Algebraic Topology

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CHAPTER 1

Introduction

Remark. Algebraic Topology concerns the connectivity properties of topological spaces.

Definition. A space X is path-connected if for $p, q \in X$ there is a continuous map $\gamma\colon [0,1] \to X$ such that $\gamma(0) = p$, $\gamma(1) = q$.

Example. $\mathbb{R}$ is path-connected, but $\mathbb{R} \setminus \{0\}$ is not.

Corollary (Intermediate Value Theorem). If $f\colon \mathbb{R} \to \mathbb{R}$ is continuous and $x < y$ satisfy $f(x) < 0$, $f(y) > 0$, then f takes the value 0 on $[x, y]$.

Proof. Otherwise, restricting f to $[x, y]$ we have that $f^{-1}((-\infty, 0)) \cup f^{-1}((0, \infty))$ is a decomposition of $[x, y]$ into two disjoint open sets. □

Definition. Let X, Y be topological spaces. Maps $f_0, f_1\colon Y \to X$ are homotopic if there is a continuous map $F\colon Y \times [0,1] \to X$ such that $F(\cdot, 0) = f_0$, $F(\cdot, 1) = f_1$. Write $f_0 \simeq f_1$ (or $f_0 \underset{F}{\simeq} f_1$).

Exercise. Show that $\simeq$ is an equivalence relation on the set of maps from Y to X.

Remark. X is path-connected if and only if any two maps $\{\star\} \to X$ are homotopic.

Remark. If X and Y are spaces, the set $\mathrm{Hom}(X, Y)$ of maps $Y \to X$ can be equipped with a natural topology called the compact-open topology. Under some mild conditions on X and Y, a homotopy between to maps f_0 and f_1 is the same as a path from f_0 to f_1 in $\mathrm{Hom}(X, Y)$.

Definition. We say that X is simply connected if every two maps $S^1 \to X$ are homotopic.

Example. $\mathbb{R}^2$ is simply connected; $\mathbb{R}^2 \setminus \{0\}$ is not.

From complex analysis you know that a map $\gamma\colon S^1 \to \mathbb{R}^2 \setminus \{0\}$ has a winding number or degree $\deg \gamma \in \mathbb{Z}$, for which

(i) If $\gamma_n(t) \coloneqq e^{2\pi i n t}$ then $\deg \gamma_n = n$,
(ii) If $\gamma_1 \simeq \gamma_2$ (in $\mathbb{R}^2 \setminus \{0\}$), then $\deg \gamma_1 = \deg \gamma_2$.

Complex analysis tells us that for differentiable γ we have $\deg \gamma = \int_\gamma \frac{1}{z} \mathrm{d}z$.

Corollary (Fundamental Theorem of Algebra). Every nonconstant complex polynomial has a root.

Proof. Let $f(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ be non-constant. Suppose that $\forall z \in C\colon f(z) \neq 0$. Define $\gamma_R\colon S^1 \to \mathbb{R}^2 \setminus \{0\}$ via $\gamma_R(t) \coloneqq f(Re^{2\pi i t})$.

γ_0 is a constant map, so $\deg \gamma_0 = 0$. Since all γ_R are homotopic, we have the $\forall R, \deg \gamma_R = 0$.

On the other hand, if $R \gg \sum_i |a_i|$, define

$$f_s(z) \coloneqq z^n + s(a_1 z^{n-1} + \ldots + a_n)$$

for $0 \le s \le 1$. By choice of R, we must have $\forall s \in [0,1], z \in \mathbb{C}, |z| = R \implies f_s(z) \neq 0$. If we now define $\gamma_{R,s}(t) := f_s(Re^{2\pi it})$, then $\gamma_{R,0} \simeq \gamma_{R,1}$, but $\gamma_{R,1} = \gamma_R$ has degree 0, whereas $\gamma_{R,0}$ is given by $t \mapsto R^n e^{2\pi int}$, so $\deg \gamma_{R,0} = n$, where $n \neq 0$, a contradiction. □

Definition. We say that X is k-connected if every two maps $S^i \to X$ are homotopic for all $i \le k$.

Example. $\mathbb{R}^n$ is $(n-1)$-connected, $\mathbb{R}^n \setminus \{0\}$ is not. Maps $S^{n-1} \to \mathbb{R}^n \setminus \{0\}$ have a homotopy-invariant degree, which is an integer such that the degree of the inclusion is 1 and the degree of a constant map is 0. We will define it later.

Corollary (Brouwer fixed point theorem). If $\overline{B}^n := \{x \in \mathbb{R}^n \mid \|x\| \le 1\}$ is the closed n-ball, then every map $f\colon \overline{B}^n \to \overline{B}^n$ has a fixed point.

Proof. Suppose f has no fixed point.

Given $0 \le R \le 1$, define $\gamma_R\colon S^{n-1} = \partial\overline{B}^n \to \mathbb{R}^n \setminus \{0\}$ via $v \mapsto Rv - f(Rv)$. This has the advertised codomain since f has no fixed point. γ_0 is a constant map, so $\deg \gamma_0 = 0$, but again all γ_R homotopic, so we conclude that $\deg(\gamma_1) = 0$.

On the other hand, given $0 \le S \le 1$, define $\gamma_{1,S}(v) := v - sf(v)$ for $v \in \partial\overline{B}^n$. We have $\gamma_{1,1} = \gamma_1$. Furthermore, if $s < 1$ and given $v \in \partial\overline{B}^n$, note that

$$\|v - sf(v)\| \ge |\|v\| - \|sf(v)\|| = |1 - |s|\|f(v)\|| = 1 - |s|\|f(v)\| > 1 - 1 \cdot 1 = 0,$$

so $\gamma_{1,s}$ is a map $S^{n-1} \to \mathbb{R}^n \setminus \{0\}$. We conclude that $\gamma_{1,0} \simeq \gamma_{1,1}$, which contradicts homotopy invariance of degree since $\gamma_{1,0}$ is the inclusion and thus has degree 1, whereas $\gamma_{1,1} = \gamma_1$ has degree 0 as seen earlier. □

Definition. We say that $f\colon X \to Y$ is a homotopy equivalence if there is some $g\colon Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$, $g \circ f \simeq \mathrm{id}_X$. The map g is called a homotopy inverse for f. If there is a homotopy equivalence, we say that X and Y are homotopy equivalence and write $X \simeq Y$. This is an equivalence relation.

Example. If $X \cong Y$, then $X \simeq Y$, since homotopy of maps is reflexive.

Example. Defining $f\colon \mathbb{R}^n \to S^{n-1}$ via $v \mapsto v/\|v\|$ and letting $g\colon S^{n-1} \to \mathbb{R}^n \setminus \{0\}$ be the inclusion yields a homotopy equivalence $\mathbb{R}^n \setminus \{0\} \simeq S^{n-1}$, since $f \circ g = \mathrm{id}_{S^{n-1}}$ and $g \circ f \simeq \mathrm{id}_{\mathbb{R}^n \setminus \{0\}}$ via $F(t,v) := tv + (1-t)v/\|v\|$ (linear interpolation).

Example. The inclusion $f\colon \{0\} \to \mathbb{R}^n$ is a homotopy equivalence, the homotopy inverse being the constant map $g\colon \mathbb{R}^n \to \{0\}$. Indeed, $f \circ g \simeq \mathrm{id}_{\mathbb{R}^n}$ via $F(t,v) := tv$ and $g \circ f = \mathrm{id}_{\{0\}}$.

A space that is homotopy equivalent to a point is called contractible.

Remark. Algebraic Topology concerns the connectivity properties of topological spaces.

Remark. Algebraic Topology is the study of topological spaces up to homotopy equivalence.

Idea: homeomorphism is too delicate a relation; homotopy equivalence keeps track of most "essential" topologyical information.

More precisely, we try to find functors from the category of topological spaces into the category of groups. Therefore, the algebraic invariants we study are defined for all spaces, but frequently have more structure for spaces which themselves have more structure (e.g., manifolds).

Classically, the first attempt to do this was homotopy theory. The basic observation is that given two loops at the same base point, we can concatenate them. Quotienting out loops which are related by basepoint-preserving homotopies, we obtain the fundamental group $\pi_1(X, x_0)$.

Similarly, there is a map $c\colon (S^n, p) \to (S^n, p) \vee (S^n, p)$ by collapsing the equator. Using this map, we can combine $f, g\colon (S^n, p) \to (X, x_0)$ into $f \vee g\colon (S^n, p) \to (X, x_0)$ by setting $f \vee g \coloneqq (f, g) \circ c$. Again, after quotienting out homotopic maps, this defines the operation of a group $\pi_n(X, x_0)$, the n-th homotopy group of X.

The issue with homotopy groups is that they are very hard to compute. For example, not all hoptopy groups of S^2 are known at the moment. In fact, there is no simply connected manifold of dimension greater than zero such that all homotopy groups of X are known.

Instead, we will focus on something else: (co)homology. These functors are slightly more difficult to construct, but turn out to be vastly easier to compute. Note though, computing the cohomology of complicated spaces is still very hard.

Some general remarks:

- The whole point of algebraic topology is being able to compute. Examples play a central role.
- There will be a particular focus on manifolds and smooth manifolds. There will be overlap with the Differential Geometry course.

Exercises

Example Sheet 1

Exercise 2.

EXERCISE. Compute $H^0(X;\mathbb{Z})$ for a topological space X. Give an example of a space X for which $H_0(X;\mathbb{Z})$ and $H^0(X;\mathbb{Z})$ are not isomorphic.

SOLUTION. $H^0(X;\mathbb{Z})$ is isomorphic to the kernel of the map

$$\begin{aligned}\partial^* \colon \operatorname{Hom}(C_0,\mathbb{Z}) &\to \operatorname{Hom}(C_1,\mathbb{Z})\\ f &\mapsto (\sigma \mapsto f(\partial\sigma)).\end{aligned}$$

We may think of 1-simplices as paths $\sigma\colon [0,1] \to X$, so $f \in \ker\partial^*$ is equivalent to the fact that for every $\sigma\colon [0,1]\to X$ we have

$$0 = \partial^*(f)(\sigma) = f(\partial\sigma) = f(\sigma(1)) - f(\sigma(0)).$$

Invoking the universal property of free abelian groups,

$$\operatorname{Hom}(C_0,\mathbb{Z}) \cong \operatorname{Hom}(X,\mathbb{Z}),$$

so $H^0(X;\mathbb{Z})$ is in bijection with the functions $X \to \mathbb{Z}$ which are constant on the path components of X.

Let X be a space with a countably infinite number of path components (e.g., $\mathbb{N}$ with the discrete topology) and let P denote the set of path components. Then $H_0(X;\mathbb{Z}) \cong \bigoplus_{p\in P}\mathbb{Z}$, while $H_0(X;\mathbb{Z}) \cong \prod_{p\in P}\mathbb{Z}$. The former is countable, while the latter is not, so in particular they are not isomorphic. $\square$

Exercise 3.

EXERCISE. What can you say about the group G and/or the homomorphism α in an exact sequence of the shape

(a) $0 \to \mathbb{Z}/2 \to G \to \mathbb{Z} \to 0$;
(b) $0 \to G \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z}/2 \to 0$;
(c) $0 \to \mathbb{Z}/4 \xrightarrow{\alpha} G \oplus \mathbb{Z}/2 \to \mathbb{Z}/4 \to 0$?