Category Theory

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These notes, taken by Markus Himmel, will at times differ significantly from what was lectured. In particular, all errors are almost certainly my own.

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Exercises

Chapter 1

Exercise 17.

Exercise. A morphism $e\colon A \to A$ is called idempotent if $ee = e$. An idempotent e is said to split if it can be factored as fg where gf is an identity morphism.

(i) Let $\mathcal{E}$ be a collection of idempotents in a category $\mathcal{C}$: show that there is a category $\mathcal{C}[\check{\mathcal{E}}]$ whose objects are the members of $\mathcal{E}$, whose morphisms $e \to d$ are those morphisms $f\colon \operatorname{dom} e \to \operatorname{dom} d$ in $\mathcal{C}$ for which $dfe = f$, and whose composition coincides with composition in $\mathcal{C}$. [Hint: first show that the single equation $dfe = f$ is equivalent to the two equations $df = f = fe$. Note that the identity morphism on an object e is not $1_{\operatorname{dom} e}$ in general.]

(ii) If $\mathcal{E}$ contains all identity morphisms of $\mathcal{C}$, show that there is a full and faithful functor $I\colon \mathcal{C} \to \mathcal{C}[\check{\mathcal{E}}]$, and that an arbitrary functor $T\colon \mathcal{C} \to \mathcal{D}$ can be factored as $\widehat{T}I$ for some $\widehat{T}$ iff it sends the members of $\mathcal{E}$ to split idempotents in $\mathcal{D}$.

(iii) Deduce that if all idempotents split in $\mathcal{D}$, then the functor categories $[\mathcal{C}, \mathcal{D}]$ and $[\widehat{\mathcal{C}}, \mathcal{D}]$ are equivalent, where $\widehat{\mathcal{C}} = \mathcal{C}[\check{\mathcal{E}}]$ for $\mathcal{E}$ the class of all idempotents in $\mathcal{C}$.

Solution. We will first show that if $f : C \to D$ is any morphism and $c : C \to C$ and $d : D \to D$ are idempotents, then $dfe = f \iff df = f = fe$.

Indeed, if $df = f = fe$, then $dfe = fe = f$. Conversely, if $dfe = f$, then $f = dfe = ddfe = df$ and $f = dfe = dfee = fe$.

To show that $\mathcal{C}[\check{\mathcal{E}}]$ is a category, we need to show that the composition of two morphisms is indeed a morphism and that there are identity morphism.

Assume that $c\colon C \to C$, $d\colon D \to D$, $e\colon E \to E$ are idempotents and that $f\colon C \to D$ and $g\colon D \to E$ satisfy $dfc = f$ and $egd = g$. We need to show that $egfc = gf$. Using the lemma, we have $egf = (eg)f = gf$ and $gfc = g(fc) = gf$, so, again by the lemma, the claim follows.

If $e\colon E \to E$ is an idempotent, define $1_e \coloneqq e \xrightarrow{e} e$. By idempotency of e, this is indeed a morphism. If $f\colon d \to e$ is a morphism, then the morphism $f1_d$ is the morphism $fd = f$ (here we use the lemma again) in $\mathcal{C}$, so $f1_d = f$ as required. Similarly, $1_e f = f$. This completes part (i).

Next, assume that $\mathcal{E}$ contains all identity morphisms of $\mathcal{C}$. Define the functor I via

$$\begin{aligned} I\colon \mathcal{C} &\to \mathcal{C}[\check{\mathcal{E}}] \\ A &\mapsto 1_A \\ (f\colon A \to B) &\mapsto (f\colon 1_A \to 1_B) \end{aligned}$$

This is indeed a functor and since the data of a morphism $A \to B$ in $\mathcal{C}$ is precisely the same as the data of a morphism $1_A \to 1_B$ in $\mathcal{C}[\check{\mathcal{E}}]$, I is fully faithful.

Now let $T\colon \mathcal{C} \to \mathcal{D}$ be any functor.

First, assume that there is some functor $\widehat{T}\colon \mathcal{C}[\check{\mathcal{E}}] \to \mathcal{D}$ such that $T = \widehat{T}I$. Let $e : A \to A \in \mathcal{E}$ be an idempotent. Then we have

$$\begin{aligned} Te &= \widehat{T}(1_A \xrightarrow{e} 1_A) \\ &= \widehat{T}(1_A \xrightarrow{e} e \xrightarrow{e} 1_A) \\ &= \widehat{T}(e \xrightarrow{e} 1_A) \circ \widehat{T}(1_A \xrightarrow{e} e), \end{aligned}$$

and we also have

$$\begin{aligned} \widehat{T}(1_A \xrightarrow{e} e) \circ \widehat{T}(e \xrightarrow{e} 1_A) &= \widehat{T}(e \xrightarrow{e} 1_A \xrightarrow{e} e) \\ &= \widehat{T}(e \xrightarrow{ee} e) \\ &= \widehat{T}(e \xrightarrow{e} e) \\ &= \widehat{T}(1_e) \\ &= 1_{\widehat{T}e}, \end{aligned}$$

which shows that Te is split.

Next, assume that Te is split for any $e \in \mathcal{E}$. For any $e \in \mathcal{E}$, choose a splitting

$$TA \underset{f_e}{\overset{g_e}{\rightleftarrows}} B_e,$$

i.e., $f_e \circ g_e = Te$, $g_e \circ f_e = 1_{B_e}$. For identity morphisms 1_A (A an object of $\mathcal{C}$), choose the specific splitting given by $B_{1_A} \coloneqq TA$, $f_{1_A} \coloneqq 1_{TA}$, $g_{1_A} \coloneqq 1_{TA}$.

Now define the functor $\widehat{T}$ via

$$\begin{aligned} \widehat{T}\colon \mathcal{C}[\check{\mathcal{E}}] &\to \mathcal{D} \\ (e\colon A \to A) &\mapsto B_e \\ (f\colon d \to e) &\mapsto g_e \circ Tf \circ f_d. \end{aligned}$$

If $e \in \mathcal{E}$, then we have

$$\begin{aligned} \widehat{T}(1_e) &= g_e \circ Te \circ f_e \\ &= g_e \circ f_e \circ g_e \circ f_e \\ &= 1_{B_e} \circ 1_{B_e} = 1_{B_e} \end{aligned}$$

Furthermore, if $f\colon c \to d$ and $g\colon d \to e$, then we have

$$\begin{aligned} \widehat{T}(g \circ f) &= g_e \circ T(g \circ f) = f_c \\ &= g_e \circ Tg \circ Tf = f_c \\ &= g_e \circ Tg \circ T(d \circ f) \circ f_c \\ &= g_e \circ Tg \circ Td \circ Tf \circ f_c \\ &= g_e \circ Tg \circ f_d \circ g_d \circ Tf \circ f_c \\ &= \widehat{T}g \circ \widehat{T}f. \end{aligned}$$

So $\widehat{T}$ is indeed a functor. If A is an object of $\mathcal{C}$, then

$$\widehat{T}IA = \widehat{T}1_A = B_{1_A} = TA$$

and if $f\colon C \to D$ is a morphism in $\mathcal{C}$, then

$$\widehat{T}If = \widehat{T}(1_C \xrightarrow{f} 1_D) = g_{1_D} \circ Tf \circ f_{1_C} = 1_{TD} \circ Tf \circ 1_{TC} = Tf,$$

so $\widehat{T}$ is the required factorisation, completing part (ii).

Define a functor $\Phi\colon [\widehat{\mathcal{C}}, \mathcal{D}] \to [\mathcal{C}, D]$ via $F \mapsto F \circ I$, $\eta \mapsto I\eta$, where $I\eta$ is defined cia $I\eta_C \coloneqq \eta_{IC} = \eta_{1_C}$. Naturality of $I\eta$ immediately follows from naturality of η. Functoriality is also clear.

We will show that this functor is full, faithful and essentially surjective.

Indeed, let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. Then $\widehat{F}$ as defined in the previous part satisfies $\Phi\widehat{F} = F$, so Φ is essentially surjective.

Next, let $F, G\colon \widehat{\mathcal{C}} \to \mathcal{D}$ be functors and $\eta\colon F \circ I \to G \circ I$ a natural transformation. For an idempotent $e\colon A \to A$ in $\mathcal{C}$, define $\hat{\eta}_e$ to be the composite

$$Fe \xrightarrow{F(e \xrightarrow{e} 1_A)} F1_A = (F \circ I)A \xrightarrow{\quad \eta_A \quad} (G \circ I)A = G1_A \xrightarrow{G(1_A \xrightarrow{e} e)} Ge.$$

We claim that this defines a natural transformation $\hat{\eta}\colon F \to G$. Indeed, if $f\colon d \to e$ is a morphism, then

$$\begin{aligned}
\hat{\eta}_e \circ Ff &= G(1_A \xrightarrow{e} e) \circ \eta_E \circ F(e \xrightarrow{e} 1_E) \circ F(d \xrightarrow{f} e) \\
&= G(1_E \xrightarrow{e} e) \circ \eta_E \circ F(d \xrightarrow{f} e \xrightarrow{e} 1_E) \\
&= G(1_E \xrightarrow{e} e) \circ \eta_E \circ F(d \xrightarrow{d} 1_D \xrightarrow{d} d \xrightarrow{f} e \xrightarrow{e} 1_E) \\
&= G(1_E \xrightarrow{e} e) \circ \eta_E \circ F(d \xrightarrow{d} 1_D \xrightarrow{d} d \xrightarrow{f} e \xrightarrow{e} 1_E) \\
&= G(1_E \xrightarrow{e} e) \circ \eta_E \circ F(1_D \xrightarrow{efd} 1_E) \circ F(d \xrightarrow{d} 1_D) \\
&= G(1_E \xrightarrow{e} e) \circ \eta_E \circ F(efd) \circ F(d \xrightarrow{d} 1_D) \\
&= G(1_E \xrightarrow{e} e) \circ G(efd) \circ \eta_D \circ F(d \xrightarrow{d} 1_D),
\end{aligned}$$

and doing the whole thing backwards we conclude that $\hat{\eta}_e \circ Ff = Gf \circ \hat{\eta}_d$, so $\hat{\eta}$ is indeed a natural transformation.

For any $A \in \mathcal{C}$ we have

$$\begin{aligned}
(I\hat{\eta})_A = \hat{\eta}_{IA} = \hat{\eta}_{1_A} &= G(1_A \xrightarrow{1_A} 1_A) \circ \eta_A \circ F(1_A \xrightarrow{1_A} 1_A) \\
&= G(1_{1_A}) \circ \eta_A \circ F(1_{1_A}) = \eta_A,
\end{aligned}$$

which means that $\Phi(\hat{\eta}) = \eta$, so Φ is full.

Finally, let $F, G\colon \widehat{\mathcal{C}} \to \mathcal{D}$ be functors and $\eta, \eta'\colon F \to G$ be natural transformations such that $\Phi(\eta) = \Phi(\eta')$. To show that Φ is faithful, we need to prove that $\eta = \eta'$. The assumption $\Phi(\eta) = \Phi(\eta')$ means that for all $A \in \mathcal{C}$ we have $\eta_{IA} = \eta'_{IA}$, so $\eta_{1_A} = \eta'_{1_A}$.

Let $e\colon A \to A$ be any idempotent in $\mathcal{C}$. We need to show that $\eta_e = \eta'_e$. Indeed, we have

$$\begin{aligned}
\eta_e &= G(1_e) \circ \eta_e \\
&= G(e \xrightarrow{e} e) \circ \eta_e \\
&= G(e \xrightarrow{ee} e) \circ \eta_e \\
&= G(e \xrightarrow{e} 1_A \xrightarrow{e} e) \circ \eta_e \\
&= G(1_A \xrightarrow{e} e) \circ G(e \xrightarrow{e} 1_A) \circ \eta_e \\
&= G(1_A \xrightarrow{e} e) \circ \eta_{1_A} \circ F(e \xrightarrow{e} 1_A) \\
&= G(1_A \xrightarrow{e} e) \circ \eta'_{1_A} \circ F(e \xrightarrow{e} 1_A),
\end{aligned}$$

and the same argument in backwards direction shows that $\eta_e = \eta'_e$, completing the proof. □

Chapter 2

Exercise 13.

Exercise. The inner automorphisms of $\mathcal{C}$ form a normal subgroup of the group of all automorphisms of $\mathcal{C}$. [Don't worry about whether these groups are sets or proper classes!]

Solution. Let $F, G\colon \mathcal{C} \to \mathcal{C}$ be automorphisms and let $\alpha\colon F \to 1_{\mathcal{C}}$ be a natural isomorphism.

Let $A \in \mathcal{C}$. Define $\beta\colon GFG^{-1} \to 1_A$ via $\beta_A := G(\alpha_{G^{-1}A})$ (so $\beta_A\colon GFG^{-1}A \to GG^{-1}A = A \to GG^{-1}A = 1_{\mathcal{C}}A$.

This is indeed a natural transformation: let $f\colon A \to B \in \mathcal{C}$, then we can write the naturality square in a funny way,

$$\begin{tikzcd} GFG^{-1}A \arrow[r, "G(\alpha_{G^{-1}A})"] \arrow[d, "GFG^{-1}(f)"] & G1_C G^{-1}A \arrow[d, "G1_C G^{-1}f"] \\ GFG^{-1}B \arrow[r, "G(\alpha_{G^{-1}B})"] & G1_C G^{-1}B \end{tikzcd}$$

and we see that it is just the functor G applied to the naturality diagram for α and the morphism $G^{-1}f$.

Therefore, β is a natural transformation, and since functors map isomorphisms to isomorphisms, it is also a natural isomorphism. So GFG^{-1} is an inner automorphism as required. □

Lemma. Let $1 \in \mathcal{C}$ be a terminal object and $F\colon C \to C$ an automorphism. Then $F1$ is a terminal object.

Proof. If $A \in \mathcal{C}$, the functor F, which is fully faithful, induces a bijection between the collection of morphisms $F^{-1}A \to 1$ and the collection of morphisms $A \to F1$. Since 1 is terminal, there is exactly one morphism $A \to F1$. □

Exercise. If $F\colon \mathsf{Set} \to \mathsf{Set}$ is an automorphism, then there is a unique natural isomorphism $1_{\mathcal{C}} \to F$.

Solution. Of course, the terminal object in the category of sets is just the one-element set $1 = \{\star\}$. Since $F1$ is also terminal, it is in bijection with 1. We write $F1 = \{\star_{F1}\}$.

By the Yoneda lemma, the set of natural transformations $\mathsf{Set}(1, -) \to F$ is in bijection with $F1$, so there is a unique natural transformation $\eta\colon \mathsf{Set}(1, -) \to F$. Examining the proof, we see that the components of this natural transformation are given by

$$\eta_A\colon \mathsf{Set}(1, A) \to FA$$
$$f \mapsto Ff(\star_{F1})$$

for any object A of $\mathcal{C}$. Let A be an object of $\mathcal{C}$. We will show that η_A is an isomorphism, i.e., a bijection.

First, let $x \in FA$. Then $\eta_A(F^{-1}(\star_{F1} \mapsto x)) = x$, so η_A is surjective.

Additionally, let $f, g\colon 1 \to A$ such that $\eta_A(f) = \eta_A(g)$. Since a map $F1 \to FA$ is completely determined by its value at $\star_{F1}$, we must have $Ff = Fg$. But then $f = F^{-1}F(f) = F^{-1}F(g) = g$.

This means that η_A is an isomorphism, so η is in fact a natural isomorphism.

We define a natural transformation $\alpha\colon 1_{\mathsf{Set}} \to \mathsf{Set}(1, -)$ by setting

$$\alpha_A(a)(\star) := a.$$

The naturality square for $f\colon A \to B$ is

$$\begin{tikzcd} A \arrow[r, "\alpha_A"] \arrow[d, "f"] & \mathsf{Set}(1, A) \arrow[d, "g \mapsto f \circ g"] \\ B \arrow[r, "\alpha_B"] & \mathsf{Set}(1, B) \end{tikzcd}$$

Both paths are just $a \mapsto (\star \mapsto f(a))$, so α is natural. It is also clear that α_A is bijective, so α is a natural isomorphism. In other words, $\star$ is a universal element of the identity functor.

In particular, this tells is that composition with α and its inverse exhibits a bijection between the collection of natural transformations

$$1_{\mathsf{Set}} \to F$$

and the collection of natural transformations

$$\mathsf{Set}(1, -) \to F.$$

This means that there is a unique natural transformation $1_{\mathsf{Set}} \to F$, and it is given by $\alpha \circ \eta$, and since α and η are both natural isomorphisms, so is $\alpha \circ \eta$, completing the proof. □

Exercise. The Sierpiński space S is, up to isomorphism, the unique topological space which has precisely three endomorphisms.

Solution. Let X be a topological space. Then for any $x \in X$, the constant map $c_x\colon X \to X$ sending $y \in X$ to x is continuous. Furthermore, the identity on X is continuous. This, if X is infinite, then X has infinitely many endomorphisms, and if X is finite, then X has at least $|X| + 1$ endomorphisms.

Now assume that X has precisely three endomorphisms. Then X is finite and has at most two points. Clearly, if X has zero or one point, then there is only one endomorphism. So X has two points, say $X = \{a, b\}$. There are four set-functions $\{a, b\} \to \{a, b\}$, three of which (the identity and the two constant maps) are continuous regardless of the topology. The final map interchanges a and b and is not continuous.

The empty set and all of X are open. If X had the trivial or the discrete topology, then the interchange would be continuous, a contradiction. Hence, precisely one of the sets $\{a\}$ and $\{b\}$ is open. Sending the member of that set to 1 and the other element to 0 describes a homeomorphism with S. □

Exercise. Let $\mathcal{C}$ be a full subcategory of Top containing the singleton space 1 and the Siperpiński space S and let F be an automorphism of $\mathcal{C}$. Then

(i) we have $FS \cong S$,
(ii) there is a unique natural isomorphism $\alpha\colon U \to UF$, where $U\colon \mathcal{C} \to \mathsf{Set}$ is the forgetful functor,
(iii) if $\mathcal{C}$ contains a space in which not every union of closed sets is closed, then α_S is continuous, and
(iv) F is uniquely naturally isomorphic to the identity functor.

Solution. For (i), we just need to notice that F is fully faithful, so it induces a bijection between the sets of morphisms $S \to S$ and $FS \to FS$. Since S is determined up to isomorphism by having exactly, three endomorphisms, the claim follows.

□