

# Category Theory

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These notes, taken by Markus Himmel, will at times differ significantly from what was lectured. In particular, all errors are almost certainly my own.

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# Exercises

## Chapter 1

### Exercise 17.

EXERCISE. A morphism  $e: A \rightarrow A$  is called idempotent if  $ee = e$ . An idempotent  $e$  is said to split if it can be factored as  $fg$  where  $gf$  is an identity morphism.

- (i) Let  $\mathcal{E}$  be a collection of idempotents in a category  $\mathcal{C}$ : show that there is a category  $\mathcal{C}[\check{\mathcal{E}}]$  whose objects are the members of  $\mathcal{E}$ , whose morphisms  $e \rightarrow d$  are those morphisms  $f: \text{dom } e \rightarrow \text{dom } d$  in  $\mathcal{C}$  for which  $dfe = f$ , and whose composition coincides with composition in  $\mathcal{C}$ . [Hint: first show that the single equation  $dfe = f$  is equivalent to the two equations  $df = f = fe$ . Note that the identity morphism on an object  $e$  is not  $1_{\text{dom } e}$  in general.]
- (ii) If  $\mathcal{E}$  contains all identity morphisms of  $\mathcal{C}$ , show that there is a full and faithful functor  $I: \mathcal{C} \rightarrow \mathcal{C}[\check{\mathcal{E}}]$ , and that an arbitrary functor  $T: \mathcal{C} \rightarrow \mathcal{D}$  can be factored as  $\hat{T}I$  for some  $\hat{T}$  iff it sends the members of  $\mathcal{E}$  to split idempotents in  $\mathcal{D}$ .
- (iii) Deduce that if all idempotents split in  $\mathcal{D}$ , then the functor categories  $[\mathcal{C}, \mathcal{D}]$  and  $[\hat{\mathcal{C}}, \mathcal{D}]$  are equivalent, where  $\hat{\mathcal{C}} = \mathcal{C}[\check{\mathcal{E}}]$  for  $\mathcal{E}$  the class of all idempotents in  $\mathcal{C}$ .

SOLUTION. We will first show that if  $f: C \rightarrow D$  is any morphism and  $c: C \rightarrow C$  and  $d: D \rightarrow D$  are idempotents, then  $dfe = f \iff df = f = fe$ .

Indeed, if  $df = f = fe$ , then  $dfe = fe = f$ . Conversely, if  $dfe = f$ , then  $f = dfe = ddfe = df$  and  $f = dfe = dfee = fe$ .

To show that  $\mathcal{C}[\check{\mathcal{E}}]$  is a category, we need to show that the composition of two morphisms is indeed a morphism and that there are identity morphism.

Assume that  $c: C \rightarrow C$ ,  $d: D \rightarrow D$ ,  $e: E \rightarrow E$  are idempotents and that  $f: C \rightarrow D$  and  $g: D \rightarrow E$  satisfy  $dfe = f$  and  $egd = g$ . We need to show that  $egfc = gf$ . Using the lemma, we have  $egf = (eg)f = gf$  and  $gfc = g(fc) = gf$ , so, again by the lemma, the claim follows.

If  $e: E \rightarrow E$  is an idempotent, define  $1_e := e \xrightarrow{e} e$ . By idempotency of  $e$ , this is indeed a morphism. If  $f: d \rightarrow e$  is a morphism, then the morphism  $f1_d$  is the morphism  $fd = f$  (here we use the lemma again) in  $\mathcal{C}$ , so  $f1_d = f$  as required. Similarly,  $1_e f = f$ . This completes part (i).

Next, assume that  $\mathcal{E}$  contains all identity morphisms of  $\mathcal{C}$ . Define the functor  $I$  via

$$\begin{aligned} I: \mathcal{C} &\rightarrow \mathcal{C}[\check{\mathcal{E}}] \\ A &\mapsto 1_A \\ (f: A \rightarrow B) &\mapsto (f: 1_A \rightarrow 1_B) \end{aligned}$$

This is indeed a functor and since the data of a morphism  $A \rightarrow B$  in  $\mathcal{C}$  is precisely the same as the data of a morphism  $1_A \rightarrow 1_B$  in  $\mathcal{C}[\check{\mathcal{E}}]$ ,  $I$  is fully faithful.

Now let  $T: \mathcal{C} \rightarrow \mathcal{D}$  be any functor.

First, assume that there is some functor  $\widehat{T}: \mathcal{C}[\check{\mathcal{E}}] \rightarrow \mathcal{D}$  such that  $T = \widehat{T}I$ . Let  $e: A \rightarrow A \in \mathcal{E}$  be an idempotent. Then we have

$$\begin{aligned} Te &= \widehat{T}(1_A \xrightarrow{e} 1_A) \\ &= \widehat{T}(1_A \xrightarrow{e} e \xrightarrow{e} 1_A) \\ &= \widehat{T}(e \xrightarrow{e} 1_A) \circ \widehat{T}(1_A \xrightarrow{e} e), \end{aligned}$$

and we also have

$$\begin{aligned} \widehat{T}(1_A \xrightarrow{e} e) \circ \widehat{T}(e \xrightarrow{e} 1_A) &= \widehat{T}(e \xrightarrow{e} 1_A \xrightarrow{e} e) \\ &= \widehat{T}(e \xrightarrow{ee} e) \\ &= \widehat{T}(e \xrightarrow{e} e) \\ &= \widehat{T}(1_e) \\ &= 1_{\widehat{T}e}, \end{aligned}$$

which shows that  $Te$  is split.

Next, assume that  $Te$  is split for any  $e \in \mathcal{E}$ . For any  $e \in \mathcal{E}$ , choose a splitting

$$TA \xrightleftharpoons[f_e]{g_e} B_e,$$

i.e.,  $f_e \circ g_e = Te$ ,  $g_e \circ f_e = 1_{B_e}$ . For identity morphisms  $1_A$  ( $A$  an object of  $\mathcal{C}$ ), choose the specific splitting given by  $B_{1_A} := TA$ ,  $f_{1_A} := 1_{TA}$ ,  $g_{1_A} := 1_{TA}$ .

Now define the functor  $\widehat{T}$  via

$$\begin{aligned} \widehat{T}: \mathcal{C}[\check{\mathcal{E}}] &\rightarrow \mathcal{D} \\ (e: A \rightarrow A) &\mapsto B_e \\ (f: d \rightarrow e) &\mapsto g_e \circ Tf \circ f_d. \end{aligned}$$

If  $e \in \mathcal{E}$ , then we have

$$\begin{aligned} \widehat{T}(1_e) &= g_e \circ Te \circ f_e \\ &= g_e \circ f_e \circ g_e \circ f_e \\ &= 1_{B_e} \circ 1_{B_e} = 1_{B_e} \end{aligned}$$

Furthermore, if  $f: c \rightarrow d$  and  $g: d \rightarrow e$ , then we have

$$\begin{aligned} \widehat{T}(g \circ f) &= g_e \circ T(g \circ f) = f_c \\ &= g_e \circ Tg \circ Tf = f_c \\ &= g_e \circ Tg \circ T(d \circ f) \circ f_c \\ &= g_e \circ Tg \circ Td \circ Tf \circ f_c \\ &= g_e \circ Tg \circ f_d \circ g_d \circ Tf \circ f_c \\ &= \widehat{T}g \circ \widehat{T}f. \end{aligned}$$

So  $\widehat{T}$  is indeed a functor. If  $A$  is an object of  $\mathcal{C}$ , then

$$\widehat{T}IA = \widehat{T}1_A = B_{1_A} = TA$$

and if  $f: C \rightarrow D$  is a morphism in  $\mathcal{C}$ , then

$$\widehat{T}If = \widehat{T}(1_C \xrightarrow{f} 1_D) = g_{1_D} \circ Tf \circ f_{1_C} = 1_{TD} \circ Tf \circ 1_{TC} = Tf,$$

so  $\widehat{T}$  is the required factorisation, completing part (ii).

Define a functor  $\Phi: [\widehat{\mathcal{C}}, \mathcal{D}] \rightarrow [\mathcal{C}, \mathcal{D}]$  via  $F \mapsto F \circ I$ ,  $\eta \mapsto I\eta$ , where  $I\eta$  is defined via  $I\eta_C := \eta_{IC} = \eta_{1_C}$ . Naturality of  $I\eta$  immediately follows from naturality of  $\eta$ . Functoriality is also clear.

We will show that this functor is full, faithful and essentially surjective.

Indeed, let  $F: \mathcal{C} \rightarrow \mathcal{D}$  be a functor. Then  $\widehat{F}$  as defined in the previous part satisfies  $\Phi\widehat{F} = F$ , so  $\Phi$  is essentially surjective.

Next, let  $F, G: \widehat{\mathcal{C}} \rightarrow \mathcal{D}$  be functors and  $\eta: F \circ I \rightarrow G \circ I$  a natural transformation. For an idempotent  $e: A \rightarrow A$  in  $\mathcal{C}$ , define  $\hat{\eta}_e$  to be the composite

$$Fe \xrightarrow{F(e \xrightarrow{e} 1_A)} F1_A = (F \circ I)A \xrightarrow{\eta_A} (G \circ I)A = G1_A \xrightarrow{G(1_A \xrightarrow{e} e)} Ge.$$

We claim that this defines a natural transformation  $\hat{\eta}: F \rightarrow G$ . Indeed, if  $f: d \rightarrow e$  is a morphism, then

$$\begin{aligned} \hat{\eta}_e \circ Ff &= G(1_A \xrightarrow{e} e) \circ \eta_E \circ F(e \xrightarrow{e} 1_E) \circ F(d \xrightarrow{f} e) \\ &= G(1_E \xrightarrow{e} e) \circ \eta_E \circ F(d \xrightarrow{f} e \xrightarrow{e} 1_E) \\ &= G(1_E \xrightarrow{e} e) \circ \eta_E \circ F(d \xrightarrow{d} 1_D \xrightarrow{d} d \xrightarrow{f} e \xrightarrow{e} 1_E) \\ &= G(1_E \xrightarrow{e} e) \circ \eta_E \circ F(d \xrightarrow{d} 1_D \xrightarrow{d} d \xrightarrow{f} e \xrightarrow{e} 1_E) \\ &= G(1_E \xrightarrow{e} e) \circ \eta_E \circ F(1_D \xrightarrow{efd} 1_E) \circ F(d \xrightarrow{d} 1_D) \\ &= G(1_E \xrightarrow{e} e) \circ \eta_E \circ F(efd) \circ F(d \xrightarrow{d} 1_D) \\ &= G(1_E \xrightarrow{e} e) \circ G(efd) \circ \eta_D \circ F(d \xrightarrow{d} 1_D), \end{aligned}$$

and doing the whole thing backwards we conclude that  $\hat{\eta}_e \circ Ff = Gf \circ \hat{\eta}_d$ , so  $\hat{\eta}$  is indeed a natural transformation.

For any  $A \in \mathcal{C}$  we have

$$\begin{aligned} (I\hat{\eta})_A &= \hat{\eta}_{IA} = \hat{\eta}_{1_A} = G(1_A \xrightarrow{1_A} 1_A) \circ \eta_A \circ F(1_A \xrightarrow{1_A} 1_A) \\ &= G(1_{1_A}) \circ \eta_A \circ F(1_{1_A}) = \eta_A, \end{aligned}$$

which means that  $\Phi(\hat{\eta}) = \eta$ , so  $\Phi$  is full.

Finally, let  $F, G: \widehat{\mathcal{C}} \rightarrow \mathcal{D}$  be functors and  $\eta, \eta': F \rightarrow G$  be natural transformations such that  $\Phi(\eta) = \Phi(\eta')$ . To show that  $\Phi$  is faithful, we need to prove that  $\eta = \eta'$ . The assumption  $\Phi(\eta) = \Phi(\eta')$  means that for all  $A \in \mathcal{C}$  we have  $\eta_{IA} = \eta'_{IA}$ , so  $\eta_{1_A} = \eta'_{1_A}$ .

Let  $e: A \rightarrow A$  be any idempotent in  $\mathcal{C}$ . We need to show that  $\eta_e = \eta'_e$ . Indeed, we have

$$\begin{aligned} \eta_e &= G(1_e) \circ \eta_e \\ &= G(e \xrightarrow{e} e) \circ \eta_e \\ &= G(e \xrightarrow{ee} e) \circ \eta_e \\ &= G(e \xrightarrow{e} 1_A \xrightarrow{e} e) \circ \eta_e \\ &= G(1_A \xrightarrow{e} e) \circ G(e \xrightarrow{e} 1_A) \circ \eta_e \\ &= G(1_A \xrightarrow{e} e) \circ \eta_{1_A} \circ F(e \xrightarrow{e} 1_A) \\ &= G(1_A \xrightarrow{e} e) \circ \eta'_{1_A} \circ F(e \xrightarrow{e} 1_A), \end{aligned}$$

and the same argument in backwards direction shows that  $\eta_e = \eta'_e$ , completing the proof.  $\square$

## Chapter 2

### Exercise 13.

EXERCISE. The inner automorphisms of  $\mathcal{C}$  form a normal subgroup of the group of all automorphisms of  $\mathcal{C}$ . [Don't worry about whether these groups are sets or proper classes!]

SOLUTION. Let  $F, G: \mathcal{C} \rightarrow \mathcal{C}$  be automorphisms and let  $\alpha: F \rightarrow 1_{\mathcal{C}}$  be a natural isomorphism.

Let  $A \in \mathcal{C}$ . Define  $\beta: GFG^{-1} \rightarrow 1_A$  via  $\beta_A := G(\alpha_{G^{-1}A})$  (so  $\beta_A: GFG^{-1}A \rightarrow GG^{-1}A = A \rightarrow GG^{-1}A = 1_{\mathcal{C}}A$ ).

This is indeed a natural transformation: let  $f: A \rightarrow B \in \mathcal{C}$ , then we can write the naturality square in a funny way,

$$\begin{array}{ccc} GFG^{-1}A & \xrightarrow{G(\alpha_{G^{-1}A})} & G1_{\mathcal{C}}G^{-1}A \\ \downarrow GFG^{-1}(f) & & \downarrow G1_{\mathcal{C}}G^{-1}f \\ GFG^{-1}B & \xrightarrow{G(\alpha_{G^{-1}B})} & G1_{\mathcal{C}}G^{-1}B \end{array}$$

and we see that it is just the functor  $G$  applied to the naturality diagram for  $\alpha$  and the morphism  $G^{-1}f$ .

Therefore,  $\beta$  is a natural transformation, and since functors map isomorphisms to isomorphisms, it is also a natural isomorphism. So  $GFG^{-1}$  is an inner automorphism as required.  $\square$

LEMMA. Let  $1 \in \mathcal{C}$  be a terminal object and  $F: \mathcal{C} \rightarrow \mathcal{C}$  an automorphism. Then  $F1$  is a terminal object.

PROOF. If  $A \in \mathcal{C}$ , the functor  $F$ , which is fully faithful, induces a bijection between the collection of morphisms  $F^{-1}A \rightarrow 1$  and the collection of morphisms  $A \rightarrow F1$ . Since  $1$  is terminal, there is exactly one morphism  $A \rightarrow F1$ .  $\square$

EXERCISE. If  $F: \mathbf{Set} \rightarrow \mathbf{Set}$  is an automorphism, then there is a unique natural isomorphism  $1_{\mathcal{C}} \rightarrow F$ .

SOLUTION. Of course, the terminal object in the category of sets is just the one-element set  $1 = \{\star\}$ . Since  $F1$  is also terminal, it is in bijection with  $1$ . We write  $F1 = \{\star_{F1}\}$ .

By the Yoneda lemma, the set of natural transformations  $\mathbf{Set}(1, -) \rightarrow F$  is in bijection with  $F1$ , so there is a unique natural transformation  $\eta: \mathbf{Set}(1, -) \rightarrow F$ . Examining the proof, we see that the components of this natural transformation are given by

$$\begin{aligned} \eta_A: \mathbf{Set}(1, A) &\rightarrow FA \\ f &\mapsto Ff(\star_{F1}) \end{aligned}$$

for any object  $A$  of  $\mathcal{C}$ . Let  $A$  be an object of  $\mathcal{C}$ . We will show that  $\eta_A$  is an isomorphism, i.e., a bijection.

First, let  $x \in FA$ . Then  $\eta_A(F^{-1}(\star_{F1} \mapsto x)) = x$ , so  $\eta_A$  is surjective.

Additionally, let  $f, g: 1 \rightarrow A$  such that  $\eta_A(f) = \eta_A(g)$ . Since a map  $F1 \rightarrow FA$  is completely determined by its value at  $\star_{F1}$ , we must have  $Ff = Fg$ . But then  $f = F^{-1}F(f) = F^{-1}F(g) = g$ .

This means that  $\eta_A$  is an isomorphism, so  $\eta$  is in fact a natural isomorphism.

We define a natural transformation  $\alpha: 1_{\mathbf{Set}} \rightarrow \mathbf{Set}(1, -)$  by setting

$$\alpha_A(a)(\star) := a.$$



The naturality square for  $f: A \rightarrow B$  is

$$\begin{array}{ccc} A & \xrightarrow{\alpha_A} & \mathbf{Set}(1, A) \\ \downarrow f & & \downarrow g \mapsto f \circ g \\ B & \xrightarrow{\alpha_B} & \mathbf{Set}(1, B) \end{array}$$

Both paths are just  $a \mapsto (\star \mapsto f(a))$ , so  $\alpha$  is natural. It is also clear that  $\alpha_A$  is bijective, so  $\alpha$  is a natural isomorphism. In other words,  $\star$  is a universal element of the identity functor.

In particular, this tells us that composition with  $\alpha$  and its inverse exhibits a bijection between the collection of natural transformations

$$1_{\mathbf{Set}} \rightarrow F$$

and the collection of natural transformations

$$\mathbf{Set}(1, -) \rightarrow F.$$

This means that there is a unique natural transformation  $1_{\mathbf{Set}} \rightarrow F$ , and it is given by  $\alpha \circ \eta$ , and since  $\alpha$  and  $\eta$  are both natural isomorphisms, so is  $\alpha \circ \eta$ , completing the proof.  $\square$

**EXERCISE.** The Sierpiński space  $S$  is, up to isomorphism, the unique topological space which has precisely three endomorphisms.

**SOLUTION.** Let  $X$  be a topological space. Then for any  $x \in X$ , the constant map  $c_x: X \rightarrow X$  sending  $y \in X$  to  $x$  is continuous. Furthermore, the identity on  $X$  is continuous. This, if  $X$  is infinite, then  $X$  has infinitely many endomorphisms, and if  $X$  is finite, then  $X$  has at least  $|X| + 1$  endomorphisms.

Now assume that  $X$  has precisely three endomorphisms. Then  $X$  is finite and has at most two points. Clearly, if  $X$  has zero or one point, then there is only one endomorphism. So  $X$  has two points, say  $X = \{a, b\}$ . There are four set-functions  $\{a, b\} \rightarrow \{a, b\}$ , three of which (the identity and the two constant maps) are continuous regardless of the topology. The final map interchanges  $a$  and  $b$  and is not continuous.

The empty set and all of  $X$  are open. If  $X$  had the trivial or the discrete topology, then the interchange would be continuous, a contradiction. Hence, precisely one of the sets  $\{a\}$  and  $\{b\}$  is open. Sending the member of that set to 1 and the other element to 0 describes a homeomorphism with  $S$ .  $\square$

**EXERCISE.** Let  $\mathcal{C}$  be a full subcategory of **Top** containing the singleton space 1 and the Sierpiński space  $S$  and let  $F$  be an automorphism of  $\mathcal{C}$ . Then

- (a) we have  $FS \cong S$ ,
- (b) there is a unique natural isomorphism  $\alpha: U \rightarrow UF$ , where  $U: \mathcal{C} \rightarrow \mathbf{Set}$  is the forgetful functor,
- (c) if  $\mathcal{C}$  contains a space in which not every union of closed sets is closed, then  $\alpha_S$  is continuous, and
- (d)  $F$  is uniquely naturally isomorphic to the identity functor.

**SOLUTION.** For (a), we just need to notice that  $F$  is fully faithful, so it induces a bijection between the sets of morphisms  $S \rightarrow S$  and  $FS \rightarrow FS$ . Since  $S$  is determined up to isomorphism by having exactly three endomorphisms, the claim follows.

The proof of (b) is entirely analogous to the proof of 2.13(ii).  $\square$

**Exercise 14.**

EXERCISE. Let  $e: A \rightarrow A$  be an idempotent. Then the following are equivalent:

- (i)  $e$  is split,
- (ii) the pair  $(e, 1_A)$  has an equaliser,
- (iii) the pair  $(e, 1_A)$  has a coequaliser.

SOLUTION. We will show that (i) is equivalent to (ii). By duality, this implies that (i) is equivalent to (iii).

Assume that there are  $f: B \rightarrow A$  and  $g: A \rightarrow B$  such that  $fg = e$  and  $gf = 1_B$ . We claim that  $f$  is an equaliser of  $e$  and  $1_A$ . We must show that any  $h: C \rightarrow A$  satisfying  $he = h$  factors uniquely through  $f$ .

$$\begin{array}{ccccc} & & C & & \\ & \swarrow h' & \downarrow h & & \\ B & \xleftarrow{f} & A & \xrightarrow[e]{1_A} & A \end{array}$$

Indeed, given such  $h$ . Then  $fg h = e h = h$ , hence  $gh$  is one such factoring. If  $h': C \rightarrow B$  is another factoring such that  $fh' = h$ , then  $h' = gh' = gh$ , so the factoring is unique.

Conversely, assume that the pair  $(e, 1_A)$  admits an equaliser  $f: B \rightarrow A$ . Since  $ee = e = 1_A e$ ,  $e$  factors through  $f$  via some  $g: A \rightarrow B$ . Hence,  $fg = e$ . On the other hand,  $fgf = ef = f$ , and by a result from the lecture,  $f$  is monic, so  $gf = 1_A$ , so  $e$  is split.  $\square$

EXERCISE. A split monomorphism is regular.

SOLUTION. If  $f: A \rightarrow B$  is a split monomorphism, then there is some  $g: B \rightarrow A$  such that  $gf = 1_A$ . Then  $fgfg = f1_A g = fg$ , so  $fg$  is a split idempotent. By what we just saw, this means that  $f$  is an equaliser of  $(fg, 1_A)$ , hence  $f$  is a regular monomorphism.  $\square$

**Exercise 15.**

EXERCISE. Every regular monomorphism is strong.

SOLUTION. Let  $f$  be the equaliser of  $u$  and  $v$  and take a commutative square as in the definition of strongness.

$$\begin{array}{ccccc} C & \xrightarrow{h} & A & & \\ \downarrow g & \nearrow t & \downarrow f & & \\ D & \xrightarrow{k} & B & \xrightarrow[u]{v} & E \end{array}$$

We have  $ukg = ufh = vfh = vkg$ . Since  $g$  is epi, this means that  $uk = vk$ , and since  $f$  is the equaliser of  $u$  and  $v$ , we find  $t: D \rightarrow A$  such that  $ft = k$ . Now  $ftg = kg = fh$ . Since  $f$  is mono, we conclude that  $tg = h$ , so  $t$  has the desired properties. Hence,  $f$  is a strong monomorphism.  $\square$

EXERCISE. Let  $\mathcal{C}$  be the finite category whose non-identity morphisms are represented by the diagram

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xleftarrow{g} & C \\ & \searrow l & \downarrow h & \downarrow k & \swarrow m \\ & & D & & \end{array}$$

The morphism  $f$  is strong monic but not regular monic.

SOLUTION. The strongness condition for  $f$  is actually vacuous: if we have a diagram

$$\begin{array}{ccc} \bullet & \xrightarrow{u} & A \\ \downarrow v & & \downarrow f \\ \bullet & \xrightarrow{w} & B, \end{array}$$

then we must have  $u = 1_A$ . The morphism  $f$  is not an epimorphism, as witnessed by the fact that  $hf = kf$ , but  $h \neq k$ , so we must have  $v = l$ . Then  $w$  is a morphism  $D \rightarrow B$ , but such a morphism does not exist. Hence, the square does not exist, so  $f$  is strong.

However, the only pairs of morphisms that  $f$  can be an equaliser of are  $(1_B, 1_B)$ ,  $(k, k)$ ,  $(h, h)$  and  $(h, k)$ . If  $f$  was the equaliser of any of these pairs,  $g$  would factor through  $f$ , but there is no morphism  $C \rightarrow A$ , hence that is not the case. So we conclude that  $f$  is not regular.  $\square$

### Exercise 16.

EXERCISE. Let  $f: A \rightarrow B$ ,  $g: B \rightarrow C$  be two morphisms.

- (a) If  $f$  and  $g$  are monic, then  $gf$  is monic,
- (b) If  $f$  and  $g$  are strong monic, then  $gf$  is strong monic,
- (c) If  $f$  and  $g$  are split monic, then  $gf$  is split monic,
- (d) If  $gf$  is monic, then  $f$  is monic,
- (e) If  $gf$  is strong monic, then  $f$  is strong monic,
- (f) If  $gf$  is split monic, then  $f$  is split monic.
- (g) If  $gf$  is regular monic and  $g$  is monic, then  $f$  is regular monic.

SOLUTION. (a) If  $gf u = gf v$ , then  $f u = f v$  since  $g$  is monic, and  $u = v$ , since  $f$  is monic.

(b) Consider the diagram

$$\begin{array}{ccc} D & \xrightarrow{h} & A \\ \downarrow l & & \downarrow f \\ E & \xrightarrow{k} & C \end{array} \quad \begin{array}{c} \nearrow u \\ \nearrow t \end{array} \quad \begin{array}{c} \nearrow \\ \nearrow \end{array}$$

Since  $g$  is strong monic, using the square  $(fh, g, l, k)$ , we find  $t: E \rightarrow B$  such that  $gt = k$  and  $tl = fh$ . Since  $f$  is strong epic, using the square  $(h, f, l, t)$ , we find  $u: E \rightarrow A$  such that  $fu = t$  and  $ul = h$ . Then we have  $gf u = gt = k$ , so  $u$  is the required morphism.

- (c) If  $u: B \rightarrow A$  satisfies  $uf = 1_A$  and  $v: C \rightarrow B$  satisfies  $vg = 1_B$ , then  $uv$  is the desired retraction, as  $uv g f = u 1_B f = u f = 1_A$ .
- (d) If  $f u = f v$ , then trivially,  $gf u = gf v$ , so  $u = v$ .

(e) Consider the diagram

$$\begin{array}{ccc}
 D & \xrightarrow{h} & A \\
 \downarrow l & \nearrow t & \downarrow f \\
 E & \xrightarrow{k} & B \\
 & \searrow gk & \downarrow g \\
 & & C
 \end{array}$$

Since  $gf$  is strong monic, using the square  $(h, gf, l, gk)$  we find  $t: E \rightarrow A$  such that  $tl = h$  (and  $gft = gk$ , but that is not important). We have  $ftl = fh = kl$ , so since  $l$  is epi, we have  $ft = k$ , so  $t$  is indeed the required diagonal morphism, so  $f$  is strong monic.

- (f) If  $u: C \rightarrow A$  satisfies  $ugf = 1_A$ , then  $(ug)f = 1_A$ , so  $f$  is split monic.  
 (g) Say  $gf$  is an equalizer of  $u$  and  $v$ .

$$\begin{array}{ccccc}
 & & T & & \\
 & \swarrow \ell & \downarrow h & & \\
 A & \xrightarrow{f} & B & \xrightarrow[g]{u} & C & \xrightarrow[v]{u} & D
 \end{array}$$

If  $h: T \rightarrow B$  satisfies  $ugh = vgh$ , then since  $gf$  is an equaliser of  $u$  and  $v$ , we find a unique  $\ell: T \rightarrow A$  such that  $gfl = gh$ . Since  $g$  is monic, we have  $f\ell = h$ . The morphism  $\ell$  is the unique morphism satisfying  $f\ell = h$ , since if  $\hat{\ell}$  also satisfies  $f\hat{\ell} = h$ , then certainly  $gfl = gh$ , hence  $\ell = \hat{\ell}$ .  $\square$

EXERCISE. Let  $\mathcal{C}$  be the full subcategory of  $\mathbf{Ab}$  whose objects are groups having no elements of order 4 (though they may have elements of order 2). Then

- (i) multiplication by 2 is a regular monomorphism  $\mathbb{Z} \rightarrow \mathbb{Z}$ ,
- (ii) multiplication by 4 is not a regular monomorphism  $\mathbb{Z} \rightarrow \mathbb{Z}$ ,
- (iii) there is a pair of morphisms  $(f, g)$  such that  $gf$  is regular monix but  $f$  is not.

SOLUTION. (i) We claim that multiplication by 2 is an equalizer in  $\mathcal{C}$  of the projection  $\pi: \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$  and the zero map  $0: \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$ .

$$\begin{array}{ccc}
 & G & \\
 \swarrow \cdot 2 & \downarrow f & \\
 \mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Z} \xrightarrow[\pi]{0} \mathbb{Z}/2\mathbb{Z}
 \end{array}$$

Indeed, if  $f: G \rightarrow \mathbb{Z}$  equalizes  $\pi$  and 0, then its image is contained in  $2\mathbb{Z}$ , hence it factors uniquely through multiplication by 2 via the map  $g \mapsto f(g)/2$ .

- (ii) Assume that multiplication by 4 is an equalizer in  $\mathcal{C}$  of  $f$  and  $g$ .

$$\begin{array}{ccc}
 & \ker(f - g) & \\
 \swarrow \cdot 4 & \downarrow \iota & \\
 \mathbb{Z} & \xrightarrow{\cdot 4} & \mathbb{Z} \xrightarrow[g]{f} G
 \end{array}$$

Clearly, the kernel of  $f - g$  has no elements of order 4 and the inclusion equalizes  $f$  and  $g$ , hence it factors through multiplication by 4. Consider the element  $\alpha := f(1) - g(1) \in G$ . We know that  $\alpha + \alpha + \alpha + \alpha = f(4) - g(4) = 0$ ,

since multiplication by 4 equalises  $f$  and  $g$ . Since  $G$  is an object of  $\mathcal{C}$ , the order of  $\alpha$  is 2 or 1. In either case, we have  $2 \in \ker(f - g)$ , which is not in the image of multiplication by 4, hence  $\iota$  cannot factor through multiplication by 4, so multiplication by 4 is not an equalizer of  $f$  and  $g$ .  $\square$

**Exercise 17.**

EXERCISE. The functor  $F$  is irreducible if and only if there is an epimorphism  $\mathcal{C}(A, -) \rightarrow F$  for some object  $A$  of  $\mathcal{C}$ .

SOLUTION. If  $F$  is irreducible, then applying the irreducibility property to the epimorphism constructed in 2.12 gives the desired result.

Conversely, if  $A$  is an object of  $\mathcal{C}$  such that there is an epimorphism  $\beta: \mathcal{C}(A, -) \rightarrow F$ , then by 2.11 we get a factoring  $\gamma: \mathcal{C}(A, -) \rightarrow \coprod_{i \in I} G_i$ . Define  $x := f_A(1_A) \in G_j(A)$  for some  $j \in I$ . By Yoneda, we know that for any object  $B$  and morphism  $f: A \rightarrow B$  we have

$$\gamma_B(f) = \left( \coprod_{i \in I} G_i \right) (f)(x) = G_j(f)(x),$$

i.e., the image of  $\gamma_B$  is completely contained in  $G_j(B)$  for every  $B$ . Hence we have a commutative diagram

$$\begin{array}{ccc} G_j & \xrightarrow{\alpha|_{G_j}} & F \\ & \nwarrow \gamma & \uparrow \beta \\ & & \mathcal{C}(A, -), \end{array}$$

and by the dual of Exercise 2.16(ii), the natural transformation  $\alpha|_{G_j}$  must be an epimorphism.  $\square$

EXERCISE. A functor  $F: \mathcal{C} \rightarrow \mathbf{Set}$  is irreducible and projective if and only if there is a split epimorphism  $e: \mathcal{C}(A, -) \rightarrow F$  for some  $A$ .

SOLUTION. If  $F$  is irreducible and projective, then by part (i) we find an epimorphism  $e: \mathcal{C}(A, -) \rightarrow F$  for some  $A$ . Applying the projectivity of  $F$  to the diagram

$$\begin{array}{ccc} & & F \\ & \swarrow s & \downarrow 1_F \\ \mathcal{C}(A, -) & \xrightarrow{e} & F \end{array}$$

yields  $s: F \rightarrow \mathcal{C}(A, -)$  such that  $es = 1$ , so  $e$  is split.

Conversely, if  $e: \mathcal{C}(A, -) \rightarrow F$  admits a section  $s: F \rightarrow \mathcal{C}(A, -)$  such that  $es = 1$ , then  $F$  is irreducible by part (i). Suppose we have a morphism  $f: F \rightarrow R$  and an epimorphism  $g: Q \rightarrow R$ .

$$\begin{array}{ccc} \mathcal{C}(A, -) & \xrightleftharpoons[s]{e} & F \\ \downarrow h & & \downarrow f \\ Q & \xrightarrow{g} & R \end{array}$$

Since  $\mathcal{C}(A, -)$  is projective by 2.11, we find some  $h: \mathcal{C}(A, -) \rightarrow Q$  such that  $fe = gh$ . But then  $ghs = fes = f$ , hence  $hs: F \rightarrow Q$  solves the lifting problem, and  $F$  is projective.  $\square$

EXERCISE. If all idempotents in  $\mathcal{C}$  split, then the irreducible projectives in  $[\mathcal{C}, \mathbf{Set}]$  are exactly the representable functors.

SOLUTION. If  $F$  is representable, then we have a natural isomorphism  $\mathcal{C}(A, -) \rightarrow F$ , which in particular is a split epimorphism, hence  $F$  is irreducible and projective by part (ii).

Conversely, if  $F$  is irreducible and projective, by (ii) we find an epimorphism  $e: \mathcal{C}(A, -) \rightarrow F$  and a section  $s: F \rightarrow \mathcal{C}(A, -)$  such that  $es = 1$ .  $se$  is a natural transformation  $\mathcal{C}(A, -) \rightarrow \mathcal{C}(A, -)$ . Define  $f := (se)_A(1_A)$ . By Yoneda, for any  $u: A \rightarrow B$ , we have

$$(se)_B(u) = \mathcal{C}(A, u)(f) = uf.$$

Since  $se$  is idempotent, in particular we get

$$f = (se)_A(1_A) = (se se)_A(1_A) = (se)_A((se)_A(1_A)) = (se)_A(f) = ff,$$

so  $f$  is idempotent. By assumption,  $f$  is split, so we find some object  $B$ ,  $g: B \rightarrow A$  and  $h: A \rightarrow B$  such that  $f = gh$ ,  $hg = 1_B$ . Defining

$$\begin{aligned} x: \mathcal{C}(A, -) &\rightarrow \mathcal{C}(B, -) & y: \mathcal{C}(B, -) &\rightarrow \mathcal{C}(A, -) \\ x_C: u &\mapsto ug & y_C: u &\mapsto uh, \end{aligned}$$

(these are natural, which we can see either using Yoneda or by noticing that the naturality squares are just associativity of composition), we find that  $yx = se$  and  $xy = 1$ . But then we have  $xsey = xyxy = 1$ ,  $eyxs = eses = 1$ , so  $xs: F \rightarrow \mathcal{C}(B, -)$  and  $ey: \mathcal{C}(B, -) \rightarrow F$  are two-sided inverses of each other, hence  $F$  is representable.  $\square$

### Exercise 18.

EXERCISE. Let  $\mathcal{D}$  be the full subcategory of the category  $\mathcal{C}$  in Exercise 2.15 with objects  $A$ ,  $B$  and  $D$ , and let  $2$  be the category with objects  $0$  and  $1$  and one non-identity morphism  $0 \rightarrow 1$ . Find an example of a morphism in the functor category  $[2, \mathcal{D}]$  which is epic but not pointwise epic.

SOLUTION. Let  $G: 2 \rightarrow \mathcal{D}$  be the functor that sends the morphism  $0 \rightarrow 1$  to  $h$ . Consider any functor  $H: 2 \rightarrow \mathcal{D}$  and a natural transformation  $\eta: G \rightarrow H$ .

$$\begin{array}{ccc} B & \xrightarrow{\eta_0} & H0 \\ \downarrow h & & \downarrow H(0 \rightarrow 1) \\ D & \xrightarrow{\eta_1} & H1. \end{array}$$

Clearly,  $H_1 = D$ ,  $\eta_1 = 1_D$ .  $H_0$  is either  $B$  or  $D$ . If  $H_0 = B$ , then  $\eta_0 = 1_B$  and  $H(0 \rightarrow 1) = h$ . If  $H_0 = D$ , then  $\eta_0 = h$  and  $H(0 \rightarrow 1) = 1_D$ . In both cases, there is only one natural transformation  $G \rightarrow H$ . Hence, any natural transformation  $\alpha: F \rightarrow G$  is automatically epic. Choose  $F$  to be the functor that sends  $0 \rightarrow 1$  to  $f$  and set  $\alpha_0 := f$ ,  $\alpha_1 := h$ . Then  $\alpha$  is a natural transformation. By what we have just seen, it is epic, but  $\alpha_0 = f$  is not an epimorphism, hence  $\alpha$  is not pointwise epic.  $\square$