# Category Theory

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These notes, taken by Markus Himmel, will at times differ significantly from what was lectured. In particular, all errors are almost certainly my own.

# Contents

# Exercises

## Chapter 1

### Exercise 17.

Exercise. A morphism $e\colon A \to A$ is called idempotent if $ee = e$. An idempotent $e$ is said to split if it can be factored as $fg$ where $gf$ is an identity morphism.

(i) Let $\mathcal{E}$ be a collection of idempotents in a category $\mathcal{C}$: show that there is a category $\mathcal{C}[\check{\mathcal{E}}]$ whose objects are the members of $\mathcal{E}$, whose morphisms $e \to d$ are those morphisms $f\colon \operatorname{dom} e \to \operatorname{dom} d$ in $\mathcal{C}$ for which $dfe = f$, and whose composition coincides with composition in $\mathcal{C}$. [Hint: first show that the single equation $dfe = f$ is equivalent to the two equations $df = f = fe$. Note that the identity morphism on an object $e$ is not $1_{\operatorname{dom} e}$ in general.]

(ii) If $\mathcal{E}$ contains all identity morphisms of $\mathcal{C}$, show that there is a full and faithful functor $I\colon \mathcal{C} \to \mathcal{C}[\check{\mathcal{E}}]$, and that an arbitrary functor $T\colon \mathcal{C} \to \mathcal{D}$ can be factored as $\widehat{T}I$ for some $\widehat{T}$ iff it sends the members of $\mathcal{E}$ to split idempotents in $\mathcal{D}$.

(iii) Deduce that if all idempotents split in $\mathcal{D}$, then the functor categories $[\mathcal{C}, \mathcal{D}]$ and $[\widehat{\mathcal{C}}, \mathcal{D}]$ are equivalent, where $\widehat{\mathcal{C}} = \mathcal{C}[\check{\mathcal{E}}]$ for $\mathcal{E}$ the class of all idempotents in $\mathcal{C}$.

Solution. We will first show that if $f\colon C \to D$ is any morphism and $c : C \to C$ and $d : D \to D$ are idempotents, then $dfe = f \iff df = f = fe$.

Indeed, if $df = f = fe$, then $dfe = fe = f$. Conversely, if $dfe = f$, then $f = dfe = ddfe = df$ and $f = dfe = dfee = fe$.

To show that $\mathcal{C}[\check{\mathcal{E}}]$ is a category, we need to show that the composition of two morphisms is indeed a morphism and that there are identity morphism.

Assume that $c\colon C \to C$, $d\colon D \to D$, $e\colon E \to E$ are idempotents and that $f\colon C \to D$ and $g\colon D \to E$ satisfy $dfc = f$ and $egd = g$. We need to show that $egfc = gf$. Using the lemma, we have $egf = (eg)f = gf$ and $gfc = g(fc) = gf$, so, again by the lemma, the claim follows.

If $e\colon E \to E$ is an idempotent, define $1_e \coloneqq e \xrightarrow{e} e$. By idempotency of $e$, this is indeed a morphism. If $f\colon d \to e$ is a morphism, then the morphism $f1_d$ is the morphism $fd = f$ (here we use the lemma again) in $\mathcal{C}$, so $f1_d = f$ as required. Similarly, $1_e f = f$. This completes part (i).

Next, assume that $\mathcal{E}$ contains all identity morphisms of $\mathcal{C}$. Define the functor $I$ via

$$\begin{gathered} I\colon \mathcal{C} \to \mathcal{C}[\check{\mathcal{E}}] \\ A \mapsto 1_A \\ (f\colon A \to B) \mapsto (f\colon 1_A \to 1_B) \end{gathered}$$

This is indeed a functor and since the data of a morphism $A \to B$ in $\mathcal{C}$ is precisely the same as the data of a morphism $1_A \to 1_B$ in $\mathcal{C}[\check{\mathcal{E}}]$, $I$ is fully faithful.

Now let $T\colon \mathcal{C} \to \mathcal{D}$ be any functor.

First, assume that there is some functor $\widehat{T}\colon \mathcal{C}[\check{\mathcal{E}}] \to \mathcal{D}$ such that $T = \widehat{T}I$. Let $e : A \to A \in \mathcal{E}$ be an idempotent. Then we have

$$\begin{aligned} Te &= \widehat{T}(1_A \xrightarrow{e} 1_A) \\ &= \widehat{T}(1_A \xrightarrow{e} e \xrightarrow{e} 1_A) \\ &= \widehat{T}(e \xrightarrow{e} 1_A) \circ \widehat{T}(1_A \xrightarrow{e} e), \end{aligned}$$

and we also have

$$\begin{aligned} \widehat{T}(1_A \xrightarrow{e} e) \circ \widehat{T}(e \xrightarrow{e} 1_A) &= \widehat{T}(e \xrightarrow{e} 1_A \xrightarrow{e} e) \\ &= \widehat{T}(e \xrightarrow{ee} e) \\ &= \widehat{T}(e \xrightarrow{e} e) \\ &= \widehat{T}(1_e) \\ &= 1_{\widehat{T}e}, \end{aligned}$$

which shows that $Te$ is split.

Next, assume that $Te$ is split for any $e \in \mathcal{E}$. For any $e \in \mathcal{E}$, choose a splitting

$$TA \underset{f_e}{\overset{g_e}{\rightleftarrows}} B_e,$$

i.e., $f_e \circ g_e = Te$, $g_e \circ f_e = 1_{B_e}$. For identity morphisms $1_A$ (A an object of $\mathcal{C}$), choose the specific splitting given by $B_{1_A} \coloneqq TA$, $f_{1_A} \coloneqq 1_{TA}$, $g_{1_A} \coloneqq 1_{TA}$.

Now define the functor $\widehat{T}$ via

$$\begin{aligned} \widehat{T}\colon \mathcal{C}[\check{\mathcal{E}}] &\to \mathcal{D} \\ (e\colon A \to A) &\mapsto B_e \\ (f\colon d \to e) &\mapsto g_e \circ Tf \circ f_d. \end{aligned}$$

If $e \in \mathcal{E}$, then we have

$$\begin{aligned} \widehat{T}(1_e) &= g_e \circ Te \circ f_e \\ &= g_e \circ f_e \circ g_e \circ f_e \\ &= 1_{B_e} \circ 1_{B_e} = 1_{B_e} \end{aligned}$$

Furthermore, if $f\colon c \to d$ and $g\colon d \to e$, then we have

$$\begin{aligned} \widehat{T}(g \circ f) &= g_e \circ T(g \circ f) = f_c \\ &= g_e \circ Tg \circ Tf = f_c \\ &= g_e \circ Tg \circ T(d \circ f) \circ f_c \\ &= g_e \circ Tg \circ Td \circ Tf \circ f_c \\ &= g_e \circ Tg \circ f_d \circ g_d \circ Tf \circ f_c \\ &= \widehat{T}g \circ \widehat{T}f. \end{aligned}$$

So $\widehat{T}$ is indeed a functor. If $A$ is an object of $\mathcal{C}$, then

$$\widehat{T}IA = \widehat{T}1_A = B_{1_A} = TA$$

and if $f\colon C \to D$ is a morphism in $\mathcal{C}$, then

$$\widehat{T}If = \widehat{T}(1_C \xrightarrow{f} 1_D) = g_{1_D} \circ Tf \circ f_{1_C} = 1_{TD} \circ Tf \circ 1_{TC} = Tf,$$

so $\widehat{T}$ is the required factorisation, completing part (ii).

Define a functor $\Phi\colon [\widehat{\mathcal{C}}, \mathcal{D}] \to [\mathcal{C}, D]$ via $F \mapsto F \circ I$, $\eta \mapsto I\eta$, where $I\eta$ is defined cia $I\eta_C \coloneqq \eta_{IC} = \eta_{1_C}$. Naturality of $I\eta$ immediately follows from naturality of $\eta$. Functoriality is also clear.

We will show that this functor is full, faithful and essentially surjective.

Indeed, let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. Then $\widehat{F}$ as defined in the previous part satisfies $\Phi\widehat{F} = F$, so $\Phi$ is essentially surjective.

Next, let $F, G\colon \widehat{\mathcal{C}} \to \mathcal{D}$ be functors and $\eta\colon F \circ I \to G \circ I$ a natural transformation. For an idempotent $e\colon A \to A$ in $\mathcal{C}$, define $\hat{\eta}_e$ to be the composite

$$Fe \xrightarrow{F(e \xrightarrow{e} 1_A)} F1_A = (F \circ I)A \xrightarrow{\quad \eta_A \quad} (G \circ I)A = G1_A \xrightarrow{G(1_A \xrightarrow{e} e)} Ge.$$

We claim that this defines a natural transformation $\hat{\eta}\colon F \to G$. Indeed, if $f\colon d \to e$ is a morphism, then

$$\begin{aligned}
\hat{\eta}_e \circ Ff &= G(1_A \xrightarrow{e} e) \circ \eta_E \circ F(e \xrightarrow{e} 1_E) \circ F(d \xrightarrow{f} e) \\
&= G(1_E \xrightarrow{e} e) \circ \eta_E \circ F(d \xrightarrow{f} e \xrightarrow{e} 1_E) \\
&= G(1_E \xrightarrow{e} e) \circ \eta_E \circ F(d \xrightarrow{d} 1_D \xrightarrow{d} d \xrightarrow{f} e \xrightarrow{e} 1_E) \\
&= G(1_E \xrightarrow{e} e) \circ \eta_E \circ F(d \xrightarrow{d} 1_D \xrightarrow{d} d \xrightarrow{f} e \xrightarrow{e} 1_E) \\
&= G(1_E \xrightarrow{e} e) \circ \eta_E \circ F(1_D \xrightarrow{efd} 1_E) \circ F(d \xrightarrow{d} 1_D) \\
&= G(1_E \xrightarrow{e} e) \circ \eta_E \circ F(efd) \circ F(d \xrightarrow{d} 1_D) \\
&= G(1_E \xrightarrow{e} e) \circ G(efd) \circ \eta_D \circ F(d \xrightarrow{d} 1_D),
\end{aligned}$$

and doing the whole thing backwards we conclude that $\hat{\eta}_e \circ Ff = Gf \circ \hat{\eta}_d$, so $\hat{\eta}$ is indeed a natural transformation.

For any $A \in \mathcal{C}$ we have

$$\begin{aligned}
(I\hat{\eta})_A = \hat{\eta}_{IA} = \hat{\eta}_{1_A} &= G(1_A \xrightarrow{1_A} 1_A) \circ \eta_A \circ F(1_A \xrightarrow{1_A} 1_A) \\
&= G(1_{1_A}) \circ \eta_A \circ F(1_{1_A}) = \eta_A,
\end{aligned}$$

which means that $\Phi(\hat{\eta}) = \eta$, so $\Phi$ is full.

Finally, let $F, G\colon \widehat{\mathcal{C}} \to \mathcal{D}$ be functors and $\eta, \eta'\colon F \to G$ be natural transformations such that $\Phi(\eta) = \Phi(\eta')$. To show that $\Phi$ is faithful, we need to prove that $\eta = \eta'$. The assumption $\Phi(\eta) = \Phi(\eta')$ means that for all $A \in \mathcal{C}$ we have $\eta_{IA} = \eta'_{IA}$, so $\eta_{1_A} = \eta'_{1_A}$.

Let $e\colon A \to A$ be any idempotent in $\mathcal{C}$. We need to show that $\eta_e = \eta'_e$. Indeed, we have

$$\begin{aligned}
\eta_e &= G(1_e) \circ \eta_e \\
&= G(e \xrightarrow{e} e) \circ \eta_e \\
&= G(e \xrightarrow{ee} e) \circ \eta_e \\
&= G(e \xrightarrow{e} 1_A \xrightarrow{e} e) \circ \eta_e \\
&= G(1_A \xrightarrow{e} e) \circ G(e \xrightarrow{e} 1_A) \circ \eta_e \\
&= G(1_A \xrightarrow{e} e) \circ \eta_{1_A} \circ F(e \xrightarrow{e} 1_A) \\
&= G(1_A \xrightarrow{e} e) \circ \eta'_{1_A} \circ F(e \xrightarrow{e} 1_A),
\end{aligned}$$

and the same argument in backwards direction shows that $\eta_e = \eta'_e$, completing the proof. □

## Chapter 2

**Exercise 13.**

Exercise. The inner automorphisms of $\mathcal{C}$ form a normal subgroup of the group of all automorphisms of $\mathcal{C}$. [Don't worry about whether these groups are sets or proper classes!]

Solution. Let $F, G\colon \mathcal{C} \to \mathcal{C}$ be automorphisms and let $\alpha\colon F \to 1_{\mathcal{C}}$ be a natural isomorphism.

Let $A \in \mathcal{C}$. Define $\beta\colon GFG^{-1} \to 1_A$ via $\beta_A \coloneqq G(\alpha_{G^{-1}A})$ (so $\beta_A\colon GFG^{-1}A \to GG^{-1}A = A \to GG^{-1}A = 1_{\mathcal{C}}A$.

This is indeed a natural transformation: let $f\colon A \to B \in \mathcal{C}$, then we can write the naturality square in a funny way,

$$\begin{tikzcd} GFG^{-1}A \arrow[r, "G(\alpha_{G^{-1}A})"] \arrow[d, "GFG^{-1}(f)"] & G1_CG^{-1}A \arrow[d, "G1_CG^{-1}f"] \\ GFG^{-1}B \arrow[r, "G(\alpha_{G^{-1}B})"] & G1_CG^{-1}B \end{tikzcd}$$

and we see that it is just the functor $G$ applied to the naturality diagram for $\alpha$ and the morphism $G^{-1}f$.

Therefore, $\beta$ is a natural transformation, and since functors map isomorphisms to isomorphisms, it is also a natural isomorphism. So $GFG^{-1}$ is an inner automorphism as required. □

Lemma. Let $1 \in \mathcal{C}$ be a terminal object and $F\colon C \to C$ an automorphism. Then $F1$ is a terminal object.

Proof. If $A \in \mathcal{C}$, the functor $F$, which is fully faithful, induces a bijection between the collection of morphisms $F^{-1}A \to 1$ and the collection of morphisms $A \to F1$. Since 1 is terminal, there is exactly one morphism $A \to F1$. □

Exercise. If $F\colon \mathsf{Set} \to \mathsf{Set}$ is an automorphism, then there is a unique natural isomorphism $1_{\mathcal{C}} \to F$.

Solution. Of course, the terminal object in the category of sets is just the one-element set $1 = \{\star\}$. Since $F1$ is also terminal, it is in bijection with 1. We write $F1 = \{\star_{F1}\}$.

By the Yoneda lemma, the set of natural transformations $\mathsf{Set}(1, -) \to F$ is in bijection with $F1$, so there is a unique natural transformation $\eta\colon \mathsf{Set}(1, -) \to F$. Examining the proof, we see that the components of this natural transformation are given by

$$\begin{aligned} \eta_A\colon \mathsf{Set}(1, A) &\to FA \\ f &\mapsto Ff(\star_{F1}) \end{aligned}$$

for any object $A$ of $\mathcal{C}$. Let $A$ be an object of $\mathcal{C}$. We will show that $\eta_A$ is an isomorphism, i.e., a bijection.

First, let $x \in FA$. Then $\eta_A(F^{-1}(\star_{F1} \mapsto x)) = x$, so $\eta_A$ is surjective.

Additionally, let $f, g\colon 1 \to A$ such that $\eta_A(f) = \eta_A(g)$. Since a map $F1 \to FA$ is completely determined by its value at $\star_{F1}$, we must have $Ff = Fg$. But then $f = F^{-1}F(f) = F^{-1}F(g) = g$.

This means that $\eta_A$ is an isomorphism, so $\eta$ is in fact a natural isomorphism.

We define a natural transformation $\alpha\colon 1_{\mathsf{Set}} \to \mathsf{Set}(1, -)$ by setting

$$\alpha_A(a)(\star) \coloneqq a.$$

The naturality square for $f\colon A \to B$ is

$$\begin{array}{ccc} A & \xrightarrow{\alpha_A} & \mathsf{Set}(1,A) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g\mapsto f\circ g} \\ B & \xrightarrow{\alpha_B} & \mathsf{Set}(1,B) \end{array}$$

Both paths are just $a \mapsto (\star \mapsto f(a))$, so $\alpha$ is natural. It is also clear that $\alpha_A$ is bijective, so $\alpha$ is a natural isomorphism. In other words, $\star$ is a universal element of the identity functor.

In particular, this tells is that composition with $\alpha$ and its inverse exhibits a bijection between the collection of natural transformations

$$1_{\mathsf{Set}} \to F$$

and the collection of natural transformations

$$\mathsf{Set}(1,-) \to F.$$

This means that there is a unique natural transformation $1_{\mathsf{Set}} \to F$, and it is given by $\alpha \circ \eta$, and since $\alpha$ and $\eta$ are both natural isomorphisms, so is $\alpha \circ \eta$, completing the proof. □

Exercise. The Sierpiński space $S$ is, up to isomorphism, the unique topological space which has precisely three endomorphisms.

Solution. Let $X$ be a topological space. Then for any $x \in X$, the constant map $c_x\colon X \to X$ sending $y \in X$ to $x$ is continuous. Furthermore, the identity on $X$ is continuous. This, if $X$ is infinite, then $X$ has infinitely many endomorphisms, and if $X$ is finite, then $X$ has at least $|X| + 1$ endomorphisms.

Now assume that $X$ has precisely three endomorphisms. Then $X$ is finite and has at most two points. Clearly, if $X$ has zero or one point, then there is only one endomorphism. So $X$ has two points, say $X = \{a, b\}$. There are four set-functions $\{a, b\} \to \{a, b\}$, three of which (the identity and the two constant maps) are continuous regardless of the topology. The final map interchanges $a$ and $b$ and is not continuous.

The empty set and all of $X$ are open. If $X$ had the trivial or the discrete topology, then the interchange would be continuous, a contradiction. Hence, precisely one of the sets $\{a\}$ and $\{b\}$ is open. Sending the member of that set to 1 and the other element to 0 describes a homeomorphism with $S$. □

Exercise. Let $\mathcal{C}$ be a full subcategory of $\mathsf{Top}$ containing the singleton space 1 and the Siperpiński space $S$ and let $F$ be an automorphism of $\mathcal{C}$. Then

(a) we have $FS \cong S$,
(b) there is a unique natural isomorphism $\alpha\colon U \to UF$, where $U\colon \mathcal{C} \to \mathsf{Set}$ is the forgetful functor,
(c) if $\mathcal{C}$ contains a space in which not every union of closed sets is closed, then $\alpha_S$ is continuous, and
(d) $F$ is uniquely naturally isomorphic to the identity functor.

Solution. For (a), we just need to notice that $F$ is fully faithful, so it induces a bijection between the sets of morphisms $S \to S$ and $FS \to FS$. Since $S$ is determined up to isomorphism by having exactly three endomorphisms, the claim follows.

The proof of (b) is entirely analogous to the proof of 2.13(ii). □

**Exercise 14.**

Exercise. Let $e\colon A \to A$ be an idempotent. Then the following are equivalent:

(i) $e$ is split,
(ii) the pair $(e, 1_A)$ has an equaliser,
(iii) the pair $(e, 1_A)$ has a coequaliser.

Solution. We will show that (i) is equivalent to (ii). By duality, this implies that (i) is equivalent to (iii).

Assume that there are $f\colon B \to A$ and $g\colon A \to B$ such that $fg = e$ and $gf = 1_B$. We claim that $f$ is an equaliser of $e$ and $1_A$. We must show that any $h\colon C \to A$ satisfying $he = h$ factors uniquely through $f$.

$$\begin{tikzcd}
 & C \arrow[d, "h"] \arrow[dl, "h'"'] & \\
B \arrow[r, shift left, "f"] & A \arrow[l, shift left, "g"] \arrow[r, shift left, "e"] \arrow[r, shift right, "1_A"'] & A
\end{tikzcd}$$

Indeed, given such $h$. Then $fgh = eh = h$, hence $gh$ is one such factoring factoring. If $h'\colon C \to B$ is another factoring such that $fh' = h$, then $h' = gfh' = gh$, so the factoring is unique.

Conversely, assume that the pair $(e, 1_A)$ admits an equaliser $f\colon B \to A$. Since $ee = e = 1_A e$, $e$ factors through $f$ via some $g\colon A \to B$. Hence, $fg = e$. On the other hand, $fgf = ef = f$, and by a result from the lecture, $f$ is monic, so $gf = 1_A$, so $e$ is split. $\square$

Exercise. A split monomorphism is regular.

Solution. If $f\colon A \to B$ is a split monomorphism, then there is some $g\colon B \to A$ such that $gf = 1_A$. Then $fgfg = f1_Ag = fg$, so $fg$ is a split idempotent. By what we just saw, this means that $f$ is an equaliser of $(fg, 1_A)$, hence $f$ is a regular monomorphism. $\square$

**Exercise 15.**

Exercise. Every regular monomorphism is strong.

Solution. Let $f$ be the equaliser of $u$ and $v$ and take a commutative square as in the definition of strongness.

$$\begin{tikzcd}
C \arrow[r, "h"] \arrow[d, two heads, "g"] & A \arrow[d, hook, "f"] & \\
D \arrow[r, "k"] \arrow[ur, dotted, "t"] & B \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & E
\end{tikzcd}$$

We have $ukg = ufh = vfh = vkg$. Since $g$ is epi, this means that $uk = vk$, and since $f$ is the equaliser of $u$ and $v$, we find $t\colon D \to A$ such that $ft = k$. Now $ftg = kg = fh$. Since $f$ is mono, we conclude that $tg = h$, so $t$ has the desired properties. Hence, $f$ is a strong monomorphism. $\square$

Exercise. Let $\mathcal{C}$ be the finite category whose non-identity morphisms are represented by the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "l"'] & B \arrow[d, shift right, "h"'] \arrow[d, shift left, "k"] & C \arrow[l, "g"'] \arrow[dl, "m"] \\
 & D. &
\end{tikzcd}$$

The morphism $f$ is strong monic but not regular monic.

SOLUTION. The strongness condition for $f$ is actually vacuous: if we have a diagram

$$\begin{tikzcd} \bullet \arrow[r, "u"] \arrow[d, two heads, "v"'] & A \arrow[d, hook, "f"] \\ \bullet \arrow[r, "w"] & B, \end{tikzcd}$$

then we must have $u = 1_A$. The morphism $f$ is not an epimorphism, as witnessed by the fact that $hf = kf$, but $h \neq k$, so we must have $v = l$. Then $w$ is a morphism $D \to B$, but such a morphism does not exist. Hence, the square does not exist, so $f$ is strong.

However, the only pairs of morphisms that $f$ can be an equaliser of are $(1_B, 1_B)$, $(k, k)$, $(h, h)$ and $(h, k)$. If $f$ was the equaliser of any of these pairs, $g$ would factor through $f$, but there is no morphism $C \to A$, hence that is not the case. So we conclude that $f$ is not regular. □

**Exercise 16.**

EXERCISE. Let $f : A \to B$, $g : B \to C$ be two morphisms.

(a) If $f$ and $g$ are monic, then $gf$ is monic,
(b) If $f$ and $g$ are strong monic, then $gf$ is strong monic,
(c) If $f$ and $g$ are split monic, then $gf$ is split monic,
(d) If $gf$ is monic, then $f$ is monic,
(e) If $gf$ is strong monic, then $f$ is strong monic,
(f) If $gf$ is split monic, then $f$ is split monic.
(g) If $gf$ is regular monic and $g$ is monic, then $f$ is regular monic.

SOLUTION. (a) If $gfu = gfv$, then $fu = fv$ since $g$ is monic, and $u = v$, since $f$ is monic.
(b) Consider the diagram

$$\begin{tikzcd} D \arrow[r, "h"] \arrow[dd, two heads, "l"'] & A \arrow[d, hook, "f"] \\ & B \arrow[d, hook, "g"] \\ E \arrow[r, "k"] \arrow[uur, dotted, "u"] \arrow[ur, dotted, "t"] & C. \end{tikzcd}$$

Since $g$ is strong monic, using the square $(fh, g, l, k)$, we find $t : E \to B$ such that $gt = k$ and $tl = fh$. Since $f$ is strong epic, using the square $(h, f, l, t)$, we find $u : E \to A$ such that $fu = t$ and $ul = h$. Then we have $gfu = gt = k$, so $u$ is the required morphism.
(c) If $u : B \to A$ satisfies $uf = 1_A$ and $v : C \to B$ satisfies $vg = 1_B$, then $uv$ is the desired retraction, as $uvgf = u1_Bf = uf = 1_A$.
(d) If $fu = fv$, then trivially, $gfu = gfv$, so $u = v$.

(e) Consider the diagram

$$\begin{tikzcd}
D \arrow[r, "h"] \arrow[d, two heads, "l"] & A \arrow[d, "f"] \\
E \arrow[r, "k"] \arrow[ur, dotted, "t"] \arrow[dr, "gk"'] & B \arrow[d, "g"] \\
& C.
\end{tikzcd}$$

Since $gf$ is strong monic, using the square $(h, gf, l, gk)$ we find $t\colon E \to A$ such that $tl = h$ (and $gft = gk$, but that is not important). We have $ftl = fh = kl$, so since $l$ is epi, we have $ft = k$, so $t$ is indeed the required diagonal morphism, so $f$ is strong monic.

(f) If $u\colon C \to A$ satisfies $ugf = 1_A$, then $(ug)f = 1_A$, so $f$ is split monic.

(g) Say $gf$ is an equalizer of $u$ and $v$.

$$\begin{tikzcd}
& T \arrow[dl, dotted, "\ell"'] \arrow[d, "h"] \\
A \arrow[r, "f"] & B \arrow[r, hook, "g"] & C \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & D
\end{tikzcd}$$

If $h\colon T \to B$ satisfies $ugh = vgh$, then since $gf$ is an equaliser of $u$ and $v$, we find a unique $\ell\colon T \to A$ such that $gf\ell = gh$. Since $g$ is monic, we have $f\ell = h$. The morphism $\ell$ is the unique morphism satisfying $f\ell = h$, since if $\hat{\ell}$ also satisfies $f\hat{\ell} = h$, then certainly $gf\hat{\ell} = gh$, hence $\ell = \hat{\ell}$.

□

Exercise. Let $\mathcal{C}$ be the full subcategory of $\mathsf{Ab}$ whose objects are groups having no elements of order 4 (though they may have elements of order 2). Then

(i) multiplication by 2 is a regular monomorphism $\mathbb{Z} \to \mathbb{Z}$,
(ii) multiplication by 4 is not a regular monomorphism $\mathbb{Z} \to \mathbb{Z}$,
(iii) there is a pair of morphisms $(f, g)$ such that $gf$ is regular monix but $f$ is not.

Solution. (i) We claim that multiplication by 2 is an equalizer in $\mathcal{C}$ of the projection $\pi\colon \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ and the zero map $0\colon \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$.

$$\begin{tikzcd}
& G \arrow[dl, dotted] \arrow[d, "f"] \\
\mathbb{Z} \arrow[r, "\cdot 2"] & \mathbb{Z} \arrow[r, shift left, "\pi"] \arrow[r, shift right, "0"'] & \mathbb{Z}/2\mathbb{Z}
\end{tikzcd}$$

Indeed, if $f\colon G \to \mathbb{Z}$ equalizes $\pi$ and 0, then its image is contained in $2\mathbb{Z}$, hence it factors uniquely through multiplication by 2 via the map $g \mapsto f(g)/2$.

(ii) Assume that multiplication by 4 is an equalizer in $\mathcal{C}$ of $f$ and $g$.

$$\begin{tikzcd}
& \ker(f-g) \arrow[dl, dotted] \arrow[d, "\iota"] \\
\mathbb{Z} \arrow[r, "\cdot 4"] & \mathbb{Z} \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & G
\end{tikzcd}$$

Clearly, the kernel of $f - g$ has no elements of order 4 and the inclusion equalizes $f$ and $g$, hence it factors through multiplication by 4. Consider the element $\alpha := f(1) - g(1) \in G$. We know that $\alpha + \alpha + \alpha + \alpha = f(4) - g(4) = 0$,

since multiplication by 4 equalises $f$ and $g$. Since $G$ is an object of $\mathcal{C}$, the order of $\alpha$ is 2 or 1. In either case, we have $2 \in \ker(f - g)$, which is not in the image of multiplication by 4, hence $\iota$ cannot factor through multiplication by 4, so multiplication by 4 is not an equalizer of $f$ and $g$.
$\square$