7-144}$$

The open-circuit impedance has components

$$R_{oc} = \frac{R_0 \alpha l}{1 - (1 - \alpha^2 l^2) \cos^2 (2\pi l/\lambda)} \tag{7-145}$$

$$X_{oc} = \frac{-R_0(1 - \alpha^2 l^2) \sin (2\pi l/\lambda) \cos (2\pi l/\lambda)}{1 - (1 - \alpha^2 l^2) \cos^2 (2\pi l/\lambda)} \tag{7-146}$$

For values of l between 0 and $\lambda/4$ this reactance is capacitive; for values of l between $\lambda/4$ and $\lambda/2$ it is inductive and can be seen to oscillate in sign every quarter wavelength, always being opposite in sign to the reactance of the short-circuited line.

The input admittance of the open-circuited line is

$$Y_{oc} = G_0 \left[\frac{\alpha l + j(1 - \alpha^2 l^2) \sin (2\pi l/\lambda) \cos (2\pi l/\lambda)}{1 - (1 - \alpha^2 l^2) \sin^2 (2\pi l/\lambda)} \right] \tag{7-147}$$

At very high frequencies the distributed inductive reactance of the leads of a capacitor may be greater than its capacitive reactance,

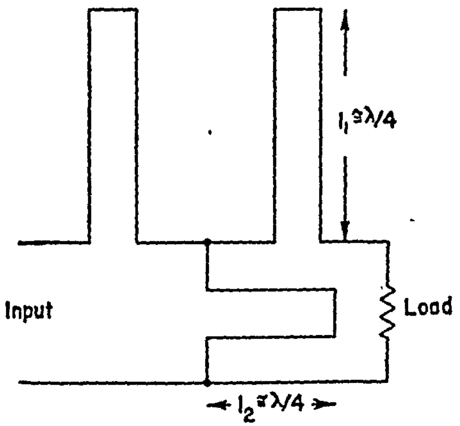


Fig. 7-32. Use of lines as reactive elements in a filter section. Lengths adjusted to give desired capacitive or inductive reactance.

so that the capacitor may appear as an inductor. An inductor may have so much distributed shunting capacitance that it becomes a parallel-resonant circuit or even behaves as a capacitor. Analysis by the methods of distributed-constant circuits has shown a way to avoid these difficulties by replacing the conventional low-frequency forms of lumped inductors or capacitors with distributed reactive elements and resonant circuits.

Lines may be used as efficient impedance matching sections and as reactive elements in wave filters. At very high frequencies, lines of appropriate length may be used to replace any reactive element called for by the circuit design. Such an application is shown in Fig. 7-32. A wave filter of this nature will be a multiband-pass type because of the multiple resonances of the elements. As will be shown in Section 7-26, the dissipation is lower, or the Q factor greater, for lines than for equivalent lumped reactances, so that filter performance more closely approaches the ideal.

7-25. Quarter- and half-wave lines of small dissipation

In the analysis of the dissipationless line it was indicated that the input reactance of a short-circuited line was infinite at $l = \lambda/4$ and zero at $l = \lambda/2$. Both such values are obviously impossible, and it is interesting to discover the more nearly true state of affairs by analysis of the line of small dissipation.

The input impedance of a *quarter-wave short-circuited line* may be found by insertion of $l = \lambda/4$ in the trigonometric functions of Eq. 7-139, giving

$$Z_{sc} = \frac{R_0}{\alpha l} \tag{7-148}$$

If in general the line is made any odd number of quarter wavelengths long by setting

$$l = (2n - 1)\frac{\lambda}{4} \quad (n = 1, 2, 3, \ldots)$$

then
$$Z_{sc} = \frac{4R_0}{\alpha(2n - 1)\lambda}$$

and upon substitution of the value of α,

$$Z_{sc} = \frac{8R_0^2}{R(2n - 1)\lambda} \tag{7-149}$$

is obtained as the input impedance of a *shorted line an odd number of quarter waves long*.

The result is obviously resistive and represents an impedance maximum. Because of the factor n present, the highest impedance value will be obtained with the shortest line or with $n = 1$. It has been shown (Reference 2) that since both R_0 and R are functions of spacing and wire diameter for open-wire lines, or of b/a for coaxial lines, the greatest value of impedance will be obtained if $d/a = 8.0$ for open-wire lines and $b/a = 9.2$ for coaxial lines. The performance of the $\lambda/4$ short-circuited line with varying frequency approximates closely that of an antiresonant circuit *near resonance.*

For a *short-circuited line any number n of half wavelengths long,* the input impedance may be computed as

$$Z_{sc} = R_0 \alpha l = \frac{Rn\lambda}{4} \tag{7-150}$$

This is again resistive and represents an impedance minimum. The $n\lambda/2$ short-circuited line behaves in approximately the same manner as a series resonant circuit when the frequency is varied *near resonance.*

For the open-circuited line the analysis starts with Eq. 7-144. Minima in the impedance characteristics occur with $l = (2n - 1)\lambda/4$, or odd numbers of quarter wavelengths. The impedance then is

$$Z_{oc} = R_0 \alpha l = \frac{R(2n-1)\lambda}{8} \tag{7-151}$$

for the *open-circuited, odd quarter-wavelength line.* The performance of the odd $\lambda/4$ open line is comparable to a series resonant circuit *near resonance.*

The maximum resistive impedance of an *open-circuited line* occurs at $l = n\lambda/2$ and is

$$Z_{oc} = \frac{R_0}{\alpha l} = \frac{4R_0^2}{Rn\lambda} \tag{7-152}$$

Such a line is comparable to the parallel resonant circuit.

It should be noted that the short-circuited $\lambda/4$ line gives a higher input impedance than the open-circuit $\lambda/2$ line for any n. The open-circuited $\lambda/4$ line gives a lower input impedance than the short-circuited $\lambda/2$ line. That is, that line which is physically shortest will usually be the most desirable. The factor n appearing in the equations has the value 1, 2, 3, ... It definitely influences the resistive impedances, lowering the maxima and raising the minima obtainable as n increases. Physically this is reasonable, since if the line were very long, its input impedance would be R_0, regardless of the termination.

A common application of the shorted $\lambda/4$ line is as an insulator, as mentioned in Section 7-12. Equation 7-149 permits calculation of the *insulation resistance* of such a line. Though not infinite, it will ordinarily be found to have a value of some hundreds of thousands of ohms, which usually is sufficiently high to be neglected in comparison with an R_0 of only a few hundred ohms or less. The copper insulator is mechanically rugged and maintains the line at low potential to ground for all frequencies except the one desired, or its odd harmonics.

7-26. The tapped quarter-wave line as an impedance transformer

The resistive impedance seen at the open end of a *short-circuited* line $\lambda/4$ long is given by Eq. 7-149 as

$$Z_{sc} = \frac{8R_0^2}{R\lambda} = \frac{4R_0}{\alpha\lambda}$$

The impedance seen at the short-circuited end of the line is obviously zero. Intermediate points along the line will present impedances

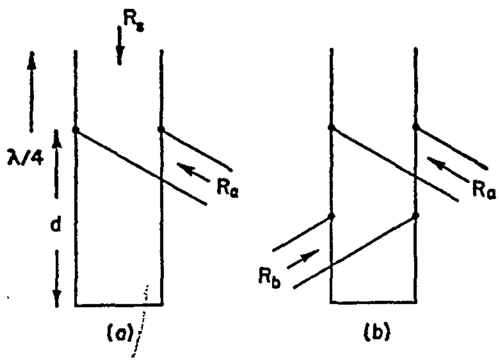


Fig. 7-33. The tapped quarter-wave line.

intermediate to the two values. Some line having a resistance R_a can then be matched to some load having resistance R_s by use of a $\lambda/4$ line as in (a), Fig. 7-33. The line having R_a impedance is connected at a point distant d meters from the short circuit. The load having resistance R_s is connected at the open end of the $\lambda/4$ line. The line then appears as an autotransformer.

In some cases neither of the devices to be coupled has a resistance equal to R_s. The two devices having resistances R_a and R_b may be connected at appropriate points on the line at which the line impedance matches that of the connected sources or loads, as in (b), Fig. 7-33. Loads or sources having impedances above the value of R_s, the input resistance at the open end of the line, cannot be matched by this system.

An expression by which the impedance of an unloaded line at any distance d from the short circuit can be found, can be obtained by writing Y_{sc} and Y_{oc} from Eqs. 7-138 and 7-143 as

$$Y_{sc} = G_0 \left[\frac{1 + j\alpha l \tan (2\pi l/\lambda)}{\alpha l + j \tan (2\pi l/\lambda)} \right] \tag{7-153}$$

$$Y_{oc} = G_0 \left[\frac{\alpha l + j \tan 2\pi l/\lambda}{1 + j\alpha l \tan 2\pi l/\lambda} \right] \tag{7-154}$$

The admittance Y_a at distance d from the short circuit is the sum of two admittances in parallel; one a short-circuited line d meters long, the second an open-circuited line $(\lambda/4 - d)$ meters long. Then

$$Y_a = G_0 \left[\frac{1 + j\alpha d \tan (2\pi d/\lambda)}{\alpha d + j \tan (2\pi d/\lambda)} + \frac{\alpha(\lambda/4 - d) + j \tan (2\pi/\lambda)(\lambda/4 - d)}{1 + j\alpha(\lambda/4 - d) \tan (2\pi/\lambda)(\lambda/4 - d)} \right]$$

By use of the trigonometric identities

$$\tan \frac{2\pi}{\lambda}\left(\frac{\lambda}{4} - d\right) = \cot \frac{2\pi d}{\lambda}$$

$$\tan \frac{2\pi d}{\lambda} + \cot \frac{2\pi d}{\lambda} = \frac{2}{\sin (4\pi d/\lambda)}$$

the above may be reduced to

$$Y_a = G_0 \frac{-j\alpha\lambda/[2 \sin (4\pi d/\lambda)]}{\alpha(2d - \lambda/4) + j[\tan (2\pi d/\lambda) + \alpha^2 d(\lambda/4 - d) \cot (2\pi d/\lambda)]} \tag{7-155}$$

Since α will be small, the denominator terms involving α may be considered negligible in magnitude and dropped. In the case of the first term in the denominator, this is equivalent to neglecting a small susceptance component. Inversion and rationalization then produces

$$Z_a = \frac{2R_0}{\alpha\lambda} \sin \frac{4\pi d}{\lambda} \tan \frac{2\pi d}{\lambda}$$

$$= \frac{4R_0}{\alpha\lambda} \sin^2 \frac{2\pi d}{\lambda} \tag{7-156}$$

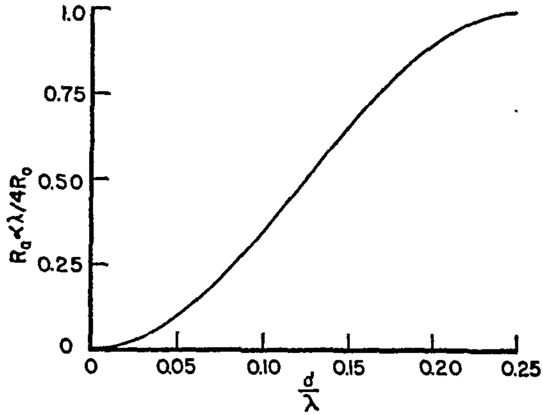


Fig. 7-34. Variation of resistance with distance d from short-circuit point.

The coefficient is recognizable as the resistive impedance at the open end of the $\lambda/4$ line. Thus the resistance at any point varies as the square of the sine of the distance function. The result is plotted in Fig. 7-34.

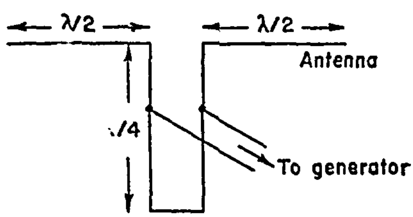


Fig. 7-35. Application of tapped quarter-wave line as an impedance transformer in matching a line to an antenna of length λ

A load connected at the open end of the line will parallel the line impedance, and if this load has a resistance R_L, the resultant resistive impedance seen at the tap at any distance d will vary as

$$Z_a = \frac{4R_0R_L}{\alpha\lambda(R_L + 4R_0/\alpha\lambda)} \sin^2 \frac{2\pi d}{\lambda}$$

$$= \frac{4R_0}{\alpha\lambda} \left(\frac{1}{1 + 4R_0/\alpha\lambda R_L}\right) \sin^2 \frac{2\pi d}{\lambda} \tag{7-157}$$

The tapped quarter-wave line is frequently employed for impedance matching between an antenna, or several antennas, and a transmission line, as shown in Fig. 7-35. While an open-line has theoretical impedance-matching properties it is rarely used because of radiation losses from the open end, end capacitance effects, and the difficulty of a smooth adjustment of length.

7-27. Voltage step-up on the resonant line

A quarter-wave open-circuited line has the property of producing a voltage step-up between its ends. By writing the voltage relation of Eq. 7-135 as

$$\frac{E_s}{E_R} = \left(1 + \alpha l \frac{Z_0}{Z_R}\right) \cos \frac{2\pi l}{\lambda} + j\left(\alpha l + \frac{Z_0}{Z_R}\right) \sin \frac{2\pi l}{\lambda}$$

and setting $Z_R = \infty$ and $l = \lambda/4$, there is obtained

$$\frac{E_R}{E_s} = \frac{1}{\alpha l} = \frac{4}{\alpha\lambda} \tag{7-158}$$

as the voltage ratio between the ends of the line. Since α may be quite small, the voltage step-up can be large.

7-28. Q of a line as a circuit element; band width

As has been pointed out in the preceding sections, lines may be employed as resonant circuits. When they are so used, it is convenient to be able to analyze and proportion them to achieve desired frequency selection properties. This may be done conveniently in terms of *band width,* which was defined in Chapter 2 as the width of the resonant curve, in cycles, at the point at which the power in the circuit is one-half the maximum power, expressed as a percentage of the resonant frequency.

It was discovered in Chapter 2 that at the frequency f_2 at which the power was one-half of the value at the resonant frequency f_r, the reactance of the circuit was equal to the resistance. In an effort to find the frequency at which the power is one-half the maximum value (current = 0.707 maximum), the reactance and resistance of an open-circuited line as given by Eqs. 7-145 and 7-146 may be equated. That is, at frequency f_2 or phase-constant value β_2,

$$R_0\alpha l = -R_0(1 - \alpha^2 l^2) \sin \beta_2 l \cos \beta_2 l$$

$$\frac{\sin 2\beta_2 l}{2} = \frac{-\alpha l}{1 - \alpha^2 l^2}$$

If αl is small, its square may be neglected with respect to unity in the denominator, so that

$$2\beta_2 l = \sin^{-1}(-2\alpha l)$$

Again, since αl is small, the sine may be replaced by the angle, so that

$$\beta_2 l = \frac{\pi}{2} + \alpha l \tag{7-159}$$

The frequency f_2 at which reactance and resistance are equal is then available by substitution of the value of β, giving

$$f_2 = \frac{v}{2\pi l}\left(\frac{\pi}{2} + \alpha l\right) \tag{7-160}$$

The chosen open-circuited line behaves like a series resonant circuit when its length is equal to $\lambda/4$. The value of the resonant frequency f_r is then available from the fact that, for $\lambda/4$ length,

$$\beta l = \frac{\pi}{2} = \frac{2\pi f_r l}{v}$$

$$f_r = \frac{v}{4l} \tag{7-161}$$

There will be a second frequency f_1, below f_r, at which the resistance equals the reactance. This frequency is

$$f_1 = \frac{v}{2\pi l}\left(\frac{\pi}{2} - \alpha l\right) \tag{7-162}$$

The band width is then given by

$$\frac{\Delta f}{f_r} = \frac{f_2 - f_1}{f_r} = \frac{(v/2\pi l)(\pi/2 + \alpha l - \pi/2 + \alpha l)}{f_r} \tag{7-163}$$

Since $\beta = 2\pi f/v$,

$$\frac{\Delta f}{f_r} = \frac{2\alpha}{\beta} \tag{7-164}$$

It has also been shown that the circuit Q is equal to the reciprocal of the band width, so that the merit factor of the line is given by

$$Q = \frac{\beta}{2\alpha} \tag{7-165}$$

Since α is ordinarily small, the value of Q for such a line may be very high. For the usual line designs it may approximate 1000 to 4000, which is considerably higher than the Q possible for lower-frequency lumped-constant circuits. The selectivity possible with lines as resonant circuits is correspondingly greater.

Since Q is increased if α is made small, the design of lines for minimum attenuation is of interest and will be considered in the next section.

7-29. Optimum design of the open-wire resonant line

In previous sections it has been found that the Q of a resonant line, the voltage step-up ratio, and the magnitudes of impedances or resistances available when lines are used as reactive or resonant elements all depend on the value of α. It is usually desirable that α be small. Since for a line of small losses

$$\alpha = \frac{R}{2R_0}$$

and both R and R_0 depend on the radius of the wires used, it is desirable to determine that value of the radius of the wire in an open-wire line which will make α a minimum.

From Eqs. 7-10 and 7-15 for a *copper line*, the value of α is

$$\alpha = \frac{(8.33 \times 10^{-8}\sqrt{f})/a}{2 \times 120 \ln (d/a)} \tag{7-166}$$

where d is the center-to-center spacing and a is the radius of the conductors of the open-wire line, in meters. Then

$$\alpha = \frac{3.47 \times 10^{-10}\sqrt{f}}{a \ln (d/a)} \text{ nepers/m} \tag{7-167}$$

Considering d constant and minimizing α with respect to a gives $d/a = \epsilon = 2.718$ as the value of the ratio for minimum attenuation on an open-wire line. However, this value of d/a is so small that the assumption of large d/a ratio used to derive Eq. 5-30 for the capacity of an open line does not hold. Consequently Eq. 7-167 should be rewritten, giving consideration to the fact that the charges will not be uniformly distributed around the conductor periphery, thereby altering the capacitance value to that of Eq. 5-33

$$C = \frac{\pi\epsilon}{\cosh^{-1}(d/2a)}$$

and adding a proximity factor to the resistance term. With these considerations the value of d/a for minimum attenuation is approximately 3.6 for an open-wire air-insulated line.

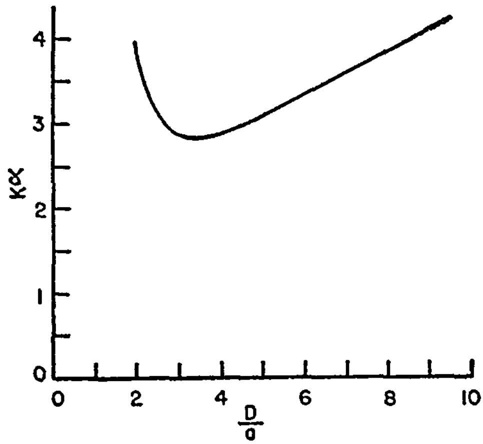


Fig. 7-36. Variation of α of an open-wire line with d/a ratio of the line.

The variation of a dimensionless factor, derivable by this method and proportional to α, as a function of d/a is shown in Fig. 7-36. This curve displays the expected minimum, but it is apparent that the minimum is quite broad, thereby permitting considerable latitude in the choice of element sizes for resonant lines.

Qualitatively, the minimum value provides a balance between R_0 and the size of conductor. If the conductor is made smaller in radius, then R_0 increases, thereby reducing the value of current flowing in a given line and reducing the losses. For the same change the conductor resistance rises, increasing the losses. The value $d/a =$ 3.6 achieves a balance of these opposing actions.

With a resonant line designed for a minimum value of α, the highest value of Q and other desirable properties of resonant lines are obtained.

7-30. Design considerations for the coaxial line

For optimum performance, the design of a high-frequency coaxial line is dependent on desirable values of the ratio b/a, where b is the inner radius of the outer conductor and a the radius of the inner conductor. If α is written as

$$\alpha = \frac{R}{2R_0}$$

it is found, by use of Eqs. 7-15 and 7-21, that α for a *copper coaxial line* may be written as

$$\alpha = \frac{4.16 \times 10^{-8} \sqrt{f}\,(1/b + 1/a)}{[2 \times 60 \ln (b/a)]/\sqrt{\epsilon_r}}$$

$$= 3.47 \times 10^{-10} \sqrt{f\epsilon_r}\, \frac{[(1/b) + (1/a)]}{\ln (b/a)} \text{ nepers/m} \qquad (7\text{-}168)$$

all dimensions being in meters. The term in parentheses may be written

$$= \frac{1}{b} \frac{[1 + (b/a)]}{\ln (b/a)} \qquad (7\text{-}169)$$

and if the outer conductor radius b is considered constant, the expression for α may be minimized, giving

$$\frac{dx}{d\left(\frac{b}{a}\right)} = \ln \frac{b}{a} - \frac{a}{b}\left(1 + \frac{b}{a}\right) = 0 \qquad (7\text{-}170)$$

$$\ln \frac{b}{a} = \frac{a}{b} + 1$$

This may be solved graphically, giving a value of b/a for minimum attenuation:

$$\frac{b}{a} = 3.6$$

The variation in attenuation for the coaxial line with the ratio b/a is shown in Fig. 7-37. The minimum occurring at $b/a = 3.6$ is broad, so that the dimensions to produce minimum attenuation are not critical. The characteristic impedance for an air-spaced coaxial line with dimensions for minimum attenuation is 77 ohms.

A second factor of importance in the design of coaxial lines

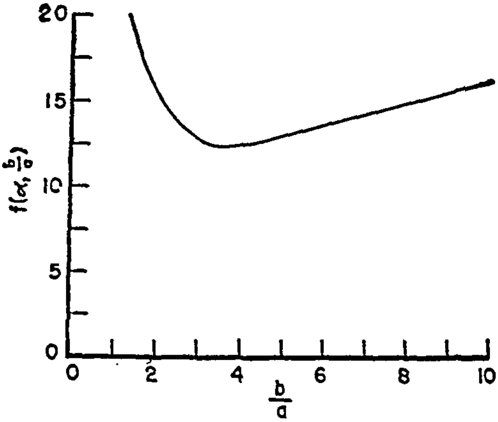


Fig. 7-37. Variation of α of a coaxial line with b/a ratio of the line.

required to handle considerable amounts of power is the design for minimum voltage gradient in the dielectric, or maximum voltage breakdown of the line. It may be argued that if a nearly equals b, the full line voltage appears across a thin layer of dielectric and the voltage gradient is large. If a is very small, the field at the central conductor surface is also very large. Between these extremes should be some value of a that will produce a minimum voltage gradient.

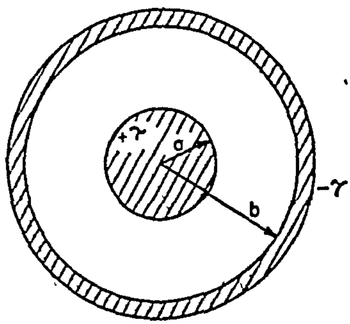


Fig. 7-38. Cross section of a coaxial line.

In a practical case the voltage between conductors will be held constant, with line charges $+\tau$ and $-\tau$ appearing on the conductors as shown in Fig. 7-38. The charges in coulombs per meter of length are related to the voltage between the cylinders according to

$$\tau = CV$$

where C is the line capacitance per meter of length. Insertion of Eq. 5-35 for the capacitance of the coaxial line gives

$$\tau = \frac{2\pi\epsilon V}{\ln (b/a)} \text{ coulombs/m} \tag{7-171}$$

According to Eq. 5-26, the electric field intensity in the dielectric is

$$\mathcal{E} = \frac{\tau}{2\pi r\epsilon} \text{ v/m}$$

The maximum value of field intensity will obviously occur at the surface of the inner conductor at $r = a$, so that

$$\max \mathcal{E} = \frac{\tau}{2\pi a\epsilon} \text{ v/m} \tag{7-172}$$

From these equations, the line charge τ may be eliminated, giving

$$\max \mathcal{E} = \frac{V}{a \ln (b/a)} \text{ v/m} \tag{7-173}$$

To obtain the dimensions that result in minimum field, the above expression may be minimized, holding the voltage V and dimension b

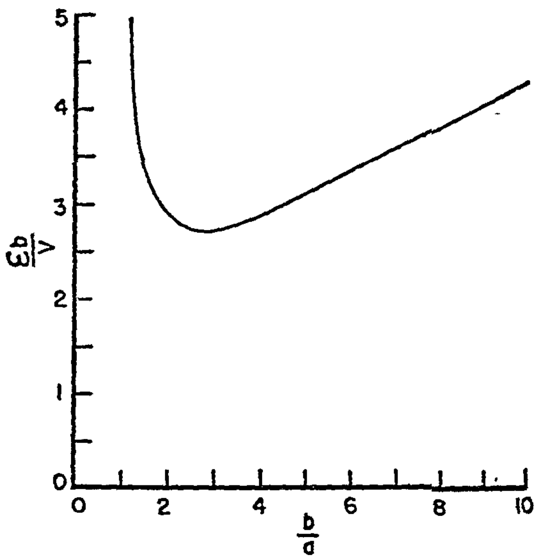


Fig. 7-39. Variation of maximum field intensity with b/a ratio for a coaxial line.

constant. Then

$$\frac{d\mathcal{E}}{da} = -V\,(\ln b - \ln a - 1) = 0$$

$$\ln \frac{b}{a} = 1 \tag{7-174}$$

For a minimum value of voltage gradient in the region between the conductors, the ratio of conductor radii should be

$$\frac{b}{a} = \epsilon = 2.718$$

The variation of field intensity at the inner conductor surface is expressed in a dimensionless plot as a function of b/a in Fig. 7-39. It can be seen from Figs. 7-37 and 7-39 that the difference between the values of b/a for minimum attenuation and minimum field intensity can be readily compromised.

Optimum design conditions for lines are summarized in Table 6.

TABLE 6

OPTIMUM DESIGN CONDITIONS FOR HIGH-FREQUENCY LINES

$\epsilon_r = 1$	Coaxial		Open-wire	
	$\frac{b}{a}$	Z_0	$\frac{d}{a}$	Z_0
Minimum attenuation	3.6	77	3.6	154
Maximum resonant impedance	9.2	133	8	250
Maximum Q	3.6	77	3.6	154
Maximum voltage breakdown	2.7	60	5.4	202
Maximum power transfer	1.6	30		

PROBLEMS

7-1. A 50-megacycle open-wire line is to be built of No. 8 copper wire and to have $Z_0 = 425$ ohms.

(a) Find the desired spacing D.

(b) Calculate the total L and C of 5 meters of this line.

7-2. An air-filled coaxial line of copper is to have a capacitance of 22 $\mu\mu$f per meter. The inner conductor has a diameter of 0.1 cm.

(a) Calculate the inductance of the line.

(b) Find the inner radius of the outer conductor.

(c) What is the characteristic impedance of the line?

(d) Find the phase constant and wavelength at a frequency of 25 megacycles, neglecting dissipation.

7-3. (a) Find Z_0, L, and C of an air-spaced coaxial line having a No. 12 inner copper conductor and $b/a = 10$, at a frequency of 40 megacycles.

(b) Repeat (a) if the line is filled with polyethylene as a dielectric.

(c) Find the velocity of propagation and λ for (a) and (b), neglecting resistance.

7-4. A line and load have the following values: $R_0 = 150$ ohms, $Z_L = 25 + j40$ ohms. Plot curves of the magnitude and phase angle of the input impedance as the value of βl varies from 0 to 2π.

7-5. (a) Calculate the reflection coefficient and standing-wave ratio for a line of open wire having No. 10 conductors spaced 3 cm, if the line is terminated in $50 + j75$ ohms at 7.3 megacycles. Neglect line resistance.

(b) If the line is supplied by a generator of 100 v, 200 ohms internal resistance, find the power delivered to the load at 7.3 megacycles if $l = 30$ m.

(c) Calculate the power that could be delivered to the load if it were matched to the line.

7-6. A transmission line is terminated in Z_L. Measurements indicate that the standing-wave minima are 102 cm apart and that the last minimum is 35 cm from the load end of the line. The value of S is 2.4 and $R_0 = 250$ ohms.

(a) Find Z_L in terms of real and reactive components.

(b) What frequency is being transmitted if line dissipation is neglected?

7-7. A certain line of $R_0 = 400$ ohms is $\frac{7}{16}\lambda$ long and open at both ends. Find the impedance seen by a generator connected at a point $\lambda/4$ from one end.

7-8. Find the input impedance of a coaxial line having $R_0 = 95$ ohms. The line is 20 m long, short-circuited at the far end, and operated at 10 megacycles. Neglect the line dissipation.

7-9. Given a dissipationless line of $R_0 = 300$ ohms, what lengths are needed to obtain an inductance of 7 μh at 75 megacycles, with both open- and short-circuit terminations.

7-10. (a) Find the b/a ratio for a coaxial line used to match a load of 300 ohms resistance to a line having $R_0 = 95$ ohms.

(b) If a coaxial matching section is used as above and made with a 0.2-cm-diameter internal conductor, find the internal diameter of the outer conductor.

7-11. A 200-m length of line of No. 8 copper wire is to be used to supply a load of 100 ohms and is to be operated at 30×10^6 c. Under short-circuit conditions the maximum voltage on the line is 310 v, and the maximum current is 0.584 amp. Find the spacing of 1-cm-diameter conductors needed for a $\lambda/4$ matching section to match this line to its load, neglecting conductor proximity.

7-12. An air-insulated coaxial line has an inside conductor of 3.6 mm copper (No. 7) and a copper outside conductor of 10 mm inside diameter. Find the input impedance of a quarter-wave short-circuited line of this material operating at 1500 megacycles.

7-13. An antenna, as load on a transmission line, produces a standing-wave ratio of 2.8, with a voltage minimum 0.12λ from the antenna terminals. Find the antenna impedance and the reflection factor at the antenna, if $R_0 = 300$ ohms for the line.

7-14. A lossless line $\frac{7}{16}\lambda$ long has an input impedance $Z_s/R_0 = 1.2 + j0.95$. Find the load impedance and the standing-wave ratio.

7-15. A prototype T-section filter is designed to operate at 10^8 c. One inductance of 5.2 μh and two capacitances of 7.8 $\mu\mu$f, are needed. The completed filter is to transmit direct current as well as the 10^8-c signal. Calculate the line lengths required and state the terminations to be used. Draw the completed circuit, if R_0 of the line is 200 ohms.

7-16. In Fig. 7-40(a), $R = 50$ ohms and $R_0 = 100$ ohms. Find the length L of an open stub needed to make the input impedance at A,B purely resistive.

7-17. An impedance $Z_R = 41 + j15$ ohms is connected as in Fig. 7-40(b) to a coaxial line of $R_0 = 100$ ohms. Find the lengths d_1 and d_2 to match the line to the load, by use of the circle diagram.

7-18. Considering dissipation, derive an expression for the impedance Z_{in} seen at distance d in Fig. 7-40(c). Plot the variation of Z_{in} against distance.

7-19. A coaxial line has $b = 0.55$ cm, $a = 0.10$ cm, and polystyrene dielectric.

(a) With this line terminated in 100 ohms resistance at 144 megacycles, find the shortest length that will have a reactive component of input impedance equal to $+j25$ ohms.

(b) Find the standing-wave ratio.

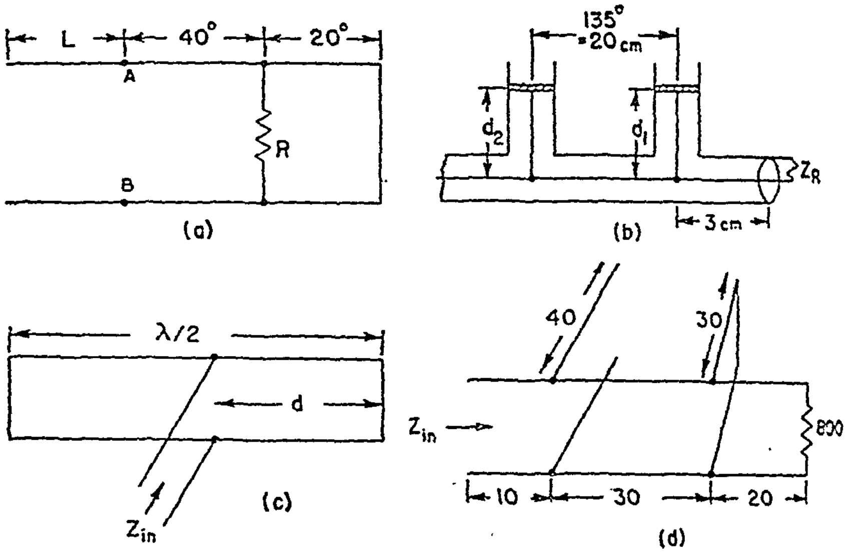


Fig. 7-40.

7-20. A standard coaxial line is designed as follows: central conductor No. 22 copper, polyethylene dielectric ($\epsilon_r = 2.25$), $b = 0.073$ in., design $Z_0 = 73$ ohms.

(a) Check the Z_0 by calculation from the dimensions.

(b) If the line is terminated in a reactance of $j150$ ohms, find the input impedance of a section 25 cm long at a frequency of 250 megacycles.

7-21. By use of the circle diagram find the input impedance of the line in Fig. 7-40(d), if the line and stubs are constructed of No. 4 copper wire spaced 10 cm. The frequency of operation is 220 megacycles. All dimensions are in centimeters.

7-22. A load having impedance of $Z_R = 140$ ohms is to be connected to a line of $R_0 = 100$ ohms by a quarter-wave matching transformer.

(a) Find the Z_0 of the matching transformer.

(b) What is S for the transformer?

(c) If the input voltage to the line is 100 v, find the load voltage, (dissipationless line).

(d) Find the location and magnitude of the maximum voltage on the quarter-wave transformer.

7-23. A line has a standing-wave-ratio of 4. The R_0 is 150 ohms and the maximum voltage measured on the line is 135. Find the power being delivered to the load.

7-24. A line of $R_0 = 300$ ohms is connected to a load of 73 ohms resistance. For a frequency of 45 megacycles, find the length, termination, and location nearest the load of a single stub to produce an impedance match.

7-25. The line of Prob. 7-24 is to be matched with double stubs, one located as near the load as possible. Specify length of both stubs, termination, and location of the second stub, using $\lambda/4$ spacing.

7-26. For a load of $Z_R/Z_0 = 0.8 + j1.2$, design a double-stub tuner, making the distance between stubs $\frac{3}{8}\lambda$. Specify the stub lengths and the distance from load to first stub. Stubs are short-circuited.

7-27. A load has admittance $Y_R/G_0 = 1.25 + j0.25$. Find the length and location for a single-stub tuner, short-circuited.

7-28. A line is made up to have $R = 0.008$ ohm, $L = 2.0$ μh, and $C = 5.0$ $\mu\mu$f, all per meter of line length. Frequency = 8 megacycles.

(a) Find Z_0.

(b) Determine the true velocity and wavelength.

(c) Calculate the Q and band width of a quarter wave of this line open-circuited.

7-29. An air-spaced coaxial line is shorted and used as a quarter-wave resonant circuit. If the line is 1.2 m long and has an outer copper conductor with inner radius of 4 cm and inner conductor radius of 0.8 cm, calculate the resonant frequency. Determine also the impedance, Q, and band width of the line.

7-30. The line of Prob. 7-29 is tapped at a point 0.3 m up from the short circuit. Find the impedance seen at the tap.

7-31. What is the maximum value of conductance that can be

8-1. Real and reactive power

The flow of *real* or *active power* over a transmission system is measured in terms of an average value over a cycle, and makes its presence known in the form of an in-phase current component. Active power is computed as the product of voltage, current, and the cosine of the phase angle, and is expressed in *watts*, *kilowatts*, or *megawatts*. It may be considered as having a direction of flow or of moving from a prime mover or energy source to a load or energy sink.

In addition there is a transfer or circulation of energy on an instantaneous basis between electric and magnetic fields present along a line or in connected equipment, and this transfer makes its presence known through a quadrature current component, and thus by the presence of reactive volt-amperes. These reactive volt-amperes are stated in *vars* (volt-amperes reactive), *kilovars*, or *megavars*, and may be measured on reactive-power meters or varmeters.

It is found that these meters will deflect in opposite directions when connected to capacitive and to inductive loads, and a *concept* of positive and negative vars, or generation and absorption of vars, has grown up by analogy to the reversal of a true wattmeter when indicating real power out of a source or generated, and power into a load or absorbed. Since this reactive energy merely circulates and is neither absorbed nor generated, then the idea of positiveness or negativeness, or of direction, should be carried only as a concept which may be of aid in system understanding. It is apparent, of course, that all energy stored in an electric field at the peak of the voltage cycle, must be transferred and appear as energy in a magnetic field somewhere in the circuit at the voltage zero. Thus the idea of flow has originated, and vars are frequently considered as flowing from or being generated in, devices predominantly capacitive in nature, and flowing to or being absorbed in, devices which are predominantly inductive.*

* It is just as logical to assume the vars being absorbed in a capacitive load, and generated in an inductive load. For this reason the algebraic sign to be assigned to Q, the reactive volt-amperes, has received considerable discussion in various standardizing bodies in recent years. The positive sign on Q indicating positive vars into a capacitor or leading load seems most commonly employed by practicing engineers and will be used here.

It is customary to express system or load volt-amperes in terms of the sum of the active power P and the reactive volt-amperes Q as

$$V\bar{I} = VI(\cos\theta + j\sin\theta) = P + jQ$$

where $V\bar{I}$ is called the *vector power* and $\bar{I}$ represents I conjugate, and positive Q is in accordance with the assumption of the footnote. Reactive volt-amperes are then assumed as positive when supplied to a synchronous capacitor (an overexcited synchronous motor), a bank of static capacitors, or by the capacitive susceptance of a transmission line. Negative vars are assumed as absorbed in the inductive reactance of a line, or an inductive load such as an induction motor.

Thus a transmission line takes positive reactive volt-amperes or vars in its capacitance, and takes negative vars in its inductive reactance. The amount of positive vars taken is dependent on the line voltage, the negative var absorption being dependent on line current. In the case of Z_0 termination of a line, the energy stored in the line capacitance is equal to that stored in the line inductance. Characteristic impedance or Z_0 termination then represents a condition in which a line supplies its own vars, that is, the positive vars taken by the line capacitance are equal to the negative vars taken by the line inductance, and no external reactive volt-amperes need be supplied.

The impedance which the terminal load presents to the line is determined by the output transformer ratio, and this in turn is controlled by the operating line voltage. With a high line voltage chosen in the design, the output load impedance may appear in a range well above Z_0, and the line will take or "generate" more positive vars than the negative vars taken by its own inductance, and these excess positive vars are available to "supply" or balance the negative vars taken by an inductive load. With a line voltage chosen low, the usual load impedance may take on values well below Z_0, causing heavy line currents and consumption of more negative vars in the line inductance than there are positive vars taken in line capacitance. The needed additional positive vars then must be introduced by operation of additional generating capacity at a leading angle, or by use of synchronous capacitors on the line.

Flow of both in-phase and quadrature current components produces power losses in the line resistance. In addition to this power loss, the supply of vars to a line requires operation of additional equipment and this is expensive to operate and maintain. Thus it is desirable that a line not be loaded too heavily, or that loading of Z_0 or above be used, so that a line does not require a supply of vars, or may actually contribute to the supply of reactive volt-amperes to the usual inductive loads. Line economics may not permit such ideal operation, however, and lines of high first cost must carry heavier loads.

8-2. Regulation of load voltage

The series line inductance and the shunt line capacitance are in series, and represent a partially resonant circuit, capable of being

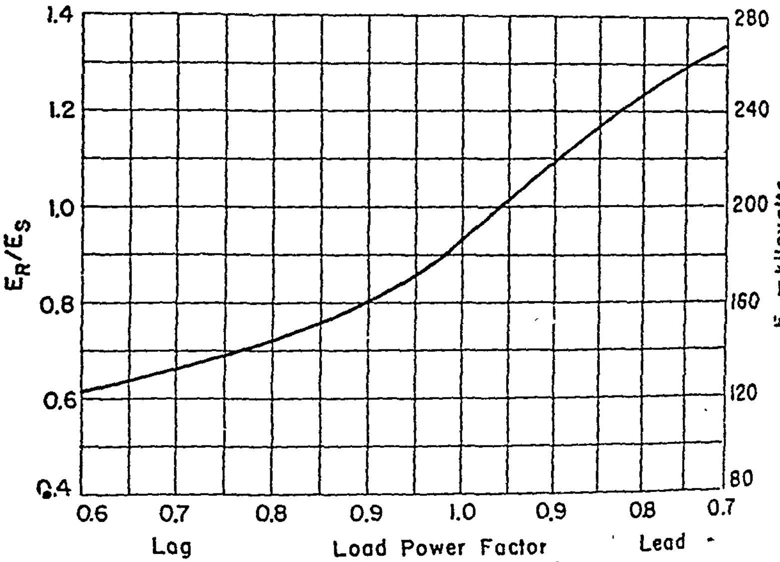


Fig. 8-1. Variation of load voltage with load power factor. $l = 100$ miles, $Zl = 200\ (80°)$, $Yl = 0.0013\ (90°)$.

tuned or detuned at the supply frequency, by the addition of adjustable capacitance in shunt with the line.

The output voltage is found to be a sensitive function of the additional shunted capacitance. Voltage regulation at the load end of a line then becomes relatively easy on lines handling large amounts of power, where a synchronous capacitor at the load may not introduce too much additional cost. Regulation is then supplied by adjusting the capacitance (or the combined load and capacitor power factor) by control of the field current of the synchronous capacitor. Automatic control of the load voltage is then also possible.

The possible variation of load voltage on a typical line, provided by shifting of the load-end power factor for constant kilowatts, is presented in Fig. 8-1. This shows the effectiveness of the addition of positive vars in controlling the line regulation.

On small-capacity lines where the more elaborate equipment cannot be justified, induction voltage regulators or tapped transformers are used for regulation.

8-3. The equivalent circuit

The transmission line equations may be used to compute performance of power lines, employing line currents, line-to-neutral voltages, and per-phase loads. However it is more usual to employ

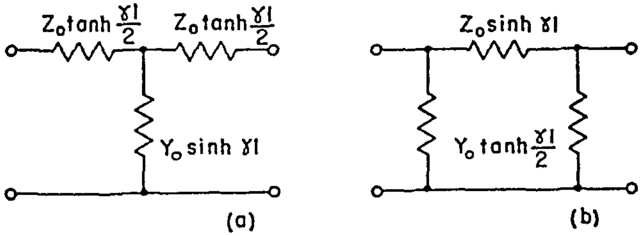


Fig. 8-2. Equivalent T and π for a transmission line.

an equivalent circuit for the line, this circuit being derived from the line equations as in Section 6-17 for long lines. For short lines the circuit is often arrived at by simply lumping the total series quantities, and introducing half the shunt susceptance at each end to form a π section.

The equivalent circuits, as derived from the long-line equations, are reproduced in Fig. 8-2, and the π section will be further discussed as an example. While such equivalent circuits are correct for only one frequency, this is the case for power transmission.

Using $Z = R + j\omega L$ ohms as the series impedance per unit length or per mile, and $Y = j\omega C$ mhos per unit length, on the assumption of negligible leakage conductance, then

$$\gamma = \sqrt{ZY} = \sqrt{(R + j\omega L)(j\omega C)} = \sqrt{-\omega^2 LC + j\omega CR} \quad (8\text{-}1)$$

Since $Z_0 = \sqrt{Z/Y}$, then

$$Z_0 = \sqrt{\frac{L}{C} - j\frac{R}{\omega C}} \quad (8\text{-}2)$$

and these are the same values as were obtained for the line of small dissipation in Section 7-22.

Since a line will be represented by a single equivalent π section, it is necessary to have the total line parameters for design of the equivalent circuit, and these will be defined as $\mathcal{Z} = Zl$, $\mathcal{Y} = Yl$, where l is the length of the line in appropriate units.

Referring to Fig. 8-2(b) for the π network, it is seen that

$$Z_B = Z_0 \sinh \gamma l = \sqrt{\frac{\mathcal{Z}}{\mathcal{Y}}} \sinh \sqrt{\mathcal{Z}\mathcal{Y}}$$

which may also be written

$$Z_B = \mathcal{Z}\,\frac{\sinh \sqrt{\mathcal{Z}\mathcal{Y}}}{\sqrt{\mathcal{Z}\mathcal{Y}}} \quad (8\text{-}3)$$

Also for the shunt element,

$$Y_A = Y_C = Y_0 \tanh \frac{\gamma l}{2} = \sqrt{\frac{\mathcal{Y}}{\mathcal{Z}}} \tanh \frac{\sqrt{\mathcal{Y}\mathcal{Z}}}{2}$$

which may be written

$$Y_A = Y_C = \frac{\mathcal{Y}}{2}\,\frac{\tanh \dfrac{\sqrt{\mathcal{Y}\mathcal{Z}}}{2}}{\dfrac{\sqrt{\mathcal{Y}\mathcal{Z}}}{2}} \quad (8\text{-}4)$$

Computation of sinh and tanh for complex arguments involves considerable labor or the use of charts (Reference 2). Because of the short electric length of power lines it is permissible to employ the series forms for these functions, cutting them off after the second term. Accuracy so obtained is better than 1 per cent for all usual line lengths and parameters. Then

$$\frac{\sinh \sqrt{\mathcal{Z}\mathcal{Y}}}{\sqrt{\mathcal{Z}\mathcal{Y}}} = 1 + \frac{\mathcal{Z}\mathcal{Y}}{3!} + \frac{(\mathcal{Z}\mathcal{Y})^2}{5!} + \cdots \tag{8-5}$$

$$\frac{\tanh \frac{\sqrt{\mathcal{Z}\mathcal{Y}}}{2}}{\sqrt{\mathcal{Z}\mathcal{Y}}/2} = 1 - \frac{\mathcal{Z}\mathcal{Y}}{12} + \frac{(\mathcal{Z}\mathcal{Y})^2}{120} - \cdots \tag{8-6}$$

The elements of the equivalent π section then become those of Fig. 8-3(a), or

$$Z_B = \mathcal{Z}\left(1 + \frac{\mathcal{Z}\mathcal{Y}}{6}\right) \tag{8-7}$$

$$Y_A = Y_C = \frac{\mathcal{Y}}{2}\left(1 - \frac{\mathcal{Z}\mathcal{Y}}{12}\right) \tag{8-8}$$

The equivalent T network may be obtained in a similar manner and appears in Fig. 8-3(b).

Networks *involving many transmission lines may have the lines* replaced with π sections according to Eqs. 8-7 and 8-8, the method

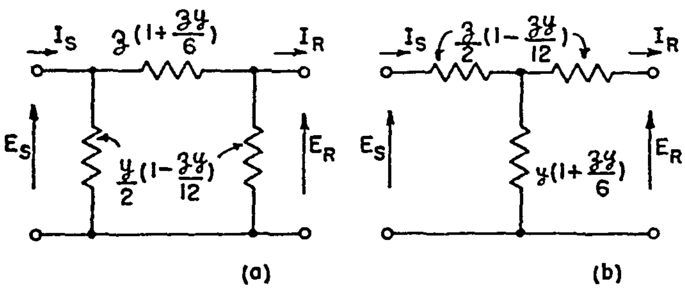


Fig. 8-3. Equivalent π and T, using line parameters for calculation.

of solution then being that of solving a complicated network, although based on the long-lines equations. An alternative method of arriving at equivalent results is given in the next section.

8-4. General circuit constants

The transmission line equations of Chapter 6 may be written

$$E_S = E_R \cosh \sqrt{\mathcal{Z}\mathcal{Y}} + I_R Z_0 \sinh \sqrt{\mathcal{Z}\mathcal{Y}}$$

$$I_S = I_R \cosh \sqrt{\mathcal{Z}\mathcal{Y}} + \frac{E_R}{Z_0} \sinh \sqrt{\mathcal{Z}\mathcal{Y}}$$

which in matrix form becomes

$$\begin{bmatrix} E_S \\ I_S \end{bmatrix} = \begin{bmatrix} \cosh \sqrt{\mathcal{Z}\mathcal{Y}}, & Z_0 \sinh \sqrt{\mathcal{Z}\mathcal{Y}} \\ \frac{1}{Z_0} \sinh \sqrt{\mathcal{Z}\mathcal{Y}}, & \cosh \sqrt{\mathcal{Z}\mathcal{Y}} \end{bmatrix} \begin{bmatrix} E_R \\ I_R \end{bmatrix} \tag{8-9}$$

In Chapter 1 it was shown that for any linear, passive, bilateral network it is possible to relate the input and output quantities as

$$E_S = AE_R + BI_R \tag{8-10}$$

$$I_S = CE_R + DI_R \tag{8-11}$$

or

$$\begin{bmatrix} E_S \\ I_S \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E_R \\ I_R \end{bmatrix} \tag{8-12}$$

where A, B, C, D are called the *general circuit constants.* Direct term by term comparison of the impedance matrices of Eqs. 8-9 and 8-12 shows that these constants will have values for a *transmission line* as follows:

$$A = \cosh \sqrt{\mathcal{Z}\mathcal{Y}}$$

$$B = Z_0 \sinh \sqrt{\mathcal{Z}\mathcal{Y}}$$

$$C = \frac{1}{Z_0} \sinh \sqrt{\mathcal{Z}\mathcal{Y}}$$

$$D = \cosh \sqrt{\mathcal{Z}\mathcal{Y}}$$

It should be noted that $A = D$ and this is true in general for symmetrical networks. Likewise

$$\Delta = AD - BC = 1 \tag{8-13}$$

is also true in general for linear, passive, bilateral networks, either symmetrical or unsymmetrical.

Equation 8-12 may be solved for the load quantities so that

$$E_R = DE_S - BI_S \tag{8-14}$$

$$I_R = AI_S - CE_S \tag{8-15}$$

or

$$\begin{bmatrix} E_R \\ I_R \end{bmatrix} = \begin{bmatrix} D & B \\ C & A \end{bmatrix} \begin{bmatrix} E_S \\ -I_S \end{bmatrix} \tag{8-16}$$

and this is the inverse relation.

It is, of course, possible to replace the hyperbolic functions in the definitions of A, B, C, D by their series expressions. If the first two terms are taken as an approximation as before, it is possible to write

$$
\begin{aligned}
A &= 1 + \frac{\mathcal{Z}\mathcal{Y}}{2} \\
B &= \mathcal{Z}\left(1 + \frac{\mathcal{Z}\mathcal{Y}}{6}\right) \\
C &= \mathcal{Y}\left(1 + \frac{\mathcal{Z}\mathcal{Y}}{6}\right) \\
D &= A
\end{aligned}
\tag{8-17}
$$

as the values of the general constants in terms of the total line parameters $\mathcal{Z}$, and $\mathcal{Y}$.

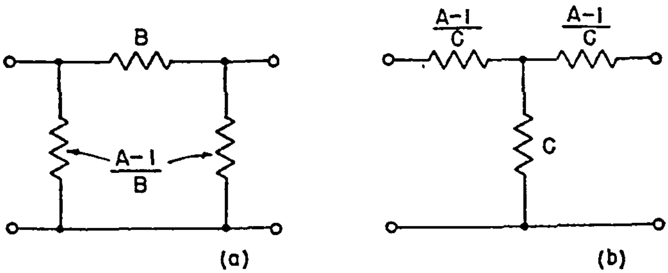


Fig. 8-4. Equivalent circuits for the line in terms of the A, B, C, D constants.

Given the values of A, B, C, D of a transmission line, it is possible to write expressions for the equivalent symmetrical π network for the line as

$$Z_B = B \tag{8-18}$$

$$Y_A = Y_C = \frac{A - 1}{B} \tag{8-19}$$

and for the equivalent symmetrical T as

$$Z_1 = Z_2 = \frac{A - 1}{C} \tag{8-20}$$

$$Y_3 = C \tag{8-21}$$

and these relations are illustrated in Fig. 8-4.

Example: A 150-mile single-circuit three-phase line has a resistance of 0.133 ohm per mile, series reactance of 0.827 ohm per mile, and shunt susceptance of 5.2×10^{-6} mho per mile, with conductance negligible. Find the A, B, C, D constants for the line, and the sending end voltage E_S for a total load of 90,000 kw at 0.85 lagging power factor, with $E_R = 127$ kv, line to neutral.

The line constants are then

$$\mathcal{Z} = 150(0.133 + j0.827) = 20 + j124 = 126\underline{/80.8^\circ} \text{ ohms}$$

$$\mathcal{Y} = 150(0 + j5.2 \times 10^{-6}) = 0.00078\underline{/90^\circ} \text{ mhos}$$

$$\mathcal{ZY} = 0.0984\underline{/170.8^\circ}$$

By Eq. 8-17,

$$A = 1 + \frac{\mathcal{ZY}}{2} = 1\underline{/0^\circ} + 0.0492\underline{/170.8^\circ} = 0.953\underline{/0.5^\circ}$$

$$B = \mathcal{Z}\left(1 + \frac{\mathcal{ZY}}{6}\right) = 125\underline{/80.8^\circ}(1 + 0.0164\underline{/170.8^\circ})$$

$$= 123\underline{/81.0^\circ} \text{ ohms}$$

$$C = \mathcal{Y}\left(1 + \frac{\mathcal{ZY}}{6}\right) = 0.00078\underline{/90^\circ}(1 + 0.0164\underline{/170.8^\circ})$$

$$= 0.000768\underline{/90.2^\circ} \text{ mho}$$

$$D = A = 0.953\underline{/0.5^\circ}$$

As a check it may be found that $AD - BC = 1 + j0$ as required.

The phase current at the load is

$$I_R = \frac{\text{kw}}{3E_R \times \text{pf}} = \frac{90{,}000}{3 \times 127 \times 0.85} = 278 \text{ amperes}$$

at an angle of 31.5° behind E_R. Then by use of Eq. 8-10,

$$E_S = AE_R + BI_R$$

$$E_S = 0.953\underline{/0.5^\circ} \times 127{,}000 + 123\underline{/81.0^\circ} \times 278\underline{/-31.5^\circ}$$

$$= 143{,}200 + j25{,}900 = 145{,}500\underline{/10.5^\circ} \text{ volts}$$

Thus 145.5 kv must be supplied to deliver 127 kv at the load.

Certain matrix manipulations in the solution of networks have been previously discussed in Chapter 1 but will be repeated here.

In particular it is helpful to find the A', B', C', D' constants for the composite line resulting when two lines, having constants A_1, B_1, C_1, D_1 and A_2, B_2, C_2, D_2, are connected in cascade. Writing the matrix relations for each line as illustrated in Fig. 8-5, gives

$$\begin{bmatrix} E_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} E_2 \\ I_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} E_2 \\ I_2 \end{bmatrix} = \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} E_3 \\ I_3 \end{bmatrix}$$

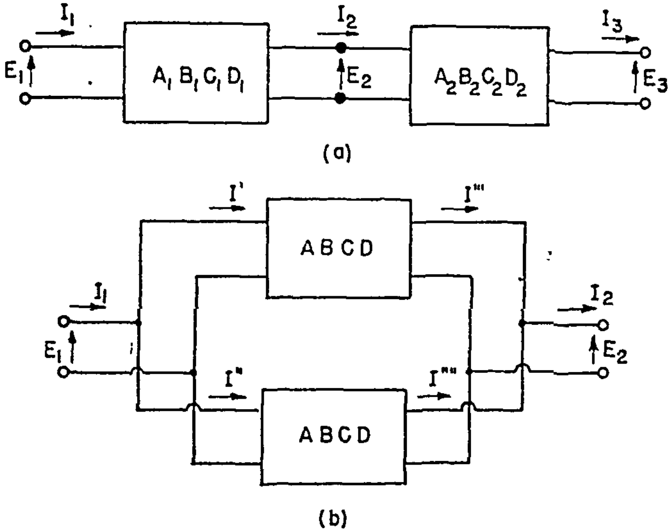


Fig. 8-5. (a) Two networks in cascade; (b) two networks in parallel.

and making the obvious substitution

$$\begin{bmatrix} E_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} E_3 \\ I_3 \end{bmatrix} \tag{8-22}$$

which relates E_1, I_1 and E_3, I_3, or the over-all input and output quantities. Performing the indicated multiplication gives

$$\begin{bmatrix} E_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A_1A_2 + B_1C_2, & A_1B_2 + B_1D_2 \\ C_1A_2 + D_1C_2, & C_1B_2 + D_1D_2 \end{bmatrix} \begin{bmatrix} E_3 \\ I_3 \end{bmatrix} \tag{8-23}$$

from which it is apparent that the A', B', C', D' quantities for the cascaded combination are

$$\begin{aligned} A' &= A_1A_2 + B_1C_2, \qquad & C' &= C_1A_2 + D_1C_2 \\ B' &= A_1B_2 + B_1D_2, \qquad & D' &= C_1B_2 + D_1D_2 \end{aligned} \tag{8-24}$$

The method may be readily extended to any number of networks or pieces of apparatus, whose A, B, C, D constants are individually known, when connected in cascade.

It is also common to parallel two transmission lines, in particular when the characteristics of the two lines are identical. Then writing for each network, from Fig. 8-5(b):

$$\begin{bmatrix} E_1 \\ I' \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E_2 \\ I''' \end{bmatrix}$$

and

$$\begin{bmatrix} E_1 \\ I'' \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E_2 \\ I'''' \end{bmatrix}$$

and solving for the currents

$$\begin{bmatrix} I'' \\ I'''' \end{bmatrix} = \frac{1}{B} \begin{bmatrix} D & BC-AD \\ 1 & -A \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}$$

and

$$\begin{bmatrix} I' \\ I''' \end{bmatrix} = \frac{1}{B} \begin{bmatrix} D & BC-AD \\ 1 & -A \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}$$

Since $I_1 = I' + I'''$ and $I_2 = I'' + I''''$, addition of the matrix equations gives

$$\begin{bmatrix} I' + I'' \\ I''' + I'''' \end{bmatrix} = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \frac{}{B} \begin{bmatrix} D & BC-AD \\ 1 & -A \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \tag{8-25}$$

from which it is possible to return to the original form as

$$\begin{bmatrix} E_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & \frac{B}{2} \\ 2C & D \end{bmatrix} \begin{bmatrix} E_2 \\ I_2 \end{bmatrix} \tag{8-26}$$

so that the A', B', C', D' constants for the parallel combination of the two identical lines are

$$A' = A, \qquad C' = 2C$$

$$B' = \frac{B}{2}, \qquad D' = D$$

in terms of the constants of a single line.

The methods of developing equivalent circuits for lines, demonstrated in this and the preceding section, provide means by which the transmission system may be represented as a network, and performance may be calculated by normal circuit procedures. These equivalent circuits also provide transmission-line models for use in network analyzer solutions of systems too complicated to be solved by direct mathematical computation.

8-5. Receiver power-circle diagram

A graphical method of solution for power and reactive volt-ampere conditions on a line is also available. It is assumed that the

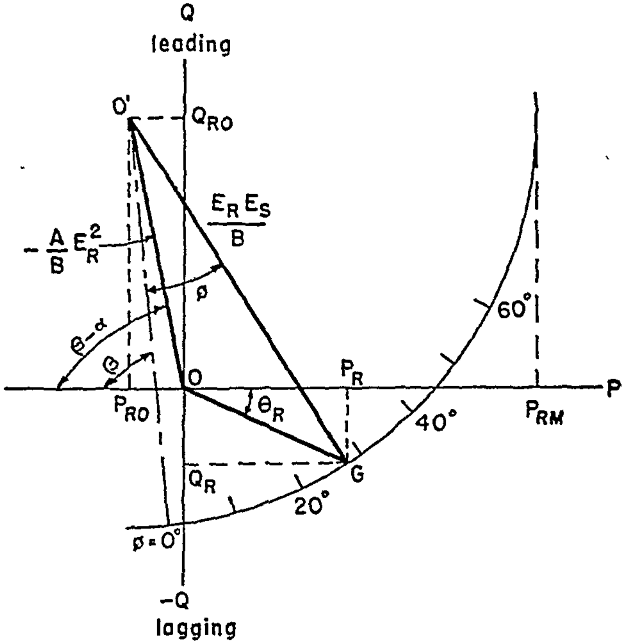


Fig. 8-6. Receiving-end power-circle diagram.

sending and receiving voltages of the line will be constant, but since this is in accord with usual operating practice, it is not a restriction on the solution. Consider Eq. 8-10 employing the general circuit constants:

$$E_S = AE_R + BI_R$$

Multiplying by E_R/B gives

$$\frac{E_R E_S}{B} - \frac{A}{B} E_R^2 = E_R I_R = P_R + jQ_R \tag{8-27}$$

If E_R is the receiving end line-to-neutral kilovolts and I_R is line current, phase quantities are given in megawatts and megavars in Eq. 8-27. Total three-phase quantities may be used if the left side of the equation is multiplied by three.

If ϕ is the angle between E_S and E_R, with E_R as reference, then

$$P_R + jQ_R = \frac{|E_R E_S|\underline{/\phi}}{B} - \frac{A}{B} E_R^2 \tag{8-28}$$

It has been assumed that E_R and E_S are fixed, so that the second term on the right represents a line from the origin to a point in the plane, the angle of this line being determined by the angles of A and B. The first term on the right is constant in magnitude for a given line but variable in angle. The locus of the right side of the equation is then a circle, in which the second term locates the center on the coordinate system, and the first term is the radius. The circle is the locus of received P_R and Q_R values for the fixed E_R and E_S, as the angle ϕ varies. A scale of angles representing ϕ may be laid off around the circle, starting from the point which may be found for $\phi = 0$.

Figure 8-6 illustrates such an equation by the triangle $OO'G$. The line OO' is drawn as $-(A/B)E_R^2$ from the origin and fixes the point O' as the center of the locus circle. The line $O'G$ is the radius of the circle, proportional to $|E_R E_S|/B$. Line OG is then the sum of OO' and $O'G$ and represents $P_R + jQ_R$ as $|E_R E_S|/B$ sweeps around the circle with changes in angle ϕ between E_R and E_S.

The dimensions for the graph may be readily obtained if

$$A = a\underline{/\alpha}, \qquad B = b\underline{/\beta}$$

and Eq. 8-28 be written

$$P_R + jQ_R + \frac{a}{b} E_R^2[\cos(\beta - \alpha) - j \sin(\beta - \alpha)] = \frac{|E_R E_S|}{B}\underline{/\phi}$$

The angles are written as $\beta - \alpha$ because β is ordinarily much larger than α. Taking the square gives

$$\left[P_R + \frac{a}{b} E_R^2 \cos(\beta - \alpha)\right]^2 + \left[Q_R - \frac{a}{b} E_R^2 \sin(\beta - \alpha)\right]^2 = \frac{E_R^2 E_S^2}{B^2} \tag{8-29}$$

Since E_R, E_S, and B are constants, this is the equation of a circle as expected.

The radius is then

$$R_R = \frac{|E_R E_S|}{|B|} \text{ megawatt units} \tag{8-30}$$

where the actual length is measured in megawatts. The coordinates of the circle center are

$$P_{RO} = -\frac{a}{b} E_R^2 \cos(\beta - \alpha) \text{ megawatts} \tag{8-31}$$

$$Q_{RO} = \frac{a}{b} E_R^2 \sin(\beta - \alpha) \text{ megavars} \tag{8-32}$$

and are measured along the equal scales of megawatts and megavars laid out on the abscissa and ordinate.

The position of the reference line for $\phi = 0°$ is found by laying off angle β from the reference axis as seen in Fig. 8-6. This is possible since E_R was originally chosen as reference but multiplication of the expression by E_R/B rotated the diagram by the angle $-\beta$. A scale of values of ϕ can then be added, starting from the reference line on the circle, and these will indicate the angle between E_S and E_R for all possible values of P_R and Q_R, at the assumed voltages.

As a result, the circle shows the required values of Q for particular desired P values, and indicates the reactive power which must be available at the load if the desired E_R and E_S values are to be met. In particular the circle shows that there is a maximum value of real power which may be transmitted over the line for the given E_S and E_R values, this maximum value being equal to the greatest abscissa value of the circle, indicated as P_{RM} in Fig. 8-6. This will be further discussed in Section 8-7.

Example: The line discussed in Section 8-4 is operated at fixed $E_S = 145.5$ kv, $E_R = 127$ kv, line to neutral. With

$$A = a\underline{/\alpha} = 0.953\underline{/0.5°} \quad \text{and} \quad B = b\underline{/\beta} = 123\underline{/81.0°}$$

the values for the receiving circle diagram of Fig. 8-7 are obtained as

$$R_R = \frac{|E_R E_S|}{|B|} = \frac{127 \times 145.5}{123} = 150.3 \text{ megawatt units radius}$$

$$P_{RO} = -\frac{a}{b} E_R^2 \cos(\beta - \alpha)$$

$$= -\frac{0.953}{123} \times 127^2 \times \cos 80.5° = -20.7 \text{ megawatts}$$

$$Q_{RO} = \frac{a}{b} E_R^2 \sin(\beta - \alpha)$$

$$= \frac{0.953}{123} \times 127^2 \times \sin 80.5° = 123.0 \text{ megavars}$$

Since the voltages used were line to neutral, all P_R and Q_R values will be per phase. The receiving circle diagram for this problem is drawn in Fig. 8-7.

The results of the example of Section 8-4 may be checked, since the voltages chosen above are those previously calculated. Using the load of 90 megawatts at 0.85 power factor, the P_R and Q_R values per phase are

$$P_R = \frac{90}{3} = 30.0 \text{ megawatts}$$

$$Q_R = \frac{90 \times \sin(\cos^{-1} 0.85)}{3 \times 0.85} = -18.6 \text{ megavars}$$

These values are located on the circle at L, and it is seen that the line will carry this load at the assumed voltages, with the angle ϕ between E_S and E_R equal to 10.5°, which checks the computed value.

It may be necessary to carry a load of 0.80 power factor lagging, under the same voltage conditions. Such a power factor *load line* drawn with appropriate slope is shown in Fig. 8-7, and is the locus of all P_R and Q_R having the required power factor. The circle diagram immediately shows that under the assumed voltage conditions, a load of 26 megawatts and 20 megavars lagging will be carried with $\phi = 8.5°$. However it may be desired that the line carry 40 megawatts 0.80 power factor lagging, and this load locates point M having $P_M + jQ_M = 40 - j30$. If M is projected up to the

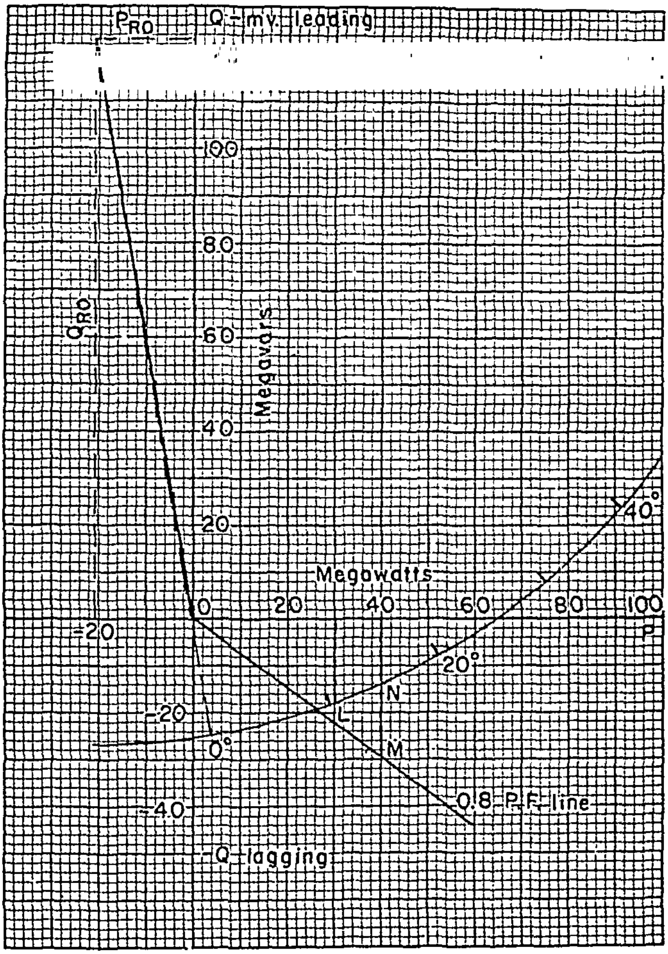


Fig. 8-7. Receiving circle-diagram for example of Section 8-5.

circle at N, it is found that the coordinates of N are $P_N + jQ_N = 40 - j14$. Thus if $30 - 14 = 16$ megavars of leading reactive power were supplied at the load by a synchronous capacitor, the specified load condition could be carried at the specified voltages, with an angle of $\phi = 14.5°$. Thus actual loads can be made to meet desired line operating voltages by providing reactive power at the load.

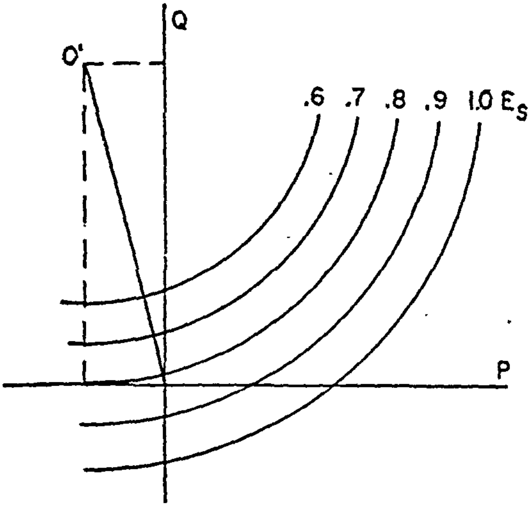


Fig. 8-8. Alterations in the circle diagram for varying E_S.

Since the circle center is independent of E_S, but the radius is a linear function of E_S, it is possible to obtain a family of receiver power circles representing fixed E_R and variable E_S as in Fig. 8-8, by changing the radius in proportion to the changed value of E_S. This broadens the application of this circle diagram.

8-6. Sending power-circle diagram

In a manner similar to that of the preceding section, and starting with Eq. 8-14

$$E_R = DE_S - BI_S$$

it is possible to develop a circle diagram showing sending end or input conditions for fixed E_R and E_S. Multiplying by E_S/B

$$\frac{D}{B}E_S{}^2 - \frac{E_RE_S}{B} = E_SI_S = P_S + jQ_S \tag{8-33}$$

Again these quantities are in megawatts and megavars per phase, if E_R and E_S are in kilovolts.

If ϕ is the angle between E_S and E_R, with E_S as reference, then

$$P_S + jQ_S = \frac{D}{B}E_S{}^2 - \frac{|E_RE_S|}{B}\underline{/\phi} \tag{8-34}$$

and this is again recognizable as a circle with center located by the first term on the right, the radius determined by the second term on the right. Defining $D = d/\delta$, $B = b/\beta$, then the radius is

$$R_S = \frac{|E_R E_S|}{|B|} \text{ megawatt units} \tag{8-35}$$

with center at

$$P_{SO} = \frac{d}{b} E_S^2 \cos(\beta - \delta) \text{ megawatts} \tag{8-36}$$

$$Q_{SO} = -\frac{d}{b} E_S^2 \sin(\beta - \delta) \text{ megavars} \tag{8-37}$$

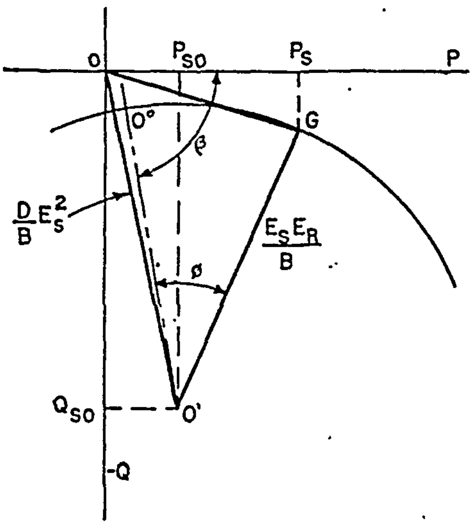


Fig. 8-9. Sending-end power-circle diagram.

The reference for $\phi = 0$ is laid off at an angle β from the reference axis as before, this time resulting in the $\phi = 0°$ point to the right of the radial line to the circle center. Figure 8-9 shows the sending-end circle diagram.

It is possible to draw the sending and receiving circles on the same P and Q axes, and obtain additional performance information. This is done in Fig. 8-10 for the line used as an example in Section 8-5. It was therein found that the line operated at $\phi = 10.5°$, and operating information may be obtained by drawing a line included, as properly reflected into the line circuit by the transformer ratios.

For the case of a line connecting two generators or two large power systems, where the power capacity of both may be considered very large, and where the voltage is regulated at both ends of the line, the possible power limit can be obtained from the largest abscissa value of the receiving end circle diagram. This can be expressed as

$$P_{RM} = R_R - P_{RO}$$

$$P_{RM} = \frac{|E_R E_S|}{|B|} - \frac{a}{b} E_R^2 \cos(\beta - \alpha) \text{ megawatts/phase} \quad (8\text{-}41)$$

which gives the limit as set by the line impedances.

Transient stability of a line or system under transient conditions of sudden load application or switching is indicated by the ability of the system to damp out any oscillations in angular displacement between the connected machines. If unstable, these transient oscillations will increase in magnitude to the point where the instantaneous power flows between machines become so great as to completely disrupt system operation, and damage equipment. *Stability is thus distinct from power limit in that it includes the* dynamic characteristics of the connected prime movers and synchronous motor loads.

These external stability factors involve the prime-mover torque, the inertia of the prime mover and generator, and inertia and torque conditions imposed by motor loads on the system. In addition, the generated voltages of generators and synchronous motors, and the line and machine reactances are important. Since the line characteristics are only one element in determination of transient stability, and since the consideration of machine characteristics is beyond the scope of this book, the matter will not be gone into further here.

8-8. Network analyzers

Many complex transmission systems can only be studied by use of system models in electric analogue computors known as *network analyzers*. These devices use the transmission line equivalent circuit to represent lines, and have power sources and loads to set up a low-power model of a complete system. They are employed in system studies of power flow and generating plant loading of existing and contemplated future systems, for studies to determine proper protective breaker settings under short-circuit conditions, and for determination of stability limits.

A network analyzer employs adjustable R, L, and C networks, usually arranged as π sections, to simulate lines, although short

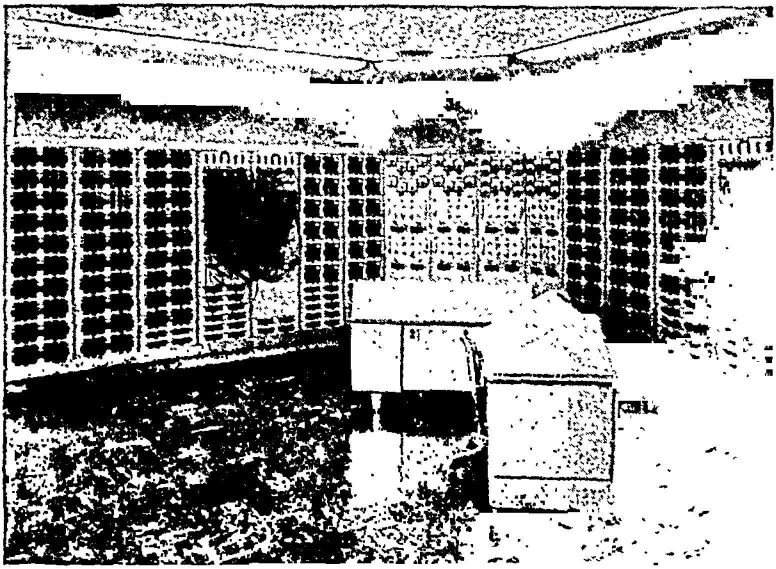


Fig. 8-11. Partial view of large network analyzer.

lines may be represented just by their series quantities. In addition there are provided adjustable R and L series or parallel circuits to serve as inductive loads, capacitors for representation of leading loads or of series capacitors, variable-ratio transformers, and 1:1 transformers for representing mutual coupling between phases or parallel lines in short-circuit studies. The generators or power sources have output voltage controls analogous to field current adjustments in actual generators, and phase angle controls analogous to governor settings on driving turbines.

For usual three-phase system studies, a system map is drawn on a single-phase line-to-neutral basis, with impedance expressed as ohms to neutral on a common voltage base, obtained by properly reflecting impedances through any connected transformers. Line impedances are obtained by calculation based on the geometry of the line as outlined in Chapter 5, although tables are often used to shorten the work. The analyzer units are then connected according to the system map by cords and plugs as in a telephone switchboard. Voltage, current, watts, vars, and phase angles may be read for any unit at a central operating desk, where the master instrument set may be connected to the desired line or load point by push buttons and relays. To prevent errors due to the electric burden imposed by the instruments, the latter are usually operated through precision electronic amplifiers of high input impedance. A partial view of a large analyzer is shown in Fig. 8-11.

Actual system operating voltages, currents, and impedances are scaled to small analyzer base quantities. A convenient system operating voltage, for which the map was prepared, such as 100 kv (line to neutral), is represented by the unit analyzer base voltage, say 100 volts. The analyzer base current of possibly 1 ampere may be allowed to represent 1000 system amperes. Then 100 megawatts in the system appears as 100 watts on the analyzer, and 100 ohms system impedance is represented by 100 ohms on the analyzer. These analyzer base quantities vary in different analyzer designs, but the principle is the same.

Since the system components are calculated in terms of pure R, L, and C quantities and ideal transformers, it is desirable that the analyzer components be equally ideal. In the case of resistors and capacitors no difficulties are usually encountered, but for inductors and transformers the specification is not so easily met. This difficulty of pure inductors, or at least very high Q inductors, has lead to a variety of analyzer designs, each fostered by an attempt to provide higher accuracy through reduction of losses in inductors and transformers; reduced size and cost are also of major importance. Some of the more interesting of these methods will be mentioned.

The first analyzer was built in 1929 at Massachusetts Institute of Technology by H. L. Hazen and others, and employed a 60-cycle operating frequency. At this frequency iron losses were high and high quality inductors and transformers were large and expensive. Soon the Westinghouse Electric Corporation at Pittsburgh and the General Electric Company at Schenectady constructed units, operating at 440 and 480 cycles per second, respectively, to improve the Q of reactors by employment of less copper and less and better steels. In these analyzers the reactor losses are of the order of 4 per cent, or most reactors have Q values of 25, and for high accuracy work it is necessary to reduce series resistor values to compensate for reactor losses. The base values for the Westinghouse design are 100 volts, 1 ampere, 100 ohms, and 100 watts; for the General Electric unit they are 50 volts, 50 milliamperes, 1000 ohms, and 2.5 watts. A number of similar analyzers of both types are in use elsewhere.

One method employed to reduce the size, weight, and cost of the needed reactors was introduced by E. B. Phillips in 1946, and such a unit is installed at the University of Kansas. In principle the reference impedance axes are shifted 5° so that a pure inductor will then be represented by one having an 85° angle, and a pure resistor by one having a $-5°$ angle. The resistors are brought to $-5°$ by a parallel capacitor. The inductors, adjusted to 85° by a series resistor, then appear to have an infinite Q by this method. Representation of shunt capacitance is not so easy, since this requires units having $-95°$ angles. However, this is handled by introduction of a properly phased auxiliary voltage on the capacitor bus. Proper metering is insured by shifting of wattmeter current and voltage axes. Because of the high inductor Q obtained by this method, it is possible to operate this analyzer at 60 cycles, but transformers are not improved by this method and some losses are encountered. The base values are 100 volts, 25 milliamperes, 4000 ohms, and 2.5 watts.

Also in 1946, J. D. Ryder and W. B. Boast designed and installed at Iowa State College an analyzer operating in conventional manner but at 10,000 cycles per second. This insured Q values above 100, or losses less than 1 per cent for all reactors, and transformer characteristics extremely close to the ideal. No operating difficulties are encountered due to the high frequency, and size and cost are materially reduced. Because of complete electronic generation and metering, the base values for this analyzer are 10 volts, 0.1 ampere, 100 ohms, and 1 watt. A second unit incorporating this design is at the University of Illinois.

In 1951, E. W. Kimbark, J. H. Starr, and J. E. Van Ness installed at Northwestern University an analyzer in which the impedance axes are shifted 180°, and conjugate impedances are used. Thus capacitors, easily obtained with high Q, are used to represent inductors, and resistors represent resistors. Shunt capacitances in the system must be represented by inductors, but since such capacitances have only nominal effect on system operation, an error due to low Q here is not so serious as when it occurs in series quantities. Base values used are 50 volts, 10 milliamperes, 5000 ohms, 0.5 watt, and operation is at 120 cycles.

There are several installations using other compromise principles. It should be noted however, that regardless of operating frequency and base values, the actual employment in system studies is similar on all. The use of network analyzers to solve system problems is now very common, and some acquaintanceship with this form of computer is advisable for anyone studying power systems.

PROBLEMS

8-1. A three-phase 60-c line is 150 miles long. The conductors are 900,000 cir mil, 37-strand copper, with an outside diameter of 1.092 in. They are equilaterally spaced 38 ft. Find the equivalent π circuit for the line, and the A, B, C, D constants.

8-2. A power station generates at 13,200 v line to line, and is carrying a load of 60,000 kva at 0.86 lagging power factor.

(a) Find P, Q, and generator line current to load.

(b) How many kva leading must be supplied to bring the station to unity power factor?

8-3. The line of Prob. 8-1 with input at 220 kv line to line, supplies a total load of 100 mva (megavolt-amperes) at 0.8 lagging power factor. Calculate the load voltage, and the input megawatts and megavars, total.

8-4. A three-phase 60-c line has three 650,000 cir mil conductors with 61 strands and self-GMR of 0.3586 in., spaced 30 ft each in a horizontal plane. The line is 150 miles long and has load values of 200 kv line-to-line, 60 mw (megawatts), 0.9 lagging power factor, total. Find E_S, I_S, and sending end megawatts and megavars. What is the efficiency of transmission?

8-5. Compute the A, B, C, D constants for the line of Prob. 8-4. Find the equivalent constants if two such lines are operated in parallel. Show that $AD - BC = 1$ as a check.

8-6. A 60-c line has total constants of $\mathcal{Z} = 80\underline{/80°}$ ohms and $\mathcal{Y} = 500 \times 10^{-6}\underline{/90°}$ mhos. Draw the equivalent π and T networks.

8-7. Find E_R for the line of Prob. 8-6, if $E_S = 96$ kv line-to-neutral, and the received power is 40 megawatts at 0.85 power factor, lagging. Use the A, B, C, D constants.

8-8. Draw on 11- $\times$ 17-in. graph paper sending end and receiving end circle diagrams for the line of Prob. 8-6, assuming sending line voltage of 125 kv and receiving voltage of 115 kv. If the load has a power factor of 0.95 lagging, find the reactive kva which must be supplied at the load, the angle between E_S and E_R, and P_S and Q_S for loads of 20,000, 40,000, and 60,000 kw total.

8-9. The Boulder Dam-Los Angeles three-phase line has tubular conductors of 1.40 in. outside diameter and 512,000 cir mils, spaced 32.5 ft horizontally. Draw a large-scale receiving and sending circle diagram, for $E_s = 287{,}000$ and $E_R = 275{,}000$ v. Plot curves of P_S and Q_S, angle between E_S and E_R, Q_R, and line loss, against received power.

8-10. What is the power limit for the line of Prob. 8-9?

8-11. Draw circle diagrams on 11- $\times$ 17-in. graph paper in terms of the ratio of E_S/E_R for the line of Prob. 8-9.

(a) For $E_S = E_R = 166$ kv (line-to-neutral) and load of 120 megawatts, find Q_R, P_S, Q_S, and line loss.

(b) What synchronous capacitance is required at the load to maintain 166 kv (line to neutral) and load of 100 megawatts total, if load power factor varies from 0.80 lagging to unity?

REFERENCES

1. Tarboux, J. G., *Introduction to Electric Power Systems*, International Textbook Company, Scranton, Pa., 1949.
2. Woodruff, L. F., *Principles of Electric Power Transmission*, 2d ed., John Wiley & Sons, Inc., New York, 1938.
3. *Electrical Transmission and Distribution Reference Book*, 3d ed., Westinghouse Electric Corp., East Pittsburgh, Pa., 1944.
4. Hazen, H. L., Schurig, O. R., and Gardner, M. F., "The M.I.T. Network Analyzer: Design and Application to Power Systems," *Trans. AIEE*, 49, 1102 (1930).

5. Travers, H. A. and Parker, W. W., "An Alternating-Current Calculating Board," *Elec. J.*, **27**, 266 (1930).
6. Kuehni, H. P. and Lorraine, R. G., "A New A-C Network Analyzer," *Trans. AIEE*, **57**, 67 (1938).
7. Parker, W. W., "The Modern A-C Network Calculator," *Trans. AIEE*, **60**, 977 (1941).
8. Phillips, E. B., "A New Principle Is Employed for 60 Cycle A-C Network Analyzers," *Trans. AIEE*, **71**, 18 (1952).
9. Ryder, J. D. and Boast, W. B., "A New Design for the A-C Network Analyzer," *Trans. AIEE*, **65**, 674 (1946).
10. ———, "Design Improvements and Operating Experience with 10 Kc Network Analyzers," *Communications and Electronics, AIEE*, No. 4, 437, 1953.
11. Kimbark, E. W., Starr, J. H., and Van Ness, J. E., "A Compact Inexpensive A-C Network Analyzer," *Trans. AIEE*, 71, 122 (1952).

Chapter 9

EQUATIONS OF THE ELECTROMAGNETIC FIELD; RADIATION

Up to this point, consideration has been given to the currents and voltages present in circuits and on lines, and only incidentally to the electric and magnetic fields with which the currents and voltages are always associated. At several points discussion of the currents or voltages present in the circuit has not provided an entirely satisfactory explanation, and it has been necessary to refer to the fields present, without having a groundwork sufficient for complete treatment.

Obviously, the problem is of "the-chicken-and-the-egg" variety, since currents and voltages are never found without their associated fields and vice versa. However, in many problems a knowledge of the electromagnetic fields present provides a solution that is more easily obtained and understood than one obtained by consideration of currents or voltages. Particularly is this true in circuits which are physically large with respect to the wavelength or in which the time of propagation of the electrical disturbance from one part of the circuit to another is appreciable with respect to the time of a cycle. Such circuits are encountered in lines, wave guides, and in the radiation of electromagnetic energy.

The *electromagnetic field equations* were first concisely stated by James Clerk Maxwell in 1864, in an effort to show that light is electromagnetic in nature. His correction of a mistake of the early experimenters in overlooking the necessity for the displacement current, or in fact implying that electromagnetic forces propagate at an infinite velocity, led to theoretical proof of the possibility of radiation of electromagnetic energy and the correlation of such radiation with light. Experimental proof that such radiation does occur from electric circuits was not available until 1886, when Heinrich Hertz demonstrated the fact and thereby laid the groundwork for radio, upon which Marconi was to build a decade later.

The electromagnetic field equations, or Maxwell's equations, as they are frequently called, are in fact merely restatements of basic *experimental* electromagnetic knowledge and may be looked upon as the basis of all electrical engineering.

9-1. Multiplication of vectors

Certain mathematical symbolism or shorthand associated with operations on vectors is quite common in the study of electric and magnetic fields. This vector symbolism will be introduced and discussed where required.

For instance, if a force F acts on a body of mass m and moves it through a distance s, the work done is

$$W = sF \cos \theta$$

where θ is the angle between the direction of the force and the positive direction of movement s. This type of multiplication is quite common in physical problems, where F and s are vectors and involve direction, but their product W is a scalar. The operation is called a *scalar product*, and the cos θ is not written but implied in the dot symbol as

$$W = sF \cos \theta = \boldsymbol{s} \cdot \boldsymbol{F} \tag{9-1}$$

Because of the symbol the result is also known as the *dot product* of the two vectors. In general then, the dot product of two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ is defined

$$\boldsymbol{A} \cdot \boldsymbol{B} = AB \cos \theta \tag{9-2}$$

and the result is scalar.

Another type of multiplication of two vectors arises in other physical problems, for example, the force F on a current I in a path of unit length in a magnetic field of density B. The magnitude of the result is known to be

$$F = IB \sin \theta$$

where θ is the angle between direction of $\boldsymbol{I}$ and the direction of $\boldsymbol{B}$. The result here, however, is a force with direction and this is a vector. It is known, of course, that this force will be perpendicular to the plane containing $\boldsymbol{I}$ and $\boldsymbol{B}$, and directed as by the movement of a right hand screw as $\boldsymbol{I}$ is turned into $\boldsymbol{B}$.

It is again convenient to employ a shorthand symbol which, in itself, tells all this, and this is the implication of the $\times$ symbol in the *vector product* where

$$F = I \times B \tag{9-3}$$

Because of the symbol chosen, the operation is also called the *cross product.*

In general, the cross or vector product of two vectors A and B is another vector, in magnitude equal to $AB \sin \theta$, directed perpendicular to the plane containing A and B, and having positive direction as that of the advance of a right-hand screw as A is turned into B.

Because of this last statement, it is apparent that the order of statement of the vectors is important and that

$$B \times A = -(A \times B) \tag{9-4}$$

It is *not correct* to write

$$A \times B = AB \sin \theta$$

since the right side is a magnitude only, without direction. However, it is convenient to define a *unit vector* n, which has a magnitude of unity and the direction of the cross product result, and then it *is correct* to write

$$A \times B = nAB \sin \theta \tag{9-5}$$

as a general definition for the vector product.

In space, a vector may be considered the sum of three components along the reference axes. However, to write

$$A = A_x + A_y + A_z$$

is not correct because A_x, A_y, A_z are merely scalar magnitudes measured in the x, y, and z directions, respectively. Again, to simplify all this language, it is possible to call on the unit vector, and define as three vectors of unit magnitude, i along the x axis, j along the y axis, and k along the z axis. It is then proper to write A as the sum of three vectors

$$A = iA_x + jA_y + kA_z \tag{9-6}$$

as the expression in terms of rectangular components.

Products of vectors written in the above form may occur, and it is necessary to inquire into the meaning of terms like $i \cdot i$, and $i \cdot j$. Because the angle between i and i is zero, the dot product $i \cdot i$

they are frequently called, are in fact merely restatements of basic *experimental* electromagnetic knowledge and may be looked upon as the basis of all electrical engineering.

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Certain mathematical symbolism or shorthand associated with operations on vectors is quite common in the study of electric and magnetic fields. This vector symbolism will be introduced and discussed where required.

For instance, if a force F acts on a body of mass m and moves it through a distance s, the work done is

$$W = sF \cos \theta$$

where θ is the angle between the direction of the force and the positive direction of movement s. This type of multiplication is quite common in physical problems, where F and s are vectors and involve direction, but their product W is a scalar. The operation is called a *scalar product,* and the cos θ is not written but implied in the dot symbol as

$$W = sF \cos \theta = s \cdot F \tag{9-1}$$

Because of the symbol the result is also known as the *dot product* of the two vectors. In general then, the dot product of two vectors A and B is defined

$$A \cdot B = AB \cos \theta \tag{9-2}$$

and the result is scalar.

Another type of multiplication of two vectors arises in other physical problems, for example, the force F on a current I in a path of unit length in a magnetic field of density B. The magnitude of the result is known to be

$$F = IB \sin \theta$$

where θ is the angle between direction of I and the direction of B. The result here, however, is a force with direction and this is a vector. It is known, of course, that this force will be perpendicular to the plane containing I and B, and directed as by the movement of a right hand screw as I is turned into B.

It is again convenient to employ a shorthand symbol which, in itself, tells all this, and this is the implication of the $\times$ symbol

B. In fact, if point A is at an infinite distance, the work done on a *unit* positive charge in moving it from infinity to B in the electric field of a charge Q is used as the definition of the potential of the point B. In Fig. 9-1, if q is the unit test charge the force on q is, by definition, equal to $\mathcal{E}$, the electric field intensity. This force or field intensity is

$$\mathcal{E} = \frac{Q}{4\pi r^2 \epsilon} \text{ v/m}$$

in the field of the point charge of value Q. The work done on the unit charge q in moving a distance dl against a component of $\mathcal{E}$ along any path in the field of Q is the differential potential dV:

$$dV = -\mathcal{E} \cos \theta \, dl$$

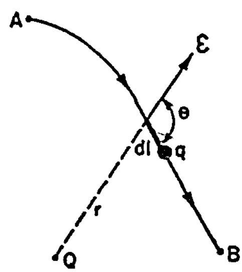

Fig. 9-1. Motion of charge q in an electric field established by charge Q.

but this is recognizable as the dot product so that

$$dV = -\boldsymbol{\mathcal{E}} \cdot \mathbf{dl}$$

The total work done in moving from A to B is the potential

$$V_{AB} = -\int_A^B \boldsymbol{\mathcal{E}} \cdot \mathbf{dl} \tag{9-12}$$

If the integral is taken along any closed path the limits would be set as A to A, a condition obviously resulting in a value of zero for the work done. Hence it is possible to write

$$V = -\oint \boldsymbol{\mathcal{E}} \cdot \mathbf{dl} = 0 \tag{9-13}$$

indicating that no net work is done on the charge q in following a closed path in the electric field of Q.

9-3. Potential in a changing magnetic field; Faraday's law

In 1831, Michael Faraday discovered one of the basic experimental laws on which all electrical engineering theory rests. This was the *induction law*, which states that

$$V = -\frac{\partial \phi}{\partial t} \tag{9-14}$$

where V is the potential induced in a one-turn circuit and ϕ is the flux in webers linking the circuit. In words, the *induction law states* will be a unit scalar. The product $i \cdot j$ involves the cosine of 90° and the result will be zero. In general, the dot product of like unit vectors will be unity, the dot product of unlike unit vectors will be zero. Then

$$\begin{aligned} A \cdot B &= (iA_x + jA_y + kA_z) \cdot (iB_x + jB_y + kB_z) \\ &= A_xB_x + A_yB_y + A_zB_z \end{aligned} \tag{9-7}$$

It is also necessary to consider $i \times i$ and $i \times j$ and similar terms. The cross product $i \times i$ involves the sine of 0° and the result is zero, while the cross product $i \times j$ involves the sine of 90° and the result is a unit vector in the direction of a right-hand screw turned from i into j, this being in the direction of k, or $i \times j = k$. The cross product of two like unit vectors is zero, of two unlike unit vectors the result is the third unit vector, positive if the normal rotation is used, negative if the rotation is reversed.

A table of such products may be formed, as

$$\begin{aligned} i \cdot i &= 1, \qquad & i \times i &= 0 \\ i \cdot j &= 0, & i \times j &= k \\ i \cdot k &= 0, & i \times k &= -j \end{aligned}$$

The vector product operation may be systematized by use of matrix notation, since it is found that the product $A \times B$ can be written as the determinant of

$$A \times B = \begin{vmatrix} i & j & k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \tag{9-8}$$

$$= i(A_yB_z - B_yA_z) + j(A_zB_x - B_zA_x) + k(A_xB_y - B_xA_y) \tag{9-9}$$

Two other multiple products will be stated as

$$(A \times B) \cdot C = (B \times C) \cdot A = (C \times A) \cdot B = \text{scalar} \tag{9-10}$$

$$(A \times B) \times C = B(C \cdot A) - A(B \cdot C) = \text{vector} \tag{9-11}$$

These results may be readily proved by performing the indicated operations.

9-2. Potential in an electric field

It is a well-known fact that work is done if an electric charge is moved from a point A through the field of another charge to a point

B. In fact, if point A is at an infinite distance, the work done on a *unit* positive charge in moving it from infinity to B in the electric field of a charge Q is used as the definition of the potential of the point B. In Fig. 9-1, if q is the unit test charge the force on q is, by definition, equal to $\mathcal{E}$, the electric field intensity. This force or field intensity is

$$\mathcal{E} = \frac{Q}{4\pi r^2 \epsilon} \text{ v/m}$$

in the field of the point charge of value Q. The work done on the unit charge q in moving a distance dl against a component of $\mathcal{E}$ along any path in the field of Q is the differential potential dV:

$$dV = -\mathcal{E} \cos \theta \, dl$$

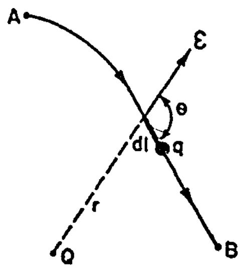

Fig. 9-1. Motion of charge q in an electric field established by charge Q.

but this is recognizable as the dot product so that

$$dV = -\mathcal{E} \cdot dl$$

The total work done in moving from A to B is the potential

$$V_{AB} = -\int_A^B \mathcal{E} \cdot dl \tag{9-12}$$

If the integral is taken along any closed path the limits would be set as A to A, a condition obviously resulting in a value of zero for the work done. Hence it is possible to write

$$V = -\oint \mathcal{E} \cdot dl = 0 \tag{9-13}$$

indicating that no net work is done on the charge q in following a closed path in the electric field of Q.

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In 1831, Michael Faraday discovered one of the basic experimental laws on which all electrical engineering theory rests. This was the *induction law*, which states that

$$V = -\frac{\partial \phi}{\partial t} \tag{9-14}$$

where V is the potential induced in a one-turn circuit and ϕ is the flux in webers linking the circuit. In words, the *induction law states* *that the voltage induced around any closed path is given by the negative of the time rate of change of magnetic flux through that path.*

It may appear that a fundamental inconsistency exists between Eqs. 9-13 and 9-14, since the first states that the voltage around a closed path is zero, and the second states that a voltage exists around a closed path. The difficulty is readily cleared if it is recognized that Eqs. 9-12 and 9-13 are not entirely general, since they define a situation existing in a static electric field without providing in any way for the possible simultaneous presence of a magnetic field. If it is recognized that both types of fields may be, and frequently are, simultaneously present in a region, a more general expression for the potential rise is

$$V = -\oint \mathcal{E} \cdot dl - \frac{\partial \phi}{\partial t}$$

If the path is closed, as it would be by starting at point A and returning to A, the net rise or work done per unit charge must be zero and consequently

$$\oint \mathcal{E} \cdot dl = -\frac{\partial \phi}{\partial t} \tag{9-15}$$

This equation states that an electric field exists around any path surrounding a time-varying magnetic field. It should be noted that this electric field exists and is independent of the resistance of the path chosen. The resistance of the path merely determines the value of current that may flow as a result of the induced electric field. Since current flow is not necessary for an electric field to exist, an electric field may be set up even along a path in space. In a region of static fields, Eq. 9-15 reduces to the form of 9-13.

In the preceding section it was shown that *in the electric field of a static electric charge,* the integral of the electric field around a closed path is zero. Equation 9-15 states that *in a time-varying magnetic field* the integral of the electric field around a path that encloses the varying flux is not zero.

The two cases are thus fundamentally unlike. For an understanding of either case, the moving test charge may be considered as constituting a current. Accompanying this current is a magnetic field. However, in the first case there is no interaction between the static electric charge Q and any magnetic field, so that when the test charge is carried completely around the closed path and returned to the starting point, no work has been done. Such an electric system is said to be *conservative*. In the second case the magnetic field due to the moving test charge adds to or subtracts from the time-varying magnetic field, thus changing the energy in the magnetic field. Since the system is again conservative, the change in energy in the original time-varying field is exactly the amount needed to equal the energy transferred to the test charge. In this manner the information in Faraday's law may be found to lead to Lenz's law.

9-4. The first electromagnetic field equation

Faraday's induction law may be put in a more convenient form for use in regions of varying magnetic and electric fields. Consider a rectangular path arbitrarily oriented in space and enclosing an elemental area. Through the region passes a magnetic flux of density

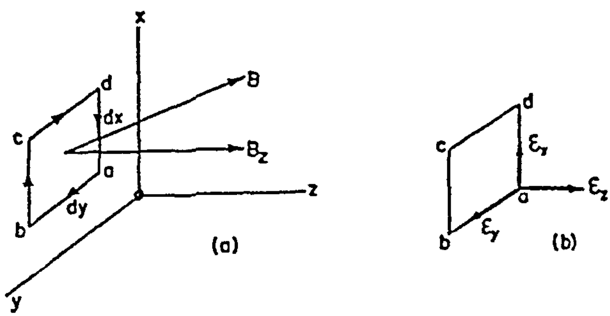


Fig. 9-2. (a) Illustrating Faraday's induction law; (b) the electric field vectors at point *a*.

B. The component B_z of this flux density passes normally through a projection of the circuit, parallel to the x,y plane, which may be considered as the rectangle *abcd* of Fig. 9-2(a). The area of the elemental rectangle is $dx\,dy = dA$.

From Eq. 9-15, the emf induced around the circuit *abcd*, if the magnetic field is caused to vary with time, is

$$\oint \mathcal{E} \cdot dl = -\frac{\partial \phi_z}{\partial t} = -\frac{\partial B_z}{\partial t}\,dA \qquad (9\text{-}16)$$

and since $B = \mu H$,

$$\oint \mathcal{E} \cdot dl = -\frac{\partial(\mu H_z)}{\partial t}\, dx\, dy$$

If μ is assumed constant or not a function of time, then

$$\oint \mathcal{E} \cdot dl = -\mu \frac{\partial H_z}{\partial t}\, dx\, dy \tag{9-17}$$

Such an assumption imposes no restriction on possible variation of μ at different space points in a region or in a material.

The left side of Eq. 9-17 is the summation of all the elemental electric field components times the respective distances through

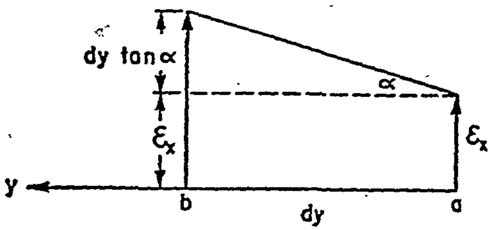


Fig. 9-3. Relation of field intensity $\mathcal{E}_x$ at points a and b, separated a distance dy.

which they act, and is accordingly the negative of the total emf rise around the path *abcd*. Consideration of the right-hand rule indicates *the induced emf will appear as a rise in potential in the direction abcd*; or as indicated by the arrows in Fig. 9-2(a). By summing the increments of field contributed by each elemental side of the rectangle, the total may be obtained for insertion in Eq. 9-17. This summation should be performed in the direction *abcd*, or in the direction of the electric field.

The electric field existing in the region is not known in general but at point a may be considered as having some value $\mathcal{E}$ with components $\mathcal{E}_x$, $\mathcal{E}_y$, $\mathcal{E}_z$, as shown in Fig. 9-2(b). Thus the electric field existing along ba is $-\mathcal{E}_y$. That existing along bc may be readily determined from Fig. 9-3 if it is remembered that dy is an infinitesimal distance.

At point a the electric field in the x direction is $\mathcal{E}_x$, and if the x-directed field varies along y, then at b it may be expected to have some different value. Since the distance dy is infinitesimal, the variation may be plotted as a straight line at some angle α or having a slope equal to $\tan \alpha$. Thus the x-directed field at b can be seen to have the value $\mathcal{E}_x + dy \tan \alpha$ (if the field decreases, $\tan \alpha$ becomes negative). However, $\tan \alpha$ is the slope of the curve of variation of $\mathcal{E}_x$ with distance y or is expressible as $\partial \mathcal{E}_x / \partial y$. The fields acting along two sides of the $abcd$ rectangle may then be summarized as follows:

$$\text{acting along } ba \text{ is the field } -\mathcal{E}_y$$

$$\text{acting along } cb \text{ is the field } -(\mathcal{E}_x + \partial \mathcal{E}_x/\partial y \, dy)$$

The field along dc may be obtained by an analysis similar to the above for that along bc, and it may be stated that

$$\text{acting along } dc \text{ is the field } \mathcal{E}_y + \partial \mathcal{E}_y/\partial x \, dx$$

$$\text{acting along } ad \text{ is the field } \mathcal{E}_x$$

These are simply expressions for the electric field intensity or voltage per unit length on the various sides of the $abcd$ rectangle, all expressed in terms of a specified set of field components at point a. If each of these field intensities is multiplied by the appropriate path length,

$$\begin{aligned} \text{along } ba &= -\mathcal{E}_y \, dy \\ \text{along } cb &= -(\mathcal{E}_x + \partial \mathcal{E}_x/\partial y \, dy) \, dx \\ \text{along } dc &= (\mathcal{E}_y + \partial \mathcal{E}_y/\partial x \, dx) \, dy \\ \text{along } ad &= \mathcal{E}_x \, dx \end{aligned}$$

The summation of these terms supplies the value for the integral of Eq. 9-17, so that it is possible to write

$$-\mathcal{E}_y \, dy - \left(\mathcal{E}_x + \frac{\partial \mathcal{E}_x}{\partial y} dy\right) dx + \left(\mathcal{E}_y + \frac{\partial \mathcal{E}_y}{\partial x} dx\right) dy + \mathcal{E}_x \, dx = -\mu \frac{\partial H_z}{\partial t} dx \, dy$$

Cancellation of terms gives

$$\frac{\partial \mathcal{E}_y}{\partial x} - \frac{\partial \mathcal{E}_x}{\partial y} = -\mu \frac{\partial H_z}{\partial t} \tag{9-18}$$

This equation states that if a z-directed magnetic field intensity is given a time variation, electric fields oriented only in the x and y directions will arise. This is merely a rather elaborate mathematical way of stating the familiar right-hand rule.

By consideration of the B_y component of B in Fig. 9-2(a), normal to a projection of the original space path parallel to the x, z plane, it is possible to arrive at a similar expression:

$$\frac{\partial \mathcal{E}_x}{\partial z} - \frac{\partial \mathcal{E}_z}{\partial x} = -\mu \frac{\partial H_y}{\partial t} \tag{9-19}$$

Likewise for the B_x component,

$$\frac{\partial \mathcal{E}_z}{\partial y} - \frac{\partial \mathcal{E}_y}{\partial z} = -\mu \frac{\partial H_x}{\partial t} \tag{9-20}$$

These expressions may be summarized as

$$\frac{\partial \mathcal{E}_z}{\partial y} - \frac{\partial \mathcal{E}_y}{\partial z} = -\mu \frac{\partial H_x}{\partial t} \tag{9-21a}$$

$$\frac{\partial \mathcal{E}_x}{\partial z} - \frac{\partial \mathcal{E}_z}{\partial x} = -\mu \frac{\partial H_y}{\partial t} \tag{9-21b}$$

$$\frac{\partial \mathcal{E}_y}{\partial x} - \frac{\partial \mathcal{E}_x}{\partial y} = -\mu \frac{\partial H_z}{\partial t} \tag{9-21c}$$

and are known as *Maxwell's first electromagnetic field equations.* They are relations between the *time rate of change* of the components of a magnetic field at a point and the *variation in space* of the components of the electric field at the same point. *A time-varying magnetic field produces an electric field.* The electric field is constant if the rate of change of the magnetic field is constant.

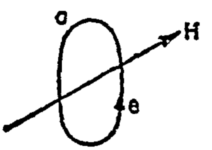


Fig. 9-4. A time-varying magnetic field produces an emf e in a path in a region.

In Fig. 9-4 a magnetic field intensity H varying with time in the region induces an emf e in a circular loop of wire placed at a. If the loop of wire be replaced by an insulated ring, an emf of the same value is still induced, although only negligible current will flow. An electric field $\mathcal{E}$, produced by variation of the magnetic field, will exist whether a physical path for current is present or not.

9-5. The displacement current

Consider the circuit of Fig. 9-5 in which a battery is connected to a slide wire on which is a sliding contact c. Points a and c are connected through ammeters A_1 and A_2 to a box inside which is a capacitor of capacitance C, with dielectric of permittivity ϵ.

Let slider c, starting from a, move toward b so as to increase the potential difference between the plates of the capacitor, thereby increasing its charge. A current i with direction as shown will flow through ammeter A_1, and an identical current will flow through A_2. If the box were closed, all external evidence would indicate a continuous circuit between the ammeters. With the capacitance C present it is evident, however, that no current actually flows between the plates of the capacitor, although the electric field between the plates has been increasing. It appears that the phenomena that happen in the dielectric between the capacitor plates are exactly the same as if a current i, called by Maxwell the *displacement current*, were really flowing between the plates.

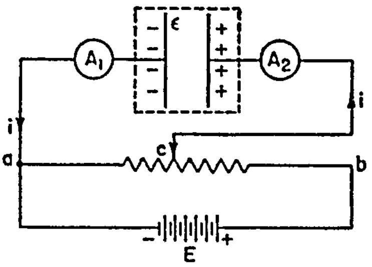


Fig. 9-5. Charging current due to change of potential on a capacity.

If the motion of slider c is stopped, the movement of charge ceases and the ammeters indicate that the current flow is zero. Hence the displacement current is seen to flow only when the electric field in the dielectric is changing. The equivalent (or displacement) current may be readily correlated with the rate of change of electric field in the dielectric. If the area of either capacitor plate is A and the charge at any instant is Q, then if fringing is neglected the electric field $\mathcal{E}$ in the dielectric is

$$\mathcal{E} = \frac{1}{\epsilon}\frac{Q}{A} \tag{9-22}$$

The variation with time may be obtained as

$$\frac{\partial \mathcal{E}}{\partial t} = \frac{1}{\epsilon A}\frac{\partial Q}{\partial t} \tag{9-23}$$

The term $\partial Q/\partial t$ can be recognized as a current i, the displacement current. Then

$$j_d = \frac{i}{A} = \epsilon \frac{\partial \mathcal{E}}{\partial t} \text{ amp/m}^2 \tag{9-24}$$

where j_d is the *equivalent displacement-current density* through the dielectric.

Thus, if at any point in a dielectric (or space) the electric field is changing at a rate $\partial\mathcal{E}/\partial t$, the phenomena resulting therefrom may be regarded as identical with what would happen if a current of density j_d were actually flowing through the same region.

It may be readily shown that the displacement current concept is consistent with ordinary circuit theory and that displacement current is really intended every time the current through a capacitance C is written as

$$i = C\frac{de}{dt}$$

For simplicity, assume that C is a parallel-plate capacitor of plate area A and spacing S with fringing neglected; then

$$C = \frac{\epsilon A}{S} \text{ farads}$$

Since the plate spacing S is constant,

$$i = \epsilon A\frac{d(e/S)}{dt}$$

and since $e/S = \mathcal{E}$ for the parallel-plate case,

$$j_d = \frac{i}{A} = \epsilon\frac{d\mathcal{E}}{dt} \text{ amp/m}^2 \tag{9-25}$$

which proves the identity of the charging-current and displacement-current concepts.

A current must then be *considered* as flowing in a dielectric through which an electric field is changing, and this current must be allowed for in determining the magnitude and direction of the resultant magnetic fields. If a circuit consists partly of conductors and partly of dielectrics, a conduction current appears, with a displacement current component also present if the electric field intensity changes with time.

A vacuum tube is an example of a device in which both currents may exist, the conduction component being due to electron flow in the space, with addition of a displacement current due to charge rearrangement at any time at which the electrode potentials change.

9-6. The second electromagnetic field equation

A current of density J amperes per square meter is flowing in a region. This current may be either the conventional conduction current density j_c or a displacement current density j_d, or both at once. Therefore, if J is to be the total current density in the region,

$$J = j_c + j_d \tag{9-26}$$

Such a current density may be considered as flowing through space with a portion enclosed by a prescribed path of which the elemental

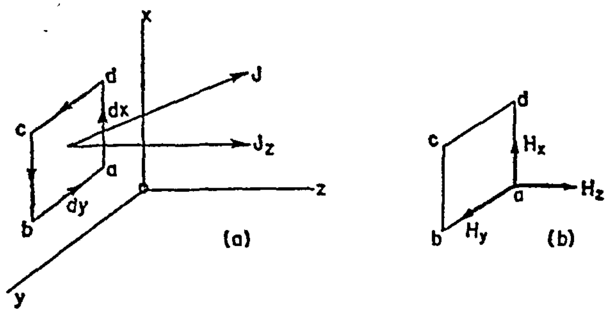


Fig. 9-6. (a) Illustrating the magnetic field due to a current density J; (b) the magnetic field vectors at point a.

rectangle *abcd*, Fig. 9-6(a), is the x,y plane projection. The current I_z enclosed by the *abcd* rectangle is

$$I_z = J_z\,dA = J_z\,dx\,dy$$

By definition, the magnetic intensity H along a path of length l enclosing a current I is related to the enclosed current as

$$Hl = NI \text{ ampere-turns/m} \tag{9-27}$$

which, since the *abcd* path encloses one turn, may be written

$$Hl = I_z = J_z\,dx\,dy \tag{9-28}$$

The term Hl represents the magnetomotive force (mmf) and is equal to the current enclosed. This value of mmf may be computed over any path, so long as it encircles the current $J_z\,dx\,dy$. It may be expressed as

$$Hl = \oint H \cdot dl$$

By writing down the value of H along any part of the path, multiplying by the length of the path over which that value of H exists, and adding up all these elemental values of mmf, the total mmf is obtained.

The magnetic field will not be specified over the whole region in general but may be defined as having components H_x, H_y, H_z at point a as illustrated in (b), Fig. 9-6. The fields and magnetomotive force in the direction indicated around the $abcd$ rectangle may be stated as follows:

acting along ad is the field H_x, and the mmf is $H_x\,dx$
acting along dc is the field $H_y + \partial H_y/\partial x\,dx$, and the mmf is $(H_y + \partial H_y/\partial x\,dx)\,dy$
acting along cb is the field $-(H_x + \partial H_x/\partial y\,dy)$, and the mmf is $-(H_x + \partial H_x/\partial y\,dy)\,dx$
acting along ba is the field $-H_y$, and the mmf is $-H_y\,dy$

The fields along dc and cb are obtained in a manner similar to that employed in Section 9-4 and Fig. 9-3. Then

$$\text{total mmf} = H_x\,dx + \left(H_y + \frac{\partial H_y}{\partial x}\,dx\right)dy - \left(H_x + \frac{\partial H_x}{\partial y}\,dy\right)dx - H_y\,dy$$

$$= \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)dx\,dy$$

But this mmf is, by Eq. 9-28, equal to the current enclosed, or

$$\left.\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)dx\,dy = J_z\,dx\,dy$$

from which

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z \tag{9-29}$$

The current-density component J_y may be considered as producing a current $I_y = J_y\,dx\,dz$ in a projection of the original path on the xz plane. By similar reasoning to that above, J_y may be found to be related to the fields as

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = J_y \tag{9-30}$$

Similarly for the J_x component,

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = J_x \tag{9-31}$$

If the electric conductivity of the elemental region is σ, it is possible to write for the conduction current density in a material

$$j_c = \sigma\mathcal{E}\ \text{amp/m}^2 \tag{9-32}$$

In view of Eqs. 9-24 and 9-32, the total current density in the region may be written as

$$J = j_c + j_d = \sigma \mathcal{E} + \epsilon \frac{\partial \mathcal{E}}{\partial t} \tag{9-33}$$

thus eliminating the need for the concept of current flow and expressing the right side of Eqs. 9-29, 9-30, and 9-31 in terms of the electric field intensity. Rewriting those equations gives

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = \sigma \mathcal{E}_x + \epsilon \frac{\partial \mathcal{E}_x}{\partial t} \tag{9-34a}$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = \sigma \mathcal{E}_y + \epsilon \frac{\partial \mathcal{E}_y}{\partial t} \tag{9-34b}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = \sigma \mathcal{E}_z + \epsilon \frac{\partial \mathcal{E}_z}{\partial t} \tag{9-34c}$$

These are known as *Maxwell's second electromagnetic field equations.*

Consider a number of electric charges in space, near a point P. If the charges are stationary, there is an electric field at P due to all the charges, but no magnetic field. Whenever the charges move with a velocity v, the electric field at P changes, and simultaneously a magnetic field is produced at P. From the current viewpoint, the moving charges constitute a current that sets up a magnetic field at P, but Eq. 9-34 indicate a direct relationship between $\mathcal{E}$ and $\boldsymbol{H}$, without need of the concept of current. Viewed from this standpoint, the production of the magnetic field around a current may be regarded as due to the change with time of the individual electric fields from the moving charges. Briefly, a *moving charge produces a magnetic field.*

The presence of both conduction and displacement currents on the right side of Eq. 9-34 need not be confusing. It is possible that the area dA enclosed by the line integral path may contain either dielectric or conductor, or both. If a dielectric is present, σ will be substantially zero; whereas if a conductor is present, ϵ will usually be negligible with respect to σ. Generally, then, in conductors only the term containing σ need be considered, and in dielectrics only the term containing ϵ is appreciable.

Maxwell's equations may be regarded merely as mathematical means of exactly specifying information contained in the conventional right-hand rule. It should be noted that the only material needed in the development was the experimentally verifiable Faraday's law, and the definition of magnetic field intensity. The foundation of the equations is seen to be solid and completely free of any limiting assumptions. Thus Maxwell's field equations are truly fundamental.

9-7. Gradient and divergence

In several of the previous sections, derivatives have been taken of components of vectors, and further study should be given to these terms which represent rates of change in space or time.

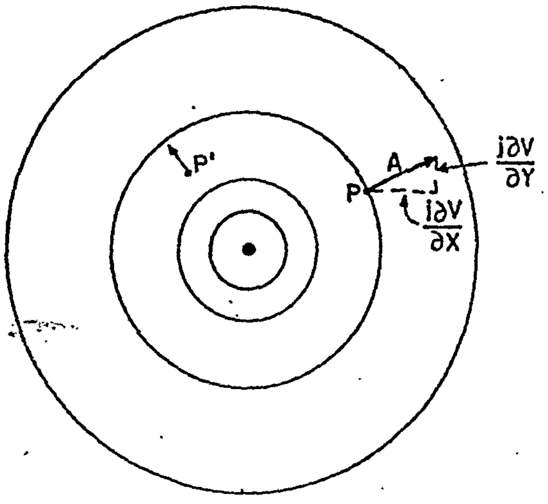


Fig. 9-7. Demonstration of gradient in a scalar field.

Consider the scalar potential field surrounding an isolated electric charge. The potential at any point P is V, and this scalar potential varies with distance from the charge. Equipotential lines may be drawn as in Fig. 9-7. Since this is a two-dimensional field, the rate of change of potential from point to point in the field can be described in terms of two directed components. Considering A in Fig. 9-7 to represent the *greatest rate of change* in potential at point P, this greatest rate has a direction and is a vector.

This vector rate of change has been given the more descriptive name of *gradient*, which implies a vector directed along and of magnitude equal to the greatest rate of change of potential at a point. Vector A or the gradient of V is abbreviated *grad* V and written

$$\text{grad } V = i\frac{\partial V}{\partial x} + j\frac{\partial V}{\partial y} \tag{9-35}$$

in a two-dimensional field. It is apparent that the gradient vector will be at right angles to the equipotential lines of the field. However, the direction of the gradient may vary from point to point

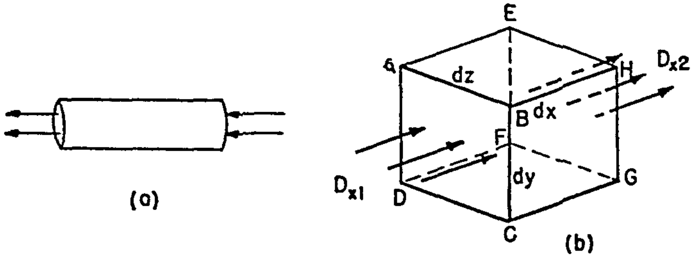


Fig. 9-8. Demonstration of divergence.

in the field, and this is apparent from consideration of some other point P' in the field.

To illustrate, if the page of Fig. 9-7 were a conductor, the current at point P in the indicated potential field would be along A. Likewise if Fig. 9-7 represented a scalar temperature field in a medium, the flow of heat would be directed along the gradient. This would also hold true if a three-dimensional field were present, a third vector component then being added to Eq. 9-35.

It is, of course, apparent that this idea of vector rate of change in a scalar electric potential field leads directly to the concept of electric field intensity. In fact electric field intensity, in three dimensions, may be defined as

$$\mathcal{E} = -\text{grad } V = -\left(i\frac{\partial V}{\partial x} + j\frac{\partial V}{\partial y} + k\frac{\partial V}{\partial z}\right) \tag{9-36}$$

This expression justifies the idea of gradient at this point.

A vector field (flux or velocity, for instance) may also have a space variation at a point, and this is called *divergence*. The velocity or rate of flow of water through a pipe is a vector. However, since water is incompressible, the vector flow into the pipe of Fig. 9-8(a), must be equal to the vector flow out of the pipe, and such a system is said to have zero divergence. If the pipe is filled with air, and momentarily more air is flowing out of the pipe than is flowing in, the divergence of the flow is not zero. Actually, as divergence is used it applies as if the pipe were infinitesimally small, giving a divergence at a point.

An electrical example is also available. Consider a region having electric flux density $\boldsymbol{D}$, with an infinitesimal box, of sides dx, dy and dz, placed at a point P, as in (b), Fig. 9-8. From the laws of electric fields, if no charge exists inside the box, the total electric flux which enters the box is equal to the total flux leaving; the box then has zero divergence. If a charge density ρ exists in the region, the flux entering and leaving may not be equal, and the divergence is not zero.

Considering components of flux density as D_x, D_y, and D_z, and setting outward flux as positive, the flux entering side $ABCD$ will be $-D_x\,dy\,dz$, and by the reasoning of Section 9-4, the flux leaving side $EFGH$ will be

$$\left(D_x + \frac{\partial D_x}{\partial x}\,dx\right) dy\,dz$$

so that the net outward flux due to $D_x = \dfrac{\partial D_x}{\partial x}\,dx\,dy\,dz$. Reasoning gives similar terms for the other flux components, so that the total outward flux from the box is

$$\text{outward flux} = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) dx\,dy\,dz$$

By Gauss's law the outward flux from a charge is equal to the charge enclosed, and ρ being the charge density, the charge enclosed by the box is $\rho\,dx\,dy\,dz$. Then from Gauss's law

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \rho$$

By definition, the left side of this expression is the divergence of vector $\boldsymbol{D}$, written div $\boldsymbol{D}$, so that

$$\operatorname{div} \boldsymbol{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \tag{9-37}$$

and Gauss's law can be stated as

$$\operatorname{div} \boldsymbol{D} = \rho \tag{9-38}$$

Since magnetic flux in homogeneous media is continuous, it is apparent that

$$\operatorname{div} B = 0 \tag{9-39}$$

and Eqs. 9-38 and 9-39 are important results in electromagnetic field theory.

It may be noted above that both gradient and divergence involve a three-dimensional operator of the form

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \tag{9-40}$$

and this is given the shorthand symbol ∇, and the name *del*, or *nabla*. Since it has the symbolism of a vector, it may be handled in dot and cross products. That is, it may be found by performing the indicated operation, that when operating on a scalar quantity P, the result is

$$\nabla P = i\frac{\partial P}{\partial x} + j\frac{\partial P}{\partial y} + k\frac{\partial P}{\partial z} = \operatorname{grad} P \tag{9-41}$$

and ∇P is another expression for the gradient of P.

Likewise performance of the indicated operation will show that the dot product of ∇ and a vector A yields

$$\nabla \cdot A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \operatorname{div} A \tag{9-42}$$

$$\nabla \cdot A = \operatorname{div} A \tag{9-43}$$

This leads to a very important theorem. Flux of an electric density D passing out through any area (the net value) is

$$\int D \cdot da$$

and the net outward flux passing through a closed surface is

$$\text{net outward flux} = \int_S D \cdot da \tag{9-44}$$

But for any element of volume dv inside the surface, the divergence or net outward flux is $\nabla \cdot D$, and for the whole volume it is

$$\text{net outward flux} = \int_v \nabla \cdot D\, dv \tag{9-45}$$

However, Eqs. 9-44 and 9-45 represent the same quantity, the net outward flux from a region, or

$$\int_S \mathbf{D} \cdot d\mathbf{a} = \int_v \nabla \cdot \mathbf{D}\, dv \tag{9-46}$$

This result, the equating of a surface integral of a vector quantity to the divergence of the same quantity within a volume, is of great importance and is known as *Gauss's theorem*, or the *divergence theorem*.

9-8. Curl of a vector; Maxwell's equations rewritten

The line integral of a vector field as

$$\oint \boldsymbol{\mathcal{E}} \cdot d\mathbf{l} \quad \text{or} \quad \oint \mathbf{H} \cdot d\mathbf{l}$$

has already been encountered. Such expressions are also known as the *circulation* of the field since they are the result of a circulation around a closed path. However, in arriving at either of these expressions, as in Eqs. 9-16 and 9-28, it was found that the result was dependent on an area. The circulation per unit area is more general and the limit of the circulation per unit area or

$$\frac{\mathbf{A} \cdot d\mathbf{l} \text{ about a closed path}}{\text{area bounded by the path}}$$

as the bounded area shrinks to a point, is called the *curl* value at the point. Thus the curl of a vector field at a point is a vector found by orienting an infinitesimal area normal to the desired direction, and finding the value of a line integral per unit enclosed area. The name should not necessarily be associated with a curvature of a field, as will be shown.

Consider Fig. 9-9(a), showing equipotential lines in a field. If the rectangle is infinitesimal, the line integral of the field, by the methods of Section 9-4, for counterclockwise tracing of the path, is

$$\begin{aligned}\text{circulation} = \oint \boldsymbol{\mathcal{E}} \cdot d\mathbf{l} &= \mathcal{E}_x\, dx + \left(\mathcal{E}_y + \frac{\partial \mathcal{E}_y}{\partial x}\, dx\right) dy \\ &\quad - \left(\mathcal{E}_x + \frac{\partial \mathcal{E}_x}{\partial y}\, dy\right) dx - \mathcal{E}_y\, dy = \left(\frac{\partial \mathcal{E}_y}{\partial x} - \frac{\partial \mathcal{E}_x}{\partial y}\right) dx\, dy\end{aligned}$$

Dividing by the infinitesimal area $dx\, dy$ gives

$$\operatorname{curl} \boldsymbol{\mathcal{E}} = \mathbf{k}\left(\frac{\partial \mathcal{E}_y}{\partial x} - \frac{\partial \mathcal{E}_x}{\partial y}\right) \tag{9-47}$$

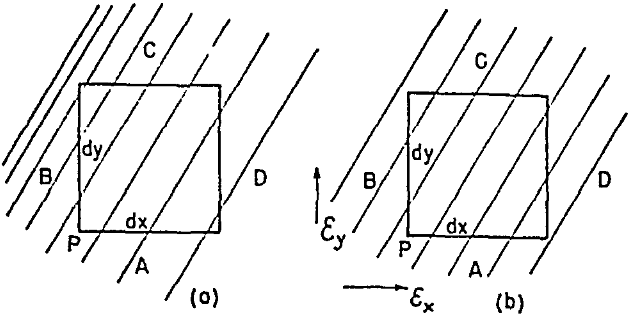


Fig. 9-9. (a) Curl is not zero; (b) curl is zero.

as the value for a two-dimensional field. The unit vector $\boldsymbol{k}$ appears because of the orientation of the area chosen for computation of the line integral, this area being perpendicular to the $\boldsymbol{k}$ vector. A three-dimensional form for the curl is

$$\text{curl}\ \mathcal{E} = \boldsymbol{i}\left(\frac{\partial \mathcal{E}_z}{\partial y} - \frac{\partial \mathcal{E}_y}{\partial z}\right) + \boldsymbol{j}\left(\frac{\partial \mathcal{E}_x}{\partial z} - \frac{\partial \mathcal{E}_z}{\partial x}\right) + \boldsymbol{k}\left(\frac{\partial \mathcal{E}_y}{\partial x} - \frac{\partial \mathcal{E}_x}{\partial y}\right) \quad (9\text{-}48)$$

If the rectangle in Fig. 9-9(b) is infinitesimal, and the equipotentials are equally spaced as shown, or the field is uniform, the curl of $\mathcal{E}$ at point P for that situation is zero, since the field components on opposite sides of the rectangle will be equal but traced in opposite directions.

The situation at (a) differs, however, since the field intensity along side C differs from that along side A, and also that along B differs from that on D. That is, the curl is a three-dimensional statement of the space-variation of a field at a point, and may be recognized as having a value, or as the field situation due to presence of a charge or a current, as examples.

It may be easily found that the above expression for curl can be written by taking the cross product of the operator ∇ and a vector, and also may be written as the determinant of a matrix form, for example:

$$\text{curl}\ \mathcal{E} = \nabla \times \mathcal{E} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \mathcal{E}_x & \mathcal{E}_y & \mathcal{E}_z \end{vmatrix} \quad (9\text{-}49)$$

If Eqs. 9-21 are multiplied by the appropriate unit vectors

$$i\left(\frac{\partial \mathcal{E}_z}{\partial y} - \frac{\partial \mathcal{E}_y}{\partial z}\right) = - i\mu \frac{\partial H_x}{\partial t}$$

$$j\left(\frac{\partial \mathcal{E}_x}{\partial z} - \frac{\partial \mathcal{E}_z}{\partial x}\right) = - j\mu \frac{\partial H_y}{\partial t}$$

$$k\left(\frac{\partial \mathcal{E}_y}{\partial x} - \frac{\partial \mathcal{E}_x}{\partial y}\right) = - k\mu \frac{\partial H_z}{\partial t}$$

then their sum becomes

$$\text{curl}\, \mathcal{E} = \nabla \times \mathcal{E} = - \mu \frac{\partial H}{\partial t} \tag{9-50}$$

Maxwell's second field equation could be handled in a similar manner, yielding

$$\text{curl}\, H = \nabla \times H = \sigma \mathcal{E} + \epsilon \frac{\partial \mathcal{E}}{\partial t} \tag{9-51}$$

These two equations are sometimes combined with Eqs. 9-38 and 9-39 as

$$\nabla \times \mathcal{E} = - \mu \frac{\partial H}{\partial t} \tag{9-52}$$

$$\nabla \times H = \sigma \mathcal{E} + \epsilon \frac{\partial \mathcal{E}}{\partial t} \tag{9-53}$$

$$\nabla \cdot B = 0 \tag{9-54}$$

$$\nabla \cdot D = \rho \tag{9-55}$$

as the four basic equations of field theory. These equations and the notation of gradient, divergence, and curl, will be used as required in what follows.

9-9. The plane wave in space—I

Maxwell's equations may now be applied to a simple case of combined electric and magnetic fields in space. The equations are usually employed in specific problems by insertion of known boundary conditions of the fields, with one or more field values usually going to zero as a result. The remaining equations are then solved simultaneously for the actual field values that must exist for the specified boundary values.

The immediate problem is to determine the fields under certain conditions *in space*. Maxwell's first equation, written in the form of Section 9-8, is

$$\nabla \times \mathcal{E} = -\mu \frac{\partial H}{\partial t} \tag{9-56}$$

The second Maxwell equation is restricted by the choice of the medium, since the conductivity σ of space is zero, so that Eq. 9-53 becomes

$$\nabla \times H = \epsilon \frac{\partial \mathcal{E}}{\partial t} \tag{9-57}$$

Consider a very large metal sheet, theoretically infinite in extent, placed in coincidence with the x,y plane as in Fig. 9-10. The rest of space in the positive-z direction is assumed empty. Permit a uniformly distributed current to flow through this sheet, parallel to the x axis. The current value is to be i amperes per meter of width of the sheet, measured along the y axis. At point P, on the z axis in space, there will be produced some magnetic intensity H. However, since the current is oriented wholly along the x axis and the current is infinite in extent, there can be no components of the magnetic field in either the H_z or H_x directions. It can then be stated that

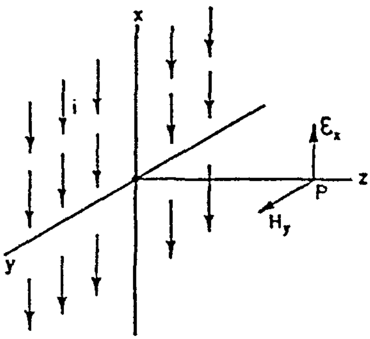


Fig. 9-10. Fields due to an infinite current sheet in the x,y plane.

$$H_z = H_x = 0 \tag{9-58}$$

and it follows that

$$\frac{\partial H_z}{\partial t} = \frac{\partial H_x}{\partial t} = \frac{\partial H_z}{\partial y} = \frac{\partial H_x}{\partial y} = \frac{\partial H_z}{\partial x} = \frac{\partial H_x}{\partial z} = 0 \tag{9-59}$$

The magnetic field at P then consists wholly of one component, H_y.

The current through the sheet may be permitted to vary with time at some rate $\partial i/\partial t$. Because of the proportionality expressed in the relation

$$H = \frac{Ni}{l}$$

there will then be a variation of H_y at some rate $\partial H_y/\partial t$. Variations of H_y at P will be delayed, however, with reference to those variations taking place in the current sheet at the origin, as the result of an implied assumption that magnetic effects travel at a finite velocity. This assumption is reasonable, since no physical phenomena have been discovered to be infinite.

Because of the infinite extent of the sheet, there can be no variation of any electric or magnetic quantity with x or y, since any events that occur at P happen simultaneously at all points in a plane through P parallel to the current sheet. Point P might exist anywhere on this plane without encountering any change in $\mathcal{E}$ or H due to position. Further boundary restrictions are then obtained as

$$\frac{\partial H_y}{\partial x} = \frac{\partial \mathcal{E}_y}{\partial x} = \frac{\partial \mathcal{E}_x}{\partial y} = \frac{\partial \mathcal{E}_z}{\partial x} = \frac{\partial \mathcal{E}_z}{\partial y} = 0 \tag{9-60}$$

The boundary values of Eqs. 9-59 and 9-60 may now be applied to Eqs. 9-56 and 9-57, or more conveniently to 9-21 and 9-34, resulting in only two equations:

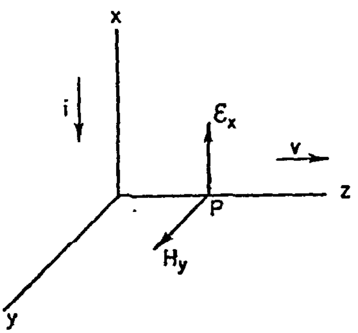


Fig. 9-11. Fields at point P on the z axis in front of the current sheet.

$$-\frac{\partial H_y}{\partial z} = \epsilon \frac{\partial \mathcal{E}_x}{\partial t} \tag{9-61}$$

$$\frac{\partial \mathcal{E}_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t} \tag{9-62}$$

for the particular plane of current, all other field components being zero or constants in time, the latter not being of interest. The field components at P are then seen to be $\mathcal{E}_x$ and H_y, and these are indicated in Fig. 9-11.

These equations may be solved by first separating the variables. Differentiating Eq. 9-62 with respect to time gives

$$\frac{\partial^2 \mathcal{E}_x}{\partial z\, \partial t} = -\mu \frac{\partial^2 H_y}{\partial t^2}$$

and differentiating Eq. 9-61 with respect to z gives

$$\frac{\partial^2 H_y}{\partial z^2} = -\varepsilon \frac{\partial^2 \mathcal{E}_x}{\partial^a \, \partial t}$$

Combining the two equations gives

$$\frac{\partial^2 H_y}{\partial t^2} = \frac{1}{\mu\epsilon} \frac{\partial^2 H_y}{\partial z^2} \tag{9-63}$$

which may also be written

$$\mu\epsilon \frac{\partial^2 H}{\partial t^2} = \nabla^2 H \tag{9-64}$$

in a three-dimensional form.

Reversing the differentiation by operating on Eq. 9-62 with respect to z and on Eq. 9-61 with respect to t gives

$$\frac{\partial^2 \mathcal{E}_x}{\partial t^2} = \frac{1}{\mu\epsilon} \frac{\partial^2 \mathcal{E}_x}{\partial z^2} \tag{9-65}$$

which, in three-dimensional general form becomes

$$\mu\epsilon \frac{\partial^2 \mathcal{E}}{\partial t^2} = \nabla^2 \mathcal{E} \tag{9-66}$$

Equations 9-63 and 9-65 are standard forms, referred to as the *wave equations* for a *plane wave*. Equations 9-64 and 9-66 are the general wave equations for any form of wave. The reason for the name of these equations will be apparent shortly.

The wave equations are partial differential equations of the form

$$\frac{\partial^2 P}{\partial t^2} = v^2 \frac{\partial^2 P}{\partial z^2} \tag{9-67}$$

where $v = 1/\sqrt{\mu\epsilon}$. A complete solution for this equation is presented in Appendix A, the method merely being indicated here for brevity. By a transformation of variable such that

$$u = z + vt, \qquad w = z - vt$$

Eq. 9-67 may be transformed to

$$\frac{\partial^2 P}{\partial u \, \partial w} = 0 \tag{9-68}$$

which is directly integrable. Integrating with respect to w,

$$\frac{\partial P}{\partial u} = f(u)$$

and with respect to u,

$$P = \int f(u)\,du + f_2(w)$$

An integral of $f(u)$ is some other function f_1 of u, so that

$$P = f_1(u) + f_2(w)$$

is the result of the integration of Eq. 9-68. By comparison, the solution of the wave equations for $\mathcal{E}_x$ and H_y yields

$$\mathcal{E}_x = F_1(z + vt) + F_2(z - vt) \tag{9-69}$$

$$H_y = F_3(z + vt) + F_4(z - vt) \tag{9-70}$$

Since z is a distance and t is time, it is necessary that v be a velocity in order that the equations be dimensionally correct. The parameter v was introduced as

$$v = \frac{1}{\sqrt{\mu\epsilon}}$$

and in space

$$\mu = \mu_v = 4\pi \times 10^{-7}$$

$$\epsilon = \epsilon_v = \frac{10^7}{4\pi c^2} = \frac{10^{-9}}{36\pi}$$

When these values are inserted, it is found that *in space*

$$v = c = 3 \times 10^8 \text{ meters/sec}$$

Because of the presence of v in the equations, it can be reasoned that there is motion of some physical entity at the velocity v, which is equal to that of light and in any medium is given by

$$v = \frac{1}{\sqrt{\mu\epsilon}} \tag{9-71}$$

It will be found that this moving physical entity is energy.

Equations 9-69 and 9-70 are in the form of the solution of the usual vibration equation. It can also be shown that the solutions previously obtained for the transmission line in terms of the incident and reflected wave are of this form. Mathematically, $F_1(z + vt)$

represents phenomena (a wave) moving in the negative-z direction in Fig. 9-10, and $F_2(z - vt)$ represents phenomena moving in the positive-z direction. These waves are analogous to the reflected and incident waves of line theory.

A wave in which the variation of the field components, $\mathcal{E}_x$ and H_y, takes place in a plane perpendicular to the direction of propagation is said to be a *transverse wave*. Consequently, the result of the time variation in the current sheet of Fig. 9-10 is a transverse electromagnetic wave moving in the z direction with the velocity of light, and with the fields $\mathcal{E}_x$ and H_y, mutually at right angles in *space*.

This particular transverse wave is called a *plane-polarized electromagnetic wave.*

9-10. The plane wave in space—II

The fields developed in the preceding section in terms of arbitrary functions may be given a more definite physical form by an assumption of a sinusoidal time function for the current of the infinite current sheet. Thus let

$$i = I_0 \sin \omega t$$

be the form of the current variation in the sheet. Because of the proportionality between i and H, the magnetic field at the *surface of the current sheet* will be

$$H_y = H_0 \sin \omega t$$

Since a finite velocity of propagation of a magnetic field was assumed and since this velocity has been found to be that of light, whatever happens to the magnetic field at the origin, Fig. 9-11, will happen to the field at P, somewhat later, and the magnetic field at any point P on the z axis may then be expressed as

$$H_y = H_0 \sin \omega(t - t_P) \tag{9-72}$$

with the current used as the time reference. The factor t_P is the time of propagation of the wave from the origin to P and is

$$t_P = \frac{z}{v}$$

where z is the distance to P and $v = 1/\sqrt{\mu\epsilon}$. Hence the magnetic field at any point P is

$$H_y = H_0 \sin \omega \left(t - \frac{z}{v}\right) \tag{9-73}$$

This is obviously a function of $(z - vt)$ and corresponds to Eq. 9-70 as a solution of the wave equation, representing a wave moving in the positive-z direction in Fig. 9-11. It is apparent that the function of $(z + vt)$ is missing, a fact explainable by the prior assumption of infinite space in the positive-z region. If there is only space in that region, there is nothing to act as a source or reflector and to start propagation in the negative-z direction. This condition is analogous to the absence of a reflected wave on an infinite line. Hence it is possible to conclude that for the conditions under discussion, the function of $(z + vt)$ is zero.

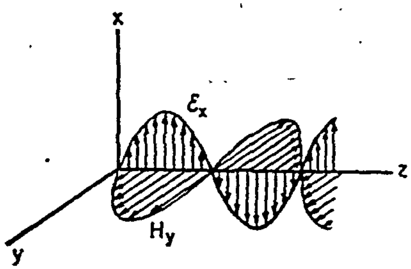


Fig. 9-12. Electric and magnetic sinusoidally varying fields at an instant in time.

The expression $\sin \omega(t - z/v)$ shows that the field is a sine function of time at any particular z, or a sine function of z at any particular time. This statement is explainable as a sine wave traveling along the z axis as illustrated in Fig. 9-12, where the accompanying electric field wave is also shown at a particular time t.

The equation for the accompanying electric field may be obtained. One of the resultants after application of the current-plane boundary conditions to Maxwell's equations was Eq. 9-61:

$$-\frac{\partial H_y}{\partial z} = \epsilon_v \frac{\partial \mathcal{E}_x}{\partial t}$$

writing ϵ_v since the point P is in space. Differentiation of Eq. 9-73 with respect to z gives

$$\frac{\partial H_y}{\partial z} = -\frac{\omega}{v} H_0 \cos \omega \left(t - \frac{z}{v}\right) \tag{9-74}$$

and substitution in Eq. 9-61 leads to

$$\frac{\partial \mathcal{E}_x}{\partial t} = \frac{\omega}{\epsilon_v v} H_0 \cos \omega \left(t - \frac{z}{v}\right) \tag{9-75}$$

Integration with respect to t gives the value for the field $\mathcal{E}_x$ that accompanies H_y:

$$\mathcal{E}_x = \frac{1}{\epsilon_v v} H_0 \sin \omega \left(t - \frac{z}{v}\right)$$

The constant of integration may be neglected, since a static field is of no interest in a study of radiation. In view of the value of v, this field may be written as

$$\mathcal{E}_x = \sqrt{\frac{\mu_v}{\epsilon_v}} H_0 \sin \omega \left(t - \frac{z}{v}\right) \text{ v/m} \tag{9-76}$$

Writing H_y for comparison,

$$H_y = H_0 \sin \omega \left(t - \frac{z}{v}\right) \text{ ampere-turns/m} \tag{9-77}$$

showing that the electric field in space is in time phase with the magnetic field.

Examination of Eqs. 9-76 and 9-77 indicates that

$$\mathcal{E}_x = \sqrt{\frac{\mu_v}{\epsilon_v}} H_y \tag{9-78}$$

The two field intensities are thus always proportional, and one cannot exist without the other. The constant of proportionality is $\sqrt{\mu_v/\epsilon_v}$, which for space is found to be

$$\sqrt{\frac{\mu_v}{\epsilon_v}} = 120\pi = 376.6.$$

by coincidence equal to the value of ω for 60 cycles. From Eq. 9-78

$$\sqrt{\frac{\mu_v}{\epsilon_v}} = \frac{\mathcal{E}_x}{H_y} = \frac{\text{volts/meter}}{\text{ampere-turns/meter}}$$

Since *turns* is a mere numeric factor without dimensions, it can be seen that $\sqrt{\mu_v/\epsilon_v}$ has the dimensions of volts/amperes and therefore is rightfully assigned the dimension of *ohms*.

The electric and magnetic fields of the *radiating wave are in time phase in space, are proportional to each other, are mutually at right angles, and are moving outward along z with a velocity equal to that of light.*

From this statement and other physical evidence it is possible to conclude, as Maxwell did, that light is an electromagnetic wave.

9-11. Energy of the radiated wave

Now that it has been shown that an electromagnetic wave is radiated outward from the sheet of time-varying current, it is of interest to obtain a measure of the energy carried or propagated through space by these waves.

The energy present in a magnetic field is

$$W_m = \frac{\mu H^2}{2} \text{ joules/m}^3 \tag{9-79}$$

and that in an electric field is

$$W_e = \frac{\epsilon \mathcal{E}^2}{2} \text{ joules/m}^3 \tag{9-80}$$

It is possible to generalize Eq. 9-78 for any medium by writing μ and ϵ in place of μ_v and ϵ_v. Rearranging gives

$$\sqrt{\epsilon}\, \mathcal{E}_x = \sqrt{\mu}\, H_y$$

and after squaring both sides and dividing by 2,

$$\frac{\epsilon \mathcal{E}_x^2}{2} = \frac{\mu H_y^2}{2} \tag{9-81}$$

By comparison with Eqs. 9-79 and 9-80, it can be seen that these terms express the electric and magnetic energy per unit volume at any point in the wave and indicate that

$$W_e = W_m$$

or the energy carried by one field is equal to that of the other. Hence the total energy carried by the waves is

$$W = \frac{\epsilon \mathcal{E}_x^2}{2} + \frac{\mu H_y^2}{2} = \epsilon \mathcal{E}_x^2 = \mu H_y^2 = \sqrt{\mu\epsilon}\, \mathcal{E}_x H_y \text{ joules/m}^3 \tag{9-82}$$

Consider a rectangular volume of dimensions $1 \times 1 \times dz$ with faces parallel to the x,y plane as shown in Fig. 9-13.
The energy in this element of volume is

$$dW = \sqrt{\mu\epsilon}\, \mathcal{E}_x H_y\, dz$$

The *rate of flow of energy* is then

$$\frac{dW}{dt} = \sqrt{\mu\epsilon}\, \mathcal{E}_x H_y \frac{dz}{dt} \quad (9\text{-}83)$$

This energy is moving with a velocity

$$v = \frac{dz}{dt}$$

so that

$$\frac{dW}{dt} = v\sqrt{\mu\epsilon}\, \mathcal{E}_x H_y$$

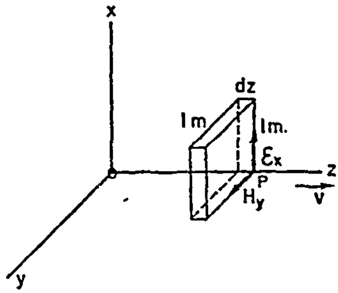


Fig. 9-13. Illustrating energy-density flow past a point P.

and since $v = 1/\sqrt{\mu\epsilon}$, the rate of energy flow or the *power passing through a surface at point* P *per square meter* is

$$P = \frac{dW}{dt} = \mathcal{E}_x H_y \text{ watts/m}^2 \quad (9\text{-}84)$$

This is *Poynting's radiation vector* as applied to a plane electromagnetic wave. In more general terms it may be proven as

$$P = \mathcal{E} \times H \text{ watts/m}^2 \quad (9\text{-}85)$$

In general terms $\mathcal{E} \times H$ delineates a parallelogram on a surface. Then $\mathcal{E}\sin\theta$ is the projection of $\mathcal{E}$ on a line normal to H, and $\mathcal{E}H \sin\theta$ is the area of the parallelogram. An area is a vector quantity, since it is oriented or faces a given direction. The area vector is at right angles to the area plane, and directed as by a right hand screw as $\mathcal{E}$ turns into H through the smallest angle. Thus the direction of energy flow is given by the $\mathcal{E} \times H$ area vector. This vector is named Poyntings' radiation vector after its discoverer, J. H. Poynting.

9-12. Poynting's theorem—the general case

From Maxwell's equations

$$\nabla \times \mathcal{E} = -\mu \frac{\partial H}{\partial t}$$

$$\nabla \times H = \sigma \mathcal{E} + \epsilon \frac{\partial \mathcal{E}}{\partial t}$$

Now by performance of the indicated operations, it may be shown that

$$\nabla \cdot (\mathcal{E} \times H) = H \cdot \nabla \times \mathcal{E} - \mathcal{E} \cdot \nabla \times H \tag{9-86}$$

and inserting the values from the field equations

$$\nabla \cdot (\mathcal{E} \times H) = H \cdot \left(-\mu \frac{\partial H}{\partial t} \right) - \mathcal{E} \cdot \left(\sigma \mathcal{E} + \epsilon \frac{\partial \mathcal{E}}{\partial t} \right)$$

If μ and ϵ are not variables in time, then

$$\epsilon \mathcal{E} \cdot \frac{\partial \mathcal{E}}{\partial t} = \frac{1}{2} \frac{\partial}{\partial t} \epsilon \mathcal{E}^2$$

$$\mu H \cdot \frac{\partial H}{\partial t} = \frac{1}{2} \frac{\partial}{\partial t} \mu H^2$$

so that

$$\nabla \cdot (\mathcal{E} \times H) = -\frac{\partial}{\partial t} \frac{\mu H^2}{2} - \frac{\partial}{\partial t} \frac{\epsilon \mathcal{E}^2}{2} - \sigma \mathcal{E}^2 \tag{9-87}$$

Taking a volume integral of this quantity gives

$$-\int_v \nabla \cdot (\mathcal{E} \times H)\, dv = \int \left[\frac{\partial}{\partial t} \left(\frac{\mu H^2}{2} + \frac{\epsilon \mathcal{E}^2}{2} \right) + \sigma \mathcal{E}^2 \right] dv$$

From the divergence theorem of Section 9-7,

$$\int_v \nabla \cdot (\mathcal{E} \times H)\, dv = \int_S (\mathcal{E} \times H) \cdot da \tag{9-88}$$

so that

$$-\int_S (\mathcal{E} \times H) \cdot da = \int_v \left[\frac{\partial}{\partial t} \left(\frac{\mu H^2}{2} + \frac{\epsilon \mathcal{E}^2}{2} \right) + \sigma \mathcal{E}^2 \right] dv \tag{9-89}$$

It can be seen that

$$\int_v \frac{\mu H^2}{2} = \text{increase in stored magnetic energy}$$

$$\int_v \frac{\epsilon \mathcal{E}^2}{2} = \text{increase in stored electric energy}$$

$$\sigma \mathcal{E}^2 = \text{energy dissipated}$$

all per unit time. These three terms represent power supplied to the system. Thus the term on the left represents the power flow into the volume through the surface S, and reversing its sign gives the power outward through the surface or

$$W = \int_S \boldsymbol{P} \cdot \boldsymbol{da} = \int_S (\mathcal{E} \times H) \cdot \boldsymbol{da} \tag{9-90}$$

Thus the outward flow of energy in an electromagnetic wave is given by a vector $(\mathcal{E} \times H)$ which gives both direction and magnitude of the power flow.

Although the concepts of this section have been developed from the plane wave, not much generality is lost, since, at reasonable distances from the source, small sections of all waves tend to appear as plane.

PROBLEMS

9-1. A plane sinusoidal electromagnetic wave traveling in space has a maximum electric intensity of 1500 μv per meter.

(a) Find the accompanying maximum magnetic intensity in ampere-terms/meter.

(b) Propagation takes place in the x direction. The magnetic intensity is directed as H_y. Find the direction of $\mathcal{E}$.

(c) Compute the average power density transmitted in the field.

9-2. A sinusoidally varying electromagnetic wave in a medium of relative permittivity $\epsilon_r = 3.0$, relative permeability $\mu = \mu_v$, is transmitting power at a density of 1.2 w per square meter. Find the maximum intensities of the electric and magnetic fields.

9-3. The plane wave of Fig. 9-11 has an electric intensity (rms) of 100 μv per meter. It passes an antenna 3 meters long oriented in the x direction.

(a) Compute the emf induced in the antenna by consideration of the electric field.

(b) Repeat, using the effect of the magnetic field.

(c) Reconcile your two results. What voltage value appears in the antenna?

(d) Repeat (a) and (b), with the antenna oriented along the y direction.

9-4. A plane sinusoidal wave in space has a magnetic intensity of 0.008 ampere-turn/meter rms.

(a) Find the average rate of energy flow in the wave.

(b) This wave enters a block of glass of relative dielectric constant = 3, and $\mu = \mu_v$. Find the velocity in the glass.

9-5. The earth receives 20,000 cal per square meter per minute from the sun at noon. Calculate the maximum electric and magnetic intensities in sunlight. The maximum rate is twice the average rate given. Is sunburn explainable as an electrical phenomenon?

9-6. A field is present in space and infinite in extent having

$$H_x = H_y = 0$$
$$H_z = A \sin \alpha y \cos \omega t$$

(a) Determine the accompanying electric field and show that the wave equation may be obtained, proving the presence of a propagating wave.

(b) Show that functions of $(y - vt)$ and $(y + vt)$ are present in the fields.

Hint: $F(y) \neq F_1(t)$ unless both functions are zero or constant.

9-7. A square circuit of side l is oriented in the x,y plane with sides centered at some points x_0 and y_0. The field of Prob. 9-6 exists in the region.

(a) Compute the emf generated in the circuit by integration of the magnetic flux through the circuit by conventional methods.

(b) Check the result of (a) by use of Maxwell's field equations.

9-8. (a) Energy must be transmitted across the air gap of an induction motor if the rotor is to rotate and deliver mechanical power. Prove that energy flow is present by use of Poynting vector considerations.

(b) For a core-type transformer with coils on opposite legs.

show that energy is transmitted through the connecting core legs, by Poynting vector methods.

9-9. By performing the indicated operations, prove the following identities:

$$\nabla \cdot (\nabla \times A) \equiv 0$$

$$\nabla \times (\nabla P) \equiv 0$$

$$\nabla \cdot \nabla P \equiv \nabla^2 P$$

$$\nabla \times \nabla \times A = \nabla(\nabla \cdot A) - \nabla^2 A$$

REFERENCES

1. Ramo, S. and Whinnery, J. R., *Fields and Waves in Modern Radio*, 2d ed., John Wiley & Sons, Inc., New York, 1953.
2. Skilling, H. H., *Fundamentals of Electric Waves*, John Wiley & Sons, Inc., New York, 1942.
3. Stratton, J. A., *Electromagnetic Theory*, McGraw-Hill Book Company, Inc., New York, 1941.
4. Rogers, W. E., *Introduction to Electric Fields*, McGraw-Hill Book Company, Inc., New York, 1954.

Chapter 10

TRANSMISSION AND REFLECTION OF PLANE WAVES AT BOUNDARIES

Problems incident to propagation and transmission of electromagnetic waves involve the study of phenomena arising at the boundaries between space and the guiding or directing conductors or dielectrics. An important example of this is *skin effect*, or the effect of a field incident on a conductor in penetrating into and producing a current flow to only a limited depth in the conductor. The confinement of fields in wave guides and in shields is entirely due to skin effect or boundary phenomena. The identity between electromagnetics and optics is well illustrated in such a study of boundary effects, and many of the relations usually considered as wholly in the field of optics can be demonstrated from the electromagnetic viewpoint.

The use of the plane-wave concept in analyses of electromagnetic transmission may seem to lack generality. However, in view of the simplicity of the analysis and the fact that at reasonable distances from the source small areas of any wave front appear as plane waves, the concept is frequently employed.

10-1. Conditions of field continuity at boundary surfaces

Let an electromagnetic wave reach a boundary surface beyond which the conductivity, permeability, or permittivity is different from that of the region in which the wave has been traveling. This surface is indicated as dividing region 1 from region 2 in Fig. 10-1. Assume a rectangular box $ABCD$ set up so that it is divided on sides BC and AD by the boundary surface. The electric field intensity in region 1 may be resolved into components normal to and tangential to the boundary surface, as $\mathcal{E}_{1n}$ and $\mathcal{E}_{1t}$. In region 2, any electric field intensity has components $\mathcal{E}_{2n}$ and $\mathcal{E}_{2t}$. The surface charge is assumed to be zero.

Now allow the box dimensions AD and BC to shrink until they become infinitesimal. Then side AB lies along and just outside the surface, side CD along and just inside the surface. The work done in moving a unit charge around the path $ABCD$, or the line integral of the field enclosing infinitesimal area, must be zero, so that

$$a\mathcal{E}_{1t} + (-a)\,\mathcal{E}_{2t} = 0$$

or

$$\mathcal{E}_{1t} = \mathcal{E}_{2t} \tag{10-1}$$

since the other two parts of the path are infinitesimal in length.

Since the path $ABCD$ would enclose only an infinitesimal current,

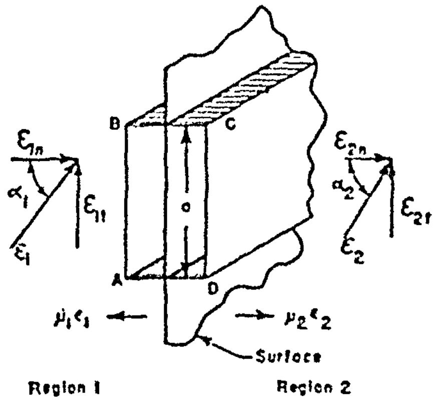


Fig. 10-1. Conditions in the electric field at a boundary between two media.

the line integral of the magnetic intensity around the path would be zero. By methods similar to the above, it can be shown that

$$H_{1t} = H_{2t} \tag{10-2}$$

It consequently becomes a rule of the electric or magnetic field that the *tangential components are continuous across a boundary.* This statement does not specify the value of the normal components.

At boundaries between dielectrics at which no free charge exists, the normal component of electric flux density in region 1 must equal that in region 2, since an electric flux line may end only on a charge. Then

$$D_{1n} = \epsilon_1\mathcal{E}_{1n} = D_{2n} = \epsilon_2\mathcal{E}_{2n} \tag{10-3}$$

If α_1 and α_2 are the angles between $\mathcal{E}_1$ and $\mathcal{E}_2$ and the normals to the respective surfaces, then

$$\tan \alpha_1 = \frac{\epsilon_1 \mathcal{E}_{1t}}{\epsilon_1 \mathcal{E}_{1n}}, \qquad \tan \alpha_2 = \frac{\epsilon_2 \mathcal{E}_{2t}}{\epsilon_2 \mathcal{E}_{2n}}$$

so that

$$\frac{\tan \alpha_1}{\tan \alpha_2} = \frac{\epsilon_1 \mathcal{E}_{1t}/\epsilon_1 \mathcal{E}_{1n}}{\epsilon_2 \mathcal{E}_{2t}/\epsilon_2 \mathcal{E}_{2n}}$$

In view of Eqs. 10-1 and 10-3, this becomes

$$\frac{\tan \alpha_1}{\tan \alpha_2} = \frac{\epsilon_1}{\epsilon_2} \tag{10-4}$$

Since magnetic flux lines are also continuous across a boundary, a similar relation that

$$\frac{\tan \alpha_1}{\tan \alpha_2} = \frac{\mu_1}{\mu_2} \tag{10-5}$$

may be developed for changes in magnetic properties as they affect the normally directed magnetic field intensities at a boundary.

Refractions of electromagnetic waves in crossing boundaries are thus indicated; they will be analyzed further in a later section.

10-2. The analogous transmission line for plane-wave propagation

At some particular point in a region of uniform properties, the plane-wave fields as developed in Chapter 9 may be written

$$H_y = H_0 \sin \omega t \tag{10-6}$$

$$\mathcal{E}_x = \sqrt{\frac{\mu}{\epsilon}}\, H_0 \sin \omega t \tag{10-7}$$

The field equations relating the existing H_y and $\mathcal{E}_x$ fields for this particular plane wave were found to be

$$\frac{\partial \mathcal{E}_x}{\partial z} = -\mu \frac{\partial H_y}{\partial t} \tag{10-8}$$

$$-\frac{\partial H_y}{\partial z} = \sigma \mathcal{E}_x + \epsilon \frac{\partial \mathcal{E}_x}{\partial t} \tag{10-9}$$

These are simply Eqs. 9-59 and 9-58 with the restoration of the conductivity term which was originally dropped because propagation was to occur in space. It is returned here so as to obtain results suitable for propagation in any medium.

Since the fields $\mathcal{E}_x$ and H_y have been stated as sinusoidal functions of time, their derivatives may be taken as

$$\frac{\partial H_y}{\partial t} = \omega H_0 \cos \omega t$$

$$\frac{\partial \mathcal{E}_x}{\partial t} = \omega \sqrt{\frac{\mu}{\epsilon}}\, H_0 \cos \omega t$$

and substituted in Eqs. 10-8 and 10-9 as

$$\frac{\partial \mathcal{E}_x}{\partial z} = -\omega\mu H_0 \cos \omega t$$

$$\frac{\partial H_y}{\partial z} = -\sigma \sqrt{\frac{\mu}{\epsilon}}\, H_0 \sin \omega t - \omega\epsilon \sqrt{\frac{\mu}{\epsilon}}\, H_0 \cos \omega t$$

In terms of the maximum values of the time functions, indicated as $\hat{\mathcal{E}}_x$ and $\hat{H}_y$, these equations become

$$\frac{\partial \hat{\mathcal{E}}_x}{\partial z} = -j\omega\mu \hat{H}_y \tag{10-10}$$

$$\frac{\partial \hat{H}_y}{\partial z} = -(\sigma + j\omega\epsilon)\hat{\mathcal{E}}_x \tag{10-11}$$

In Chapter 6 relations for voltage and current on a transmission line were developed as:

$$\frac{dE}{ds} = ZI, \qquad \frac{dI}{ds} = YE$$

with distance s measured as positive from the receiving end. When s is measured from the sending end, or in the direction of propagation of the incident wave as in the field case above, then

$$\frac{dE}{ds} = -ZI \tag{10-12}$$

$$\frac{dI}{ds} = -YE \tag{10-13}$$

It can be seen that these equations are of exactly the same form as Eqs. 10-10 and 10-11 for the field waves. *Two differential equations of the same form will have solutions of the same form.* Thus Eqs. 10-10 and 10-11 will have solutions in the form of those for the transmission line, predicting in general the presence of waves traveling in opposite directions. The presence of such waves in the propagation of fields has already been established in Chapter 9 from field theory. Basic knowledge of transmission of electric energy on lines may then be applied directly to the transmission of energy in electromagnetic field waves through use of the analogy suggested by the differential equations. In fact, transmission along wires is merely a special case of electromagnetic field waves *guided* by the wires.

The analogy can be more firmly established by differentiation of Eqs. 10-10 and 10-11 with respect to z and separation of the variables, giving

$$\frac{\partial^2 \hat{H}_y}{\partial z^2} = j\omega\mu(\sigma + j\omega\epsilon)\hat{H}_y$$

$$\frac{\partial^2 \hat{\mathcal{E}}_x}{\partial z^2} = j\omega\mu(\sigma + j\omega\epsilon)\hat{\mathcal{E}}_x$$

which may be written in the form

$$\frac{\partial^2 \hat{H}_y}{\partial z^2} = \gamma^2 \hat{H}_y \tag{10-14}$$

$$\frac{\partial^2 \hat{\mathcal{E}}_x}{\partial z^2} = \gamma^2 \hat{\mathcal{E}}_x \tag{10-15}$$

where $\gamma = \sqrt{j\omega\mu(\sigma + j\omega\epsilon)}$. These equations are of the form of the differential equations for the transmission line, in which case γ was identified as $\sqrt{ZY}$.

From comparison of Eqs. 10-10, 10-11, 10-12, and 10-13, and from dimensional considerations, it can be seen that E and $\mathcal{E}_x$, and I and H_y, are analogous variables. Also, it may be noted that Z is

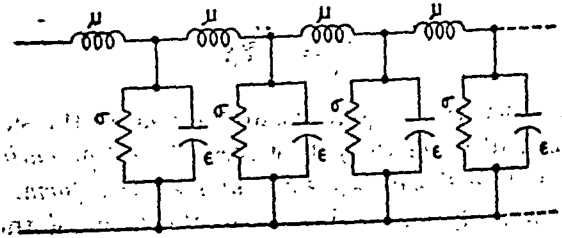


Fig. 10-2. The analogous transmission line for a plane wave in any medium.

analogous to $j\omega\mu$. Since for the line

$$Z = R + j\omega L$$

then L and μ are analogous quantities and there appears to be no field transmission-line equivalent of R, the series-line resistance. In a similar manner, Y is analogous to $\sigma + j\omega\epsilon$, so that since

$$Y = G + jB = G + j\omega C$$

σ is analogous to G and ϵ to C. A transmission-line equivalent for field waves would then appear as in Fig. 10-2.

The parameters of the transmission line representing conditions of the wave analogue are readily obtained. The propagation constant γ has already been identified above as

$$\begin{aligned}\gamma &= \sqrt{j\omega\mu(\sigma + j\omega\epsilon)}\\ &= \sqrt{-\omega^2\mu\epsilon + j\omega\mu\sigma} \qquad (10\text{-}16)\end{aligned}$$

The characteristic impedance or *intrinsic impedance* of the medium furnishes the relation between E and H in an infinite medium and is given the symbol η where

$$\eta = \sqrt{\frac{Z}{Y}} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}} \qquad (10\text{-}17)$$

Some manipulation shows that η and γ are related, since

$$\eta = \frac{j\omega\mu}{\gamma} = \frac{\gamma}{\sigma + j\omega\epsilon} \qquad (10\text{-}18)$$

For propagation *in space* where σ is considered zero, these equations reduce to

$$\eta_v = \sqrt{\frac{\mu_v}{\epsilon_v}} = 120\pi = 376.6 \text{ ohms}$$

since $\mu_v = 4\pi \times 10^{-7}$, $\epsilon_v = 1/36\ \pi \times 10^{-9}$. The units of the ratio are ohms in the rationalized m.k.s. system. The value of 376.6 ohms is considered the *intrinsic impedance of space* or the characteristic impedance of the equivalent transmission line for waves in space. The subscript v is used to indicate propagation in space.

The propagation constant *in space* is

$$\gamma_v = \alpha_v + j\beta_v = \sqrt{-\omega^2\mu_v\epsilon_v} = j\omega\sqrt{\mu_v\epsilon_v}$$

from which it is seen that *in space*

$$\alpha_v = 0; \beta_v = \omega\sqrt{\mu_v\epsilon_v}$$

In space there is no attenuation of a plane-polarized wave. For a cylindrical or spherical wave there is, of course, a reduction of power density in the wave as it propagates due to the constantly expanding

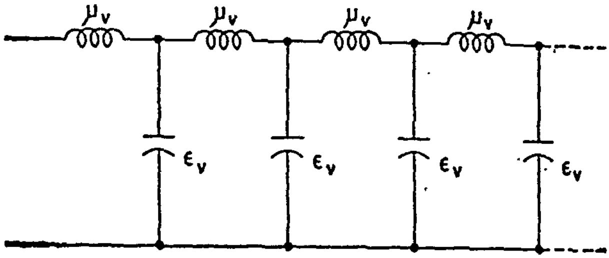


Fig. 10-3. The analogous transmission line for a plane wave in space.

surface area of the wave, but there is no dissipation of power in loss components of the region. Substitution of values for μ_v and ϵ_v gives

$$\beta_v = \frac{\omega}{3 \times 10^8}$$

and the velocity is

$$v = c = \frac{\omega}{\beta_v} = 3 \times 10^8 \text{ m/sec}$$

as would be expected.

The equivalent transmission line for plane-wave propagation in *space* is shown in Fig. 10-3. No loss elements appear.

The transmission of electromagnetic field waves inside a material having the properties of an electric conductor may also be analyzed through use of an equivalent transmission line. The intrinsic impedance and propagation constants as given by Eqs. 10-17 and 10-16 are quite complex, however, so that certain approximations concerning the relative values of σ_m and $\omega\epsilon_m$ for metals are usually made. The subscript *m* is used here to indicate values within metal conductors. The dielectric constant or relative permittivity of metals is usually taken to be that of free space, or $1/36\,\pi \times 10^{-9}$. The greatest value of the $\omega\epsilon_m$ term would occur for large values of ω. At present the shortest wavelengths generated by ultra-high-frequency electric equipment, as contrasted to optical equipment, are in the range of one millimeter, representing frequencies of the order of 3×10^{11} cycles. For that frequency, ω is $6\pi \times 10^{11}$, and the greatest value of $\omega\epsilon_m$ is therefore approximately 16.7. For copper, the conductivity σ_m is 5.75×10^7 mhos per meter. Consequently, for any frequency of ordinary interest in operation of electric equipment, $\omega\epsilon_m$ is entirely negligible with respect to σ_m for reasonably good conductors.

Using the approximation *for metals* that $\omega\epsilon_m \ll \sigma_m$,

$$\eta_m = \sqrt{\frac{j\omega\mu_m}{\sigma_m}} = (1 + j1)\sqrt{\frac{\omega\mu_m}{2\sigma_m}} \tag{10-19}$$

and

$$\gamma_m = \sqrt{j\omega\mu_m\sigma_m} = (1 + j1)\sqrt{\frac{\omega\mu_m\sigma_m}{2}} \tag{10-20}$$

from which

$$\alpha_m = \sqrt{\frac{\omega\mu_m\sigma_m}{2}}; \ \beta_m = \sqrt{\frac{\omega\mu_m\sigma_m}{2}}$$

For copper, η_m is $0.144 + j0.144$ ohms at a frequency of 3×10^{11} cycles. The values of α_m and β_m are 8.24×10^6 nepers and radians per meter, respectively, at the same frequency. For lower frequencies, η_m, α_m, and β_m are all reduced in proportion to the $\sqrt{\omega}$

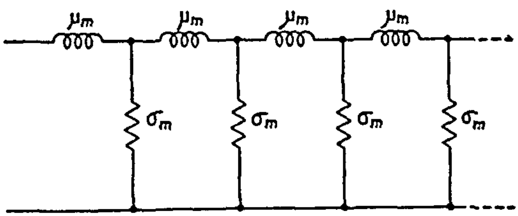


Fig. 10-4. The analogous transmission line for a plane wave in a good conductor.

factor. At a frequency of one megacycle in copper, η_m is $0.00026 + j0.00026$ ohms, $\alpha_m = 1.5 \times 10^4$ nepers per meter (13×10^4 db per meter), and β_m is 1.5×10^4 radians per meter. Compared with the values of attenuation normally expected on wire transmission lines, this is a huge attenuation, indicating that the wave is reduced to a negligible value in transmission through a few hundredths of a millimeter of copper.

The velocity of propagation in a metal is

$$v_m = \frac{\omega}{\beta_m} = \sqrt{\frac{2\omega}{\mu_m \sigma_m}} \text{ m/sec} \tag{10-21}$$

which is a function of frequency. For copper, at a frequency of 1 megacycle the velocity is only 417 meters per second, and at 60 cycles the velocity in copper is only 2.62 cm per second.

An equivalent transmission line for plane-wave propagation in a good conductor, shown in Fig. 10-4, consists simply of series inductance with a value proportional to the permeability of the metal, and shunt leakance due to the conductivity σ_m.

The parameters of equivalent lines for fields propagating in space, metals, and good dielectrics are summarized in Table 7.

TABLE 7

SUMMARY OF PARAMETERS OF EQUIVALENT TRANSMISSION LINES FOR PLANE-WAVE PROPAGATION

Parameter	General value	In space	In good conductors	In good dielectrics
η	$\sqrt{\dfrac{j\omega\mu}{\sigma + j\omega\epsilon}}$	$\sqrt{\dfrac{\mu_v}{\epsilon_v}}$	$(1 + j1)\sqrt{\dfrac{\omega\mu_m}{2\sigma_m}}$	$\sqrt{\dfrac{\mu}{\epsilon}}$
Z	$j\omega\mu$	$j\omega\mu_v$	$j\omega\mu_m$	$j\omega\mu$
Y	$\sigma + j\omega\epsilon$	$j\omega\epsilon_v$	σ_m	$j\omega\epsilon$
γ	$\sqrt{-\omega^2\mu\epsilon + j\omega\mu\sigma}$	$j\omega\sqrt{\mu_v\epsilon_v}$	$(1 + j1)\sqrt{\dfrac{\omega\mu_m\sigma_m}{2}}$	$j\omega\sqrt{\mu\epsilon}$
α		0	$\sqrt{\dfrac{\omega\mu_m\sigma_m}{2}}$	0
β		$\omega\sqrt{\mu_v\epsilon_v}$	$\sqrt{\dfrac{\omega\mu_m\sigma_m}{2}}$	$\omega\sqrt{\mu\epsilon}$
v		$\dfrac{1}{\sqrt{\mu_v\epsilon_v}} = c$	$\sqrt{\dfrac{2\omega}{\mu_m\sigma_m}}$	$\dfrac{1}{\sqrt{\mu\epsilon}}$

10-3. Reflection from a plane conductor at normal incidence

The concept of the equivalent transmission line for plane field waves has prepared the way for understanding of the effects taking place when a plane wave moving in space strikes a conducting metal sheet. Consider the plane wave with field components $\mathcal{E}_x$ and H_y traveling in space in the direction of positive z, as shown at (a), Fig. 10-5. This is the same wave previously discussed except that the origin of axes has been moved ahead of the wave to the new position

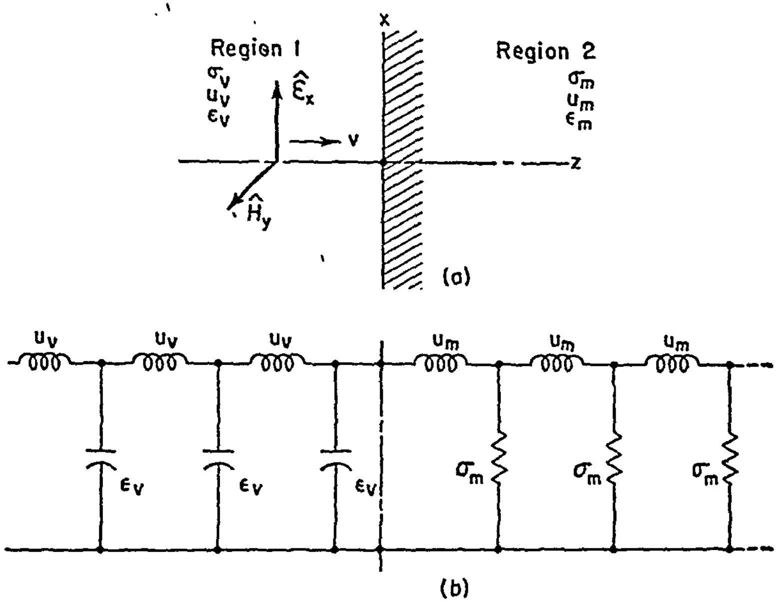


Fig. 10-5. (a) Plane-wave reflection at normal incidence from a sheet of good conductor; (b) the analogous transmission line for (a).

shown. The wave will strike at *normal incidence* on an infinite block of metal placed with its face in coincidence with the x,y plane and extending in depth to infinity on the positive-z axis.

The analogous transmission-line situation is shown at (b), Fig. 10-5. At the boundary of space and metal block at the origin, a source of reflection has been introduced, because of the junction of a space-equivalent line with one of entirely different characteristics corresponding to metal. Consequently, another component wave must be considered in region 1, namely, a reflected wave traveling back along the z axis, and of magnitude determined by the reflection coefficient at the boundary discontinuity. The magnitude of this reflection discontinuity between space and metal is apparent if it is considered that a space line of characteristic impedance equal to 377 ohms is joined to a metal line representing region 2 of characteristic impedance less than a value of the order of $0.144 + j0.144$ ohms. The equivalent metal transmission line appears as very near a short circuit on the field wave from region 1. As a result there will be present in space an incident wave, a reflected wave of almost equal magnitude, and in the metal a small transmitted wave that is rapidly attenuated. The fields present in the incident plane wave at a point in space are given by

$$H_y = H_v \sin \omega t$$

$$\mathcal{E}_x = \sqrt{\frac{\mu_v}{\epsilon_v}}\, H_v \sin \omega t$$

The total wave at any point in space may be found by recourse to the transmission-line analogue. For the condition of an arbitrary termination, the conventional transmission-line equations indicate the presence of an incident and a reflected wave, the current being expressed as the difference between the two waves, the voltage as their sum. Writing these equations for the field conditions present on the equivalent line of (b), Fig. 10-5, gives

$$\hat{H}_y = \frac{\hat{H}_R(Z_R + Z_0)}{2Z_0}\left(\epsilon^{-j\omega\sqrt{\mu_r\epsilon_r}\,z} - K\epsilon^{j\omega\sqrt{\mu_r\epsilon_r}\,z}\right)$$

$$\hat{\mathcal{E}}_x = \frac{\hat{\mathcal{E}}_R(Z_R + Z_0)}{2Z_R}\left(\epsilon^{-j\omega\sqrt{\mu_r\epsilon_r}\,z} + K\epsilon^{j\omega\sqrt{\mu_r\epsilon_r}\,z}\right)$$

since α is zero in space. The signs on the exponents are reversed from previous practice, because z is measured positive in the direction of propagation of the incident wave rather than in the direction of movement of the reflected wave as in Chapter 6. The fields are expressed in terms of maximum values, $\hat{H}_y$ and $\hat{\mathcal{E}}_x$, of their respective sinusoidal time variations, and $\hat{H}_R$ and $\hat{\mathcal{E}}_R$ are maximum values of H and $\mathcal{E}$ at the metal surface at $z = 0$.

The line in space is terminated by the input of the infinite line in the metal so that Z_R becomes η_m of the line in the metal. Also, in view of the fact that Z_0 of the line in space (region 1) is η_v, the above equations become

$$\hat{H}_y = \frac{\hat{H}_R(\eta_m + \eta_v)}{2\eta_v}\left(\epsilon^{-j\beta_v z} - K\epsilon^{j\beta_v z}\right) \tag{10-22}$$

$$\hat{\mathcal{E}}_x = \frac{\hat{\mathcal{E}}_R(\eta_m + \eta_v)}{2\eta_m}\left(\epsilon^{-j\beta_v z} + K\epsilon^{j\beta_v z}\right) \tag{10-23}$$

Since both $\hat{H}_R$ and $\hat{\mathcal{E}}_R$ are unknown, it is desirable to express them in terms of the known amplitude of the incident wave through use of the reflection coefficient definition. Since $\hat{H}_R$ and $\hat{\mathcal{E}}_R$ are the sum of the incident and reflected waves at $z = 0$ and

$$\text{reflected voltage wave} = K \times \text{incident voltage wave}$$

then the magnetic field (analogue of the current) at the metal boundary, in terms of the maximum value H_v of the original field, is

$$\hat{H}_R = (1 - K)H_v \tag{10-24}$$

The maximum electric field (analogue of the voltage) at the metal boundary is

$$\hat{\mathcal{E}}_R = (1 + K)\eta_v H_v \tag{10-25}$$

Since the reflection coefficient K may be written for the terminated line in space (region 1) as

$$K = \frac{\eta_m - \eta_v}{\eta_m + \eta_v}$$

the factor $(1 + K)$ and $(1 - K)$ may be modified to

$$1 + K = \frac{2\eta_m}{\eta_m + \eta_v}$$

$$1 - K = \frac{2\eta_v}{\eta_m + \eta_v}$$

giving for $\hat{H}_R$ and $\hat{\mathcal{E}}_R$:

$$\hat{H}_R = \frac{2\eta_v H_v}{\eta_m + \eta_v} \tag{10-26}$$

$$\hat{\mathcal{E}}_R = \frac{2\eta_v \eta_m H_v}{\eta_m + \eta_v} \tag{10-27}$$

and resulting in simplified forms of Eq. 10–22 and 10–23 for the total maximum fields at any point in region 1 as

$$\hat{H}_y = H_v(\epsilon^{-j\beta_v z} - K\epsilon^{j\beta_v z}) \tag{10-28}$$

$$\hat{\mathcal{E}}_x = \eta_v H_v(\epsilon^{-j\beta_v z} + K\epsilon^{j\beta_v z}) \tag{10-29}$$

The maximum fields at the boundary ($z = 0$) are directly given by Eqs. 10-26 and 10-27. Since these fields are wholly tangential, they are transmitted through the boundary and become the source fields for transmission within the metal. Thus at the boundary, *just inside the metal surface,*

$$\hat{H}_{y2} = \hat{H}_R = \frac{2\eta_v H_v}{\eta_m + \eta_v} \text{ amp/m} \tag{10-30}$$

$$\hat{\mathcal{E}}_{x2} = \hat{\mathcal{E}}_R = \frac{2\eta_v \eta_m H_v}{\eta_m + \eta_v} \text{ v/m} \tag{10-31}$$

Since η_m has an associated angle of 45°, the two fields in the metal differ in time phase by that amount. Since the metal is assumed of infinite depth in the z direction, there will be no reflected wave in the metal; and the fields at any point along the metal equivalent line in region 2, or the fields present at any point in the metal, are given in terms of the propagation constant γ_m as

$$\hat{H}_{y2} = \frac{2\eta_v H_v}{\eta_m + \eta_v} \epsilon^{-\gamma_m z}$$

$$\hat{\mathcal{E}}_{x2} = \frac{2\eta_m \eta_v H_v}{\eta_m + \eta_v} \epsilon^{-\gamma_m z}$$

Substitution of the value of γ_m for a conductor gives

$$\hat{H}_{y2} = \frac{2\eta_v H_v}{\eta_m + \eta_v} \epsilon^{-\sqrt{\frac{\omega\mu_m\sigma_m}{2}}\, z}\, \epsilon^{-j\sqrt{\frac{\omega\mu_m\sigma_m}{2}}\, z} \tag{10-32}$$

$$\hat{\mathcal{E}}_{x2} = \frac{2\eta_m \eta_v H_v}{\eta_m + \eta_v} \epsilon^{-\sqrt{\frac{\omega\mu_m\sigma_m}{2}}\, z}\, \epsilon^{-j\sqrt{\frac{\omega\mu_m\sigma_m}{2}}\, z} \tag{10-33}$$

as the variation of the maximum values of the fields with the depth in the metal. These equations show that the fields fall off exponentially with depth in the conductor. For materials of high conductivity, the attenuation is greater than for poor conductors. The attenuation is also proportional to the square root of the frequency, so that for frequencies in the audio range the fields may not be rapidly attenuated.

An important application of metal in electromagnetic fields is for *shielding* of one circuit or piece of equipment from another circuit or portion of the same circuit. Circuits may be separated by simple metal baffle plates or may be placed in complete metal boxes to prevent fields from one circuit from reaching another, where unwanted voltages might be induced. The efficiency of shielding in preventing this unwanted interaction of circuits is indicated above as dependent upon rapid attenuation of the field wave after entering the metal, so that no appreciable field intensity can reach the second surface of the metal sheet.

For shielding of audio-frequency apparatus or circuits, it is customary to use iron with a high value of permeability, the increase in permeability improving the shielding effect. This practice need not be carried out at high frequencies, because the high value of ω causes the field to be attenuated so rapidly that it reaches into or penetrates only a very small distance into the iron, and therefore most of the iron volume is unnecessary and might more economically be replaced with space. The difficulty encountered in shielding for low frequencies is shown by the fact that at 10^8 cycles a certain field intensity is reduced to a satisfactorily low value in passing into 0.0363 mm of copper, whereas to obtain the same shielding effect at 60 cycles requires the use of 5.4 mm of copper.

Because of the large value of ω, the attenuation is extremely rapid at high frequencies. When high-conductivity silver-plated or copper shields are used, the attenuation is so great that ordinarily no thought need be given to any appreciable field reaching the space beyond even a very thin metal sheet. It is assumed that the shield box is almost literally watertight, since some field may enter the second space if the shield is not complete or if holes or gaps exist.

10-4. Power flow in the reflected and transmitted waves

In space, where $\mathcal{E}$ and H were in time phase in the incident wave the instantaneous power density was shown to be

$$P = \mathcal{E} \times H$$

The incident plane wave in space of Fig. 10-5 is found by this rule to give a *Poynting vector* or power-flow direction resulting from $\mathcal{E}_x$ being turned through 90° to H_y, or power flow in the direction of the positive-z axis in Fig. 10-5. The reflected field wave in space is given by the exponential factors with K coefficients in Eqs. 10-28 and 10-29. For reflection from a metal surface, $|\eta_m| < |\eta_v|$, so that K becomes negative. As a result, the field vectors of the reflected wave are H_y and $-\mathcal{E}_x$. The Poynting vector then indicates power flow in the minus-z direction, resulting from $-\mathcal{E}_x$ being turned toward H_y, thus justifying the name "reflected wave."

With K negative, the value of $\hat{H}_R$ at the metal surface is the sum of the incident and reflected fields, and the direction of H_y is maintained when transmitted through the boundary as a tangential component. Hence the value of $\hat{\mathcal{E}}_R$ is the difference between incident and reflected electric fields; but since the magnitude of K is less than unity, the resultant $\hat{\mathcal{E}}_R$, though small, maintains the direction of $\mathcal{E}_x$. The Poynting vector or power-flow direction in the metal results from rotation of $\mathcal{E}_x$ into H_y, the right-hand screw moving toward positive z or into the conductor in Fig. 10-5(a).

However, because of the presence of η_m in the numerator of the coefficient of the electric field in Eq. 10-33, the electric and magnetic fields are found to be at a time angle of 45° in the metal, whereas they were in time phase in space. Therefore, a method for determining the average power flow is needed. In an electric circuit where V and I may differ in time phase, the average power is given by $P_{\text{ave}} = \frac{1}{2}\hat{V}\hat{I}\cos\phi$, where $\hat{V}$ and $\hat{I}$ are peak values and ϕ is the time phase angle between $\hat{V}$ and $\hat{I}$. By analogy the time-average power density in an electromagnetic field wave is

$$P_{\text{ave}} = \tfrac{1}{2}\hat{\mathcal{E}}\hat{H}\sin\theta\cos\phi \text{ watts/m}^2 \tag{10-34}$$

where $\hat{\mathcal{E}}$ and $\hat{H}$ are maximum values, θ is the space angle between $\hat{\mathcal{E}}$ and $\hat{H}$, and ϕ is the time-phase angle between $\hat{\mathcal{E}}$ and $\hat{H}$.

In view of the relation

$$\hat{\mathcal{E}} = \eta\hat{H} \tag{10-35}$$

between the components of a *plane wave*,

$$P_{\text{ave}} = \tfrac{1}{2}\hat{H}^2\eta\cos\phi$$

Since ϕ is the angle associated with η, then $\eta\cos\phi$ is actually the real part of η, or

$$P_{\text{ave}} = \tfrac{1}{2}\hat{H}^2 \times \text{real part of } \eta$$

For metals, where $$\eta_m = (1 + j1)\sqrt{\frac{\omega\mu_m}{2\sigma_m}}$$

$$P_{\text{ave}} = \tfrac{1}{2}\hat{H}^2\sqrt{\frac{\omega\mu_m}{2\sigma_m}} \text{ watts/m}^2 \tag{10-36}$$

10-5. Current flow in the conductor; depth of penetration; skin effect

The presence of an electric field in the metal indicates that there is a current flow, since

$$J = \sigma\mathcal{E}$$

which is merely a restatement of Ohm's law. This current in the problem under consideration is a sinusoidal time function because of its proportionality to $\mathcal{E}$. The maximum value of the current density, in amperes per square meter, at any depth z in the metal block, is obtained from Eq. 10-33 as

$$\hat{J}_x = \sigma_m \hat{\mathcal{E}}_{xm} = \frac{2\eta_r \eta_m H_r \sigma_m}{\eta_m + \eta_r} \epsilon^{-\sqrt{\frac{\omega\mu_m\sigma_m}{2}}z} \epsilon^{-j\sqrt{\frac{\omega\mu_m\sigma_m}{2}}z} \tag{10-37}$$

showing attenuation of the current density value with depth z in the metal. The variation of current density with depth in a copper

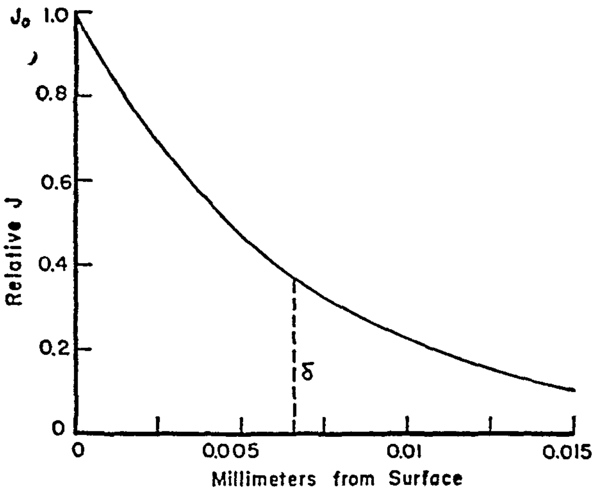


Fig. 10-6. Variation of current density with depth in copper at a frequency of 100 megacycles.

sheet for a frequency of 100 megacycles is plotted in Fig. 10-6. The greatest current density occurs at the metal surface or at $z = 0$, at which point the magnitude of the current density, signified by $\hat{J}_0$, is given by

$$\hat{J}_0 = \left| \frac{2\eta_r \eta_m H_r \sigma_m}{\eta_m + \eta_r} \right| \tag{10-38}$$

At a depth $z = \delta$ for which

$$\delta = \sqrt{\frac{2}{\omega\mu_m\sigma_m}} \text{ meters} \tag{10-39}$$

the current-density magnitude will be reduced to $1/\epsilon$ of its value at the surface. This distance δ, called the *equivalent depth of penetration*, is used as a measure of efficiency of a shielding material. It should be noted that for a medium possessing high permeability or high conductivity, or for a high frequency of the incident wave, the equivalent depth of penetration of current becomes very small. In copper, at 100 mc, Fig. 10-6 shows the depth of penetration to be only 0.0066 mm, or 6.6 microns, which is the length of a heat wave in the infrared region of the spectrum.

Since the depth to which the field penetrates is so small at high frequencies, a thin plating of a high-conductivity material, such as silver on a base metal, is quite frequently used. A plating of a few hundredths of a millimeter (a few thousandths of an inch) is quite sufficient, and the material used for the base will have very little effect on the over-all conductivity. A copper shield with a thin silver plating may then be equivalent to a solid-silver sheet, at a much lower cost.

For copper at a temperature of 20°C the equivalent depth of penetration is

$$\delta = \frac{0.0664}{\sqrt{f}} \text{ m} \tag{10-40}$$

where f is in cycles.

The conductivity of a number of common metals and alloys is given in Table 8.

The time-average power entering into the metal through the surface of the conductor is given by Eq. 10-36 as

$$P_{\text{ave}} = \tfrac{1}{2}\hat{H}_R^2 \sqrt{\frac{\omega\mu_m}{2\sigma_m}} \text{ watts/m}^2$$

where H_R is the maximum field just inside the conductor surface. In the infinite metal block, Eqs. 10-30 and 10-31 show that

$$\hat{H}_R = \frac{\hat{\mathcal{E}}_R}{\eta_m}$$

and it can be reasoned that the electric field is related to the current density as

$$\hat{\mathcal{E}}_R = \frac{\hat{J}_0}{\sigma_m}$$

TABLE 8

PROPERTIES OF CONDUCTORS AND SEMICONDUCTORS*

Material	Conductivity, mhos/meter, 20°C	Relative permittivity, ϵ_r
Aluminum	3.53×10^7	1
Copper	5.75×10^7	1
Gold	4.10×10^7	1
Iron	1.04×10^7	1
Manganin	0.22×10^7	1
Nichrome	0.09×10^7	1
Nickel	1.15×10^7	1
Silver	6.10×10^7	1
Tin	0.86×10^7	1
Water, ocean	3	80
Water, lake	10^{-3}	80

* From data in *Reference Data for Radio Engineers*, 3d ed., Federal Telephone and Radio Corporation, New York, 1949.

so that the magnetic field may be written

$$\hat{H}_R = \frac{\hat{J}_0}{\sigma_m \eta_m} \tag{10-41}$$

The average power entering the metal from the wave in space is therefore

$$\begin{aligned} P_{\text{ave}} &= \frac{\hat{J}_0^2}{2\sigma_m^2 \eta_m^2} \sqrt{\frac{\omega \mu_m}{2\sigma_m}} \\ &= \frac{\hat{J}_0^2}{2\sqrt{2}\, \sigma_m \sqrt{\omega \mu_m \sigma_m}} \end{aligned} \tag{10-42}$$

Writing this expression in terms of the depth of penetration δ gives

$$P_{\text{ave}} = \frac{\hat{J}_0^2 \delta}{4\sigma_m} \text{ watts/m}^2 \tag{10-43}$$

where $\hat{J}_0$ is the time-maximum value of the current density at the surface of the conductor. This is the power received by the conductor from the field and dissipated as I^2R losses in a thin *skin* of conductor near the surface.

Equation 10-43 may be more readily given physical interpretation by writing it as

$$P_{\text{ave}} = \frac{\hat{J}_0{}^2\delta^2}{4}\frac{1}{\sigma_m\delta}$$
$$= I_{eq}{}^2R_{eq}$$

where I_{eq} is an rms value,

$$I_{eq} = \frac{\hat{J}_0\delta}{2} \text{ amp/m width} \tag{10-44}$$

and

$$R_{eq} = \frac{1}{\sigma_m\delta} \text{ ohms/m width/m} \tag{10-45}$$

The equivalent rms current can be considered a uniformly distributed current, with a value equal to one-half of the time-maximum current density at the surface flowing in the positive-x direction in the conductor in a *skin* that is δ meters deep (see Fig. 10-7). The resistance R_{eq} is therefore the resistance of the block of metal of cross section 1 meter $\times$ δ and length 1 meter, of conductivity $\sigma_m = 1/\rho_m$, where ρ_m is the specific resistance. Thus

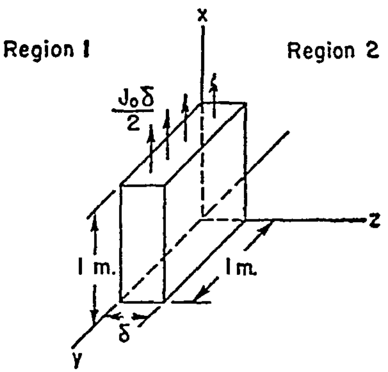


Fig. 10-7. The equivalent current flows in a skin of depth δ.

$$R_{eq} = \frac{\rho_m \times 1 \text{ meter}}{\delta \times 1 \text{ meter}} = \frac{\rho l}{A}$$

Thus for physical interpretation the current in the metal may be considered as having an rms value one-half of the surface maximum value, given by $\sigma_m\hat{\mathcal{E}}_R$, and flowing in a skin of thickness δ at the surface of the conductor.

The failure of the high-frequency fields to penetrate deeply beneath the surface of a conductor is the reason for the phenomenon of nonuniform alternating current distribution in conductors, known as *skin effect*.

Using the value of δ in Eqs. 10-44 and 10-45 gives

$$I_{eq} = \frac{\hat{J}_0}{\sqrt{2\omega\mu_m\sigma_m}} \tag{10-46}$$

$$R_{eq} = \sqrt{\frac{\omega\mu_m}{2\sigma_m}} \tag{10-47}$$

showing that the resistance increases and the current decreases with increasing frequency. It is interesting to note that the resistance value given by Eq. 10-47 corresponds exactly to the real or resistive term of the intrinsic impedance η_m of the metal, since

$$\eta_m = (1 + j1)\sqrt{\frac{\omega\mu_m}{2\sigma_m}}$$

This serves as a further tie between the fields and the transmission-line analogy.

For round conductors, if δ is small with respect to the conductor radius a, the a-c resistance may be computed by considering the effective conductor area that of a ring of thickness δ at the periphery of the conductor. The ratio of a-c to d-c resistance for conductors then becomes just the inverse ratio of the areas, or

$$\frac{R_{ac}}{R_{dc}} = \frac{\pi a^2}{2\pi a\delta} = \frac{a}{2\delta} = \frac{a\sqrt{\pi f\mu_m\sigma_m}}{2} \tag{10-48}$$

which corresponds to the expression given in Section 7-1. For conductors in which the depth of penetration is appreciable with respect to the radius, more complex relations involving Bessel functions are available in Reference (4).

In general, more current will flow in a conductor of high conductivity, but it will be distributed much closer to the surface than for a poor conductor. Hence a good conductor has greater skin effect, or its resistance changes more with frequency than does that of a poor conductor. Moreover, either higher frequency or higher permeability decreases the current flowing, although these factors also tend to cause the current to flow much nearer the surface.

10-6. The perfect-conductor concept

The conductivity of metals used as electric conductors is of the order of 10^{16} to 10^{18} times better than materials used as dielectrics. This difference is so great that when comparing wave propagation at metal boundaries with wave propagation in space or a dielectric it is frequently convenient to think of the action of the metal boundary as if it were a *perfect* conductor, of infinite conductivity. This perfect-conductor concept is so fundamental in the simplification of many field propagation problems that it needs further consideration.

Since the conductivity is infinite or resistance zero, no voltage gradient or electric field can exist in the perfect conductor, so that at a boundary between any dielectric and a perfect conductor the tangential electric field intensity must be zero. A magnetic field intensity may be present, however, as can be seen by examination of Eqs. 10-30 and 10-31. The intrinsic impedance of a conductor

$$\eta_m = (1 + j1)\sqrt{\frac{\omega\mu_m}{2\sigma_m}} \text{ ohms}$$

would reduce to zero for σ_m infinite. Thus the metal surface would appear as a short circuit on the equivalent field transmission line, and no energy could be delivered to the metal. The reflection coefficient

$$K = \frac{\eta_m - \eta_1}{\eta_m + \eta_1}$$

would become negative unity, and the entire wave energy would be sent back in the reflected wave. In a perfect conductor the velocity of propagation becomes zero and the attenuation infinite, so that even were it possible for energy to enter the perfect conductor, it could not travel; and even if it could travel, it would be attenuated at an infinite rate.

An anomalous situation arises for the surface current density

$$\mathcal{J}_0 = \left|\frac{2\eta_v\eta_m H_0\sigma_m}{\eta_m + \eta_v}\right|$$

of the perfect conductor, since it becomes indeterminate in view of $\eta_m = 0$ and $\sigma_m = \infty$. Resolution of the indeterminate form shows that for the perfect conductor the surface current density $\mathcal{J}_0$ will be infinite. Reference to Eq. 10-39 for the depth of penetration

$$\delta = \sqrt{\frac{2}{\omega\mu_m\sigma_m}}$$

shows that the depth of penetration of this infinite current will be zero. Thus infinite current density would appear in zero area, resulting in a finite total current, which by definition is equal to H.

For normal incidence on a perfect conductor, the fields given by

Eqs. 10-28 and 10-29 for the space region in front of the conductor become

$$\hat{H}_y = H_r(\epsilon^{-j\beta z} + \epsilon^{j\beta z})$$
$$\hat{\mathcal{E}}_x = \eta_r H_r(\epsilon^{-j\beta z} - \epsilon^{j\beta z})$$

since $K = -1$. These may be written

$$\hat{H}_y = 2H_r \frac{(\epsilon^{-j\beta z} + \epsilon^{j\beta z})}{2}$$

$$\hat{\mathcal{E}}_x = -2j\eta_r H_r \frac{(\epsilon^{j\beta z} - \epsilon^{-j\beta z})}{2j}$$

After insertion of the sine or cosine time functions, these equations become

$$H_y = 2H_r \cos \beta z \sin \omega t \tag{10-49}$$
$$\mathcal{E}_x = -2\eta_r H_r \sin \beta z \cos \omega t \tag{10-50}$$

Use of a trigonometric identity allows these equations to be written

$$H_y = H_r[\sin (\omega t - \beta z) + \sin (\omega t + \beta z)] \tag{10-51}$$
$$\hat{\mathcal{E}}_x = \eta_r H_r[\sin (\omega t - \beta z) - \sin (\omega t + \beta z)] \tag{10-52}$$

The form is recognizable as that indicating waves traveling in both directions, giving the incident and reflected wave as predicted by the theory of Chapter 9. Equations 10-49 and 10-50 indicate that standing waves will be present in the space region in front of the perfect conductor. The total $\mathcal{E}$ and H fields in the standing-wave system are in space quadrature.

The Poynting energy expressions for the incident and reflected waves are equal so that as much energy is returned by the reflected wave as was conveyed to the perfect-metal sheet by the incident wave.

10-7. Wave incident on a perfect conductor at an arbitrary angle

Figure 10-8 shows a boundary surface between a lossless dielectric in region 1 and a perfect conductor, with the boundary coincident with the x,y plane. A plane wave with the electric vector pointing out of the page is approaching this surface in the plane of the page and at an arbitrary angle of incidence θ_i to the z axis. Such a wave is described as polarized *normal* to the plane of incidence that is taken as the plane of the page. The direction of polarization is considered as the direction of the *electric field vector*. For the assumed direction of propagation, Poynting's vector requires that H be oriented as shown. The incident H_1 wave may be resolved into components, parallel to and normal to the boundary. The total electric field $\mathcal{E}_{y1}$ exists as parallel to the boundary and in the y direction. At some point in region 1 the incident electric and magnetic fields are

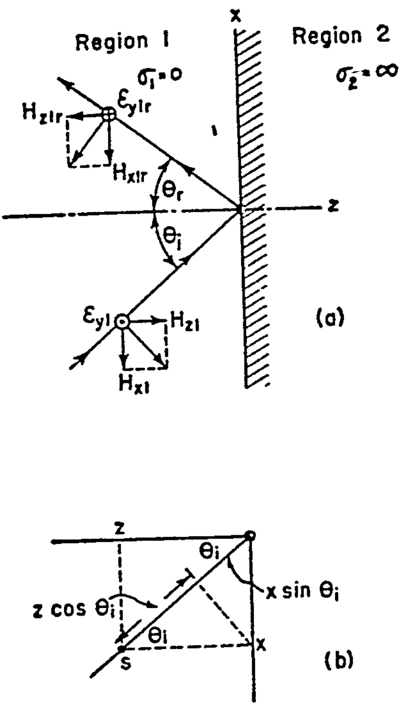


Fig. 10-8. Plane wave polarized normal to the plane of incidence, incident on a perfect conductor.

$$H_1 = \hat{H}_0 \sin \omega t$$
$$\mathcal{E}_{y1} = \hat{\mathcal{E}}_{y0} \sin \omega t$$

If distance along the path of propagation is given the symbol s, the maximum values of the two incident fields in region 1 will be given by

$$\hat{H}_1 = H_0 \epsilon^{-\gamma_1 s}$$
$$\hat{\mathcal{E}}_{y1} = \eta_1 H_0 \epsilon^{-\gamma_1 s}$$

where γ_1 is in general complex and $\epsilon^{-\gamma_1 s}$ indicates propagation of a wave in the s direction, as has been previously shown. Since region 1 has been assumed as a good dielectric with $\sigma_1 = 0$, or with σ_1 very small with respect to $\omega\epsilon_1$, the attenuation factor α_1 will be zero or negligible, and

$$\gamma_1 = j\beta_1$$

so that the maximum field values become

$$\hat{H}_1 = H_0 \epsilon^{-j\beta_1 s} = H_0(\cos \beta_1 s - j \sin \beta_1 s)$$
$$\hat{\mathcal{E}}_{y1} = \eta_1 H_0 \epsilon^{-j\beta_1 s} = \eta_1 H_0(\cos \beta_1 s - j \sin \beta_1 s)$$

Inserting these maximum values and using sine or cosine terms as called for by the j factors causes the fields as instantaneous time functions to become

$$H_1 = H_0(\cos \beta_1 s \sin \omega t - \sin \beta_1 s \cos \omega t)$$
$$\mathcal{E}_{y1} = \eta_1 H_0(\cos \beta_1 s \sin \omega t - \sin \beta_1 s \cos \omega t)$$

so that finally the instantaneous fields in the incident wave may be written

$$H_1 = H_0 \sin (\omega t - \beta_1 s)$$
$$\mathcal{E}_{y1} = \eta_1 H_0 \sin (\omega t - \beta_1 s)$$

These may be written

$$H_1 = H_0 \sin \omega \left(t - \frac{\beta_1 s}{\omega}\right)$$

$$\mathcal{E}_{y1} = \eta_1 H_0 \sin \omega \left(t - \frac{\beta_1 s}{\omega}\right)$$

and if it is remembered that $\omega/\beta = v$, the fields are seen as functions of $s - vt$, or are propagating fields as assumed.

In terms of the rectangular coordinates x and z,

$$s = x \sin \theta_i + z \cos \theta_i \tag{10-53}$$

as shown in Fig. 10-8(b). The fields in the incident wave may then be written as

$$H_1 = H_0 \sin (\omega t - \beta_1 x \sin \theta_i - \beta_1 z \cos \theta_i) \tag{10-54}$$
$$\mathcal{E}_{y1} = \eta_1 H_0 \sin (\omega t - \beta_1 x \sin \theta_i - \beta_1 z \cos \theta_i) \tag{10-55}$$

with components H_x and H_z making up H_1.

The incident wave conveys energy to the perfect conductor, which is unable to receive it, so that a reflected wave must arise. This reflected wave must propagate with a component of velocity in the minus z direction, giving rise to a $+\beta_1 z$ term in the angle. Since the reflection coefficient for a perfect conductor is -1, the electric field, being a tangential component, must be reflected with a value

$$\mathcal{E}_{y1r} = -\eta_1 H_0 \sin (\omega t - \beta_1 x \sin \theta_r + \beta_1 z \cos \theta_r) \tag{10-56}$$

the angle θ_r being arbitrary as yet, and indicated in (a), Fig. 10-8.

The plane reflecting surface is a perfect conductor, so that the total tangential electric field must be zero for all points on the metal surface, or for all values of x and time at $z = 0$. Comparison of

Eqs. 10-55 and 10-56 shows that the total electric field, due to the sum of $\mathcal{E}_{y1}$ and $\mathcal{E}_{y1r}$, can be zero at $z = 0$ for all values of x and time only if

$$\beta_1 x \sin \theta_i = \beta_1 x \sin \theta_r$$

From this reasoning it follows that the angles θ_i and θ_r must be related so that

$$\theta_i = \theta_r \tag{10-57}$$

which is the proof of the optical law that *the angle of incidence must equal the angle of reflection.*

The reflected fields may be obtained as an exercise in the application of Maxwell's field equations. The first equation may be written in general as

$$\nabla \times \mathcal{E} = -\mu \frac{\partial H}{\partial t}$$

Introducing the known field component $\mathcal{E}_{y1r}$ for the reflected field, all other electric components being zero, leaves

$$-\frac{\partial \mathcal{E}_{y1r}}{\partial z} = -\mu \frac{\partial H_{x1r}}{\partial t} \tag{10-58}$$

$$\frac{\partial \mathcal{E}_{y1r}}{\partial x} = -\mu \frac{\partial H_{z1r}}{\partial t} \tag{10-59}$$

wherein constant fields are not of interest. Taking the derivative of $\mathcal{E}_{y1r}$ with respect to z in Eq. 10-56 and inserting in Eq. 10-58 gives

$$-\mu \frac{\partial H_{x1r}}{\partial t} = \eta_1 H_0 \beta_1 \cos \theta_i \cos (\omega t - \beta_1 x \sin \theta_i + \beta_1 z \cos \theta_i)$$

Integration with respect to t gives the value of the H_{x1r} component in the reflected wave as

$$H_{x1r} = -H_0 \cos \theta_i \sin (\omega t - \beta_1 x \sin \theta_i + \beta_1 z \cos \theta_i) \tag{10-60}$$

after neglecting the constant of integration as a constant field.

Use of the value of $\mathcal{E}_{y1r}$ in Eq. 10-59 gives for the other magnetic field component H_{z1r}

$$H_{z1r} = -H_0 \sin \theta_i \sin (\omega t - \beta_1 x \sin \theta_i + \beta_1 z \cos \theta_i) \tag{10-61}$$

The field vectors in the reflected wave will appear as in Fig. 10-8 and the Poynting vector indicates power flow outward from the plane in the direction indicated, and with $\theta_i = \theta_r$.

A second possible orientation of the incident field is shown in Fig. 10-9 with the electric field in the plane of the page or the plane of incidence, or with the wave *polarized parallel to the plane of incidence.* The fields in the incident wave traveling in the direction shown in region 1 are

$$H_{y1} = H_0 \sin (\omega t - \beta_1 x \sin \theta_i - \beta_1 z \cos \theta_i) \tag{10-62}$$

$$\mathcal{E}_{x1} = \eta_1 H_0 \cos \theta_i \sin (\omega t - \beta_1 x \sin \theta_i - \beta_1 z \cos \theta_i) \tag{10-63}$$

$$\mathcal{E}_{z1} = -\eta_1 H_0 \sin \theta_i \sin (\omega t - \beta_1 x \sin \theta_i - \beta_1 z \cos \theta_i) \tag{10-64}$$

Because of the perfect conductivity of the plane metal reflecting surface, the tangential electric field component must be zero at $z = 0$ for all values of x and time. The reflected tangential electric component $\mathcal{E}_{x1r}$ will be

$$\mathcal{E}_{x1r} = -\eta_1 H_0 \cos \theta_i \sin (\omega t - \beta_1 x \sin \theta_i + \beta_1 z \cos \theta_i) \tag{10-65}$$

the positive sign on $\beta_1 z$ appearing because propagation of the reflected wave must occur with a component in the minus z direction.

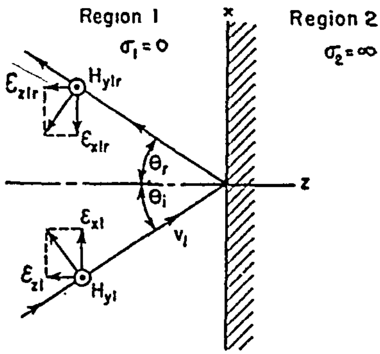


Fig. 10-9. Plane wave polarized parallel to the plane of incidence, incident on a perfect conductor.

Applying Maxwell's second field equation to the conditions present in the reflected wave, assuming zero conductivity in region 1, gives

$$\nabla \times H = \epsilon \frac{\partial \mathcal{E}}{\partial t}$$

Here H_{y1} is a wholly tangential component and therefore is the only source for H fields, so that only H_{y1r} exists. Consequently

$$\frac{\partial H_{y1r}}{\partial z} = \epsilon \frac{\partial \mathcal{E}_{x1r}}{\partial t} \tag{10-66}$$

$$\frac{\partial H_{y1r}}{\partial x} = \epsilon \frac{\partial \mathcal{E}_{z1r}}{\partial t} \tag{10-67}$$

Substitution of the time-derivative of $\mathcal{E}_{z1r}$ from Eq. 10-65 gives

$$\frac{-\partial H_{y1r}}{\partial z} = -\epsilon\eta_1 H_0\omega \cos\theta_i \cos(\omega t - \beta_1 x \sin\theta_i + \beta_1 z \cos\theta_i)$$

Upon integration, the magnetic field in the reflected wave is given by

$$H_{y1r} = H_0 \sin(\omega t - \beta_1 x \sin\theta_i + \beta_1 z \cos\theta_i) \tag{10-68}$$

neglecting the constant of integration, which is a constant field in space and not of interest.

That the above value is correct can be shown by use of the transmission-line analogue, which states that the surface tangential magnetic field (maximum) is given by

$$\hat{H}_R = (1 - K)H_0$$

The reflection coefficient K for the perfect conductor is -1, so that

$$\hat{H}_R = 2H_0$$

which indicates that the reflected magnetic component must be

$$H_{y1r} = H_0 \sin(\omega t - \beta_1 x \sin\theta_i + \beta_1 z \cos\theta_i)$$

confirming the results in Eq. 10-68.

Use of the x derivative of H_{y1r} in Eq. 10-67 gives

$$-\beta_1 \sin\theta_i H_0 \cos(\omega t - \beta_1 x \sin\theta_i + \beta_1 z \cos\theta_i) = \epsilon\frac{\partial \mathcal{E}_{z1r}}{\partial t}$$

which, after integration and dropping of the constant, becomes

$$\mathcal{E}_{z1r} = -\eta_1 H_0 \sin\theta_i \sin(\omega t - \beta_1 x \sin\theta_i + \beta_1 z \cos\theta_i) \tag{10-69}$$

Collecting the other two components of the reflected field:

$$\mathcal{E}_{x1r} = -\eta_1 H_0 \cos\theta_i \sin(\omega t - \beta_1 x \sin\theta_i + \beta_1 z \cos\theta_i) \tag{10-70}$$

$$H_{y1r} = H_0 \sin(\omega t - \beta_1 x \sin\theta_i + \beta_1 z \cos\theta_i) \tag{10-71}$$

These are the fields shown in Fig. 10-9 in the reflected wave. The Poynting vector indicates power flow away from the reflecting conductor sheet as shown, with angle $\theta_i = \theta_r$.

The field magnitudes involved in reflection at an arbitrary angle from a perfect conductor may be collected and expressed in terms of ratios of the magnitudes to that of the incident wave:

Polarization Parallel to Plane of Incidence

$$\frac{H_{y1r}}{H_{y1}} = \frac{H_0}{H_0} = +1$$

$$\frac{\mathcal{E}_{x1r}}{\mathcal{E}_{x1}} = \frac{-\eta_1 H_0 \cos \theta_i}{\eta_1 H_0 \cos \theta_i} = -1$$

$$\frac{\mathcal{E}_{z1r}}{\mathcal{E}_{z1}} = \frac{-\eta_1 H_0 \sin \theta_i}{-\eta_1 H_0 \sin \theta_i} = +1$$

Polarization Normal to Plane of Incidence

$$\frac{\mathcal{E}_{y1r}}{\mathcal{E}_{y1}} = \frac{-\eta_1 H_0}{\eta_1 H_0} = -1$$

$$\frac{H_{x1r}}{H_{x1}} = \frac{-H_0 \cos \theta_i}{-H_0 \cos \theta_i} = +1$$

$$\frac{H_{z1r}}{H_{z1}} = \frac{-H_0 \sin \theta_i}{H_0 \sin \theta_i} = -1$$

In the components chosen for the two polarizations, all possible field components of a wave with any arbitrary orientation are represented, that is, in the two polarizations there are all three possible $\mathcal{E}$ components and all three possible H components. Thus any incident wave may be considered as the sum of two waves, one polarized normal to the plane of incidence and the other polarized parallel to the plane of incidence. The two polarizations cover all possible orientations, and therefore the material of this section is general in scope.

10-8. Wave incident at an arbitrary angle on a boundary between dielectrics

Consider the plane wave of Fig. 10-8 propagating in a dielectric (region 1) where the parameters are μ_1 and ϵ_1, with $\sigma_1 = 0$. Let this wave approach a plane boundary of a second dielectric region with parameters μ_2 and ϵ_2, with $\sigma_2 = 0$. No loss in generality will occur if it be assumed that $\mu_1 = \mu_2 = \mu_v$, since no dielectrics are known in which the permeability is appreciably different from that of space. Considering the electric field oriented in the y direction or the wave polarized normal to the plane of incidence, the situation appears as in Fig. 10-10, with an incident wave, a reflected wave, and a possible wave transmitted into region 2, with angles of incidence and reflection θ_1 and angle of refraction of the transmitted wave θ_2.

The incident fields are

$$\begin{aligned}
\mathcal{E}_{y1} &= \eta_1 H_0 \sin (\omega t - \beta_1 x \sin \theta_1 - \beta_1 z \cos \theta_1) \\
H_{x1} &= -H_0 \cos \theta_1 \sin (\omega t - \beta_1 x \sin \theta_1 - \beta_1 z \cos \theta_1) \\
H_{z1} &= H_0 \sin \theta_1 \sin (\omega t - \beta_1 x \sin \theta_1 - \beta_1 z \cos \theta_1)
\end{aligned}$$

From the line analogy, the electric field at the dielectric surface will be the resultant of the incident and reflected waves; and since $\mathcal{E}_{y1}$ is

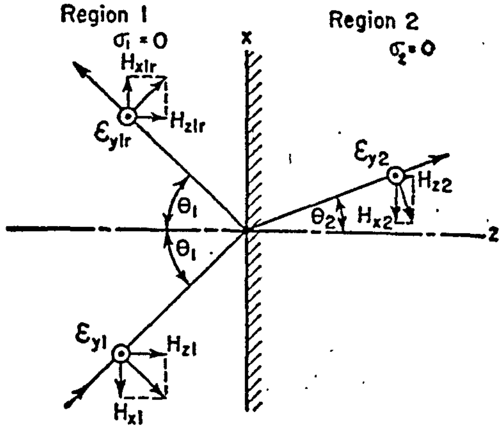


Fig. 10-10. Plane wave polarized normal to the plane of incidence, incident on a dielectric.

wholly tangential to the boundary, the maximum surface electric field is given by

$$\hat{\mathcal{E}}_R = (1 + K)\eta_1 H_0 \tag{10-72}$$

where
$$K = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

The electric field component in the reflected wave is then easily seen as

$$\mathcal{E}_{y1r} = K\eta_1 H_0 \sin(\omega t - \beta_1 x \sin\theta_1 + \beta_1 z \cos\theta_1) \tag{10-73}$$

Having one component of the reflected wave and knowing that $\mathcal{E}_{x1r} = \mathcal{E}_{z1r} = 0$, Maxwell's first field equation may be used. Inserting the zero field values leaves

$$-\frac{\partial \mathcal{E}_{y1r}}{\partial z} = -\mu \frac{\partial H_{x1r}}{\partial t} \tag{10-74}$$

$$\frac{\partial \mathcal{E}_{y1r}}{\partial x} = -\mu \frac{\partial H_{z1r}}{\partial t} \tag{10-75}$$

and upon performing the indicated operations on Eq. 10-73, the other field components that must accompany $\mathcal{E}_{y1r}$ in the reflected field, if it is to propagate, are obtained as

$$H_{x1r} = KH_0 \cos\theta_1 \sin(\omega t - \beta_1 x \sin\theta_1 + \beta_1 z \cos\theta_1) \tag{10-76}$$
$$H_{z1r} = KH_0 \sin\theta_1 \sin(\omega t - \beta_1 x \sin\theta_1 + \beta_1 z \cos\theta_1) \tag{10-77}$$

These field components of the reflected wave are shown in Fig. 10-10.

Since the electric field is wholly tangential at the boundary, Eq. 10-72 may be used to write an expression for the electric field in the second dielectric as

$$\mathcal{E}_{y2} = (1 + K)\eta_1 H_0 \sin(\omega t - \beta_2 x \sin\theta_2 - \beta_2 z \cos\theta_2) \tag{10-78}$$

showing propagation continuing with x and z velocity components but with a new phase constant β_2 and allowance for a changed direction through an angle of refraction θ_2 in the second medium.

By reason of the continuity of tangential field components at the boundary, Eq. 10-78, expressed at $z = 0$ or at the boundary in region 2, must give a value equal to the sum of $\mathcal{E}_{y1}$ and $\mathcal{E}_{y1r}$ at $z = 0$ or at the boundary in region 1, for all values of x and t. Since at $z = 0$,

$$\mathcal{E}_{y1} + \mathcal{E}_{y1r} = (1 + K)\eta_1 H_0 \sin(\omega t - \beta_1 x \sin\theta_1)$$

the equality can be seen as true at all values of x and t if

$$\beta_1 x \sin\theta_1 = \beta_2 x \sin\theta_2 \tag{10-79}$$

or

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{\beta_2}{\beta_1}$$

In a good dielectric,

$$\beta = \omega\sqrt{\mu\epsilon}$$

so that

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{\sqrt{\mu_2\epsilon_2}}{\sqrt{\mu_1\epsilon_1}} \tag{10-80}$$

and since $\mu_1 = \mu_2 = \mu_v$ for all known dielectrics,

$$\frac{\sin\theta_1}{\sin\theta_2} = \sqrt{\frac{\epsilon_2}{\epsilon_1}} \tag{10-81}$$

thus determining the angle of refraction θ_2 in terms of the angle of incidence θ_1 and the permittivities. The expression indicates that when a wave travels from a medium of low permittivity to one of high permittivity, the wave path is bent toward the normal to the surface. The reverse is true when a wave travels from a substance of high permittivity to one of low. The effect is commonly observed by noting the path a beam of light takes in passing from space into water, or vice versa.

Having Eq. 10-78 for the electric field of the traveling wave in the second dielectric, the known conditions may again be applied to Maxwell's first field equation and the other field components determined. These other components must accompany $\mathcal{E}_{y2}$ if a traveling wave is to exist in region 2 as desired. Since $\mathcal{E}_{z2} = \mathcal{E}_{x2} = 0$, the Maxwell equation reduces to

$$-\frac{\partial \mathcal{E}_{y2}}{\partial z} = -\mu \frac{\partial H_{x2}}{\partial t}$$

$$\frac{\partial \mathcal{E}_{y2}}{\partial x} = -\mu \frac{\partial H_{z2}}{\partial t}$$

from which the magnetic components in region 2 can be obtained as

$$H_{x2} = -(1 + K) \sqrt{\frac{\epsilon_2}{\epsilon_1}} \cos \theta_2 H_0 \sin (\omega t - \beta_2 x \sin \theta_2 - \beta_2 z \cos \theta_2) \tag{10-82}$$

$$H_{z2} = (1 + K) \sqrt{\frac{\epsilon_2}{\epsilon_1}} \sin \theta_2 H_0 \sin (\omega t - \beta_2 x \sin \theta_2 - \beta_2 z \cos \theta_2) \tag{10-83}$$

Field $\mathcal{E}_{y2}$ may be rewritten for reference:

$$\mathcal{E}_{y2} = (1 + K)\eta_1 H_0 \sin (\omega t - \beta_2 x \sin \theta_2 - \beta_2 z \cos \theta_2) \tag{10-84}$$

These fields of the transmitted wave are shown in Fig. 10-10 in region 2.

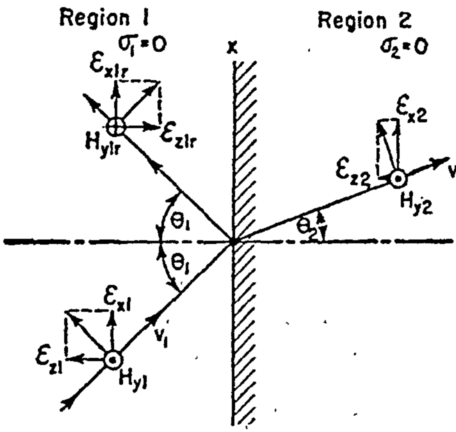


Fig. 10-11. Plane wave polarized parallel to the plane of incidence, incident on a dielectric.

With a field polarized in the plane of incidence, the situation will appear as in Fig. 10-11, with the electric field vector in the plane of the page, and the wave approaching a boundary surface between dielectric regions 1 and 2. The analysis is very similar to that of the wave polarized normal to the plane of incidence.

The incident fields are

$$H_{y1} = H_0 \sin(\omega t - \beta_1 x \sin\theta_1 - \beta_1 z \cos\theta_1)$$
$$\mathcal{E}_{x1} = \eta_1 H_0 \cos\theta_1 \sin(\omega t - \beta_1 x \sin\theta_1 - \beta_1 z \cos\theta_1)$$
$$\mathcal{E}_{z1} = -\eta_1 H_0 \sin\theta_1 \sin(\omega t - \beta_1 x \sin\theta_1 - \beta_1 z \cos\theta_1)$$

The magnetic field is wholly tangential, and the maximum field at the surface is given by the line analogy as

$$\hat{H}_R = (1 - K)H_0 \tag{10-85}$$

so that the magnetic field of the *reflected wave* is

$$H_{y1r} = -KH_0 \sin(\omega t - \beta_1 x \sin\theta_1 + \beta_1 z \cos\theta_1) \tag{10-86}$$

Maxwell's second field equation for a good dielectric may be used to obtain the other wave components. With $H_{x1r} = H_{z1r} = 0$, the Maxwell equation becomes

$$-\frac{\partial H_{y1r}}{\partial z} = \epsilon \frac{\partial \mathcal{E}_{x1r}}{\partial t}$$
$$\frac{\partial H_{y1r}}{\partial x} = \epsilon \frac{\partial \mathcal{E}_{z1r}}{\partial t}$$

Performing the indicated operations and neglecting constants of integration as constant fields,

$$\mathcal{E}_{x1r} = K\eta_1 H_0 \cos\theta_1 \sin(\omega t - \beta_1 x \sin\theta_1 + \beta_1 z \cos\theta_1) \tag{10-87}$$
$$\mathcal{E}_{z1r} = K\eta_1 H_0 \sin\theta_1 \sin(\omega t - \beta_1 x \sin\theta_1 + \beta_1 z \cos\theta_1) \tag{10-88}$$

and these fields are shown in region 1 in Fig. 10-11.

Because of the wholly tangential magnetic component, the *transmitted* magnetic field may be written from the line analogy and by comparison with Eqs. 10-85 and 10-86 as

$$H_{y2} = (1 - K)H_0 \sin(\omega t - \beta_2 x \sin\theta_2 - \beta_2 z \cos\theta_2) \tag{10-89}$$

which can be true for all values of x and t on the boundary plane only if

$$\frac{\sin\theta_1}{\sin\theta_2} = \sqrt{\frac{\epsilon_2}{\epsilon_1}} \tag{10-90}$$

as for the field polarized normal to the plane of incidence. Again using Maxwell's field equation and neglecting constants of integration as constant fields, the accompanying components must be

$$\mathcal{E}_{x2} = (1 - K)\eta_2 H_0 \cos\theta_2 \sin(\omega t - \beta_2 x \sin\theta_2 - \beta_2 z \cos\theta_2) \quad (10\text{-}91)$$

$$\mathcal{E}_{z2} = -(1 - K)\eta_2 H_0 \sin\theta_2 \sin(\omega t - \beta_2 x \sin\theta_2 - \beta_2 z \cos\theta_2) \quad (10\text{-}92)$$

in region 2 for polarization in the plane of incidence. These fields are illustrated in Fig. 10-11.

The field magnitudes for incidence on a dielectric boundary at an arbitrary angle may be collected for reference and expressed as ratios of their magnitudes to that of the incident field component:

Polarization Parallel to Plane of Incidence	*Polarization Normal to Plane of Incidence*
(a) $\dfrac{H_{y1r}}{H_{y1}} = \dfrac{-KH_0}{H_0} = -K$	(g) $\dfrac{\mathcal{E}_{y1r}}{\mathcal{E}_{y1}} = \dfrac{K\eta_1 H_0}{\eta_1 H_0} = K$
(b) $\dfrac{\mathcal{E}_{x1r}}{\mathcal{E}_{x1}} = \dfrac{K\eta_1 H_0 \cos\theta_1}{\eta_1 H_0 \cos\theta_1} = K$	(h) $\dfrac{H_{x1r}}{H_{x1}} = \dfrac{KH_0 \cos\theta_1}{-H_0 \cos\theta_1} = -K$
(c) $\dfrac{\mathcal{E}_{z1r}}{\mathcal{E}_{z1}} = \dfrac{K\eta_1 H_0 \sin\theta_1}{-\eta_1 H_0 \sin\theta_1} = -K$	(i) $\dfrac{H_{z1r}}{H_{z1}} = \dfrac{KH_0 \sin\theta_1}{H_0 \sin\theta_1} = K$
(d) $\dfrac{H_{y2}}{H_{y1}} = \dfrac{(1-K)H_0}{H_0} = 1 - K$	(j) $\dfrac{\mathcal{E}_{y2}}{\mathcal{E}_{y1}} = \dfrac{(1+K)\eta_1 H_0}{\eta_1 H_0} = 1 + K$
(e) $\dfrac{\mathcal{E}_{x2}}{\mathcal{E}_{x1}} = (1 - K)\sqrt{\dfrac{\epsilon_1}{\epsilon_2}}\dfrac{\cos\theta_2}{\cos\theta_1}$	(k) $\dfrac{H_{x2}}{H_{x1}} = (1 + K)\sqrt{\dfrac{\epsilon_2}{\epsilon_1}}\dfrac{\cos\theta_2}{\cos\theta_1}$
(f) $\dfrac{\mathcal{E}_{z2}}{\mathcal{E}_{z1}} = (1 - K)\sqrt{\dfrac{\epsilon_1}{\epsilon_2}}\dfrac{\sin\theta_2}{\sin\theta_1}$	(l) $\dfrac{H_{z2}}{H_{z1}} = (1 + K)\sqrt{\dfrac{\epsilon_2}{\epsilon_1}}\dfrac{\sin\theta_2}{\sin\theta_1}$

The symmetry between the two directions of polarization is very apparent.

It may be recalled that

$$K = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{\sqrt{\mu_2/\epsilon_2} - \sqrt{\mu_1/\epsilon_1}}{\sqrt{\mu_2/\epsilon_2} + \sqrt{\mu_1/\epsilon_1}}$$

Since the permeabilities are equal, the reflection coefficient is

$$K = \frac{\sqrt{\epsilon_1/\epsilon_2} - 1}{\sqrt{\epsilon_1/\epsilon_2} + 1} = \frac{\sqrt{\epsilon_1} - \sqrt{\epsilon_2}}{\sqrt{\epsilon_1} + \sqrt{\epsilon_2}} \quad (10\text{-}93)$$

Dielectric properties at high frequencies are summarized in Table 9.

TABLE 9

PROPERTIES OF HIGH-FREQUENCY DIELECTRICS*

Material	Relative permittivity, ϵ_r	Power factor $\cong \sigma/\omega\epsilon$	Frequency, cycles/sec
Bakelite, mineral-filled	4.3	0.012	10^8
Cellulose acetate	3.4	0.039	10^8
Glass	4.0	0.0012	10^8
Isolantite	6.0	0.0018	10^8
Lucite	2.6	0.007	10^8
Mica	5.4	0.0003	10^8
Mycalex	7.0	0.0022	10^8
Polyethylene	2.25	0.0003	10^8
Polystyrene	2.52	0.0003	10^8
Quartz	3.8	0.0002	10^8
Steatite	5.77	0.0006	10^8
Teflon	2.1	<0.0002	10^8

* From data in *Reference Data for Radio Engineers*, 3d ed., Federal Telephone and Radio Corporation, New York, 1949.

10-9. Index of refraction; Snell's law; total reflection

Equation 10-80 shows that angle of refraction θ_2 is related to the angle of incidence θ_1 as

$$\frac{\sin \theta_1}{\sin \theta_2} = \sqrt{\frac{\mu_2\epsilon_2}{\mu_1\epsilon_1}} = \frac{v_1}{v_2}$$

Dividing numerator and denominator by c, the velocity of electromagnetic waves in space, and rearranging,

$$\frac{\sin \theta_1}{\sin \theta_2} = \frac{c/v_2}{c/v_1} = \frac{N_2}{N_1}$$

The *index of refraction of a dielectric is defined as the ratio of the velocity of light in space to the velocity of light in the material,* so that N_1 and N_2 are the indices of refraction of dielectric regions 1 and 2 of the preceding section. The index is thus always greater than unity. Since $\mu_1 = \mu_2 = \mu_v$ in all known dielectrics, the above may also be written

$$\frac{\sin\,\theta_1}{\sin\,\theta_2} = \sqrt{\frac{\epsilon_2}{\epsilon_1}} = \frac{N_2}{N_1} \tag{10-94}$$

in which form it is known as *Snell's law* in the field of optics. In the present case it has been derived by use of the wave equations for propagation of electromagnetic energy.

Snell's law may be rewritten as

$$\sin\,\theta_1 = \sqrt{\frac{\epsilon_2}{\epsilon_1}}\sin\,\theta_2$$

If $\epsilon_1 > \epsilon_2$, a situation may arise in which $\sin\,\theta_2$ is required to have a value greater than unity. Such a condition is obviously impossible. The critical or limiting value of the angle of incidence θ_c is fixed by $\sin\,\theta_2 = 1$ or $\theta_2 = 90°$, as shown in Fig. 10-12. For this case, propagation of the wave in region 2 is along the boundary surface and no energy enters region 2. For angles of incidence greater than θ_c there is no propagation in region 2, and total reflection, internal to region 1, occurs. The *critical angle* of incidence, θ_c, above which total internal reflection occurs, is

$$\sin\,\theta_c = \sqrt{\frac{\epsilon_2}{\epsilon_1}} \tag{10-95}$$

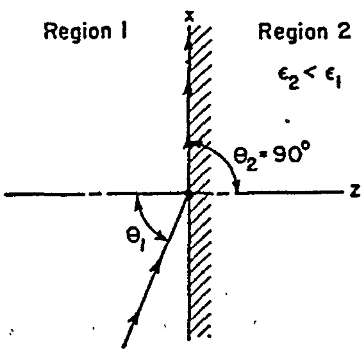


Fig. 10-12. Case of total reflection at a dielectric boundary.

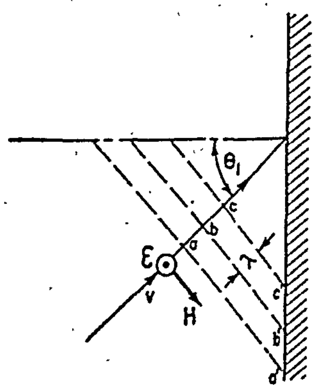


Fig. 10-13. Measurement of wave velocities.

10-10. Phase and group velocities

A wave propagating in the direction shown in Fig. 10-13 has a velocity v in the direction of propagation determined by the characteristics of the medium. When the wave is propagating at some arbitrary angle of incidence, as in the figure, the velocity v along the path may be considered as having two components, one parallel to the incident surface known as v_p, and one normal to the incident surface as v_n.

The dashed lines perpendicular to the direction of propagation in Fig. 10-13 are intended to indicate the position of successive positive maxima of the wave, being separated by the distance λ. If t_c be the time of one cycle, then during t_c seconds the wave front or crest at a will move to b, that at b to c, and so on. Hence the velocity is

$$v = \frac{\lambda}{t_c}$$

At the same time, an observer on the surface of incidence would note that in time t_c a given crest at a' would move to b', that at b' to c', and so on. According to his observations the velocity, which is that parallel to the incident surface, would be given by

$$v_p = \frac{\lambda}{t_c \sin \theta_1}$$

because a wave distance $a'b'$ is related to ab as $1/\sin \theta_1$. Since the observer bases his conclusions on observations of points of equal phase, or on the rate of phase change, the velocity v_p is also called the *phase velocity*. It is of importance because of the ease with which it may be observed or measured, occurring as it does on the bounding surface.

If the expression $\lambda/\sin \theta_1$ be considered as defining a new wavelength λ', then by reason of the relationship between frequency and wavelength

$$v_p = f\lambda' = \frac{f\lambda}{\sin \theta_1} = \frac{v}{\sin \theta_1} \qquad (10\text{-}96)$$

which relates the phase velocity, or velocity parallel to the surface, to the true velocity v in the medium.

The velocity v_n normal to the surface is not of much interest because of the difficulty of measurement in the interior of a region without disturbing the conditions of propagation by insertion of a measuring device. It may be readily defined, in a manner similar to the above, as

$$v_n = \frac{v}{\cos \theta_1}$$

It is apparent that the phase velocity will equal or exceed that of light in the medium. In fact, it becomes infinite for normal incidence, introducing a conflict with modern physical theory, which holds the velocity of light as a limiting velocity for all motion. This conflict may be readily reconciled if it is remembered that the true wave velocity is directed along the path a,b,c with a value equal to the velocity of light, as expected. This is the velocity at which the energy of the wave is propagated. No energy propagates at velocity v_p, and it serves merely as a quantity capable of measurement but without physical reality, fictitious but convenient.

In the consideration of the general transmission line, the high-frequency line with dissipation, and wave transmission in conductors, the velocity has been found to be a function of ω or of frequency. Conversely, in dissipationless lines and in wave propagation in space or lossless dielectrics, the velocity is not found to be a function of ω. A material in which the velocity is a function of ω is called *dispersive,* whereas a material in which velocity is not a function of ω is *nondispersive.* The above examples might indicate that a dispersive medium is also one with dissipation, but this statement is not necessarily true; it merely implies that up to the present point no other examples have been encountered.

The phase velocity has been defined in terms of a steady-state sine wave in which the crests or troughs are readily identified. If a complex wave form is applied to a dispersive system, since the various frequencies will travel at different rates there will be a change of relative phase and of wave form in transmission through the system, and identification of crests or troughs may be difficult. Actually, many radio waves are not sinusoidal but may be *groups* of closely related frequencies. That this must be true was shown by the discussion of the Hartley law in the Preface, which indicated that if intelligence is to be transmitted, groups or bands of frequencies must be used, rather than a discrete sinusoidal frequency.

In the case of modulated waves the whole group of waves conveys the intelligence. The point of interest is the velocity with which the intelligence travels through the system, not necessarily the velocity of a particular phase point as in the case of phase velocity. The velocity with which the intelligence travels may be shown by an example.

Consider a wave propagating with distance s and containing a group of two frequencies differing by $2\Delta\omega$, where $\Delta\omega$ is the change in frequency used to transmit certain intelligence; then

$$e = E\{\cos[(\omega - \Delta\omega)t - (\beta - \Delta\beta)s] + \cos[(\omega + \Delta\omega)t - (\beta + \Delta\beta)s]\}$$

where $\Delta\omega \ll \omega$. If the medium is dispersive, β is a function of frequency or of ω, and $\Delta\beta$ is associated with the change of frequency $\Delta\omega$. The above equation may be written as

$$e = E\{\cos[(\omega t - \beta s) - (\Delta\omega t - \Delta\beta s)] + \cos[(\omega t - \beta s) + (\Delta\omega t - \Delta\beta s)]\}$$

By use of a trigonometric identity, this equation becomes

$$e = 2E\cos(\Delta\omega t - \Delta\beta s)\cos(\omega t - \beta s)$$

This may be transformed to

$$e = 2E\cos\left[\Delta\omega\left(t - \frac{\Delta\beta}{\Delta\omega}s\right)\right]\cos\left[\omega\left(t - \frac{\beta}{\omega}s\right)\right]$$

which, taking account of term dimensions, may be written

$$e = 2E\cos\left[\Delta\omega\left(t - \frac{s}{v_g}\right)\right]\cos\left[\omega\left(t - \frac{s}{v}\right)\right] \qquad (10\text{-}97)$$

Equation 10-97 is seen to be a frequency $\omega/2\pi$ modulated or varied at a low frequency rate $\Delta\omega/2\pi$. The high-frequency component is traveling with a velocity

$$v = \frac{\omega}{\beta}$$

which is recognizable as the ordinary velocity of a traveling wave.

The low-frequency component bearing the signal or intelligence is in turn traveling with a velocity

$$v_g = \frac{\Delta\omega}{\Delta\beta}$$

which in the limit becomes

$$v_g = \frac{d\omega}{d\beta} = \frac{1}{d\beta/d\omega} \tag{10-98}$$

Since β may be a function of ω, the latter form has more meaning, both mathematically and physically. The term v_g is called the *group velocity* and may be considered as the velocity with which a signal, produced by variation of a steady-state wave or by introduction of a group of frequencies, travels through the system.

For a freely propagating wave in space, the value of β is $\omega\sqrt{\mu\epsilon}$. Taking the derivative,

$$\frac{d\beta}{d\omega} = \sqrt{\mu\epsilon}$$

so that the group velocity in such a wave is

$$v_g = \frac{1}{d\beta/d\omega} = \frac{1}{\sqrt{\mu\epsilon}}$$

The phase velocity in this case is $v_p = v$, since, for propagation in space without reflection, $\theta_1 = 90°$. Then

$$v_p = \frac{1}{\sqrt{\mu\epsilon}} = v_g$$

and therefore the group velocity is equal to the phase velocity and equal to the velocity of light in the medium for a freely propagating wave.

The concept of group velocity will be considerably amplified in the next chapter in connection with the study of transmission in wave guides, wherein a wave guide will be shown to be a dispersive medium without having dissipation present.

10-11. Elimination of reflections

The transmission-line analogy suggests a method of eliminating reflections from the boundary surface between two dielectrics by use of the quarter-wave matching section. If a dielectric section of quarter-wave thickness is introduced at the boundary as in Fig. 10-14, and if the intrinsic impedance of the dielectric section is properly chosen, reflections of a single frequency may be eliminated.

The analogy fixes the intrinsic impedance of the matching layer as

$$\eta_2 = \sqrt{\eta_1\eta_3}$$

$$\sqrt{\frac{\mu_2}{\epsilon_2}} = \sqrt[4]{\frac{\mu_1\mu_3}{\epsilon_1\epsilon_3}}$$

and if $\mu_1 = \mu_2 = \mu_3$, then

$$\epsilon_2 = \sqrt{\epsilon_1\epsilon_3} \tag{10-99}$$

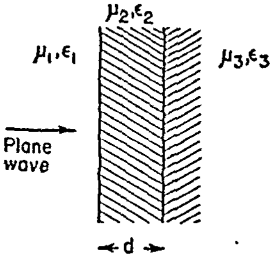


Fig. 10-14. Elimination of reflection at a dielectric boundary by a quarter-wave matching section.

The thickness d of the matching layer must be a quarter wavelength; and since

$$v = f\lambda = \frac{1}{\sqrt{\mu_2\epsilon_2}}$$

then

$$d = \frac{\lambda}{4} = \frac{1}{4f\sqrt{\mu_2\epsilon_2}} \text{ meters} \tag{10-100}$$

is the layer thickness in meters. This is the principle employed in the coating of optical lenses to prevent reflections. Obviously, reflection can be prevented for only one wavelength present in the light, but by choice of this wavelength near the middle of the spectral region to be covered, considerable reduction of reflection is obtained over a satisfactory spectral band.

Occasionally it is desirable to eliminate reflections from conductors. By analogy, a wave equivalent transmission line terminated in a conducting surface is essentially short-circuited by the conductor. Standing waves will be present in the region in front of the conductor. At a point a quarter wavelength in front of the terminating sheet, the line impedance is very high; it would be infinite if the sheet were a perfect conductor. Essentially, the line appears to be open-circuited at that point. If a load with impedance equal to the intrinsic impedance of the region in front of the conductor is connected across the analogous line at that point, as shown in Fig. 10-15, the line will appear to be terminated in its intrinsic impedance and reflections will be eliminated, or the wave will be totally absorbed. Essentially, the load is in parallel with the infinite impedance of the line at the $\lambda/4$ distance in front of the conductor.

In practice, the load is introduced as a semiconducting sheet placed a distance l in front of the reflecting surface. For a perfect terminating reflector, the distance l will be $\lambda/4$ measured to the back of the semiconducting sheet. At this point the impedance of the line looking to the right is infinite. Thus the plane wave approaching from the left in region 1 enters the sheet, which represents an impedance dependent on its thickness d, since the sheet appears to be an open-circuited line. The impedance that the wave enters is therefore

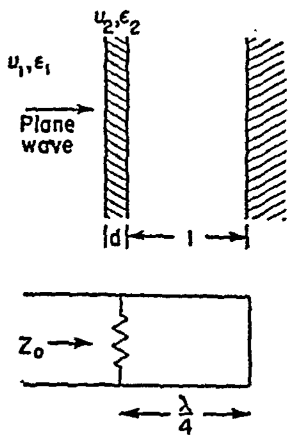


Fig. 10-15. Elimination of reflection from a conductor surface.

$$Z = Z_\infty = \frac{\eta_2}{\tanh \gamma_2 d}$$

For elimination of reflection, Z must equal η_1, so that

$$\eta_1 = \frac{\eta_2}{\tanh \gamma_2 d} \tag{10-101}$$

Assuming d small so that $\tanh \gamma_2 d = \gamma_2 d$ and using the values of γ_2 and η_2 for conducting materials, under the assumptions $\sigma_m \gg \omega\epsilon_m$, then

$$\eta_1 = \frac{(1 + j1)\sqrt{\omega\mu_m/2\sigma_m}}{d(1 + j1)\sqrt{\omega\mu_m\sigma_m/2}}$$

$$\sigma_m = \frac{1}{\eta_1 d} \tag{10-102}$$

Thus the conductivity required for the semiconducting sheet placed in the wave path is dependent on the thickness of the sheet and the intrinsic impedance of the medium from which the wave approaches.

Such a semiconducting sheet may be placed in front of a metal surface and reflections may be eliminated, making a region with metallic boundaries appear as a space region. If appreciable power is present in the incident wave, some means must be provided for removing the heat generated due to absorption of the power.

Lossy materials are also available to serve as absorbers of radio-frequency energy and prevent reflection from surface. These may take the form of a loose fibrous material coated with a solution of carbon black in neoprene. By varying the amount of coating with depth, a gradual transition is made from a space condition to a lossy condition. In effect the wave is attenuated in a lossy tapered transmission line which transforms the wave impedance from that of free space to that of the medium. The thickness of the material should exceed a quarter wavelength of the radiation.

PROBLEMS

10-1. An incident plane electromagnetic wave varying sinusoidally at a frequency of 10^7 c and having an electric field $\mathcal{E}_x = 3.0$ v/m and accompanying H value traveling on the positive-y axis in space strikes an aluminum sheet at normal incidence.

(a) Find the direction and amplitude of the magnetic field in the incident wave.

(b) Calculate the average power density in the incident wave.

(c) Calculate the average power in the reflected wave and in the wave entering the metal. Check your answer by the law of energy conservation.

10-2. A plane electromagnetic wave in space is normally incident on an iron block. The incident wave has a maximum potential gradient of 1.23 v/m and frequency of 60 c. The iron has specific resistance of 10^{-7} ohm-meter, a relative μ of 250, and relative permittivity of unity.

(a) Find the resultant electric and magnetic fields at the shield surface.

(b) Find the depth in the iron at which the magnetic field is reduced to 0.1 per cent of its value at the shield surface.

(c) Find the total power dissipated in the iron plate.

10-3. Repeat Problem 10-2 with a frequency of 10^6 c.

10-4. An incident plane wave of 600 megacycles frequency with a maximum electric intensity of 500 microvolts (μv) per meter impinges normally on salt (ocean) water (see Table 8, page 441).

(a) Determine the reflection factor for the boundary between space and the salt water.

(b) What is the time phase angle between the electric and magnetic fields that penetrate the salt water?

(c) Determine the field intensities (maximum) in the reflected and transmitted waves.

10-5. (a) For the wave of Prob. 10-4 sketch the incident electric field, and the resultant electric field back for a distance of one wavelength from the boundary surface for a value of time $t = 0$.

(b) Repeat (a) for $\omega t = \pi/4$.

(c) Repeat (b) for $\omega t = \pi/2$.

(d) Repeat (c) for $\omega t = 3\pi/4$.

(e) Repeat (d) for $\omega t = \pi$.

(f) Sketch the penetrating electric field for a distance of 0.015 m inside the water boundary surface for each of the ωt values specified above.

10-6. (a) For equivalent shielding effect at 3×10^7 c, compare the thickness of aluminum and silver shields required.

(b) What is the ratio of the resistance of the aluminum sheet to the silver sheet for the current induced at the above frequency?

10-7. A plane electromagnetic wave of unit electric intensity in space impinges at normal incidence on an aluminum sheet of 0.05 cm thickness. The frequency is 10^7 c. Find the percentage strength of the electric vector in the field reaching the far side of the sheet.

10-8. A plane wave in air strikes a sheet of polystyrene at an angle of incidence equal to 30°, with polarization parallel to the y,z plane of incidence. The maximum electric intensity is 50 μv/m at a frequency of 3×10^8 c.

(a) Find the maximum values of the fields in the incident, reflected, and transmitted waves.

(b) Determine the angle of refraction.

(c) Write the expressions for the electric and magnetic field components in the transmitted wave as functions of time and the space coordinates, assuming the polystyrene layer to be very thick.

10-9. (a) Find the critical angle of total internal reflection for glass, polyethylene, and polystyrene to air surfaces.

(b) Compute the index of refraction for each of the above materials.

10-10. A wave in polystyrene approaches a surface at an angle of 26° incidence. Find the phase and group velocities for a wave frequency of 10^8 c.

10-11. (a) If a surface reflects 10 per cent of the incident power in an electromagnetic wave falling normally from space, calculate the value of ϵ_r and N, assuming that the material is a perfect dielectric.

(b) Repeat for surfaces reflecting 20 and 50 per cent of the incident power.

10-12. (a) Calculate the per cent increase in resistance over the d-c value for a No. 4 B & S gauge copper conductor at frequencies of 10^6, 10^9, and 10^{12} c.

(b) Repeat for No. 36 B & S copper wire and explain the difference in relative per cent change.

(c) Repeat for a No. 4 iron wire with $\mu_r = 100$.

10-13. Find the dielectric constant and thickness needed for a surface coating on a glass of $\epsilon_r = 2.25$ to eliminate reflections of a wave arriving from air, if the wavelength is that of yellow light of 5000 Ångstrom units (Å). (1Å = 10^{-8} cm.)

10-14. Explain why placement of the semiconducting film on the surface of a metal will not eliminate reflections.

10-15. For a plane wave of 3000 megacycles, traveling in polyethylene, find the phase velocity, the intrinsic impedance, and the attenuation constant.

10-16. (a) For each of the dielectrics of Table 9, compute the critical angle for waves passing from the dielectric into air.

(b) Find the index of refraction for each dielectric in Table 9.

10-17. Calculate the reflection coefficient and per cent of incident energy reflected when a plane wave is normally incident on a dielectric sheet of thickness $\frac{1}{4}$ in., $\epsilon_r = 2.8$, with air on both sides, at a frequency of 30,000 megacycles.

10-18. A quarter-wave matching coating is applied to eliminate reflections of a 3000-megacycle wave passing from space into a dielectric of $\epsilon_r = 20$. Specify the coating thickness and its ϵ_r value. Plot a curve of per cent incident energy reflected for normal incidence as a function of frequency, for 1000 to 5000 megacycle range.

10-19. A conducting film of 377 ohms impedance per square meter, is placed a quarter wavelength at 6000 megacycles, in front of a perfect conductor. Plot a curve of the ratio of reflected to incident power over a frequency range of 2000 to 10,000 megacycles.

REFERENCES

1. Schelkunoff, S. A., "The Impedance Concept and Its Application to Problems of Reflection, Refraction, Shielding and Power Absorption," *Bell System Tech. J.*, **17**, 17–48 (1938).
2. *Reference Data for Radio Engineers*, 3d ed., Federal Telephone and Radio Corporation, New York, 1949.
3. Bronwell, A. B., and Beam, R. E., *Theory and Application of Microwaves*, McGraw-Hill Book Company, Inc., New York, 1947.
4. Ramo, S., and Whinnery, J. R., *Fields and Waves in Modern Radio*, 2d ed., John Wiley & Sons, Inc., New York, 1953.
5. Howe, G. W. O., "The Application of Telephone Transmission Formulas to Skin-Effect Problems," *J. Inst. Elec. Engrs.* **54**, 473–480 (1916).

Chapter 11

GUIDED WAVES BETWEEN PARALLEL PLANES

Maxwell's equations indicate the most general conditions necessary for propagation of an electromagnetic wave in an arbitrary medium. In their normal form they may apply to conditions in a radio wave in a free region remote from other media, but for much transmission it is necessary to confine and *guide* the wave energy from point to point as desired. In such applications the fields are confined or restricted by boundaries of materials different from those of the transmission path, and the waves are said to be guided by these materials. Hence it is necessary to apply to Maxwell's equations certain mathematical restrictions or *boundary conditions* in order to fit the general equations to the particular physical problem in hand. After insertion of the boundary requirements, it may be possible to obtain a solution showing the form and type of wave transmission that can occur in the confined region.

The ordinary coaxial line is an example of such confining of the fields by the conductor boundaries, with transmission of energy wholly within the outer conductor by reason of the traveling fields that are present. The open-wire line is also a form of wave-guiding system, with the field guided by the two wires. Another example is the tube or *wave guide*, in which the field energy propagates inside a rectangular or cylindrical tube, without a central conductor.

The study of the wave guide is most easily approached by first merely confining the fields between two parallel planes of perfectly conducting material, determining the conditions necessary for propagation and the forms of the fields that may be present. Most of the physical ideas and much of the mathematical formulation developed for such a simple configuration may then be directly applied to the completely enclosed form of wave guide.

It should be noted that the method is that of applying successive restrictions to the general form of Maxwell's equations: first, choosing a particular form of time variation; second, requiring propagation in the z direction; and third, restricting the wave to travel between parallel conducting planes parallel to the y,z plane. In Chapter 10 parallel conducting planes parallel to the x,z plane are added to restrict the wave further, forming the complete tube or wave guide.

For simplicity the planes first will be assumed as perfect conductors and later modified to develop the case of finite conductivity.

11-1. Application of the restrictions to Maxwell's equations

The wave equations were obtained in Chapter 9 as

$$\mu\epsilon \frac{\partial^2 \mathcal{E}}{\partial t^2} = \nabla^2 \mathcal{E}, \qquad \mu\epsilon \frac{\partial^2 H}{\partial t^2} = \nabla^2 H$$

and these show that time variation of the electromagnetic field is necessary for propagation of electric field waves to occur. Because of the wide use of sinusoidal variation with time in electrical theory,

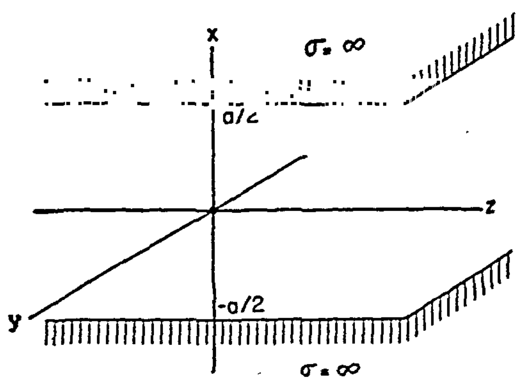


Fig. 11-1. Placement of the conducting planes as wave boundaries.

it will be required here. In a uniform or homogeneous dielectric medium the $\mathcal{E}$ and H fields will also be in time phase, so that, as initial assumed field conditions for a propagating plane wave,

$$\mathcal{E} = \hat{\mathcal{E}} \sin \omega t \tag{11-1}$$

$$H = \hat{H} \sin \omega t \tag{11-2}$$

Variations in time phase may arise later as these waves are introduced into a nonhomogeneous region.

It is desired to study the transmission of this electromagnetic field when it is constricted between two parallel sheets of a perfect conductor, infinite in extent, parallel to the y,z plane, and intersecting the x axis at distance $a/2$ and $-a/2$ from the origin.

It is desirable to orient the axes so that propagation, if there be any, will occur in some desired direction. Thus it is convenient to require the wave energy to propagate in the positive-z direction in Fig. 11-1, since this will correspond to the direction of propagation of some of the field waves discussed in the preceding chapter. This assumption causes no loss of generality in this case, since the axes may be oriented at will, and the conducting planes of the figure were assumed infinite in extent in both y and z directions.

If a propagation constant γ, possibly complex in nature, is assumed to determine the manner of variation of the wave in the z direction, then Eqs. 11-1 and 11-2 become

$$\mathcal{E} = \hat{\mathcal{E}}\epsilon^{-\gamma z} \sin \omega t \tag{11-3}$$

$$H = \hat{H}\epsilon^{-\gamma z} \sin \omega t \tag{11-4}$$

It may be shown that this is just another way of indicating that there is a $(z - vt)$ or $(z + vt)$ function, or that a traveling wave is present.

It should not be immediately assumed that because of the z-directed propagation the electric and magnetic fields will lie wholly in the x,y plane, as was found true in the previous chapter for plane waves in a homogeneous region. The region under discussion is no longer homogeneous, as a result of the introduction of the conducting planes. Therefore it is much safer to permit the solution of the equations to indicate the direction of the fields present.

Now that the fields have been restricted to sinusoidal time variation and to propagation of energy in the z direction, and have been bounded in the x direction by infinite perfectly conducting planes, it is time to insert these conditions in Maxwell's field equations and to determine the field components that may exist in a propagating wave. Writing Maxwell's two field equations in terms of the ordinary values, $\mathcal{E}$ and H of the fields, results in

$$\nabla \times \mathcal{E} = -\mu \frac{\partial H}{\partial t} \tag{11-5}$$

$$\nabla \times H = \sigma\mathcal{E} + \epsilon \frac{\partial \mathcal{E}}{\partial t} \tag{11-6}$$

Because of the infinite extent of the planes in the y direction and to the assumed z-propagation it may be reasoned that there will be no variation of any of the field magnitudes with y. Consequently,

$$\frac{\partial H_z}{\partial y} = \frac{\partial H_x}{\partial y} = \frac{\partial \mathcal{E}_z}{\partial y} = \frac{\partial \mathcal{E}_x}{\partial y} = 0$$

and Eqs. 11-5 and 11-6 become

$$\left.\begin{aligned} -\frac{\partial H_y}{\partial z} &= \sigma \mathcal{E}_x + \epsilon \frac{\partial \mathcal{E}_x}{\partial t} \\ \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} &= \sigma \mathcal{E}_y + \epsilon \frac{\partial \mathcal{E}_y}{\partial t} \\ \frac{\partial H_y}{\partial x} &= \sigma \mathcal{E}_z + \epsilon \frac{\partial \mathcal{E}_z}{\partial t} \end{aligned}\right\} \tag{11-7}$$

$$\left.\begin{aligned} -\frac{\partial \mathcal{E}_y}{\partial z} &= -\mu \frac{\partial H_x}{\partial t} \\ \frac{\partial \mathcal{E}_x}{\partial z} - \frac{\partial \mathcal{E}_z}{\partial x} &= -\mu \frac{\partial H_y}{\partial t} \\ \frac{\partial \mathcal{E}_y}{\partial x} &= -\mu \frac{\partial H_z}{\partial t} \end{aligned}\right\} \tag{11-8}$$

If the appropriate derivatives of Eqs. 11-3 and 11-4 are taken and with $\hat{\mathcal{E}}$ and $\hat{H}$ as maximum time values, there results

$$\left.\begin{aligned} \gamma \hat{H}_y &= (\sigma + j\omega\epsilon)\hat{\mathcal{E}}_x \\ -\gamma \hat{H}_x - \frac{\partial \hat{H}_z}{\partial x} &= (\sigma + j\omega\epsilon)\hat{\mathcal{E}}_y \\ \frac{\partial \hat{H}_y}{\partial x} &= (\sigma + j\omega\epsilon)\hat{\mathcal{E}}_z \end{aligned}\right\} \tag{11-9}$$

$$\left.\begin{aligned} \gamma \hat{\mathcal{E}}_y &= -j\omega\mu \hat{H}_x \\ -\gamma \hat{\mathcal{E}}_x - \frac{\partial \hat{\mathcal{E}}_z}{\partial x} &= -j\omega\mu \hat{H}_y \\ \frac{\partial \hat{\mathcal{E}}_y}{\partial x} &= -j\omega\mu \hat{H}_z \end{aligned}\right\} \tag{11-10}$$

These equations are the result of introduction of restrictions on time, and on the form and direction of propagation, as limited by the perfectly conducting infinite planes. It remains to find the form of the fields and the types of propagation possible under the restrictions

11-2. Types of propagation; TM, TE, and TEM waves

Equations 11-9 and 11-10 indicate that, in general, electric and magnetic field components may exist along all axes. The equations may be further simplified, without loss of generality, by orientation of the fields or by appropriate excitation from the wave source, such that either the magnetic field lies wholly along the y axis or the electric field lies wholly along the y axis.

In the first case, $\hat{H}_x = \hat{H}_z = 0$, and the magnetic field is seen to be wholly *transverse* to the direction of propagation on z in Fig. 11-2(a); hence the electric field has components $\hat{\varepsilon}_x$ and $\hat{\varepsilon}_z$, with

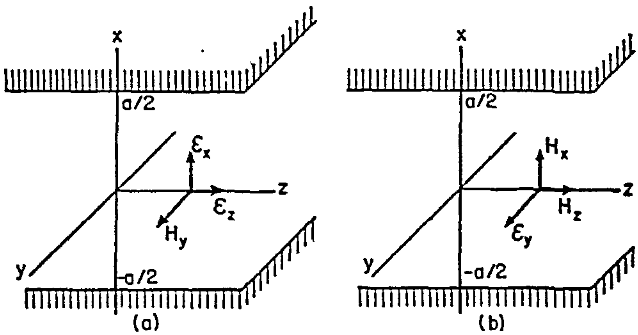


Fig. 11-2. (a) Fields in the transverse-magnetic (TM) wave; (b) fields in the transverse-electric (TE) wave.

$\hat{\varepsilon}_y = 0$, since no electric field can exist in the direction of the magnetic field. Such a wave is said to be *transverse magnetic* and is known as a TM wave. Equations 11-9 and 11-10 then reduce to

$$\left.\begin{aligned} \gamma\hat{H}_y &= (\sigma + j\omega\epsilon)\hat{\varepsilon}_x && \text{(a)} \\ \frac{\partial\hat{H}_y}{\partial x} &= (\sigma + j\omega\epsilon)\hat{\varepsilon}_z && \text{(b)} \\ \gamma\hat{\varepsilon}_x + \frac{\partial\hat{\varepsilon}_z}{\partial_x} &= j\omega\mu\hat{H}_y && \text{(c)} \end{aligned}\right\}\text{TM waves} \qquad (11\text{-}11)$$

In the second case, illustrated by Fig. 11-2(b), $\hat{\varepsilon}_x = \hat{\varepsilon}_z = 0$, and the electric field is made wholly transverse to the direction of propagation on z. The magnetic field has components $\hat{H}_x$ and $\hat{H}_z$, and $H_y = 0$. The wave so oriented is said to be *transverse electric* and is known as a TE wave. For the TE wave, Eqs. 11-9 and 11-10 reduce to

$$\left.\begin{aligned} \gamma\hat{H}_x + \frac{\partial \hat{H}_z}{\partial x} &= -(\sigma + j\omega\epsilon)\hat{\mathcal{E}}_y && \text{(a)} \\ \gamma\hat{\mathcal{E}}_y &= -j\omega\mu\hat{H}_x && \text{(b)} \\ \frac{\partial \hat{\mathcal{E}}_y}{\partial x} &= -j\omega\mu\hat{H}_z && \text{(c)} \end{aligned}\right\} \text{TE waves} \qquad (11\text{-}12)$$

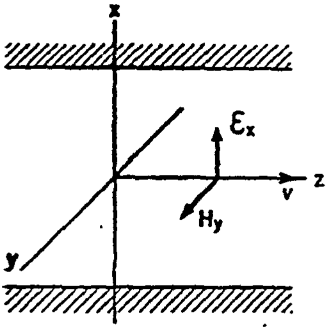


Fig. 11-3. The transverse-electromagnetic (TEM) field vectors.

A third case may exist in which the magnetic field is wholly along y while the electric field is wholly along x, no fields existing in the z direction. In this case, which represents a special form of TM wave with $\mathcal{E}_z = 0$, both the electric and magnetic field components are transverse to the direction of propagation on z, and the wave is said to be *transverse electromagnetic* or of the TEM type.

Equations 11-9 and 11-10 reduce, for the TEM condition, to

$$\left.\begin{aligned} \gamma\hat{H}_y &= (\sigma + j\omega\epsilon)\hat{\mathcal{E}}_x && \text{(a)} \\ \frac{\partial \hat{H}_y}{\partial x} &= 0 && \text{(b)} \\ \gamma\hat{\mathcal{E}}_x &= j\omega\mu\hat{H}_y && \text{(c)} \end{aligned}\right\} \text{TEM waves} \qquad (11\text{-}13)$$

11-3. Transmission of TM waves between parallel planes

The three differential equations describing TM waves (Eq. 11-11) may be readily solved for expressions for the three field components present. Differentiation of Eq. 11-11(b) and substitution of the result and Eq. 11-11(a) into Eq. 11-11(c) yields an expression containing H_y only:

$$\frac{\partial^2 \hat{H}_y}{\partial x^2} = [-\gamma^2 + (\sigma + j\omega\epsilon)j\omega\mu]\,\hat{H}_y$$

If the space between the parallel planes is considered a good dielectric of properties μ_1 and ϵ_1, and with σ_1 very small with respect to $\omega\epsilon_1$,

$$\frac{\partial^2 \hat{H}_y}{\partial x^2} = -(\gamma^2 + \omega^2\mu_1\epsilon_1)\hat{H}_y \qquad (11\text{-}14)$$

This equation may be readily solved; and the solution may be written

$$\hat{H}_y = A_1\epsilon^{j\sqrt{\gamma^2+\omega^2\mu_1\epsilon_1}\,x} + A_2\epsilon^{-j\sqrt{\gamma^2+\omega^2\mu_1\epsilon_1}\,x}$$

where A_1 and A_2 are arbitrary constants whose values are determined by the source of field excitation.

The two exponential functions of x indicate the presence of an incident and reflected wave system for H_y, the two waves propagating in the plus and minus x directions. This result is not surprising, because of the bounding of the field in the x directions by the perfectly conducting planes. The above equation can be put in a form in which the field may be more easily visualized by use of the expressions $\epsilon^{j\theta} = \cos\theta + j\sin\theta$ and $\epsilon^{-j\theta} = \cos\theta - j\sin\theta$. Then

$$\hat{H}_y = B_1 \sin\left(\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1}\right)x + B_2 \cos\left(\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1}\right)x \qquad (11\text{-}15)$$

By insertion of this result in Eq. 11-11(a), the value of $\hat{\mathcal{E}}_x$ can be obtained as

$$\hat{\mathcal{E}}_x = \frac{-j\gamma}{\omega\epsilon_1}\left[B_1 \sin\left(\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1}\right)x + B_2 \cos\left(\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1}\right)x\right] \qquad (11\text{-}16)$$

The $\hat{\mathcal{E}}_z$ component of field may be obtained from Eq. 11-11(b) as

$$\hat{\mathcal{E}}_z = \frac{-j\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1}}{\omega\epsilon_1}\left[B_1 \cos\left(\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1}\right)x - B_2 \sin\left(\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1}\right)x\right] \qquad (11\text{-}17)$$

For each of these fields it is obvious that the value along the x axis is the resultant of two fields traveling in opposite directions, and that because of the lack of attenuation in this direction, a true standing-wave pattern of field distribution in the x direction will result.

One further bit of boundary information may now be employed to gain further knowledge of the arbitrary constants B_1 and B_2. Because of the perfect conductivity of the planes, it is known that at $x = a/2$ and $x = -a/2$ the tangential component of electric field $\mathcal{E}_z$ must be zero. Examination of Eq. 11-17 shows that this value may occur either if $B_1 = B_2 = 0$ (which is of no importance since the field would be zero everywhere), or if the angles have values of $m\pi/2$ at $a/2$ and $-a/2$. Accordingly it can be said that the conditions at the boundary planes require that

$$\left(\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1}\right)\frac{a}{2} = \frac{m\pi}{2} \qquad (m = 0, 1, 2, 3, \ldots)$$

from which

$$\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1} = \frac{m\pi}{a} \tag{11-18}$$

and the expression for $\hat{\mathcal{E}}_z$ may be written

$$\hat{\mathcal{E}}_z = \frac{-jm\pi}{a\omega\epsilon_1}\left[B_1 \cos\left(\frac{m\pi x}{a}\right) - B_2 \sin\left(\frac{m\pi x}{a}\right)\right] \tag{11-19}$$

This expression may be studied further. When the field is examined at the planes or at $x = a/2$ or $x = -a/2$, three possible conditions leading to zero field intensity are discovered. These are:

If $m = 0$, then $\mathcal{E}_z = 0$ everywhere.
If m is even, then $B_1 = 0$.
If m is odd, then $B_2 = 0$.

Thus the configuration of the fields between the planes becomes dependent on the value of the integer m. To designate the particular type of wave under discussion, it is customary to refer to waves having $m = 0$, $m = 1$, $m = 2$, . . . , as TM_0, TM_1, TM_2, . . . , waves. These various configurations of the field intensities in the guide are called *modes of propagation*. The field distributions are sinusoidal or cosinusoidal in the x direction, the number of maxima depending on the value of m.

As previously required, the fields are to propagate in the positive z direction according to the relation $\epsilon^{-\gamma z}$. The constant γ is complex in general, and it has been customary to express it in terms of the attenuation constant α and phase constant β as

$$\gamma = \alpha + j\beta$$

The value of γ for the particular field under consideration may be found from Eq. 11-18 as

$$\gamma_m = \sqrt{\left(\frac{m\pi}{a}\right)^2 - \omega^2\mu_1\epsilon_1} \tag{11-20}$$

Since all the values under the radical are real, the value of γ must be wholly real or wholly imaginary, depending on the relative magnitudes of $m\pi/a$ or $\omega^2\mu_1\epsilon_1$. Thus two conditions arise, with α equal to zero or with β equal to zero. It is possible to have propagation with zero attenuation or high attenuation with β equal to zero. The angular velocity ω or the frequency chosen is the factor determining whether or not propagation takes place for a given spacing a of the planes. The critical frequency f_c at which the attenuation condition changes to the propagation condition can be determined from

$$\sqrt{\left(\frac{m\pi}{a}\right)^2 - \omega_c^2\mu_1\epsilon_1} = 0$$

$$f_c = \frac{m}{2a}\sqrt{\frac{1}{\mu_1\epsilon_1}} \tag{11-21}$$

A value for m/a in terms of f_c permits Eq. 11-20 for γ_m to be written as

$$\gamma_m = \frac{m\pi}{a}\sqrt{1 - \frac{f^2}{f_c^2}} \tag{11-22}$$

$$= \omega_c\sqrt{\mu_1\epsilon_1}\sqrt{1 - \frac{f^2}{f_c^2}} \tag{11-23}$$

For frequencies *below cutoff* where $f < f_c$ and γ_m is real, the fields are attenuated and $\gamma_m = \alpha_m$, $\beta_m = 0$. The phase angles will be constant, and the field amplitudes will decrease very rapidly with distance z, because of the attenuation term $\epsilon^{-\gamma_m z} = \epsilon^{-\alpha_m z}$.

At frequencies *above cutoff*, $f > f_c$, γ_m is imaginary, and $\alpha_m = 0$. Thus propagation will occur and

$$\gamma_m = j\beta_m = j\frac{m\pi}{a}\sqrt{\frac{f^2}{f_c^2} - 1} \tag{11-24}$$

$$= j\omega_c\sqrt{\mu_1\epsilon_1}\sqrt{\frac{f^2}{f_c^2} - 1} \tag{11-25}$$

A condition of propagation without attenuation exists in the z direction; this condition is normally the one of interest. In the practical case, owing to the finite conductivity of the plates, some attenuation accompanies propagation at frequencies above cutoff.

The action of the planes in only propagating frequencies above a certain cutoff value bears a definite resemblance to the operation of the high-pass filter. This similarity can be further borne out by study of relations for cutoff frequency, Z_0, attenuation in the stop band, as functions of f/f_c.

The critical frequency of cutoff may be given some physical significance by writing the expression for f_c as

$$f_c = \frac{v_1}{2a/m}$$

from which the critical wavelength can be seen as

$$\lambda_c = \frac{2a}{m} \tag{11-26}$$

The critical wavelength is related to the plate spacing a as

$$\frac{m\lambda_c}{2} = a$$

so that the critical wavelength is fixed as that at which the distance between the planes becomes exactly m half waves. For waves longer than this value (lower frequencies), the planes serve to attenuate the fields. Shorter wavelengths are propagated without loss. The integer m can be interpreted as the number of field maximums occurring in the x direction between the planes.

The expressions developed above now permit the three field components to be written by inclusion of the time and z-propagation function $\epsilon^{-\gamma z} \sin \omega t$ as

$$H_y = \left[B_1 \sin\left(\frac{m\pi x}{a}\right) + B_2 \cos\left(\frac{m\pi x}{a}\right)\right] \epsilon^{-j\beta_m z} \sin \omega t \tag{11-27}$$

$$\mathcal{E}_x = \frac{\beta_m}{\omega\epsilon_1}\left[B_1 \sin\left(\frac{m\pi x}{a}\right) + B_2 \cos\left(\frac{m\pi x}{a}\right)\right] \epsilon^{-j\beta_m z} \sin \omega t \tag{11-28}$$

$$\mathcal{E}_z = \frac{-m\pi}{a\omega\epsilon_1}\left[B_1 \cos\left(\frac{m\pi x}{a}\right) - B_2 \sin\left(\frac{m\pi x}{a}\right)\right] \epsilon^{-j\beta_m z} \cos \omega t \tag{11-29}$$

for the TM_m waves under the conditions of propagation. The form is obtained by use of the value for $\gamma_m = j\beta_m$ in the propagation region and consideration of the meaning of the j coefficient appearing before the magnitude function in some cases.

11-4. Transmission of TE waves between parallel planes

The TE wave equations given as Eq. 11-12 may be combined to yield

$$\frac{\partial^2 \hat{\mathcal{E}}_y}{\partial x^2} = -(\gamma^2 + \omega^2\mu_1\epsilon_1)\hat{\mathcal{E}}_y \tag{11-30}$$

for the case of a good dielectric between the planes. This equation is of the same form as Eq. 11-14 for $\hat{H}_y$ in the TM wave case, and a similar solution follows directly, giving

$$\hat{\mathcal{E}}_y = A_3\epsilon^{j\sqrt{\gamma^2+\omega^2\mu_1\epsilon_1}\,x} + A_4\epsilon^{-j\sqrt{\gamma^2+\omega^2\mu_1\epsilon_1}\,x}$$

As for the TM case, the two exponential x functions indicate two $\mathcal{E}_y$ field wave components traveling oppositely in the x direction, as a result of reflections from the parallel conducting planes.

The above equation may be written in terms of sines and cosines as

$$\hat{\mathcal{E}}_y = B_3 \sin\left(\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1}\right)x + B_4 \cos\left(\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1}\right)x \tag{11-31}$$

indicating a standing-wave form of distribution in the x direction between the planes.

Use of Eq. 11-12(b) gives for $\hat{H}_x$,

$$\hat{H}_x = \frac{j\gamma}{\omega\mu_1}\left[B_3 \sin\left(\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1}\right)x + B_4 \cos\left(\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1}\right)x\right] \tag{11-32}$$

and, after performing the indicated differentiation, Eq. 11-12(c) gives

$$\hat{H}_z = \frac{j\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1}}{\omega\mu_1}\left[B_3 \cos\left(\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1}\right)x - B_4 \sin\left(\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1}\right)x\right] \tag{11-33}$$

The three field intensities for the TE waves between parallel planes are thus determined.

The perfect conductivity of the bounding parallel planes requires that the tangential electric field, $\mathcal{E}_y$, be zero at $x = a/2$ and $x = -a/2$. This condition may be satisfied by proper choice of the angle involved as

$$\left(\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1}\right)\frac{a}{2} = \frac{m\pi}{2} \qquad (m = 0, 1, 2, 3, \ldots) \tag{11-34}$$

and by assignment of either B_3 or B_4 as zero.

The propagation constant γ_m for TE transmission may be determined from Eq. 11-34 as

$$\gamma_m = \sqrt{\left(\frac{m\pi}{a}\right)^2 - \omega^2\mu_1\epsilon_1} \tag{11-35}$$

Since this result is identical with that for TM waves, the conditions of propagation are the same and the parallel planes become a high-pass filter with cutoff frequency

$$f_c = \frac{m}{2a}\sqrt{\frac{1}{\mu_1\epsilon_1}} \tag{11-36}$$

and with constants in the pass-band of frequencies as

$$\gamma_m = j\beta_m = j\,\frac{m\pi}{a}\sqrt{\frac{f^2}{f_c^2} - 1} \tag{11-37}$$

$$\alpha_m = 0$$

Use of the relations developed above and the time and propagation function permits simplification of the field equations for TE_m waves between parallel planes to

$$\mathcal{E}_y = \left[B_3 \sin\left(\frac{m\pi x}{a}\right) + B_4 \cos\left(\frac{m\pi x}{a}\right)\right]\epsilon^{-j\beta_m z}\sin\omega t \tag{11-38}$$

$$H_x = \frac{-\beta_m}{\omega\mu_1}\left[B_3 \sin\left(\frac{m\pi x}{a}\right) + B_4 \cos\left(\frac{m\pi x}{a}\right)\right]\epsilon^{-j\beta_m z}\sin\omega t \tag{11-39}$$

$$H_z = \frac{m\pi}{a\omega\mu_1}\left[B_3 \cos\left(\frac{m\pi x}{a}\right) - B_4 \sin\left(\frac{m\pi x}{a}\right)\right]\epsilon^{-j\beta_m z}\cos\omega t \tag{11-40}$$

Further consideration of the boundary conditions imposed on $\mathcal{E}_y$ indicates that three possible conditions may arise, dependent on the choice of m:

If $m = 0$, then $B_4 = 0$ and all fields are zero.
If m is even, then $B_4 = 0$.
If m is odd, then $B_3 = 0$.

The configuration of the fields between the planes is seen as dependent on the value chosen for m; and for values of $m = 1$, $m = 2$, . . . , the various field modes designated as TE_1, TE_2, . . . , arise.

11-5. Transmission of TEM waves between parallel planes

The TEM form of field was arrived at in Section 11-2 as a special case of TM propagation in which $\mathcal{E}_z$ was zero. In Section 11-3 it was found that this condition on $\mathcal{E}_z$ was obtained if m were made zero. Accordingly, the TEM wave becomes a TM wave with $m = 0$, and the field equations may be written from Eqs. 11-27, 11-28, and 11-29 as

$$H_y = B_2\epsilon^{-j\beta_0 z} \sin \omega t \tag{11-41}$$

$$\mathcal{E}_x = \frac{\beta_0}{\omega\epsilon_1} B_2\epsilon^{-j\beta_0 z} \sin \omega t \tag{11-42}$$

$$\mathcal{E}_z = 0$$

for conditions of TEM propagation. The same results could have been obtained from the differential equations for TEM waves, Eq. 11-13, under the good-dielectric assumption. One additional piece of information may be gained from Eq. 11-13(b), which states

$$\frac{\partial \hat{H}_y}{\partial x} = 0$$

This equation states that $\hat{H}_y$ is a constant for all values of x or for all positions between the planes, for a particular z value.

From Eq. 11-13 it is also possible to write that for the good-dielectric condition,

$$\frac{\gamma^2}{j\omega\mu_1} \hat{\mathcal{E}}_x = j\omega\epsilon_1\hat{\mathcal{E}}_x$$

from which
$$\gamma^2 + \omega^2\mu_1\epsilon_1 = 0 \tag{11-43}$$

which is Eq. 11-18, previously obtained for TM waves, with $m = 0$ With the above equation, the propagation constant γ for TEM waves is

$$\gamma = j\beta = j\omega \sqrt{\mu_1\epsilon_1} \tag{11-44}$$

and
$$\alpha = 0$$

Since $m = 0$, the cutoff frequency given for TM waves becomes zero, so that TEM waves propagate without attenuation between the perfectly conducting plates for all frequencies above that of zero.

Use of Eq. 11-44 permits the expression for $\mathcal{E}_x$ to be reduced to

$$\mathcal{E}_x = \sqrt{\frac{\mu_1}{\epsilon_1}}\, B_2\epsilon^{-j\beta z} \sin \omega t$$

$$= \eta_1 B_2 \epsilon^{-j\beta z} \sin \omega t \tag{11-45}$$

Comparison with Eq. 11-41 for H_y,

$$H_y = B_2\epsilon^{-j\beta z} \sin \omega t$$

shows $\mathcal{E}$ and H to be related by the factor η_1, the intrinsic impedance of the dielectric, which is a condition analogous to that occurring in wave propagation in free space. The value of β also corresponds to free-space transmission. Since there is no component of electric field tangential to the conducting planes, it can be seen that these planes have no effect except to limit the area of the wave.

Because of the relation of TEM waves to the previously studied space waves and in turn the analogy of space waves to transmission-line waves, it would seem possible to deduce an analogy between TEM waves and transmission-line waves. In fact, it is found that TEM waves are actually those which would appear in the fields present along a dissipationless transmission line.

Equations 11-41 and 11-45 show that for the TEM wave,

$$\mathcal{E}_x = \eta_1 H_y$$

The power propagating in the z direction between the planes, per unit width in the y direction, is given by the *Poynting radiation theorem from field concepts* as

$$P = \tfrac{1}{2}\eta_1 \hat{H}_y^2 \times a \times 1 \text{ watts/meter of width} \tag{11-46}$$

the area of the path being $a \times 1$ meter. The Poynting vector direction confirms the positive-z direction of this power flow. The area under consideration is outlined in Fig. 11-4. The maximum value of potential *drop* between the bottom and top plates is given by the integral of the electric field as

$$\hat{V} = \int \hat{\mathcal{E}} \cos\theta\, dl = \int_{-a/2}^{a/2} \eta_1 B_2\, dx$$

$$\hat{V} = \eta_1 B_2 a \text{ volts} \tag{11-47}$$

The current flowing in either top or bottom plate per unit y width may be found by consideration of the path $ABCD$ in Fig. 11-4. Because of the perfect conductivity, all current will flow on the plate surface and a field intensity H_y will exist between A and B in the dielectric. Since H_y does not vary with x, the exact location of the path AB is immaterial. Since the magnetic field will not penetrate into the perfect conductor, no magnetic-field intensity will exist

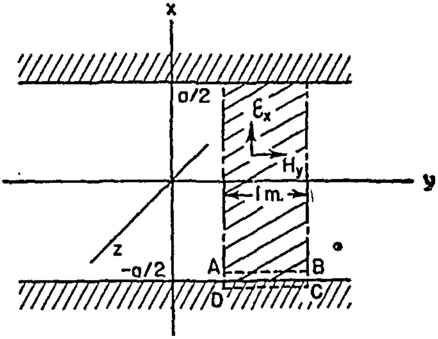


Fig. 11-4. The cross section considered for energy flow in the TEM wave.

along the path DC in the perfect conductor. Hence the current flowing per meter of width in the y direction is the current enclosed by the path $ABCD$. Consequently,

$$I = \oint H \cdot dl = \int_A^B H_y \, dy + \int_B^C 0 \, dx + \int_C^D 0 \, dy + \int_D^A 0 \, dx$$

there being no normal H component. The maximum value of current per unit width of plate then is

$$\hat{I} = B_2$$

Since the planes are lossless, the power being transmitted along the planes and supplied to the load is

$$P = \tfrac{1}{2}\hat{V}\hat{I} = \tfrac{1}{2}\eta_1 B_2{}^2 a \qquad (11\text{-}48)$$

per unit width in the y direction. Since the maximum value of H_y is B_2 the power delivered to the load, *in terms of current and voltage concepts*, is

$$P = \tfrac{1}{2}\eta_1 H_y{}^2 a \text{ watts/meter of width} \qquad (11\text{-}49)$$

and this is identical with the power shown to be conveyed by the electromagnetic fields by use of Poynting's vector, given by Eq. 11-46. It may then be concluded that the energy transmitted along a lossless conductive path is conveyed by the electric and magnetic fields in the region, the voltage and current being present merely as evidence of existence of the fields.

11-6. Manner of wave travel

Propagation of TM and TE waves has been shown to be similar, except for polarization of the fields. The presence of the z component of field introduces an apparent inconsistency, since propagation has been assumed to occur in the z direction; yet Fig. 11-2 would

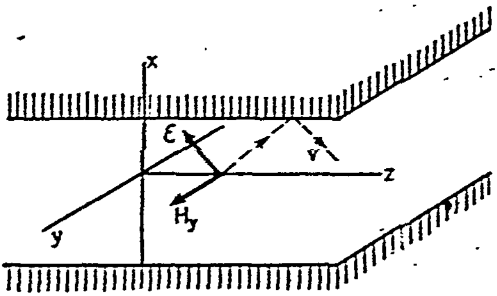


Fig. 11-5. TM field components.

seem to indicate propagation at an angle, because of the presence of the z-directed field component. This point can be readily clarified.

The electric field components of an arbitrarily chosen TM wave are obtainable from Eqs. 11-28 and 11-29 as

$$\mathcal{E}_z = \frac{\beta_m}{\omega\epsilon_1}\left[B_1 \sin\left(\frac{m\pi x}{a}\right) + B_2 \cos\left(\frac{m\pi x}{a}\right)\right]\epsilon^{-j\beta_m z} \sin \omega t$$

$$\mathcal{E}_x = \frac{-m\pi}{a\omega\epsilon_1}\left[B_1 \cos\left(\frac{m\pi x}{a}\right) - B_2 \sin\left(\frac{m\pi x}{a}\right)\right]\epsilon^{-j\beta_m z} \cos \omega t$$

and the wave is illustrated in Fig. 11-5.

For simplicity, an m = odd mode of operation may be selected, making $B_2 = 0$. The electric field components then become

$$\mathcal{E}_z = \frac{\beta_m}{\omega\epsilon_1} B_1\epsilon^{-j\beta_m z} \sin \frac{m\pi x}{a} \sin \omega t \tag{11-50}$$

$$\mathcal{E}_x = \frac{-m\pi}{a\omega\epsilon_1} B_1\epsilon^{-j\beta_m z} \cos \frac{m\pi x}{a} \cos \omega t \tag{11-51}$$

By use of the trigonometric identities,

$$\sin a \sin b = \tfrac{1}{2}[\cos (a - b) - \cos (a + b)]$$
$$\cos a \cos b = \tfrac{1}{2}[\cos (a - b) + \cos (a + b)]$$

the electric fields may be converted to

$$\mathcal{E}_x = \frac{\beta_m}{2\omega\epsilon_1} B_1 \epsilon^{-j\beta_m z} \left[\cos\left(\omega t - \frac{m\pi x}{a}\right) - \cos\left(\omega t + \frac{m\pi x}{a}\right)\right] \quad (11\text{-}52)$$

$$\mathcal{E}_z = \frac{-m\pi}{2a\omega\epsilon_1} B_1 \epsilon^{-j\beta_m z} \left[\cos\left(\omega t - \frac{m\pi x}{a}\right) + \cos\left(\omega t + \frac{m\pi x}{a}\right)\right] \quad (11\text{-}53)$$

If the directions of the field components are indicated by unit vectors, $\bar{a}_x$ for the x direction and $\bar{a}_z$ for the z direction, then the total electric field intensity is given by the vector sum of $\mathcal{E}_x$ and $\mathcal{E}_z$ as

$$\boldsymbol{\mathcal{E}} = \bar{a}_x \mathcal{E}_x + \bar{a}_z \mathcal{E}_z$$

Then

$$\boldsymbol{\mathcal{E}} = \frac{B_1}{2\omega\epsilon_1} \epsilon^{-j\beta_m z} \left\{\left[\bar{a}_x\beta_m - \bar{a}_z \frac{m\pi}{a}\right] \cos\left(\omega t - \frac{m\pi x}{a}\right) - \left[\bar{a}_x\beta_m + \bar{a}_z \frac{m\pi}{a}\right] \cos\left(\omega t + \frac{m\pi x}{a}\right)\right\} \quad (11\text{-}54)$$

The H_y component of the field may be written from Eq. 11-27 for the TM_m (m = odd) mode as

$$H_y = B_1 \epsilon^{-j\beta_m z} \sin \frac{m\pi x}{a} \sin \omega t$$

which may be converted to

$$H_y = \frac{B_1}{2} \epsilon^{-j\beta_m z} \left[\cos\left(\omega t - \frac{m\pi x}{a}\right) - \cos\left(\omega t + \frac{m\pi x}{a}\right)\right] \quad (11\text{-}55)$$

Equations 11-54 and 11-55 show that propagation is the resultant of two waves at an angle in the region between the plates as represented in Fig. 11-6, one wave at a particular z value having an upward-directed x component of velocity, $\cos (\omega t - m\pi x/a)$ and the other at the same point having a downward-directed x component of velocity, $\cos (\omega t + m\pi x/a)$.

When the electric field having a magnitude $(\bar{a}_x\beta_m - \bar{a}_z m\pi/a)$ and upward-directed velocity is combined with its companion H_y term, Poynting's vector indicates a power-density flow up to the right, as in Fig. 11-6(b). Consideration of the electric field component having a magnitude $-(\bar{a}_x\beta_m + \bar{a}_z m\pi/a)$ and downward-directed x velocity

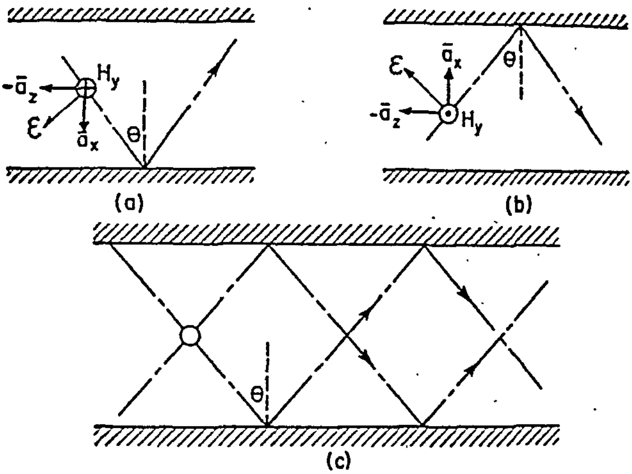


Fig. 11-6. (a) Component wave with velocity component in downward x direction; (b) component wave with velocity component in upward x direction; (c) path of both waves between the parallel planes.

component combined with the appropriate H_y field (which is negative) gives a Poynting vector indicating a power flow down to the right as in Fig. 11-6(a). These fields represent simultaneous propagation of crossed waves between the plane guides, the wave paths making an angle of $\pi - 2\theta$ with each other as in (c). Both waves are propagated equally in the z direction according to $\epsilon^{j\beta_m z}$.

From the figures it is apparent that the angle of incidence θ on the guiding planes is related to the magnitude of the field vectors in the $\bar{a}_x$ and $\bar{a}_z$ directions as

$$\tan\theta = \frac{\beta_m}{m\pi/a} = \frac{a\beta_m}{m\pi}$$

$$= \sqrt{\frac{f^2}{f_c^2} - 1} \tag{11-56}$$

It may be shown that this result, though derived for TM modes with m = *odd, is perfectly general for any TM or TE modes.* As the frequency is reduced and nears the critical frequency, $\tan\theta$ and θ approach zero. At $\theta = 0°$ there is no transmission in the z direction, since $\beta_m = 0$, and the waves simply bounce back and forth between the upper and lower planes. For frequencies much above f_c the angle of incidence θ becomes large and the waves propagate between the guiding planes by a succession of glancing reflections.

11-7. Velocities of the waves

The velocity of propagation of the component waves *along* their respective paths in Fig. 11-6 is determined by the dielectric material between the plates and is

$$v_1 = \frac{1}{\sqrt{\mu_1\epsilon_1}} \text{ m/sec}$$

In Eq. 11-56 the angle of incidence of the bouncing waves on the planes was found to be specified by

$$\tan\theta = \sqrt{\frac{f^2}{f_c^2} - 1}$$

The other trigonometric functions of θ may be calculated as

$$\sin\theta = \frac{\sqrt{f^2/f_c^2 - 1}}{f/f_c} = \sqrt{1 - \frac{f_c^2}{f^2}} \tag{11-57}$$

$$\cos\theta = \frac{1}{f/f_c} = \frac{f_c}{f} \tag{11-58}$$

In Eq. 10-100 the phase velocity of a wave was found to be

$$v_p = \frac{v_1}{\sin\theta} \tag{11-59}$$

By use of the value of $\sin\theta$ from Eq. 11-57, this may be written

$$v_p = \frac{v_1}{\sqrt{1 - f_c^2/f^2}} \tag{11-60}$$

Equation 11-60 indicates that the phase velocity is above v_1, the velocity of light along the wave path, and that at cutoff the phase velocity becomes infinite. Since the phase velocity measures the rate of change of phase along the surface of a conducting plane and the wave at cutoff is bouncing back and forth between the planes with $\theta = 0$ deg, the phase angle changes simultaneously at all points along one of the conducting planes and an infinite phase velocity is indicated.

The phase constant β is given by

$$\beta = \frac{m\pi}{a}\sqrt{\frac{f^2}{f_c^2} - 1}$$

The derivative $d\beta/d\omega$ may be taken as

$$\frac{d\beta}{d\omega} = \frac{m\pi}{a}\frac{\omega/\omega_c^2}{\sqrt{\omega^2/\omega_c^2 - 1}} = \frac{m}{2af_c\sqrt{1 - f_c^2/f^2}} \tag{11-61}$$

By taking the reciprocal the group velocity is obtained as

$$v_g = \frac{1}{d\beta/d\omega} = \frac{2af_c}{m}\sqrt{1 - \frac{f_c^2}{f^2}} \tag{11-62}$$

Using the expression for f_c,

$$f_c = \frac{mv_1}{2a}$$

the group velocity becomes

$$v_g = v_1\sqrt{1 - \frac{f_c^2}{f^2}} \tag{11-63}$$

$$= v_1 \sin\theta \tag{11-64}$$

By reference to Fig. 11-7, the expression $v_1 \sin\theta$ can be seen as simply the component of v_1 in the z direction, and thus the group velocity is merely the *average rate of travel* of energy in the z direction for the wave guided between the conducting planes.

The wave-front configuration indicated in Fig. 11-7 for one of the crossed waves can be supported through use of the expression for $\cos\theta$

$$\cos\theta = \frac{f_c}{f} = \frac{m}{2af\sqrt{\mu_1\epsilon_1}}$$

which, for $m = 1$, becomes

$$\cos\theta = \frac{\lambda/2}{a} \tag{11-65}$$

a statement for which Fig. 11-7 is drawn.

It should be noted that the dielectric between the planes is a dispersive medium as indicated by the fact that β is a function of frequency. The region is lossless by assumption, however, so that the implications of Section 10-10 concerning dispersive and nondispersive media are supported. In general, it may be stated that a

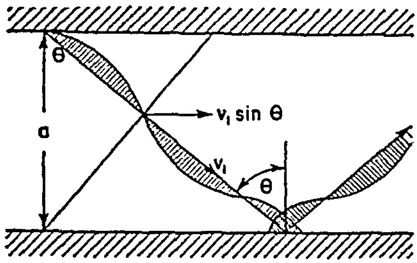


Fig. 11-7. Manner of travel of one wave component between parallel planes.

dissipationless medium will appear dispersive when the phase and group velocities are measured in directions different from the direction of travel of the wave.

From Eqs. 11-59 and 11-64 for v_p and v_g, it is possible to write

$$v_1 = \sqrt{v_p v_g} \tag{11-66}$$

showing that v_1 is always the geometric mean of v_p and v_g.

11-8. Characteristic impedance of the planes

The *characteristic wave impedance* or intrinsic impedance of a traveling electromagnetic field has been defined as a result of the transmission-line analogy. In Eq. 10-18 it was found that this Z_0 or η was related to γ of the medium as

$$Z_0 = \eta = \frac{j\omega\mu_1}{\gamma} = \frac{\gamma}{\sigma + j\omega\epsilon_1} \tag{11-67}$$

This result was found to be the ratio between $\hat{\mathcal{E}}$ and $\hat{H}$ in the medium. Reference to Eqs. 11-15 and 11-16 for $\mathcal{E}_x$ and H_y, the transverse field components of the TM wave, gives for the ratio of their maximum values

$$\frac{\hat{\mathcal{E}}_x}{\hat{H}_y} = \frac{\gamma}{j\omega\epsilon_1}$$

which is equivalent to the result of Eq. 11-67, since $\sigma = 0$. It is thus reasonable to define the characteristic impedance of the guiding planes as the ratio of the transverse electric to the transverse magnetic field. Use of the value of $\gamma_m = j\beta_m$ from Eq. 11-25 allows Z_0 for TM waves to be modified to

$$Z_0 = \eta_1 \sqrt{1 - \frac{f_c^2}{f^2}} \qquad \text{TM waves} \tag{11-68}$$

where $\eta_1 = \sqrt{\mu_1/\epsilon_1}$, the intrinsic impedance of the dielectric.

For TE waves the transverse fields may be obtained from Eqs. 11-31 and 11-32, giving Z_0 for TE waves between the planes

$$Z_0 = \frac{\hat{\mathcal{E}}_y}{\hat{H}_x} = \frac{j\omega\mu_1}{\gamma}$$

and again using the appropriate value for γ permits Z_0 for TE waves to be stated as

$$Z_0 = \frac{\eta_1}{\sqrt{1 - f_c^2/f^2}} \qquad \text{TE waves} \tag{11-69}$$

The ratio of the transverse fields for TEM waves is easily obtained from Eqs. 11-41 and 11-42 as

$$Z_0 = \frac{\hat{\mathcal{E}}_x}{\hat{H}_y} = \frac{\beta}{\omega\epsilon_1}$$

Use of the value $\beta = \omega\sqrt{\mu_1\epsilon_1}$ for TEM waves gives

$$Z_0 = \eta_1 \qquad \text{TEM waves} \tag{11-70}$$

which is just that of the same waves propagating in an unbounded medium. This result further confirms the previous statement that the planes merely serve to confine and limit the area of the fields in transmission of TEM waves.

11-9. Attenuation with planes of finite conductivity—TEM case

To this point the effects of the guiding planes have been studied by considering their conductivity to be infinite and the attenuation zero above cutoff frequency. The conductivities of silver, gold, or copper, which might be the plane materials, are high; and losses of energy that occur in the planes will be small compared with the energy transmitted in the dielectric. Because of the smallness of the losses, it is reasonable to assume that the field intensities near the conducting surfaces will be essentially unchanged because of the small component of energy flow into the planes. That is, the ratio of the power transmitted between the planes to the power transmitted into the planes as losses is very great. Field magnitudes required to transmit the power along the guide are large; those required to transmit the losses into the planes are very small. Therefore the fields due to the transmission of power into the planes will have no appreciable effect on the magnitude of the fields transmitting the power along the planes.

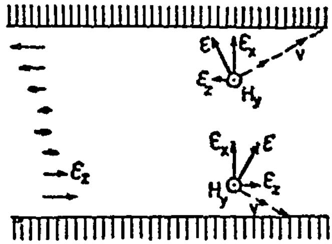


Fig. 11-8. Effect of finite wall conductivity in adding an $\mathcal{E}_z$ component in the TEM wave.

The magnitude field existing at the surface of the planes for the TEM wave has been determined as

$$H_y = B_2\epsilon^{-\gamma z} \sin \omega t$$

and this is assumed unchanged in magnitude by the finite plane conductivity. There will now exist a small $\mathcal{E}_z$ component of field due to the value of J/σ present in the metal. This will cause the wave to tip and no longer lie entirely in the x,y plane. The $\mathcal{E}_z$ component will be directed on $+z$ on one plate and $-z$ on the other plate. The tipping of the wave toward the plane indicates that a component of the field is incident on, and entering, the metal plane. It is this component of field that conveys to the metal the energy required to supply the conduction losses.

Since H_y is wholly tangential, it is continuous across the boundary and represents the magnetic field of the wave entering the metal. The power conveyed by this field into the metal per unit area at some value of z is given by the average power expression for metals:

$$P_M = \frac{1}{2} \hat{H}^2 \sqrt{\frac{\omega\mu_m}{2\sigma_m}}$$

$$= \frac{1}{2} B_2{}^2 \sqrt{\frac{\omega\mu_m}{2\sigma_m}} \tag{11-71}$$

The loss in both planes will be twice this value.

The power being transmitted in the z direction by the TEM wave past some plane was stated in Eq. 11-46 as

$$\begin{aligned} P_T &= \tfrac{1}{2}\eta_1 \hat{H}_y{}^2 a \\ &= \tfrac{1}{2}\eta_1 B_2{}^2 a \end{aligned} \tag{11-72}$$

It is now necessary to restate the definition of the attenuation factor α. This factor was originally defined by

$$\frac{|I_T|}{|I_1|} = \epsilon^{-\alpha z}$$

where α was given as nepers per meter of line with I_T the current at z, and I_1 the current at $z = 0$. For equal impedances a power ratio may be stated as

$$\frac{P_T}{P_1} = \epsilon^{-2\alpha z}$$

or

$$P_T = P_1 \epsilon^{-2\alpha z}$$

The rate of decrease of power with distance is

(11-73)

and this rate represents the power loss per unit length, where P_T is the power being transmitted past any point. In the case of the guided wave, P_M is the power loss per unit length in the z direction per meter of width in the y direction. P_T is the power being transmitted between the planes per meter of width in the y direction. Then

$$\begin{aligned} P_M &= 2\alpha P_T \\ \alpha &= \frac{1}{2}\frac{P_M}{P_T} \text{ nepers/m} \end{aligned} \tag{11-74}$$

Thus α is defined in terms of the ratio of the power lost to the power transmitted per unit width in the y direction. With this definition for α and the values of P_M and P_T from Eqs. 11-71 and 11-72, the attenuation for the TEM wave traveling between parallel planes is found to be

$$\alpha_{TEM} = \frac{1}{a\eta_1}\sqrt{\frac{\pi f \mu_m}{\sigma_m}} \tag{11-75}$$

in terms of nepers per meter in the z direction, per meter of width in the y direction.

11-10. Attenuation with planes of finite conductivity—TM case

As an example of attenuation with planes of finite conductivity confining a TM wave, consider that m = odd, for which the value of H_y in the TM$_m$ wave can be written

$$\hat{H}_y = B_1 \sin\left(\frac{m\pi x}{a}\right)$$

At the upper plate, for which $x = a/2$, this is

$$\hat{H}_y = B_1 \tag{11-76}$$

The power entering the metal plane per square meter at some z value is then obtainable from

$$P = \frac{1}{2}\hat{H}_y^2\sqrt{\frac{\omega\mu_m}{2\sigma_m}}$$

as

$$P_M = \frac{1}{2}B_1^2\sqrt{\frac{\omega\mu_m}{2\sigma_m}} \text{ watts/m}^2 \tag{11-77}$$

The power lost in both planes is twice this value per unit area of one plane.

The z-directed power flowing in the wave past a point in the dielectric between the planes is given by

$$\begin{aligned} P_T &= \tfrac{1}{2}\hat{\mathcal{E}}_x\hat{H}_y \\ &= \frac{1}{2}\frac{\beta_m}{\omega\epsilon_1}B_1^2\sin^2\left(\frac{m\pi x}{a}\right) \end{aligned} \tag{11-78}$$

using the field values from Eqs. 11-27 and 11-28. The total power flowing in the z direction through a strip extending from top to bottom plane and one meter wide in the y direction is

$$\begin{aligned} P_T &= \frac{1}{2}\frac{\beta_m}{\omega\epsilon_1}B_1^2\int_{-a/2}^{a/2}\sin^2\left(\frac{m\pi x}{a}\right)dx \\ &= \frac{1}{2}\frac{\beta_m}{\omega\epsilon_1}B_1^2\frac{a}{2} \end{aligned} \tag{11-79}$$

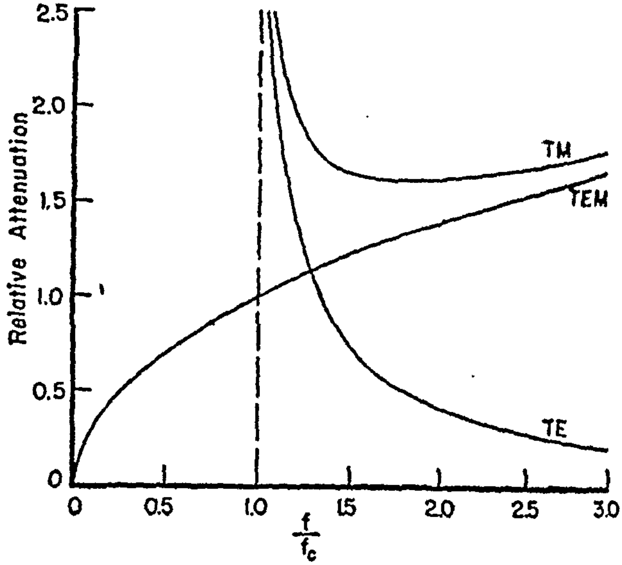


Fig. 11-9. Variation of attenuation of TM, TE, and TEM modes between parallel planes ($m = 1$).

The attenuation factor, expressed in terms of power lost, P_M, and power transmitted, P_T, is

$$\alpha_{TM} = \frac{1}{2}\frac{P_M}{P_T} = \frac{2\omega\epsilon_1}{a\beta_m}\sqrt{\frac{\omega\mu_m}{2\sigma_m}}$$

Introduction of Eq. 11-21 for f_c and considerable manipulation give

$$\alpha_{TM} = \frac{\sqrt{m}}{a^{3/2}}\sqrt{\frac{2\pi\mu_m}{\sigma_m\mu_1\eta_1}}\frac{\sqrt{f/f_c}}{\sqrt{1 - f_c^2/f^2}} \tag{11-80}$$

for the attenuation in the pass band for the TM$_m$ mode between parallel planes of finite conductivity, m being odd. The result, a rather involved function of the ratio f/f_c, is plotted in Fig. 11-9.

It may be shown that a similar result is obtainable with m even.

11-11. Attenuation with planes of finite conductivity—TE case

Attenuation of the wave for planes of finite conductivity for TE waves may be investigated for m = odd, for example, in a manner similar to that of the preceding section. The tangential component of magnetic field in the TE$_m$ wave is given by Eq. 11-40.

$$\hat{H}_z = -\frac{m\pi}{a\omega\mu_1} B_4 \sin\left(\frac{m\pi x}{a}\right)$$

Evaluation of the fields at the bounding planes and use of the Poynting radiation vector give, for the power loss per unit area in both planes,

$$P_M = \left(\frac{m\pi}{a\omega\mu_1}\right)^2 B_4^2 \sqrt{\frac{\omega\mu_m}{2\sigma_m}} \text{ watts/m}^2$$

The power flowing through the dielectric at any point is given by

$$P_T = \tfrac{1}{2}\hat{\mathcal{E}}_y \hat{H}_x$$

and since

$$\hat{\mathcal{E}}_y = B_4 \cos\left(\frac{m\pi x}{a}\right)$$

$$\hat{H}_x = -\frac{\beta_m}{\omega\mu_1} B_4 \cos\left(\frac{m\pi x}{a}\right)$$

the power transmitted through a plane strip extending from plane to plane and one meter wide in the y direction is obtained by integration as

$$P_T = \frac{1}{4}\frac{\beta_m}{\omega\mu_1} B_4^2 a \tag{11-81}$$

The attenuation then may be written, after some labor,

$$\alpha_{TE} = \frac{\sqrt{m}}{a^{3/2}}\sqrt{\frac{2\pi\mu_m}{\sigma_m\mu_1\eta_1}}\frac{(f_c/f)^{3/2}}{\sqrt{1 - f_c^2/f^2}} \text{ nepers/m} \tag{11-82}$$

This is also an involved function of f/f_c, as well as being dependent on m and on the metal plane parameters. The same expression is obtained for m = even.

The variation in value of attenuation with frequency for TM, TE, and TEM modes is plotted in Fig. 11-9 as a function of f/f_c. The minimum in the attenuation curve for the TM mode may be shown to occur at $f = \sqrt{3}\, f_c$.

It may be noted that

$$\alpha_{TE} = \left(\frac{f_c}{f}\right)^2 \alpha_{TM}$$

and that the attenuation of the TE mode decreases with increasing frequency, becoming zero for infinite frequency.

PROBLEMS

11-1. Sketch the field distributions for $\mathcal{E}$ and H between the parallel planes for the TE_1 and TM_1 modes, for one wavelength in the z direction.

11-2. Sketch the field distribution for $\mathcal{E}$ and H between the parallel planes for the TE_4 and TM_4 modes, for one wavelength in the z direction.

11-3. By analogy between Z_0 and γ for the TM_1 wave and for a high-pass filter, find an analogous electric circuit for the TM_1 wave between parallel planes.

11-4. A pair of perfectly conducting planes are separated 8 cm in air. For a frequency of 5000 megacycles with the TM_1 mode excited, find the following:

(a) Cutoff frequency.
(b) Characteristic impedance.
(c) β.
(d) Attenuation constant for $f = 0.95f_c$.
(e) Phase and group velocity.
(f) Wavelength measured along the guiding walls.

11-5. A parallel plane guide is transmitting an average power of 1000 kw per meter of width. The plane separation is 4 cm and the frequency is 10,000 megacycles, with TE_1 transmission. Compute the maximum values of electric and magnetic intensity in the space between planes and show where these maxima occur.

11-6. (a) If the planes of Prob. 11-5 are made of copper, find the values and direction of the total current flowing in the planes per meter of width for both TM_1 and TE_1 propagation, with power as stated flowing.

(b) Find the power loss per unit area of both planes, for TE_1.
(c) Compute the attenuation.
(d) What percentage of reduction in power loss may be achieved by changing to silver planes for the TE_1 case?

11-7. Prove that Eq. 11-80 is the attenuation for m = even with the TM_m mode.

11-8. Prove that Eq. 11-82 applies equally well for attenuation with m = even, for the TE_m mode.

11-9. Find the frequency of minimum attenuation for the TM mode, in terms of f_c.

11-10. For a frequency of 6000 megacycles and plane separation $= 7$ cm, find the following for the TE_1 mode:

(a) Cutoff frequency.
(b) Angle of incidence on the planes.
(c) Phase and group velocity.
(d) Is it possible to propagate the TE_3 mode?

11-11. Show that the propagation of the TE_1 wave takes place by reason of two crossed waves and find the value of the angle between the waves.

11-12. Consider the possibilities inherent in the use of an infinite slab of solid good dielectric, of permittivity ϵ_1, oriented parallel to the x,z plane in space and of thickness a, as a wave-guiding medium. Discover the cutoff frequency and expressions for phase and group velocities.

11-13. A field $\mathcal{E}_x = A \sin(\pi y/b) \sin(\omega t - \beta z)$ is the only electric component in a region. Show that div $\mathcal{E} = 0$.

REFERENCES

1. Ramo, S., and Whinnery, J. R., *Fields and Waves in Modern Radio*, 2d ed., John Wiley & Sons, Inc., New York, 1953.
2. Slater, J. C., *Microwave Transmission*, McGraw-Hill Book Company, Inc., New York, 1942.
3. Skilling, H. H., *Fundamentals of Electric Waves*, John Wiley & Sons, Inc., New York, 1942.
4. Barrow, W. L., "Transmission of Electromagnetic Waves in Hollow Tubes of Metal," *Proc. I.R.E.*, **24**, 1298–1328 (1936).
5. Schelkunoff, S. A., "Transmission Theory of Plane Electromagnetic Waves," *Proc. I.R.E.*, **25**, 1457–1492 (1937).

Chapter 12

WAVE GUIDES

Wave guides or hollow metal tubes are now employed extensively for transmission of the very high frequencies where the attenuation caused by the wave guide is smaller than that for any other form of transmission line. In order to obtain an understanding or physical picture of electric wave transmission through hollow metal tubes, the previous chapter analyzed the directing of electromagnetic waves between infinite parallel metal planes. It is now possible to complete the picture by adding two more sides to form a complete tube. It will be demonstrated that the approach of the previous chapter was helpful, since many of the wave properties will be unaffected and unchanged by the presence of the sides on the guide.

The guides will first be assumed made of perfect metallic conductors. If it is desired to determine the attenuation due to finite conductivity, the methods of the preceding section may be employed.

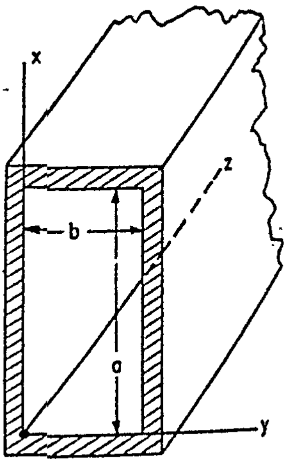


Fig. 12-1. Cross section of the rectangular wave guide.

12-1. Application of Maxwell's equations to the rectangular wave guide

Conducting side planes may be added to the parallel planes of the preceding chapter to form a rectangular wave guide, as in Fig. 12-1. Note that the guide is oriented with two faces in the planes of the axes, and that the dimension a is the height in the x direction, b being the width in the y direction. The guide is assumed made of perfectly conducting metal walls and to be filled with a good dielectric with constants μ_1 and ϵ_1, and with $\sigma_1 = 0$.

It is desired that propagation of energy take place in the z direction, as before, with the length of the guide being infinite in the z direction.

Thus it is reasonable to assume that all fields which may be present will vary with time and distance z according to

$$\epsilon^{-\gamma z} \sin \omega t$$

where γ is again complex in general. Using this form of variation the maximum values of the fields, indicated by the carets, may be written from Maxwell's equations as

$$\left.\begin{aligned}
\frac{\partial \hat{H}_z}{\partial y} + \gamma \hat{H}_y &= (\sigma_1 + j\omega\epsilon_1)\hat{\mathcal{E}}_x \\
-\gamma \hat{H}_x - \frac{\partial \hat{H}_z}{\partial x} &= (\sigma_1 + j\omega\epsilon_1)\hat{\mathcal{E}}_y \\
\frac{\partial \hat{H}_y}{\partial x} - \frac{\partial \hat{H}_x}{\partial y} &= (\sigma_1 + j\omega\epsilon_1)\hat{\mathcal{E}}_z
\end{aligned}\right\} \qquad (12\text{-}1)$$

$$\left.\begin{aligned}
\frac{\partial \hat{\mathcal{E}}_z}{\partial y} + \gamma \hat{\mathcal{E}}_y &= -j\omega\mu_1 \hat{H}_x \\
-\gamma \hat{\mathcal{E}}_x - \frac{\partial \hat{\mathcal{E}}_z}{\partial x} &= -j\omega\mu_1 \hat{H}_y \\
\frac{\partial \hat{\mathcal{E}}_y}{\partial x} - \frac{\partial \hat{\mathcal{E}}_x}{\partial y} &= -j\omega\mu_1 \hat{H}_z
\end{aligned}\right\} \qquad (12\text{-}2)$$

After selection of a field configuration having certain field components, these equations will give the other field components that must exist in the chosen configuration for a propagating wave to exist. The result will then be a wave propagating in the z direction and varying sinusoidally with time as assumed. As in the preceding chapter, two possible arbitrary field configurations will meet the imposed conditions: one with an entirely transverse electric field and one with an entirely transverse magnetic field. For the transverse electric wave it is then possible to require that $\mathcal{E}_z = 0$, and for the transverse magnetic wave that $H_z = 0$. Because of lack of advance knowledge on the effects of the two sides added to the parallel planes, it is not possible to impose any other field restrictions. The boundary restriction that $\mathcal{E} = 0$ along any of the perfectly conducting planes is the remaining physical information of use in defining the fields. To this may be added intuition based on the analysis of the parallel-plane case.

12-2. The $TM_{m,n}$ wave in the rectangular guide

The TM wave is obtained if it be assumed that the magnetic field is wholly transverse to the z direction of propagation and thus $H_z = 0$. Maxwell's equations for propagation in the z direction, as given in Eqs. 12-1 and 12-2, may then be modified for TM transmission in a good dielectric inside the guide where $\sigma_1 = 0$, under the assumption $H_z = 0$, and in terms of maximum values, as

$$\left.\begin{aligned} \gamma \hat{H}_y &= j\omega\epsilon_1 \hat{\mathcal{E}}_x \\ -\gamma \hat{H}_x &= j\omega\epsilon_1 \hat{\mathcal{E}}_y \\ \frac{\partial \hat{H}_y}{\partial x} - \frac{\partial \hat{H}_x}{\partial y} &= j\omega\epsilon_1 \hat{\mathcal{E}}_z \end{aligned}\right\} \tag{12-3}$$

$$\left.\begin{aligned} \frac{\partial \hat{\mathcal{E}}_z}{\partial y} + \gamma \hat{\mathcal{E}}_y &= -j\omega\mu_1 \hat{H}_x \\ -\gamma \hat{\mathcal{E}}_x - \frac{\partial \hat{\mathcal{E}}_z}{\partial x} &= -j\omega\mu_1 \hat{H}_y \\ \frac{\partial \hat{\mathcal{E}}_y}{\partial x} - \frac{\partial \hat{\mathcal{E}}_x}{\partial y} &= 0 \end{aligned}\right\} \tag{12-4}$$

These equations indicate the possible presence of field components H_x, H_y, $\mathcal{E}_x$, $\mathcal{E}_y$, $\mathcal{E}_z$, in the TM wave, representing a consistent set of

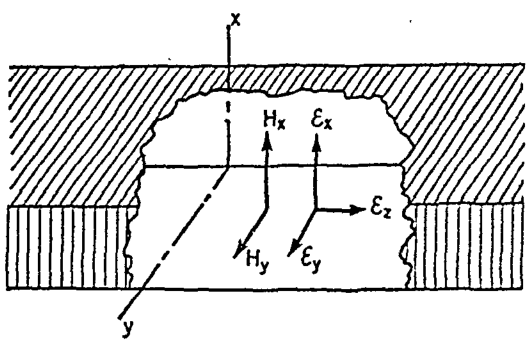


Fig. 12-2. Possible fields present in the TM wave.

physically realizable fields that will propagate in the z direction, with sinusoidal variation in time, and with $H_z = 0$ in the guide. Assumption of a value for any one field component and insertion in the above equations lead to values for the accompanying field components.

In the previous discussion of parallel conducting planes it was found that the fields varied as a sinusoidal function of distance between the planes. The tangential electric field components were also required to have zero values at the boundaries. From Fig. 12-2 it can be seen that the requirement of zero values at the boundaries must be placed on $\mathcal{E}_z$ in both x and y directions. If the field is to vary sinusoidally in the dielectric between the boundaries, it seems reasonable that $\mathcal{E}_z$ may be expressed as

$$\mathcal{E}_z = \mathcal{E}_0 \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \epsilon^{-\gamma z} \sin \omega t \tag{12-5}$$

where m and n are integers having values 0, 1, 2, 3, . . . , and a and b are the guide dimensions shown in Fig. 12-1. With the origin located as in that figure, Eq. 12-5 would satisfy the boundary conditions that $\mathcal{E}_z = 0$ at the guide walls or at $x = a$ or $y = b$.

Equations 12-3 and 12-4 may be manipulated to give

$$\frac{\partial^2 \hat{\mathcal{E}}_z}{\partial x^2} + \frac{\partial^2 \hat{\mathcal{E}}_z}{\partial y^2} = -(\gamma^2 + \omega^2 \mu_1 \epsilon_1)\hat{\mathcal{E}}_z \tag{12-6}$$

The value assumed for $\mathcal{E}_z$ may then be substituted in the above and found to satisfy the equation and in turn to satisfy Maxwell's equations if

$$\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 = \gamma^2 + \omega^2 \mu_1 \epsilon_1 \tag{12-7}$$

If these conditions are met, the assumption for $\mathcal{E}_z$ is a possible solution (actually the only one) for the z-directed electric component of a physically realizable wave propagating as required in a TM mode in the rectangular guide.

From Eq. 12-7,

$$\gamma_{m,n} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2 \mu_1 \epsilon_1} \tag{12-8}$$

and thus the propagation constant for the z direction is determined in terms of frequency, guide dimensions, and arbitrary integers m and n. As in the case of parallel planes, $\gamma_{m,n}$ may be either wholly real or imaginary, dependent on the value of ω. If it is real, the value of $\mathcal{E}_z$ in Eq. 12-5 attenuates according to the factor $\epsilon^{-\gamma_{m,n} z}$; if it is imaginary, the field propagates without attenuation in the z direction along the guide according to the factor $\epsilon^{-j\beta_{m,n}z}$. The rectangular wave guide acts as a high-pass filter, in a manner similar to the parallel-plane case and in the propagation region

$$\alpha_{m,n} = 0$$

$$\gamma_{m,n} = j\beta_{m,n} = j\sqrt{\omega^2\mu_1\epsilon_1 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \qquad (12\text{-}9)$$

The cutoff frequency for TM waves occurs when $\beta_{m,n} = 0$ and is

$$f_c = \frac{v_1}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \qquad (12\text{-}10)$$

and the cutoff wavelength may be obtained as

$$\lambda_c = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}}$$

The minimum cutoff frequency, or maximum cutoff wavelength occurs for $m = n = 1$. As will be shown, values of $m = 0$ or $n = 0$ are not possible with the TM mode.

It is possible to write $\gamma_{m,n}$ in the propagation or pass band as

$$\gamma_{m,n} = j\beta_{m,n} = j\,\frac{2\pi f_c}{v_1}\sqrt{\frac{f^2}{f_c^2} - 1} \qquad (12\text{-}11)$$

The phase velocity for TM waves can be obtained from $\omega/\beta_{m,n}$ and is

$$v_p = \frac{v_1}{\sqrt{1 - \frac{f_c^2}{f^2}}} \qquad (12\text{-}12)$$

Since it has been demonstrated that the phase and group velocities are related according to $\sqrt{v_p v_g} = v_1$, the group velocity can be readily written as

$$v_g = v_1\sqrt{1 - \frac{f_c^2}{f^2}} \qquad (12\text{-}13)$$

The same result could have been obtained by taking $1/(d\beta/d\omega)$. By analogy with the case of the parallel planes, it may be reasoned that propagation inside the wave guide takes place by successive glancing reflections of the waves from the conducting side walls, the actual wave path being zigzag in form. Another viewpoint may be that standing-wave patterns are set up on the x and y axes due to reflections from the conducting side walls and that these standing-wave patterns then propagate down the guide in the z direction.

Use of the value for $\mathcal{E}_z$ in Eqs. 12-3 and 12-4 and some manipulation allow the complete field components of the $\mathrm{TM}_{m,n}$ wave to be written for the pass band where $\gamma_{m,n} = j\beta_{m,n}$ as

$$H_x = \frac{n\pi}{b}\frac{\omega\epsilon_1}{\omega_c^2\mu_1\epsilon_1}\mathcal{E}_0 \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\epsilon^{-j\beta_{m,n}z}\cos\omega t \tag{12-14}$$

$$H_y = -\frac{m\pi}{a}\frac{\omega\epsilon_1}{\omega_c^2\mu_1\epsilon_1}\mathcal{E}_0 \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\epsilon^{-j\beta_{m,n}z}\cos\omega t \tag{12-15}$$

$$H_z = 0 \tag{12-16}$$

$$\mathcal{E}_x = -\frac{m\pi}{a}\frac{\beta_{m,n}}{\omega_c^2\mu_1\epsilon_1}\mathcal{E}_0 \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\epsilon^{-j\beta_{m,n}z}\cos\omega t \tag{12-17}$$

$$\mathcal{E}_y = -\frac{n\pi}{b}\frac{\beta_{m,n}}{\omega_c^2\mu_1\epsilon_1}\mathcal{E}_0 \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\epsilon^{-j\beta_{m,n}z}\cos\omega t \tag{12-18}$$

$$\mathcal{E}_z = \mathcal{E}_0 \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\epsilon^{-j\beta_{m,n}z}\sin\omega t \tag{12-19}$$

It should be noted that if either m or n is zero, all the fields are zero. Thus a $\mathrm{TM}_{m,0}$ or a $\mathrm{TM}_{0,n}$ mode of propagation cannot exist in the rectangular wave guide. This result also rules out a TEM mode in a guide.

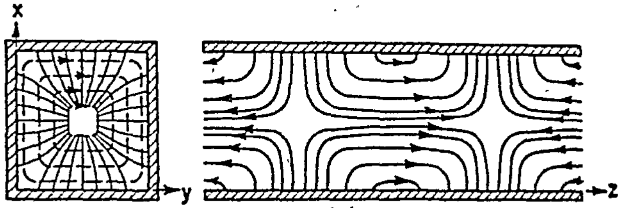
For the parallel planes, a Z_0 or characteristic wave impedance was defined as the ratio of transverse electric to transverse magnetic field. If a similar definition is used for the rectangular wave guide

$$\frac{\mathcal{E}}{H} = \frac{\sqrt{\mathcal{E}_x^2 + \mathcal{E}_y^2}}{\sqrt{H_x^2 + H_y^2}} = \frac{\beta_{m,n}}{\omega\epsilon_1} = Z_{0(TM)} \tag{12-20}$$

Use of Eq. 12-11 for $\beta_{m,n}$ allows the expression for the characteristic impedance of a guide with the $\mathrm{TM}_{m,n}$ mode existing to be written

$$Z_{0(TM)} = \eta_1\sqrt{1 - \frac{f_c^2}{f^2}} \tag{12-21}$$

The field configurations for the rectangular guide for the $TM_{1,1}$ and $TM_{2,1}$ modes are shown in Fig. 12-3. The pattern as shown in the longitudinal section is drawn for the center line of the guide and parallel to the x,z plane. The field pattern may be visualized as traveling along the infinite length guide in the z direction. Should the guide be of finite length and not properly terminated in its Z_0



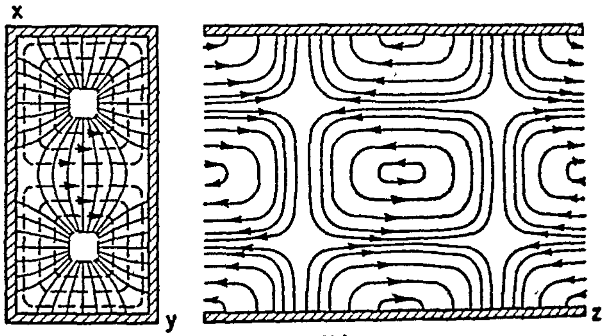


Fig. 12-3. (a) Configuration of $TM_{1,1}$ fields; (b) configuration of $TM_{2,1}$ fields.

value, then the pattern will be set up as a standing wave because of reflection of part of the incident energy.

Consideration of the field patterns will give meaning to the n and m subscripts employed. The m subscript indicates the number of maxima of field intensity in the x direction between the walls, whereas n indicates the number of such field-intensity maxima between the walls in the y direction. Modes in which the subscripts are complementary, such as $TM_{1,2}$ and $TM_{2,1}$, simply imply that the guide has been turned on its side.

It is customary to assume the y coordinate coincident with the smaller transverse dimension, the x coordinate coincident with the larger transverse dimension.

12-3. The $TE_{m,n}$ wave in the rectangular guide

If it be assumed that the electric field is wholly transverse to the z direction of propagation, the TE mode is obtained in the rectangular guide. For this case, $\mathcal{E}_z = 0$. Considering the dielectric in the guide to have $\sigma_1 = 0$. Eqs. 12-1 and 12-2 become, for the TE mode,

$$\left.\begin{aligned} \frac{\partial \hat{H}_z}{\partial y} + \gamma \hat{H}_y &= j\omega\epsilon_1 \hat{\mathcal{E}}_x \\ -\gamma \hat{H}_x - \frac{\partial \hat{H}_z}{\partial x} &= j\omega\epsilon_1 \hat{\mathcal{E}}_y \\ \frac{\partial \hat{H}_y}{\partial x} - \frac{\partial \hat{H}_x}{\partial y} &= 0 \end{aligned}\right\} \tag{12-22}$$

$$\left.\begin{aligned} \gamma \hat{\mathcal{E}}_y &= -j\omega\mu_1 \hat{H}_x \\ -\gamma \hat{\mathcal{E}}_x &= -j\omega\mu_1 \hat{H}_y \\ \frac{\partial \hat{\mathcal{E}}_y}{\partial x} - \frac{\partial \hat{\mathcal{E}}_x}{\partial y} &= -j\omega\mu_1 \hat{H}_z \end{aligned}\right\} \tag{12-23}$$

As before, these equations indicate fields with components H_x, H_y, H_z, $\mathcal{E}_x$, and $\mathcal{E}_y$, representing a consistent set with propagation in the z direction down the guide.

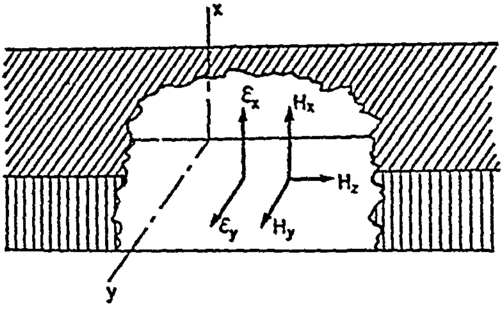


Fig. 12-4. Possible fields present in the TE wave.

In the previous TM case it was convenient to assume a value for $\mathcal{E}_z$, since it was symmetrically affected by both sets of guide walls. For the same reason it will be convenient to assume a value for H_z for the TE case. It appears reasonable to assume that

$$H_z = H_0 \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \epsilon^{-\gamma z} \sin \omega t \tag{12-24}$$

the use of cosine functions being dictated by the physical knowledge that since the currents flow in the side walls, the fields will be largest when near the currents. If the intensities for the various components obtained under this assumption satisfy the boundary requirements set up by the conducting walls, the assumption will have been proved correct.

Equations 12-22 and 12-23 may be manipulated to give

$$\frac{\partial^2 \hat{H}_z}{\partial x^2} + \frac{\partial^2 \hat{H}_z}{\partial y^2} = -(\gamma^2 + \omega^2\mu_1\epsilon_1)\hat{H}_z \tag{12-25}$$

into which Eq. 12-24 for H_z may be substituted. The assumed H_z value will satisfy this equation, and thus will satisfy Maxwell's equations, if

$$\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 = \gamma^2 + \omega^2\mu_1\epsilon_1 \tag{12-26}$$

thus leading to the same propagation condition as for TM waves.

The propagation constant $\gamma_{m,n}$ for TE waves is the same as for TM waves:

$$\gamma_{m,n} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2\mu_1\epsilon_1}$$

and becomes, in the propagation or pass-band region,

$$\gamma_{m,n} = j\beta_{m,n} = j\sqrt{\omega^2\mu_1\epsilon_1 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \tag{12-27}$$

with $$\alpha_{m,n} = 0$$

Obviously, the rectangular wave guide again behaves as a high-pass filter. The cutoff frequency, cutoff wavelength, and phase and group velocities will be the same for TE waves as TM waves.

Insertion of the assumed H_z value into Eqs. 12-22 and 12-23

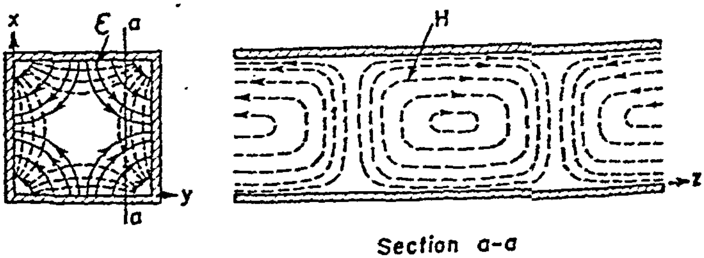


Fig. 12-5. Configuration of the $TE_{1,1}$ field.

permits the TE field components to be written for the pass band as

$$H_x = \frac{m\pi}{a}\frac{\beta_{m,n}}{\omega_c{}^2\mu_1\epsilon_1} H_0 \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\epsilon^{-j\beta_{m,n}z}\cos\omega t \tag{12-28}$$

$$H_y = \frac{n\pi}{b}\frac{\beta_{m,n}}{\omega_c{}^2\mu_1\epsilon_1} H_0 \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\epsilon^{-j\beta_{m,n}z}\cos\omega t \tag{12-29}$$

$$H_z = H_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\epsilon^{-j\beta_{m,n}z}\sin\omega t \tag{12-30}$$

$$\mathcal{E}_x = \frac{n\pi}{b}\frac{\omega\mu_1}{\omega_c{}^2\mu_1\epsilon_1} H_0 \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\epsilon^{-j\beta_{m,n}z}\cos\omega t \tag{12-31}$$

$$\mathcal{E}_y = -\frac{m\pi}{a}\frac{\omega\mu_1}{\omega_c{}^2\mu_1\epsilon_1} H_0 \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\epsilon^{-j\beta_{m,n}z}\cos\omega t \tag{12-32}$$

$$\mathcal{E}_z = 0 \tag{12-33}$$

A field configuration for the $TE_{1,1}$ mode in the rectangular guide is drawn in Fig. 12-5.

The characteristic wave impedance of the rectangular guide for TE waves can be readily found as the ratio of the transverse electric field to the transverse magnetic field:

$$\frac{\mathcal{E}}{H} = \frac{\sqrt{\mathcal{E}_x{}^2 + \mathcal{E}_y{}^2}}{\sqrt{H_x{}^2 + H_y{}^2}} = \frac{\omega\mu_1}{\beta_{m,n}} = Z_{0(TE)} \tag{12-34}$$

and substitution for β gives

$$Z_{0(TE)} = \frac{\eta_1}{\sqrt{1 - f_c{}^2/f^2}} \tag{12-35}$$

which is *not* identical with that obtained for TM waves. In fact, comparison with Eq. 12-21 shows

$$\eta_1 = \sqrt{Z_{0(TM)}Z_{0(TE)}}$$

or that the intrinsic impedance in the dielectric in the unbounded case is the geometric mean of the wave impedance to TE and TM propagation in the rectangular guide.

It can be seen from these equations that waves of the $TE_{m,0}$ type are possible except for the $TE_{0,0}$ mode in which case all fields go to zero. The $TE_{1,0}$ wave is of very considerable interest because of its simplicity. For this wave, $m = 1$, $n = 0$.

Since $$\gamma_{1,0} = j\beta_{1,0} = j\sqrt{\omega^2\mu_1\epsilon_1 - \left(\frac{\pi}{a}\right)^2}$$

the field expressions for the $TE_{1,0}$ wave reduce to

$$H_z = \frac{a}{\pi}\beta_{m,n}H_0 \sin\left(\frac{\pi x}{a}\right)\epsilon^{-j\beta_{1,0}z}\cos\omega t \tag{12-36}$$

$$H_x = H_0 \cos\left(\frac{\pi x}{a}\right)\epsilon^{-j\beta_{1,0}z}\sin\omega t \tag{12-37}$$

$$\mathcal{E}_y = -\frac{a}{\pi}\omega\mu_1 H_0 \sin\left(\frac{\pi x}{a}\right)\epsilon^{-j\beta_{1,0}z}\cos\omega t \tag{12-38}$$

all other fields being zero. None of the field components vary with dimension b, so that guides of different b dimensions but equal a

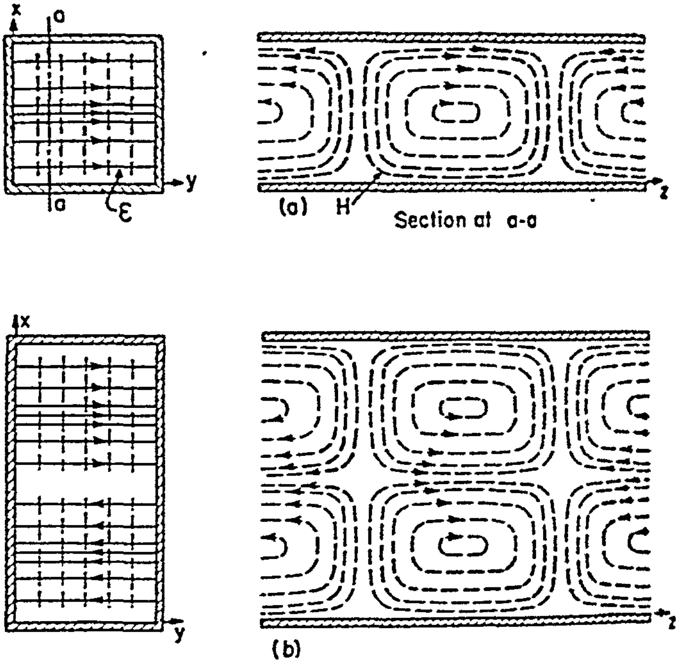


Fig. 12-6. (a) The arrangement of the $TE_{1,0}$ fields in the rectangular guide; (b) the $TE_{2,0}$ field.

dimensions will have the same value of cutoff frequency. The field configuration for a $TE_{1,0}$ wave is shown in Fig. 12-6(a), wherein the simplicity of the field may be noted.

The $TE_{1,0}$ critical frequency is given by

$$f_c = \frac{v_1}{2a}$$

and it can be seen that at the cutoff wavelength the a dimension is just one-half wavelength, or the value of λ_c is

$$\frac{\lambda_c}{2} = a$$

bearing out the physical concept developed for the parallel-plane cutoff wavelength.

Since a $TM_{1,0}$ wave cannot exist, the $TE_{1,0}$ wave is the simplest possible form and has the lowest cutoff frequency for a given guide

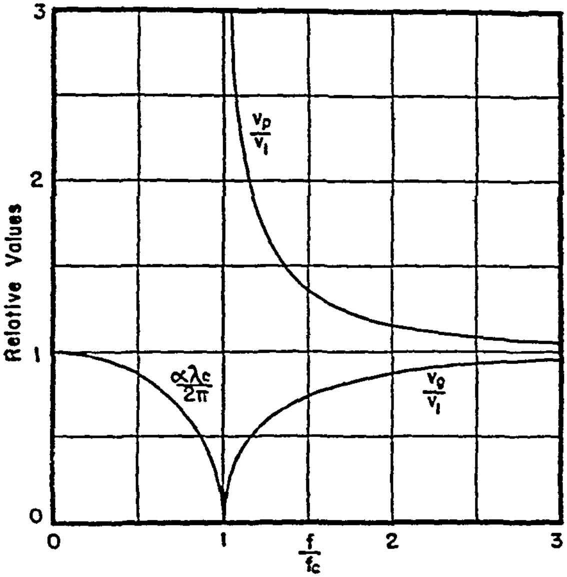


Fig. 12-7. Variation of attenuation below cutoff, and of the velocities above cutoff, for the rectangular wave guide.

dimension in the x direction. The $TE_{1,0}$ mode is called the *dominant mode* for the rectangular guide, the dominant mode being the only wave which will convey energy through the guide when the excitation frequency is between the lowest cutoff and that due to the next higher mode. Ordinarily it is desired that only one mode be propagated, and the guide is so designed that for a $TE_{1,0}$ wave for example, dimension a is longer than half a wavelength at the operating frequency, and shorter than two half wavelengths. The dimension is thus too small to support a $TE_{2,0}$ wave. For dominant-mode operation the frequency range of a rectangular guide is restricted to less than 2 to 1. For circular guides this is 1.25 to 1 or less. Ordinarily a rectangular guide is used with the $TE_{1,0}$ mode and the circular guide with the $TE_{1,1}$ mode excited.

The smaller dimension of the rectangular guide is kept definitely less than a half wavelength at any operating frequency in order to avoid exciting the $TM_{1,1}$ or $TE_{1,1}$ mode. Where rotational symmetry is required, as in coupling a guide to a rotating antenna, higher modes may be employed. For the $TE_{1,0}$ mode the lowest cutoff frequency in a rectangular guide is

$$f_c = \frac{v_1}{2a}$$

Since all tangential components of electric field and all normal components of magnetic field go to zero at the perfectly conducting guide walls, the derived fields satisfy the boundary conditions and the original assumption of a value for H_z is proved satisfactory.

12-4. Cylindrical wave guides

In order to determine the conditions for propagation of waves inside a hollow, perfectly conducting cylinder as in Fig. 12-8, it is convenient to employ cylindrical coordinates r, ϕ, and z, instead of the previously used rectangular coordinates x, y, and z. As shown in Appendix B, Maxwell's equations may be converted to cylindrical coordinates, resulting in

$$\left.\begin{aligned} \frac{1}{r}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z} &= \sigma \mathcal{E}_r + \epsilon \frac{\partial \mathcal{E}_r}{\partial t} \\ \frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} &= \sigma \mathcal{E}_\phi + \epsilon \frac{\partial \mathcal{E}_\phi}{\partial t} \\ \frac{1}{r}\frac{\partial}{\partial r}(rH_\phi) - \frac{1}{r}\frac{\partial H_r}{\partial \phi} &= \sigma \mathcal{E}_z + \epsilon \frac{\partial \mathcal{E}_z}{\partial t} \end{aligned}\right\} \tag{12-39}$$

$$\left.\begin{aligned} \frac{1}{r}\frac{\partial \mathcal{E}_z}{\partial \phi} - \frac{\partial \mathcal{E}_\phi}{\partial z} &= -\mu \frac{\partial H_r}{\partial t} \\ \frac{\partial \mathcal{E}_r}{\partial z} - \frac{\partial \mathcal{E}_z}{\partial r} &= -\mu \frac{\partial H_\phi}{\partial t} \\ \frac{1}{r}\frac{\partial}{\partial r}(r\mathcal{E}_\phi) - \frac{1}{r}\frac{\partial \mathcal{E}_r}{\partial \phi} &= -\mu \frac{\partial H_z}{\partial t} \end{aligned}\right\} \tag{12-40}$$

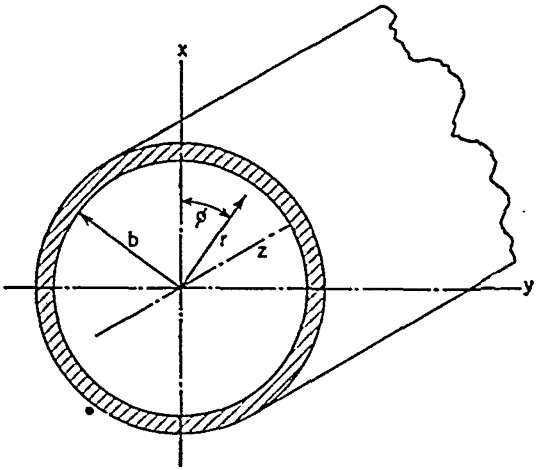


Fig. 12-8. The cylindrical wave guide.

If it is required that the waves propagate in the z direction and be a sinusoidal time function, then

$$\mathcal{E} = \hat{\mathcal{E}}\epsilon^{-\gamma z} \sin \omega t$$
$$H = \hat{H}\epsilon^{-\gamma z} \sin \omega t$$

The proper derivatives may be taken, after which Maxwell's equations for a good dielectric ($\sigma_1 = 0$), in terms of maximum values, become

$$\left.\begin{aligned} \frac{1}{r}\frac{\partial \hat{H}_z}{\partial \phi} + \gamma \hat{H}_\phi &= j\omega\epsilon_1\hat{\mathcal{E}}_r \\ -\gamma \hat{H}_r - \frac{\partial \hat{H}_z}{\partial r} &= j\omega\epsilon_1\hat{\mathcal{E}}_\phi \\ \frac{1}{r}\frac{\partial}{\partial r}(r\hat{H}_\phi) - \frac{1}{r}\frac{\partial \hat{H}_r}{\partial \phi} &= j\omega\epsilon_1\hat{\mathcal{E}}_z \end{aligned}\right\} \tag{12-41}$$

$$\left.\begin{aligned} \frac{1}{r}\frac{\partial \hat{\mathcal{E}}_z}{\partial \phi} + \gamma \hat{\mathcal{E}}_\phi &= -j\omega\mu_1\hat{H}_r \\ -\gamma \hat{\mathcal{E}}_r - \frac{\partial \hat{\mathcal{E}}_z}{\partial r} &= -j\omega\mu_1\hat{H}_\phi \\ \frac{1}{r}\frac{\partial}{\partial r}(r\hat{\mathcal{E}}_\phi) - \frac{1}{r}\frac{\partial \hat{\mathcal{E}}_r}{\partial \phi} &= -j\omega\mu_1\hat{H}_z \end{aligned}\right\} \tag{12-42}$$

As before, it is possible to have TE modes with $\mathcal{E}_z = 0$, and TM modes with $H_z = 0$.

Solution of Maxwell's equations in rectangular coordinates gave fields in terms of trigonometric functions of the tube dimensions. Operation on Maxwell's equations in cylindrical coordinate systems leads to Bessel's equation and field solutions in terms of Bessel functions. After considerable manipulation and the use of an assumed field value

$$\mathcal{E}_z = \mathcal{E}_0 J_n(r\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1}) \cos n\phi \epsilon^{-\gamma z} \sin \omega t \tag{12-43}$$

it is possible to write as the field values for the TM wave that will fulfill the boundary conditions of the cylindrical guide:

$$\mathcal{E}_r = -\frac{\gamma}{\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1}} \mathcal{E}_0 J_n'(r\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1}) \cos n\phi\, \epsilon^{-\gamma z} \sin \omega t \tag{12-44}$$

$$\mathcal{E}_\phi = \frac{n\gamma}{r(\gamma^2 + \omega^2\mu_1\epsilon_1)} \mathcal{E}_0 J_n(r\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1}) \sin n\phi\, \epsilon^{-\gamma z} \sin \omega t \tag{12-45}$$

$$\mathcal{E}_z = \mathcal{E}_0 J_n(r\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1}) \cos n\phi\, \epsilon^{-\gamma z} \sin \omega t \tag{12-46}$$

$$H_r = -\frac{n\omega\epsilon_1}{r(\gamma^2 + \omega^2\mu_1\epsilon_1)} \mathcal{E}_0 J_n(r\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1}) \sin n\phi\, \epsilon^{-\gamma z} \cos \omega t \tag{12-47}$$

$$H_\phi = -\frac{\omega\epsilon_1}{\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1}} \mathcal{E}_0 J_n'(r\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1}) \cos n\phi\, \epsilon^{-\gamma z} \cos \omega t \tag{12-48}$$

$$H_z = 0$$

The fields $\mathcal{E}_r$ and H_r are directed along r. The $\mathcal{E}_\phi$ and H_ϕ fields have positive directions in the direction of increasing ϕ and are therefore at right angles to the radially directed fields. The z components are at right angles to both the r and ϕ components.

So that $\mathcal{E}_z$ shall have the same value for $\cos n\phi = \cos n(\phi + 2\pi)$, that is, that $\mathcal{E}_z$ be single-valued at a given space point, it is necessary that n be an integer. Therefore the Bessel function J_n and its derivative J_n' are only of integer orders. That is, $n = 0, 1, 2, 3, \ldots$.

For the TM wave it is necessary that $\mathcal{E}_z$ and $\mathcal{E}_\phi$ be zero at the pipe wall where $r = b$, because of the infinite conductivity of the wall. This requirement fixes the value of the Bessel function

$$J_n(b\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1}) = 0$$

and a listing of a few of the roots of J_n is given in Table 10. The manner of variation of J_0, J_1, J_2, and J_3 is indicated in Fig. 12-9,

TABLE 10

ROOTS OF $J_n = 0$

m	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$
1	2.40	3.83	5.14	6.38	7.59
2	5.52	7.02	8.42	9.76	11.06
3	8.65	10.17	11.62	13.02	14.37
4	11.79	13.32	14.80	16.22	17.62

which gives significance to the value m as designating the particular root meant, since there are an infinite number of roots. The subscripts n and m then have definite meaning in terms of the field

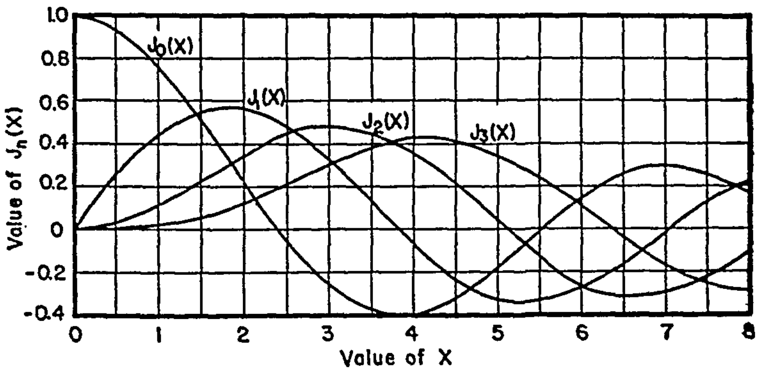


FIG. 12-9. Variation of the Bessel functions $J_0(x)$, $J_1(x)$, $J_2(x)$, $J_3(x)$.

configuration and are used to designate the mode of field propagation as in $\mathrm{TM}_{n,m}$. Thus the root of $J_n = 0$ for $n = 1$, $m = 2$ is equal to 7.02, and this corresponds to a $\mathrm{TM}_{1,2}$ wave. Physically, Eq. 12-43 shows that n is the number of cycles of variation of $\mathcal{E}_z$ found as ϕ varies around the complete cylinder, or through 2π radians.

The subscript m indicates the number of zeros of electric field along a radial path from the center to the inside surface of the outer guide wall.

If $\tau_{n,m}$ is a root of $J_n = 0$, then

$$b\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1} = \tau_{n,m}$$

and

$$\gamma_{n,m} = \sqrt{\left(\frac{\tau_{n,m}}{b}\right)^2 - \omega^2\mu_1\epsilon_1} \tag{12-49}$$

Following the analysis of preceding sections, for propagation to occur it is necessary that $\gamma_{n,m}$ be imaginary or that

$$\gamma_{n,m} = j\beta_{n,m} = j\sqrt{\omega^2\mu_1\epsilon_1 - \left(\frac{\tau_{n,m}}{b}\right)^2} \tag{12-50}$$

which makes the attenuation $\alpha_{n,m}$ equal to zero in the pass band. The critical or cutoff frequency, above which transmission takes place, is

$$f_c = \frac{\tau_{n,m}v_1}{2\pi b} \tag{12-51}$$

and the critical wavelength is

$$\lambda_c = \frac{2\pi b}{\tau_{n,m}}$$

Since $2\pi b$ is the inner circumference of the outer wall of the guide, the critical wavelength is that at which the circumference is equal to $\tau_{n,m}$ wavelengths.

The phase velocity is

$$v_p = \frac{\omega}{\beta_{n,m}} = \frac{v_1}{\sqrt{1 - \frac{f_c^2}{f^2}}} \tag{12-52}$$

and the group velocity is

$$v_g = \frac{1}{\frac{d\beta_{n,m}}{d\omega}} = v_1\sqrt{1 - \frac{f_c^2}{f^2}} \tag{12-53}$$

as for the rectangular guide. The characteristic impedances of the cylindrical guide may also be found identical with those of the rectangular guide for both TM and TE modes.

Choice of an assumed field value

$$H_z = H_0 J_n(r\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1}) \cos n\phi\, \epsilon^{-\gamma z} \sin \omega t$$

for H_z, a value selected because of similarity to the preceding TM case, and its use in Eqs. 12-41 and 12-42 permit an equation for H_r in the TE field to be written as

$$H_r = \frac{-\beta_{n,m}}{\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1}} H_0 J_n'(r\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1}) \cos n\phi\, \epsilon^{-j\beta_{n,m}z} \cos \omega t \tag{12-54}$$

It is known that at $r = b$, the H_r or normal field value must be zero, which shows that

$$J_n'(b\sqrt{\gamma^2 + \omega^2\mu_1\epsilon_1}) = 0$$

is a condition of propagation for the TE mode. If $\tau'_{n,m}$ is a root of $J_n' = 0$, the field equations for the TE case in the cylindrical guide may be written

$$H_r = \frac{-\beta_{n,m}}{\left(\dfrac{\tau'_{n,m}}{b}\right)} H_0 J_n'\left(r\,\frac{\tau'_{n,m}}{b}\right) \cos n\phi\, \epsilon^{-j\beta_{n,m}z} \cos \omega t \tag{12-55}$$

$$H_\phi = \frac{n\beta_{n,m}}{r\left(\dfrac{\tau'_{n,m}}{b}\right)^2} H_0 J_n\left(r\,\frac{\tau'_{n,m}}{b}\right) \sin n\phi\, \epsilon^{-j\beta_{n,m}z} \cos \omega t \tag{12-56}$$

$$H_z = H_0 J_n\left(r\,\frac{\tau'_{n,m}}{b}\right) \cos n\phi\, \epsilon^{-j\beta_{n,m}z} \sin \omega t \tag{12-57}$$

$$\mathcal{E}_r = \frac{n\omega\mu_1}{r\left(\dfrac{\tau'_{n,m}}{b}\right)^2} H_0 J_n\left(r\,\frac{\tau'_{n,m}}{b}\right) \sin n\phi\, \epsilon^{-j\beta_{n,m}z} \cos \omega t \tag{12-58}$$

$$\mathcal{E}_\phi = \frac{\omega\mu_1}{\left(\dfrac{\tau'_{n,m}}{b}\right)} H_0 J_n'\left(r\,\frac{\tau'_{n,m}}{b}\right) \cos n\phi\, \epsilon^{-j\beta_{n,m}z} \cos \omega t \tag{12-59}$$

$$\mathcal{E}_z = 0 \tag{12-60}$$

The values for $\beta_{n,m}$, $\alpha_{n,m}$, f_c, v_p, and v_g are the same as those obtained for the TM mode except for the use of $\tau'_{n,m}$ in place of $\tau_{n,m}$.

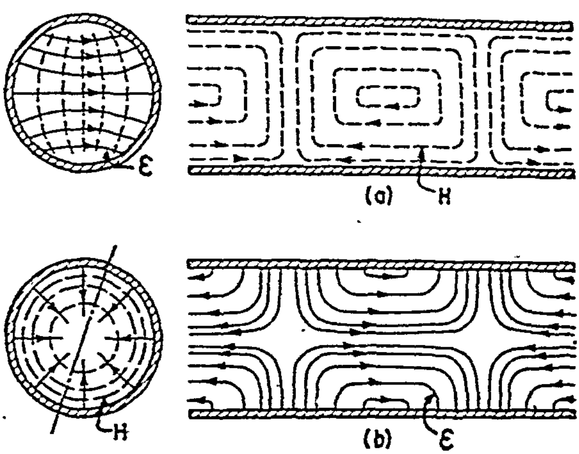


Fig. 12-10. (a) The $TE_{1,1}$ field in the cylindrical guide; (b) the $TM_{0,1}$ field in the cylindrical guide.

The derivative J_n' with respect to r of the Bessel function may be obtained from

$$\frac{\partial J_n(kr)}{\partial r} = J_n'(kr) = \frac{n}{kr} J_n(kr) - J_{n+1}(kr)$$

and a few roots of $J_n' = 0$ are given in Table 11.

TABLE 11

VALUES OF $J_n' = 0$

m	$n = 0$	$n = 1$	$n = 2$
1	3.83	1.84	3.05
2	7.02	5.33	6.71
3	10.17	8.54	9.97

Since there is no root $r_{0,0}$ of the Bessel function J_0, $TM_{0,0}$ waves cannot exist. Waves of order $TM_{1,0}$ and $TE_{1,0}$ are possible, as well as higher orders. The field patterns for several modes are shown in Fig. 12-10.

12-5. The TEM wave in the coaxial line

The TEM mode (sometimes called the *principal mode*) is the approximate form of propagation encountered on parallel-wire and coaxial lines at low frequencies. Actually, it exists only in theory on the dissipationless form of those lines; but since the losses are small, the true condition closely approximates the TEM field.

To analyze the TEM wave in the coaxial line, it is desirable to start with all fields varying in time and propagating in the z direction according to

$$H = \hat{H}\epsilon^{-\gamma z} \sin \omega t$$

$$\mathcal{E} = \hat{\mathcal{E}}\epsilon^{-\gamma z} \sin \omega t$$

The TEM mode is a special case of the TM mode with $\mathcal{E}_z = H_z = 0$. It may also be established that $\mathcal{E}_\phi = H_r = 0$. Under these conditions, Eqs. 12-41 and 12-42 reduce to

$$\gamma \hat{H}_\phi = j\omega\epsilon_1\mathcal{E}_r \tag{12-61}$$

$$\frac{\partial}{\partial r}(r\hat{H}_\phi) = 0 \tag{12-62}$$

$$\gamma\hat{\mathcal{E}}_r = j\omega\mu_1\hat{H}_\phi \tag{12-63}$$

$$\frac{\partial\hat{\mathcal{E}}_r}{\partial\phi} = 0 \tag{12-64}$$

and it is then possible to write

$$\hat{H}_\phi = \frac{j\omega\epsilon_1}{\gamma}\hat{\mathcal{E}}_r = \frac{\gamma}{j\omega\mu_1}\hat{\mathcal{E}}_r$$

indicating that

$$\gamma^2 = -\omega^2\mu_1\epsilon_1$$

or that

$$\gamma = j\beta = j\omega\sqrt{\mu_1\epsilon_1} \tag{12-65}$$

which is identical with the propagation constant obtained for the dissipationless line from transmission-line theory. It shows that under the special TEM conditions, propagation will occur in the z direction with $\alpha = 0$.

The cutoff frequency is found by setting β equal to zero. Such an operation indicates that

$$f_c = 0$$

or that all frequencies are propagated on the coaxial line by the TEM mode.

The Poynting radiation vector resulting from $\hat{\mathcal{E}}_r$ and $\hat{H}_\phi$ also shows that energy will propagate along the positive-z axis. The electric field is totally radial and the magnetic field is concentric to the inner conductor. Equation 12-64 shows that $\mathcal{E}_r$ does not vary with ϕ, or that circular symmetry of radial electric field exists.

Equation 12-62 indicates that

$$H_\phi = \frac{k}{r}$$

where k is not a function of r, and thus the H_ϕ field is an inverse function of r between the conductors. This is exactly the situation to be expected from the static field configuration in the coaxial cable. However,

$$\oint H \cdot dl = i$$

by Ampere's law, where i is the instantaneous current enclosed by the path and H_ϕ is the corresponding component instantaneous magnetic field. If the circumference of the inner conductor of radius a is chosen as a path, then if I_0 is the value of instantaneous current in the inner conductor,

$$2\pi a H_\phi = I_0$$

Evaluating k at $r = a$,

$$k = \frac{I_0}{2\pi} \tag{12-66}$$

and thus the magnetic field intensity is expressible in terms of the inner conductor current as

$$H_\phi = \frac{I_0}{2\pi r} \tag{12-67}$$

which is certainly true by reason of the definition of H, the field intensity along a path of radius r.

From Eqs. 12-63 and 12-65,

$$\mathcal{E}_r = \sqrt{\frac{\mu_1}{\epsilon_1}}\, H_\phi$$

so that

$$\mathcal{E}_r = \eta_1 \frac{I_0}{2\pi r} \tag{12-68}$$

showing the electric field also to be inversely proportional to r, as expected from the cylindrical symmetry of the situation.

The voltage *drop* from the center conductor to the outer conductor of the coaxial line may be written in terms of the maximum value $\hat{V}$ as

$$\hat{V} = \int_a^b \hat{\mathcal{E}}_r \, dr = \eta_1 \frac{\hat{I}_0}{2\pi} \ln \frac{b}{a}$$

The characteristic impedance of such an infinite length line is the ratio of V/I by circuit methods, so that

$$Z_0 = \frac{1}{2\pi} \eta_1 \ln \frac{b}{a} \text{ ohms} \tag{12-69}$$

which, for a dielectric of relative permittivity ϵ_r between the conductors, is

$$Z_0 = \frac{60}{\sqrt{\epsilon_r}} \ln \frac{b}{a} \text{ ohms}$$

and this checks the value obtained through circuit considerations in Eq. 7-21.

Inserting the z-propagation and time functions gives the instantaneous voltage as

$$V = \eta_1 \frac{\hat{I}_0 \ln (b/a)}{2\pi} \epsilon^{-j\beta z} \sin \omega t \tag{12-70}$$

Taking the z derivative of the voltage,

$$\begin{aligned} \frac{\partial V}{\partial z} &= -j\beta\eta_1\hat{I}_0 \ln \frac{b}{a} \epsilon^{-j\beta z} \sin \omega t \\ &= -\frac{j\omega\mu_1}{2\pi} I_0 \ln \frac{b}{a} \end{aligned} \tag{12-71}$$

in terms of the effective current and voltage. Since $I_0 = \hat{I}_0\epsilon^{-j\beta z}$ $\sin \omega t$,

$$\begin{aligned} \frac{\partial I_0}{\partial z} &= -j\beta\hat{I}_0\epsilon^{-j\beta z} \sin \omega t \\ &= -\frac{j2\pi\omega\epsilon_1}{\ln (b/a)} V \end{aligned} \tag{12-72}$$

It has been previously shown that

$$L = \frac{\mu}{2\pi} \ln \frac{b}{a} \qquad C = \frac{2\pi\epsilon}{\ln (b/a)}$$

in Chapter 7 for the high-frequency line, so that Eqs. 12-71 and 12-72 become

$$\frac{\partial V}{\partial z} = -j\omega LI \tag{12-73}$$

$$\frac{\partial I}{\partial z} = -j\omega CV \tag{12-74}$$

which, obtained from field theory, are identical with the transmission-line equations as previously obtained for a dissipationless line by use of circuit parameters and current and voltage concepts. This analysis then serves to show once more that circuit concepts and field concepts are really one and the same thing.

In addition to the TEM wave in the coaxial line, it is possible for higher-order forms of TM and TE waves to exist with components of electric or magnetic field in the direction of the line axis. However, for the usual coaxial lines the dimensions are small enough that the lines are operating at frequencies far below cutoff for these modes, and these modes will not be considered here.

12-6. Attenuation in the coaxial line

For the TEM mode in the coaxial line, the tangential magnetic field is

$$H_\phi = \frac{\hat{I}_0}{2\pi r} \epsilon^{-j\beta z} \sin \omega t$$

from the preceding section. Again it may be assumed that the currents flowing in conductors of finite conductivity are essentially the same as those for the perfect-conductor case, or that the losses are small and the fields and currents are affected only slightly thereby. At some particular point of observation along the z axis, the average power entering through the surface of the inner conductor of the coaxial cable of radius a, due to the tangential magnetic field H evaluated at $r = a$, is by Eqs. 12-67 and 12-68,

$$P_{M1} = \frac{1}{2}\left(\frac{\hat{I}_0}{2\pi a}\right)^2 \sqrt{\frac{\omega\mu_m}{2\sigma_m}} \text{ watts/m}^2$$

The area of surface of inner conductor is $2\pi a$ square meters per meter of length, so that the power supplied to the inner conductor per meter of length is

$$P_{M1} = \frac{1}{2} \frac{\hat{I}_0^2}{2\pi a} \sqrt{\frac{\omega \mu_m}{2\sigma_m}} \tag{12-75}$$

The power supplied to the outer conductor of radius b per meter of length is

$$P_{M2} = \frac{1}{2} \frac{\hat{I}_0^2}{2\pi b} \sqrt{\frac{\omega \mu_m}{2\sigma_m}} \tag{12-76}$$

The total power loss into the conductors, per meter of line, is

$$P_M = \frac{1}{4\pi} \hat{I}_0^2 \sqrt{\frac{\omega \mu_m}{2\sigma_m}} \left(\frac{1}{a} + \frac{1}{b}\right) \text{ watts} \tag{12-77}$$

The power transmitted past a given point in the guide is readily obtained by integrating the Poynting radiation vector or product of the $\mathcal{E}_r$ and H_ϕ fields over the space between conductors and is

$$P_T = \int_0^{2\pi} \int_a^b \frac{1}{2} \hat{\mathcal{E}}_r \hat{H}_\phi r \, d\phi \, dr$$

By use of the field relations of the preceding section, the above may be written

$$P_T = \frac{\hat{I}_0^2}{4\pi} \eta_1 \ln \frac{b}{a} \text{ watts} \tag{12-78}$$

It has already been shown that the attenuation is given by

$$\alpha = \frac{1}{2} \frac{P_M}{P_T}$$

so that for the coaxial line, under approximate TEM conditions, the attenuation due to side-wall, or conductor, losses is

$$\alpha = \frac{\sqrt{\pi \mu_m / \sigma_m}\,(1/a + 1/b)}{2\eta_1 \ln (b/a)} \sqrt{f} \quad \text{nepers/m} \tag{12-79}$$

showing that this attenuation is proportional to the square root of frequency.

Equation 12-77, for the total power conveyed into the metal by the fields, may be written for a line with copper walls as

$$P_M = \frac{\hat{I}_0^2}{2} \left[4.16 \times 10^{-8} \sqrt{f} \left(\frac{1}{a} + \frac{1}{b}\right)\right] \text{watts}$$

The term in brackets is the resistance expression for a coaxial line as previously obtained in Section 7-2. Since $\hat{I}_0$ is the maximum value of current flowing in the conductors, it can be seen that the power delivered from the field to the side walls is exactly the amount of power which is dissipated as I^2R losses in the conductor. It has previously been shown that the power transmitted along a line to the load is conveyed in the fields; and now it is evident that as the power is transmitted along the line, the fields supply power to the conductors, this power then appearing as the heat losses in the conductors.

12-7. Attenuation in guides due to imperfect conductors

For the TM mode in the rectangular guide the H_x field is the tangential component on the sides of the guide parallel to the x,z plane, or of length a, and the H_y component is tangential on the side parallel to the y,z plane, or of length b. For the TM$_{m,n}$ mode in the rectangular guide, the maximum values of these fields at $y = 0$ or $y = b$ and at $x = 0$ or $x = a$, respectively, are

$$\hat{H}_x = \frac{n\pi}{b} \frac{\omega\epsilon_1}{\omega_c^2\mu_1\epsilon_1} \mathcal{E}_0 \sin\left(\frac{m\pi x}{a}\right) \epsilon^{-j\beta_{m,n}z} \tag{12-80}$$

$$\hat{H}_y = -\frac{m\pi}{a} \frac{\omega\epsilon_1}{\omega_c^2\mu_1\epsilon_1} \mathcal{E}_0 \sin\left(\frac{n\pi y}{b}\right) \epsilon^{-j\beta_{m,n}z} \tag{12-81}$$

As previously stated, the power conveyed into a metal of finite conductivity by an electromagnetic field is

$$P_M = \frac{1}{2}\hat{H}^2\sqrt{\frac{\omega\mu_m}{2\sigma_m}} \text{ watts/m}^2$$

Assuming that the rectangular guide walls are conductors of finite conductivity and that since the metal losses are small, the fields are approximately the same as for the case of perfect conductivity, an assumption supported by the inequality $\sigma_m \gg \omega\epsilon_m$, then the power delivered to the guide walls by the above TM$_{m,n}$ fields is

$$P_M = \sqrt{\frac{\omega\mu_m}{2\sigma_m}}\left(\int_0^a H_x^2\,dx + \int_0^b H_y^2\,dy\right) \text{ watts/meter of length}$$

the $\frac{1}{2}$ factor being removed by the fact that there are two faces acted upon by each field.

Substituting for the fields gives

$$P_M = \sqrt{\frac{\omega\mu_m}{2\sigma_m}} \left(\frac{\omega\epsilon_1}{\omega_c^2\mu_1\epsilon_1} \mathcal{E}_0\right)^2 \left[\left(\frac{n\pi}{b}\right)^2 \int_0^a \sin^2\left(\frac{m\pi x}{a}\right) dx + \left(\frac{m\pi}{a}\right)^2 \int_0^b \sin^2\left(\frac{n\pi y}{b}\right) dy\right]$$

$$= \sqrt{\frac{\omega\mu_m}{2\sigma_m}} \frac{v_1^2 f^2 \mathcal{E}_0^2}{8\eta_1^2 f_c^4} \left(\frac{n^2 a}{b^2} + \frac{m^2 b}{a^2}\right) \tag{12-82}$$

This is the loss in watts per meter of length of guide.

The power transmitted along the guide may be considered as due to two waves traveling in the z direction simultaneously. One wave is composed of the $\mathcal{E}_x$ and H_y fields, the other of the $-\mathcal{E}_y$ and H_x fields. Consequently, the power passing any point in the guide is

$$P = \frac{1}{2} (\hat{\mathcal{E}}_x \hat{H}_y + \hat{\mathcal{E}}_y \hat{H}_x)$$

and the total power transmitted may be obtained by integration of this result over the cross-sectional area of the guide. With the field values for the $\mathrm{TM}_{m,n}$ wave from Eqs. 12-14 to 12-19, the power transmitted past a given point by each of the two component waves is

$$P_1 = \frac{1}{2} \left(\frac{m\pi\mathcal{E}_0}{a\omega_c^2\mu_1\epsilon_1}\right)^2 \omega\epsilon_1\beta_{m,n} \int_0^a \int_0^b \cos^2\left(\frac{m\pi x}{a}\right) \sin^2\left(\frac{n\pi y}{b}\right) dx\, dy$$

$$= \frac{1}{2} \left(\frac{m\pi\mathcal{E}_0}{a\omega_c^2\mu_1\epsilon_1}\right)^2 \omega\epsilon_1\beta_{m,n} \frac{ab}{4} \tag{12-83}$$

$$P_2 = \frac{1}{2} \left(\frac{n\pi\mathcal{E}_0}{b\omega_c^2\mu_1\epsilon_1}\right)^2 \omega\epsilon_1\beta_{m,n} \int_0^a \int_0^b \sin^2\left(\frac{m\pi x}{a}\right) \cos^2\left(\frac{n\pi y}{b}\right) dx\, dy$$

$$= \frac{1}{2} \left(\frac{n\pi\mathcal{E}_0}{b\omega_c^2\mu_1\epsilon_1}\right)^2 \omega\epsilon_1\beta_{m,n} \frac{ab}{4} \tag{12-84}$$

The total power transmitted through the guide being the sum of P_1 and P_2, it may be written, after insertion of values for $\beta_{m,n}$ and f_c, as

$$P_T = \frac{ab\mathcal{E}_0^2}{8\eta_1} \left(\frac{f}{f_c}\right)^2 \sqrt{1 - \frac{f_c^2}{f^2}} \text{ watts} \tag{12-85}$$

this being the value for the $\mathrm{TM}_{m,n}$ mode in the *rectangular guide.*

Similar methods give the following value for the TE mode in the *rectangular guide:*

$$P_T = \frac{ab\eta_1 H_0^2}{8}\left(\frac{f}{f_c}\right)^2\sqrt{1-\frac{f_c^2}{f^2}} \text{ watts} \tag{12-86}$$

Likewise for the TM mode, *cylindrical guide:*

$$P_T = \frac{\pi b^2 \mathcal{E}_0^2}{4\eta_1}[J_n'(\tau_{m,n})]^2\left(\frac{f}{f_c}\right)^2\sqrt{1-\frac{f_c^2}{f^2}} \text{ watts} \tag{12-87}$$

For the TE mode, *cylindrical guide:*

$$P_T = \frac{\pi b^2 \eta_1 H_0^2}{4}[J_n(\tau_{m,n})]^2\left(1-\frac{n^2}{\tau_{m,n}^2}\right)\left(\frac{f}{f_c}\right)^2\sqrt{1-\frac{f_c^2}{f^2}} \text{ watts} \tag{12-88}$$

Since the attenuation is given by

$$\alpha = \frac{1}{2}\frac{P_M}{P_T}$$

then the attenuation for the TM mode in the *rectangular guide* with side walls of finite conductivity is

$$\alpha_{TM} = \sqrt{\frac{\pi\mu_m}{\sigma_m}}\frac{v_1^2\sqrt{f}}{2\eta_1 f_c^2}\frac{(n^2/b^3+m^2/a^3)}{\sqrt{1-f_c^2/f^2}} \text{ nepers/m} \tag{12-89}$$

By the same methods, the attenuations for other forms of guide are as follows: Rectangular guide, $TE_{0,n}$ mode:

$$\alpha_{TE(0,n)} = \sqrt{\frac{\pi\mu_m}{\sigma_m}}\frac{\sqrt{f}}{a\eta_1\sqrt{1-f_c^2/f^2}}\left[1+\frac{2a}{b}\left(\frac{f_c}{f}\right)^2\right] \text{ nepers/m} \tag{12-90}$$

Rectangular guide, $TE_{m,n}$ mode, $(m \neq 0)$:

$$\alpha_{TEm,n} = \sqrt{\frac{\pi\mu_m}{\sigma_m}}\frac{2\sqrt{f}}{b\eta_1\sqrt{1-f_c^2/f^2}}$$

$$\left\{\left[1-\left(\frac{f_c}{f}\right)^2\right]\frac{bm^2/an^2+1}{bm^2/an^2+a/b}+\left(1+\frac{b}{a}\right)\left(\frac{f_c}{f}\right)^2\right\} \text{nepers/m} \tag{12-91}$$

Cylindrical guide, $TE_{n,m}$ mode:

$$\alpha_{TE} = \sqrt{\frac{\pi\mu_m}{\sigma_m}}\frac{\sqrt{f}}{b\eta_1\sqrt{1-f_c^2/^2f}}\left[\left(\frac{f_c}{f}\right)^2+\frac{n^2}{\tau_{m,n}^2-n^2}\right] \text{ nepers/m} \tag{12-92}$$

Cylindrical guide, $TM_{n,m}$ mode:

$$\alpha_{TM} = \sqrt{\frac{\pi\mu_m}{\sigma_m}} \frac{\sqrt{f}}{b\eta_1 \sqrt{1 - f_c^2/f^2}} \text{ nepers/m} \qquad (12\text{-}93)$$

From these expressions the attenuation due to imperfect conductors in the guide walls can be readily found. Attenuation due to

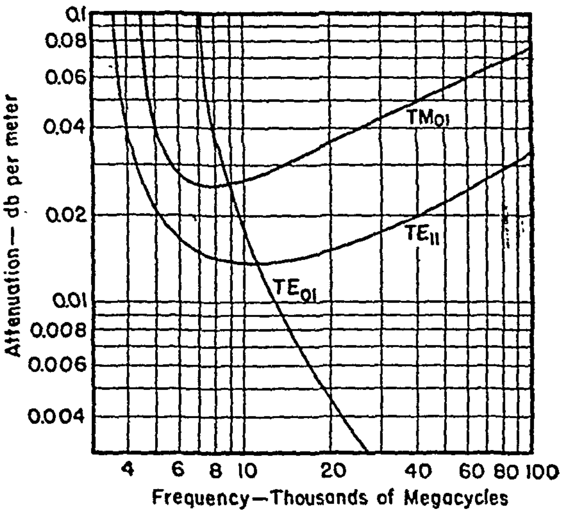


Fig. 12-11. Variation in attenuation of various modes in a cylindrical copper guide of 12.7 cm diameter.

imperfect dielectrics inside the guide may also occur, although most guides operate with gaseous dielectric of zero conductivity.

The values of attenuation for the $TE_{0,1}$, $TM_{0,1}$, and $TE_{1,1}$ waves in cylindrical metal guides are plotted in Fig. 12-11 for a guide diameter of 12.7 cm. It can be seen here, and also from Eq. 12-92 that the attenuation of the $TE_{0,1}$ wave decreases indefinitely with increasing frequency. This statement is true only if the guide cross section is perfectly circular, a very difficult situation to maintain.

12-8. Excitation of wave guides

The form or mode of propagation is determined by the type and location of the exciting device. Although either loops (magnetic) or rods (electric) may be used as excitation sources, the rod antenna is normally preferred because of its simplicity. If a guide is closed at one end by a conducting wall and an appropriate exciting antenna rod is inserted through the end or side of the guide, as in Fig. 12-12, the end of the guide serves as a reflector, and if the distance between rod and end wall is properly adjusted, the reflected wave arrives at

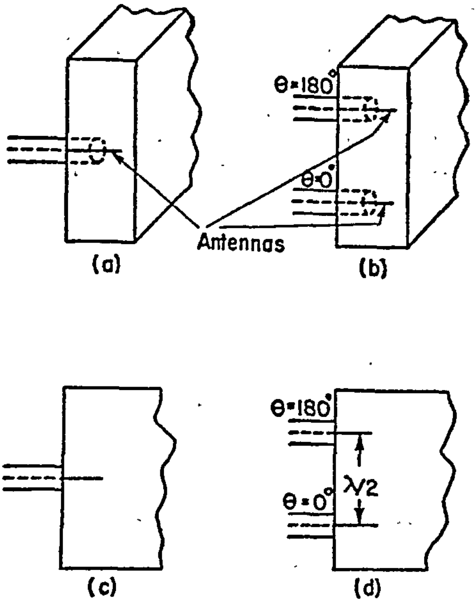


Fig. 12-12. Methods of exciting various modes by setting up the appropriate electric fields: (a) $TE_{1,0}$; (b) $TE_{2,0}$; (c) $TM_{1,1}$; (d) $TM_{2,1}$.

the antenna in phase with the emitted wave, and the two propagate down the guide as one wave.

It can be seen that the rods should coincide with the positions of maximum electric intensity in the fields that they are intended to excite, with attention being given to the proper phasing of the potentials supplied to the rods in accordance with the phasing of the fields to be excited. If current loops are introduced for excitation, the plane of the loop will be made normal to the magnetic field and the loop will be located at a point of maximum magnetic field intensity.

These sources will not usually excite just the waves desired but will also excite higher-order modes. By choice of guide dimensions it is possible to have only the desired wave above cutoff frequency, the other waves then being attenuated and not propagated. A set of customary guide dimensions is given in Table 12.

TABLE 12

USUAL WAVE-GUIDE DIMENSIONS

Dimensions, in.	λ_c, cm	Useful range, λ-cm	Attenuation—brass walls, db per meter
1.5 × 3.0	14.4	7.6–11.8	0.038 at 10 cm λ
1.0 × 2.0	9.5	5.0– 7.6	0.068 at 6 cm λ
0.75 × 1.5	6.97	3.7– 5.7	0.118 at 5 cm λ
0.625 × 1.25	5.7	3.0– 4.7	0.164 at 3.6 cm λ
0.5 × 1.0	4.57	2.4– 3.7	0.25 at 3.2 cm λ

12-9. Guide terminations

A wave guide is a form of transmission line and must be properly terminated to avoid reflection losses. The termination should provide a wave impedance equal to that of the transmission mode in the guide. Since loads are not always of suitable values, various forms of couplings have been developed for transformation of the impedances to the desired values.

One of the most common forms of loads that are coupled to guides is free space. That is, it is desired that the energy being conveyed by the guide be radiated or transmitted as a space wave, usually in a narrow, well-defined beam. This requirement involves coupling the guide impedance to a wave impedance of 377 ohms and is usually done by expanding the area of the guide in an appropriate direction. The terminating transformer section thus approximates an acoustical horn and in properties is similar to the exponential line. At the throat of the horn the impedance approximates that of the guide, whereas at the mouth the wave impedance is that of free space. If the horn is several wavelengths long, good transformation without reflection is obtained.

Dissipative loads or nonreflecting terminations are also available. They again employ the principle of the tapered or exponential line and use dielectrics having considerable conductivity to provide the power-absorbing properties. The simplest form is that of a wave guide entering a tank of water at a small angle, the guide then being partially filled with a wedge of water. The wedge shape provides the approximate exponential taper, and the water furnishes dielectric loss and a ready means for conveying away the generated heat.

Materials consisting of graphite-bearing porcelains, or graphite- or metallic-bearing plastics, are also available in wedge shapes to replace the water as other types of power-dissipating or nonreflecting loads.

12-10. Resonant cavities

If an excited wave guide is closed by a perfectly conducting sheet at some point, a standing-wave pattern will be set up along the axis of propagation. Since the tangential electric intensity must be zero at the end-wall closure, it will also be zero at half wavelength points back along the guide, wavelength being measured along the guide wall and thus in terms of phase velocity. Since the electric field is zero at any half-wave point, a perfectly conducting plane may be inserted across the guide at any such point without changing the field distribution present between the plane and the end wall, the exciting antenna being considered as present between the shorting planes. A volume or cavity is then formed and is discovered to have resonant properties similar to those of the parallel-resonant circuit, since only definite frequencies can produce the necessary standing-wave field conditions. The *dominant mode* is that field configuration having the lowest resonant frequency. Since such a cavity has other dimensions also terminated in conducting planes, there is a possibility of resonance at other frequencies, depending on excitation and orientation of the cavity.

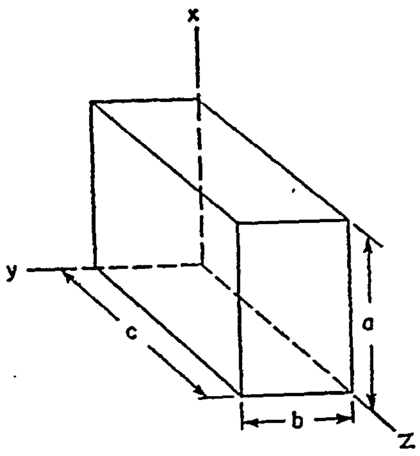


Fig. 12-13. Dimensions of the resonant cavity.

Assume that initially there existed in the rectangular cavity of Fig. 12-13 a $TE_{m,n}$ field propagating in the z direction. The cavity has walls that are perfect conductors and is filled with a good dielectric. The initial electric fields would be

$$\hat{\mathcal{E}}_x = \frac{n\pi}{b}\frac{\omega\mu_1}{\omega_c^2\mu_1\epsilon_1} H_0 \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \epsilon^{-j\beta_{m,n}z} \tag{12-94}$$

$$\hat{\mathcal{E}}_y = -\frac{m\pi}{a}\frac{\omega\mu_1}{\omega_c^2\mu_1\epsilon_1} H_0 \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \epsilon^{-j\beta_{m,n}z} \tag{12-95}$$

$$\hat{\mathcal{E}}_z = 0 \tag{12-96}$$

Because of reflections at the walls after the guide is closed off to become a cavity, the field components present will consist of incident and reflected waves, and

$$\mathcal{E}_{xs} = (1 + K)\mathcal{E}_x$$
$$\mathcal{E}_{ys} = (1 + K)\mathcal{E}_y$$

For reflection from perfect conductors, $K = -1$ so that, after considering the reversed sign in the exponential $\epsilon^{-j\beta z}$ for the reflected wave, the standing-wave fields $\mathcal{E}_{xs}$ and $\mathcal{E}_{ys}$ are

$$\hat{\mathcal{E}}_{xs} = \frac{n\pi}{b}\frac{\omega\mu_1}{\omega_c^2\mu_1\epsilon_1} H_0 \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) [\epsilon^{-j\beta_{m,n}z} - \epsilon^{j\beta_{m,n}z}] \tag{12-97}$$

$$\hat{\mathcal{E}}_{ys} = -\frac{m\pi}{a}\frac{\omega\mu_1}{\omega_c^2\mu_1\epsilon_1} H_0 \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) [\epsilon^{-j\beta_{m,n}z} - \epsilon^{j\beta_{m,n}z}] \tag{12-98}$$

If part of the coefficient is collected as H_0', then

$$\mathcal{E}_{xs} = \frac{n\pi}{b} H_0' \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \sin(\beta_{m,n}z) \sin \omega t \tag{12-99}$$

$$\mathcal{E}_{ys} = -\frac{m\pi}{a} H_0' \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \sin(\beta_{m,n}z) \sin \omega t \tag{12-100}$$

$$\mathcal{E}_{zs} = 0$$

At $z = 0$ and $z = c$, the boundary conditions require that $\mathcal{E}_y \doteq 0$. To satisfy this condition it is necessary that

$$\beta_{m,n} = \frac{p\pi}{c}$$

where p is an integer and c is the dimension of the cavity in the z direction.

Maxwell's field equations, as given in Eqs. 12-1 and 12-2, give the wave equation in terms of $\mathcal{E}_z$, or any other component. In terms of $\hat{\mathcal{E}}_z$ this might be

$$\frac{\partial^2\hat{\mathcal{E}}_z}{\partial x^2} + \frac{\partial^2\hat{\mathcal{E}}_z}{\partial y^2} + \frac{\partial^2\hat{\mathcal{E}}_z}{\partial z^2} = -(\gamma^2 + \omega^2\mu_1\epsilon_1)\hat{\mathcal{E}}_z \qquad (12\text{-}101)$$

Use of the $\mathcal{E}_{zs}$ value in the cavity (Eq. 12-99) in this form of the wave equation gives as a requirement for satisfaction of the wave equation, and in turn for the existence of the fields, that

$$\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{c}\right)^2 = \gamma^2 + \omega^2\mu_1\epsilon_1$$

It is then seen that for the fields to exist, the value of γ must be given by

$$\gamma = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{c}\right)^2 - \omega^2\mu_1\epsilon_1} \qquad (12\text{-}102)$$

which is closely related to the value for the wave guide. For the fields to propagate in the cavity, γ must be imaginary, or

$$\gamma_{m,n,p} = j\beta_{m,n,p} = j\sqrt{\omega^2\mu_1\epsilon_1 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2 - \left(\frac{p\pi}{c}\right)^2}$$

The resonant frequency f_r of the cavity is that frequency at which β is zero, so from the above radical,

$$f_r = \frac{v_1}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{c}\right)^2} \qquad (12\text{-}103)$$

This is analogous to the cutoff frequency in the wave guide.

Use of Maxwell's equations permits the accompanying magnetic field components for the TE mode in the resonant rectangular cavity to be written as

$$H_{xs} = -\frac{m\pi}{a}\frac{p\pi}{c}\frac{H_0'}{\omega\mu_1}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\cos\left(\frac{p\pi z}{c}\right)\cos\omega t \qquad (12\text{-}104)$$

$$H_{ys} = \frac{n\pi}{b}\frac{p\pi}{c}\frac{H_0'}{\omega\mu_1}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\cos\left(\frac{p\pi z}{c}\right)\cos\omega t \qquad (12\text{-}105)$$

$$H_{zs} = -\frac{H_0'}{\omega\mu_1}\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\sin\left(\frac{p\pi z}{c}\right)\cos\omega t \qquad (12\text{-}106)$$

Any one of the integers m, n, or p may be zero but not two, since in that case all the fields would go to zero. Because of the form of the resonant-frequency expression, a wide variety of resonant frequencies are possible, depending on the choice of integers. For a cubical resonator with all dimensions equal, the resonant wavelength is equal to a diagonal of one of the cube faces for the $TE_{0,1,1}$ mode.

Exactly similar expressions for γ, and f_r can be obtained for the TM mode of oscillation in the cavity. When field expressions are written it is found that any zero subscript mode is impossible.

Since the cavity behaves and is employed as a resonant circuit or tuning element, its frequency selectivity is of interest and may be evaluated in terms of its Q at resonance. The Q value was originally defined according to

$$Q = 2\pi \times \frac{\text{energy stored per cycle}}{\text{energy lost per cycle}}$$

The energy loss is due to finite conductivity of the cavity walls and is therefore proportional to the area of the walls. The energy stored is considered as scattered through the cavity volume and consequently is proportional to that volume. If it be assumed that the tangential magnetic field at the walls is twice the average value of magnetic-field intensity in the cavity volume, it is possible to obtain an approximate and simple expression for the Q of a rectangular cavity as

$$\begin{aligned} Q &= \frac{\pi f \mu_1 \times \text{volume}}{\sqrt{\pi f \mu_m / \sigma_m} \times \text{area}} \\ &= \sqrt{\frac{\pi f \sigma_m}{\mu_m}}\, \frac{\mu_1 \times \text{volume}}{\text{area}} \end{aligned} \tag{12-107}$$

Cylindrical cavities are frequently used because of the ease with which they may be manufactured, and their ease of adjustment to resonance by a sliding circular piston. The resonant frequency of such a cavity is given by

$$f_r = \frac{v_1}{2\pi} \sqrt{\left(\frac{p\pi}{c}\right)^2 + \frac{\tau^2}{b^2}} \tag{12-108}$$

where τ means any of the roots of $J_n(b\sqrt{\gamma^2 + \omega^2 \mu_1 \epsilon_1}) = 0$ for TM modes and of $J_n'(b\sqrt{\gamma^2 + \omega^2 \mu_1 \epsilon_1})$ for TE modes. The mode designations become $TE_{m,n,p}$ and $TM_{m,n,p}$. The $TM_{1,0,0}$ mode is the simplest of the TM type, whereas $TE_{1,0,1}$ is the simplest for the TE modes, since p cannot be zero.

The Q of a cylindrical resonator in the $TM_{n,0,0}$ mode is

$$Q = \mu_1 \eta_1 \sqrt{\frac{\pi f_r \sigma_m}{\mu_m}} \left(\frac{bc}{b + c} \right) \tag{12-109}$$

By careful design and plating of internal surfaces with silver or gold, it is possible to obtain Q values up to 100,000, far above those obtained by coils at the lower frequencies or by lines at intermediate frequencies. Tuning means may be supplied by the use of sliding pistons in one end of the cavity, provided good spring contact is made between the end and side walls at all points of the periphery

PROBLEMS

12-1. (a) Find the cutoff frequencies for a $TE_{1,0}$ wave in air in a rectangular guide measuring 5 cm by 2.5 cm.

(b) Calculate the phase and group velocities for the above waves at a frequency of 6000 megacycles.

(c) Find the attenuation to be expected for a frequency at 0.95 times the cutoff value. Neglect conductivity.

12-2. (a) Find the attenuation of a 900-megacycle wave if applied to a square guide with $f_c = 1000$ megacycles, $TE_{1,0}$ mode, air dielectric.

(b) What would be the dimensions of such a guide?

12-3. A wave guide is 8 cm on a side. Tabulate all modes which may be propagated at frequencies of 1500, 3000, and 6000 megacycles.

12-4. Find the characteristic wave impedance for the TM and TE modes in the cylindrical guide.

12-5. In a coaxial line of dimensions $a = 0.5$ cm, $b = 3$ cm, the radial electric field in TEM transmission is equal to 1000 v per centimeter at the surface of the inner conductor. If the dielectric is air, find the magnetic flux density at the surface of the inner conductor, with $f = 10$ megacycles.

(a) Compute the voltage across the line, the conductor currents and the power being transmitted.

(b) By integration of the value of $\mathcal{E} \times H$, or Poynting's vector, over the cross-sectional area of the line, check the value of power transmitted as obtained in (a), above.

12-6. (a) If the line of Prob. 12-5 is made of copper, find the power loss per meter when transmitting the field intensity $\mathcal{E}$ of that problem, with a frequency of 1.4×10^8 c.

(b) Find the attenuation in decibels per meter.

(c) If the line is silver-plated to a depth considerably greater than the depth of penetration, compute the attenuation in decibels per meter.

12-7. Prove that Eq. 12-86 for power transmitted by the TE mode, in a rectangular guide, is correct.

12-8. Compute the power loss for the $TE_{m,0}$ mode in a rectangular guide with walls of finite conductivity.

12-9. (a) A square guide 10 cm on a side is transmitting a 5000-megacycle signal. If it is of silver, air-filled, find the attenuation in decibels per 100 m of guide.

(b) Repeat, if of copper.

(c) Repeat for iron of relative permeability = 100.

12-10. A cylindrical copper tube of inside diameter 3 cm is air-filled. Calculate the cutoff frequencies in the $TE_{1,0}$, $TM_{1,0}$, $TE_{1,1}$, and $TM_{2,1}$ modes.

12-11. If a frequency of 10^{10} c is transmitted in the tube of Prob. 12-10 in the $TE_{1,1}$ mode calculate the attenuation in decibels per 100 ft.

12-12. A metal box is $3 \times 4 \times 5$ cm in dimensions. If it is filled with air, find the resonant frequencies for $TE_{1,0,2}$ fields in the various possible directions.

12-13. Show that for all $TM_{m,n}$ waves in a rectangular guide the minimum attenuation arising from imperfect side-wall conductors occurs at a frequency $f = \sqrt{3}f_c$.

REFERENCES

1. *Standards on Radio Wave Propagation: Definitions of Terms Relating To Guided Waves*, Institute of Radio Engineers, New York, 1945.

2. Sarbacher, R. I., and Edson, W. A., *Hyper- and Ultra-high Frequency Engineering*, John Wiley & Sons, Inc., New York, 1943.

3. Slater, J. C., *Microwave Transmission*, McGraw-Hill Book Company, Inc., New York, 1942.

4. Bronwell, A. B., and Beam, R. E., *Theory and Application of Microwaves*, McGraw-Hill Book Company, Inc., New York, 1947.

5. Ramo, S., and Whinnery, J. R., *Fields and Waves in Modern Radio*, 2d ed., John Wiley & Sons, Inc., New York, 1953.

6. Barrow, W. L., "Transmission of Electromagnetic Waves in Hollow Tubes of Metal," *Proc. I.R.E.*, 24, 1298–1328 (1936).

7. Southworth, G. C., "Hyper-Frequency Wave Guides—General Considerations and Experimental Results," *Bell System Tech. J.*, 15, 284–309 (1936).

8. Chu, L. J., and Barrow, W. L., "Electromagnetic Waves in Hollow Metal Tubes of Rectangular Cross Section," *Proc. I.R.E.*, 26, 1520–1555 (1938).

Chapter 13

RADIATION INTO SPACE

In a previous chapter it was shown that it was possible to set up a traveling electromagnetic wave carrying energy into space. This chapter will deal with a few of the practical means for coupling circuits with space, or the *antennas* for radiating such waves, and with methods of calculating the power radiated, the input impedance, the effect of the earth, and methods for controlling the direction in which the energy is radiated.

A collection of antennas is shown in Fig. 13-1, only a few of which will be discussed here.

13-1. Vector potential

In static electric fields it is relatively easy to determine the scalar potentials present, and to obtain the field about a configuration of charges by differentiating the potential, or taking the gradient. An analogous method is useful in finding the magnetic fields about currents, since in general it is easier to obtain the magnetic potentials from the known currents than it is to write the fields directly.

The static electric potential in a system of charges having density ρ is given by

$$\phi = \frac{1}{\epsilon} \int_v \frac{\rho \, dv}{4\pi r}$$

and by analogy a *magnetic vector potential* is assumed as

$$A = \mu \int_v \frac{J \, dv}{4\pi r} \tag{13-1}$$

where J is the current density throughout the volume dv. This expression is a vector because it depends on the orientation of the current density J.

In a thin filament of cross sectional area da,

$$dv = da \, dl$$

and

$$i \, dl = J \, dv$$

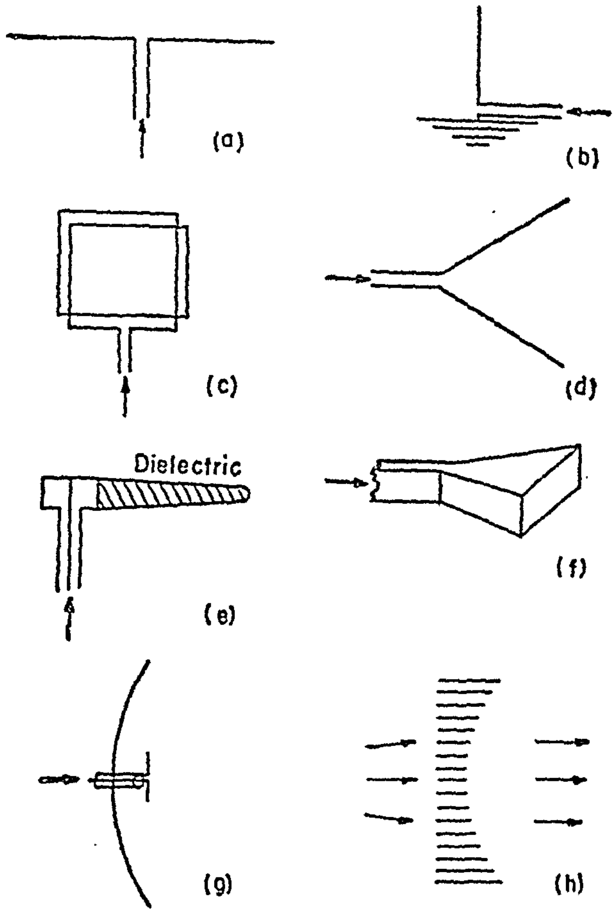


Fig. 13-1. (a) Dipole antenna; (b) grounded antenna; (c) loop; (d) vee antenna; (e) dielectric rod; (f) horn; (g) parabolic reflector; (h) metal lens.

where $i = J\,da$ is the total current in the thin filament. Then

$$A = \mu \int \frac{i\,dl}{4\pi r} \tag{13-2}$$

This magnetic vector potential is in the direction, or everywhere parallel to, the current-carrying element dl.

The potential A would obviously not serve its intended purpose if it failed to produce the field upon proper differentiation. That it will do so, and is therefore the desired form of potential function, can be shown by relating it to the magnetic field intensity as

$$\mu H = \nabla \times A \tag{13-3}$$

or
$$H = \frac{i}{4\pi}\int \nabla \times \left(\frac{dl}{r}\right) = \frac{i}{4\pi}\int a_r \times \frac{\partial}{\partial r}\left(\frac{dl}{r}\right) \tag{13-4}$$

where a_r is a unit vector along r from the current element dl to the point P at which the field intensity is computed. From this expression it is possible to write

$$dH = \frac{i\,dl\sin\theta}{4\pi r^2} \tag{13-5}$$

as the field magnitude due to the current element dl, where θ is the smallest angle between dl and the radius r.

The above expression, however, is Ampere's law for the field intensity near a current element. Thus the magnetic vector potential A has been shown to lead to the usual solution for the field, and the validity of the function A has been established.

13-2. Retarded vector potential

The integration of Eq. 13-2 implies the summing up of potentials due to various elements of current $i\,dl$. Let the current in the element be a time function

$$i = I_m \sin \omega t$$

and reconsider the meaning of Eq. 13-2. This equation expresses the superposition, at some remote point P, at distance r, of potentials due to different current elements $i\,dl$. If these are simply added up, as by the process that lead to Eq. 13-5, an assumption is made that these field effects which are superposed at time t all started from the current elements at the same value of current and time, even though they have traveled varying distances r. Obviously, this can be correct only if the velocity of propagation is infinite, whereas it has already been shown that the velocity of propagation is actually finite and that of light in the medium.

It is then necessary to introduce the idea of *retardation*, or that the effect reaching P from a given element dl at an instant t is due to a current value which flowed at an earlier time, or that the current effective in producing a field from a given element was that which flowed at an earlier time, this time being dependent on the distance travelled from dl to P. The current in the element must then be written

$$i = I_m \sin \omega \left(t - \frac{r}{c}\right)$$

where c is the velocity of propagation or of light in the medium.

The magnetic vector potential A then is called a *retarded potential*, since it is a potential that appears at a point in space later than the current which caused it; A is written

$$A_{(t-r/c)} = \mu \int \frac{i_{(t-r/c)}\, dl}{4\pi r} \tag{13-6}$$

wherein it is implied that i is a function of $(t - r/c)$. For sinusoidal element current the retarded vector potential is

$$A_{(t-r/c)} = \mu \int \frac{I_m \sin \omega(t - r/c)\, dl}{4\pi r}. \tag{13-7}$$

where the function of $(t - r/c)$ introduces the proper retardation of potential, dependent on the distance r from dl to P.

If r is small with respect to ct, there is no difference between the retarded and static field functions. This is the case close to a current, or when the changes in current take place very slowly. This point will be discussed further in Section 13-3.

13-3. Radiation from a current element

It is now possible to continue the use of the retarded vector potential in calculating the $\mathcal{E}$ and H fields surrounding the element of current $i\, dl$. If these dH fields be found, it then follows that finite-sized antennas may have their fields computed by integrating over all the current elements present. Thus the simple current element becomes the building block for larger systems. It is also the classical charge-separation dipole of electrostatic theory.

Figure 13-2 illustrates the element of current of length dl at the origin and in spherical coordinates, r, θ, and ϕ. For element current with sinusoidal time variation, the retarded vector potential at P is

$$A_y = \mu \int \frac{I_m \sin \omega(t - r/c)\, dl}{4\pi r} \tag{13-8}$$

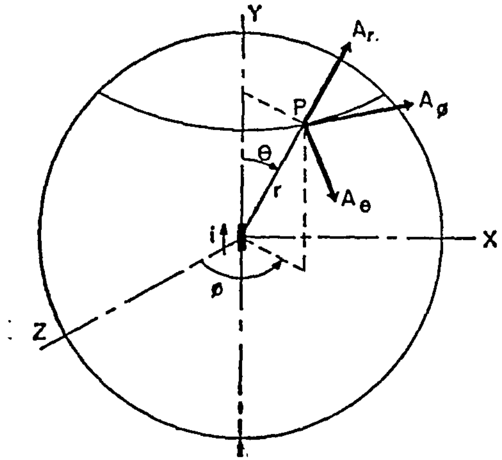


Fig. 13-2. A current element at the origin in spherical coordinates.

in the y direction, since that is the direction of the current producing it. By use of

$$H = \nabla \times A$$

and noting that

$$A_x = A_z = 0$$

then

$$H_x = -\frac{\partial A_y}{\partial z} \tag{13-9}$$

$$H_y = 0$$

$$H_z = \frac{\partial A_y}{\partial x} \tag{13-10}$$

are the magnetic field intensity components present.

From Fig. 13-2, it can be seen that

$$r = (x^2 + y^2 + z^2)^{1/2}$$

$$x = r \sin \theta \sin \phi$$

$$y = r \cos \theta$$

$$z = r \sin \theta \cos \phi$$

Use of these expressions in the indicated derivatives of Eqs. 13-9 and 13-10 gives

$$H_z = -\frac{I_m\,dl}{4\pi}\left[\frac{-z}{r^3}\sin\omega(t - r/c) - \frac{\omega z}{cr^2}\cos\omega(t - r/c)\right] \qquad (13\text{-}11)$$

$$H_x = \frac{I_m\,dl}{4\pi}\left[\frac{-y}{r^3}\sin\omega(t - r/c) - \frac{\omega y}{cr^2}\cos\omega(t - r/c)\right] \qquad (13\text{-}12)$$

Since H_z and H_x lie in a plane, the only spherical coordinate field component will be H_ϕ, given by

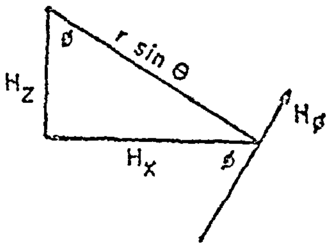


Fig. 13-3. Components of H_ϕ.

$$H_\phi = H_z\cos\phi - H_x\sin\phi \qquad (13\text{-}13)$$

after referring to Fig. 13-3. Also noting that

$$\frac{z}{r} = \sin\theta\cos\phi, \qquad \frac{y}{r} = \sin\theta\sin\phi$$

then Eqs. 13-11 and 13-12 may be combined to yield H_ϕ as

$$H_\phi = \frac{I_m\,dl}{4\pi}\left[\frac{\sin\theta\cos^2\phi}{r^2}\sin\omega(t - r/c) + \frac{\omega\sin\theta\cos^2\phi}{cr}\cos\omega(t - r/c) + \frac{\sin\theta\sin^2\phi}{r^2}\sin\omega(t - r/c) + \frac{\omega\sin\theta\sin^2\phi}{cr}\cos\omega(t - r/c)\right]$$

Writing $\delta = \omega(t - r/c)$,

$$H_\phi = \frac{I_m\sin\theta\,dl}{4\pi}\left[\frac{\sin\delta}{r^2} + \frac{\omega}{cr}\cos\delta\right] \qquad (13\text{-}14)$$

and this is the total magnetic field intensity at P.

It is interesting to note that the first term yields the Ampere's law result of Eq. 13-5, which was derived without consideration of a finite velocity of propagation. The second term is present because of consideration of this finite velocity. Since the first term varies as $1/r^2$ its effect is felt only near the current element, and this is the field measured near conductors and coils, usually referred to as the *induction field*. The second field term is small near the conductor, but since it varies as $1/r$ it becomes the predominant field component at great distances and it is known as the *radiation field*.

The relations between the two fields may be further explored if

ω/c is replaced with $2\pi/\lambda$ so that

$$H_\phi = \frac{I_m \sin\theta\, dl}{2\lambda r}\left[\frac{\lambda}{2\pi r}\sin\delta + \cos\delta\right] \tag{13-15}$$

Thus at distances r where the ratio λ/r is small with respect to unity, the induction field term is negligible compared to the magnitude of the radiation term. Equality of the two field magnitudes occurs at

$$r = \frac{\lambda}{2\pi} = 0.159\lambda$$

or a distance of one-sixth wavelength. Beyond this point the radiation field is the greater.

The accompanying electric fields may be found by use of Eq. 13-14 in Maxwell's equations. These equations must be placed in spherical coordinates and the results are

$$\left.\begin{aligned}
\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\mathcal{E}_\phi \sin\theta) - \frac{\partial \mathcal{E}_\theta}{\partial\phi}\right] &= -\mu\frac{\partial H_r}{\partial t}\\
\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial \mathcal{E}_r}{\partial\phi} - \frac{\partial}{\partial r}(r\mathcal{E}_\phi)\right] &= -\mu\frac{\partial H_\theta}{\partial t}\\
\frac{1}{r}\left[\frac{\partial}{\partial r}(r\mathcal{E}_\theta) - \frac{\partial \mathcal{E}_r}{\partial\theta}\right] &= -\mu\frac{\partial H_\phi}{\partial t}
\end{aligned}\right\} \tag{13-16}$$

$$\left.\begin{aligned}
\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(H_\phi \sin\theta) - \frac{\partial H_\theta}{\partial\phi}\right] &= \sigma\mathcal{E}_r + \epsilon\frac{\partial \mathcal{E}_r}{\partial t}\\
\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial H_r}{\partial\phi} - \frac{\partial}{\partial r}(rH_\phi)\right] &= \sigma\mathcal{E}_\theta + \epsilon\frac{\partial \mathcal{E}_\theta}{\partial t}\\
\frac{1}{r}\left[\frac{\partial}{\partial r}(rH_\theta) - \frac{\partial H_r}{\partial\phi}\right] &= \sigma\mathcal{E}_\phi + \epsilon\frac{\partial \mathcal{E}_\phi}{\partial t}
\end{aligned}\right\} \tag{13-17}$$

Field components $H_\theta = H_r = 0$, and because of symmetry there is no variation of any quantity with ϕ. Then in space the field conditions due to the current element will be given by

$$\frac{1}{r}\left[\frac{\partial}{\partial r}(r\mathcal{E}_\theta) - \frac{\partial \mathcal{E}_r}{\partial\theta}\right] = -\mu\frac{\partial H_\phi}{\partial t} \tag{13-18}$$

$$\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(H_\phi\sin\theta)\right] = \epsilon\frac{\partial \mathcal{E}_r}{\partial t} \tag{13-19}$$

$$-\frac{1}{r}\frac{\partial}{\partial r}(rH_\phi) = \epsilon\frac{\partial \mathcal{E}_\theta}{\partial t} \tag{13-20}$$

Using H_ϕ in the form

$$H_\phi = \frac{I_m \sin\theta\, dl}{4\pi}\left[\frac{\omega}{cr}\cos\omega(t - r/c) + \frac{\sin\omega(t - r/c)}{r^2}\right] \quad (13\text{-}21)$$

and performing the operation indicated on the left of Eq. 13-19 gives

$$\epsilon\frac{\partial \mathcal{E}_r}{\partial t} = \frac{2I_m \sin\theta\, dl}{4\pi}\left[\frac{\omega}{cr^2}\cos\omega(t - r/c) + \frac{\sin\omega(t - r/c)}{r^3}\right]$$

Integrating with respect to time, and again letting $\delta = \omega(t - r/c)$, leads to

$$\mathcal{E}_r = \frac{I_m \cos\theta\, dl}{2\pi\epsilon}\left(\frac{\sin\delta}{cr^2} - \frac{\cos\delta}{\omega r^3}\right) \quad (13\text{-}22)$$

after dropping the constant field obtained. Since the terms in brackets vary as $1/r^2$ and $1/r^3$, the $\mathcal{E}_r$ field will be negligible at any distance from the current element where $r \gg \lambda$. Thus no radial $\mathcal{E}$ component will be present in the radiation field.

Using H_ϕ again, and performing the operation indicated on the left in Eq. 13-20, followed by integration with respect to time, and use of $\delta = \omega(t - r/c)$, gives

$$\mathcal{E}_\theta = \frac{I_m \sin\theta\, dl}{4\pi\epsilon}\left(\frac{\omega}{c^2 r}\cos\delta + \frac{\sin\delta}{cr^2} - \frac{\cos\delta}{\omega r^3}\right) \quad (13\text{-}23)$$

Two of the terms in the bracket belong to the induction field and will become negligible when $r \gg \lambda$, so for the radiation field $\mathcal{E}_\theta$ is given by

$$\mathcal{E}_\theta = \frac{\omega I_m \sin\theta\, dl}{4\pi\epsilon c^2 r}\cos\omega(t - r/c) \quad (13\text{-}24)$$

Rewriting H_ϕ for $r \gg \lambda$

$$H_\phi = \frac{\omega I_m \sin\theta\, dl}{4\pi cr}\cos\omega(t - r/c) \quad (13\text{-}25)$$

Equations 13-24 and 13-25 constitute the fields present in the radiating wave from an element dl carrying a sinusoidal current of maximum value I_m. The field intensities increase with frequency, are in time phase, indicating energy transfer, and vary as the sine of the angle θ. being maximum at $\theta = 90°$.

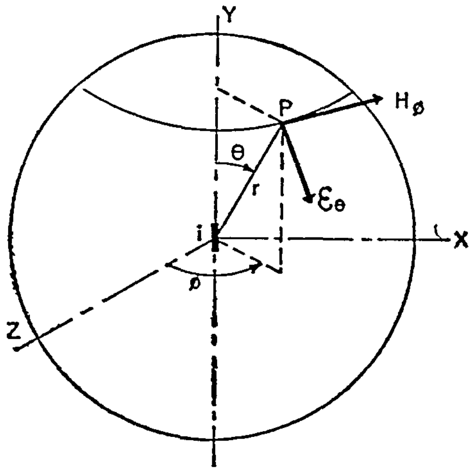


Fig. 13-4. Radiating fields in space from an element dl.

The geometry of the situation is illustrated in Fig. 13-4, where it is seen that H_ϕ is tangent to a parallel of latitude on a sphere, and $\mathcal{E}_\theta$ is tangent to a meridian. They both lie on the spherical surface, and for a small area would appear as a plane wave traveling outward. This is further substantiated by taking

$$\frac{\mathcal{E}_\theta}{H_\phi} = \frac{1}{c\epsilon} = 120\pi = \eta_v$$

or the wave has the same intrinsic impedance as was obtained for the plane wave in space.

13-4. Radiation resistance

Since the radiation field components from the current element, $\mathcal{E}_\theta$ and H_ϕ, are tangential to a spherical surface, the Poynting vector will be radial everywhere, indicating radial flow of power from the current element. The average Poynting power flow is one half the product of $\mathcal{E}_\theta$ and H_ϕ,

$$P = \frac{\omega^2 I_m{}^2 \sin^2 \theta \, dl^2}{32\pi^2 r^2 c^3 \epsilon} = \frac{\eta_v I_m{}^2 \sin^2 \theta \, dl^2}{8r^2\lambda^2} \text{ watts/m}^2 \qquad (13\text{-}26)$$

The total power being radiated from the current element is given by the surface integral of the Poynting vector over any surrounding surface, usually taken as a sphere. With an area element as shown in Fig. 13-5, having an area

$$da = 2\pi r^2 \sin\theta \, d\theta$$

then

$$W = \int_S P \cdot da = \frac{\pi \eta_v I_m^2 \, dl^2}{4\lambda^2} \int_0^\pi \sin^3\theta \, d\theta \tag{13-27}$$

$$= 40\pi^2 I_m^2 \left(\frac{dl}{\lambda}\right)^2 \text{ watts} \tag{13-28}$$

The value of resistance which would dissipate this amount of power with the same value of current is called the *radiation resistance*. For this current element

$$R_R = \frac{W}{I_m^2/2} = 80\pi^2 \left(\frac{dl}{\lambda}\right)^2 \text{ ohms} \tag{13-29}$$

Radiation resistance is due to power radiated. The total power input to an antenna will be increased by the losses, and consequently the total resistance of an antenna is made up of the radiation resistance plus the resistance due to the other power losses. For high efficiency the value of radiation resistance should be large with respect to the loss resistance.

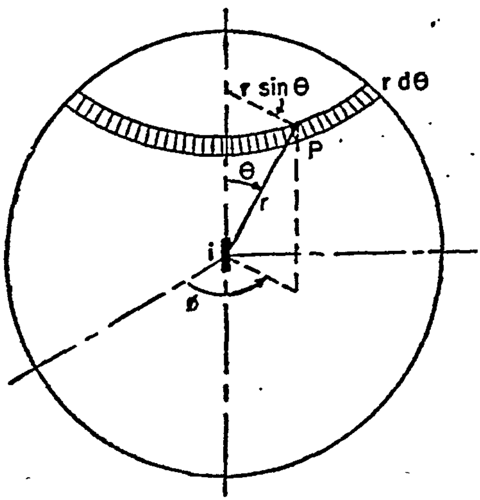


Fig. 13-5. Area element for Poynting vector integration.

13-5. The half-wave antenna in space

One of the simplest antennas, and one which is frequently employed as an element of a more complex directional system, is the half-wave long, thin wire or dipole. To simplify the problem, the antenna will be assumed as a number of wavelengths removed from the earth or other obstacle.

Because of the anticipated use of the dl current element, a *thin* or fine wire antenna is assumed, this requirement going back to the

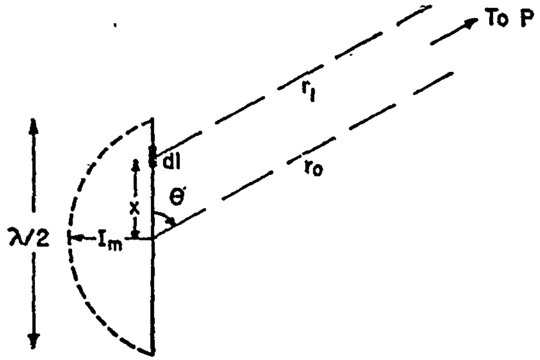


Fig. 13-6. The half-wave antenna.

argument preceding Eq. 13-2. The current at the ends of the wire must have a zero value, since there is nowhere for current to flow. A sinusoidal distribution is found to exist for a thin straight wire, as in Fig. 13-6, and using the center of the wire as reference, the current along the antenna is

$$i = I_m \cos 2\pi \frac{x}{\lambda} \tag{13-30}$$

where x has values from 0 to $\pm \lambda/4$.

Choosing a point P a long distance away, radius vectors r_0 and r_1 may be drawn at angle θ, and may be considered parallel, with

$$r_1 = r_0 - x \cos \theta$$

Inserting values for the constants, Eq. 13-25 may be written for the field at P, due to current i in the element dl, as

$$dH_\phi = \frac{I_m \cos (2\pi x/\lambda)}{2\lambda(r_0 - x \cos \theta)} \sin \theta \, dl \cos \omega \left(t - \frac{r_0 - x \cos \theta}{c}\right)$$

The term $x \cos \theta$ in the denominator will be small with respect to r_0 and may be dropped. However, in the angle the distance $x \cos \theta$ may represent an appreciable part of a wavelength and cannot be dropped, *since it may alter the phase of the field contribution from a given element.*

By expanding the time function the total magnetic field at P may be written

$$H_\phi = \frac{I_m}{2\lambda r_0} \sin \theta \left[\cos \omega \left(t - \frac{r_0}{c} \right) \int_{-\lambda/4}^{\lambda/4} \cos \frac{2\pi x}{\lambda} \cos \frac{2\pi x \cos \theta}{\lambda}\, dl \right.$$
$$\left. - \sin \omega \left(t - \frac{r_0}{c} \right) \int_{-\lambda/4}^{\lambda/4} \cos \frac{2\pi x}{\lambda} \sin \frac{2\pi x \cos \theta}{\lambda}\, dl \right] \quad (13\text{-}31)$$

Using the integrals

$$\int \cos mx \cos nx\, dx = \frac{\sin (m-n)x}{2(m-n)} + \frac{\sin (m+n)x}{2(m+n)} \quad (13\text{-}32)$$

$$\int \sin mx \cos nx\, dx = \frac{-\cos (m-n)x}{2(m-n)} - \frac{\cos (m+n)x}{2(m+n)} \quad (13\text{-}33)$$

it is found that the second term in the bracket of Eq. 13-31 disappears, and the first term leads to

$$H_\phi = \frac{I_m}{4\pi r_0} \sin \theta \cos \omega \left(t - \frac{r_0}{c} \right) \left[\frac{\sin (\frac{1}{2}\pi - \frac{1}{2}\pi \cos \theta)}{1 - \cos \theta} \right.$$
$$\left. + \frac{\sin (\frac{1}{2}\pi + \frac{1}{2}\pi \cos \theta)}{1 + \cos \theta} \right]$$

Further simplifying and writing I_{rms} for the effective value of current at the antenna center, gives

$$\text{rms } H_\phi = \frac{I_{rms}}{2\pi r_0} \left[\frac{\cos (\frac{1}{2}\pi \cos \theta)}{\sin \theta} \right] \text{ amp turns/m} \quad (13\text{-}34)$$

The accompanying electric field can be found by multiplying by η_v, or 120π,

$$\text{rms } \mathcal{E}_\theta = \frac{60 I_{rms}}{r_0} \left[\frac{\cos (\frac{1}{2}\pi \cos \theta)}{\sin \theta} \right] \text{ v/m} \quad (13\text{-}35)$$

These fields are mutually at right angles and in time phase. Both contain a magnitude term and a second term in brackets, which states the pattern of radiation as a function of θ, and is called the *directivity term.*

The received signal from an antenna is proportional to the power radiated in a given direction, or to the magnitude of the Poynting vector. The Poynting vector using rms values is

$$P = |\mathcal{E}_\theta||H_\phi| = \frac{30I_{rms}^2}{\pi r_0^2}\left[\frac{\cos\left(\frac{1}{2}\pi\cos\theta\right)}{\sin\theta}\right]^2 \text{ watts/m}^2 \quad (13\text{-}36)$$

The power directivity function, or the square of the field directivity, appears in Fig. 13-7 as a polar plot about the antenna. Because of

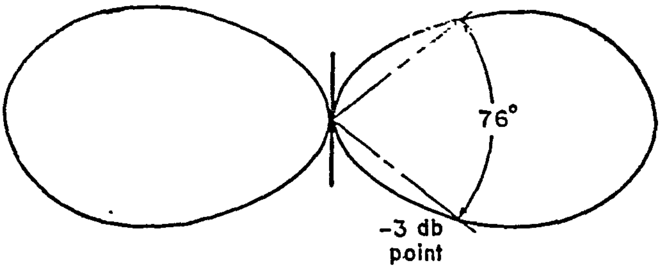


Fig. 13-7. Power directivity of a $\lambda/2$ antenna in space.

the symmetry with ϕ, the complete space radiation pattern is a doughnut-shaped figure of revolution, of which the polar plot is a cross section.

If the $\lambda/2$ dipole is vertical, the radiation will be predominantly horizontal. If mounted horizontally, the radiation will be directed off the sides of the antenna with a null at the ends. The effect of earth or other obstructions is neglected in these statements.

13-6. Radiation resistance of the $\lambda/2$ dipole

The radiation resistance of the half-wave dipole may be found by integrating the Poynting vector over the surface of a surrounding sphere. Using the band indicated in Fig. 13-5 as an area element, where

$$da = 2\pi r_0^2 \sin\theta\, d\theta$$

the total power radiated from the antenna will be found by use of Eq. 13-36, as

$$W = \int_S P \cdot da = 60I_{rms}^2 \int_0^\pi \frac{\cos^2\left(\frac{1}{2}\pi\cos\theta\right)}{\sin\theta}\, d\theta \quad (13\text{-}37)$$

This integration is carried out in Appendix C, and by several changes of variable leads to

$$W = 60I_{rms}^2 \times 1.221 = 73.26I_{rms}^2 \text{ watts} \tag{13-38}$$

The radiation resistance of the half-wave dipole far removed from earth is then 73.26 ohms. Since the current is measured at the antenna center, this value is the resistance into which a transmission line would supply power, if the line were connected into the center of the antenna as in Fig. 13-1(a).

The actual radiation resistance measured for a dipole over earth will be a function of height, tending to the above value as a limit. A transmission line with a characteristic impedance of 73 ohms would be matched by such a load, and various line designs are available with this impedance.

Because of the complexity of the integrals involved, most solutions for radiation resistance must be by graphical methods.

13-7. The effect of ground; the vertical antenna above earth

When an antenna is within a few wavelengths of earth, the field distribution will be altered by the effect of the earth as a conductive reflector. Assuming the earth is a perfect reflector, then radiation from an element dl of the antenna will reach point P over two paths,

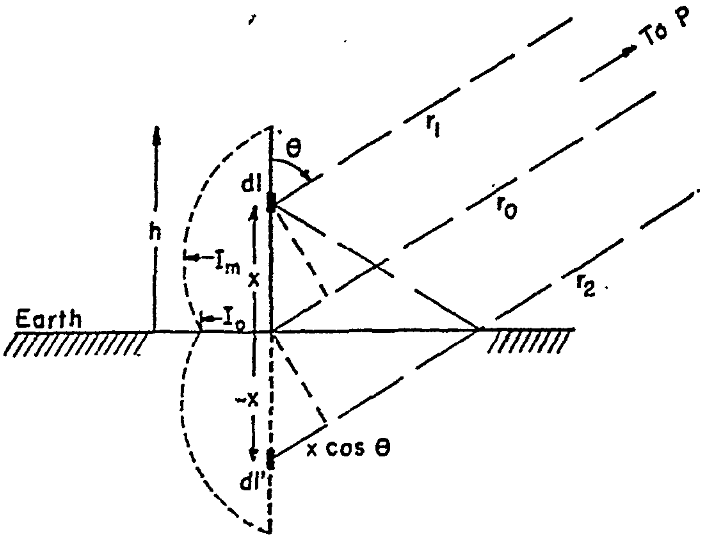


Fig. 13-8. Vertical antenna above conducting earth.

as shown in Fig. 13-8, one path direct along r_1, the other by reflection from the earth over path r_2.

The effect of the perfectly conducting earth may be introduced by use of an image antenna below the earth's surface and an image element dl', situated $-x$ units below the reflecting plane, where dl is $+x$ units above the surface. For other configurations it is possible to find the image element located by consideration of the mirror action of the earth, projected back to the image antenna.

Radiation is assumed as reaching P from elements dl and dl', over paths r_1 and r_2, these paths differing in length such that

$$r_1 = r_0 - x \cos \theta$$

$$r_2 = r_0 + x \cos \theta$$

As before, these length differentials will be negligible in their effect on elemental field magnitudes, but not on phase angles.

The antenna of Fig. 13-8 is grounded, and of length $h > \lambda/4$. The maximum current magnitude I_m occurs at a distance $\lambda/4$ down from the top of the antenna, and the current distribution is

$$i = I_m \sin \frac{2\pi(h - x)}{\lambda} \sin \omega t \tag{13-39}$$

In terms of the more easily measured current at the antenna base

$$I_m = \frac{I_0}{\sin (2\pi h/\lambda)}$$

where I_0 is the maximum value at the base.

The field at point P due to current in the element dl is then

$$dH_\phi = \frac{I_m}{2\lambda r_0} \sin \theta \sin \frac{2\pi(h - x)}{\lambda} \cos \omega \left(t - \frac{r_0 - x \cos \theta}{c}\right) dl \tag{13-40}$$

Due to reflection of the field from the earth's surface, an additional component of field from the element dl' appears at P. This additional field is

$$dH_\phi' = \frac{I_m}{2\lambda r_0} \sin \theta \sin \frac{2\pi(h - x)}{\lambda} \cos \omega \left(t - \frac{r_0 + x \cos \theta}{c}\right) dl \tag{13-41}$$

The magnetic field at P due to both elements is then

$$dH_\phi = \frac{I_m}{2\lambda r_0} \sin\theta \sin\frac{2\pi(h-x)}{\lambda}\left[\cos\omega\left(t - \frac{r_0 - x\cos\theta}{c}\right) + \cos\omega\left(t - \frac{r_0 + x\cos\theta}{c}\right)\right] dl \quad (13\text{-}42)$$

Using $\omega/c = 2\pi/\lambda$ and

$$\cos(a+b) + \cos(a-b) = 2\cos a \cos b$$

it is possible to write for the magnetic field at P, due to the whole antenna

$$\begin{aligned} H_\phi &= \frac{I_m}{\lambda r_0}\sin\theta\cos\omega\left(t - \frac{r_0}{c}\right)\int_0^h \sin\frac{2\pi(h-x)}{\lambda}\cos\left(\frac{2\pi x\cos\theta}{\lambda}\right) dl \\ &= \frac{I_m}{\lambda r_0}\sin\theta\cos\omega\left(t - \frac{r_0}{c}\right)\left[\sin\frac{2\pi h}{\lambda}\int_0^h \cos\frac{2\pi x}{\lambda}\cos\left(\frac{2\pi x\cos\theta}{\lambda}\right) dl \right. \\ &\qquad \left. - \cos\frac{2\pi h}{\lambda}\int_0^h \sin\frac{2\pi x}{\lambda}\cos\left(\frac{2\pi x\cos\theta}{\lambda}\right) dl\right] \end{aligned} \quad (13\text{-}43)$$

Proceeding as in Section 13-6, the rms magnetic field is finally obtained, in terms of I_{rms} at the maximum current point, as

$$\text{rms } H_\phi = \frac{I_{\text{rms}}}{2\pi r_0}\left[\frac{\cos\left(\frac{2\pi h\cos\theta}{\lambda}\right) - \cos\frac{2\pi h}{\lambda}}{\sin\theta}\right] \text{ amp turn/m} \quad (13\text{-}44)$$

The accompanying electric field is

$$\text{rms } \mathcal{E}_\theta = \frac{60 I_{\text{rms}}}{r_0}\left[\frac{\cos\left(\frac{2\pi h\cos\theta}{\lambda}\right) - \cos\frac{2\pi h}{\lambda}}{\sin\theta}\right] \text{ v/m} \quad (13\text{-}45)$$

The Poynting vector or power density would then be

$$P = \frac{30 I_{\text{rms}}^2}{\pi r_0^2}\left[\frac{\cos\left(\frac{2\pi h\cos\theta}{\lambda}\right) - \cos\frac{2\pi h}{\lambda}}{\sin\theta}\right]^2 \text{ watts/m}^2 \quad (13\text{-}46)$$

Once again, it is possible to state the result in terms of a magnitude and a directivity function. This power directivity function is plotted in Fig. 13-9 for values of $h = \frac{1}{4}$, $\frac{5}{8}$, and $\frac{3}{4}$ wavelength. These are distributions of power density in the vertical plane, and are cross sections of figures of revolution in space above the earth's surface. The horizontal directivity is circular, centered on the antenna.

For much broadcasting service it is desirable that the power radiated horizontally, or along the earth's surface, be a maximum.

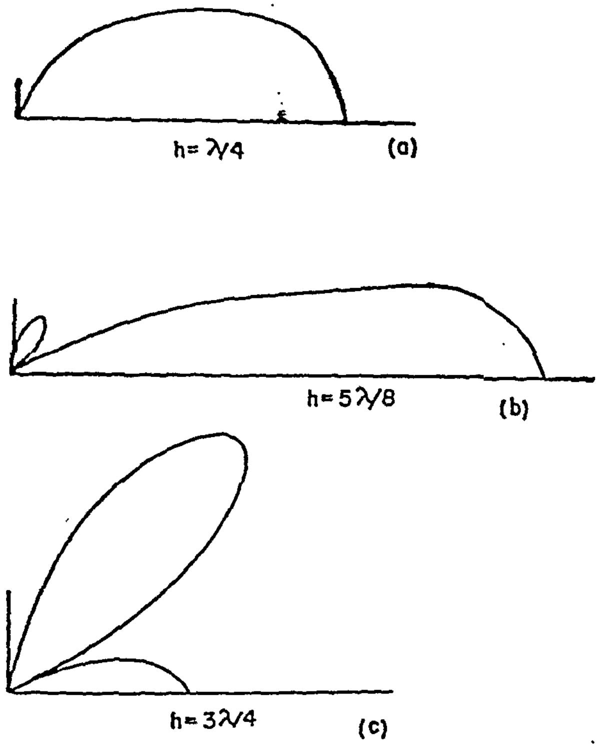


Fig. 13-9. Vertical directivity pattern for grounded antenna: (a) $h = \lambda/4$; (b) $h = 5\lambda/8$; (c) $h = 3\lambda/4$. (Pattern symmetric about antenna.)

It can be shown that this occurs for $h = 5\lambda/8$ for this antenna, greater heights tending to force some power into a high-angle lobe which is not useful for surface radiation. At some distances and some frequencies, this high-angle power may be useful for communication by the so-called "sky wave" which is refracted from the ionosphere.

13-8. The grounded quarter-wave antenna

A quarter-wave vertical grounded antenna is frequently employed, since the resistive impedance at the ground end is low enough that transmission lines can be easily matched, the antenna radiation is at a desirable low angle, and for some frequencies a higher antenna is prohibitive in cost.

The fields and power density may be obtained from the results of the preceding section by making $h = \lambda/4$. The Poynting vector then has a magnitude

$$P = \frac{30 I_{rms}^2}{\pi r_0^2} \left[\frac{\cos\left(\frac{1}{2}\pi \cos\theta\right)}{\sin\theta} \right]^2 \text{ watts/m}^2 \qquad (13\text{-}47)$$

and this is recognizable as exactly the expression obtained for the $\lambda/2$ antenna in space, given in Eq. 13-36. Since the antenna with its image constitutes a $\lambda/2$ dipole, the result is not surprising.

The radiation resistance may be obtained by the method of Eq. 13-37, but the reflective action of the earth confines the radiation to only the upper hemisphere; thus the limits of integration are from 0 to $\pi/2$, and the result for power radiated is

$$W = 36.6 I_{rms}^2 \text{ watts} \qquad (13\text{-}48)$$

and the radiation resistance of the antenna at the base is 36.6 ohms, one-half the value for the $\lambda/2$ dipole in space. Since the resistance at the end of the antenna is infinite, all values of resistance between 36.6 ohms and infinity are present at points on the antenna for feed line matching purposes. In this respect, the antenna and its image appear as an open-circuited line.

The distribution pattern will be that plotted in Fig. 13-9(a). In using any form of grounded antenna, the space pattern will be distorted by discontinuities in earth conductivity, and if the earth conductivity is low, the radiation angles will usually be elevated by a few degrees. Earth conductivity under the antenna can be increased, and power losses in the ground reduced, if a *counterpoise* of wires is strung just above the earth's surface, or buried just below, and connected as the ground element of the antenna. These wires should be at least $\lambda/4$ long, and uniformly spaced at 5° to 10° radial intervals.

13-9. The two-element array

One of the simplest antenna systems which may be designed to have a variety of directive patterns, is the phased array of two $\lambda/2$ elements of Fig. 13-10. The antennas carry equal currents, are separated a distance $n\lambda$, where n is usually fractional, and are

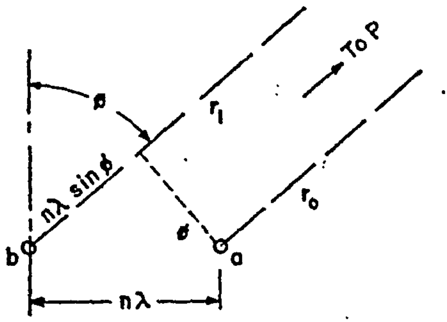


Fig. 13-10. Two-element directional array.

phased such that the current in a leads and that in b by an arbitrary angle δ. The field at point P due to antenna a will be

$$\mathcal{E}_a = \mathcal{E}_m \cos\omega\left(t - \frac{r_0}{c}\right) \tag{13-49}$$

where

$$\mathcal{E}_m = \frac{60I_m}{r_0}\left[\frac{\cos\left(\frac{1}{2}\pi\cos\theta\right)}{\sin\theta}\right]$$

The field due to antenna b, under the usual assumptions regarding the magnitude of r_0 and r_1, will be

$$\mathcal{E}_b = \mathcal{E}_m \cos\omega\left(t - \frac{r_0 + n\lambda\sin\phi}{c} - \frac{\delta}{\omega}\right) \tag{13-50}$$

Adding the fields at P, and using

$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$$

$$\mathcal{E}_a + \mathcal{E}_b = 2\mathcal{E}_m \cos\omega\left(t - \frac{r_0}{c} - \frac{n\lambda\sin\phi}{2c}\right)\cos\left(\pi n\sin\phi + \frac{\delta}{2}\right)$$

The rms value of the resultant electric field is

$$\mathcal{E}_r = \mathcal{E}_{rms} \cos\left(\pi n \sin \phi + \frac{\delta}{2}\right) \tag{13-51}$$

where angle ϕ is measured from the normal to the line of the antennas, and δ is positive for the current in a leading that in b.

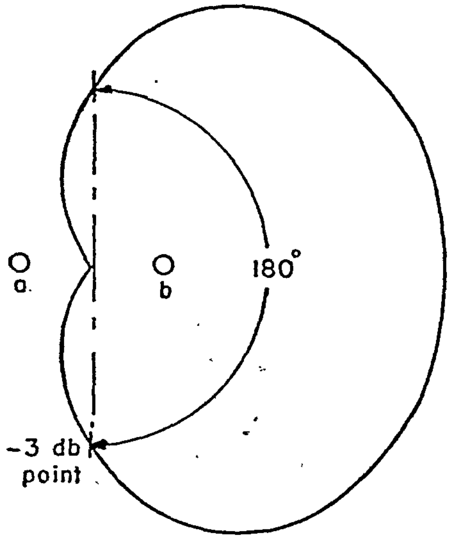


Fig. 13-11. Power density pattern for a two-element array spaced $\lambda/4$ and phased at $\pi/2$; (a leads b).

The accompanying magnetic field can be obtained by use of $\eta_v = 120\pi$, so that the Poynting power density is

$$P_r = \frac{60 I_{rms}^2}{\pi r_0^2}\left[\frac{\cos\left(\frac{1}{2}\pi \cos\theta\right)}{\sin\theta}\right]^2 \cos^2\left(\pi n \sin\phi + \frac{\delta}{2}\right) \tag{13-52}$$

The terms in brackets is, of course, the directivity as a function of θ for one antenna element. The term following the brackets is the directivity introduced by the combination of the two elements. Thus the over-all pattern of an array or combination of antenna elements is found as the product of an array directivity function times the directivity of an element of the array.

If the antenna elements are vertical, the horizontal directivity around the array will be given for $\theta = 90°$, when the bracket of the above equation goes to unity and

$$P_r = P \cos^2 \left(\pi n \sin \phi + \frac{\delta}{2}\right) \tag{13-53}$$

and this condition is frequently employed to give a wide variety of horizontal patterns by choice of n and δ.

In particular, the antennas may be spaced $\lambda/4$ or $n = \frac{1}{4}$, and driven with equal currents $\pi/2$ out of time phase. Equation 13-53 becomes, for this special case

$$P_r = P \cos^2 \left[\frac{\pi}{4} (\sin \phi + 1)\right] \tag{13-54}$$

and this horizontal distribution, for a pair of vertical antennas, is plotted in Fig. 13-11. It becomes a cardioid, with the radiation unidirectional in the direction of the antenna with the lagging current.

13-10. Parasitic elements

Only one element of the antenna of Fig. 13-10 need be driven or supplied with current, it being possible to induce a current of proper phase in the other element. The magnitude and phase of the induced current depends on the spacing between the driven and the *parasitic element*, and on the length or tuning of the latter. Close spacing between 0.1λ and 0.25λ is desirable to assure adequate current magnitude. The length of the parasitic element must be adjusted for proper phasing of the current, which must be leading the induced emf for a *reflector*, and lagging for a *director* or element in the direction of the radiation lobe. The input resistance of the driven element will be reduced by reason of the mutual coupling between driven and parasitic element, and matching for the transmission line will have to be designed accordingly.

A series of reflectors or directors may be assembled in line with a driven element to increase the directivity along the line of the antennas, such an assembly being called a *Yagi antenna*.

13-11. Broadside and end-fire arrays

Additional elements may be placed in line with the simple antennas of Section 13-9, and Fig. 13-12 represents a view looking down on an array of N uniformly spaced antennas, each of which may be considered a $\lambda/2$ dipole. The field at a remote point P will be a superposition of the fields due to each antenna, with proper phase relations introduced due to the varying distances traveled.

Assume all antennas with equal current, but that the current in antenna b leads that in a by some arbitrary angle δ, and that each

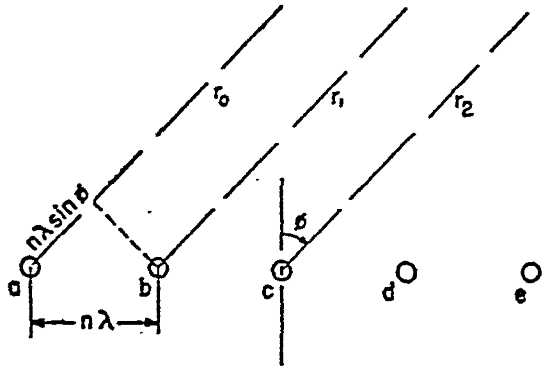


Fig. 13-12. An array of antennas.

antenna in the line is similarly phased with respect to the antenna on its left. This phasing may be achieved by connecting appropriate lengths of transmission line between antennas. If it is assumed that $r_0 \gg n\lambda \sin \phi$, so that the difference in path length to P for different antennas is negligible in its effect on field magnitudes, the fields can be written as in Section 13-9.

It is apparent that the field at P due to each antenna differs from the one before by a phase angle $\alpha + \delta$, where $\alpha = (\omega n\lambda \sin \phi)/c = 2\pi n \sin \phi$. The total field due to all N antennas is then

$$\mathcal{E}_r = \mathcal{E}_m\{\cos\beta + \cos(\beta + \alpha + \delta) + \cos[\beta + 2(\alpha + \delta)] + \cos[\beta + 3(\alpha + \delta)] + \cdots + \cos[\beta + (N-1)(\alpha + \delta)]\} \tag{13-55}$$

where $\beta = \omega(t - r_0/c)$. This series has a sum given by

$$\mathcal{E}_r = \mathcal{E}_m \left\{ \frac{\cos\left[\beta + \dfrac{N-1}{2}(\alpha + \delta)\right] \sin \dfrac{N(\alpha + \delta)}{2}}{\sin \dfrac{\alpha + \delta}{2}} \right\} \tag{13-56}$$

This may be further simplified by expansion of the cosine term and passing to rms values as

$$\mathcal{E}_r = \mathcal{E}_{\text{rms}} \left[\frac{j \cos \dfrac{N-1}{2} (\alpha + \delta) - \sin \dfrac{N-1}{2} (\alpha + \delta)}{\sin \dfrac{\alpha + \delta}{2}} \right] \sin \frac{N(\alpha + \delta)}{2} \tag{13-57}$$

The power density radiated will be proportional to the square of this function so that

$$P_r = P \frac{\sin^2 \dfrac{N(\alpha + \delta)}{2}}{\sin^2 \dfrac{(\alpha + \delta)}{2}} \text{ watts/m}^2 \tag{13-58}$$

$$\alpha = 2\pi n \sin \phi$$

The resultant directivity pattern is a function of both angle of observation ϕ, and of δ the angle of lead of the antenna currents.

The result may be applied to several special cases. Assume first that all antennas are supplied equally and in phase, or $\delta = 0$. It is evident that at a great distance on the line of $\phi = 0$, the fields will arrive essentially in phase. However, at a distance at $\phi = 90°$, the fields will be out of phase by varying amounts, depending on the element spacing, and the sum will tend to zero. Such an array of antennas radiates normal to the line of antennas and is called a *broadside array*. For the in-phase broadside condition Eq. 13-58 becomes

$$P_r = P \frac{\sin^2 (\frac{1}{2}N\pi n \sin \phi)}{\sin^2 (\frac{1}{2}\pi n \sin \phi)} \tag{13-59}$$

for which a power density pattern is plotted in Fig. 13-13, for $N = 10$ and $n = \frac{1}{2}$.

The pattern has a main lobe and several side lobes. The width of the main lobe is usually stated in terms of the angles at which a first zero occurs, and this is more easily determined from a rectangular plot as at (b). Calling these angles of zero value $\phi_0/2$, then a zero occurs where

$$N\pi n \sin \frac{\phi_0}{2} = \pm\pi$$

$$\phi_0 = 2 \sin^{-1} \frac{1}{Nn}$$

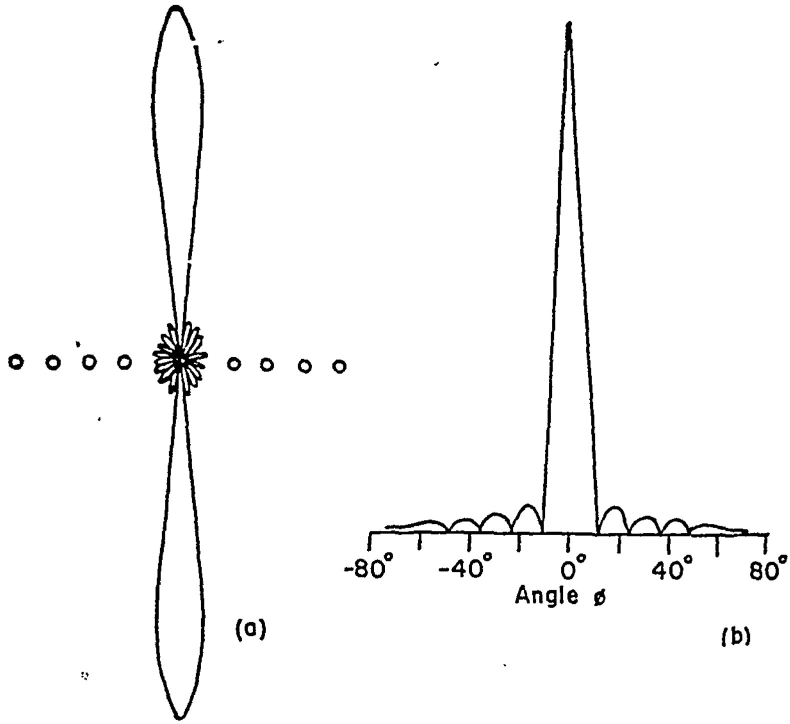


Fig. 13-13. Patterns for a broadside array, $N = 10$: (a) polar; (b) rectangular.

But $(N - 1)n\lambda$ is the width w of the array in wavelengths, or $(N - 1)n\lambda = w$. If N is large, the unity term may be dropped and

$$Nn \cong \frac{w}{\lambda}$$

so that

$$\phi_0 \cong 2 \sin^{-1} \frac{\lambda}{w} \cong \frac{2\lambda}{w}$$

for narrow beams or small angles. The beam becomes narrower for increasing array width. Beam width may also be specified as the angle between half-power or 3-db points on the directivity pattern.

As another case, the antennas may be supplied equal currents, but with each element delayed in phase by the amount of propagation time from element to element. That is, the spacing will be $n\lambda$, with the elements phased at $2\pi n$. It would be expected that along the line of the array the radiation will add up, but that broadside the field tends to zero for an array of many elements, and the result is an *end-fire array*. For the end-fire case, with $\delta = 2\pi n$, Eq. 13-58 becomes

$$P_r = P \frac{\sin^2 N\pi n(\sin \phi + 1)}{\sin^2 \pi n(\sin \phi + 1)} \tag{13-60}$$

It can be seen that the pattern is essentially the same shape as for the broadside case, rotated by 90°.

The width of the main lobe between zeros can be found by again calling the angle $\phi_0/2$. The first zeros will occur on each side of the main lobe, or at $-\pi/2 \pm \phi_0/2$. Then

$$N\pi n\left[-\sin\left(\frac{\pi}{2} \pm \frac{\phi_0}{2}\right) + 1\right] = \pm\pi$$

$$1 - \cos\frac{\phi_0}{2} = \frac{1}{Nn}$$

For small angles the cosine may be replaced by the first two terms of its series

$$1 - 1 + \frac{\phi_0^2}{8} \cong \frac{1}{Nn}$$

and using $1/Nn \cong \lambda/w$, then

$$\phi_0 \cong 2\sqrt{\frac{2\lambda}{w}}$$

is the beam width for the end-fire array.

It is common to place an array of parasitic elements $\lambda/4$ behind a broadside array, thereby making it unidirectional. Each driven and parasitic element then represents a pair whose directivity function is given in Eq. 13-53. The over-all characteristic of the array with reflectors is obtainable by multiplying Eq. 13-58 for the broadside case, by the directivity function of Eq. 13-53 for the reflector. For small arrays at high frequencies it becomes practical to convert the reflector dipoles to a continuous screen.

The above has been developed in terms of uniform current in all antenna elements. Obviously another parameter is available if the current is varied, and this is done to further reduce side lobes. The method is beyond the scope of this book, however.

13-12. The colinear array

Another form of array is of the colinear type in which $\lambda/2$ antennas are arranged end-to-end and in phase as in Fig. 13-14(a). The in-phase condition can be achieved by feeding the antennas

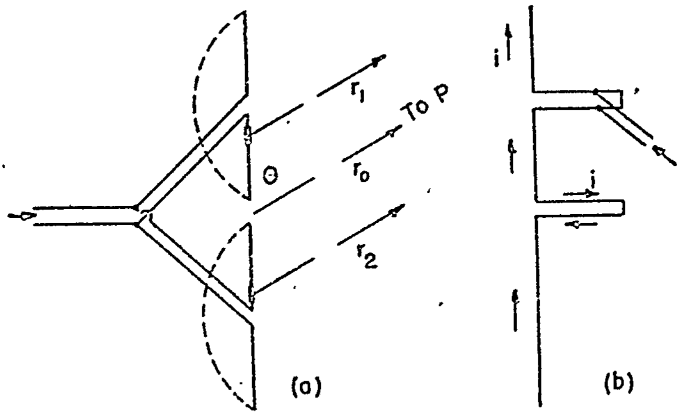


Fig. 13-14. Two colinear $\lambda/2$ antennas.

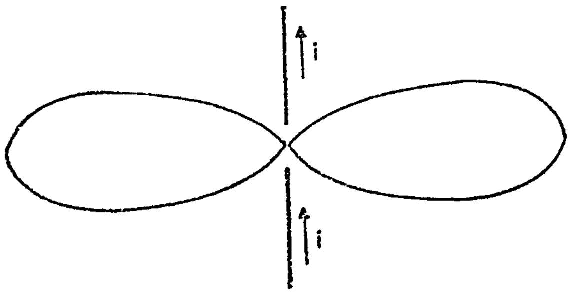


Fig. 13-15. Power density pattern for a two-element colinear array.

in parallel from a line, as in (a). Another method appears in (b) wherein one $\lambda/2$ antenna is supplied at the center, or at its low-impedance point through a $\lambda/4$ transformer, and the second antenna is coupled to the first by a short-circuited $\lambda/4$ section of line. Current directions are indicated by the arrows, with both antennas in phase.

The field pattern for such an array involving only two elements can be found from the result for the arbitrary length antenna above ground, since the physical picture is the same if h in Fig. 13-8 be made $\lambda/2$. The power density expression for the two-antenna colinear array may then be written from Eq. 13-46 as

$$P = \frac{30 I_{rms}^2}{\pi r_0^2} \left[\frac{\cos(\pi \cos\theta + 1)}{\sin\theta} \right]^2 \text{ watts/m}^2 \qquad (13\text{-}61)$$

The radiation pattern turns out to be broadside to the line of the antennas, and use of additional $\lambda/2$ elements increases the sharpness of the lobe of radiation. The pattern for a two-element array is shown in Fig. 13-15, for a plane through the antenna axis.

13-13. Antenna gain

Directional antennas or arrays are used to increase the power radiated in desired directions, or to decrease the power radiated in undesired directions. For instance, a radio broadcasting station located on a sea coast would have little use for a circular horizontal directivity pattern, but could use a multielement array so spaced and phased as to produce a cardioid pattern, with the null or low-radiation side toward the ocean. The power in the desired direction is thereby increased as well.

The ratio of the Poynting vector produced by an actual antenna in a particular direction, to the value of Poynting vector generated in all directions by an isotropic source of equal power, is defined as the power *gain* of the actual antenna.* The particular direction chosen is usually that of the maximum intensity produced by the actual antenna.

For the isotropic radiator of equal power input W, the Poynting vector is

$$P_i = \frac{W}{4\pi r^2}$$

and the gain of an antenna is

$$G = \frac{P_r}{P_i} = \frac{4\pi r^2 P_r}{W} \qquad (13\text{-}62)$$

The Poynting vector power density for the half-wave dipole is maximum at $\theta = \pi/2$, and from Eq. 13-36

$$P_r = \frac{30 I_{rms}^2}{\pi r^2}$$

* Gain may also be stated with respect to a $\lambda/2$ dipole as reference, in which case the dipole must be oriented to produce *its maximum in the same* direction as that of the test antenna.

The power input to the dipole is $R_R I_{rms}^2 = 73.26 I_{rms}^2$, and this is W, the input to the comparison isotropic source. The power gain of the half-wave dipole over the isotropic source is then

$$G = \frac{4\pi r^2 \dfrac{30 I_{rms}^2}{\pi r^2}}{73.26 I_{rms}^2} = 1.63 = 2.12 \text{ db}$$

in the direction of the maximum field intensity, or at $\theta = \pi/2$.

Averaging the area inside the power-density polar chart for a given antenna gives the power density for the isotropic radiator of equal power input. The ratio of the maximum power density from the plot of the actual antenna, to the average value obtained by measurement of the plot, is the power gain, usually expressed in decibels.

13-14. Paraboloid reflectors and lenses

A simple metal reflector behind a dipole may be used to turn forward the usual backward-directed energy, and to double the antenna gain in the forward direction, in accordance with the

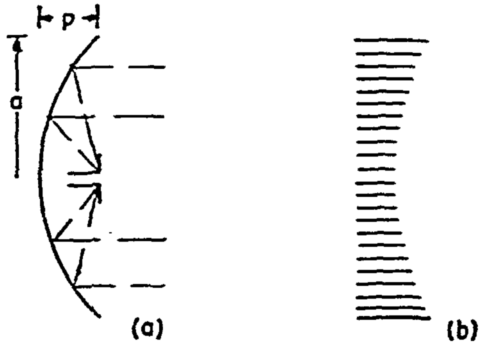


Fig. 13-16. (a) Paraboloid reflector with dipole feed; (b) metal lens.

principles of Chapter 10. It is possible to increase further the gain and narrow the angle of the forward beam by use of a paraboloidal metal reflector uniformly illuminated by a small antenna or a wave guide at the focus. Radiation from such a reflector approximates that from a simple current sheet as previously discussed. The paraboloid becomes practical only for very short wavelengths where the aperture may be many wavelengths, since the sharpness of the beam depends on the aperture width in wavelengths.

The gain is maximum when the focus and source lie in a plane defined by the face of the aperture, and $p = a/2$ as in Fig. 13-16. Then

$$G = \frac{4\pi(\pi a^2)}{\lambda^2} = \frac{4\pi\ (\text{area})}{\lambda^2} \tag{13-63}$$

If the illumination of the reflector is not uniform, or if the antenna or wave-guide feed distorts the field, this value may not be achieved. The equation shows that the gain is proportional to the aperture area-to-wavelength ratio. Beam angles of 1° to 2° are possible.

The metal lens of (b), Fig. 13-16 uses wave-guide sections to achieve focusing through usual optical principles, the outer sections of parallel-plane wave guide having a phase velocity greater than the center section, thereby advancing the outer wave components, just as would a converging glass lens in optics. Very narrow beam angles are possible.

PROBLEMS

13-1. Calculate the total H and $\mathcal{E}$ fields at a point P for an antenna λ in length in space. Compute the Poynting energy density and plot the space distribution of power in the plane of the antenna (horizontal pattern for a horizontal antenna).

13-2. Find the ratio of current magnitudes required to radiate 1 kw from the $\lambda/2$ dipole and from the elemental dipole, of height 0.01λ.

13-3. Compute the gain of the elemental dipole as an antenna.

13-4. Make a rectangular plot of the one-way radiation from a broadside array of six dipoles, spaced $\lambda/2$ and fed in phase.

13-5. Plot the polar distribution for two $\lambda/2$ dipoles spaced $3\lambda/8$ and fed 180° out of phase.

13-6. Show the location of the elemental dipoles, and the current directions, for horizontal antenna above perfectly conducting earth, and its image.

13-7. Plot the polar power density pattern for four vertical $\lambda/2$ dipoles placed at the corners of a square, with opposite corners excited in phase, with the two pairs of antennas excited at 180°.

13-8. Find the beam width for a broadside antenna with 16 elements spaced $\lambda/2$, also for the same arrangement with elements phased at 180°.

13-9. Compute and plot the vertical plane pattern for a vertical antenna $3\lambda/8$ long above a perfect earth.

13-10. By graphical means compute the gain of each of the antennas of Prob. 13-8, with respect to an isotropic source.

13-11. Compute the vertical pattern in a plane through the center and normal to the antenna, for a $\lambda/2$ horizontal dipole spaced $\lambda/2$ above a perfect earth.

13-12. Two vertical antennas $\lambda/2$ long are spaced $5\lambda/8$ and fed out of phase by 45°. Plot the horizontal power pattern.

13-13. Repeat Prob. 13-12 but with spacing $\lambda/2$ and a 90° phase relationship.

13-14. Two $\lambda/2$ vertical antennas are spaced $\lambda/4$, but fed with unequal in-phase currents such that current in $A = 3$, in $B = 1$. Calculate and plot the horizontal power density pattern.

13-15. An antenna in free space in $3\lambda/2$ long. Calculate and plot the power density pattern in a plane containing the antenna.

13-16. Calculate the gain over an isotropic source of two vertical $\lambda/2$ antennas, spaced $\lambda/2$ and fed at 180°.

13-17. Calculate and plot in rectangular form the power density pattern of an end-fire array of eight $\lambda/2$ antennas, spaced 0.2λ, and phased accordingly.

13-18. Calculate and plot the polar distribution for two $\lambda/2$ dipoles spaced $3\lambda/8$ and fed 135° out of phase.

REFERENCES

1. Dwight, H. B., *Tables of Integrals and Other Mathematical Data* The Macmillan Company, New York, 1934.
2. Ramo, S. and Whinnery, J. R., *Fields and Waves in Modern Radio*, 2d ed., John Wiley & Sons, Inc., New York, 1953.
3. Jordan, E. C., *Electromagnetic Waves and Radiating Systems*, Prentice-Hall, Inc., New York, 1950.
4. Kraus, J. D., *Antennas*, McGraw-Hill Book Company, Inc., New York, 1950.
5. Stratton, J. A., *Electromagnetic Theory*, McGraw-Hill Book Company, Inc., New York, 1941.

Appendix A

SOLUTION OF THE WAVE EQUATION

The wave equation, as obtained in Chapter 9 for the plane wave, is in the form

$$\frac{\partial^2 P}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 P}{\partial t^2} \tag{A-1}$$

where P is any desired field vector, or component, and

$$v = \frac{1}{\sqrt{\mu\epsilon}}$$

This partial differential equation may be solved by introducing a change of variable such that

$$P_{(z,t)} = \psi_{(u,w)}$$

wherein the transformation equations are assumed as

$$u = z + vt \tag{A-2}$$

$$w = z - vt \tag{A-3}$$

The first partial derivative of ψ with respect to z is

$$\frac{\partial P_{(z,t)}}{\partial z} = \frac{\partial \psi_{(u,w)}}{\partial z} = \frac{\partial \psi}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial \psi}{\partial w}\frac{\partial w}{\partial z}$$

But from Eqs. A-2 and A-3,

$$\frac{\partial u}{\partial z} = 1, \qquad \frac{\partial w}{\partial z} = 1$$

Then
$$\frac{\partial P}{\partial z} = \frac{\partial \psi}{\partial z} = \frac{\partial \psi}{\partial u} + \frac{\partial \psi}{\partial w}$$

Taking the second partial derivative,

$$\frac{\partial^2 P}{\partial z^2} = \frac{\partial\left(\dfrac{\partial \psi}{\partial z}\right)}{\partial z} = \frac{\partial^2 \psi}{\partial u^2}\frac{\partial u}{\partial z} + \frac{\partial^2 \psi}{\partial u\,\partial w}\frac{\partial u}{\partial z} + \frac{\partial^2 \psi}{\partial u\,\partial w}\frac{\partial w}{\partial z} + \frac{\partial^2 \psi}{\partial w^2}\frac{\partial w}{\partial z}$$

which reduces to

$$\frac{\partial^2 P}{\partial z^2} = \frac{\partial^2 \psi}{\partial u^2} + 2\frac{\partial^2 \psi}{\partial u\, \partial w} + \frac{\partial^2 \psi}{\partial w^2} \tag{A-4}$$

It is then necessary to perform the same operations with respect to t:

$$\frac{\partial P_{(z,t)}}{\partial t} = \frac{\partial \psi_{(u,w)}}{\partial t} = \frac{\partial \psi}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial \psi}{\partial w}\frac{\partial w}{\partial t}$$

From Eqs. A-2 and A-3,

$$\frac{\partial u}{\partial t} = v; \qquad \frac{\partial w}{\partial t} = -v \tag{A-5}$$

and so

$$\frac{\partial P}{\partial t} = \frac{\partial \psi}{\partial t} = v\frac{\partial \psi}{\partial u} - v\frac{\partial \psi}{\partial w}$$

Taking the second partial derivative,

$$\frac{\partial^2 P}{\partial t^2} = \frac{\partial\left(\frac{\partial \psi}{\partial t}\right)}{\partial t} = v\frac{\partial^2 \psi}{\partial u^2}\frac{\partial u}{\partial t} + v\frac{\partial^2 \psi}{\partial u\, \partial w}\frac{\partial w}{\partial t} - v\frac{\partial^2 \psi}{\partial u\, \partial w}\frac{\partial u}{\partial t} - v\frac{\partial^2 \psi}{\partial w^2}\frac{\partial w}{\partial t}$$

In view of Eq. A-5, this reduces to

$$\frac{\partial^2 P}{\partial t^2} = v^2\frac{\partial^2 \psi}{\partial u^2} - 2v^2\frac{\partial^2 \psi}{\partial u\, \partial w} + v^2\frac{\partial^2 \psi}{\partial w^2} \tag{A-6}$$

Substitution of Eqs. A-4 and A-6 into A-1 gives for the transformed wave equation

$$4\frac{\partial^2 \psi}{\partial u\, \partial w} = 0$$

and since 4 obviously is not zero,

$$\frac{\partial^2 \psi}{\partial u\, \partial w} = 0 \tag{A-7}$$

This is a form of equation that may be readily integrated. Performing the operations, and bearing in mind that the *arbitrary constant* obtained in integration of a partial derivative is a function of all variables except the one being operated upon, then integrating with respect to w,

$$\frac{\partial \psi}{\partial u} = f(u)$$

and integrating with respect to u,

$$\psi = \int f(u)\,du + f_2(w)$$

But the integral of a function wholly dependent on u is some other function f_1 of u, so that

$$\psi = f_1(u) + f_2(w) \tag{A-8}$$

Transforming back to the function $P\ (z,t)$,

$$P = f_1(z + vt) + f_2(z - vt) \tag{A-9}$$

which is the solution of the wave equation.

Appendix B

TRANSFORMATION OF MAXWELL'S EQUATIONS FROM RECTANGULAR TO CYLINDRICAL COORDINATES

Using the usual transformations,

$$r = \sqrt{x^2 + y^2}$$
$$x = r \cos \phi$$
$$y = r \sin \phi$$
$$z = z$$

for conversion from rectangular to cylindrical coordinates, the original fields may be written in terms of the new components in

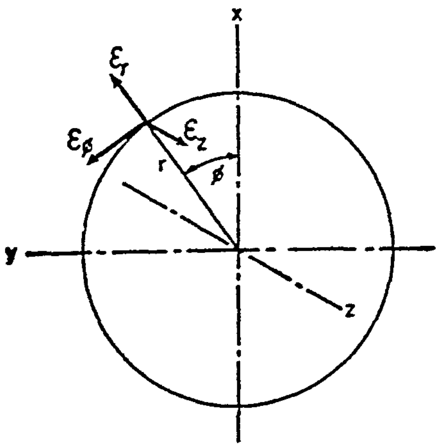


Fig. B-1. Cylindrical coordinate system employed.

directions r, ϕ, and z by observation of Fig. B-1 as

$$\left.\begin{aligned} \mathcal{E}_x &= \mathcal{E}_r \cos \phi - \mathcal{E}_\phi \sin \phi \\ \mathcal{E}_y &= \mathcal{E}_r \sin \phi + \mathcal{E}_\phi \cos \phi \\ \mathcal{E}_z &= \mathcal{E}_z \end{aligned}\right\} \qquad \text{(B-1)}$$

$$\left.\begin{aligned} H_x &= H_r \cos\phi - H_\phi \sin\phi \\ H_y &= H_r \sin\phi + H_\phi \cos\phi \\ H_z &= H_z \end{aligned}\right\} \tag{B-2}$$

Derivatives of the coordinates may be obtained as

$$\frac{\partial r}{\partial y} = \sin\phi, \qquad \frac{\partial \phi}{\partial y} = \frac{\cos\phi}{r}$$

$$\frac{\partial r}{\partial x} = \cos\phi, \qquad \frac{\partial \phi}{\partial x} = -\frac{\sin\phi}{r}$$

As an example use Maxwell's first equation

$$\left.\begin{aligned} \frac{\partial \mathcal{E}_z}{\partial y} - \frac{\partial \mathcal{E}_y}{\partial z} &= -\mu\frac{\partial H_x}{\partial t} \quad \text{(a)} \\ \frac{\partial \mathcal{E}_x}{\partial z} - \frac{\partial \mathcal{E}_z}{\partial x} &= -\mu\frac{\partial H_y}{\partial t} \quad \text{(b)} \\ \frac{\partial \mathcal{E}_y}{\partial x} - \frac{\partial \mathcal{E}_x}{\partial y} &= -\mu\frac{\partial H_z}{\partial t} \quad \text{(c)} \end{aligned}\right\} \tag{B-3}$$

The partial derivatives may be prepared as follows:

$$\begin{aligned} \frac{\partial \mathcal{E}_z}{\partial y} &= \frac{\partial \mathcal{E}_z}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial \mathcal{E}_z}{\partial \phi}\frac{\partial \phi}{\partial y} \\ &= \frac{\partial \mathcal{E}_z}{\partial r}\sin\phi + \frac{1}{r}\frac{\partial \mathcal{E}_z}{\partial \phi}\cos\phi \end{aligned} \tag{B-4}$$

$$\frac{\partial \mathcal{E}_y}{\partial z} = \frac{\partial \mathcal{E}_r}{\partial z}\sin\phi + \frac{\partial \mathcal{E}_\phi}{\partial z}\cos\phi \tag{B-5}$$

$$\frac{\partial \mathcal{E}_x}{\partial z} = \frac{\partial \mathcal{E}_r}{\partial z}\cos\phi - \frac{\partial \mathcal{E}_\phi}{\partial z}\sin\phi \tag{B-6}$$

$$\begin{aligned} \frac{\partial \mathcal{E}_z}{\partial x} &= \frac{\partial \mathcal{E}_z}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial \mathcal{E}_z}{\partial \phi}\frac{\partial \phi}{\partial x} \\ &= \frac{\partial \mathcal{E}_z}{\partial r}\cos\phi - \frac{1}{r}\frac{\partial \mathcal{E}_z}{\partial \phi}\sin\phi \end{aligned} \tag{B-7}$$

$$\begin{aligned} \frac{\partial \mathcal{E}_y}{\partial x} &= \frac{\partial \mathcal{E}_y}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial \mathcal{E}_y}{\partial \phi}\frac{\partial \phi}{\partial x} \\ &= \frac{\partial \mathcal{E}_y}{\partial r}\cos\phi - \frac{1}{r}\frac{\partial \mathcal{E}_y}{\partial \phi}\sin\phi \end{aligned} \tag{B-8}$$

$$\frac{\partial \mathcal{E}_z}{\partial y} = \frac{\partial \mathcal{E}_z}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial \mathcal{E}_z}{\partial \phi}\frac{\partial \phi}{\partial y}$$

$$= \frac{\partial \mathcal{E}_z}{\partial r} \sin \phi + \frac{1}{r}\frac{\partial \mathcal{E}_z}{\partial \phi} \cos \phi \tag{B-9}$$

$$\frac{\partial H_x}{\partial t} = \frac{\partial H_r}{\partial t} \cos \phi - \frac{\partial H_\phi}{\partial t} \sin \phi \tag{B-10}$$

$$\frac{\partial H_y}{\partial t} = \frac{\partial H_r}{\partial t} \sin \phi + \frac{\partial H_\phi}{\partial t} \cos \phi \tag{B-11}$$

$$\frac{\partial H_z}{\partial t} = \frac{\partial H_z}{\partial t} \tag{B-12}$$

The appropriate values may be substituted in Eq. B-3(a):

$$\frac{\partial \mathcal{E}_z}{\partial r} \sin \phi + \frac{1}{r}\frac{\partial \mathcal{E}_z}{\partial \phi} \cos \phi - \frac{\partial \mathcal{E}_r}{\partial z} \sin \phi - \frac{\partial \mathcal{E}_\phi}{\partial z} \cos \phi$$
$$= -\mu \frac{\partial H_r}{\partial t} \cos \phi + \mu \frac{\partial H_\phi}{\partial t} \sin \phi \tag{B-13}$$

and in Eq. B-3(b):

$$\frac{\partial \mathcal{E}_r}{\partial z} \cos \phi - \frac{\partial \mathcal{E}_\phi}{\partial z} \sin \phi - \frac{\partial \mathcal{E}_z}{\partial r} \cos \phi + \frac{1}{r}\frac{\partial \mathcal{E}_z}{\partial \phi} \sin \phi$$
$$= -\mu \frac{\partial H_r}{\partial t} \sin \phi - \mu \frac{\partial H_\phi}{\partial t} \cos \phi \tag{B-14}$$

Multiply Eq. B-13 by $\cos \phi$ and Eq. B-14 by $\sin \phi$ and add, obtaining:

$$\frac{1}{r}\frac{\partial \mathcal{E}_z}{\partial \phi} - \frac{\partial \mathcal{E}_\phi}{\partial z} = -\mu \frac{\partial H_r}{\partial t} \tag{B-15}$$

Multiply Eq. B-13 by $\sin \phi$ and Eq. B-14 by $\cos \phi$ and subtract, obtaining

$$\frac{\partial \mathcal{E}_r}{\partial z} - \frac{\partial \mathcal{E}_z}{\partial r} = -\mu \frac{\partial H_\phi}{\partial t} \tag{B-16}$$

In similar fashion, Eqs. B-8 and B-9 may be combined with Eq. B-1 as

$$\frac{\partial \mathcal{E}_y}{\partial x} = \left(\frac{\partial \mathcal{E}_r}{\partial r} \sin \phi + \frac{\partial \mathcal{E}_\phi}{\partial r} \cos \phi\right) \cos \phi - \frac{1}{r}\left(\frac{\partial \mathcal{E}_r}{\partial \phi} \sin \phi + \mathcal{E}_r \cos \phi\right.$$
$$\left. + \frac{\partial \mathcal{E}_\phi}{\partial \phi} \cos \phi - \mathcal{E}_\phi \sin \phi\right) \sin \phi$$

$$\frac{\partial \mathcal{E}_z}{\partial y} = \left(\frac{\partial \mathcal{E}_r}{\partial r} \cos \phi - \frac{\partial \mathcal{E}_\phi}{\partial r} \sin \phi\right) \sin \phi + \frac{1}{r}\left(\frac{\partial \mathcal{E}_r}{\partial \phi} \cos \phi - \mathcal{E}_r \sin \phi - \frac{\partial \mathcal{E}_\phi}{\partial \phi} \sin \phi - \mathcal{E}_\phi \cos \phi\right) \cos \phi$$

before substitution in Eq. B-3(c) to give:

$$\frac{1}{r}\mathcal{E}_\phi - \frac{1}{r}\frac{\partial \mathcal{E}_r}{\partial \phi} + \frac{\partial \mathcal{E}_\phi}{\partial r} = -\mu \frac{\partial H_z}{\partial t}$$

which may be written

$$\frac{1}{r}\frac{\partial}{\partial r}(r\mathcal{E}_\phi) - \frac{1}{r}\frac{\partial \mathcal{E}_r}{\partial \phi} = -\mu \frac{\partial H_z}{\partial t} \tag{B-17}$$

Collecting Eqs. B-15, B-16, and B-17 allows Maxwell's first field equation to be written in cylindrical coordinates as

$$\left.\begin{aligned} \frac{1}{r}\frac{\partial \mathcal{E}_z}{\partial \phi} - \frac{\partial \mathcal{E}_\phi}{\partial z} &= -\mu \frac{\partial H_r}{\partial t} \\ \frac{\partial \mathcal{E}_r}{\partial z} - \frac{\partial \mathcal{E}_z}{\partial r} &= -\mu \frac{\partial H_\phi}{\partial t} \\ \frac{1}{r}\frac{\partial}{\partial r}(r\mathcal{E}_\phi) - \frac{1}{r}\frac{\partial \mathcal{E}_r}{\partial \phi} &= -\mu \frac{\partial H_z}{\partial t} \end{aligned}\right\} \tag{B-18}$$

The second field equation may be written, after similar manipulation, as

$$\left.\begin{aligned} \frac{1}{r}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z} &= \sigma\mathcal{E}_r + \epsilon \frac{\partial \mathcal{E}_r}{\partial t} \\ \frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} &= \sigma\mathcal{E}_\phi + \epsilon \frac{\partial \mathcal{E}_\phi}{\partial t} \\ \frac{1}{r}\frac{\partial}{\partial r}(rH_\phi) - \frac{1}{r}\frac{\partial H_r}{\partial \phi} &= \sigma\mathcal{E}_z + \epsilon \frac{\partial \mathcal{E}_z}{\partial t} \end{aligned}\right\} \tag{B-19}$$

Appendix C

THE EVALUATION OF THE INTEGRAL

$$\int_0^\pi \frac{\cos^2(\frac{1}{2}\pi\cos\theta)}{\sin\theta}\,d\theta \tag{C-1}$$

proceeds by writing

$$\cos^2\theta = \tfrac{1}{2}(1+\cos 2\theta)$$

whereby

$$\int_0^\pi \frac{\cos^2(\frac{1}{2}\pi\cos\theta)}{\sin\theta}\,d\theta = \frac{1}{2}\int_0^\pi \frac{1+\cos(\pi\cos\theta)\,d\theta}{\sin\theta} \tag{C-2}$$

The variable is changed to $u = \cos\theta$, whence

$$du = \sin\theta\,d\theta, \qquad \theta = \cos^{-1} u$$

$$\sin\theta = \sqrt{1-u^2}, \qquad d\theta = -\frac{du}{\sqrt{1-u^2}}$$

with limits of

$$\theta = \pi, \qquad u = -1$$

$$\theta = 0, \qquad u = 1$$

so that Eq. C-2 becomes

$$\frac{1}{2}\int_1^{-1} \frac{-(1+\cos\pi u)}{\sqrt{1-u^2}}\,\frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\int_{-1}^{1} \frac{1+\cos\pi u}{1-u^2}\,du \tag{C-3}$$

and since

$$\frac{1}{1-u^2} = \frac{1}{2}\left(\frac{1}{1+u} + \frac{1}{1-u}\right)$$

C-3 becomes

$$\frac{1}{4}\int_{-1}^{1} \frac{1+\cos\pi u}{1+u}\,du + \frac{1}{4}\int_{-1}^{1} \frac{1+\cos\pi u}{1-u}\,du\,. \tag{C-4}$$

If a further variable change be made in the second integral such that $u = -x$, $du = -dx$, with limits

$$u = 1, \qquad x = -1$$

$$u = -1, \qquad x = 1$$

the second integral becomes

$$\frac{1}{4}\int_1^{-1} \frac{-(1+\cos\pi x\,dx)}{1+x} = \frac{1}{4}\int_{-1}^{1} \frac{1+\cos\pi x\,dx}{1+x} \tag{C-5}$$

and this is of the same form as the first integral of C-4, so that C-4 then becomes

$$= \frac{1}{2} \int_{-1}^{1} \frac{1 + \cos \pi u \, du}{1 + u} \tag{C-6}$$

Now let

$$1 + u = \frac{v}{\pi}, \qquad du = \frac{dv}{\pi}, \qquad u = \frac{v - \pi}{\pi}$$

with limits

$$u = 1, \qquad v = 2\pi$$
$$u = -1, \qquad v = 0$$

Then Eq. C-6 becomes

$$= \frac{1}{2} \int_{0}^{2\pi} \frac{1 + \cos (v - \pi)}{v/\pi} \frac{dv}{\pi} = \frac{1}{2} \int_{0}^{2\pi} \frac{1 - \cos v \, dv}{v} \tag{C-7}$$

However,

$$\frac{1}{2} \int_{0}^{2\pi} \frac{(1 - \cos v)dv}{v} = \frac{1}{2} \left[\int_{1}^{2\pi} \frac{dv}{v} + \int_{0}^{1} \frac{(1 - \cos v)dv}{v} - \int_{1}^{\infty} \frac{\cos v \, dv}{v} + \int_{2\pi}^{\infty} \frac{\cos v \, dv}{v} \right] \tag{C-8}$$

in which

$$\int_{1}^{2\pi} \frac{dv}{v} = \ln 2\pi = 1.83788$$

$$\int_{0}^{1} \frac{(1 - \cos v)dv}{v} = \int_{0}^{1} \frac{1 - \left(1 - \frac{v^2}{2!} + \frac{v^4}{4!} - \frac{v^6}{6!} + \cdots \right) dv}{v}$$
$$= 0.23961$$

$$\int_{2\pi}^{\infty} \frac{\cos v \, dv}{v} = \text{Ci } 1 = 0.3374$$

where Ci = cosine integral from tables.

$$\int_{2\pi}^{\infty} \frac{\cos v \, dv}{v} = \text{Ci } 2\pi = 0.02716$$

Therefore

$$\frac{1}{2} \int_{0}^{2\pi} \frac{(1 - \cos v)dv}{v} = 1.221 \tag{C-9}$$

which is the result for Eq. C-1.

Appendix D: TABLES

Table A1: Exponentials

Value of ϵ^n

n	0	1	2	3	4	5	6	7	8	9
0.0	1.000	1.010	1.020	1.030	1.041	1.051	1.062	1.073	1.083	1.094
0.1	1.105	1.116	1.127	1.139	1.150	1.162	1.174	1.185	1.197	1.209
0.2	1.221	1.234	1.246	1.259	1.271	1.284	1.297	1.310	1.323	1.336
0.3	1.350	1.363	1.377	1.391	1.405	1.419	1.433	1.448	1.462	1.477
0.4	1.492	1.507	1.522	1.537	1.553	1.568	1.584	1.600	1.616	1.632
0.5	1.649	1.665	1.682	1.699	1.716	1.733	1.751	1.768	1.786	1.804
0.6	1.822	1.840	1.859	1.878	1.896	1.916	1.935	1.954	1.974	1.994
0.7	2.014	2.034	2.054	2.075	2.096	2.117	2.138	2.160	2.181	2.203
0.8	2.226	2.248	2.270	2.293	2.316	2.340	2.363	2.387	2.411	2.435
0.9	2.460	2.484	2.509	2.535	2.560	2.586	2.612	2.638	2.664	2.691
1.0	2.718	2.746	2.773	2.801	2.829	2.858	2.886	2.915	2.945	2.974
.1	3.004	3.034	3.065	3.096	3.127	3.158	3.190	3.222	3.254	3.287
.2	3.320	3.353	3.387	3.421	3.456	3.490	3.525	3.561	3.597	3.633
.3	3.669	3.706	3.743	3.781	3.819	3.857	3.896	3.935	3.975	4.015
.4	4.055	4.096	4.137	4.179	4.221	4.263	4.306	4.349	4.393	4.437
.5	4.482	4.527	4.572	4.618	4.665	4.711	4.759	4.807	4.855	4.904
.6	4.953	5.003	5.053	5.104	5.155	5.207	5.259	5.312	5.366	5.419
.7	5.474	5.529	5.585	5.641	5.697	5.755	5.812	5.871	5.930	5.989
.8	6.050	6.110	6.172	6.234	6.297	6.360	6.424	6.488	6.554	6.619
.9	6.686	6.753	6.821	6.890	6.959	7.029	7.099	7.171	7.243	7.316
2.0	7.389	7.463	7.538	7.614	7.691	7.768	7.846	7.925	8.001	8.085
.1	8.166	8.248	8.331	8.415	8.499	8.585	8.671	8.758	8.846	8.935
.2	9.025	9.116	9.207	9.300	9.393	9.488	9.583	9.679	9.777	9.875
.3	9.974	10.07	10.18	10.28	10.38	10.49	10.59	10.70	10.81	10.91
.4	11.02	11.13	11.25	11.36	11.47	11.59	11.71	11.82	11.94	12.06
.5	12.18	12.31	12.43	12.55	12.68	12.81	12.94	13.07	13.20	13.33
.6	13.46	13.60	13.74	13.87	14.01	14.15	14.30	14.44	14.59	14.73
.7	14.88	15.03	15.18	15.33	15.49	15.64	15.80	15.96	16.12	16.28
.8	16.45	16.61	16.78	16.95	17.12	17.29	17.46	17.64	17.81	17.99
.9	18.17	18.36	18.54	18.73	18.92	19.11	19.30	19.49	19.69	19.89
3.0	20.09	20.29	20.49	20.70	20.91	21.12	21.33	21.54	21.76	21.98
.1	22.20	22.42	22.65	22.87	23.10	23.34	23.57	23.81	24.05	24.29
.2	24.53	24.78	25.03	25.28	25.53	25.79	26.05	26.31	26.58	26.84
.3	27.11	27.39	27.66	27.94	28.22	28.50	28.79	29.08	29.37	29.67
.4	29.96	30.27	30.57	30.88	31.19	31.50	31.82	32.14	32.46	32.79
.5	33.11	33.45	33.78	34.12	34.47	34.81	35.16	35.52	35.87	36.23
.6	36.60	36.97	37.34	37.71	38.09	38.47	38.86	39.25	39.65	40.05
.7	40.45	40.85	41.26	41.68	42.10	42.52	42.95	43.38	43.82	44.26
.8	44.70	45.15	45.60	46.06	46.53	46.99	47.47	47.94	48.42	48.91
.9	49.40	49.90	50.40	50.91	51.42	51.93	52.46	52.99	53.52	54.05

TABLE A1 (Cont.): EXPONENTIALS

Value of ϵ^{-n}

$-n$	0	1	2	3	4	5	6	7	8	9
0.0	1.0000	0.9900	0.9802	0.9705	0.9608	0.9512	0.9418	0.9324	0.9231	0.9139
.1	.9048	.8958	.8869	.8781	.8694	.8607	.8521	.8437	.8353	.8270
.2	.8187	.8106	.8025	.7945	.7866	.7788	.7711	.7634	.7558	.7483
.3	.7408	.7334	.7262	.7189	.7118	.7047	.6977	.6907	.6839	.6771
.4	.6703	.6637	.6571	.6505	.6440	.6376	.6313	.6250	.6188	.6126
.5	.6065	.6005	.5945	.5886	.5828	.5770	.5712	.5655	.5599	.5543
.6	.5488	.5434	.5379	.5326	.5273	.5221	.5169	.5117	.5066	.5016
.7	.4966	.4916	.4868	.4819	.4771	.4724	.4677	.4630	.4584	.4538
.8	.4493	.4449	.4404	.4361	.4317	.4274	.4232	.4190	.4148	.4107
.9	.4066	.4025	.3985	.3946	.3906	.3867	.3829	.3791	.3753	.3716
1.0	.3679	.3642	.3606	.3570	.3535	.3499	.3465	.3430	.3396	.3362
.1	.3329	.3296	.3263	.3230	.3198	.3166	.3135	.3104	.3073	.3042
.2	.3012	.2982	.2952	.2923	.2894	.2865	.2837	.2808	.2780	.2753
.3	.2725	.2698	.2671	.2645	.2619	.2592	.2567	.2541	.2516	.2491
.4	.2466	.2441	.2417	.2393	.2369	.2346	.2322	.2299	.2276	.2254
.5	.2231	.2209	.2187	.2165	.2144	.2123	.2001	.2081	.2060	.2039
.6	.2019	.1999	.1979	.1959	.1940	.1921	.1901	.1883	.1864	.1845
.7	.1827	.1809	.1791	.1773	.1755	.1738	.1720	.1703	.1686	.1670
.8	.1653	.1637	.1620	.1604	.1588	.1572	.1557	.1541	.1526	.1511
.9	.1496	.1481	.1466	.1451	.1437	.1423	.1409	.1395	.1381	.1367
2.0	.1353	.1340	.1327	.1313	.1300	.1287	.1275	.1262	.1249	.1237
.1	.1225	.1212	.1200	.1188	.1177	.1165	.1153	.1142	.1130	.1119
.2	.1108	.1097	.1086	.1075	.1065	.1054	.1043	.1033	.1023	.1013
.3	.1003	.0993	.0983	.0973	.0963	.0954	.0944	.0935	.0925	.0916
.4	.0907	.0898	.0889	.0880	.0872	.0863	.0854	.0846	.0837	.0829
.5	.0821	.0813	.0805	.0797	.0789	.0781	.0773	.0765	.0758	.0750
.6	.0743	.0735	.0728	.0721	.0714	.0707	.0699	.0693	.0686	.0679
.7	.0672	.0665	.0659	.0652	.0646	.0639	.0633	.0627	.0620	.0614
.8	.0608	.0602	.0596	.0590	.0584	.0578	.0563	.0567	.0551	.0556
.9	.0550	.0545	.0539	.0534	.0529	.0523	.0518	.0513	.0508	.0503
3.0	.0498	.0493	.0488	.0483	.0478	.0474	.0469	.0464	.0460	.0455
.1	.0450	.0446	.0442	.0437	.0433	.0429	.0424	.0420	.0416	.0412
.2	.0408	.0404	.0400	.0396	.0392	.0388	.0384	.0380	.0376	.0373
.3	.0369	.0365	.0361	.0358	.0354	.0351	.0347	.0344	.0340	.0337
.4	.0334	.0330	.0327	.0324	.0321	.0317	.0314	.0311	.0308	.0305
.5	.0302	.0299	.0296	.0293	.0290	.0287	.0284	.0281	.0279	.0276
.6	.0273	.0271	.0268	.0265	.0263	.0260	.0257	.0255	.0252	.0250
.7	.0247	.0245	.0242	.0240	.0238	.0235	.0233	.0231	.0228	.0226
.8	.0224	.0221	.0219	.0217	.0215	.0213	.0211	.0209	.0207	.0204
.9	.0202	.0200	.0198	.0196	.0194	.0193	.0191	.0189	.0187	.0185

TABLE A2: DEGREES TO RADIANS

Degs.	0	1	2	3	4	5	6	7	8	9
0	0.0000	0.01745	0.03491	0.05236	0.06981	0.08727	0.1047	0.1222	0.1396	0.1571
10	.1745	.1920	.2094	.2269	.2443	.2618	.2793	.2967	.3141	.3316
20	.3491	.3665	.3840	.4014	.4189	.4363	.4538	.4712	.4887	.5061
30	.5236	.5410	.5585	.5760	.5934	.6109	.6283	.6458	.6632	.6807
40	.6981	.7156	.7330	.7505	.7679	.7854	.8028	.8203	.8378	.8552
50	.8727	.8901	.9076	.9250	.9425	.9599	.9774	.9948	1.012	1.030
60	1.047	1.065	1.082	1.100	1.117	1.134	1.152	1.169	1.187	1.204
70	1.222	1.239	1.257	1.274	1.292	1.309	1.326	1.344	1.361	1.379
80	1.396	1.414	1.431	1.449	1.466	1.484	1.501	1.518	1.536	1.553
90	1.571	1.588	1.606	1.623	1.641	1.658	1.676	1.693	1.710	1.728
100	1.745	1.763	1.780	1.798	1.815	1.833	1.850	1.868	1.885	1.902
110	1.920	1.937	1.955	1.972	1.990	2.007	2.025	2.042	2.059	2.077
120	2.094	2.112	2.129	2.147	2.164	2.182	2.199	2.217	2.234	2.251
130	2.269	2.286	2.304	2.321	2.339	2.356	2.374	2.391	2.409	2.426
140	2.443	2.461	2.478	2.496	2.513	2.531	2.548	2.566	2.583	2.601
150	2.618	2.635	2.653	2.670	2.688	2.705	2.723	2.740	2.758	2.775
160	2.793	2.810	2.827	2.845	2.862	2.880	2.897	2.915	2.932	2.950
170	2.967	2.985	3.002	3.019	3.037	3.054	3.072	3.089	3.107	3.124
180	3.1416									

TABLE A3: RADIANS TO DEGREES

Radians	0	1	2	3	4	5	6	7	8	9
0.0	0.0	0.57	1.15	1.72	2.29	2.86	3.44	4.01	4.58	5.16
.1	5.73	6.30	6.88	7.45	8.02	8.59	9.17	9.74	10.31	10.90
.2	11.46	12.03	12.61	13.18	13.75	14.32	14.90	15.47	16.04	16.62
.3	17.19	17.76	18.33	18.91	19.48	20.05	20.63	21.20	21.77	22.35
.4	22.92	23.49	24.06	24.64	25.21	25.78	26.36	26.93	27.50	28.07
.5	28.65	29.22	29.79	30.37	30.94	31.51	32.09	32.66	33.23	33.80
.6	34.38	34.95	35.52	36.10	36.67	37.24	37.81	38.39	38.96	39.53
.7	40.10	40.68	41.25	41.83	42.40	42.97	43.54	44.12	44.69	45.26
.8	45.84	46.41	46.98	47.55	48.13	48.70	49.27	49.85	50.42	50.99
.9	51.57	52.14	52.71	53.29	53.86	54.43	55.00	55.58	56.15	56.72
1.0	57.30	57.87	58.44	59.01	59.59	60.16	60.73	61.30	61.88	62.45
.1	63.03	63.60	64.17	64.74	65.32	65.89	66.46	67.04	67.61	68.18
.2	68.75	69.33	69.90	70.47	71.05	71.62	72.19	72.77	73.34	73.91
.3	74.48	75.06	75.63	76.20	76.78	77.35	77.92	78.50	79.07	79.64
.4	80.21	80.79	81.36	81.93	82.50	83.08	83.65	84.22	84.80	85.37
.5	85.94	86.52	87.09	87.66	88.24	88.81	89.38	89.95	90.53	91.10
.6	91.67	92.25	92.82	93.39	93.97	94.54	95.11	95.68	96.26	96.83
.7	97.40	97.98	98.55	99.12	99.69	100.3	100.8	101.4	102.0	102.6
.8	103.1	103.7	104.3	104.8	105.4	106.0	106.6	107.1	107.7	108.3
.9	108.9	109.4	110.0	110.6	111.1	111.7	112.3	112.9	113.4	114.0
2.0	114.6	115.2	115.7	116.3	116.9	117.5	118.0	118.6	119.2	119.7
.1	120.3	120.9	121.5	122.0	122.6	123.2	123.8	124.3	124.9	125.5
.2	126.0	126.6	127.2	127.8	128.3	128.9	129.5	130.1	130.6	131.2
.3	131.8	132.4	132.9	133.5	134.1	134.6	135.2	135.8	136.4	136.9
.4	137.5	138.1	138.7	139.2	139.8	140.4	140.9	141.5	142.1	142.7
.5	143.2	143.8	144.4	145.0	145.5	146.1	146.7	147.2	147.8	148.4
.6	149.0	149.5	150.1	150.7	151.3	151.8	152.4	153.0	153.6	154.1
.7	154.7	155.3	155.8	156.4	157.0	157.6	158.1	158.7	159.3	159.9
.8	160.4	161.0	161.6	162.1	162.7	163.3	163.9	164.4	165.0	165.6
.9	166.2	166.7	167.3	167.9	168.4	169.0	169.6	170.2	170.7	171.3
3.0	171.9	172.5	173.0	173.6	174.2	174.7	175.3	175.9	176.5	177.0
.1	177.6	178.2	178.8	179.3	179.9	180.5				

TABLE A4: TRIGONOMETRIC FUNCTIONS

Deg	Sin	Cos	Tan	
0.0	.00000	1.0000	.00000	90
0.5	.00873	1.0000	.00873	89.5
1.0	.01745	0.9998	.01746	89.0
1.5	.02618	.9997	.02619	88.5
2.0	.03490	.9994	.03492	88.0
2.5	.04362	.9990	.04366	87.5
3.0	.05234	.9986	.05241	87.0
3.5	.06105	.9981	.06116	86.5
4.0	.06976	.9976	.06993	86.0
4.5	.07846	.9969	.0787	85.5
5.0	.0872	.9962	.0875	85.0
5.5	.0959	.9954	.0963	84.5
6.0	.1045	.9945	.1051	84.0
6.5	.1132	.9936	.1139	83.5
7.0	.1219	.9925	.1228	83.0
7.5	.1305	.9914	.1316	82.5
8.0	.1392	.9903	.1405	82.0
8.5	.1478	.9890	.1494	81.5
9.0	.1564	.9877	.1584	81.0
9.5	.1651	.9863	.1673	80.5
10.0	.1736	.9848	.1763	80.0
10.5	.1822	.9833	.1853	79.5
11.0	.1908	.9816	.1944	79.0
11.5	.1994	.9799	.2034	78.5
12.0	.2079	.9781	.2126	78.0
12.5	.2164	.9763	.2217	77.5
13.0	.2250	.9744	.2309	77.0
13.5	.2334	.9724	.2401	76.5
14.0	.2419	.9703	.2493	76.0
14.5	.2504	.9681	.2586	75.5
15.0	.2588	.9659	.2679	75.0
15.5	.2672	.9636	.2773	74.5
16.0	.2756	.9613	.2867	74.0
16.5	.2840	.9588	.2962	73.5
17.0	.2924	.9563	.3057	73.0
17.5	.3007	.9537	.3153	72.5
18.0	.3090	.9511	.3249	72.0
18.5	.3173	.9483	.3346	71.5
19.0	.3256	.9455	.3443	71.0
19.5	.3338	.9426	.3541	70.5
20.0	.3420	.9397	.3640	70.0
20.5	.3502	.9367	.3739	69.5
21.0	.3584	.9336	.3839	69.0
21.5	.3665	.9304	.3939	68.5
22.0	.3746	.9272	.4040	68.0
	Cos	Sin	Cot	Deg

TABLE A4 (Cont.): TRIGONOMETRIC FUNCTIONS

Deg	Sin	Cos	Tan	
22.5	.3827	.9239	.4142	67.5
23.0	.3907	.9205	.4245	67.0
23.5	.3987	.9171	.4348	66.5
24.0	.4067	.9135	.4452	66.0
24.5	.4147	.9100	.4557	65.5
25.0	.4226	.9063	.4663	65.0
25.5	.4305	.9026	.4770	64.5
26.0	.4384	.8988	.4877	64.0
26.5	.4462	.8949	.4986	63.5
27.0	.4540	.8910	.5095	63.0
27.5	.4617	.8870	.5206	62.5
28.0	.4695	.8829	.5317	62.0
28.5	.4772	.8788	.5430	61.5
29.0	.4848	.8746	.5543	61.0
29.5	.4924	.8704	.5658	60.5
30.0	.5000	.8660	.5773	60.0
30.5	.5075	.8616	.5890	59.5
31.0	.5150	.8572	.6009	59.0
31.5	.5225	.8526	.6128	58.5
32.0	.5299	.8480	.6249	58.0
32.5	.5373	.8434	.6371	57.5
33.0	.5446	.8387	.6494	57.0
33.5	.5519	.8339	.6619	56.5
34.0	.5592	.8290	.6745	56.0
34.5	.5664	.8241	.6873	55.5
35.0	.5736	.8191	.7002	55.0
35.5	.5807	.8141	.7133	54.5
36.0	.5878	.8090	.7265	54.0
36.5	.5948	.8039	.7400	53.5
37.0	.6018	.7986	.7535	53.0
37.5	.6088	.7933	.7673	52.5
38.0	.6157	.7880	.7813	52.0
38.5	.6225	.7826	.7954	51.5
39.0	.6293	.7771	.8098	51.0
39.5	.6361	.7716	.8243	50.5
40.0	.6428	.7660	.8391	50.0
40.5	.6494	.7604	.8541	49.5
41.0	.6561	.7547	.8693	49.0
41.5	.6626	.7490	.8847	48.5
42.0	.6691	.7431	.9004	48.0
42.5	.6756	.7373	.9163	47.5
43.0	.6820	.7313	.9325	47.0
43.5	.6884	.7254	.9490	46.5
44.0	.6947	.7193	.9657	46.0
44.5	.7009	.7132	.9827	45.5

Table A5: Hyperbolic Functions*

Hyperbolic sines [$\sinh x = \frac{1}{2}(e^x - e^{-x})$]

x	0	1	2	3	4	5	6	7	8	9	avg diff
0.0	0.0000	0.0100	0.0200	0.0300	0.0400	0.0500	0.0600	0.0701	0.0801	0.0901	100
.1	0.1002	0.1102	0.1203	0.1304	0.1405	0.1506	0.1607	0.1708	0.1810	0.1911	101
.2	0.2013	0.2115	0.2218	0.2320	0.2423	0.2526	0.2629	0.2733	0.2837	0.2941	103
.3	0.3045	0.3150	0.3255	0.3360	0.3466	0.3572	0.3678	0.3785	0.3892	0.4000	106
.4	0.4108	0.4216	0.4325	0.4434	0.4543	0.4653	0.4764	0.4875	0.4986	0.5098	110
0.5	0.5211	0.5324	0.5438	0.5552	0.5666	0.5782	0.5897	0.6014	0.6131	0.6248	116
.6	0.6367	0.6485	0.6605	0.6725	0.6846	0.6967	0.7090	0.7213	0.7336	0.7461	122
.7	0.7586	0.7712	0.7838	0.7966	0.8094	0.8223	0.8353	0.8484	0.8615	0.8748	130
.8	0.8881	0.9015	0.9150	0.9286	0.9423	0.9561	0.9700	0.9840	0.9981	1.012	138
.9	1.027	1.041	1.055	1.070	1.085	1.099	1.114	1.129	1.145	1.160	15
1.0	1.175	1.191	1.206	1.222	1.238	1.254	1.270	1.286	1.303	1.319	16
.1	1.336	1.352	1.369	1.386	1.403	1.421	1.438	1.456	1.474	1.491	17
.2	1.509	1.528	1.546	1.564	1.583	1.602	1.621	1.640	1.659	1.679	19
.3	1.698	1.718	1.738	1.758	1.779	1.799	1.820	1.841	1.862	1.883	21
.4	1.904	1.926	1.948	1.970	1.992	2.014	2.037	2.060	2.083	2.106	22
1.5	2.129	2.153	2.177	2.201	2.225	2.250	2.274	2.299	2.324	2.350	25
.6	2.376	2.401	2.428	2.454	2.481	2.507	2.535	2.562	2.590	2.617	27
.7	2.646	2.674	2.703	2.732	2.761	2.790	2.820	2.850	2.881	2.911	30
.8	2.942	2.973	3.005	3.037	3.069	3.101	3.134	3.167	3.200	3.234	33
.9	3.268	3.303	3.337	3.372	3.408	3.443	3.479	3.516	3.552	3.589	36
2.0	3.627	3.665	3.703	3.741	3.780	3.820	3.859	3.899	3.940	3.981	39
.1	4.022	4.064	4.106	4.148	4.191	4.234	4.278	4.322	4.367	4.412	44
.2	4.457	4.503	4.549	4.596	4.643	4.691	4.739	4.788	4.837	4.887	48
.3	4.937	4.988	5.039	5.090	5.142	5.195	5.248	5.302	5.356	5.411	53
.4	5.466	5.522	5.578	5.635	5.693	5.751	5.810	5.869	5.929	5.989	58
2.5	6.050	6.112	6.174	6.237	6.300	6.365	6.429	6.495	6.561	6.627	64
.6	6.695	6.763	6.831	6.901	6.971	7.042	7.113	7.185	7.258	7.332	71
.7	7.406	7.481	7.557	7.634	7.711	7.789	7.868	7.948	8.028	8.110	79
.8	8.192	8.275	8.359	8.443	8.529	8.615	8.702	8.790	8.879	8.969	87
.9	9.060	9.151	9.244	9.337	9.431	9.527	9.623	9.720	9.819	9.918	96
3.0	10.02	10.12	10.22	10.32	10.43	10.53	10.64	10.75	10.86	10.97	11
.1	11.08	11.19	11.30	11.42	11.53	11.65	11.76	11.88	12.00	12.12	12
.2	12.25	12.37	12.49	12.62	12.75	12.88	13.01	13.14	13.27	13.40	13
.3	13.54	13.67	13.81	13.95	14.09	14.23	14.38	14.52	14.67	14.82	14
.4	14.97	15.12	15.27	15.42	15.58	15.73	15.89	16.05	16.21	16.38	16
3.5	16.54	16.71	16.88	17.05	17.22	17.39	17.57	17.74	17.92	18.10	17
.6	18.29	18.47	18.66	18.84	19.03	19.22	19.42	19.61	19.81	20.01	19
.7	20.21	20.41	20.62	20.83	21.04	21.25	21.46	21.68	21.90	22.12	21
.8	22.34	22.56	22.79	23.02	23.25	23.49	23.72	23.96	24.20	24.45	24
.9	24.69	24.94	25.19	25.44	25.70	25.96	26.22	26.48	26.75	27.02	25
4.0	27.29	27.56	27.84	28.12	28.40	28.69	28.98	29.27	29.56	29.86	29
.1	30.16	30.47	30.77	31.08	31.39	31.71	32.03	32.35	32.68	33.00	32
.2	33.34	33.67	34.01	34.35	34.70	35.05	35.40	35.75	36.11	36.48	35
.3	36.84	37.21	37.59	37.97	38.35	38.73	39.12	39.52	39.91	40.31	39
.4	40.72	41.13	41.54	41.96	42.38	42.81	43.24	43.67	44.11	44.56	43
4.5	45.00	45.46	45.91	46.37	46.84	47.31	47.79	48.27	48.75	49.24	47
.6	49.74	50.24	50.74	51.25	51.77	52.29	52.81	53.34	53.88	54.42	52
.7	54.97	55.52	56.08	56.64	57.21	57.79	58.37	58.96	59.55	60.15	58
.8	60.75	61.36	61.98	62.60	63.23	63.87	64.51	65.16	65.81	66.47	64
.9	67.14	67.82	68.50	69.19	69.88	70.58	71.29	72.01	72.73	73.46	71
5.0	74.20										

If $x > 5$, $\sinh x = \frac{1}{2}(e^x)$ and $\log_{10} \sinh x = (0.4343)x + 0.6990 - 1$, correct to four significant figures.

* Tables on pages 580–582 reproduced by permission from *Reference Data for Radio Engineers*, 3d ed., Federal Telephone and Radio Corporation, New York, 1949.

Hyperbolic cosines $[\cosh x = \frac{1}{2}(e^x + e^{-x})]$

x	0	1	2	3	4	5	6	7	8	9	avg diff
0.0	1.000	1.000	1.000	1.000	1.001	1.001	1.002	1.002	1.003	1.004	1
.1	1.005	1.006	1.007	1.008	1.010	1.011	1.013	1.014	1.016	1.018	2
.2	1.020	1.022	1.024	1.027	1.029	1.031	1.034	1.037	1.039	1.042	3
.3	1.045	1.048	1.052	1.055	1.058	1.062	1.066	1.069	1.073	1.077	4
.4	1.081	1.085	1.090	1.094	1.098	1.103	1.108	1.112	1.117	1.122	5
0.5	1.128	1.133	1.138	1.144	1.149	1.155	1.161	1.167	1.173	1.179	6
.6	1.185	1.192	1.198	1.205	1.212	1.219	1.226	1.233	1.240	1.248	7
.7	1.255	1.263	1.271	1.278	1.287	1.295	1.303	1.311	1.320	1.329	8
.8	1.337	1.346	1.355	1.365	1.374	1.384	1.393	1.403	1.413	1.423	10
.9	1.433	1.443	1.454	1.465	1.475	1.486	1.497	1.509	1.520	1.531	11
1.0	1.543	1.555	1.567	1.579	1.591	1.604	1.616	1.629	1.642	1.655	13
.1	1.669	1.682	1.696	1.709	1.723	1.737	1.752	1.766	1.781	1.796	14
.2	1.811	1.826	1.841	1.857	1.872	1.888	1.905	1.921	1.937	1.954	16
.3	1.971	1.988	2.005	2.023	2.040	2.058	2.076	2.095	2.113	2.132	18
.4	2.151	2.170	2.189	2.209	2.229	2.249	2.269	2.290	2.310	2.331	20
1.5	2.352	2.374	2.395	2.417	2.439	2.462	2.484	2.507	2.530	2.554	23
.6	2.577	2.601	2.625	2.650	2.675	2.700	2.725	2.750	2.776	2.802	25
.7	2.828	2.855	2.882	2.909	2.936	2.964	2.992	3.021	3.049	3.078	28
.8	3.107	3.137	3.167	3.197	3.228	3.259	3.290	3.321	3.353	3.385	31
.9	3.418	3.451	3.484	3.517	3.551	3.585	3.620	3.655	3.690	3.726	34
2.0	3.762	3.799	3.835	3.873	3.910	3.948	3.987	4.026	4.065	4.104	38
.1	4.144	4.185	4.226	4.267	4.309	4.351	4.393	4.436	4.480	4.524	42
.2	4.568	4.613	4.658	4.704	4.750	4.797	4.844	4.891	4.939	4.988	47
.3	5.037	5.087	5.137	5.188	5.239	5.290	5.343	5.395	5.449	5.503	52
.4	5.557	5.612	5.667	5.723	5.780	5.837	5.895	5.954	6.013	6.072	58
2.5	6.132	6.193	6.255	6.317	6.379	6.443	6.507	6.571	6.636	6.702	64
.6	6.769	6.836	6.904	6.973	7.042	7.112	7.183	7.255	7.327	7.400	70
.7	7.473	7.548	7.623	7.699	7.776	7.853	7.932	8.011	8.091	8.171	78
.8	8.253	8.335	8.418	8.502	8.587	8.673	8.759	8.847	8.935	9.024	86
.9	9.115	9.206	9.298	9.391	9.484	9.579	9.675	9.772	9.869	9.968	95
3.0	10.07	10.17	10.27	10.37	10.48	10.58	10.69	10.79	10.90	11.01	11
.1	11.12	11.23	11.35	11.45	11.57	11.69	11.81	11.92	12.04	12.16	12
.2	12.29	12.41	12.53	12.66	12.79	12.91	13.04	13.17	13.31	13.44	13
.3	13.57	13.71	13.85	13.99	14.13	14.27	14.41	14.56	14.70	14.85	14
.4	15.00	15.15	15.30	15.45	15.61	15.77	15.92	16.08	16.25	16.41	16
3.5	16.57	16.74	16.91	17.08	17.25	17.42	17.60	17.77	17.95	18.13	17
.6	18.31	18.50	18.68	18.87	19.06	19.25	19.44	19.64	19.84	20.03	19
.7	20.24	20.44	20.64	20.85	21.06	21.27	21.49	21.70	21.92	22.14	21
.8	22.36	22.59	22.81	23.04	23.27	23.51	23.74	23.98	24.22	24.47	23
.9	24.71	24.96	25.21	25.46	25.72	25.98	26.24	26.50	26.77	27.04	25
4.0	27.31	27.58	27.86	28.14	28.42	28.71	29.00	29.29	29.58	29.88	29
.1	30.18	30.48	30.79	31.10	31.41	31.72	32.04	32.37	32.69	33.02	32
.2	33.35	33.69	34.02	34.37	34.71	35.06	35.41	35.77	36.13	36.49	35
.3	36.86	37.23	37.60	37.98	38.36	38.75	39.13	39.53	39.93	40.33	39
.4	40.73	41.14	41.55	41.97	42.39	42.82	43.25	43.68	44.12	44.57	43
4.5	45.01	45.47	45.92	46.38	46.85	47.32	47.80	48.28	48.76	49.25	47
.6	49.75	50.25	50.75	51.26	51.78	52.30	52.82	53.35	53.89	54.43	52
.7	54.98	55.53	56.09	56.65	57.22	57.80	58.38	58.96	59.56	60.15	58
.8	60.76	61.37	61.99	62.61	63.24	63.87	64.52	65.16	65.82	66.48	64
.9	67.15	67.82	68.50	69.19	69.89	70.59	71.30	72.02	72.74	73.47	71
5.0	74.21										

If $x > 5$, $\cosh x = \frac{1}{2}(e^x)$, and $\log_{10} \cosh x = (0.4343)x + 0.6990 - 1$, correct to four significant figures.

Hyperbolic tangents [$\tanh x = (e^x - e^{-x})/(e^x + e^{-x}) = \sinh x/\cosh x$]

x	0	1	2	3	4	5	6	7	8	9	avg diff
0.0	0000	0100	.0200	.0300	.0400	.0500	.0599	.0699	.0798	.0898	100
1	0997	1096	1194	.1293	.1391	.1489	.1587	.1684	.1781	.1878	98
2	1974	2070	.2165	.2260	.2355	.2449	.2543	.2636	.2729	.2821	94
3	2913	.3004	3095	.3185	.3275	.3364	.3452	.3540	.3627	.3714	87
.4	.3800	.3885	3969	.4053	.4136	.4219	.4301	.4382	.4462	.4542	82
0.5	4621	4700	4777	.4854	.4930	.5005	.5080	.5154	.5227	.5299	75
.6	5370	.5441	5511	.5581	.5649	.5717	.5784	.5850	.5915	.5980	67
.7	.6044	.6107	.6169	.6231	.6291	.6352	.6411	.6469	.6527	.6584	60
.8	6640	.6696	6751	.6805	.6858	.6911	.6963	.7014	.7064	.7114	52
9	7163	7211	7259	.7306	.7352	.7398	.7443	.7487	.7531	.7574	45
1.0	7616	7658	7699	.7739	.7779	.7818	.7857	.7895	.7932	.7967	39
1	8005	8041	.8076	.8110	.8144	.8178-	.8210	.8243	.8275	.8306	33
.2	8337	8367	.8397	.8426	.8455	.8483	.8511	.8538	.8565	.8591	28
.3	8617	8643	8668	.8693	.8717	.8741	.8764	.8787	.8810	.8832	24
.4	.8854	.8875	8896	.8917	.8937	.8957	.8977	.8996	.9015	.9033	20
1.5	9052	9069	9087	.9104	.9121	.9138	.9154	.9170	.9186	.9202	17
.6	9217	9232	9246	.9261	.9275	.9289	.9302	.9316	.9329	.9342	14
.7	9354	9367	9379	.9391	.9402	.9414	.9425	.9436	.9447	.9458	11
.8	9468	9478	9488	.9498	.9508	.9518	.9527	.9536	.9545	.9554	9
9	9562	9571	9579	.9587	.9595	.9603	.9611	.9619	.9626	.9633	8
2.0	9640	9647	9654	.9661	.9668	.9674	.9680	.9687	.9693	.9699	6
1	9705	.9710	9716	.9722	.9727	.9732	.9738	.9743	.9748	.9753	5
.2	9757	·9762	9767	.9771	.9776	.9780	.9785	.9789	.9793	.9797	4
.3	9801	9805	9809	.9812	.9816	.9820	.9823	.9827	.9830	.9834	4
.4	9837	.9840	.9843	.9846	.9849	.9852	.9855	.9858	.9861	.9863	3
2.5	9866	9869	9871	.9874	.9876	.9879	.9881	.9884	.9886	.9888	2
.6	9890	.9892	9895	.9897	.9899	.9901	.9903	.9905	.9906	.9908	2
.7	9910	9912	.9914	.9915	.9917	.9919	.9920	.9922	.9923	.9925	2
8	9926	.9928	.9929	.9931	.9932	.9933	.9935	.9936	.9937	.9938	1
9	9940	.9941	.9942	.9943	.9944	.9945	.9946	.9947	.9949	.9950	1
3.0	.9951	.9959	.9967	.9973	.9978	.9982	.9985	.9988	.9990	.9992	4
4.0	.9993	.9995	.9996	.9996	.9997	.9998	.9998	.9998	.9999	.9999	1
5.0	.9999										

If $x > 5$, $\tanh x = 1.0000$ to four decimal places.

Table A6

Copper Wire—American Wire Gauge (B & S)

Gauge	Diameter, mils	Diameter, mm.	Area, cir. mils	Ohms/ft. 20° C D.C.	Ohms/meter 20° C D.C.
0000	460.0	11.68	211,600	0.00004901	0.0001608
000	409.6	10.40	167,800	.00006180	.0002027
00	364.8	9.266	133,100	.00007793	.0002557
0	324.9	8.253	105,500	.00009827	.0003224
1	289.3	7.349	83,690	.0001239	.0004065
2	257.6	6.543	66,370	.0001563	.0005128
3	229.4	5.827	52,640	.0001970	.0006463
4	204.3	5.189	41,740	.0002485	.0008153
5	181.9	4.620	33,100	.0003133	.001028
6	162.0	4.115	26,250	.0003951	.001296
7	144.3	3.665	20,820	.0004982	.001634
8	128.5	3.264	16,510	.0006282	.002061
9	114.4	2.906	13,090	.0007921	.002599
10	101.9	2.588	10,380	.0009989	.003277
11	90.74	2.305	8,234	.001260	.004134
12	80.81	2.053	6,530	.001588	.005216
13	71.96	1.828	5,178	.002003	.006571
14	64.08	1.628	4,107	.002525	.008284
15	57.07	1.450	3,257	.003184	.01045
16	50.82	1.291	2,583	.004016	.01318
17	45.26	1.150	2,048	.005064	.01661
18	40.30	1.024	1,624	.006385	.02095
19	35.89	0.9116	1,288	.008051	.02641
20	31.96	.8118	1,022	.01015	.03330
21	28.46	.7229	810.1	.01280	.04199
22	25.35	.6439	642.4	.01614	.05295
23	22.57	.5733	509.5	.02036	.06680
24	20.10	.5106	404.0	.02567	.08422
25	17.90	.4548	320.4	.03237	.1062
26	15.94	.4049	254.1	.04081	.1339
27	14.20	.3607	202.5	.05147	.1689
28	12.64	.3211	159.8	.06490	.2129
29	11.26	.2860	126.7	.08183	.2685
30	10.03	.2548	100.5	.1032	.3386
31	8.928	.2268	79.70	.1301	.4268

TABLE A6 (Cont.)

COPPER WIRE—AMERICAN WIRE GAUGE (B & S)

Gauge	Diameter, mils	Diameter, mm.	Area, cir. mils	Ohms/ft. 20° C D.C.	Ohms/meter 20° C D.C.
32	7.950	.2029	63.21	.1641	.5384
33	7.080	.1798	50.13	.2069	.6788
34	6.305	.1603	39.75	.2609	.8559
35	5.615	.1428	31.52	.3290	1.079
36	5.000	.1270	25.00	.4148	1.361
37	4.453	.1130	19.83	.5231	1.716
38	3.965	.1006	15.72	.6596	2.164
39	3.531	.0867	12.47	.8318	2.729
40	3.145	.0800	9.89	1.049	3.441

TABLE A7

CHARACTERISTICS OF COPPER CONDUCTORS, HARD DRAWN, 97.3% CONDUCTIVITY*

Size of Conductor		No. of Strands	Dia. of Strand, in.	Over-all Dia.	Current capacity at 75°C amperes	GMR, ft	Resistance ohms/mile	
Cir Mils	B & S						d-c 25°C	60 cycles
1,000,000		37	0.1644	1.151	1300	0.0368	0.0585	0.0634
900,000		37	.1560	1.092	1220	.0349	.0650	.0695
800,000		37	.1470	1.029	1130	.0329	.0731	.0772
750,000		37	.1424	0.997	1090	.0319	.0780	.0818
700,000		37	.1375	.963	1040	.0308	.0836	.0871
600,000		37	.1273	.891	940	.0285	.0975	.1006
500,000		37	.1162	.814	840	.0260	.1170	.1196
500,000		19	.1622	.811	840	.0256	.1170	.1196
450,000		19	.1539	.770	780	.0243	.1300	.1323
400,000		19	.1451	.726	730	.0229	.1462	.1484
350,000		19	.1357	.679	670	.0214	.1671	.1690
350,000		12	.1708	.710	670	.0225	.1671	.1690
300,000		19	.1257	.629	610	.01987	.1950	.1966
300,000		12	.1581	.657	610	.02080	.1950	.1966
250,000		19	.1147	.574	540	.01813	.234	.235
250,000		12	.1443	.600	540	.01902	.234	.235
211,600	4/0	19	.1055	.528	480	.01668	.276	.278
211,600	4/0	12	.1328	.552	480	.01750	.276	.278
211,600	4/0	7	.1739	.522	480	.01579	.276	.278
167,800	3/0	12	.1183	.492	420	.01559	.349	.350
167,800	3/0	7	.1548	.464	420	.01404	.349	.350
133,100	2/0	7	.1379	.414	360	.01252	.440	.440
105,500	1/0	7	.1228	.368	310	.01113	.555	.555
83,690	1	7	.1093	.328	270	.00992	.699	.699
83,690	1	3	.1670	.360	270	.01016	.692	.692
66,370	2	7	.0974	.292	230	.00883	.881	.881
66,370	2	3	.1487	.320	240	.00903	.873	.873
66,370	2	1		.258	220	.00836	.864	.864
52,630	3	7	.0867	.260	200	.00787	1.112	1.112
52,630	3	3	.1325	.285	200	.00805	1.101	1.101
52,630	3	1		.229	190	.00745	1.090	1.090
41,740	4	3	.1180	.254	180	.00717	1.388	1.388
41,740	4	1		.204	170	.00663	1.374	1.374
33,100	5	3	.1050	.226	150	.00638	1.750	1.750
33,100	5	1		.181	140	.00590	1.733	1.733
26,250	6	3	.0935	.201	130	.00568	2.21	2.21
26,250	6	1		.162	120	.00526	2.18	2.18
20,820	7	1		.144	110	.00468	2.75	2.75
16,510	8	1		.129	90	.00417	3.47	3.47

* From *Electrical Transmission and Distribution Reference Book*, Westinghouse Electric Corp., E. Pittsburgh, Pa.

TABLE A8

CHARACTERISTICS OF CERTAIN STANDARD COAXIAL CABLES
(American Phenolic Corp., Chicago, Ill.)

Type	Z_0, ohms	O.D., in.	O.D. of Di-electric	Inner Conductor	Velocity of Prop. % of c	Cap. $\mu\mu f$/ft	Attenuation Db/100 ft at megacycles		
							15	150	1000
RG-8U.	52	0.405	0.285	7/#21	65.9	29.5	0.660	2.75	8.50
-11U	75	.405	.285	7/#26	65.9	20.5	.600	2.40	7.70
-14U	52	.545	.370	#10	65.9	29.5	.440	1.80	6.10
-18U	52	.945	.680	0.188	65.9	29.5	.270	1.10	4.00
-19U	52	1.120	.910	.250	65.9	29.5	.200	.92	3.40
-34U	71	.625	.455	7/#21	65.9	21.5	.600	2.40	7.70
-59U	73	.242	.146	#22	65.9	21.0	1.33	4.60	14.3
-83U	35	.405	.240	#10	65.9	44	.980	4.0	12.8
-89U	125	.632	.285	#22	84	10	.73	2.4	7.0
-114U	185	.405	.285	#33	86	6.5			
Twinax (double conductor) cables									
-22U	95	0.405	0.285	Two 7/.0152	65.9	16	1.13	4.40	
-57U	95	.625	.472	Two 7/#21	65.9	17	.880	3.80	
-111U	95	.490	.285	Two 7/.0152	65.9	16	1.13	4.40	

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