Algebraic Geometry

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These notes, taken by Markus Himmel, will at times differ significantly from what was lectured. In particular, all errors are almost certainly my own.

Contents

Introduction

DEFINITION 0.1. Let A be a ring. Then $\operatorname{Spec} A := \{p \subseteq A \mid p \text{ a prime ideal}\}$. For $I \subseteq A$ an ideal, define

$$V(I) := \{p \subseteq A \mid p \text{ prime}, p \supseteq I\}.$$

PROPOSITION 0.2. The sets $V(I)$ form the closed sets of a topology on $\operatorname{Spec} A$, called the Zariski topology.

PROOF.

(1) $V(A) = \varnothing$
(2) $V(0) = \operatorname{Spec} A$
(3) If $\{I_i\}_{i \in J}$ is a collection of ideals, then $V(\sum_{i \in J} I_j) = \bigcap V(I_i)$.
(4) We claim: $V(I_1 \cap I_2) = V(I_1) \cup V(I_2)$.
"$\supseteq$" is obvious.
"$\subseteq$": Follows from the fact that $p \supseteq I_1 \cap I_2$ is prime, then $p \supseteq I_1$ or $p \supseteq I_2$.

□

EXAMPLE 0.3. Let $A = k[X_1, \ldots, X_n]$ with k algebraically closed. Let $I \subseteq A$ be an ideal. Then the maximal ideals m of A containing I are in one-to-one correspondence with $V(I)$ in $\mathbb{A}^n(k)$: by Nulstellensatz, every maximal ideal is of the form $(X_1 - a_1, \ldots, X_n - a_n)$, which corresponds to $(a_1, \ldots, a_n)$ in the old $V(I)$.

The new $V(I)$ now extends this notion of zero set by including other prime ideals.

EXAMPLE 0.4. If k is a field, then $\operatorname{Spec} k = \{0\}$, so the topological space cannot see the field. We fix this by also thinking about what functions are on these spaces.

CHAPTER 1

Sheaves

REMARK. Fix a topological space X.

DEFINITION 1.1. A presheaf $\mathcal{F}$ on X consists of

(1) For every open set $U \subseteq X$ an abelian group $\mathcal{F}U$,
(2) for every inclusion $V \subseteq U \subseteq X$ a restriction map $\rho_{UV} : \mathcal{F}U \to \mathcal{F}V$ such that $\rho_{UU} = \mathrm{id}_{\mathcal{F}U}$ and $p_{UW} = \rho_{VW} \circ \rho_{UV}$.

REMARK 1.2. A presheaf is just a contravariant functor from the poset category of open sets of X to the catgegory of abelian groups.

We can generalize this to any contravariant functor $X^{\mathrm{op}} \to \mathcal{C}$ for some category $\mathcal{C}$.

DEFINITION 1.3. A morphism of presheaves $f : \mathcal{F} \to \mathcal{G}$ on X is a collection of morphisms $f_U : \mathcal{F}U \to \mathcal{G}U$ such that for all $V \subseteq U$ the diagram

$$\begin{tikzcd} \mathcal{F}U \arrow[r, "f_U"] \arrow[d, "\rho_{UV}"] & \mathcal{G}U \arrow[d, "\rho_{UV}"] \\ \mathcal{F}V \arrow[r, "f_V"] & \mathcal{G}V \end{tikzcd}$$

commutes.

DEFINITION 1.4. A presheaf $\mathcal{F}$ is called a sheaf if it satisfies additional axioms:

(S1) If $U \subseteq X$ is covered by an open cover $\{U_i\}$ and $s \in \mathcal{F}U$ satsfies $s|_{U_i} := \rho_{UU_i}(s) = 0$ for all i, then $s = 0$
(S2) If U, and U_i are as before, and if $s_i \in \mathcal{F}U_i$ such that for all i and j we have $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, then there is some $s \in \mathcal{F}U$ such that $s|_{U_i} = s_i$ for all i.

REMARK 1.5. (1) If $\mathcal{F}$ is a sheaf, then $\varnothing \subseteq X$ is covered by the empty covering; hence $\mathcal{F}(\varnothing) = 0$.
(2) The two sheaf axioms can be described as saying that given $U, \{U_i\}$,

$$0 \longrightarrow \mathcal{F}U \xrightarrow{\alpha} \prod_i \mathcal{F}U_i \underset{\beta_2}{\overset{\beta_1}{\rightrightarrows}} \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

is exact, where $\alpha(s) = (s|_{U_i})_{i \in I}$, $\beta_1((s_i)_{i \in I}) = (s_i|_{U_i \cap U_j})$, $\beta_2((s_i)_{i \in I}) = (s_j|_{U_i \cap U_j})_{i,j}$.

Exactness means that α is injective, $\beta_1 \circ \alpha = \beta_2 \circ \alpha$, and for any $(s_i) \in \prod_{i \in I} \mathcal{F}U_i$, with $\beta_1((s_i)) = \beta_2((s_i))$, there exists $s \in \mathcal{F}U$ with $\alpha(s) = (s_i)$.

This is all subsumed by saying that α is the equalizer of β_1 and β_2.

EXAMPLE. (1) Let X be any topological space, $\mathcal{F}U$ the continuous functions $U \to \mathbb{R}$.

This is a sheaf: $\rho_{UV} : \mathcal{F}U \to \mathcal{F}V$ is just the restriction.

The first sheaf axiom says that a contiunous function is zero if it is zero on every open set of cover.

The second sheaf axiom says that continuous functions can be glued.

(2) Let $X = \mathbb{C}$ with the Euclidean topology.

Define $\mathcal{F}U$ to be the set of bounded analytic functions $f\colon U \to \mathbb{C}$.

This is a presheaf, since the restriction of bounded analytic functions is bounded analytic. It also satisfies the first sheaf axiom. However, it does not satisfy the second sheaf axiom.

For example, consider the cover $\{U_i\}_{i\in\mathbb{N}}$ of $\mathbb{C}$ given by $U_i = \{z \in \mathbb{C} \mid |z| < i\}$. Define $s_i\colon U_i \to \mathbb{C}$ by $z \mapsto z$. Note that if $i < j$, then $U_i \cap U_j = U_i$ and $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$. However, gluing yields the identity function on $\mathbb{C}$, which is not bounded (note that complex analysis tells us that $\mathcal{F}\mathbb{C} = \mathbb{C}$.

The underlying problem is that sheafs can only track properties that can be tested locally.

(3) Let G be a group and set $\mathcal{F}U \coloneqq G$ for any open set U. This is called the constant presheaf. This is in general not a sheaf (unless G is trivial).

Take U to be an disjoint union of open sets $U_1 \cup U_2$. If $\mathcal{F}U_1 = G$ and $\mathcal{F}U_2 = G$, then we need $\mathcal{F}(U_1 \cap U_2) = 0$.

If the second sheaf axiom was to be satisfied, we would want $s_1 \in \mathcal{F}U_1$ and $s_2 \in \mathcal{F}U_2$ to glue, so we should have $\mathcal{F}U = G \times G$.

Now give G the discrete topology, and define instead $\mathcal{F}U$ to be the set of continuous maps $f\colon U \to G$. By our choice of topology, this means that f is locally constant, i.e., for every $x \in U$ we have a neighborhood $V \subseteq U$ of x such that $f|_V$ is constant.

This is called the constant sheaf and if U is nonempty and connected then $\mathcal{F}U = G$.

(4) If X is an algebraic variety, $U \subseteq X$ a Zariski open subset, then define $\mathcal{O}_X(U)$ to be the regular functions $f\colon U \to k$.

Roughly, f regular means that every point of U has an open neighborhood on which f is expressed as a ratio of polynomials g/h with h nonvanishing on the neighborhood.

$\mathcal{O}_X$ is a sheaf, called the structure sheaf of X.

Definition 1.6. Let $\mathcal{F}$ be a presheaf on X and let $x \in X$. Then the stalk of $\mathcal{F}$ at x is $\mathcal{F}_x \coloneqq \{(U, s) \mid U \subseteq X \text{ open neighborhood at } x, s \in \mathcal{F}U\}/\sim$, w here $(U, s) \sim (V, s')$ if there is a neighborhood $W \subseteq U \cap V$ of x such that $s|_W = s'|_W$. An equivalence class of a pair (U, s) is called a germ.

Remark. $\mathcal{F}_x$ is just the colimit of $\mathcal{F}U$ where U ranges over the open neighborhoods of x.

Note that a morphism $f\colon \mathcal{F} \to \mathcal{G}$ of presheaves induces a morphism $f_p\colon \mathcal{F}_p \to \mathcal{G}_p$ via $f_p(U, s) \coloneqq (U, f_U(s))$.

Proposition 1.7. Let $f\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then f is an isomorphism if and only if f_p is an isomorphism for every $p \in X$.

Proof. "$\Longrightarrow$" is obvious.

"$\Longleftarrow$": Assume that f_p is an isomorphism for all $p \in X$. Need to show that $f_U\colon \mathcal{F}U \to \mathcal{G}U$ is an isomorphism for all $U \subseteq X$, as then we can define $(f^{-1})_U = (f_U)^{-1}$. This defines a morphism of sheaves, as

$$\begin{aligned}
\rho^{\mathcal{F}}_{UV} \circ f_U^{-1} &= f_V^{-1} \circ f_V \circ \rho^{\mathcal{F}}_{UV} \circ f_U^{-1} \\
&= f_V^{-1} \circ \rho^{\mathcal{G}}_{UV} \circ f_U \circ f_U^{-1} \\
&= f_V^{-1} \circ \rho^{\mathcal{G}}_{UV}.
\end{aligned}$$

We will first check that f_U is injective. Suppose $s \in \mathcal{F}U$ and $f_U(s) = 0$. Then for all $p \in U$, we have $f_p(U, s) = (U, f_U(s)) = (U, 0) = 0 \in \mathcal{F}_p$. Since f_p is injective, this means that $(U, s) = 0$ in $\mathcal{F}_p$. This means that there is an open neighborhood V_p of p in U such that $s|_{V_p} = 0$. Since the sets $\{V_p\}_{p\in U}$ cover U, we see by sheaf axiom 1 that we have $s = 0$.

Next, we will show that f_U is surjective. Let $t \in \mathcal{G}U$ and write $t_p := (U, t) \in \mathcal{G}_p$. Since f_p is surjective, we find $s_p \in \mathcal{F}_p$ with $f_p(s_p) = t_p$. This means that we find an open neighborhood $V_p \subseteq U$ of p and a germ (V_p, s_p) such that $(V_p, f_{V_p}(s_p)) \sim (U, t)$. By shrinking V_p if necessary wen can assume that $t|_{V_p} = f_{V_p}(s_p)$.

Now on $V_p \cap V_q$, $f_{V_p\cap V_q}(s_p|_{V_p\cap V_q} - s_q|_{V_p\cap V_q}) = t_{V_p\cap V_q} - t|_{V_p\cap V_q} = 0$ and hence by injectivity of $f_{V_p\cap V_q}$ already proved, we have $s_p|_{V_p\cap V_q} = s_q|_{V_p\cap V_q}$. By the second sheaf axiom, the s_p glue to give an element $s \in \mathcal{F}U$ with $s|_{V_p} = s_p$ for every $p \in U$.

Now $f_U(s)|_{V_p} = f_{V_p}(s|_{V_p}) = f_{V_p}(s_p) = t|_{V_p}$. By the first sheaf axiom applied to $f_U(s) - t$ we get $f_U(s) = t$. This shows surjectivity of f_U, completing the proof. □

THEOREM 1.8. Given a presheaf $\mathcal{F}$ there is a sheaf $\mathcal{F}^+$ and a morphism $\theta\colon \mathcal{F} \to \mathcal{F}^+$ satisfying the following universal propert:

For any sheaf $\mathcal{G}$ and morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ there is a unique morphism $\varphi^+\colon \mathcal{F}^+ \to \mathcal{G}$ such that $\varphi^+ \circ \theta = \varphi$.

The pair $(\mathcal{F}^+, \theta)$ is unique up to unique isomorphism and is called the sheafification of $\mathcal{F}$.

PROOF. See exercises. □

DEFINITION. Let $f\colon \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves on a space X. We define

(1) The presheaf kernel of f, $\ker f$, is the presheaf given by
$$(\ker f)(U) := \ker f_U.$$
One should check that this is a presheaf.

(2) The presheaf cokernel of f, $\operatorname{coker} f$, is the presheaf given by
$$(\operatorname{coker} f)(U) := \operatorname{coker} f_U.$$

(3) The presheaf image $\operatorname{im} f$ is the presheaf given by
$$(\operatorname{im} f)(U) = \operatorname{im} f_U.$$

REMARK. If $f\colon \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, then $\ker f$ is also a sheaf. The identity axiom is certainly satisfied: If $s \in (\ker f)(U) \subseteq \mathcal{F}U$ satisfies $s|_{U_i} = 0$ for all U_i in a cover of U, then we use the identity axiom for $\mathcal{F}$ to find that $s = 0$.

Given $s_i \in (\ker f)(U_i)$ with $\{U_i\}$ an open cover of U, and with $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$, then we find $s \in \mathcal{F}U$ with $s|_{U_i} = s_i$. But $f_U(s) = 0$ since
$$f_U(s)|_{U_i} = f_{U_i}(s|_{U_i}) = f_{U_i}(s_i) = 0,$$
and we can use the identity axiom to conclude that $f_U(s) = 0$.

EXAMPLE. Let $X = \mathbb{P}^1$ (or think of the Riemann sphere). Let $P, Q \in X$ be distinct points. Let $\mathcal{G}$ be the sheaf of regular functions on X (alternatively, think of holomorphic functions on the Riemann sphere). Next, let $\mathcal{F}$ be the sheaf of regular functions which vanish on P and Q. Notice that $\mathcal{F}U = \mathcal{G}U$ if $U \cap \{P, Q\} = \varnothing$.

Let $U := \mathbb{P}^1 \setminus \{P\}$, $V = \mathbb{P}^1 \setminus \{Q\}$.

Note that $\mathcal{F}(\mathbb{P}^1) = 0$, $\mathcal{G}(\mathbb{P}^1) = k$, because regular functions on $\mathbb{P}^1$ are constants. Let $f\colon \mathcal{F} \to \mathcal{G}$ be the inclusion.

Then $(\operatorname{coker} f)(\mathbb{P}^1) \cong k$, $(\operatorname{coker} f)(U) = \mathcal{G}U/\mathcal{F}U = k[X]/(X) \cong k\mathrm{a}$, $(\operatorname{coker} f)(V) \cong k$. However, $(\operatorname{coker} f)(U \cap V) = \mathcal{G}(U \cap V)/\mathcal{F}(U \cap V) \cong 0$.

Therefore, if the gluing axiom held, then we could need to have
$$(\operatorname{coker} f)(\mathbb{P}^1) \cong k \oplus k.$$

Note that this failure to be a sheaf is not a bug, but a feature!

Definition. Let $f\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. The sheaf kernel of f is just the presheaf kernel.

The sheaf cokernel is the sheaf associated to the presheaf cokernel of f.

The sheaf image is the sheaf associated to the presheaf image of f.

We can check that these notions give kernels, cokernels and images in the category of sheaves.

Exercise. The sheaf image $\operatorname{im} f$ is a subsheaf of $\mathcal{G}$, where $\mathcal{F}$ is called a subsheaf of $\mathcal{G}$ if we have a morphism $f\colon \mathcal{F} \to \mathcal{G}$ such that f_U is a monomorphism for every open set U.

Solution. See exercises. □

Definition. We say that f is injective if $\ker f = 0$. We say that f is surjective if $\operatorname{im} f = \mathcal{G}$.

Note that surjectivity does not imply that f_U is surjective for every U.

We say that a sequence of morphisms of sheaves

$$\dots \longrightarrow \mathcal{F}^{i-1} \xrightarrow{f^i} \mathcal{F}^i \xrightarrow{f^{i+1}} \mathcal{F}^{i+1} \longrightarrow \dots$$

is exact if $\ker f^{i+1} = \operatorname{im} f^i$ for all i.

If $\mathcal{F}' \subseteq \mathcal{F}$ is a subsheaf, then we write $\mathcal{F}/\mathcal{F}'$ for the sheaf associated to the presheaf $U \mapsto \mathcal{F}U/\mathcal{F}'U$, so $\mathcal{F}/\mathcal{F}'$ is the cokernel of the inclusion $\mathcal{F}' \to \mathcal{F}$.

Lemma 1.9. Let $f\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then for all $p \in X$ we have

$$(\ker f)_p = \ker(f_p\colon \mathcal{F}_p \to \mathcal{G}_p)$$
$$(\operatorname{im} f)_p = \operatorname{im} f_p$$

Proof. We first define a map $(\ker f)_p \to \ker f_p$. If $(U, s) \in (\ker f)_p$, then $(U, s) \in \mathcal{F}_p$ and

$$f_p(U, s) = (U, f_U(s)) = (U, 0) = 0 \in \mathcal{G}_p.$$

Therefore, $(U, s) \in \ker f_p$.

We will check injectivity and surjectivity of this map.

For injectivity, assume that $(U, s) = 0$ in $\mathcal{F}_p$, then there is $V \subseteq U$ of p such that $s|_V = 0$. Then we also have the equality

$$(U, s) = (V, s|_V) = (V, 0) = 0$$

in $(\ker f)_p$.

For surjectivity, assume that $(U, s) \in \ker f_p$. This means that $(U, f_U(s)) = 0$ in $\mathcal{G}_p$, so there is $p \in V \subseteq U$ such that $0 = f_U(s)|_V = f_V(s|_V)$. Thus, $s|_V \in (\ker f)(V)$, and $(V, s|_V) \in (\ker f)_p$, and $(V, s|_V)$ maps to the element in $\ker f_p$ represented by (U, s).

For images: Let $\operatorname{im}' f$ be the presheaf image.

From the exercises we know that $\theta_p\colon \mathcal{F}_p \to \mathcal{F}_p^+$ is an isomorphism for every p.

Therefore $(\operatorname{im} f)_p \cong (\operatorname{im}' f)_p$, so we need to show that $(\operatorname{im}' f)_p \cong \operatorname{im} f_p$. Define a map $(\operatorname{im}' f)_p \to \operatorname{im} f_p$ by

$$(U, s) \in (\operatorname{im}' f)_p \mapsto (U, s) \in \operatorname{im} f_p.$$

Once again, we will check that this is injective and surjective.

For injectivity: if $(U, s) = 0$ in $\mathcal{G}_p$ then there is a neighborhood $V \subseteq U$ of p such that $s|_V = 0$. Then $(U, s) = (V, 0)$ in $(\operatorname{im}' f)_p$.

For surjectivity: if $(U, s) \in \operatorname{im} f_p$, then there is $(V, t) \in \mathcal{F}_p$ with $(V, f_V(t)) = f_p(V, t) = (U, s)$, so after shrinking U and V if necessary, then we can take $U = V$ and $f_U(t) = s$. Then $(U, s) \in (\operatorname{im}' f)_p$. □

PROPOSITION. Let $f\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then f is injective if and only if for every $p \in X$ the map $f_p\colon \mathcal{F}_p \to \mathcal{G}_p$ is injective and f is surjective if and only if for every $p \in X$ the map $f_p\colon \mathcal{F}_p \to \mathcal{G}_p$ is surjective.

PROOF. f_p is injective for every p if and only if $\ker f_p = 0$ for every p if and only if $(\ker f)_p = 0$ for every p.

In the exercises, we show that for any sheaf $\mathcal{F}$, the map

$$\mathcal{F}U \to \prod_{p \in U} \mathcal{F}_p$$

is injective. Now if all of the $\mathcal{F}_p$ are trivial, then so is $\mathcal{F}U$.

Therefore $(\ker f)_p = 0$ for every p if and only if $\ker f = 0$.

Similarly, f_p is surjective for every p iff $\operatorname{im} f_p = \mathcal{G}_p$ for every p. Now consider the diagram

$$\begin{tikzcd}
\operatorname{im} f_p \arrow[r] \arrow[d, "\cong"] & \mathcal{G}_p \\
(\operatorname{im}' f)_p \arrow[r, "\cong"] \arrow[ur] & (\operatorname{im} f)_p, \arrow[u]
\end{tikzcd}$$

where

- the top arrow is the inclusion,
- the left arrow is the isomorphism defined in Lemma 1.9,
- the bottom arrow is the isomorphism on stalks induced by the inclusion into the associated sheaf,
- the diagonal arrow is the morphism on stalks induced by the inclusion of the presheaf image, and
- the right arrow is induced by the arrow making the sheaf image into a subsheaf.

The upper triangle commutes trivially, and the lower triangle commutes because by construction the right arrow is induced by the unique arrow making the non-stalk version of the triangle commute. Thus, since the bottom and left arrows are isomorphisms and the diagram commutes, we have that $\operatorname{im} f_p \to \mathcal{G}_p$ is an isomorphism (which just means that $\operatorname{im} f_p = \mathcal{G}_p$) if and only if $(\operatorname{im} f)_p \to \mathcal{G}_p$ is an isomorphism.

Now, the arrow $(\operatorname{im} f)_p \to \mathcal{G}_p$ is an isomorphism for every p if and only if $\operatorname{im} f \to \mathcal{G}$ is an isomorphism (Proposition 1.7), and this is the definition of surjectivity. $\square$

EXERCISE. Given $f\colon \mathcal{F} \to \mathcal{G}$, then we have $\mathcal{G}/\operatorname{im} f \cong \operatorname{coker} f$.

1. Passing between spaces

REMARK. Let $f\colon X \to G$ be a continuous map between topological spaces, $\mathcal{F}$ a sheaf on X, $\mathcal{G}$ a sheaf on Y.

DEFINITION. Define $f_*\mathcal{F}$ by setting

$$(f_*\mathcal{F}) \coloneqq \mathcal{F}(f^{-1}(U))$$

for $U \subseteq Y$ open.

EXERCISE. $f_*\mathcal{F}$ is a sheaf on Y.

SOLUTION. TODO $\square$

DEFINITION. Define $f^{-1}\mathcal{G}$ to be the sheaf associated to the presheaf

$$U \mapsto \{(V, s) \mid f(U) \subseteq V, s \in \mathcal{G}V\}/\sim$$

where $(V, s) \sim (v', s')$ if there is some $W \subseteq V \cap V'$ such that $f(U) \subseteq W$ and $s|_W = s'|_W$.

EXAMPLE. If $f\colon \{p\} \to X$ is an inclusion of a point, then $f^{-1}\mathcal{G}$ is the sheaf on the one-point space given by $\mathcal{G}_p$.

More generally, if $i\colon Z \to X$ is the inclusion of a subspace, we often write $\mathcal{F}|_Z \coloneqq i^{-1}\mathcal{F}$. If Z is open in X, then we have

$$F|_Z(U) \cong \mathcal{F}U.$$

NOTATION 1.10. If $s \in \mathcal{F}U$, then we say that s is a section of $\mathcal{F}$ over U.

We often write $\mathcal{F}U = \Gamma(U, \mathcal{F})$. This allows us to think of $\Gamma(U, \cdot)$ as a functor from the category of presheaves on X to the category of abelian groups.

CHAPTER 2

Affine schemes

Remark. Our goal is to construct a sheaf $\mathcal{O}$ on $\operatorname{Spec} A$, analogous to the sheaf of regular functions on a variety.

$\mathcal{O}$ will be a sheaf of rings, i.e., $\mathcal{O}U$ will be a ring for each open set U and restriction maps will be ring homomorphisms.

Remark. Let A be a ring and let $S \subseteq A$ be a multiplicative subset (i.e., $1 \in S$ and S is closed under multiplication). We define a ring

$$S^{-1}A = \{(a, s) \mid a \in A, s \in S\}/\sim,$$

where $(a, s) \sim (a', s')$ iff there is $s'' \in S$ such that $s''(as' - a's) = 0$.

We write a/s for the equivalence class of (a, s).

Observe that the usual equivalence relation on fractions suggests that we should have $a/s = a'/s' \iff as' = a's$. We need the extra possibility of killing $as' - a's$ with s'' if A is not an integral domain.

The ring $S^{-1}A$ is called the localization of A at S.

Example. (1) Take $f \in A$ and $S := \{f^n \mid n \in \mathbb{N}_0\}$. Then we write $A_f := S^{-1}A$.

This example will correspond to open subsets.

(2) Let $p \subseteq A$ be a prime ideal of A. Then $S := A \setminus p$ satisfies $1 \in S$ and is closed under multiplication since p is prime. We define $A_p := S^{-1}A$. This is the localization of A at (or rather, away from?) p.

This example will correspond to taking stalks.

Definition. $\mathcal{O}$ should satisfy $\mathcal{O}_p = A_p$.

Define

$$\mathcal{O}U := \{s \colon U \to \coprod_{p \in U} A_p \mid (\star)\},$$

where $(\star)$ means that

(1) $\forall p \in U \colon s(p) \in A_p$,
(2) for each $p \in U$ there is some $p \in V \subseteq U$ with V open and $a, f \in A$ such that for all $q \in V \colon f \notin q \wedge s(q) = a/f$.

Lemma. For any $p \in \operatorname{Spec} A$, we have $\mathcal{O}_p \cong A_p$.

Proof. We define a map

$$\begin{aligned} \mathcal{O}_p &\to A_p \\ (U, s) &\mapsto s(p) \end{aligned}$$

and will show that it is injective and surjective.

For surjectivity, notice that every element of A_p can be written as a/f for some $a \in A$, $f \notin p$. Then

$$D(f) := \operatorname{Spec} A \setminus V(f) = \{p \in \operatorname{Spec} A \mid f \notin p\}$$

is an open set (in fact it is called a standard open). Now a/f defines an element of $s \in \mathcal{O}(D(f))$ given by $q \mapsto a/f \in A_q$. In particular, $s(p) = a/f \in A_p$.

For injectivity, let $p \in U \subseteq \operatorname{Spec} A$, $s \in \mathcal{O}U$ with $s(p) = 0$ in A_p. We need to show that $(U, s) = 0$ in $\mathcal{O}_p$. By shrinking U we can assume that s is given by $a, f \in A$ with $s(q) = a/f$ for all $q \in U$. In particular $f \notin q$ for every $q \in U$.

Thus, $a/f = 0/1$ in A_p. By definition of localization, this means that there is $h \in A \setminus p$ such that $h \cdot (a \cdot 1 + f \cdot 0) = 0$ in A, so we have $ah = 0$.

Now let $V = D(f) \cap D(h)$. Then $(V, s|_V) = 0$ in $\mathcal{O}_p$, since for $q \in V$, $s|_v(q) = s(q) = a/f \in A_q$ and $ha = 0$, $h \notin A \setminus q$, so $ha = 0$ implies $a/f = 0/1$ in A_q. Thus $(U, s) = 0$ in $\mathcal{O}_p$. $\square$

Lemma. For any $f \in A$, we have $\mathcal{O}(D(f)) \cong A_f$.
In particular, since $\operatorname{Spec} A = D(1)$, we have $\mathcal{O}(\operatorname{Spec} A) \cong A_1 \cong A$.

Proof. Define

$$\begin{aligned} \Psi \colon A_f &\to \mathcal{O}(D(f)) \\ a/f^n &\mapsto (p \mapsto a/f^n). \end{aligned}$$

This makes sense since if $f \notin p$, then $f^n \notin p$. As usual, we will verify injectivity and surjectivity.

For injectivity, assume that $\Psi(a/f^n) = 0$. Then for all $p \in D(f)$, we have $a/f^n = 0$ in A_p, i.e., there is $h \in A \setminus p$ such that $ha = 0$ in A.

Let $I = \{q \in A \mid q \cdot a = 0\}$ (the annihilator of a). So $h \in I$, but $h \notin p$, so $I \nsubseteq p$. This is true for all $p \in D(f)$, so $V(I) \cap D(f) = \varnothing$. Thus $f \in \bigcap_{p \in V(I)} p = \sqrt{I}$, as we know from commutative algebra. This means that $f^n \in I$ for some $n > 0$. Thus $f^n \cdot a = 0$, so $a/f^n = 0$ in A_f, so Ψ is injective.

Next, we will prove surjectivity. Let $s \in \mathcal{O}(D(f))$. Cover $D(f)$ with open sets V_i on which s is represented by as a_i/g_i with $a_i, g_i \in A$, $g_i \notin p$ whenever $p \in V_i$. Thus $V_i \subseteq D(g_i)$. By question 1 on the first example sheet, the sets of the form $D(h)$ form a base for the Zariski topology on $\operatorname{Spec} A$. Thus we can assume $V_i = D(h_i)$ for some $h_i \in A$. Since $D(h_i) \subseteq D(g_i)$, we have $V(h_i) \supseteq V(g_i)$, so $\sqrt{(h_i)} \subseteq \sqrt{(g_i)}$, since the radical is the intersection of all the primes of $V(\cdot)$. Hence, $h_i^n \in (g_i)$ for some n, say $h_i^n = c_i g_i$, so we have $\frac{a_i}{g_i} = \frac{c_i a_i}{h_i^n}$. Now replace h_i by h_i^n. This does not change the open sets because in general $D(h_i) = D(h_i^n)$ and replace a_i by $c_i a_i$.

The situation so far is that we may assume that $D(f)$ is covered by sets $D(h_i)$ such that s is represented by a_i/h_i on $D(h_i)$.

We now claim that $D(f)$ can be covered by a finite number of the $D(h_i)$, i.e., $D(f)$ is quasicompact. Indeed, $D(f) \subseteq \bigcup_i D(h_i)$, which is equivalent to $V(f) \supseteq \bigcap_i V(h_i) = V(\sum(h_i))$. This in turn is equivalent to $f \in \sqrt{\sum_i (h_i)}$ (because it just says that f is in every prime ideal containing $\sum_i (h_i)$), which is equivalent to there being some n such that $f^n \in \sum_i (h_i)$. Hence, we can write $f^n = \sum_{i \in I} b_i h_i$ for some finite set I.

Reversing this argument yields that $D(f) \subseteq \bigcup_{i \in I} D(h_i)$ as required, completing the proof of the claim.

We now pass to this finite subcover $\{D(h_i)\}_{i \in I}$. On $D(h_i) \cap D(h_j) = D(h_i h_j)$, note a_i/h_i and a_j/h_j both represent s. Since we have already shown injectivity, this means that $a_i h_j / h_i h_j = a_j h_i / h_i h_j$ in $A_{h_i h_j}$.

Thus, for some n, $(h_i h_j)^n (h_j a_i - h_i a_j) = 0$ in A. We can pick an n sufficiently large to work for all pairs i, j (since there are only finitely many such pairs).

We rewrite this equality as $h_j^{n+1}(h_i^n a_i) - h_i^{n+1}(h_j^n a_j) = 0$. Now replace h_i by h_i^{n+1}, and a_i by $h_i^n a_i$ (this is allowed because $\frac{a_i}{h_i} = \frac{a_i h_i^n}{h_i^{n+1}}$. Thus we can assume that s is still represented on $D(h_i)$ by a_i/h_i but also for each i, j we have $h_i a_j = h_j a_i$.

Since $D(f) \subseteq \bigcup_{i \in I} D(h_i)$, we have $V(\sum(h_i)) = \bigcap_{i \in I} V(h_i) \subseteq V(f)$, hence $f^n = \sum b_i h_i$ for some h_i. Define $a \coloneqq b_i a_i$.

Then for any j, we have

$$h_j a = \sum_i b_i a_i h_j = \sum_i b_i a_j h_i = f^n a_j.$$

This means that $a/f^n = a_j/h_j$ on $D(h_j)$. Hence $\Psi(a/f^n) = s$, completing the proof of surjectivity. □

Remark. We now have a topological space $\operatorname{Spec} A$ equipped with a sheaf of rings $\mathcal{O}$.

Definition. A ringed space is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X.

A morphism of ringed spaces $f: (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ consists of a continuous map $X \to Y$ and a morphism of shaves of rings $f^\#: \mathcal{O}_Y \to f_*\mathcal{O}_X$, i.e., for every open $O \subseteq Y$, a homomorphism of rings $f^\#_U: \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$.

Example. (1) Let X, Y be topological spaces and $\mathcal{O}_X$ and $\mathcal{O}_Y$ the sheaf of continuous $\mathbb{R}$-valued functions. Given $f: X \to Y$, we get $f^\#: \mathcal{O}_Y \to f_*\mathcal{O}_X$ defined by $f^\#_U(\varphi) = \varphi \circ f$.

(2) Let X be a variety and $\mathcal{O}_X$ the sheaf of regular functions on X. A morphism of varieties $f: X \to Y$ is a continuous map inducing

$$\mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U)),$$
$$\varphi \mapsto \varphi \circ f.$$

Definition. A locally ringed space $(X, \mathcal{O}_X)$ is a ringed space such that $\mathcal{O}_{X,p}$ is a local ring (i.e., has a unique maximal ideal) for every $p \in X$.

A morphism $f: (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of locally ringed spaces is a morphism of ringed spaces such that such that the induced map $f^\#_p: \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$ is a local homomorphism. Here,

- the map $f^\#_p$ is defined by $(U, s) \mapsto (f^{-1}(U), f^\#_U(s))$ for a section $s \in \mathcal{O}_Y(U)$, and
- a local homomorphism $\varphi: (A, m_A) \to (B, m_B)$ is a ring homomorphism between local rings such that $\varphi^{-1}(m_B) = m_A$. Note that $\varphi(A \setminus m_A) = \varphi(A^\times) \subseteq B^\times = B \setminus m_B$. Hence, $\varphi^{-1}(m_B) \subseteq m_A$ is always true, and the opposite inclusion is what makes a ring homomorphism local.

Remark. In the case of varieties, $\mathcal{O}_{X,p}$ has a unique maximal ideal $\{(U, f) \in \mathcal{O}_X(U) \mid f(p) = 0\}/\sim$, i.e., if $f(p) \neq 0$, then f is nowhere vanishing on some neighborhood of p, so after shrinking U, we can invert f.

The local homomorphism condition just follows from the pullback of a function φ vanishing at $f(p)$ vanishes at p.

Example. $(\operatorname{Spec} A, \mathcal{O})$ is a locally ringed space; which we call an affine scheme.

Theorem. The category of affine schemes with locally ringed morphisms is equivalent to the opposite of the category of rings.

Proof. We need to show the following things.

(1) If $\varphi: A \to B$ is a ring homomorphism, we obtain an induced morhphism

$$(f, f^\#): (\operatorname{Spec} B, \mathcal{O}_B) \to (\operatorname{Spec} A, \mathcal{O}_A).$$

(2) Any morphism of affine schemes as locally ringed spaces arises in this way.

For the first part, let $\varphi: A \to B$ be a ring homomorphism and define

$$f: \operatorname{Spec} B \to \operatorname{Spec} A$$
$$p \mapsto \varphi^{-1}(p),$$

where we use that $\varphi^{-1}(p)$ is prime: if $ab \in \varphi^{-1}(p)$, then $\varphi(ab) = \varphi(a)\varphi(b) \in p$. Hence $\varphi(a) \in p$ or $\varphi(b) \in p$, hence $a \in \varphi^{-1}(p)$ or $b \in \varphi^{-1}(p)$.

We also need to show that f is continuous. Any closed set is of the form $V(I)$. We calculate

$$\begin{aligned} f^{-1}(V(I)) &= f^{-1}(\{p \in \operatorname{Spec} A \mid p \supseteq I\}) \\ &= \{q \in \operatorname{Spec} B \mid f(q) \supseteq I\} \\ &= \{q \in \operatorname{Spec} B \mid \varphi^{-1}(q) \supseteq I\} \\ &= \{q \in \operatorname{Spec} B \mid q \supseteq \varphi(I)\} \\ &= V(\varphi(I)). \end{aligned}$$

Hence the preimage of a closed set is closed, so f is continuous.

We need to construct a morphism of sheaves

$$f_{\#} \colon \mathcal{O}_{\operatorname{Spec} A} \to f_* \mathcal{O}_{\operatorname{Spec} B}.$$

For $p \in \operatorname{Spec} B$, we obtain a natural homomorphism

$$\begin{aligned} \varphi_p \colon A_{\varphi^{-1}(p)} &\to B_p \\ \frac{a}{s} &\mapsto \frac{\varphi(a)}{\varphi(s)}, \end{aligned}$$

where $a \in A$, $s \notin \varphi^{-1}(p)$. This makes sense since $\varphi(a) \in B$ and $\varphi(s) \notin p$.

The maximal ideal pB_p of B_p is generated by the image of p under the map $B \to B_p$. The maximal ideal $\varphi^{-1}(p)A_{\varphi^{-1}(p)}$ of $A_{\varphi^{-1}(p)}$ is generated by the image of $\varphi^{-1}(p)$ under the map $A \to A_p$.

We have a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d] & B \arrow[d] \\ A_{\varphi^{-1}(p)} \arrow[r, "\varphi_p"] & B_p. \end{tikzcd}$$

Thus $\varphi_p^{-1}(pB_p) = \varphi^{-1}(p)A_{\varphi^{-1}(p)}$. Given $V \subseteq \operatorname{Spec} A$ open, we may define

$$\begin{aligned} f_V^{\#} \colon \mathcal{O}_{\operatorname{Spec} A}(V) &\to \mathcal{O}_{\operatorname{Spec} B}(f^{-1}(V)) \\ s &\mapsto (q \mapsto \varphi_q(s(f(q)))) \end{aligned}$$

We now have to check the local coherence condition of $\mathcal{O}$, i.e., if s is locally given by a/h, then $f_V^{\#}(s)$ is locally given by $\frac{\varphi(a)}{\varphi(h)}$ (this is obvious, but should be checked carefully).

This gives the desired map $f^{\#} \colon \mathcal{O}_{\operatorname{Spec} A} \to f_* \mathcal{O}_{\operatorname{Spec} B}$ and the induced map on stalks $f_p^{\#} \colon O_{\operatorname{Spec} A, f(p)} \to \mathcal{O}_{\operatorname{Spec} B, p}$ agrees with $\varphi_p \colon A_{\varphi^{-1}(p)} \to B_p$ by construction (this should be checked carefully). Hence, the pair $(f, f^{\#})$ is a morphism of locally ringed spaces.

Now suppose given a morphism $(f, f^{\#}) \colon \operatorname{Spec} B \to \operatorname{Spec} A$ of locally ringed spaces. We have

$$f^{\#}_{\operatorname{Spec} A} \colon \Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \to \Gamma(\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}),$$

but since the glocal sections of $\operatorname{Spec} R$ are just R, we get $\varphi \colon A \to B$.

We need to show that φ gives rise to $(f, f^{\#})$. We have a local homomorphism

$$f_p^{\#} \colon A_{f(p)} \cong \mathcal{O}_{\operatorname{Spec} A, f(p)} \to \mathcal{O}_{\operatorname{Spec} B, p} \cong B_p.$$

This is compatible with the corresponding map on glocal sections in the sense that

$$\begin{tikzcd} \Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \arrow[r, "f^{\#}_{\operatorname{Spec} A}"] \arrow[d] & \Gamma(\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \arrow[d] \\ \mathcal{O}_{\operatorname{Spec} A, f(p)} \arrow[r, "f^{\#}_p"] & \mathcal{O}_{\operatorname{Spec} B, p} \end{tikzcd}$$

is a commutative diagram. By applying our calculations, this yields a diagram

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d] & B \arrow[d] \\ A_{f(p)} \arrow[r, "f^{\#}_p"] & B_p \end{tikzcd}$$

and we find that $f^{\#}_p$ is a local homomorphism. Thus $(f^{\#}_p)^{-1}(pB_p) = f(p)A_{f(p)}$. Along the lower left path, the maximal ideal pB_p is pulled back to $f(p)A_{f(p)}$ and then to $f(p)$. Along the upper right path, it gets pulled back to p and then to $\varphi^{-1}(p)$. By commutativity, we conclude that $f(p) = \varphi^{-1}(p)$.

Thus f is induced by φ and by commutativity, $f^{\#}_p = \varphi_p$. Then $f^{\#}$ is as constructed previously (this needs to be checked). □

Remark. Note that demanding that $(f, f^{\#})$ is a morphism of locally ringed spaces rather than ringed spaces was crucial to make the proof work.

Definition 2.1. An affine scheme is a locally ringed space that is isomorphic as a locally ringed space to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ for some ring A.

A scheme is a locally ringed space $(X, \mathcal{O}_X)$ with an open cover $\{(U_i, \mathcal{O}_X|_{U_i})\}$ such that each $(U_i, \mathcal{O}_X|_{U_i})$ is an affine scheme. Recall that we have $\mathcal{O}_X|_{U_i}(V) = \mathcal{O}_X(V)$ for $V \subseteq U$ open.

Example. (1) Let k be a field. Then $\operatorname{Spec} k = (\{0\}, k)$.

What does giving a morphism $f\colon \operatorname{Spec} k \to X$ a scheme mean?

First, we need to choose a point $x \in X$, the image of f. Second, we get a local ring homomorphism

$$f^{\#}_x\colon \mathcal{O}_{X,x} \to \mathcal{O}_{\operatorname{Spec} k, 0} \cong k,$$

i.e., $(f^{\#}_x)^{-1}(0) = m_x \subseteq \mathcal{O}_{X,x}$, the maximal ideal of $\mathcal{O}_{X,x}$. Thus we get a factorization $f^{\#}_x\colon \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}/m_x \to k$. The middle quotient is a field denoted as $\kappa(x)$, the residue field of X at x.

Thus f induces an inclusion $\kappa(x) \to k$.

Conversely, given an inclusion $\iota\colon \kappa(x) \to k$ we get a morphism of schemes $\operatorname{Spec} k \to X$ by defining $f(0) = x$ and $f^{\#}\colon \mathcal{O}_X \to f_* k$ by defining $s \mapsto \iota(s(x)) \in k$, where $s(x)$ means taking the stalk of s at x.

Moral: Giving a morphism $f\colon \operatorname{Spec} k \to X$ is equivalent to giving a point $x \in X$ and an inclusion $\kappa(x) \to k$.

Exercises

Example Sheet 1

Exercise 1.

Exercise. Let A be a ring. Show that the sets $D(f) \coloneqq \{\mathfrak{p} \in \operatorname{Spec} A \mid f \notin \mathfrak{p}\}$ with f ranging over elements of A form a basis of the topology on $\operatorname{Spec} A$.

Solution. We have $\operatorname{Spec} A = D(1)$ and for $f, g \in A$ we have

$$\begin{aligned} D(f) \cap D(g) &= \{\mathfrak{p} \in \operatorname{Spec} A \mid f \notin \mathfrak{p} \wedge g \notin \mathfrak{p}\} \\ &= \{\mathfrak{p} \in \operatorname{Spec} A \mid fg \notin \mathfrak{p}\} \\ &= D(fg), \end{aligned}$$

so the collection $\{D(f)\}$ forms the basis of *a* topology, and it remains to show that the topology generated by the $D(f)$ is the Zariski topology. Firstly, for any $f \in A$ we have

$$\begin{aligned} \operatorname{Spec} A \setminus D(f) &= \{\mathfrak{p} \in \operatorname{Spec} A \mid f \in p\} \\ &= \{\mathfrak{p} \in \operatorname{Spec} A \mid (f) \subseteq \mathfrak{p}\} \\ &= V((f)), \end{aligned}$$

so each $D(f)$ is open. It remains to show that every open set is the union of sets of the form $D(f)$. Indeed, if I is any ideal of A, then

$$\begin{aligned} \operatorname{Spec} A \setminus V(I) &= \{\mathfrak{p} \in \operatorname{Spec} A \mid I \nsubseteq \mathfrak{p}\} \\ &= \{\mathfrak{p} \in \operatorname{Spec} A \mid \exists f \in I \colon f \notin \mathfrak{p}\} \\ &= \bigcup_{f \in I} D(f) \end{aligned}$$

as required. □

Exercise. An element $f \in A$ is nilpotent if and only if $D(f) = \varnothing$.

Solution. If f is nilpotent, say $f^n = 0$, and $\mathfrak{p}$ is a prime ideal, then we have $f^n = 0 \in \mathfrak{p}$, so $f \in \mathfrak{p}$. Hence, $D(f) = \varnothing$.

If f is not nilpotent, then define $\mathcal{S}$ to be the collection of all ideals I such that $f^n \notin I$ for every $n > 0$. Since f is not nilpotent, $(0) \in \mathcal{S}$. The set $\mathcal{S}$ is partially ordered by inclusion and admits upper bounds, since the increasing union of ideals disjoint from $\{f^n\}$ is still an ideal disjoint from $\{f^n\}$. Hence $\mathcal{S}$ admits a maximal member I. We will show that I is prime.

Let $x, y \in A$ such that $xy \in I$ and suppose that $x \notin I$, $y \notin I$. Then $I + Ax$ and $I + Ay$ are not disjoint from $\{f^n\}$ so we find $n, m \in \mathbb{N}$, $i, j \in I$ and $a, b \in A$ such that $f^n = i + ax$, $f^m = j + by$. But then $f^{n+m} = ij + iby + jax + abxy \in I$, a contradiction, so $x \in I$ or $y \in I$ and I is prime. Hence, $I \in D(f)$, so $D(f) \neq \varnothing$. □

Exercise 4.

NOTATION. For $s \in \mathcal{F}U$ and $p \in U$ we will write $s_p := (U, s) \in \mathcal{F}_p$.

DEFINITION. Let $\mathcal{F}$ be a presheaf and $U \subseteq X$ an open set. Define

$$\mathcal{F}^+U := \{s \colon U \to \coprod_{p \in U} \mathcal{F}_p \mid \forall p \in U \colon s(p) \in \mathcal{F}_p, (\star)\},$$

where $(\star)$ is the following statement: for every $p \in U$ there is an open $p \in V_p \subseteq U$ and a section $s_{V_p} \in \mathcal{F}U$ such that for ever $q \in V_p$ we have $(s_{V_p})_q = s(q)$.

EXERCISE. $\mathcal{F}^+$ together with the obvious restriction maps forms a sheaf.

SOLUTION. $\mathcal{F}^+U$ is an abelian group with pointwise addition, as the sum of $s, t \in F^+U$ still satisfies $(\star)$ by taking the intersection of the V_p obtained from s and t.

It is obvious that $\mathcal{F}^+$ is a presheaf.

Next, let $s \in \mathcal{F}^+U$ and $\{U_i\}$ an open cover such that $\forall i, s|_{U_i} = 0$. Let $p \in U$. Then $p \in U_i$ for some i and we have $s(p) = (s|_{U_i})(p) = 0$, so $s = 0$, so the identity axiom is satisfied.

Next, let $\{U_i\}_{i \in I}$ be a cover, $s_i \in \mathcal{F}^+U_i$ such that $\forall i, j \colon s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$. Given $p \in U$, define $s(p) := s_i(p)$ for $p \in U_i$. This is well-defined because of the compatibility condition. We need to show that $s \in \mathcal{F}^+U$. Indeed, let $p \in U$. Then $s(p) = s_i(p)$ for some i, and since $s_i \in \mathcal{F}^+U_i$ and taking stalks is compatible with restrictions, we get a neighborhood that satisfies the required condition. It remains to show that for all i, $s|_{U_i} = s_i$, but that is true by definition. $\square$

DEFINITION. For a presheaf $\mathcal{F}$ and an open set U, define

$$\theta_U \colon \mathcal{F}U \to \mathcal{F}^+U; \qquad s \mapsto (p \mapsto s_p).$$

This is obviously a homomorphism of groups. It also defines a morphism of shaves, because for $s \in \mathcal{F}U$, $V \subseteq U$ and $p \in V$ we have

$$\theta_U(s)|_V(p) = \theta_U(s)(p) = s_p = (s|_V)_p = \theta_V(s|V)(p).$$

LEMMA. Let $\mathcal{F}$ be a sheaf and U an open set. Then the natural map

$$\mathcal{F}U \to \prod_{p \in U} \mathcal{F}_p$$

is injective.

PROOF. Let $s, t \in \mathcal{F}U$ such that $s_p = t_p$ for every p. Let $p \in U$. By definition of a stalk, $s_p = t_p$ means that there is an open $p \in V_p \subseteq U$ such that $s|_{V_p} = t|_{V_p}$. These V_p cover U so by the identity axiom we have $s = t$. $\square$

LEMMA. Let $\mathcal{F}$ be a sheaf. Let U be an open set. Let $s \colon U \to \coprod_{p \in U} \mathcal{F}_p$ such that for every $p \in U$ we have $s(p) \in \mathcal{F}_p$ and there is an open $p \in V_p \subseteq U$ together with $s_{V_p} \in \mathcal{F}V_p$ such that for every $q \in V_p$ we have $(s_{V_p})_q = s(q)$. Then there is a unique $t \in \mathcal{F}U$ such that $t_q = s(q)$ for every $q \in U$.

PROOF. Uniqueness follows from the previous lemma. For existence, notice that the V_p cover U. Let $p, q \in U$. The s_{V_p} are gluable because their stalks agree on the intersection, so the conditions of the gluing axiom are satisfied by the previous lemma. Since talking stalks is compatible with restrictions, the glued section has the correct stalks. $\square$

EXERCISE. Let $\mathcal{F}$ be a presheaf, $\mathcal{G}$ a sheaf and $\varphi \colon \mathcal{F} \to \mathcal{G}$ a morphism of presheaves. Then there is a unique morphism of sheaves $\varphi^+ \colon \mathcal{F}^+ \to \mathcal{G}$ such that $\varphi = \varphi^+ \circ \theta$.

Solution. Let U be an open and let $s \in \mathcal{F}^+U$. Cover U with the V_p from the definition of $\mathcal{F}^+$ and obtain the associated $s_{V_p} \in \mathcal{F}V_p$. Define $t_{V_p} \coloneqq \varphi_{V_p}(s_{V_p}) in \mathcal{G}V_p$. We can calulate that for $q \in V_p$ we have

$$(t_{V_p})_q = (\varphi_{V_p}(s_{V_p}))_q = \varphi_q((s_{V_p})_q) = \varphi_q(s(q)).$$

Therefore, Lemma 2 gives us a unique $t_U \in \mathcal{G}_U$ such that

$$(\star) \qquad \forall q \in U \colon (t_U)_q = \varphi_q(s(q)).$$

We define $\varphi_U^+(s) = t_U$.

This is indeed a morphism of sheaves: if $V \subseteq U$ and $s \in \mathcal{F}^+U$, then

$$\varphi^+(s|_V) = \varphi^+(s)|_V$$

follows from the fact that, using $(\star)$, the germ of both sides at $p \in V$ is just $\varphi_p(s(p))$. By Lemma 1, the two sides are equal.

Similarly, if $s \in \mathcal{F}U$ and $p \in U$, then

$$(\varphi_U^+\theta_U(s))_p \overset{(\star)}{=} \varphi_q(\theta(s)(q)) = \varphi_q(s_q) = (\varphi_U(s))_q,$$

so $\varphi_U^+ \circ \theta_U = \varphi_U$ by Lemma 1, so $\varphi^+ \circ \theta = \varphi$.

Finally, to see uniqueness, assume that $\varphi^\#$ satisfies $\varphi^\# \circ \theta = \varphi$. Let $s \in \mathcal{F}^+U$ and $p \in U$. By definition of $\mathcal{F}^+$ there is $p \in V_p \subseteq U$, $s_{V_p} \in \mathcal{F}V_p$ such that $\forall q \in V_p \colon (s_{V_p})_q = s(q)$. The condition can be reprased as $s|_{V_p} = \theta(s_{V_p})$ and we calculate

$$\begin{aligned}(\varphi_U^\#(s))_p &= (\varphi_U^\#(s)|_{V_p})_p = (\varphi_{V_p}^\#(s|_{V_p}))_p = (\varphi_{V_p}^\#(\theta(s_{V_p})))_p \\ &= (\varphi_{V_p}^+(\theta(s_{V_p})))_p = \ldots = (\varphi_U^+(s))_p,\end{aligned}$$

so by Lemma 1, we have $\varphi_U^+ = \varphi_U^\#$, so $\varphi^+ = \varphi^\#$, completing the proof of uniqueness. □

Exercise. We have $(\mathcal{F}^+)_p = \mathcal{F}_p$ for $p \in X$. Show that if $f \colon \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves, then there is an induced morphism $f^+ \colon \mathcal{F}^+ \to \mathcal{G}^+$ with $(f^+)_p = f_p$.

Solution. Let $p \in X$. Of course, $(\mathcal{F}^+)_p$ and $\mathcal{F}_p$ cannot be literally equal. Instead, we show the following more precise statement: The map $\theta_p \colon \mathcal{F}_p \to \mathcal{F}_p^+$ is an isomorphism.

Indeed, we define $g_p \colon \mathcal{F}_p^+ \to \mathcal{F}_p$ as follows: for an open U and $s \in \mathcal{F}^+U$ we define $g_p(s_p) \coloneqq s(p)$. This is well-defined because sections $s \in \mathcal{F}^+U$, $t \in \mathcal{F}^+V$ that have the same germ at p must satisfy $s|_W = t|_W$ for some W that contains p, so $s(p) = s|_W(p) = t|_W(p) = t(p)$.

Next, let U be an open and $s \in \mathcal{F}_p^+$. By definition of $\mathcal{F}^+$, there is some $p \in V_p \subseteq U$ open, $s_{V_p} \in \mathcal{F}V_p$ such that for all $q \in V_p$ we have $(s_{V_p})_q = s(q)$. This is equivalent to saying that $s|_{V_p} = \theta_{V_p}(s_{V_p})$, so in particular, in $\mathcal{F}_p^+$, we have $s_p = (\theta_{V_p}(s_{V_p}))_p$. This lets us calculate

$$\theta_p(g_p(s_p)) = \theta_p(s(p)) = \theta_p((s_{V_p})_p) = (\theta_{V_p}(s_{V_p}))_p = s_p,$$

so we have $\theta_p \circ g_p = \mathrm{id}_{\mathcal{F}_p^+}$.

Next, let U be an open and $s \in \mathcal{F}U$. Then we have

$$g_p(\theta_p(s_p)) = g_p(\theta_U(s)_p) = g_p((q \mapsto s_q)_p) = (q \mapsto s_q)(p) = s_p,$$

so $g_p \circ \theta_p = \mathrm{id}_{\mathcal{F}_p}$, and θ_p is an isomorphism as required.

Next, let $\mathcal{F}$ and $\mathcal{G}$ be presheaves and let $\theta \colon \mathcal{F} \to \mathcal{F}^+$ and $\iota \colon \mathcal{G} \to \mathcal{G}^+$ denote the natural maps to the associated sheaf. If $f \colon \mathcal{F} \to \mathcal{G}$ is a map of presheaves, we can invoke the universal property of $\mathcal{F}^+$ on the composite $\iota \circ f$ and find a morphism $f^+ \colon \mathcal{F}^+ \to \mathcal{G}^+$ making the diagram

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\theta"] \arrow[d, "f"] & \mathcal{F}^+ \arrow[d, dotted, "f^+"] \\
\mathcal{G} \arrow[r, "\iota"] & \mathcal{G}^+
\end{tikzcd}$$

commute.

On stalks, we have

$$f_p^+ \circ \theta_p = (f^+ \circ \theta)_p = (\iota \circ f)_p = \iota_p \circ f_p,$$

and since θ_p is an isomorphism, we have

$$f_p^+ = \iota_p \circ f_p \circ \theta_p^{-1},$$

which is how we should interpret the “equality” $(f^+)_p = f_p$ under the natural identifications θ_p and ι_p. $\square$

Exercise 5.

Exercise. Show that if $f \colon \mathcal{F} \to \mathcal{G}$ is a morphism between sheaves, then the sheaf image $\operatorname{im} f$ can be naturally identified with a subsheaf of $\mathcal{G}$.

Solution. We will prove the following more general statement: if $\mathcal{F}$ is a presheaf satisfying the identity axiom, $\mathcal{G}$ is a sheaf and $f \colon \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves such that f_U is injective for every U, then the induced morphism $f_U^+ \colon \mathcal{F}^+U \to \mathcal{G}U$ is injective for every U.

Indeed, the inclusion of the presheaf image into $\mathcal{G}$ satisfies these conditions. It satisfies sheaf axiom 1 for the same reason that the presheaf kernel does.

We will now prove the claim. Let U be an open set, $s \in \mathcal{F}^+U$ such that $f_U^+(s) = 0$. From the construction of the associated sheaf we see that $f_U^+(s) = f_U(t)$ where t is the unique element of $\mathcal{F}U$ such that $\forall q \in U \colon t_q = f_q(s(q))$.

So we have $0 = f_U^+(s) = f_U(t)$, so since f_U is injective we have $t = 0$. Let $q \in U$. Then $f_q(s(q)) = t_q = 0_q = 0$. The element $s(q)$ of $\mathcal{F}_q$ is represented by some open set V and a section $u \in \mathcal{F}V$. Thus $0 = f_q(s(q)) = f_q(V, u) = (V, f_V(u))$. Thus, there is some open $W \subseteq V$ such that $0 = f_V(u)|_W = f_W(u|_W)$. Since f_W is injective, we conclude $u|_W = 0$, and $u|_W$ represents the same element in $\mathcal{F}_q$ as u, but that element is just $s(q)$, so $s(q) = 0$. Since q was arbitrary, we conclude $s = 0$. $\square$

Exercise 6.

Exercise. A sequence of sheaves is exact if and only if for every $p \in X$ the corresponding sequence of maps of abelian groups is exact.

Solution. Assume that $f \colon \mathcal{F} \to \mathcal{G}$ and $g \colon \mathcal{G} \to \mathcal{H}$ are morphisms of sheaves such that $g \circ f = 0$. Consider the diagram

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "f"] \arrow[d] & \mathcal{G} \arrow[r, "g"] & \mathcal{H} \\
\operatorname{im}' f \arrow[ur] \arrow[r, "\theta"] \arrow[rr, bend right, "\iota"'] & \operatorname{im} f \arrow[u] \arrow[r, "\varphi"] & \ker g, \arrow[ul]
\end{tikzcd}$$

where the map ι is an inclusion of subsheaves of $\mathcal{G}$ and φ is induced by ι. We say that $\operatorname{im} f = \ker g$ if φ is an isomorphism. By a result of the lecture, this is the case if and only if forall $p \in X$, the induced map $\varphi_p \colon (\operatorname{im} f)_p \to (\ker g)_p$ is an isomorphism. Since θ induces isomorphisms on stalks and the bottom triangle commutes, this is the case if and only if $\iota_p \colon (\operatorname{im}' f)_p \to (\ker g)_p$ is an isomorphism for every $p \in X$. Now consider the diagram

$$\begin{tikzcd} (\operatorname{im}' f)_p \arrow[r, "\iota_p"] \arrow[d, "\cong"] & (\ker g)_p \arrow[d, "\cong"] \\ \operatorname{im} f_p \arrow[r, "i"] & \ker g_p, \end{tikzcd}$$

where the left and right maps are the isomorphisms defined in the proof of a result from the lecture and the bottom map is just the inclusion (this makes sense since $g \circ f = 0 \iff \forall p \in X \colon g_p \circ f_p = 0$ as stalks characterize morphisms). The diagram commutes since none of the maps actually does anything. Since the left and right maps are isomorphisms, we have that the top map is an isomorphism if and only if the bottom map is an isomophism.

But the bottom map is an isomorphism if and only if the sequence

$$\mathcal{F}_p \xrightarrow{f_p} \mathcal{G}_p \xrightarrow{g_p} \mathcal{H}_p$$

is exact, so putting everything together, we find that (f, g) is exact if and only if (f_p, g_p) is exact for every $p \in X$. $\square$

Exercise 7.

Exercise. Show that a morphism of sheaves is an isomorphism if and only if it is injective and surjective.

Solution. Let $f \colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. By a result from the lecture, f is an isomorphism if and only if $f_p \colon \mathcal{F}_p \to \mathcal{G}_p$ is an isomorphism for every p. Since f_p is a morphism of abelian groups, this is the case if and only if f_p is injective and surjective for every p. By another result from the lecture, this is the case if and only if f is injective and surjective. $\square$

Exercise 8.

Exercise. Let $\mathcal{F}'$ be a subsheaf of a sheaf $\mathcal{F}$. Then the natural map $\mathcal{F} \to \mathcal{F}/\mathcal{F}'$ is surjective and has kernel $\mathcal{F}'$ so that there is an exact sequence

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}/\mathcal{F}' \longrightarrow 0.$$

Solution. The natural map $e \colon \mathcal{F} \to \mathcal{F}/\mathcal{F}'$ is given as the composite $\theta \circ \hat{e}$, where $\hat{e}$ is the map $\mathcal{F} \to \operatorname{coker}' i$, where $i \colon \mathcal{F}' \to \mathcal{F}$ is the inclusion and $\operatorname{coker}' i$ is the presheaf cokernel of i, and θ is the natural map into the sheafification.

For every $p \in X$, θ_p is surjective because θ induces isomorphisms on stalks, and $\hat{e}_p$ is surjective, because $\hat{e}$ is surjective on open sets, which in particular implies surjectivity on stalks. Hence e_p is surjective as the composite of two open maps. By a result from the lecture, this implies that e is surjective.

Additionally, for any open $U \subseteq X$, we have $(\ker e)U = \ker p_U = \mathcal{F}'U$, so $\ker e$ and $\mathcal{F}'$ are equal as subsheaves of $\mathcal{F}$.

Now we have $\operatorname{im} i = \mathcal{F}'$ as subsheaves of $\mathcal{F}$, since the map $\theta \colon \mathcal{F}' = \operatorname{im}' i \to \operatorname{im} f$ induces isomorphisms on stalks, but since the domain already is a sheaf this forces θ to be an isomorphism. Hence $\operatorname{im} i = \ker e$ as subsheaves of $\mathcal{F}$, so the sequence is exact at $\mathcal{F}$. Exactness at $\mathcal{F}'$ and $\mathcal{F}/\mathcal{F}'$ is trivially checked on stalks using Exercise 6. $\square$

Exercise. If

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

is an exact sequence, then $\mathcal{F}'$ is isomorphic to a subsheaf of $\mathcal{F}$ and $\mathcal{F}''$ is isomorphic to the quotient of $\mathcal{F}$ by this subsheaf.

Solution. We have a commutative diagram

$$\begin{tikzcd} \mathcal{F}' \arrow[r, "i"] \arrow[d, "\hat{\imath}"] & \mathcal{F} \\ \operatorname{im}' i \arrow[r, "\theta"] \arrow[ur] & \operatorname{im} i, \arrow[u, "\iota"'] \end{tikzcd}$$

where the diagonal arrow is the inclusion, the bottom arrow is the natural map into the associated sheaf, and the right arrow is induced by the diagonal arrow. By Exercise 5, $\operatorname{im} i$ can be regarded as a subsheaf of $\mathcal{F}$. Since i is injective, for every $p \in X$, i_p is injective, so by commutativity, $\hat{\imath}_p$ is injective. Furthermore, for every $p \in X$, $\hat{\imath}$ is surjective, because it is surjective on open sets. Hence, the composite $\theta \circ \hat{\imath}$ is an isomorphism on stalks. Since it is a map between sheaves, this means that is is an isomorphism. Hence, $\mathcal{F}$ is isomorphic to the subsheaf $\operatorname{im} i$.

Next, consider the diagram

$$\begin{tikzcd} & \operatorname{coker}' \iota \arrow[r, "\eta"] \arrow[dr, "\hat{p}"] & \operatorname{coker} \iota \arrow[d, "\hat{p}^+"] \\ \operatorname{im} i \arrow[r, "\iota"] & \mathcal{F} \arrow[u, "\pi"] \arrow[r, "p"] & \mathcal{F}'', \end{tikzcd}$$

where the map $\hat{p}$ is defined on open sets using the fact that $p \circ \iota = 0$, hence $p_U \circ \iota_U = 0$, hence $(\operatorname{im} i)(U) \subseteq (\ker p)(U)$. The map, η is the natural map into the associated sheaf and $\hat{p}^+$ is obtained from the universal property. Since p is surjective, it is surjective on stalks, hence by commutatity $\hat{p}^+$ must also be surjective on stalks.

TODO: show that $\hat{p}^+$ is injective on stalks. □

Exercise 9.

Exercise. If $U \subseteq X$ is an open subset and

$$0 \longrightarrow \mathcal{F}' \xrightarrow{i} \mathcal{F} \xrightarrow{p} \mathcal{F}''$$

is exact, then

$$0 \longrightarrow \Gamma(U, \mathcal{F}') \xrightarrow{i_U} \Gamma(U, \mathcal{F}) \xrightarrow{p_U} \Gamma(U, \mathcal{F}'')$$

is exact.

Solution. Since $(0, i)$ is exact, $(0, i_x)$ is exact for every $x \in U$, hence i_x is injective for every $x \in U$, hence i is injective, hence $\ker i = 0$, hence $0 = (\ker i)(U) = \ker i_U$, hence i_U is injective, so the sequence is exact at $\Gamma(U, \mathcal{F}')$.

It remains to show exactness at $\Gamma(U, \mathcal{F})$. Since $p_U \circ i_U = (p \circ i)_U = 0_U = 0$, we have $\operatorname{im} i_U \subseteq \ker i_U$.

Conversely, let $s \in \ker p_U$, i.e., $p_U(s) = 0$. By Exercise 6 we know that (i_x, p_x) is exact for every $x \in U$. Since $p_U(s) = 0$, we have $p_x(U, s) = 0$ for every $x \in U$, hence $(U, s) \in \operatorname{im} i_x$ for all $x \in U$, i.e., we find $(V_x, t_x) \in \mathcal{F}'_x$ such that $i_x(V_x, t_x) = (U, s)$. If necessary, shrink V_x such that $i_{V_x}(t_x) = s|_{V_x}$.

For $x, y \in U$, we have

$$i_{V_x \cap V_y}(t_x|_{V_x \cap V_y} - t_y|_{V_x \cap V_y}) = s|_{V_x \cap V_y} - s|_{V_x \cap V_y} = 0.$$

Since i is injective, we conclude $t_x|_{V_x \cap V_y} = t_y|_{V_x \cap V_y}$, hence we can glue the t_x to a $t \in \mathcal{F}'U$. For any $x \in U$ we have

$$\mathcal{F}_x \ni (U, i_U(t)) = (V_x, i_{V_x}(t|_{V_x})) = (V_x, i_{V_x}(t_x)) = (V_x, s|_{V_x}) = (U, s),$$

and since stalks characterize sections, this implies that $i_U(t) = s$, hence $s \in \operatorname{im} i_U$ as required. $\square$