Category Theory

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These notes, taken by Markus Himmel, will at times differ significantly from what was lectured. In particular, all errors are almost certainly my own.

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CHAPTER 3

Adjunctions

Theorem 3.3. If $g\colon FA \to B$, then consider the square

$$\begin{tikzcd} (FA \to FA) \arrow[r] \arrow[d] & (A \to GFA) \arrow[d] \\ (FA \to B) \arrow[r] & A \to GB. \end{tikzcd}$$

Along the top right 1_{FA} is mapped to η_A and then to $Gg \circ \eta_A$. Along the bottom left, 1_{FA} is mapped to g and then to the morphism corresponding to g. Hence we have the the morphism corresponding with g is precisely $Gg \circ \eta_A$, i.e., if $f = Gg \circ \eta_A$, then f must correspond to g.

Corollary 3.4. From the initial objects we obtain the compontents:

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "\eta'_A"'] & GFA \arrow[d, shift left, "G\theta_A"] \\ & GF'A \arrow[u, shift left, "G\theta_A^{-1}"] \end{tikzcd}$$

It remains to show naturality. Let $f\colon A \to A'$ be a morphism. By initiality, there is a unique morphism $\alpha\colon FA \to FA'$ making the square

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"] & GFA \arrow[d, dotted, "G\alpha"] \\ A' \arrow[r, "\eta_{A'}"] & GFA' \end{tikzcd}$$

commute. Recall that in the proof of 3.3 we saw that the morphism corresponding to $GFf \circ \eta_A\colon A \to GFA'$ is $Ff\colon FA \to FA'$. On the other hand, consider the adjunction square

$$\begin{tikzcd} (FA' \to FA') \arrow[r] \arrow[d] & (A' \to GFA') \arrow[d] \\ (FA \to FA') \arrow[r] & (A \to GFA'). \end{tikzcd}$$

Along the top right path, $1_{FA'}$ is mapped to $\eta_{A'}$ and then to $\eta_{A'} \circ f$. Along the bottom left path $1_{FA'}$ is mapped to Ff and then to the morphism corresponding with Ff. Hence, Ff corresponds to $\eta_{A'} \circ f$. But Ff also corresponds to $GFf \circ \eta_A$, so we must have $\eta_{A'} \circ f = GFf \circ \eta_A$, which just means that η is a natural transformation, and in particular, $\alpha = Ff$.

On the other hand, we may calculate that

$$\begin{aligned} G\theta_{A'}^{-1} \circ GF'f \circ G\theta_A \circ \eta_A &= G\theta_{A'}^{-1} \circ GF'f \circ \eta'_A \\ &= G\theta_{A'}^{-1} \circ \eta'_{A'} \circ f \\ &= \eta_{A'} \circ f, \end{aligned}$$

where we use that η' is a natural transformation for the same reason as η and that the triangle at the start commutes. Therefore, we find that $\alpha = \theta_{A'}^{-1} \circ F'f \circ \theta_A$. Rearranging, this yields $\theta_{A'} \circ Ff = F'f \circ \theta_A$, so θ is natural, which is what we wanted to show.

THEOREM 3.7. Let A be an object of $\mathcal{C}$. The naturality in the first variable of an adjunction asserts that

$$\begin{tikzcd}
(GFA \to GFA) \arrow[r] \arrow[d] & (FGFA \to FA) \arrow[d] \\
(A \to GFA) \arrow[r] & (FA \to FA)
\end{tikzcd}$$

is a commutative diagram, where the horizontal arrows are the adjunction and the vertical arrows are given by precomposition with η_A resp. $F\eta_A$.

Starting with 1_{GFA}, along the top right way we map to ϵ_{FA} and then to $\epsilon_{FA} \circ F\eta_A$. Along the bottom left way we map to $\eta_A \circ 1_{GFA}\circ = \eta_A$ and then to 1_{FA}, since this is how we defined η_A. Thus $\epsilon_{FA} \circ F\eta_A = 1_{FA}$, so the first triangular identity holds.

Exercises

Chapter 1

Exercise 17.

Exercise. A morphism $e\colon A \to A$ is called idempotent if $ee = e$. An idempotent e is said to split if it can be factored as fg where gf is an identity morphism.

(i) Let $\mathcal{E}$ be a collection of idempotents in a category $\mathcal{C}$: show that there is a category $\mathcal{C}[\check{\mathcal{E}}]$ whose objects are the members of $\mathcal{E}$, whose morphisms $e \to d$ are those morphisms $f\colon \operatorname{dom} e \to \operatorname{dom} d$ in $\mathcal{C}$ for which $dfe = f$, and whose composition coincides with composition in $\mathcal{C}$. [Hint: first show that the single equation $dfe = f$ is equivalent to the two equations $df = f = fe$. Note that the identity morphism on an object e is not $1_{\operatorname{dom} e}$ in general.]

(ii) If $\mathcal{E}$ contains all identity morphisms of $\mathcal{C}$, show that there is a full and faithful functor $I\colon \mathcal{C} \to \mathcal{C}[\check{\mathcal{E}}]$, and that an arbitrary functor $T\colon \mathcal{C} \to \mathcal{D}$ can be factored as $\widehat{T}I$ for some $\widehat{T}$ iff it sends the members of $\mathcal{E}$ to split idempotents in $\mathcal{D}$.

(iii) Deduce that if all idempotents split in $\mathcal{D}$, then the functor categories $[\mathcal{C}, \mathcal{D}]$ and $[\widehat{\mathcal{C}}, \mathcal{D}]$ are equivalent, where $\widehat{\mathcal{C}} = \mathcal{C}[\check{\mathcal{E}}]$ for $\mathcal{E}$ the class of all idempotents in $\mathcal{C}$.

Solution. We will first show that if $f\colon C \to D$ is any morphism and $c : C \to C$ and $d : D \to D$ are idempotents, then $dfe = f \iff df = f = fe$.

Indeed, if $df = f = fe$, then $dfe = fe = f$. Conversely, if $dfe = f$, then $f = dfe = ddfe = df$ and $f = dfe = dfee = fe$.

To show that $\mathcal{C}[\check{\mathcal{E}}]$ is a category, we need to show that the composition of two morphisms is indeed a morphism and that there are identity morphism.

Assume that $c\colon C \to C$, $d\colon D \to D$, $e\colon E \to E$ are idempotents and that $f\colon C \to D$ and $g\colon D \to E$ satisfy $dfc = f$ and $egd = g$. We need to show that $egfc = gf$. Using the lemma, we have $egf = (eg)f = gf$ and $gfc = g(fc) = gf$, so, again by the lemma, the claim follows.

If $e\colon E \to E$ is an idempotent, define $1_e \coloneqq e \xrightarrow{e} e$. By idempotency of e, this is indeed a morphism. If $f\colon d \to e$ is a morphism, then the morphism $f1_d$ is the morphism $fd = f$ (here we use the lemma again) in $\mathcal{C}$, so $f1_d = f$ as required. Similarly, $1_e f = f$. This completes part (i).

Next, assume that $\mathcal{E}$ contains all identity morphisms of $\mathcal{C}$. Define the functor I via

$$\begin{gathered} I\colon \mathcal{C} \to \mathcal{C}[\check{\mathcal{E}}] \\ A \mapsto 1_A \\ (f\colon A \to B) \mapsto (f\colon 1_A \to 1_B) \end{gathered}$$

This is indeed a functor and since the data of a morphism $A \to B$ in $\mathcal{C}$ is precisely the same as the data of a morphism $1_A \to 1_B$ in $\mathcal{C}[\check{\mathcal{E}}]$, I is fully faithful.

Now let $T\colon \mathcal{C} \to \mathcal{D}$ be any functor.

First, assume that there is some functor $\widehat{T}\colon \mathcal{C}[\check{\mathcal{E}}] \to \mathcal{D}$ such that $T = \widehat{T}I$. Let $e : A \to A \in \mathcal{E}$ be an idempotent. Then we have

$$\begin{aligned} Te &= \widehat{T}(1_A \xrightarrow{e} 1_A) \\ &= \widehat{T}(1_A \xrightarrow{e} e \xrightarrow{e} 1_A) \\ &= \widehat{T}(e \xrightarrow{e} 1_A) \circ \widehat{T}(1_A \xrightarrow{e} e), \end{aligned}$$

and we also have

$$\begin{aligned} \widehat{T}(1_A \xrightarrow{e} e) \circ \widehat{T}(e \xrightarrow{e} 1_A) &= \widehat{T}(e \xrightarrow{e} 1_A \xrightarrow{e} e) \\ &= \widehat{T}(e \xrightarrow{ee} e) \\ &= \widehat{T}(e \xrightarrow{e} e) \\ &= \widehat{T}(1_e) \\ &= 1_{\widehat{T}e}, \end{aligned}$$

which shows that Te is split.

Next, assume that Te is split for any $e \in \mathcal{E}$. For any $e \in \mathcal{E}$, choose a splitting

$$TA \underset{f_e}{\overset{g_e}{\rightleftarrows}} B_e,$$

i.e., $f_e \circ g_e = Te$, $g_e \circ f_e = 1_{B_e}$. For identity morphisms 1_A (A an object of $\mathcal{C}$), choose the specific splitting given by $B_{1_A} \coloneqq TA$, $f_{1_A} \coloneqq 1_{TA}$, $g_{1_A} \coloneqq 1_{TA}$.

Now define the functor $\widehat{T}$ via

$$\begin{aligned} \widehat{T}\colon \mathcal{C}[\check{\mathcal{E}}] &\to \mathcal{D} \\ (e\colon A \to A) &\mapsto B_e \\ (f\colon d \to e) &\mapsto g_e \circ Tf \circ f_d. \end{aligned}$$

If $e \in \mathcal{E}$, then we have

$$\begin{aligned} \widehat{T}(1_e) &= g_e \circ Te \circ f_e \\ &= g_e \circ f_e \circ g_e \circ f_e \\ &= 1_{B_e} \circ 1_{B_e} = 1_{B_e} \end{aligned}$$

Furthermore, if $f\colon c \to d$ and $g\colon d \to e$, then we have

$$\begin{aligned} \widehat{T}(g \circ f) &= g_e \circ T(g \circ f) = f_c \\ &= g_e \circ Tg \circ Tf = f_c \\ &= g_e \circ Tg \circ T(d \circ f) \circ f_c \\ &= g_e \circ Tg \circ Td \circ Tf \circ f_c \\ &= g_e \circ Tg \circ f_d \circ g_d \circ Tf \circ f_c \\ &= \widehat{T}g \circ \widehat{T}f. \end{aligned}$$

So $\widehat{T}$ is indeed a functor. If A is an object of $\mathcal{C}$, then

$$\widehat{T}IA = \widehat{T}1_A = B_{1_A} = TA$$

and if $f\colon C \to D$ is a morphism in $\mathcal{C}$, then

$$\widehat{T}If = \widehat{T}(1_C \xrightarrow{f} 1_D) = g_{1_D} \circ Tf \circ f_{1_C} = 1_{TD} \circ Tf \circ 1_{TC} = Tf,$$

so $\widehat{T}$ is the required factorisation, completing part (ii).

Define a functor $\Phi\colon [\widehat{\mathcal{C}}, \mathcal{D}] \to [\mathcal{C}, D]$ via $F \mapsto F \circ I$, $\eta \mapsto I\eta$, where $I\eta$ is defined cia $I\eta_C \coloneqq \eta_{IC} = \eta_{1_C}$. Naturality of $I\eta$ immediately follows from naturality of η. Functoriality is also clear.

We will show that this functor is full, faithful and essentially surjective.

Indeed, let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. Then $\widehat{F}$ as defined in the previous part satisfies $\Phi\widehat{F} = F$, so Φ is essentially surjective.

Next, let $F, G\colon \widehat{\mathcal{C}} \to \mathcal{D}$ be functors and $\eta\colon F \circ I \to G \circ I$ a natural transformation. For an idempotent $e\colon A \to A$ in $\mathcal{C}$, define $\hat{\eta}_e$ to be the composite

$$Fe \xrightarrow{F(e \xrightarrow{e} 1_A)} F1_A = (F \circ I)A \xrightarrow{\quad\eta_A\quad} (G \circ I)A = G1_A \xrightarrow{G(1_A \xrightarrow{e} e)} Ge.$$

We claim that this defines a natural transformation $\hat{\eta}\colon F \to G$. Indeed, if $f\colon d \to e$ is a morphism, then

$$\begin{aligned}
\hat{\eta}_e \circ Ff &= G(1_A \xrightarrow{e} e) \circ \eta_E \circ F(e \xrightarrow{e} 1_E) \circ F(d \xrightarrow{f} e) \\
&= G(1_E \xrightarrow{e} e) \circ \eta_E \circ F(d \xrightarrow{f} e \xrightarrow{e} 1_E) \\
&= G(1_E \xrightarrow{e} e) \circ \eta_E \circ F(d \xrightarrow{d} 1_D \xrightarrow{d} d \xrightarrow{f} e \xrightarrow{e} 1_E) \\
&= G(1_E \xrightarrow{e} e) \circ \eta_E \circ F(d \xrightarrow{d} 1_D \xrightarrow{d} d \xrightarrow{f} e \xrightarrow{e} 1_E) \\
&= G(1_E \xrightarrow{e} e) \circ \eta_E \circ F(1_D \xrightarrow{efd} 1_E) \circ F(d \xrightarrow{d} 1_D) \\
&= G(1_E \xrightarrow{e} e) \circ \eta_E \circ F(efd) \circ F(d \xrightarrow{d} 1_D) \\
&= G(1_E \xrightarrow{e} e) \circ G(efd) \circ \eta_D \circ F(d \xrightarrow{d} 1_D),
\end{aligned}$$

and doing the whole thing backwards we conclude that $\hat{\eta}_e \circ Ff = Gf \circ \hat{\eta}_d$, so $\hat{\eta}$ is indeed a natural transformation.

For any $A \in \mathcal{C}$ we have

$$\begin{aligned}
(I\hat{\eta})_A = \hat{\eta}_{IA} = \hat{\eta}_{1_A} &= G(1_A \xrightarrow{1_A} 1_A) \circ \eta_A \circ F(1_A \xrightarrow{1_A} 1_A) \\
&= G(1_{1_A}) \circ \eta_A \circ F(1_{1_A}) = \eta_A,
\end{aligned}$$

which means that $\Phi(\hat{\eta}) = \eta$, so Φ is full.

Finally, let $F, G\colon \widehat{\mathcal{C}} \to \mathcal{D}$ be functors and $\eta, \eta'\colon F \to G$ be natural transformations such that $\Phi(\eta) = \Phi(\eta')$. To show that Φ is faithful, we need to prove that $\eta = \eta'$. The assumption $\Phi(\eta) = \Phi(\eta')$ means that for all $A \in \mathcal{C}$ we have $\eta_{IA} = \eta'_{IA}$, so $\eta_{1_A} = \eta'_{1_A}$.

Let $e\colon A \to A$ be any idempotent in $\mathcal{C}$. We need to show that $\eta_e = \eta'_e$. Indeed, we have

$$\begin{aligned}
\eta_e &= G(1_e) \circ \eta_e \\
&= G(e \xrightarrow{e} e) \circ \eta_e \\
&= G(e \xrightarrow{ee} e) \circ \eta_e \\
&= G(e \xrightarrow{e} 1_A \xrightarrow{e} e) \circ \eta_e \\
&= G(1_A \xrightarrow{e} e) \circ G(e \xrightarrow{e} 1_A) \circ \eta_e \\
&= G(1_A \xrightarrow{e} e) \circ \eta_{1_A} \circ F(e \xrightarrow{e} 1_A) \\
&= G(1_A \xrightarrow{e} e) \circ \eta'_{1_A} \circ F(e \xrightarrow{e} 1_A),
\end{aligned}$$

and the same argument in backwards direction shows that $\eta_e = \eta'_e$, completing the proof. □

Chapter 2

Exercise 13.

Exercise. The inner automorphisms of $\mathcal{C}$ form a normal subgroup of the group of all automorphisms of $\mathcal{C}$. [Don't worry about whether these groups are sets or proper classes!]

Solution. Let $F, G\colon \mathcal{C} \to \mathcal{C}$ be automorphisms and let $\alpha\colon F \to 1_{\mathcal{C}}$ be a natural isomorphism.

Let $A \in \mathcal{C}$. Define $\beta\colon GFG^{-1} \to 1_A$ via $\beta_A \coloneqq G(\alpha_{G^{-1}A})$ (so $\beta_A\colon GFG^{-1}A \to GG^{-1}A = A \to GG^{-1}A = 1_{\mathcal{C}}A$.

This is indeed a natural transformation: let $f\colon A \to B \in \mathcal{C}$, then we can write the naturality square in a funny way,

$$\begin{array}{ccc}
GFG^{-1}A & \xrightarrow{G(\alpha_{G^{-1}A})} & G1_CG^{-1}A \\
\Big\downarrow{\scriptstyle GFG^{-1}(f)} & & \Big\downarrow{\scriptstyle G1_CG^{-1}f} \\
GFG^{-1}B & \xrightarrow{G(\alpha_{G^{-1}B})} & G1_CG^{-1}B
\end{array}$$

and we see that it is just the functor G applied to the naturality diagram for α and the morphism $G^{-1}f$.

Therefore, β is a natural transformation, and since functors map isomorphisms to isomorphisms, it is also a natural isomorphism. So GFG^{-1} is an inner automorphism as required. $\square$

Lemma. Let $1 \in \mathcal{C}$ be a terminal object and $F\colon C \to C$ an automorphism. Then $F1$ is a terminal object.

Proof. If $A \in \mathcal{C}$, the functor F, which is fully faithful, induces a bijection between the collection of morphisms $F^{-1}A \to 1$ and the collection of morphisms $A \to F1$. Since 1 is terminal, there is exactly one morphism $A \to F1$. $\square$

Exercise. If $F\colon \mathsf{Set} \to \mathsf{Set}$ is an automorphism, then there is a unique natural isomorphism $1_{\mathcal{C}} \to F$.

Solution. Of course, the terminal object in the category of sets is just the one-element set $1 = \{\star\}$. Since $F1$ is also terminal, it is in bijection with 1. We write $F1 = \{\star_{F1}\}$.

By the Yoneda lemma, the set of natural transformations $\mathsf{Set}(1, -) \to F$ is in bijection with $F1$, so there is a unique natural transformation $\eta\colon \mathsf{Set}(1, -) \to F$. Examining the proof, we see that the components of this natural transformation are given by

$$\begin{aligned}
\eta_A\colon \mathsf{Set}(1, A) &\to FA \\
f &\mapsto Ff(\star_{F1})
\end{aligned}$$

for any object A of $\mathcal{C}$. Let A be an object of $\mathcal{C}$. We will show that η_A is an isomorphism, i.e., a bijection.

First, let $x \in FA$. Then $\eta_A(F^{-1}(\star_{F1} \mapsto x)) = x$, so η_A is surjective.

Additionally, let $f, g\colon 1 \to A$ such that $\eta_A(f) = \eta_A(g)$. Since a map $F1 \to FA$ is completely determined by its value at $\star_{F1}$, we must have $Ff = Fg$. But then $f = F^{-1}F(f) = F^{-1}F(g) = g$.

This means that η_A is an isomorphism, so η is in fact a natural isomorphism.

We define a natural transformation $\alpha\colon 1_{\mathsf{Set}} \to \mathsf{Set}(1, -)$ by setting

$$\alpha_A(a)(\star) \coloneqq a.$$

The naturality square for $f\colon A \to B$ is

$$\begin{array}{ccc} A & \xrightarrow{\alpha_A} & \mathsf{Set}(1, A) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g \mapsto f \circ g} \\ B & \xrightarrow{\alpha_B} & \mathsf{Set}(1, B) \end{array}$$

Both paths are just $a \mapsto (\star \mapsto f(a))$, so α is natural. It is also clear that α_A is bijective, so α is a natural isomorphism. In other words, $\star$ is a universal element of the identity functor.

In particular, this tells is that composition with α and its inverse exhibits a bijection between the collection of natural transformations

$$1_{\mathsf{Set}} \to F$$

and the collection of natural transformations

$$\mathsf{Set}(1, -) \to F.$$

This means that there is a unique natural transformation $1_{\mathsf{Set}} \to F$, and it is given by $\alpha \circ \eta$, and since α and η are both natural isomorphisms, so is $\alpha \circ \eta$, completing the proof. □

Exercise. The Sierpiński space S is, up to isomorphism, the unique topological space which has precisely three endomorphisms.

Solution. Let X be a topological space. Then for any $x \in X$, the constant map $c_x\colon X \to X$ sending $y \in X$ to x is continuous. Furthermore, the identity on X is continuous. This, if X is infinite, then X has infinitely many endomorphisms, and if X is finite, then X has at least $|X| + 1$ endomorphisms.

Now assume that X has precisely three endomorphisms. Then X is finite and has at most two points. Clearly, if X has zero or one point, then there is only one endomorphism. So X has two points, say $X = \{a, b\}$. There are four set-functions $\{a, b\} \to \{a, b\}$, three of which (the identity and the two constant maps) are continuous regardless of the topology. The final map interchanges a and b and is not continuous.

The empty set and all of X are open. If X had the trivial or the discrete topology, then the interchange would be continuous, a contradiction. Hence, precisely one of the sets $\{a\}$ and $\{b\}$ is open. Sending the member of that set to 1 and the other element to 0 describes a homeomorphism with S. □

Exercise. Let $\mathcal{C}$ be a full subcategory of Top containing the singleton space 1 and the Siperpiński space S and let F be an automorphism of $\mathcal{C}$. Then

(a) we have $FS \cong S$,
(b) there is a unique natural isomorphism $\alpha\colon U \to UF$, where $U\colon \mathcal{C} \to \mathsf{Set}$ is the forgetful functor,
(c) if $\mathcal{C}$ contains a space in which not every union of closed sets is closed, then α_S is continuous, and
(d) F is uniquely naturally isomorphic to the identity functor.

Solution. For (a), we just need to notice that F is fully faithful, so it induces a bijection between the sets of morphisms $S \to S$ and $FS \to FS$. Since S is determined up to isomorphism by having exactly three endomorphisms, the claim follows.

The proof of (b) is entirely analogous to the proof of 2.13(ii).

For (c), write $FS = \{\tilde{0}, \tilde{1}\}$ such that $\{\tilde{1}\}$ is open. Suppose that α_S is not continuous. Then α_S must send $0 \mapsto \tilde{1}$ and $1 \mapsto \tilde{0}$. Now let $U \subseteq X$ be an open set of some topological space in $\mathcal{C}$. Consider the map $g\colon X \to S$ which sends $x \in X$ to 1 if and only if $x \in U$. This map is continuous. Define $f := F^{-1}g$, then by naturality we have

$$(\alpha_{F^{-1}X})^{-1}(U) = \alpha^{-1}_{F^{-1}X}((UFf)^{-1}(\{\tilde{1}\})) = (Uf)^{-1}((\alpha_S)^{-1}(\tilde{1})) = (Uf)^{-1}(\{0\}).$$

Since f is continuous, the right hand side is closed. Hence the preimage under $\alpha_{F^{-1}X}$ of an open set is closed. In analogous fashion and using the fact that F^{-1} is also an automorphism (noting that α^{-1} must be the unique natural isomorphism $U \to UF^{-1}$), we find that for any space X in $\mathcal{C}$ we have

- the preimage under α_X of an open set is closed,
- the preimage under α_X of a closed set is open,
- the image under α_X of an open set is closed,
- the image under α_X of a closed set is open.

Now let X be a space in $\mathcal{C}$ and a collection U_i closed sets such that $\bigcup U_i$ is not closed. We have

$$\alpha_X^{-1}(\bigcup U_i) = \bigcup \alpha_X^{-1}(U_i),$$

where the left hand side is not open, since otherwise $\bigcup U_i$ would be closed, but the right hand side is open, since $\alpha_X^{-1}(U_i)$ is open for every i. This is a contradiction, so α_S is continuous.

For (d), we can now carry out the same calculation as above to find that α_X and α_X^{-1} are continuous for every X, so α lifts to a natural isomorphism $1_C \to F$, which must be unique since the forgetful functor $[\mathcal{C}, \mathcal{C}] \to [\mathsf{Set}, \mathsf{Set}]$ is faithful. □

Exercise 14.

Exercise. Let $e\colon A \to A$ be an idempotent. Then the following are equivalent:

(i) e is split,
(ii) the pair $(e, 1_A)$ has an equaliser,
(iii) the pair $(e, 1_A)$ has a coequaliser.

Solution. We will show that (i) is equivalent to (ii). By duality, this implies that (i) is equivalent to (iii).

Assume that there are $f\colon B \to A$ and $g\colon A \to B$ such that $fg = e$ and $gf = 1_B$. We claim that f is an equaliser of e and 1_A. We must show that any $h\colon C \to A$ satisfying $he = h$ factors uniquely through f.

$$\begin{tikzcd}
& C \arrow[dl, "h'"'] \arrow[d, "h"] & \\
B \arrow[r, shift left, "f"] & A \arrow[l, shift left, "g"] \arrow[r, shift left, "e"] \arrow[r, shift right, "1_A"'] & A
\end{tikzcd}$$

Indeed, given such h. Then $fgh = eh = h$, hence gh is one such factoring factoring. If $h'\colon C \to B$ is another factoring such that $fh' = h$, then $h' = gfh' = gh$, so the factoring is unique.

Conversely, assume that the pair $(e, 1_A)$ admits an equaliser $f\colon B \to A$. Since $ee = e = 1_A e$, e factors through f via some $g\colon A \to B$. Hence, $fg = e$. On the other hand, $fgf = ef = f$, and by a result from the lecture, f is monic, so $gf = 1_A$, so e is split. □

Exercise. A split monomorphism is regular.

Solution. If $f\colon A \to B$ is a split monomorphism, then there is some $g\colon B \to A$ such that $gf = 1_A$. Then $fgfg = f1_Ag = fg$, so fg is a split idempotent. By what we just saw, this means that f is an equaliser of $(fg, 1_A)$, hence f is a regular monomorphism. □

Exercise 15.

Exercise. Every regular monomorphism is strong.

Solution. Let f be the equaliser of u and v and take a commutative square as in the definition of strongness.

$$\begin{tikzcd} C \arrow[r, "h"] \arrow[d, two heads, "g"'] & A \arrow[d, hook, "f"] & \\ D \arrow[r, "k"] \arrow[ur, dotted, "t"] & B \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & E \end{tikzcd}$$

We have $ukg = ufh = vfh = vkg$. Since g is epi, this means that $uk = vk$, and since f is the equaliser of u and v, we find $t\colon D \to A$ such that $ft = k$. Now $ftg = kg = fh$. Since f is mono, we conclude that $tg = h$, so t has the desired properties. Hence, f is a strong monomorphism. □

Exercise. Let $\mathcal{C}$ be the finite category whose non-identity morphisms are represented by the diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "l"'] & B \arrow[d, shift right, two heads, "h"'] \arrow[d, shift left, two heads, "k"] & C \arrow[l, "g"'] \arrow[dl, "m"] \\ & D. & \end{tikzcd}$$

The morphism f is strong monic but not regular monic.

Solution. The strongness condition for f is actually vacuous: if we have a diagram

$$\begin{tikzcd} \bullet \arrow[r, "u"] \arrow[d, two heads, "v"'] & A \arrow[d, hook, "f"] \\ \bullet \arrow[r, "w"] & B, \end{tikzcd}$$

then we must have $u = 1_A$. The morphism f is not an epimorphism, as witnessed by the fact that $hf = kf$, but $h \neq k$, so we must have $v = l$. Then w is a morphism $D \to B$, but such a morphism does not exist. Hence, the square does not exist, so f is strong.

However, the only pairs of morphisms that f can be an equaliser of are $(1_B, 1_B)$, (k, k), (h, h) and (h, k). If f was the equaliser of any of these pairs, g would factor through f, but there is no morphism $C \to A$, hence that is not the case. So we conclude that f is not regular. □

Exercise 16.

Exercise. Let $f\colon A \to B$, $g\colon B \to C$ be two morphisms.

(a) If f and g are monic, then gf is monic,
(b) If f and g are strong monic, then gf is strong monic,
(c) If f and g are split monic, then gf is split monic,
(d) If gf is monic, then f is monic,
(e) If gf is strong monic, then f is strong monic,
(f) If gf is split monic, then f is split monic.
(g) If gf is regular monic and g is monic, then f is regular monic.

Solution. (a) If $gfu = gfv$, then $fu = fv$ since g is monic, and $u = v$, since f is monic.

(b) Consider the diagram

$$\begin{tikzcd}
D \arrow[r, "h"] \arrow[dd, two heads, "l"'] & A \arrow[d, hook, "f"] \\
 & B \arrow[d, hook, "g"] \\
E \arrow[r, "k"] \arrow[uur, dotted, "u"] \arrow[ur, dotted, "t"] & C.
\end{tikzcd}$$

Since g is strong monic, using the square (fh, g, l, k), we find $t\colon E \to B$ such that $gt = k$ and $tl = fh$. Since f is strong epic, using the square (h, f, l, t), we find $u\colon E \to A$ such that $fu = t$ and $ul = h$. Then we have $gfu = gt = k$, so u is the required morphism.

(c) If $u\colon B \to A$ satisfies $uf = 1_A$ and $v\colon C \to B$ satisfies $vg = 1_B$, then uv is the desired retraction, as $uvgf = u1_Bf = uf = 1_A$.

(d) If $fu = fv$, then trivially, $gfu = gfv$, so $u = v$.

(e) Consider the diagram

$$\begin{tikzcd}
D \arrow[r, "h"] \arrow[d, two heads, "l"'] & A \arrow[d, "f"] \\
E \arrow[r, "k"] \arrow[ur, dotted, "t"] \arrow[dr, "gk"'] & B \arrow[d, "g"] \\
 & C.
\end{tikzcd}$$

Since gf is strong monic, using the square (h, gf, l, gk) we find $t\colon E \to A$ such that $tl = h$ (and $gft = gk$, but that is not important). We have $ftl = fh = kl$, so since l is epi, we have $ft = k$, so t is indeed the required diagonal morphism, so f is strong monic.

(f) If $u\colon C \to A$ satisfies $ugf = 1_A$, then $(ug)f = 1_A$, so f is split monic.

(g) Say gf is an equalizer of u and v.

$$\begin{tikzcd}
 & T \arrow[d, "h"] \arrow[dl, dotted, "\ell"'] & & \\
A \arrow[r, "f"] & B \arrow[r, hook, "g"] & C \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & D
\end{tikzcd}$$

If $h\colon T \to B$ satisfies $ugh = vgh$, then since gf is an equaliser of u and v, we find a unique $\ell\colon T \to A$ such that $gf\ell = gh$. Since g is monic, we have $f\ell = h$. The morphism ℓ is the unique morphism satisfying $f\ell = h$, since if $\hat{\ell}$ also satisfies $f\hat{\ell} = h$, then certainly $gf\hat{\ell} = gh$, hence $\ell = \hat{\ell}$.

□

Exercise. Let $\mathcal{C}$ be the full subcategory of Ab whose objects are groups having no elements of order 4 (though they may have elements of order 2). Then

(i) multiplication by 2 is a regular monomorphism $\mathbb{Z} \to \mathbb{Z}$,
(ii) multiplication by 4 is not a regular monomorphism $\mathbb{Z} \to \mathbb{Z}$,
(iii) there is a pair of morphisms (f, g) such that gf is regular monix but f is not.

Solution. (i) We claim that multiplication by 2 is an equalizer in $\mathcal{C}$ of the projection $\pi\colon \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ and the zero map $0\colon \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$.

$$\begin{tikzcd} & G \arrow[d, "f"] \arrow[dl, dotted] & \\ \mathbb{Z} \arrow[r, "\cdot 2"] & \mathbb{Z} \arrow[r, shift left, "\pi"] \arrow[r, shift right, "0"'] & \mathbb{Z}/2\mathbb{Z} \end{tikzcd}$$

Indeed, if $f\colon G \to \mathbb{Z}$ equalizes π and 0, then its image is contained in $2\mathbb{Z}$, hence it factors uniquely through multiplication by 2 via the map $g \mapsto f(g)/2$.

(ii) Assume that multiplication by 4 is an equalizer in $\mathcal{C}$ of f and g.

$$\begin{tikzcd} & \ker(f-g) \arrow[d, "\iota"] \arrow[dl, dotted] & \\ \mathbb{Z} \arrow[r, "\cdot 4"] & \mathbb{Z} \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & G \end{tikzcd}$$

Clearly, the kernel of $f-g$ has no elements of order 4 and the inclusion equalizes f and g, hence it factors through multiplication by 4. Consider the element $\alpha \coloneqq f(1) - g(1) \in G$. We know that $\alpha+\alpha+\alpha+\alpha = f(4)-g(4) = 0$, since multiplication by 4 equalises f and g. Since G is an object of $\mathcal{C}$, the order of α is 2 or 1. In either case, we have $2 \in \ker(f-g)$, which is not in the image of multiplication by 4, hence ι cannot factor through multiplication by 4, so multiplication by 4 is not an equalizer of f and g. □

Exercise 17.

Exercise. The functor F is irreducible if and only if there is an epimorphism $\mathcal{C}(A,-) \to F$ for some object A of $\mathcal{C}$.

Solution. If F is irreducible, then applying the irreducibility property to the epimorphism constructed in 2.12 gives the desired result.

Conversely, if A is an object of $\mathcal{C}$ such that there is an epimorphism $\beta\colon C(A,-) \to F$, then by 2.11 we get a factoring $\gamma\colon \mathcal{C}(A,-) \to \coprod_{i\in I} G_i$. Define $x \coloneqq f_A(1_A) \in G_j(A)$ for some $j \in I$. By Yoneda, we know that for any object B and morphism $f\colon A \to B$ we have

$$\gamma_B(f) = \left(\coprod_{i\in I} G_i\right)(f)(x) = G_j(f)(x),$$

i.e., the image of γ_B is completely contained in $G_j(B)$ for every B. Hence we have a commutative diagram

$$\begin{tikzcd} G_j \arrow[r, "\alpha|_{G_j}"] & F \\ & \mathcal{C}(A,-), \arrow[ul, "\gamma"] \arrow[u, two heads, "\beta"'] \end{tikzcd}$$

and by the dual of Exercise 2.16(ii), the natural transformation $\alpha|_{G_i}$ must be an epimorphism. □

Exercise. A functor $F\colon \mathcal{C} \to \mathsf{Set}$ is irreducible and projective if and only if there is a split epimorphism $e\colon \mathcal{C}(A,-) \to F$ for some A.

Solution. If F is irreducible and projective, then by part (i) we find an epimorphism $e\colon \mathcal{C}(A,-) \to F$ for some A. Applying the projectivity of F to the diagram

$$\begin{tikzcd} & F \arrow[dl, dotted, "s"'] \arrow[d, "1_F"] \\ \mathcal{C}(A,-) \arrow[r, two heads, "e"] & F \end{tikzcd}$$

yields $s\colon F \to \mathcal{C}(A,-)$ such that $es = 1$, so e is split.

Conversely, if $e\colon \mathcal{C}(A,-) \to F$ admits a section $s\colon F \to \mathcal{C}(A,-)$ such that $es = 1$, then F is irreducible by part (i). Suppose we have a morphism $f\colon F \to R$ and an epimorphism $g\colon Q \to R$.

$$\begin{tikzcd} \mathcal{C}(A,-) \arrow[r, two heads, shift left, "e"] \arrow[d, dotted, "h"] & F \arrow[l, shift left, "s"] \arrow[d, "f"] \\ Q \arrow[r, two heads, "g"] & R \end{tikzcd}$$

Since $\mathcal{C}(A,-)$ is projective by 2.11, we find some $h\colon \mathcal{C}(A,-) \to Q$ such that $fe = gh$. But then $ghs = fes = f$, hence $hs\colon F \to Q$ solves the lifting problem, and F is projective. $\square$

Exercise. If all idempotents in $\mathcal{C}$ split, then the irreducible projectives in $[\mathcal{C}, \mathsf{Set}]$ are exactly the representable functors.

Solution. If F is representable, then we have a natural isomorphism $\mathcal{C}(A,-) \to F$, which in particular is a split epimorphism, hence F is irreducible and projective by part (ii).

Conversely, if F is irreducible and projective, by (ii) we find an epimorphism $e\colon \mathcal{C}(A,-) \to F$ and a section $s\colon F \to \mathcal{C}(A,-)$ such that $es = 1$. se is a natural transformation $\mathcal{C}(A,-) \to \mathcal{C}(A,-)$. Define $f \coloneqq (se)_A(1_A)$. By Yoneda, for any $u\colon A \to B$, we have

$$(se)_B(u) = \mathcal{C}(A,u)(f) = uf.$$

Since se is idempotent, in particular we get

$$f = (se)_A(1_A) = (sese)_A(1_A) = (se)_A((se)_A(1_A)) = (se)_A(f) = ff,$$

so f is idempotent. By assumption, f is split, so we find some object B, $g\colon B \to A$ and $h\colon A \to B$ such that $f = gh$, $hg = 1_B$. Defining

$$\begin{aligned} x\colon \mathcal{C}(A,-) &\to \mathcal{C}(B,-) & \qquad y\colon \mathcal{C}(B,-) &\to \mathcal{C}(A,-) \\ x_C\colon u &\mapsto ug & y_C\colon u &\mapsto uh, \end{aligned}$$

(these are natural, which we can see either using Yoneda or by noticing that the naturality squares are just associativity of composition), we find that $yx = se$ and $xy = 1$. But then we have $xsey = xyxy = 1$, $eyxs = eses = 1$, so $xs\colon F \to \mathcal{C}(B,-)$ and $ey\colon \mathcal{C}(B,-) \to F$ are two-sided inverses of each other, hence F is representable. $\square$

Exercise 18.

Exercise. Let $\mathcal{D}$ be the full subcategory of the category $\mathcal{C}$ in Exercise 2.15 with objects A, B and D, and let 2 be the category with objects 0 and 1 and one non-identity morphism $0 \to 1$. Find an example of a morphism in the functor category $[2, D]$ which is epic but not pointwise epic.

SOLUTION. Let $G\colon 2 \to D$ be the functor that sends the morphism $0 \to 1$ to h. Consider any functor $H\colon 2 \to D$ and a natural transformation $\eta\colon G \to H$.

$$\begin{tikzcd} B \arrow[r, "\eta_0"] \arrow[d, "h"] & H0 \arrow[d, "H(0\to 1)"] \\ D \arrow[r, "\eta_1"] & H1. \end{tikzcd}$$

Clearly, $H_1 = D$, $\eta_1 = 1_D$. H_0 is either B or D. If $H_0 = B$, then $\eta_0 = 1_B$ and $H(0 \to 1) = h$. If $H_0 = D$, then $\eta_0 = h$ and $H(0 \to 1) = 1_D$. In both cases, there is only one natural transformation $G \to H$. Hence, any natural transformation $\alpha\colon F \to G$ is automatically epic. Choose F to be the functor that sends $0 \to 1$ to f and set $\alpha_0 \coloneqq f$, $\alpha_1 \coloneqq h$. Then α is a natural transformation. By what we have just seen, it is epic, but $\alpha_0 = f$ is not an epimorphism, hence α is not pointwise epic. □

Chapter 3

Exercise 13.

EXERCISE. If $\mathcal{C}$ is a small category, then the functor category $[\mathcal{C}, \mathsf{Set}]$ is cartesian closed.

SOLUTION. Let $F, G\colon \mathcal{C} \to \mathsf{Set}$ be functors and let A be an object of $\mathcal{C}$. Define

$$F^G(A) \coloneqq \operatorname{Hom}_{[\mathcal{C},\mathsf{Set}]}(\mathcal{C}(A,-) \times G, F).$$

(TODO: Why is the thing on the right a set?)

If $f\colon A \to A'$ is a morphism in $\mathcal{C}$, $\eta\colon \mathcal{C}(A,-) \times G \to F$ a natural transformation, B an object of $\mathcal{C}$, $g\colon A' \to B$ and $x \in G(B)$, define

$$F^G(f)(\eta)_B(g, x) \coloneqq \eta_B(g \circ f, x).$$

It is immediate this this makes F^G into a functor $F^G\colon \mathcal{C} \to \mathsf{Set}$.

Furthermore, if $H\colon \mathcal{C} \to \mathsf{Set}$ is a functor and $\varphi\colon F \to H$ is natural, we declare $\varphi^G\colon F^G \to H^G$ via

$$(\varphi^G)_A\colon F^G(A) \to H^G(A), \quad \alpha \mapsto \varphi \circ \alpha.$$

This is clearly a natural transformation, and it behaves well under identities and composition, hence we have a functor

$$-^G\colon [\mathcal{C}, \mathsf{Set}] \to [\mathcal{C}, \mathsf{Set}].$$

It remains to verify that $- \times G \dashv -^G$. We apply Theorem 3.7. Let

Our first goal will be to define a natural transformation

$$\eta\colon 1_{[\mathcal{C},\mathsf{Set}]} \to (- \times G)^G.$$

Let $F\colon \mathcal{C} \to \mathsf{Set}$, A an object of $\mathcal{C}$, $x \in F(A)$, B an object of $\mathcal{C}$, $g\colon A \to B$ and $y \in G(B)$. Define

$$\eta_{F,A}(x)_B(f, y) \coloneqq (F(f)(x), y).$$

By the Yoneda lemma, this defines a natural transformation

$$\eta_{F,A}(x)\colon \mathcal{C}(A,-) \times G \to F \times G$$

and hence we have a morphism of sets

$$\eta_{F,A}\colon F(A) \to \operatorname{Hom}_{[\mathcal{C},\mathsf{Set}]}(\mathcal{C}(A,-) \times G, F \times G).$$

Let A' be an object of $\mathcal{C}$, $f\colon A \to A'$, $x \in F(A)$, B an object of $\mathcal{C}$, $g\colon A' \to B$, and $y \in G(B)$. We can calculate

$$(F \times G)^G(f)(\eta_{F,A}(x))_B(g, y) = \eta_{F,A}(x)_B(g \circ f, y)$$

$$\begin{aligned}
&= (F(g \circ f)(x), y) \\
&= (F(g)(F(f)(x)), y) \\
&= \eta_{F,A'}(F(f)(x))_B(g, y).
\end{aligned}$$

In other words,

$$\eta_F \colon F \to (F \times G)^G$$

is a natural transformation. Next, let $H \colon \mathcal{C} \to \mathsf{Set}$ be a functor and $\varphi \colon F \to H$ be a natural transformation. Also, let A an object of $\mathcal{C}$, $x \in F(A)$, B an object of $\mathcal{C}$, $f \colon A \to B$, $y \in G(B)$. We have

$$\begin{aligned}
((\varphi \times G)^G \circ \eta_F)_A(x)_B(f, y) &= (((\varphi \times G)^G)_A \times \eta_{F,A})(x)_B(f, y) \\
&= ((\varphi \times G)^G_A(\eta_{F,A}(x)))_B(f, y) \\
&= ((\varphi \times G) \circ \eta_{F,A}(x))_B(f, y) \\
&= (\varphi \times G)_B \circ \eta_{F,A}(x)_B(f, y) \\
&= (\varphi \times G)_B(F(f)(x), y) \\
&= (\varphi_B(F(f)(x)), y) \\
&= (H(f)(\varphi_A(x)), y) \\
&= \eta_{H,A}(\varphi_A(x))_B(f, y),
\end{aligned}$$

so η is indeed a natural transformation as promised.

Next, we need to define a natural transformation

$$\epsilon \colon -^G \times G \to 1_{[\mathcal{C},\mathsf{Set}]}.$$

Indeed, let $F \colon \mathcal{C} \to \mathsf{Set}$ be a functor, A an object of $\mathcal{C}$ and $\alpha \colon \mathcal{C}(A, -) \times G \to F$ be a natural transformation and $x \in G(A)$. Define

$$\epsilon_{F,A}(\alpha, x) \coloneqq \alpha_A(1_A, x).$$

Let A' be an object of $\mathcal{C}$, $f \colon A \to A'$ and $x \in G(A)$. We have

$$\begin{aligned}
\epsilon_{F,A'} \circ (F^G \times G)(f)(\alpha, x) &= \epsilon_{F,A'}(F^G(f)(a), G(f)(x)) \\
&= F^G(f)(\alpha)_{A'}(1_{A'}, G(f(x))) \\
&= \alpha_{A'}(\mathcal{C}(A, f)(1_A), G(f)(x)) \\
&= \alpha_{A'}((\mathcal{C}(A, -) \times G)(f)(1_A, x)) \\
&= F(f)(\alpha_A(1_A, x)) \\
&= F(f)(\epsilon_{F,A}(\alpha, x)),
\end{aligned}$$

so $\epsilon_F \colon F^G \times G \to F$ is a natural transformation. Next, if $H \colon \mathcal{C} \to \mathsf{Set}$ is a functor and $\varphi \colon F \to H$ is a natural transformation, A is an object of $\mathcal{C}$, $\alpha \colon \mathcal{C}(A, -) \times G \to F$ is natural and $x \in G(A)$, then we have

$$\begin{aligned}
(\epsilon_H \circ (\varphi^G \times G))_A(\alpha, x) &= \epsilon_{H,A}((\varphi^G \times G)_A(\alpha, x)) \\
&= \epsilon_{H,A}((\varphi^G)_A(\alpha), x) \\
&= \epsilon_{H,A}(\varphi \circ \alpha, x) \\
&= (\varphi \circ \alpha)_A(1_A, x) \\
&= (\varphi_A \circ \alpha_A(1_A, x)) \\
&= \varphi_A(\epsilon_{F,A}(1_A, x)) \\
&= (\varphi \circ \epsilon_F)_A(\alpha, x).
\end{aligned}$$

Hence, $\epsilon \colon -^G \times G \to 1_{[\mathcal{C},\mathsf{Set}]}$ is a natural transformation.

It remains to verify the triangle identities. For the first triangle identity, let $F\colon \mathcal{C} \to \mathsf{Set}$ be a functor, A an object of $\mathcal{C}$, $x \in F(A)$ and $y \in G(A)$. Then

$$\begin{aligned}\epsilon_{F\times G,A}((\eta_F \times G)_A(x,y)) &= \epsilon_{F\times G,A}(\eta_{F,A}(x),y) = \eta_{F,A}(x)_A(1_A,y)\\ &= (F(1_A)(x),y) = (x,y),\end{aligned}$$

so the first triangle identity holds.

Finally, let $F\colon \mathcal{C} \to \mathsf{Set}$ be a functor, $\alpha\colon \mathcal{C}(A,-)\times G \to F$ a natural transformation, B an object of $\mathcal{C}$, $f\colon A \to B$ and $x \in G(B)$. Then

$$\begin{aligned}((\epsilon_F)^G \circ \eta_{F^G})_A(\alpha)_B(f,x) &= (\epsilon_F \circ \eta_{F^G,A}(\alpha))_B(f,x)\\ &= \epsilon_{F,B}(\eta_{F^G,A}(\alpha)_B(f,x))\\ &= \epsilon_{F,B}(F^G(f)(\alpha),x)\\ &= F^G(f)(\alpha)_B(1_B,x)\\ &= \alpha_B(1_B \circ f,x)\\ &= \alpha_B(f,x).\end{aligned}$$

This completes the proof of second triangle identity, and we are done. □

Chapter 5

Exercise 15.

Exercise. If $\mathcal{C}$ is a category, then $\mathrm{End}_{[\mathcal{C},\mathcal{C}]}(1_\mathcal{C})$ is a commutative monoid.

Solution. If $\alpha,\beta\colon 1_\mathcal{C} \to 1_\mathcal{C}$ are natural transformations, then by naturality of α the square

$$\begin{tikzcd} A \arrow[r, "\alpha_A"] \arrow[d, "\beta_A"] & A \arrow[d, "\beta_A"] \\ A \arrow[r, "\alpha_A"] & B \end{tikzcd}$$

is commutative for all objects A of $\mathcal{C}$. Hence, $\alpha \circ \beta = \beta \circ \alpha$. □

Exercise. If $(1_\mathcal{C},\eta,\mu)$ is a monad, then η is an isomorphism.

Solution. Indeed, the first monad law gives $\mu\eta = 1_{1_\mathcal{C}}$. By (i) this implies $\eta\mu = 1_{1_\mathcal{C}}$, so η has the two-sided inverse μ. □

Exercise. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor having a right adjoint G such that there is a natural isomorphism $\alpha\colon 1_\mathcal{C} \to GF$. Then the unit is also an isomorphism. In particular, F is full and faithful.

Solution. We will show that $(1_\mathcal{C},\alpha^{-1}\eta,\mu)$ is a monad, where μ is the composite

$$1_\mathcal{C} \xrightarrow{\alpha} GF \xrightarrow{\alpha_{GF}} GFGF \xrightarrow{G\epsilon_F} GF \xrightarrow{\alpha^{-1}} 1_\mathcal{C}.$$

Indeed, observe that for $1_\mathcal{C}$ the first and second monad laws are identical and the associativity law is vacuous. Hence, it suffices to check the first monad law. Let A be an object of $\mathcal{C}$. Then we need to show that

$$\alpha_A^{-1} \circ G\epsilon_{FA} \circ \alpha_{GFA} \circ \alpha_A \circ \alpha_A^{-1} \circ \eta_A = \alpha_A^{-1} \circ G\epsilon_{FA} \circ \alpha_{GFA} \circ \eta_A$$

is the identity on A. Indeed, this follows from the commutative diagram

$$\begin{tikzcd} A \arrow[r, "\alpha_A"] \arrow[d, "\eta_A"'] & GFA \arrow[d, "GF\eta_A"'] \arrow[dr, "1_{GFA}"] & & \\ GFA \arrow[r, "\alpha_{GFA}"] & GFGFA \arrow[r, "G\epsilon_{FA}"] & GFA \arrow[r, "\alpha_A^{-1}"] & A, \end{tikzcd}$$

where the square commutes by naturality of α and the triangle is just G applied to the first triangle identity.

By (ii), $\alpha^{-1}\eta$ is an isomorphism, so $\eta = \alpha\alpha^{-1}\eta$ is an isomorphism. The fact that F is fully faithful follows from the dual of Lemma 3.9. $\square$

Exercise 16.

Exercise. If $\mathbb{T} = (T, \eta, \mu)$ is idempotent and (A, α) is a $\mathbb{T}$-algebra, then α is a two-sided inverse for η_A.

Solution. The identity $\alpha\eta_A = 1_A$ is part of the definition of an algebra. For the other direction, consider the diagram

$$\begin{tikzcd} TA \arrow[r, shift left, "T\eta_A"] \arrow[r, shift right, "\eta_{TA}"'] \arrow[d, "\alpha"] & TTA \arrow[r, "\mu_A"] \arrow[d, "T\alpha"] & TA \\ A \arrow[r, "\eta_A"] & TA. & \end{tikzcd}$$

The square commutes by naturality of η, and the top composites are both the identity by definition of a monad. But μ_A is an isomorphism, so $\eta_{TA} = T\eta_A$. Hence, $\eta_A\alpha = T(\alpha\eta_A) = 1_{TA}$, since (A, α) is an algebra. $\square$

Exercise. If $\mathbb{T} = (T, \eta, \mu)$ is idempotent, then the categories $\mathcal{C}^{\mathbb{T}}$ and $\mathrm{Fix}(T)$ are isomorphic.

Furthermore, the categories $\mathcal{C}_{\mathbb{T}}$ and $\mathrm{Fix}(T)$ are equivalent.

Solution. Indeed, we have a functor $\mathcal{C}^{\mathbb{T}} \to \mathrm{Fix}(T)$ that sends (A, α) to A (this is valid by the previous part) and f to f. Conversely, we have a functor $\mathrm{Fix}(T) \to \mathcal{C}^{\mathbb{T}}$ that sends $A \mapsto (A, \eta_A^{-1})$ and $f \mapsto f$. Indeed, (A, η_A^{-1}) is an algebra: the first axiom is trivially true, and the second is equivalent to $T\eta_A^{-1} = \mu_A$, which follows from the first monad law. Furthermore, f is a morphism of algebras by naturality of η. It is clear that these functors are two-sided inverses of each other.

For the second claim, note that by the second monad law and the fact that μ_A is an isomorphism for every A we must have $\eta_{TA} = \mu_A^{-1}$, so η_{TA} is an isomorphism for every A. Thus, the adjunction from Proposition 5.6 is restricts to an adjunction $F_{\mathbb{T}} \colon \mathrm{Fix}(T) \to \mathcal{C}_{\mathbb{T}} \dashv G_{\mathbb{T}} \colon \mathcal{C}_{\mathbb{T}} \to \mathrm{Fix}(T)$. But the unit of this adjunction is just η restricted to $\mathrm{Fix}(T)$, which is a natural isomorphism. If A is an object of $\mathcal{C}$, then ϵ_A is the morphism $TA \rightsquigarrow A$ represented by the identity 1_{TA} in $\mathcal{C}$. But it is readily checked (using the monad laws and the fact that $T\eta_A = \eta_{TA}$) that the morphism $A \rightsquigarrow TA$ given by the composite $\eta_{TA}\eta_A \colon A \to TA \to TTA$ is a two-sided inverse of this morphism. Hence, the counit is also a natural isomorphism, completing the proof that $\mathrm{Fix}(T)$ and $\mathcal{C}_{\mathbb{T}}$ are equivalent. $\square$