Algebraic Geometry

Mark Gross

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Introduction

DEFINITION 0.1. Let A be a ring. Then $\operatorname{Spec} A := \{p \subseteq A \mid p \text{ a prime ideal}\}$. For $I \subseteq A$ an ideal, define

$$V(I) := \{p \subseteq A \mid p \text{ prime}, p \supseteq I\}.$$

PROPOSITION 0.2. The sets $V(I)$ form the closed sets of a topology on $\operatorname{Spec} A$, called the Zariski topology.

PROOF. (1) $V(A) = \varnothing$
(2) $V(0) = \operatorname{Spec} A$
(3) If $\{I_i\}_{i \in J}$ is a collection of ideals, then $V(\sum_{i \in J} I_j) = \bigcap V(I_i)$.
(4) We claim: $V(I_1 \cap I_2) = V(I_1) \cup V(I_2)$.
"$\supseteq$" is obvious.
"$\subseteq$": Follows from the fact that $p \supseteq I_1 \cap I_2$ is prime, then $p \supseteq I_1$ or $p \supseteq I_2$.

□

EXAMPLE 0.3. Let $A = k[X_1, \ldots, X_n]$ with k algebraically closed. Let $I \subseteq A$ be an ideal. Then the maximal ideals m of A containing I are in one-to-one correspondence with $V(I)$ in $\mathbb{A}^n(k)$: by Nulstellensatz, every maximal ideal is of the form $(X_1 - a_1, \ldots, X_n - a_n)$, which corresponds to $(a_1, \ldots, a_n)$ in the old $V(I)$.

The new $V(I)$ now extends this notion of zero set by including other prime ideals.

EXAMPLE 0.4. If k is a field, then $\operatorname{Spec} k = \{0\}$, so the topological space cannot see the field. We fix this by also thinking about what functions are on these spaces.

CHAPTER 1

Sheaves

Remark. Fix a topological space X.

Definition 1.1. A presheaf $\mathcal{F}$ on X consists of

(1) For every open set $U \subseteq X$ an abelian group $\mathcal{F}U$,
(2) for every inclusion $V \subseteq U \subseteq X$ a restriction map $\rho_{UV}\colon \mathcal{F}U \to \mathcal{F}V$ such that $\rho_{UU} = \mathrm{id}_{\mathcal{F}U}$ and $p_{UW} = \rho_{VW} \circ \rho_{UV}$.

Remark 1.2. A presheaf is just a contravariant functor from the poset category of open sets of X to the catgegory of abelian groups.

We can generalize this to any contravariant functor $X^{\mathrm{op}} \to \mathcal{C}$ for some category $\mathcal{C}$.

Definition 1.3. A morphism of presheaves $f\colon \mathcal{F} \to \mathcal{G}$ on X is a collection of morphisms $f_U\colon \mathcal{F}U \to \mathcal{G}U$ such that for all $V \subseteq U$ the diagram

$$\begin{tikzcd} \mathcal{F}U \arrow[r, "f_U"] \arrow[d, "\rho_{UV}"] & \mathcal{G}U \arrow[d, "\rho_{UV}"] \\ \mathcal{F}V \arrow[r, "f_V"] & \mathcal{G}V \end{tikzcd}$$

commutes.

Definition 1.4. A presheaf $\mathcal{F}$ is called a sheaf if it satisfies additional axioms:

(S1) If $U \subseteq X$ is covered by an open cover $\{U_i\}$ and $s \in \mathcal{F}U$ satsfies $s|_{U_i} := \rho_{UU_i}(s) = 0$ for all i, then $s = 0$
(S2) If U, and U_i are as before, and if $s_i \in \mathcal{F}U_i$ such that for all i and j we have $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, then there is some $s \in \mathcal{F}U$ such that $s|_{U_i} = s_i$ for all i.

Remark 1.5. (1) If $\mathcal{F}$ is a sheaf, then $\varnothing \subseteq X$ is covered by the empty covering; hence $\mathcal{F}(\varnothing) = 0$.
(2) The two sheaf axioms can be described as saying that given $U, \{U_i\}$,

$$0 \longrightarrow \mathcal{F}U \xrightarrow{\ \alpha\ } \prod_i \mathcal{F}U_i \overset{\beta_1}{\underset{\beta_2}{\rightrightarrows}} \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

is exact, where $\alpha(s) = (s|_{U_i})_{i \in I}$, $\beta_1((s_i)_{i \in I}) = (s_i|_{U_i \cap U_j})$, $\beta_2((s_i)_{i \in I}) = (s_j|_{U_i \cap U_j})_{i,j}$.

Exactness means that α is injective, $\beta_1 \circ \alpha = \beta_2 \circ \alpha$, and for any $(s_i) \in \prod_{i \in I} \mathcal{F}U_i$, with $\beta_1((s_i)) = \beta_2((s_i))$, there exists $s \in \mathcal{F}U$ with $\alpha(s) = (s_i)$.

This is all subsumed by saying that α is the equalizer of β_1 and β_2.

Example. (1) Let X be any topological space, $\mathcal{F}U$ the continuous functions $U \to \mathbb{R}$.

This is a sheaf: $\rho_{UV}\colon \mathcal{F}U \to \mathcal{F}V$ is just the restriction.

The first sheaf axiom says that a contiunous function is zero if it is zero on every open set of cover.

The second sheaf axiom says that continuous functions can be glued.

(2) Let $X = \mathbb{C}$ with the Euclidean topology.

Define $\mathcal{F}U$ to be the set of bounded analytic functions $f: U \to \mathbb{C}$.

This is a presheaf, since the restriction of bounded analytic functions is bounded analytic. It also satisfies the first sheaf axiom. However, it does not satisfy the second sheaf axiom.

For example, consider the cover $\{U_i\}_{i\in\mathbb{N}}$ of $\mathbb{C}$ given by $U_i = \{z \in \mathbb{C} \mid |z| < i\}$. Define $s_i: U_i \to \mathbb{C}$ by $z \mapsto z$. Note that if $i < j$, then $U_i \cap U_j = U_i$ and $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$. However, gluing yields the identity function on $\mathbb{C}$, which is not bounded (note that complex analysis tells us that $\mathcal{F}\mathbb{C} = \mathbb{C}$.

The underlying problem is that sheafs can only track properties that can be tested locally.

(3) Let G be a group and set $\mathcal{F}U := G$ for any open set U. This is called the constant presheaf. This is in general not a sheaf (unless G is trivial).

Take U to be an disjoint union of open sets $U_1 \cup U_2$. If $\mathcal{F}U_1 = G$ and $\mathcal{F}U_2 = G$, then we need $\mathcal{F}(U_1 \cap U_2) = 0$.

If the second sheaf axiom was to be satisfied, we would want $s_1 \in \mathcal{F}U_1$ and $s_2 \in \mathcal{F}U_2$ to glue, so we should have $\mathcal{F}U = G \times G$.

Now give G the discrete topology, and define instead $\mathcal{F}U$ to be the set of continuous maps $f: U \to G$. By our choice of topology, this means that f is locally constant, i.e., for every $x \in U$ we have a neighborhood $V \subseteq U$ of x such that $f|_V$ is constant.

This is called the constant sheaf and if U is nonempty and connected then $\mathcal{F}U = G$.

(4) If X is an algebraic variety, $U \subseteq X$ a Zariski open subset, then define $\mathcal{O}_X(U)$ to be the regular functions $f: U \to k$.

Roughly, f regular means that every point of U has an open neighborhood on which f is expressed as a ratio of polynomials g/h with h nonvanishing on the neighborhood.

$\mathcal{O}_X$ is a sheaf, called the structure sheaf of X.

Definition 1.6. Let $\mathcal{F}$ be a presheaf on X and let $x \in X$. Then the stalk of $\mathcal{F}$ at x is $\mathcal{F}_x := \{(U, s) \mid U \subseteq X \text{ open neighborhood at } x, s \in \mathcal{F}U\}/\sim$, w here $(U, s) \sim (V, s')$ if there is a neighborhood $W \subseteq U \cap V$ of x such that $s|_W = s'|_W$. An equivalence class of a pair (U, s) is called a germ.

Remark. $\mathcal{F}_x$ is just the colimit of $\mathcal{F}U$ where U ranges over the open neighborhoods of x.

Note that a morphism $f: \mathcal{F} \to \mathcal{G}$ of presheaves induces a morphism $f_p: \mathcal{F}_p \to \mathcal{G}_p$ via $f_p(U, s) := (U, f_U(s))$.

Proposition 1.7. Let $f: \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then f is an isomorphism if and only if f_p is an isomorphism for every $p \in X$.

Proof. "$\Longrightarrow$" is obvious.

"$\Longleftarrow$": Assume that f_p is an isomorphism for all $p \in X$. Need to show that $f_U: \mathcal{F}U \to \mathcal{G}U$ is an isomorphism for all $U \subseteq X$, as then we can define $(f^{-1})_U = (f_U)^{-1}$. This defines a morphism of sheaves, as

$$\begin{aligned}\rho^{\mathcal{F}}_{UV} \circ f_U^{-1} &= f_V^{-1} \circ f_V \circ \rho^{\mathcal{F}}_{UV} \circ f_U^{-1} \\ &= f_V^{-1} \circ \rho^{\mathcal{G}}_{UV} \circ f_U \circ f_U^{-1} \\ &= f_V^{-1} \circ \rho^{\mathcal{G}}_{UV}.\end{aligned}$$

We will first check that f_U is injective. Suppose $s \in \mathcal{F}U$ and $f_U(s) = 0$. Then for all $p \in U$, we have $f_p(U, s) = (U, f_U(s)) = (U, 0) = 0 \in \mathcal{F}_p$. Since f_p is injective, this means that $(U,s) = 0$ in $\mathcal{F}_p$. This means that there is an open neighborhood V_p of p in U such that $s|_{V_p} = 0$. Since the sets $\{V_p\}_{p\in U}$ cover U, we see by sheaf axiom 1 that we have $s = 0$.

Next, we will show that f_U is surjective. Let $t \in \mathcal{G}U$ and write $t_p \coloneqq (U,t) \in \mathcal{G}_p$. Since f_p is surjective, we find $s_p \in \mathcal{F}_p$ with $f_p(s_p) = t_p$. This means that we find an open neighborhood $V_p \subseteq U$ of p and a germ (V_p, s_p) such that $(V_p, f_{V_p}(s_p)) \sim (U,t)$. By shrinking V_p if necessary wen can assume that $t|_{V_p} = f_{V_p}(s_p)$.

Now on $V_p \cap V_q$, $f_{V_p\cap V_q}(s_p|_{V_p\cap V_q} - s_q|_{V_p\cap V_q}) = t_{V_p\cap V_q} - t|_{V_p\cap V_q} = 0$ and hence by injectivity of $f_{V_p\cap V_q}$ already proved, we have $s_p|_{V_p\cap V_q} = s_q|_{V_p\cap V_q}$. By the second sheaf axiom, the s_p glue to give an element $s \in \mathcal{F}U$ with $s|_{V_p} = s_p$ for every $p \in U$.

Now $f_U(s)|_{V_p} = f_{V_p}(s|_{V_p}) = f_{V_p}(s_p) = t|_{V_p}$. By the first sheaf axiom applied to $f_U(s) - t$ we get $f_U(s) = t$. This shows surjectivity of f_U, completing the proof. $\square$

Theorem 1.8. Given a presheaf $\mathcal{F}$ there is a sheaf $\mathcal{F}^+$ and a morphism $\theta\colon \mathcal{F} \to \mathcal{F}^+$ satisfying the following universal propert:

For any sheaf $\mathcal{G}$ and morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ there is a unique morphism $\varphi^+\colon \mathcal{F}^+ \to \mathcal{G}$ such that $\varphi^+ \circ \theta = \varphi$.

The pair $(\mathcal{F}^+, \theta)$ is unique up to unique isomorphism and is called the sheafification of $\mathcal{F}$.

Proof. We define

$$\mathcal{F}^+U \coloneqq \{(s_p) \in \prod_{p\in U} \mathcal{F}_p \mid (\star)\},$$

where $(\star)$ means that for every $p \in U$ there is some open neighborhood $p \in V \subseteq U$ and $s_{(V)} \in \mathcal{F}V$ such that for all $q \in V$ we have $s_q = (V, s_{(V)}) \in \mathcal{F}_q$.

This is obviously a presheaf with the obvious restriction maps, modulo the fact that we have not checked that $\mathcal{F}^+U$ is an abelian group.

Let $s,t \in \mathcal{F}^+U$ such that there is an open cover $\{U_i\}$ such that $\forall i\colon s|_{U_i} = t|_{U_i}$. If $p \in U$, then $p \in U_i$ for some i. Then $s_p = (s|_{U_i})_p = (t|_{U_i})_p = t_p$, so $s = t$.

Next, let U_i be a cover, $s_i \in \mathcal{F}^+U_i$ such that $\forall i,j\colon s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$. Given $p \in U$, define $s_p \coloneqq (s_i)_p$ for $p \in U_i$. This is well-defined because of the compatibility condition. Let $p \in U$. Then $s_p = (s_i)_p$ for some i, and so we find an open neighborhood of p in $U_i \subseteq U$ satisfying the needed condition. This shows that $\mathcal{F}^+$ is a sheaf.

Now define

$$\theta_U\colon \mathcal{F}U \to \mathcal{F}^+U; \qquad s \mapsto ((U,s) \in \mathcal{F}_p)_{p\in U}.$$

We should check that θ_U is a homomorphism of groups.

If $V \subseteq U$ and $p \in V$ we have

$$(\rho_{UV}(\theta_U(s)))_p = (\theta_U(s))_p = (U,s) = (V, s|_V) = \theta_V(\rho_{UV}(s))_p,$$

so θ is a morphism. $\square$