Finite Dimensional Lie and Associative Algebras

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CHAPTER 1

Introduction

Definition 1.1. Let k be a field. A Lie algebra $\mathfrak{L}$ over k is a k-vector space with a bilinear map $[\cdot,\cdot]\colon \mathfrak{L}\times\mathfrak{L}\to\mathfrak{L}$ satisfying

(1) $\forall x\in\mathfrak{L}\colon [x,x]=0$, and
(2) $\forall x,y,z\in\mathfrak{L}\colon [x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$. This is the Jacobi identity.

If $\operatorname{char} k\neq 2$, then (1) is equivalent to

(1') $\forall x,y\in\mathfrak{L}\colon [x,y]=-[y,x]$.

Remark. Groups describe symmetries. Lie algebras describe infinitesimal symmetries.

For example, let $G=\mathrm{GL}_n(\mathbb{R})$. This is an example of a Lie group, i.e., an analytic manifold with continuous group operations. The associated Lie algebra is the tangent space T_1G at the identity.

The matrix exponential diffeomorphically (with inverse log) takes a neighborhood of 0, which is the same as T_1G, to a neighborhood of 1.

$\exp A\exp B=\exp(\mu(A,B))$ for sufficiently small A and B.

The Taylor series for μ is

$$\mu(A,B)=A+B+\frac{1}{2}[A,B]+\text{higher degree terms},$$

where $[A,B]=AB-BA$ (matrix multiplication).

This is an example of a Lie bracket. Note that $T_1G\times T_1G\to T_1G$, $(A,B)\mapsto[A,B]$ is bilinar, skew-symmetric.

The Lie algebra corresponding to G is often called $\mathfrak{g}$.

Note that

(1) The first approximation to the group product is addition in the Lie algebra T_1G.
(2) If $[g_1,g_2]=g_1g_2g_1^{-1}g_2^{-1}$ is the group commutator, then the Lie bracket is the first approximation of the commutator $[\exp A,\exp B]$ in G.
(3) The Jacobi identity arises from the associativity in G. Note that Lie algebras in general are non-associative.

As a further example, let $G=\mathrm{GL}_n(\mathbb{C})$. This is an example of an algebraic group, i.e., a complex algebraic variety with continuous group operations. We have $T_1G\cong M_n(\mathbb{C})$ as the tangent space at the identity. Similarly to before, we define a Lie bracket and end up with a complex Lie algebra.

Definition 1.2. (a) A Lie subalgebra $\mathfrak{J}$ of $\mathfrak{L}$ is a k-subspace such that $[x,y]\in\mathfrak{J}$ for $x,y\in\mathfrak{J}$.
(b) An ideal $\mathfrak{J}$ of $\mathfrak{L}$ is a k-subalgebra such that $[x,y]\in\mathfrak{J}$ for $x\in\mathfrak{J}$ and $y\in\mathfrak{L}$.

In a couple of lectures we will define a canonical ideal $R(\mathfrak{L})$.

Definition 1.3. (a) $\mathfrak{L}$ is semisimple if $R(\mathfrak{L})=0$. In general $\mathfrak{L}/R(\mathfrak{L})$ is semisimple.
(b) $\mathfrak{L}$ is simple if the only ideals are 0 and $\mathfrak{L}$.

We will see that semisimple Lie algebras are direct products of finitely many simple ones. In this course we will concentrate on the simple complex Lie algebras.

We will find that classifying these boils down to classifying finite root systems, which are collections of combinatorial data. Root systems have a symmetry group called the Weyl group and are labelled by Dynkin diagrams.

Root systems also appear in the representations of quivers (i.e., directed graphs) arising in algebraic geometry.

Definition 1.4. An associative ring R with unity is a k-algebra if there is a ring homomorphism $\phi\colon k \to R$ such that $\phi(k) \leq Z(R)$, where $Z = \{r \in R \mid \forall s \in R\colon rs = sr\}$ is the centre of R.

We can regard k as a subalgebra of R and R is a k-vector space.

Remark. If R is a k-algebra, we can define a Lie bracket $[r, s] = rs - sr$, where we use the associative product, so R is a Lie algebra.

Definition 1.5. (a) A k-subspace J of R is a left ideal if $\forall r \in R, s \in J\colon rs \in J$. Right ideals are define analogously. A (2-sided) ideal is both a left and a right ideal.

We'll see that in finite-dimensional k-algebras there is a canonical ideal, the Jacobson radical $J(R)$.

Definition 1.6. (a) R is semisimple if $J(R) = 0$, and in general $R/J(R)$ is semisimple.

(b) R is simple if the only ideals are 0 and R.

Exercise: $M_n(k)$ is a simple algebra (work out the left and the right ideals).

We will prove the Artin-Wedderburn theorem which says the finite-dimensional semisimple algebras are direct products of simple ones, where simple algebras are isomorphic to $M_n(D)$, where D is a division algebra, where $\dim_k D < \infty$.

An example of a skew field are the quaternions $\mathbb{H}$. They are an $\mathbb{R}$-algebra with a basis $1, i, j, k$ such that $ij = k, ji = -k$. The quaternions are not a $\mathbb{C}$-algebra.

Artin-Wedderburn applies in

(a) representation theory of finite groups,
(b) path algebras R of quivers, where R-modules correspond to representations of quivers.

Definition 1.7. An R-module M is indecomposable if one cannot express it as $M = M_1 \oplus M_2$ with $M_1, M_2 \neq 0$.

We will consider quivers where the path algebras only have finitely many isomorphism classes of indecomposable modules. These quivers are called quivers of finite representation type.

The classification due to Gabriel again involves root systems labelled by Dynkin diagrams.

CHAPTER 2

Elementary properties of Lie algebras

Remark. Assume that $\operatorname{char} k = 0$.

Example. $\mathfrak{gl}_n$ has Lie subalgebras:

(1) $\mathfrak{sl}_n$ is the subalgebra of trace zero matrices. It is associated with SL_n.

Example: $\mathfrak{sl}_2$ is a 3-dimensional k-vector space. It has a standard basis given by

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We notice that $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$.

(2) $\mathfrak{so}_n$ is the subalgebra of skew-symmetric ($A + A^T = 0$) matrices. It is associated with SO_n, the special orthogonal group (endomorphisms preserving an inner product).

Example: $\mathfrak{so}_3$ is a 3-dimensional k-vector space. It has a basis given by

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We have $[A_1, A_2] = A_3$, $[A_2, A_3] = A_1$, $[A_3, A_1] = A_2$.

(3) $\mathfrak{sp}_{2n}$ is the subalgebra of matrices A, such that $JA^TJ^{-1} + A = 0$ where J has -1s on the lower-left half of the antidiagonal and 1s on the upper-right half of the antidiagonal. It is associated with the group SP_{2n} preserving a non-degenerate skew-symmetric bilinar form (also known as a symplectic form).

(4) $\mathfrak{b}_n$ is the subalgebra of upper triangular matrices, also called the Borel subalgebra and is associated with the inverted upper triangular matrices.

(5) $\mathfrak{n}_n$ is the subalgebra of strictly upper triangular matrices with zeros on the leading diagonal. It is associated with the upper triangular matrices with ones on the leading diagonal.

We can also consider $\mathrm{End}_k(R)$, which are the k-linear maps $R \to R$, where R is an associative algebra. If $\dim R = n$, then $\mathrm{End}(R) = M_n(k)$. $\mathrm{End}(R)$ has a Lie subalgebra called $\mathrm{Der}(R)$ consisting of derivations.

Definition 2.1. A k-linear map $D \colon R \to R$ is called a derivation if it satisfies the Leibnitz rule:

$$D(rs) = D(r)s + rD(s),$$

where we are taking products in R.

Example. We have $\mathrm{Der}(k[X]) = \{fD \mid f \in k[X]\}$, where $D \colon k[X] \to k[X]$ is the differential (straightforward proof by induction).

$\mathrm{Der}(k[X, X^{-1}])$ is called the Witt Lie algebra, which is closely related to the Virasoro algebra (appears in geometry and physics). It is infinite-dimensional.

Geometrically, when R is a coordinate ring, then $\mathrm{Der}(R)$ corresponds to vector fields. However, R need not be commutative in the general case.

DEFINITION 2.2. An inner derivation is a k-linear map $R \to R$ of the form $s \mapsto [r, s]$ for some $r \in R$.

The inner derivations form a Lie subalgebra of $\operatorname{Der}(R)$ and in fact form a (Lie) ideal.

REMARK. (1) If R is commutative, then $\operatorname{Innder}(R) = 0$.
(2) At the end of the commutative algebra course you may meet Hochschild cohomology (a cohomology theory for associative algebras). The first Hochschild cohomology group $HH^1(R, R)$ is the quotient $\operatorname{Der}(R)/\operatorname{Innder}(R)$, which is a Lie algebra.
(3) Lie algebras appear as derivations of other algebraic structures. For example for the octonians one gets the Lie algebra G_2.

1. Representations

DEFINITION 2.3. (a) A morphism of Lie algebras $\rho\colon \mathfrak{L}_1 \to \mathfrak{L}_2$ is a k-linear map such that $\rho([x, y]) = [\rho(x), \rho(y)]$.
(b) A representation of $\mathfrak{L}$ is a morphism of Lie algebras $\rho_V : \mathfrak{L} \to \operatorname{End} V$, where V is a vector space. If $\dim V < \infty$, we call ρ_V a linear representation.

If $U \leq V$ and $\rho_V(\mathfrak{L})(U) \subseteq U$, then there is a subrepresentation $\rho_U : \mathfrak{L} \to \operatorname{End} U$ where $\rho_U(x)(u) \coloneqq \rho_V(x)(u)$ for $x \in \mathfrak{L}, u \in U$.

An irreducible representation is one that does not admit any proper subrepresentations.

EXAMPLE. (1) The adjoint representation $\operatorname{ad}_{\mathfrak{L}}\colon \mathfrak{L} \to \operatorname{End}\mathfrak{L}$ is given by $x \mapsto (y \mapsto [x, y])$.

It is indeed a homomorphism: if $x, y, z \in \mathfrak{L}$, then we may calculate

$$\begin{aligned}
\operatorname{ad}_{\mathfrak{L}}([x, y])(z) &= [[x, y], z] \\
&= -[z, [x, y]] \\
&= [x, [y, z]] + [y, [z, x]] \\
&= \operatorname{ad}_{\mathfrak{L}}(x)(\operatorname{ad}_{\mathfrak{L}}(y)(z)) - \operatorname{ad}_{\mathfrak{L}}(y)(\operatorname{ad}_{\mathfrak{L}}(x)(z)) \\
&= (\operatorname{ad}_{\mathfrak{L}}(x) \circ \operatorname{ad}_{\mathfrak{L}}(y) - \operatorname{ad}_{\mathfrak{L}}(y) \circ \operatorname{ad}_{\mathfrak{L}}(x))(z) \\
&= [\operatorname{ad}_{\mathfrak{L}}(x), \operatorname{ad}_{\mathfrak{L}}(y)](z),
\end{aligned}$$

where we have used the Jacobi identity.

DEFINITION 2.4. The centre of $\mathfrak{L}$ is defined to be

$$\ker \operatorname{ad}_{\mathfrak{L}} = \{x \in \mathfrak{L} \mid \forall y \in \mathfrak{L}\colon [x, y] = 0\}.$$

Note that if the centre is 0 then the adjoint representation is injective and we can regard $\mathcal{L}$ as a subalgebra of $\operatorname{End}\mathfrak{L}$. If $\mathfrak{L}$ is finite-dimensional, then $\mathfrak{L}$ is a subalgebra of $\mathfrak{gl}_n \cong \operatorname{End}\mathfrak{L}$, where $n = \dim \mathfrak{L}$.

REMARK. There is a difficult result called Ado's theorem which states that if $\operatorname{char} k = 0$ and $\mathfrak{L}$ is finite-dimensional then there is an injective morphism of Lie algebras $\mathfrak{L} \to \mathfrak{gl}_n$ for some n.

Iwasawa then extended this to characteristic $p > 0$ (quite hard).

EXAMPLE. Let $k = \mathbb{R}$. $\mathbb{R}^3$ is a Lie algebra under the cross product (have to check the Jacobi identity). If e_1, e_2, e_3 form the standard basis, then we find that

$$e_1 \times e_2 = e_3, \qquad e_2 \times e_3 = e_1, \qquad e_3 \times e_1 = e_2.$$

We have (TODO: think about this more)

$$\begin{aligned}
\operatorname{ad}_{\mathbb{R}^3} : \mathbb{R}^3 &\to \operatorname{End}\mathfrak{L} \cong M_3(\mathbb{R}) \\
e_i &\mapsto A_i \in \mathfrak{so}_3(\mathbb{R}) \subseteq \mathfrak{gl}_3
\end{aligned}$$

Hence $\ker \operatorname{ad}_{\mathfrak{L}} = 0$, $\operatorname{im} \operatorname{ad}_{\mathfrak{L}} = \mathfrak{so}_3$. Thus $\mathbb{R}^3$ with the vector product is isomorphic to $\mathfrak{so}_3$ as a Lie algebra.

Example. We define a morphism

$$\begin{aligned} \rho\colon \mathfrak{sl}_2 &\to \operatorname{Der}(k[X,Y]) \subseteq \operatorname{End}(k[X,Y]) \\ e &\mapsto X\frac{\partial}{\partial Y} \\ f &\mapsto Y\frac{\partial}{\partial X} \\ h &\mapsto X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y} \end{aligned}$$

An easy but somewhat lengthy calculation shows that this is a morhpism (notably, we use the symmetry of second partial derivatives). Note that the images of e, f, h map V_n, the span of the monomials of total degree n ($\dim V_n = n+1$; for example, V_1 has basis elements X, Y, while V_2 has basis elements X^2, XY, Y^2) to itself. So we have subrepresentations $\mathfrak{sl}_2 \to \operatorname{End} V_n$. Exercise: think about the cases $n = 1$ and $n = 2$ and show that they are irreducible.

Lemma 2.5. The subrepresentations $\rho_n\colon \mathfrak{sl}_2 \to \operatorname{End}(V_n)$ are irreducible.

Proof. Suppose $\rho_n(\mathfrak{sl}_2)(U) \subseteq U$ for a subspace U. Then if $U \neq 0$ there exists $f \in U$, where $\sum_{i+j=n} \lambda_{ij} X^i Y^j$ where not all λ_{ij} are zero. Then

$$\rho_n(e)(f) = XD_Y(f) = \sum j\lambda_{ij} X^{i+1} Y^{j-1} \in U.$$

Repeatedly applying $\rho_n(e)$ yields a nonzero scalar multiple of X^n, so $X^n \in U$. Now apply $\rho_n(f)$ repeatedly to get nonzero scalar multiples of all monomials in V_n. So if U is nonzero, then $U = V_n$ as required. $\square$

Remark. Note that $\bigoplus V_n = k[X,Y]$.

A note about terminology: Strictly speaking, the representation is the map $\mathfrak{L} \to \operatorname{End}(V)$. Often, V is also called the representation. This is an abuse of notation. In this course, we will use the term "module" for V, for example "V is a module for $\mathfrak{sl}_2$" or "V is a $\mathfrak{sl}_2$-module." Similarly, we'll sometimes use the term "simple module" to refer to irreducible representations.

We'll see later that the V_n are precisely the simple finite-dimensional $\mathfrak{sl}_2$-modules up to isomorphism.

Also any finite-dimensional $\mathfrak{sl}_2$-module is a direct sum of copies of the V_n.

However, there are infinite-dimensional $\mathfrak{sl}_2$-modules that aren't such direct sums. There will be an example on the example sheet.

Definition 2.6. A Lie algebra is called abelian if $\forall x, y \in \mathfrak{L}, [x,y] = 0$.

For example, all 1-dimensional Lie algebras are abelian.

Definition 2.7. The derived series of $\mathfrak{L}$ is defined inductively: $\mathfrak{L}^{(0)} := \mathfrak{L}$, $\mathfrak{L}^{(n+1)} := [\mathfrak{L}^{(n)}, \mathfrak{L}^{(n)}]$, where $[\mathfrak{L}, \mathfrak{L}]$ is the span (!) of the elements of the form $[x,y]$, $x, y \in \mathfrak{L}$.

We call $\mathfrak{L}^{(1)}$ the derived subalgebra of $\mathfrak{L}$.

Note that $\mathfrak{L}^{(i)}$ is a Lie ideal of $\mathfrak{L}$: this follows from induction and the Jacobi identity.

Definition 2.8. The Lie algebra $\mathfrak{L}$ is called soluble if $\mathfrak{L}^{(r)} = 0$ for some r. The derived length of $\mathfrak{L}$ is the least such r.

For example, being a non-zero abelian Lie algebra is equivalent to the derived length being 1.

Remark. If J is an ideal of $\mathfrak{L}$, then $\mathfrak{L}/J$ is a lie algebra via $[x+J, y+J] \coloneqq [x,y]+J$.

Lemma 2.9. (1) Subalgebras and quotients of soluble Lie algebras are soluble.
(2) If J is an ideal such that J and $\mathfrak{L}/J$ are soluble, then $\mathfrak{L}$ is soluble.

Example. Let $\mathfrak{L}$ be a 2-dimensional Lie algebra. Either $\mathfrak{L}$ is abelian or there are x, y such that $[x,y] \neq 0$, so $\mathfrak{L}^{(1)} \neq 0$.

However, x and y form a basis of $\mathfrak{L}$, $\mathfrak{L}^{(1)}$ is equal to the span of $[x,y]$. Therefore, the derived series of $\mathfrak{L}$ looks like

$$\mathfrak{L} \supsetneq \mathfrak{L}^{(1)} \supsetneq 0.$$

So in the first case, where $\mathfrak{L}$ is abelian, the derived length is 1, and otherwise the derived length is 2.

Annoying exercise: classify three-dimensional Lie algebras. It is done in Jacobson's book.

Definition 2.10. The lower central series is defined inductively: $\mathfrak{L}_{(1)} \coloneqq \mathfrak{L}$, $\mathfrak{L}_{(n+1)} \coloneqq [\mathfrak{L}_{(n)}, \mathfrak{L}]$.

Note $\mathfrak{L}_{(i)}$ are ideals of $\mathfrak{L}$.

We say that $\mathfrak{L}$ is nilpotent if $\mathfrak{L}_{(c+1)} = 0$ for some c. The nilpotency class of $\mathfrak{L}$ is the smallest such c.

Note that if $\mathfrak{L}$ is nilpotent, then $\mathfrak{L}$ is soluble.

Example. Recall that $\mathfrak{n}_n$ is the Lie algebra of strictly upper triangular matrices. Exercise: this is nilpotent for every n.

For example, $\mathfrak{n}_3$ is called the Heisenberg Lie algebra. It has dimension 3. There is an obvious basis

$$x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We can calculate that $[x,y] = z$, $[x,z] = 0$, $[y,z] = 0$, so $\mathfrak{n}_3$ is nonabelian and of nilpotency class 2. In general, we can show that $\mathfrak{n}_n$ is of nilpotency class $n-1$.

Example. Recall $\mathfrak{b}_n$ consists of the upper triangular matrices. We have $\mathfrak{b}_n^{(1)} = \mathfrak{n}_n$. $\mathfrak{b}_n$ is soluble but not nilpotent for $n \geq 2$.

Lemma 2.11. If $\mathfrak{L}$ is a Lie algebra and $n \in \mathbb{N}$, then $\mathfrak{L}^{(n)} \subseteq \mathfrak{L}_{(2^n)}$.

Proof. We will first show that for natural numbers i and j we have $[\mathfrak{L}_{(i)}, \mathfrak{L}_{(j)}] \subseteq \mathfrak{L}_{(i+j)}$.

We do induction on j. The case $j = 1$ is true by definition.

Now assume that for some $j \in \mathbb{N}$ and all $i \in \mathbb{N}$ we have $[\mathfrak{L}_{(i)} + \mathfrak{L}_{(j)}] \subseteq \mathfrak{L}_{(i+j)}$. Let $i \in \mathbb{N}$. We need to show that $[\mathfrak{L}_{(i)}, \mathfrak{L}_{(j+1)}] \subseteq \mathfrak{L}_{(i+j+1)}$. We will check this on generators, so let $x \in \mathfrak{L}_{(i)}$, $y \in \mathfrak{L}_{(j)}$ and $z \in \mathfrak{L}$. We need to show that $[x,[y,z]] \in \mathfrak{L}_{(i+j+1)}$.

Indeed, $[x,y] \in \mathfrak{L}_{(i+j)}$ by our inductive hypothesis, so $\alpha \coloneqq [z,[x,y]] \in \mathfrak{L}_{(i+j+1)}$ by definition. Furthermore, $[z,x]$ in $\mathfrak{L}_{(i+1)}$ by definition, so $\beta \coloneqq [y,[z,x]] \in \mathfrak{L}_{(i+j+1)}$ by inductive hypothesis. Therefore $[x,[y,z]] = -\alpha - \beta \in \mathfrak{L}_{(i+j+1)}$ as required, completing the proof of the lemma.

Now we will proceed to prove the claim, again by induction. The base case is again trivial, and for $n \in \mathbb{N}$ we have

$$\mathfrak{L}^{(n+1)} = [\mathfrak{L}^{(n)}, \mathfrak{L}^{(n)}] \subseteq [\mathfrak{L}_{(2^n)}, \mathfrak{L}_{(2^n)}] \subseteq \mathfrak{L}_{(2^{n+1})},$$

using the inductive hypothesis and our lemma. This completes the proof. □

Remark. Out next aim is to prove some theorems.

Theorem 2.12 (Engel). Suppose $\mathfrak{L} \subseteq \operatorname{End} V$ is a subalgebra with $\dim V < \infty$ and every $x \in L$ is a nilpotent endomorphism.

Then there is some $v \in V$ such that $v \neq 0$, but $\forall x \in L \colon x(v) = 0$.

Proof. We proceed by induction on $\dim \mathfrak{L}$.

Assume first that $\dim \mathfrak{L} = 1$, i.e., $\mathfrak{L} = \langle x \rangle$. Since x is nilpotent, then x has eigenvalue 0, so there is $v \neq 0$ such that $x(v) = 0$. Since x spans $\mathfrak{L}$, we have $\mathfrak{L}(v) = 0$.

Next, assume that $\dim \mathfrak{L} > 1$. We will first show that $\mathfrak{L}$ satisfies the idealiser condition. Let $A \subsetneq \mathfrak{L}$ be a proper Lie subalgebra. Consider $\rho \colon A \to \operatorname{End} \mathfrak{L}$ given by $a \mapsto \operatorname{ad}(a) = (x \mapsto [a, x])$, the restriction of the adjoint representation of $\mathfrak{L}$ to A. Since A is a subalgebra, there is a representation $\overline{\rho} \colon A \to \operatorname{End}(L/A)$ given by $a \mapsto \overline{\operatorname{ad}(a)} = (x + A \mapsto [a, x] + A)$. This is indeed a representation, because A is a subalgebra.

By (2.17) we know that if a is nilpotent, then so is $\operatorname{ad}(a)$, which implies that $\overline{\operatorname{ad}(a)}$ is also nilpotent. Note that $\dim \overline{\rho}(A) \leq \dim A < \dim \mathfrak{L}$.

By the inductive hypothesis, we find $0 \neq x' \in L/A$ such that $\forall f \in \overline{\rho}(A) \colon f(x') = 0$. In other words, we find $x \in L \setminus A$ such that for all $a \in A$ we have

$$\overline{\rho}(a)(x + A) = A.$$

By definition of $\overline{\rho}$, this just means that $[a, x] \in A$ for all $a \in A$, which implies that $[x, a] \in A$ for $a \in A$. Therefore, $x \in \operatorname{Id}_L(A) \setminus A$ and the idealiser condition is indeed satisfied.

Now, if M is a maximal proper subablgra of $\mathfrak{L}$, then $\operatorname{Id}_{\mathfrak{L}}(M) = \mathfrak{L}$ by maximality of M. This just means that M is an ideal of $\mathfrak{L}$. This means that $\mathfrak{L}/M$ is a Lie algebra and the maximality of M forces $\dim(\mathfrak{L}/M) = 1$, because every Lie algebra has subalgebras of dimension 1 (indeed, the span of any nonzero element is one) and these can be pulled back to Lie subalgebras in between M and $\mathfrak{L}$.

This means that $\mathfrak{L} = \langle M, x \rangle$ for some $x \in \mathfrak{L}$.

Consider $U := \{u \in V \mid M(u) = 0\}$. By the inductive hypothesis, since $\dim M < \dim \mathfrak{L}$, we know that $U \neq 0$.

Let $u \in U$ and $m \in M$. Then $m(x(u)) = ([m, x] + x \circ m)(u) = 0$, since $m \in M$ and $[m, x] \in M$ as M is an ideal. So $x(u) \in U$ for all $u \in U$. This means that x restricts to a nilpotent endomorphism of U and so has an eigenvector $0 \neq v \in U$ with $x(v) = 0$ (every eigenvector of a nilpotent endomorphism must be zero). But $v \in U$ and so $M(v) = 0$. As $\mathfrak{L}$ is the span of M and x, it follows that $\mathfrak{L}(v) = 0$ as required. $\square$

Theorem 2.13 (Lie). Assume that k is algebraically closed of characteristic 0. Again, let $\mathfrak{L} \subseteq \operatorname{End} V$ be a subalgebra with $\dim V < \infty$. Suppose that $\mathfrak{L}$ is soluble. Then there is some $v \in V$ such that $v \neq 0$ and for all $x \in L$ there is $\lambda_x \in k$ such that $x(v) = \lambda_x v$.

In words: all x have a common eigenvector.

Proof. Again, we use induction on $\dim \mathfrak{L}$.

If $\dim \mathfrak{L} = 1$, then we can use the fact that k is algebraically closed to find an eigenvector of x such that $\mathfrak{L} = \langle x \rangle$, and we are done.

Next, assume that $\dim \mathfrak{L} > 1$ and suppose the the theorem is true for all soluble Lie subalgebras of $\operatorname{End} W$ of smaller dimension.

Since $\mathfrak{L} \neq 0$ and $\mathfrak{L}$ is soluble, we have $\mathfrak{L}^{(1)} \subsetneq \mathfrak{L}$. Let M be a maximal Lie subalgebra containing $\mathfrak{L}^{(1)}$. Then M is an ideal of $\mathfrak{L}$ (since $[x, y] \subseteq [\mathfrak{L}, \mathfrak{L}] \subseteq M$) and $\dim L/M = 1$ (as seen in the proof of Engel's theorem). Again, pick $x \in \mathfrak{L}$

such that $\mathfrak{L}$ is the span of M and x. By induction, we find $0 \neq u \in V$ such that $\forall m \in M\colon m(u) = \lambda_m u$. Notice that the map $\lambda\colon M \to k$ given by $m \mapsto \lambda_m$ is linear.

Let $u_0 \coloneqq u$ and inductively set $u_{i+1} \coloneqq x(u_i)$. Define $U_i \coloneqq \langle u_0, \dots, u_i \rangle$. Let n be the smallest natural number such that $u_0, \dots, u_n$ are linearly dependent.

We will now prove that if $m \in M$ and $i < n$, then $m(u_i) \equiv \lambda_m u_i \pmod{U_{i-1}}$. Note that this implies $M(U_i) \subseteq U_i$.

We prove this by induction on i. It is true for $i = 0$ by definition.

Next, assume it is true for $i > 0$ and $M(U_i) \subseteq U_i$. If $m(u_i) \equiv \lambda_m u_i \pmod{U_{i-1}}$, then $x(m(u_i)) \equiv \lambda_m x(u_i) = \lambda_m u_{i+1} \pmod{U_i}$ (just write out the previous relation and apply x to both sides).

Therefore,

$$m(u_{i+1}) = m(x(u_i)) = ([m, x] + x \circ m)(u_i) \equiv \lambda_m u_{i+1} \pmod{U_1},$$

using the previous calculation and the fact that $[m, x] \in M$ (since M is an ideal) and $M(U_i) \subseteq U_i$. This completes the proof of the claim.

Using the claim, we see that $M(U_{n-1}) \subseteq U_{n-1}$. On the other hand, $x(U_{n-1}) \subseteq U_{n-1}$. This means that $\mathfrak{L}(U_{n-1} \subseteq U_{n-1}$, but we halso have $x(U_{n-1} \subseteq U_{n-1}$ (by linear dependence of $u_0, \dots, u_n$). Moreover, with respect to the basis $u_0, \dots, u_{n-1}$, the action of M is represented by upper triangular matrices (since $M(U_i) \subseteq U_i$ with diagonal entries λ_m (by the formula modulo U_{i-1}. In particular, this is true for $m \in \mathfrak{L}^{(1)} \subseteq M$.

But matrices representing elements of $\mathfrak{L}^{(1)}$ must have trace 0 (since $\operatorname{tr} XY = \operatorname{tr} YX$). So $n\lambda_m = 0$ for $m \in \mathfrak{L}^{(1)}$. Since $\operatorname{char} k = 0$, we conclude that $\lambda_m = 0$ for $m \in \mathfrak{L}^{(1)}$.

We now claim that for $i < n$ and $m \in M$ we actually have $m(u_i) = \lambda_m u_i$ (compare this to the previous claim).

We will prove this again by induction (again the base case is trivial). For the inductive step, assume that $m(u_i) = \lambda_m u_i$ for all $m \in M$.

Then

$$m(u_{i+1}) = m(x(u_i)) = ([m, x] + x \circ m)(u_i) = x(m(u_i)) = \lambda_m u_{i+1}$$

because λ is linear and $\lambda_{[m,x]} = 0$, finishing the proof of the claim.

So now we know that $m(w) = \lambda_m w$ for all $m \in M$ and $w \in U_{n-1}$. On the other hand, $x(U_{n-1}) \subseteq U_{n-1}$ (by linear dependence). Choose an eigenvector $0 \neq v \in U_{n-1}$ of the restriction of x to U_{n-1}, say $x(v) = \lambda_x v$. Thus v is a common eigenvector for M (see beginning of this paragraph) and x, and therefore for all of $\mathfrak{L}$, since $\mathfrak{L}$ is spanned by M and x. This completes the proof. □

Corollary 2.14 (Corollary of Engel and Lie).

(a) If $\mathfrak{L}$ satisfies the condition of Engel, then we can pick a basis that defines an isomorphism $\operatorname{End} V \to M_n(k)$ such that $\mathfrak{L}$ maps to a Lie subalgebra of $\mathfrak{n}_n$.

(b) If $\mathfrak{L}$ satisfies the condition of Lie, then we can pick a basis that defines an isomorphism $\operatorname{End} V \to M_n(k)$ such that $\mathfrak{L}$ maps to a Lie subalgebra of $\mathfrak{b}_n$.

Proof. Follows from the fact that $\mathfrak{n}_n$ is nilpotent. □

Corollary 2.15. If $\mathfrak{L}$ satisfies the condition of Engel, then $\mathfrak{L}$ is nilpotent as a Lie algebra.

Definition 2.16.

(a) The idealiser of a subset S of $\mathfrak{L}$ is

$$\operatorname{Id}_{\mathfrak{L}}(S) = \{y \in \mathfrak{L} \mid [y, S] \subseteq S.\}$$

If S is a Lie subalgebra of $\mathfrak{L}$, then $\operatorname{Id}_L(S)$ is also a Lie subalgebra. Furthermore, we have $S \subseteq \operatorname{Id}_L(S)$.

(b) We say that $\mathfrak{L}$ satisfies the idealiser condition if every proper Lie subalgebra of $\mathfrak{L}$ is properly contained in its idealiser.

REMARK. A note on terminology: some people, for example Serre, use the term normaliser instead of idealiser.

LEMMA 2.17. If $x \in \mathfrak{L} \subseteq \operatorname{End} V$ and $x^m = 0$, then $(\operatorname{ad}(x))^{2m} = 0$ in $\operatorname{End} \mathfrak{L}$.

PROOF. We may assume that $\mathfrak{L} = \operatorname{End} V$. Let $\theta\colon \operatorname{End} V \to \operatorname{End} V$ denote premultiplication my x, i.e., $y \mapsto x \circ y$. Similarly, let ϕ denote postmultiplication, i.e., $y \mapsto y \circ x$. Notice that $\operatorname{ad}(x) = \theta - \phi$. The maps θ and φ commute, and $\theta^m = 0 = \phi^m$. Therefore,

$$(\operatorname{ad}(x))^2 m = (\theta - \varphi)^{2m} = 0$$

by the binomial theorem. $\square$