Category Theory

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These notes, taken by Markus Himmel, will at times differ significantly from what was lectured. In particular, all errors are almost certainly my own.

Contents

Exercises

Chapter 1

Exercise 17.

Exercise. A morphism $e\colon A \to A$ is called idempotent if $ee = e$. An idempotent e is said to split if it can be factored as fg where gf is an identity morphism.

(i) Let $\mathcal{E}$ be a collection of idempotents in a category $\mathcal{C}$: show that there is a category $\mathcal{C}[\check{\mathcal{E}}]$ whose objects are the members of $\mathcal{E}$, whose morphisms $e \to d$ are those morphisms $f\colon \operatorname{dom} e \to \operatorname{dom} d$ in $\mathcal{C}$ for which $dfe = f$, and whose composition coincides with composition in $\mathcal{C}$. [Hint: first show that the single equation $dfe = f$ is equivalent to the two equations $df = f = fe$. Note that the identity morphism on an object e is not $1_{\operatorname{dom} e}$ in general.]

(ii) If $\mathcal{E}$ contains all identity morphisms of $\mathcal{C}$, show that there is a full and faithful functor $I\colon \mathcal{C} \to \mathcal{C}[\check{\mathcal{E}}]$, and that an arbitrary functor $T\colon \mathcal{C} \to \mathcal{D}$ can be factored as $\widehat{T}I$ for some $\widehat{T}$ iff it sends the members of $\mathcal{E}$ to split idempotents in $\mathcal{D}$.

(iii) Deduce that if all idempotents split in $\mathcal{D}$, then the functor categories $[\mathcal{C}, \mathcal{D}]$ and $[\widehat{\mathcal{C}}, \mathcal{D}]$ are equivalent, where $\widehat{\mathcal{C}} = \mathcal{C}[\check{\mathcal{E}}]$ for $\mathcal{E}$ the class of all idempotents in $\mathcal{C}$.

Solution. We will first show that if $f : C \to D$ is any morphism and $c : C \to C$ and $d : D \to D$ are idempotents, then $dfe = f \iff df = f = fe$.

Indeed, if $df = f = fe$, then $dfe = fe = f$. Conversely, if $dfe = f$, then $f = dfe = ddfe = df$ and $f = dfe = dfee = fe$.

To show that $\mathcal{C}[\check{\mathcal{E}}]$ is a category, we need to show that the composition of two morphisms is indeed a morphism and that there are identity morphism.

Assume that $c\colon C \to C$, $d\colon D \to D$, $e\colon E \to E$ are idempotents and that $f\colon C \to D$ and $g\colon D \to E$ satisfy $dfc = f$ and $egd = g$. We need to show that $egfc = gf$. Using the lemma, we have $egf = (eg)f = gf$ and $gfc = g(fc) = gf$, so, again by the lemma, the claim follows.

If $e\colon E \to E$ is an idempotent, define $1_e \coloneqq e \xrightarrow{e} e$. By idempotency of e, this is indeed a morphism. If $f\colon d \to e$ is a morphism, then the morphism $f1_d$ is the morphism $fd = f$ (here we use the lemma again) in $\mathcal{C}$, so $f1_d = f$ as required. Similarly, $1_e f = f$. This completes part (i).

Next, assume that $\mathcal{E}$ contains all identity morphisms of $\mathcal{C}$. Define the functor I via

$$
\begin{gathered}
I\colon \mathcal{C} \to \mathcal{C}[\check{\mathcal{E}}] \\
A \mapsto 1_A \\
(f\colon A \to B) \mapsto (f\colon 1_A \to 1_B)
\end{gathered}
$$

This is indeed a functor and since the data of a morphism $A \to B$ in $\mathcal{C}$ is precisely the same as the data of a morphism $1_A \to 1_B$ in $\mathcal{C}[\check{\mathcal{E}}]$, I is fully faithful.

Now let $T\colon \mathcal{C} \to \mathcal{D}$ be any functor.

First, assume that there is some functor $\widehat{T}\colon \mathcal{C}[\check{\mathcal{E}}] \to \mathcal{D}$ such that $T = \widehat{T}I$. Let $e : A \to A \in \mathcal{E}$ be an idempotent. Then we have

$$\begin{aligned} Te &= \widehat{T}(1_A \xrightarrow{e} 1_A) \\ &= \widehat{T}(1_A \xrightarrow{e} e \xrightarrow{e} 1_A) \\ &= \widehat{T}(e \xrightarrow{e} 1_A) \circ \widehat{T}(1_A \xrightarrow{e} e), \end{aligned}$$

and we also have

$$\begin{aligned} \widehat{T}(1_A \xrightarrow{e} e) \circ \widehat{T}(e \xrightarrow{e} 1_A) &= \widehat{T}(e \xrightarrow{e} 1_A \xrightarrow{e} e) \\ &= \widehat{T}(e \xrightarrow{ee} e) \\ &= \widehat{T}(e \xrightarrow{e} e) \\ &= \widehat{T}(1_e) \\ &= 1_{\widehat{T}e}, \end{aligned}$$

which shows that Te is split.

Next, assume that Te is split for any $e \in \mathcal{E}$. For any $e \in \mathcal{E}$, choose a splitting

$$TA \underset{f_e}{\overset{g_e}{\rightleftarrows}} B_e,$$

i.e., $f_e \circ g_e = Te$, $g_e \circ f_e = 1_{B_e}$. For identity morphisms 1_A (A an object of $\mathcal{C}$), choose the specific splitting given by $B_{1_A} \coloneqq TA$, $f_{1_A} \coloneqq 1_{TA}$, $g_{1_A} \coloneqq 1_{TA}$.

Now define the functor $\widehat{T}$ via

$$\begin{aligned} \widehat{T}\colon \mathcal{C}[\check{\mathcal{E}}] &\to \mathcal{D} \\ (e\colon A \to A) &\mapsto B_e \\ (f\colon d \to e) &\mapsto g_e \circ Tf \circ f_d. \end{aligned}$$

If $e \in \mathcal{E}$, then we have

$$\begin{aligned} \widehat{T}(1_e) &= g_e \circ Te \circ f_e \\ &= g_e \circ f_e \circ g_e \circ f_e \\ &= 1_{B_e} \circ 1_{B_e} = 1_{B_e} \end{aligned}$$

Furthermore, if $f\colon c \to d$ and $g\colon d \to e$, then we have

$$\begin{aligned} \widehat{T}(g \circ f) &= g_e \circ T(g \circ f) = f_c \\ &= g_e \circ Tg \circ Tf = f_c \\ &= g_e \circ Tg \circ T(d \circ f) \circ f_c \\ &= g_e \circ Tg \circ Td \circ Tf \circ f_c \\ &= g_e \circ Tg \circ f_d \circ g_d \circ Tf \circ f_c \\ &= \widehat{T}g \circ \widehat{T}f. \end{aligned}$$

So $\widehat{T}$ is indeed a functor. If A is an object of $\mathcal{C}$, then

$$\widehat{T}IA = \widehat{T}1_A = B_{1_A} = TA$$

and if $f\colon C \to D$ is a morphism in $\mathcal{C}$, then

$$\widehat{T}If = \widehat{T}(1_C \xrightarrow{f} 1_D) = g_{1_D} \circ Tf \circ f_{1_C} = 1_{TD} \circ Tf \circ 1_{TC} = Tf,$$

so $\widehat{T}$ is the required factorisation, completing part (ii).

Define a functor $\Phi\colon [\widehat{\mathcal{C}}, \mathcal{D}] \to [\mathcal{C}, D]$ via $F \mapsto F \circ I$, $\eta \mapsto I\eta$, where $I\eta$ is defined cia $I\eta_C \coloneqq \eta_{IC} = \eta_{1_C}$. Naturality of $I\eta$ immediately follows from naturality of η. Functoriality is also clear.

We will show that this functor is full, faithful and essentially surjective.

Indeed, let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. Then $\widehat{F}$ as defined in the previous part satisfies $\Phi\widehat{F} = F$, so Φ is essentially surjective.

Next, let $F, G\colon \widehat{\mathcal{C}} \to \mathcal{D}$ be functors and $\eta\colon F \circ I \to G \circ I$ a natural transformation. For an idempotent $e\colon A \to A$ in $\mathcal{C}$, define $\hat{\eta}_e$ to be the composite

$$Fe \xrightarrow{F(e \xrightarrow{e} 1_A)} F1_A = (F \circ I)A \xrightarrow{\quad \eta_A \quad} (G \circ I)A = G1_A \xrightarrow{G(1_A \xrightarrow{e} e)} Ge.$$

We claim that this defines a natural transformation $\hat{\eta}\colon F \to G$. Indeed, if $f\colon d \to e$ is a morphism, then

$$\begin{aligned}
\hat{\eta}_e \circ Ff &= G(1_A \xrightarrow{e} e) \circ \eta_E \circ F(e \xrightarrow{e} 1_E) \circ F(d \xrightarrow{f} e)\\
&= G(1_E \xrightarrow{e} e) \circ \eta_E \circ F(d \xrightarrow{f} e \xrightarrow{e} 1_E)\\
&= G(1_E \xrightarrow{e} e) \circ \eta_E \circ F(d \xrightarrow{d} 1_D \xrightarrow{d} d \xrightarrow{f} e \xrightarrow{e} 1_E)\\
&= G(1_E \xrightarrow{e} e) \circ \eta_E \circ F(d \xrightarrow{d} 1_D \xrightarrow{d} d \xrightarrow{f} e \xrightarrow{e} 1_E)\\
&= G(1_E \xrightarrow{e} e) \circ \eta_E \circ F(1_D \xrightarrow{efd} 1_E) \circ F(d \xrightarrow{d} 1_D)\\
&= G(1_E \xrightarrow{e} e) \circ \eta_E \circ F(efd) \circ F(d \xrightarrow{d} 1_D)\\
&= G(1_E \xrightarrow{e} e) \circ G(efd) \circ \eta_D \circ F(d \xrightarrow{d} 1_D),
\end{aligned}$$

and doing the whole thing backwards we conclude that $\hat{\eta}_e \circ Ff = Gf \circ \hat{\eta}_d$, so $\hat{\eta}$ is indeed a natural transformation.

For any $A \in \mathcal{C}$ we have

$$\begin{aligned}
(I\hat{\eta})_A = \hat{\eta}_{IA} = \hat{\eta}_{1_A} &= G(1_A \xrightarrow{1_A} 1_A) \circ \eta_A \circ F(1_A \xrightarrow{1_A} 1_A)\\
&= G(1_{1_A}) \circ \eta_A \circ F(1_{1_A}) = \eta_A,
\end{aligned}$$

which means that $\Phi(\hat{\eta}) = \eta$, so Φ is full.

Finally, let $F, G\colon \widehat{\mathcal{C}} \to \mathcal{D}$ be functors and $\eta, \eta'\colon F \to G$ be natural transformations such that $\Phi(\eta) = \Phi(\eta')$. To show that Φ is faithful, we need to prove that $\eta = \eta'$. The assumption $\Phi(\eta) = \Phi(\eta')$ means that for all $A \in \mathcal{C}$ we have $\eta_{IA} = \eta'_{IA}$, so $\eta_{1_A} = \eta'_{1_A}$.

Let $e\colon A \to A$ be any idempotent in $\mathcal{C}$. We need to show that $\eta_e = \eta'_e$. Indeed, we have

$$\begin{aligned}
\eta_e &= G(1_e) \circ \eta_e\\
&= G(e \xrightarrow{e} e) \circ \eta_e\\
&= G(e \xrightarrow{ee} e) \circ \eta_e\\
&= G(e \xrightarrow{e} 1_A \xrightarrow{e} e) \circ \eta_e\\
&= G(1_A \xrightarrow{e} e) \circ G(e \xrightarrow{e} 1_A) \circ \eta_e\\
&= G(1_A \xrightarrow{e} e) \circ \eta_{1_A} \circ F(e \xrightarrow{e} 1_A)\\
&= G(1_A \xrightarrow{e} e) \circ \eta'_{1_A} \circ F(e \xrightarrow{e} 1_A),
\end{aligned}$$

and the same argument in backwards direction shows that $\eta_e = \eta'_e$, completing the proof. □

Chapter 2

Exercise 13.

Exercise. The inner automorphisms of $\mathcal{C}$ form a normal subgroup of the group of all automorphisms of $\mathcal{C}$. [Don't worry about whether these groups are sets or proper classes!]

Solution. Let $F, G\colon \mathcal{C} \to \mathcal{C}$ be automorphisms and let $\alpha\colon F \to 1_{\mathcal{C}}$ be a natural isomorphism.

Let $A \in \mathcal{C}$. Define $\beta\colon GFG^{-1} \to 1_A$ via $\beta_A \coloneqq G(\alpha_{G^{-1}A})$ (so $\beta_A\colon GFG^{-1}A \to GG^{-1}A = A \to GG^{-1}A = 1_{\mathcal{C}}A$.

This is indeed a natural transformation: let $f\colon A \to B \in \mathcal{C}$, then we can write the naturality square in a funny way,

$$\begin{tikzcd}
GFG^{-1}A \arrow[r, "G(\alpha_{G^{-1}A})"] \arrow[d, "GFG^{-1}(f)"] & G1_C G^{-1}A \arrow[d, "G1_C G^{-1}f"] \\
GFG^{-1}B \arrow[r, "G(\alpha_{G^{-1}B})"] & G1_C G^{-1}B
\end{tikzcd}$$

and we see that it is just the functor G applied to the naturality diagram for α and the morphism $G^{-1}f$.

Therefore, β is a natural transformation, and since functors map isomorphisms to isomorphisms, it is also a natural isomorphism. So GFG^{-1} is an inner automorphism as required. $\square$

Lemma. Let $1 \in \mathcal{C}$ be a terminal object and $F\colon C \to C$ an automorphism. Then $F1$ is a terminal object.

Proof. If $A \in \mathcal{C}$, the functor F, which is fully faithful, induces a bijection between the collection of morphisms $F^{-1}A \to 1$ and the collection of morphisms $A \to F1$. Since 1 is terminal, there is exactly one morphism $A \to F1$. $\square$

Exercise. If $F\colon \mathsf{Set} \to \mathsf{Set}$ is an automorphism, then there is a unique natural isomorphism $1_{\mathcal{C}} \to F$.

Solution. Of course, the terminal object in the category of sets is just the one-element set $1 = \{\star\}$. Since $F1$ is also terminal, it is in bijection with 1. We write $F1 = \{\star_{F1}\}$.

By the Yoneda lemma, the set of natural transformations $\mathsf{Set}(1, -) \to F$ is in bijection with $F1$, so there is a unique natural transformation $\eta\colon \mathsf{Set}(1, -) \to F$. Examining the proof, we see that the components of this natural transformation are given by

$$\begin{aligned} \eta_A\colon \mathsf{Set}(1, A) &\to FA \\ f &\mapsto Ff(\star_{F1}) \end{aligned}$$

for any object A of $\mathcal{C}$. Let A be an object of $\mathcal{C}$. We will show that η_A is an isomorphism, i.e., a bijection.

First, let $x \in FA$. Then $\eta_A(F^{-1}(\star_{F1} \mapsto x)) = x$, so η_A is surjective.

Additionally, let $f, g\colon 1 \to A$ such that $\eta_A(f) = \eta_A(g)$. Since a map $F1 \to FA$ is completely determined by its value at $\star_{F1}$, we must have $Ff = Fg$. But then $f = F^{-1}F(f) = F^{-1}F(g) = g$.

This means that η_A is an isomorphism, so η is in fact a natural isomorphism.

We define a natural transformation $\alpha\colon 1_{\mathsf{Set}} \to \mathsf{Set}(1, -)$ by setting

$$\alpha_A(a)(\star) \coloneqq a.$$

The naturality square for $f\colon A \to B$ is

$$\begin{tikzcd} A \arrow[r, "\alpha_A"] \arrow[d, "f"] & \mathsf{Set}(1, A) \arrow[d, "g \mapsto f \circ g"] \\ B \arrow[r, "\alpha_B"] & \mathsf{Set}(1, B) \end{tikzcd}$$

Both paths are just $a \mapsto (\star \mapsto f(a))$, so α is natural. It is also clear that α_A is bijective, so α is a natural isomorphism. In other words, $\star$ is a universal element of the identity functor.

In particular, this tells is that composition with α and its inverse exhibits a bijection between the collection of natural transformations

$$1_{\mathsf{Set}} \to F$$

and the collection of natural transformations

$$\mathsf{Set}(1, -) \to F.$$

This means that there is a unique natural transformation $1_{\mathsf{Set}} \to F$, and it is given by $\alpha \circ \eta$, and since α and η are both natural isomorphisms, so is $\alpha \circ \eta$, completing the proof. $\square$

Exercise. The Sierpiński space S is, up to isomorphism, the unique topological space which has precisely three endomorphisms.

Solution. Let X be a topological space. Then for any $x \in X$, the constant map $c_x\colon X \to X$ sending $y \in X$ to x is continuous. Furthermore, the identity on X is continuous. This, if X is infinite, then X has infinitely many endomorphisms, and if X is finite, then X has at least $|X| + 1$ endomorphisms.

Now assume that X has precisely three endomorphisms. Then X is finite and has at most two points. Clearly, if X has zero or one point, then there is only one endomorphism. So X has two points, say $X = \{a, b\}$. There are four set-functions $\{a, b\} \to \{a, b\}$, three of which (the identity and the two constant maps) are continuous regardless of the topology. The final map interchanges a and b and is not continuous.

The empty set and all of X are open. If X had the trivial or the discrete topology, then the interchange would be continuous, a contradiction. Hence, precisely one of the sets $\{a\}$ and $\{b\}$ is open. Sending the member of that set to 1 and the other element to 0 describes a homeomorphism with S. $\square$

Exercise. Let $\mathcal{C}$ be a full subcategory of Top containing the singleton space 1 and the Siperpiński space S and let F be an automorphism of $\mathcal{C}$. Then

(a) we have $FS \cong S$,
(b) there is a unique natural isomorphism $\alpha\colon U \to UF$, where $U\colon \mathcal{C} \to \mathsf{Set}$ is the forgetful functor,
(c) if $\mathcal{C}$ contains a space in which not every union of closed sets is closed, then α_S is continuous, and
(d) F is uniquely naturally isomorphic to the identity functor.

Solution. For (a), we just need to notice that F is fully faithful, so it induces a bijection between the sets of morphisms $S \to S$ and $FS \to FS$. Since S is determined up to isomorphism by having exactly three endomorphisms, the claim follows.

The proof of (b) is entirely analogous to the proof of 2.13(ii). $\square$

Exercise 14.

Exercise. Let $e\colon A \to A$ be an idempotent. Then the following are equivalent:

(i) e is split,
(ii) the pair $(e, 1_A)$ has an equaliser,
(iii) the pair $(e, 1_A)$ has a coequaliser.

Solution. We will show that (i) is equivalent to (ii). By duality, this implies that (i) is equivalent to (iii).

Assume that there are $f\colon B \to A$ and $g\colon A \to B$ such that $fg = e$ and $gf = 1_B$. We claim that f is an equaliser of e and 1_A. We must show that any $h\colon C \to A$ satisfying $he = h$ factors uniquely through f.

$$\begin{tikzcd}
& C \arrow[dl, "h'"'] \arrow[d, "h"] & \\
B \arrow[r, shift left, "f"] & A \arrow[l, shift left, "g"] \arrow[r, shift left, "e"] \arrow[r, shift right, "1_A"'] & A
\end{tikzcd}$$

Indeed, given such h. Then $fgh = eh = h$, hence gh is one such factoring factoring. If $h'\colon C \to B$ is another factoring such that $fh' = h$, then $h' = gfh' = gh$, so the factoring is unique.

Conversely, assume that the pair $(e, 1_A)$ admits an equaliser $f\colon B \to A$. Since $ee = e = 1_A e$, e factors through f via some $g\colon A \to B$. Hence, $fg = e$. On the other hand, $fgf = ef = f$, and by a result from the lecture, f is monic, so $gf = 1_A$, so e is split. $\square$

Exercise. A split monomorphism is regular.

Solution. If $f\colon A \to B$ is a split monomorphism, then there is some $g\colon B \to A$ such that $gf = 1_A$. Then $fgfg = f1_Ag = fg$, so fg is a split idempotent. By what we just saw, this means that f is an equaliser of $(fg, 1_A)$, hence f is a regular monomorphism. $\square$

Exercise 15.

Exercise. Every regular monomorphism is strong.

Solution. Let f be the equaliser of u and v and take a commutative square as in the definition of strongness.

$$\begin{tikzcd}
C \arrow[r, "h"] \arrow[d, two heads, "g"] & A \arrow[d, hook, "f"] & \\
D \arrow[r, "k"] \arrow[ur, dotted, "t"] & B \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & E
\end{tikzcd}$$

We have $ukg = ufh = vfh = vkg$. Since g is epi, this means that $uk = vk$, and since f is the equaliser of u and v, we find $t\colon D \to A$ such that $ft = k$. Now $ftg = kg = fh$. Since f is mono, we conclude that $tg = h$, so t has the desired properties. Hence, f is a strong monomorphism. $\square$

Exercise. Let $\mathcal{C}$ be the finite category whose non-identity morphisms are represented by the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "l"'] & B \arrow[d, shift right, "h"'] \arrow[d, shift left, "k"] & C \arrow[l, "g"'] \arrow[dl, "m"] \\
& D. &
\end{tikzcd}$$

The morphism f is strong monic but not regular monic.

Solution. The strongness condition for f is actually vacuous: if we have a diagram

$$\begin{tikzcd} \bullet \arrow[r, "u"] \arrow[d, two heads, "v"'] & A \arrow[d, hook, "f"] \\ \bullet \arrow[r, "w"] & B, \end{tikzcd}$$

then we must have $u = 1_A$. The morphism f is not an epimorphism, as witnessed by the fact that $hf = kf$, but $h \neq k$, so we must have $v = l$. Then w is a morphism $D \to B$, but such a morphism does not exist. Hence, the square does not exist, so f is strong.

However, the only pairs of morphisms that f can be an equaliser of are $(1_B, 1_B)$, (k, k), (h, h) and (h, k). If f was the equaliser of any of these pairs, g would factor through f, but there is no morphism $C \to A$, hence that is not the case. So we conclude that f is not regular. $\square$

Exercise 16.

Exercise. Let $f \colon A \to B$, $g \colon B \to C$ be two morphisms.

(a) If f and g are monic, then gf is monic,
(b) If f and g are strong monic, then gf is strong monic,
(c) If f and g are split monic, then gf is split monic,
(d) If gf is monic, then f is monic,
(e) If gf is strong monic, then f is strong monic,
(f) If gf is split monic, then f is split monic.
(g) If gf is regular monic and g is monic, then f is regular monic.

Solution. (a) If $gfu = gfv$, then $fu = fv$ since g is monic, and $u = v$, since f is monic.

(b) Consider the diagram

$$\begin{tikzcd} D \arrow[r, "h"] \arrow[dd, two heads, "l"'] & A \arrow[d, hook, "f"] \\ & B \arrow[d, hook, "g"] \\ E \arrow[r, "k"] \arrow[uur, dotted, "u"] \arrow[ur, dotted, "t"] & C. \end{tikzcd}$$

Since g is strong monic, using the square (fh, g, l, k), we find $t \colon E \to B$ such that $gt = k$ and $tl = fh$. Since f is strong epic, using the square (h, f, l, t), we find $u \colon E \to A$ such that $fu = t$ and $ul = h$. Then we have $gfu = gt = k$, so u is the required morphism.

(c) If $u \colon B \to A$ satisfies $uf = 1_A$ and $v \colon C \to B$ satisfies $vg = 1_B$, then uv is the desired retraction, as $uvgf = u1_Bf = uf = 1_A$.

(d) If $fu = fv$, then trivially, $gfu = gfv$, so $u = v$.

(e) Consider the diagram

$$\begin{tikzcd}
D \arrow[r, "h"] \arrow[d, two heads, "l"] & A \arrow[d, "f"] \\
E \arrow[r, "k"] \arrow[ur, dotted, "t"] \arrow[dr, "gk"'] & B \arrow[d, "g"] \\
& C.
\end{tikzcd}$$

Since gf is strong monic, using the square (h, gf, l, gk) we find $t\colon E \to A$ such that $tl = h$ (and $gft = gk$, but that is not important). We have $ftl = fh = kl$, so since l is epi, we have $ft = k$, so t is indeed the required diagonal morphism, so f is strong monic.

(f) If $u\colon C \to A$ satisfies $ugf = 1_A$, then $(ug)f = 1_A$, so f is split monic.

(g) Say gf is an equalizer of u and v.

$$\begin{tikzcd}
& T \arrow[dl, dotted, "\ell"'] \arrow[d, "h"] \\
A \arrow[r, "f"] & B \arrow[r, hook, "g"] & C \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & D
\end{tikzcd}$$

If $h\colon T \to B$ satisfies $ugh = vgh$, then since gf is an equaliser of u and v, we find a unique $\ell\colon T \to A$ such that $gf\ell = gh$. Since g is monic, we have $f\ell = h$. The morphism ℓ is the unique morphism satisfying $f\ell = h$, since if $\hat{\ell}$ also satisfies $f\hat{\ell} = h$, then certainly $gf\hat{\ell} = gh$, hence $\ell = \hat{\ell}$.

□

Exercise. Let $\mathcal{C}$ be the full subcategory of Ab whose objects are groups having no elements of order 4 (though they may have elements of order 2). Then

(i) multiplication by 2 is a regular monomorphism $\mathbb{Z} \to \mathbb{Z}$,

(ii) multiplication by 4 is not a regular monomorphism $\mathbb{Z} \to \mathbb{Z}$,

(iii) there is a pair of morphisms (f, g) such that gf is regular monix but f is not.

Solution. (i) We claim that multiplication by 2 is an equalizer in $\mathcal{C}$ of the projection $\pi\colon \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ and the zero map $0\colon \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$.

$$\begin{tikzcd}
& G \arrow[dl, dotted] \arrow[d, "f"] \\
\mathbb{Z} \arrow[r, "\cdot 2"] & \mathbb{Z} \arrow[r, shift left, "\pi"] \arrow[r, shift right, "0"'] & \mathbb{Z}/2\mathbb{Z}
\end{tikzcd}$$

Indeed, if $f\colon G \to \mathbb{Z}$ equalizes π and 0, then its image is contained in $2\mathbb{Z}$, hence it factors uniquely through multiplication by 2 via the map $g \mapsto f(g)/2$.

(ii) Assume that multiplication by 4 is an equalizer in $\mathcal{C}$ of f and g.

$$\begin{tikzcd}
& \ker(f-g) \arrow[dl, dotted] \arrow[d, "\iota"] \\
\mathbb{Z} \arrow[r, "\cdot 4"] & \mathbb{Z} \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & G
\end{tikzcd}$$

Clearly, the kernel of $f - g$ has no elements of order 4 and the inclusion equalizes f and g, hence it factors through multiplication by 4. Consider the element $\alpha \coloneqq f(1) - g(1) \in G$. We know that $\alpha + \alpha + \alpha + \alpha = f(4) - g(4) = 0$, since multiplication by 4 equalises f and g. Since G is an object of $\mathcal{C}$, the order of α is 2 or 1. In either case, we have $2 \in \ker(f-g)$, which is not in the image of multiplication by 4, hence ι cannot factor through multiplication by 4, so multiplication by 4 is not an equalizer of f and g. $\square$

Exercise 17.

Exercise. The functor F is irreducible if and only if there is an epimorphism $\mathcal{C}(A,-) \to F$ for some object A of $\mathcal{C}$.

Solution. If F is irreducible, then applying the irreducibility property to the epimorphism constructed in 2.12 gives the desired result.

Conversely, if A is an object of $\mathcal{C}$ such that there is an epimorphism $\beta\colon C(A,-) \to F$, then by 2.11 we get a factoring $\gamma\colon \mathcal{C}(A,-) \to \coprod_{i\in I} G_i$. Define $x \coloneqq f_A(1_A) \in G_j(A)$ for some $j \in I$. By Yoneda, we know that for any object B and morphism $f\colon A \to B$ we have

$$\gamma_B(f) = \left(\coprod_{i\in I} G_i\right)(f)(x) = G_j(f)(x),$$

i.e., the image of γ_B is completely contained in $G_j(B)$ for every B. Hence we have a commutative diagram

$$\begin{tikzcd} G_j \arrow[r, "\alpha|_{G_j}"] & F \\ & \mathcal{C}(A,-), \arrow[ul, "\gamma"] \arrow[u, two heads, "\beta"'] \end{tikzcd}$$

and by the dual of Exercise 2.16(ii), the natural transformation $\alpha|_{G_i}$ must be an epimorphism. $\square$