Algebraic Geometry

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Contents

Introduction

Definition 0.1. Let A be a ring. Then $\operatorname{Spec} A \coloneqq \{p \subseteq A \mid p \text{ a prime ideal}\}$. For $I \subseteq A$ an ideal, define

$$V(I) \coloneqq \{p \subseteq A \mid p \text{ prime}, p \supseteq I\}.$$

Proposition 0.2. The sets $V(I)$ form the closed sets of a topology on $\operatorname{Spec} A$, called the Zariski topology.

Proof. (1) $V(A) = \varnothing$
(2) $V(0) = \operatorname{Spec} A$
(3) If $\{I_i\}_{i\in J}$ is a collection of ideals, then $V(\sum_{i\in J} I_j) = \bigcap V(I_i)$.
(4) We claim: $V(I_1 \cap I_2) = V(I_1) \cup V(I_2)$.
"$\supseteq$" is obvious.
"$\subseteq$": Follows from the fact that $p \supseteq I_1 \cap I_2$ is prime, then $p \supseteq I_1$ or $p \supseteq I_2$.

□

Example 0.3. Let $A = k[X_1, \ldots, X_n]$ with k algebraically closed. Let $I \subseteq A$ be an ideal. Then the maximal ideals m of A containing I are in one-to-one correspondence with $V(I)$ in $\mathbb{A}^n(k)$: by Nulstellensatz, every maximal ideal is of the form $(X_1 - a_1, \ldots, X_n - a_n)$, which corresponds to $(a_1, \ldots, a_n)$ in the old $V(I)$.

The new $V(I)$ now extends this notion of zero set by including other prime ideals.

Example 0.4. If k is a field, then $\operatorname{Spec} k = \{0\}$, so the topological space cannot see the field. We fix this by also thinking about what functions are on these spaces.

CHAPTER 1

Sheaves

Remark. Fix a topological space X.

Definition 1.1. A presheaf $\mathcal{F}$ on X consists of

(1) For every open set $U \subseteq X$ an abelian group $\mathcal{F}U$,
(2) for every inclusion $V \subseteq U \subseteq X$ a restriction map $\rho_{UV} \colon \mathcal{F}U \to \mathcal{F}V$ such that $\rho_{UU} = \mathrm{id}_{\mathcal{F}U}$ and $p_{UW} = \rho_{VW} \circ \rho_{UV}$.

Remark 1.2. A presheaf is just a contravariant functor from the poset category of open sets of X to the catgegory of abelian groups.

We can generalize this to any contravariant functor $X^{\mathrm{op}} \to \mathcal{C}$ for some category $\mathcal{C}$.

Definition 1.3. A morphism of presheaves $f \colon \mathcal{F} \to \mathcal{G}$ on X is a collection of morphisms $f_U \colon \mathcal{F}U \to \mathcal{G}U$ such that for all $V \subseteq U$ the diagram

$$\begin{tikzcd} \mathcal{F}U \arrow[r, "f_U"] \arrow[d, "\rho_{UV}"] & \mathcal{G}U \arrow[d, "\rho_{UV}"] \\ \mathcal{F}V \arrow[r, "f_V"] & \mathcal{G}V \end{tikzcd}$$

commutes.

Definition 1.4. A presheaf $\mathcal{F}$ is called a sheaf if it satisfies additional axioms:

(S1) If $U \subseteq X$ is covered by an open cover $\{U_i\}$ and $s \in \mathcal{F}U$ satsfies $s|_{U_i} := \rho_{UU_i}(s) = 0$ for all i, then $s = 0$
(S2) If U, and U_i are as before, and if $s_i \in \mathcal{F}U_i$ such that for all i and j we have $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, then there is some $s \in \mathcal{F}U$ such that $s|_{U_i} = s_i$ for all i.

Remark 1.5. (1) If $\mathcal{F}$ is a sheaf, then $\varnothing \subseteq X$ is covered by the empty covering; hence $\mathcal{F}(\varnothing) = 0$.
(2) The two sheaf axioms can be described as saying that given $U, \{U_i\}$,

$$0 \longrightarrow \mathcal{F}U \xrightarrow{\alpha} \prod_i \mathcal{F}U_i \overset{\beta_1}{\underset{\beta_2}{\rightrightarrows}} \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

is exact, where $\alpha(s) = (s|_{U_i})_{i \in I}$, $\beta_1((s_i)_{i \in I}) = (s_i|_{U_i \cap U_j})$, $\beta_2((s_i)_{i \in I}) = (s_j|_{U_i \cap U_j})_{i,j}$.

Exactness means that α is injective, $\beta_1 \circ \alpha = \beta_2 \circ \alpha$, and for any $(s_i) \in \prod_{i \in I} \mathcal{F}U_i$, with $\beta_1((s_i)) = \beta_2((s_i))$, there exists $s \in \mathcal{F}U$ with $\alpha(s) = (s_i)$.

This is all subsumed by saying that α is the equalizer of β_1 and β_2.