

# Commutative Algebra

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These notes, taken by Markus Himmel, will at times differ significantly from what was lectured. In particular, all errors are almost certainly my own.

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## CHAPTER 0

### Introduction

REMARK 0.0. Commutative algebra is the study of commutative rings developed from

- (1) algebraic geometry and
- (2) algebraic number theory

In (1) focus is on  $k[X_1, \dots, X_n]$ , the polynomial ring over the field  $k$ . In (2) focus is on  $\mathbb{Z}$ , the ring of rational integers. Modern development of (1) by Grothendieck encompasses much of (2).

Going back further, Hilbert wrote a series of papers on polynomial invariant theory, 1888-1893.

EXAMPLE 0.1. Denote by  $\Sigma_n$  the symmetric group on  $\{1, \dots, n\}$ .  $\Sigma_n$  acts on  $k[X_1, \dots, X_n]$  by permuting variables: given  $\sigma \in \Sigma_n$ ,  $f \in k[X_1, \dots, X_n]$ , we set

$$(\sigma f)(X_1, \dots, X_n) := f(X_{\sigma^{-1}(1)}, \dots, X_{\sigma^{-1}(n)}).$$

The action of  $\Sigma_n$  is via ring automorphisms so it makes sense to define the *ring of invariants*

$$S := \{f \in k[X_1, \dots, X_n] \mid \forall \sigma \in \Sigma_n: \sigma f = f\}.$$

$S$  is a ring, called the *ring of symmetric polynomials*. Consider the following elementary symmetric functions:

$$\begin{aligned} e_1(X_1, \dots, X_n) &= X_1 + \dots + X_n, \\ e_2(X_1, \dots, X_n) &= \sum_{i < j} X_i X_j, \\ &\vdots \\ e_n(X_1, \dots, X_n) &= X_1 \cdots X_n. \end{aligned}$$

It turns out that  $S$  is generated as a ring by these  $e_i$  and the canonical map  $k[Y_1, \dots, Y_n] \rightarrow S$  given by  $Y_i \mapsto e_i$  is an isomorphism of rings.

Hilbert showed that  $S$  is finitely generated for many other groups. Among the way he proved a few very deep results.

- the basis theorem,
- the Nullstellensatz,
- the polynomial nature of the Hilbert function (and beginnings of dimension theory),
- the syzygy theorem (and beginnings of the homological theory of polynomial rings).

REMARK 0.2. Emmy Noether (1921) extracted the key property that made the basis theorem work: we call a ring *noetherian* if every ideal is finitely generated. There are many properties that are equivalent to this.

THEOREM 0.3. Hilbert's basis theorem states that if  $R$  is a commutative Noetherian ring, then so is  $R[X]$ .

COROLLARY 0.4. In particular, if  $k$  is a field, then  $k[X_1, \dots, X_n]$  is noetherian.

Noether developed a theory of ideals for noetherian rings, for example the existence of a primary decomposition which generalises the factorisation into primes known from number theory.

### Links between commutative algebra and algebraic geometry

REMARK. Recall the fundamental theorem of algebra: a polynomial  $f \in \mathbb{C}[X]$  is determined up to scalar multiples by its zeros up to multiplicity.

Given  $f \in \mathbb{C}[X_1, \dots, X_n]$  we have a polynomial function  $\mathbb{C}^n \rightarrow \mathbb{C}$  given by  $(a_1, \dots, a_n) \mapsto f(a_1, \dots, a_n)$ .

Different polynomials yield different functions, so  $\mathbb{C}[X_1, \dots, X_n]$  can be viewed as the ring of polynomial functions on complex affine  $n$ -space.

Given  $I \subseteq \mathbb{C}[X_1, \dots, X_n]$ , define the set of common zeros

$$Z(I) = \{(a_1, \dots, a_n) \in \mathbb{C}^n \mid \forall f \in I: f(a_1, \dots, a_n) = 0\},$$

called an (affine) algebraic set, which is a subset of  $\mathbb{C}^n$ .

REMARK. (1) One can replace  $I$  by the ideal generated by  $I$  and get the same algebraic set. Replacing an ideal by a generating set of the ideal leaves the algebraic set unchanged. Hilbert's basis theorem asserts that any algebraic set is the set of common zeros of a finite set of polynomials.

(2)

$$\bigcap_j Z(I_j) = Z\left(\bigcup_j I_j\right),$$

$$\bigcup_{j=1}^n Z(I_j) = Z\left(\prod_{j=1}^n I_j\right)$$

for ideals  $I_j$ . Define a topology of  $\mathbb{C}^n$  with closed sets being the algebraic sets. This is the Zariski topology; it is coarser than the normal topology on  $\mathbb{C}^n$ .

(3) For  $S \subseteq \mathbb{C}^n$  define

$$I(S) := \{f \in \mathbb{C}[X_1, \dots, X_n] \mid \forall (a_1, \dots, a_n) \in S: f(a_1, \dots, a_n) = 0\}.$$

This is an ideal of  $\mathbb{C}[X_1, \dots, X_n]$  and it is radical, i.e., if  $f^r \in I(S)$  for some  $r \geq 1$ , then  $f \in I(S)$ .

The Nullstellensatz is a family of results asserting that the correspondence

$$I \mapsto Z(I)$$

$$I(S) \leftarrow S$$

gives a bijection between the radical ideals of  $\mathbb{C}[X_1, \dots, X_n]$  and the algebraic subsets of  $\mathbb{C}^n$ . In particular, the maximal ideals of  $\mathbb{C}[X_1, \dots, X_n]$  correspond to points in  $\mathbb{C}^n$ .

### Dimension

REMARK. A large section of the course treats dimension of rings:

- the maximal length of chains of prime ideals;
- in geometric context in terms of growth rates (uses Hilbert function);
- the transcendence degree of the field of fractions (of an integral domain).

Over commutative rings these all give the same answer. A fourth way uses homological algebra and gives the same answer at least for nice noetherian rings.

Most of the theory dates between 1920 and 1950.

Rings of dimension 0 are called artinian rings. In dimension 1, special things happen which are important in number theory; this is crucial in the study of algebraic curves.





## CHAPTER 1

### Noetherian Rings

REMARK. Throughout the lecture,  $R$  is a commutative unital ring.

LEMMA 1.1. Let  $M$  be a (left)  $R$ -module. The following are equivalent.

- (i) all submodules of  $M$  (including  $M$  itself) are finitely generated,
- (ii) the ascending chain condition (ACC) holds: there are no strictly increasing infinite chains of submodules.
- (iii) maximum condition in submodules holds: any nonempty set  $\mathcal{S}$  of submodules of  $M$  has a maximal element  $L$ , i.e., if  $L' \in \mathcal{S}$  and  $L \subseteq L'$ , then  $L = L'$ .

PROOF. If all submodules of  $M$  are finitely generated and  $N_1 \subseteq N_2 \subseteq \dots$  is an increasing chain of submodules of  $M$ , define  $N := \bigcup_{i=1}^{\infty} N_i$ . This is a submodule of  $M$ , so it is finitely generated with generators  $m_1, \dots, m_k$ . Each  $m_i$  lies in some  $N_{n_i}$ . If  $n$  is the maximum of all  $n_i$ , we have  $N_n = N$  and the chain is stationary.

If the ACC holds and  $\mathcal{S}$  is nonempty, let  $M_0 := \{0\}$ . Proceed inductively. If  $M_i$  is maximal, we are done. Otherwise, there is some  $M_{i+1}$  such that  $M_i \subsetneq M_{i+1}$ . By the ACC, this process must terminate after a finite number of steps.

If the maximum condition holds and  $N$  is any submodule of  $M$ , define  $\mathcal{S}$  to be the collection of finitely generated submodules of  $N$ .  $\mathcal{S}$  is nonempty as it contains the zero module. Let  $L$  be a maximal member of  $\mathcal{S}$ . Let  $x \in N$ . Then  $L + Rx$  is finitely generated and  $L \subseteq L + Rx$ , hence,  $x \in L$  and therefore  $N = L$ .  $\square$

DEFINITION 1.2. An  $R$ -module is called noetherian if all of its submodules are finitely generated.

LEMMA 1.3. Let  $N$  be a submodule of  $M$ . Then  $M$  is noetherian if and only if  $N$  and  $M/N$  are noetherian.

PROOF. If  $M$  is noetherian, then in particular all submodules of  $N$  are finitely generated. Furthermore, all submodules of  $M/N$  are of the form  $Q/N$ , where  $Q$  is submodule of  $M$  containing  $N$ .  $Q$  is finitely generated, say by  $x_1, \dots, x_r$ . Then  $Q/N$  is generated by  $x_1 + N, \dots, x_r + N$ .

Conversely, if both  $N$  and  $M/N$  are noetherian, and  $L_1 \subseteq L_2 \subseteq \dots$  is an increasing chain of submodules of  $M$ , define  $Q_i := L_i + N$  and  $N_i := L_i \cap N$ . Then  $Q_i/N$  and  $N_i$  are chains of submodules of  $M/N$  and  $N$ , respectively, so they terminate and we find  $r$  such that  $\forall i \geq r: Q_i/N = Q_r/N$  and  $s$  such that  $\forall i \geq s: N_i = N_s$ . Define  $k := \max\{r, s\}$ .

We will show that  $\forall i \geq k: L_i = L_k$ . Indeed, let  $\ell \in L_i$ . Then  $\ell + N \in Q_i/N = Q_k/N = (L_k + N)/N$ , so there are  $\tilde{\ell} \in N, \ell' \in L_k, \hat{\ell} \in N$  such that  $\ell - \tilde{\ell} = \ell' + \hat{\ell}$ . Rearranging, we find that  $\ell - \ell' = \tilde{\ell} + \hat{\ell} \in N$ , and since  $L_k \subseteq L_i$  we conclude that  $\ell - \ell' \in N \cap L_i = N \cap L_k$ . Therefore,  $\ell = (\ell - \ell') + \ell' \in L_k$  and we are done.  $\square$

ALTERNATIVE PROOF. It suffices to show that if

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is a short exact sequence of  $R$ -modules, then  $B$  is noetherian if and only if both  $A$  and  $C$  are noetherian.

If  $B$  is noetherian and  $N$  is a submodule of  $C$ , then  $g^{-1}(N)$  is a submodule of  $B$ , thus finitely generated, say by  $b_1, \dots, b_n$ . If  $c \in N$ , then

$$c = f\left(\sum_{i=1}^n r_i b_i\right) = \sum_{i=1}^n r_i f(b_i),$$

so  $N$  is finitely generated. If  $N$  is a submodule of  $A$ , then it is isomorphic to a submodule of  $B$ , which is finitely generated, hence  $N$  is also finitely generated.

Assume that  $A$  and  $C$  are finitely generated and  $N$  is a submodule of  $B$ . Then  $g(N)$  is finitely generated, say by  $c_1, \dots, c_n$ . Additionally,  $f^{-1}(N)$  is finitely generated, say by  $a_1, \dots, a_m$ . Pick preimages  $b_1, \dots, b_n$  such that  $g(b_i) = c_i$ . Now let  $x \in N$ . Then  $g(x) = \sum_{i=1}^n r_i c_i$  and therefore  $x - \sum_{i=1}^n r_i b_i \in \ker g = \operatorname{im} f$ . Thus

$$x - \sum_{i=1}^n r_i b_i = f\left(\sum_{i=1}^m r'_i a_i\right).$$

Rearranging gives

$$x = \sum_{i=1}^m r'_i f(a_i) + \sum_{i=1}^n r_i b_i$$

and we conclude that  $N = \langle b_1, \dots, b_n, f(a_1), \dots, f(a_m) \rangle$  as required.  $\square$

LEMMA 1.4. Let  $M, N, M_1, \dots$  be  $R$ -modules.

- (i)  $M \oplus N$  is noetherian if and only if both  $M$  and  $N$  are.
- (ii)  $M_1 \oplus \dots \oplus M_n$  is noetherian if and only if all  $M_i$  are.
- (iii) If  $M$  is noetherian then every homomorphic image is noetherian.
- (iv) If  $M$  can be represented as the sum  $M_1 + \dots + M_n$ , then  $M$  is noetherian if and only if each  $M_i$  is.

PROOF.

- (i) Apply the previous lemma to the split exact sequence

$$0 \longrightarrow N \xrightarrow{\iota} M \oplus N \xrightarrow{\pi} M \longrightarrow 0.$$

- (ii) Induction.
- (iii) If  $\theta: M \rightarrow N$ , apply the previous lemma to the short exact sequence

$$0 \longrightarrow \ker \theta \longrightarrow M \xrightarrow{\theta} \operatorname{im} \theta \longrightarrow 0.$$

- (iv) If  $M$  is noetherian, then so is  $M_i$  as a submodule of  $M$ . If all  $M_i$  are noetherian, then so is  $M_1 \oplus \dots \oplus M_n$ , and since the map

$$\begin{aligned} M_1 \oplus \dots \oplus M_n &\rightarrow M_1 + \dots + M_n, \\ (m_1, \dots, m_n) &\mapsto m_1 + \dots + m_n \end{aligned}$$

is surjective,  $M_1 + \dots + M_n$  is noetherian.  $\square$

DEFINITION 1.5. A ring  $R$  is called noetherian if it is noetherian as a module over itself.

LEMMA 1.6. If  $R$  is a noetherian ring and  $M$  is a finitely generated  $R$ -module. Then  $M$  is noetherian.

PROOF. Assume  $M$  is generated by  $m_1, \dots, m_n$ . Then  $R^n \cong R^{\oplus n}$  is noetherian and the map  $R^n \rightarrow M$  given by  $e_i \mapsto m_i$  is surjective, so  $M$  is noetherian.  $\square$

THEOREM 1.7. If  $R$  is a noetherian ring, then  $R[X]$  is also noetherian.

PROOF. We will show that every ideal (i.e., submodule) of  $R[X]$  is finitely generated. Let  $I$  be an ideal and let  $I_n := \{f \in I \mid \deg f \leq n\}$ .  $0 \in I_n$  and  $I_0 \subseteq I_1 \subseteq \dots$  form an ascending chain.

Define  $R_n$  to be the set of coefficients of  $X^n$  appearing in elements of  $I_n$ .

If  $a, b \in R_n$ , then  $a + b \in R_n$  and  $ra \in R_n$  for any  $r \in R$ . Therefore,  $R_n$  is an ideal of  $R$ .

Furthermore, if  $a \in R_n$ , then  $a \in R_{n+1}$  by multiplying the corresponding polynomial by  $X$ .

Since  $R$  is noetherian, the chain  $R_0 \subseteq R_1 \subseteq \dots$  terminates, so we have  $N$  such that  $\forall n \geq N: R_n = R_N$ . Each of  $R_0, \dots, R_N$  is a finitely generated ideal of  $R$ , say  $R_j$  is generated by  $a_{j1}, \dots, a_{jk_j}$ . There are polynomials  $f_{j1}, \dots, f_{jk_j}$  such that  $\deg f_{ji} = j$  and leading coefficient of  $f_{ji}$  is  $a_{ji}$ .

We will show that the finite set  $\{f_{jk} \mid 0 \leq j \leq N, 1 \leq k \leq k_j\}$  generates  $I$ .

We will use induction on  $\deg f$ , where  $f \in I$ . If  $\deg f = 0$ , then  $f = a$  for some  $a \in R$ . By definition of  $R_0$ ,  $a \in R_0$ , and  $a$  is in the ideal generated by the  $f_{0i}$ .

Assume next that  $0 < \deg f \leq N$  and that the claim is true for smaller degrees. Let  $a$  be the leading coefficient of  $f$ .  $a \in R_n$ , so we may write

$$a = \sum_j r_{nj} a_{nj}.$$

Then

$$f - \sum_j r_{nj} f_{nj}$$

is in  $I$  and of smaller degree, so is expressible as a linear combination of the  $f_{ij}$ , so  $f$  is expressible as a linear combination as well.

Finally, assume that  $\deg f > N$  and that the claim is true for smaller degrees. If  $a$  is the leading coefficient of  $f$ , then  $a \in R_n = R_N$ , so we may write

$$a = \sum_j r_{Nj} a_{Nj}.$$

Then

$$f - X^{n-N} \sum_j r_{Nj} f_{Nj}$$

is in  $I$  and of smaller degree, so is expressible as a linear combination of the  $f_{ij}$ , so  $f$  is expressible as a linear combination as well.  $\square$

REMARK. In practice one uses Gröbner bases for ideals, which are special generating sets that admit efficient algorithms.

EXAMPLE. • Fields are noetherian.

- PIDs are noetherian.
- Let  $p$  be a prime number.  $\{\frac{m}{n} \mid m, n \in \mathbb{Z}, p \nmid n\}$  is an example of a localization of  $\mathbb{Z}$  (at  $p$ ). All localizations of noetherian rings are noetherian.
- $k[X_1, X_2, \dots]$  is not noetherian, as there is an infinite chain  $(X_1) \subsetneq (X_1, X_2) \subsetneq \dots$ .
- $k[X_1, \dots, X_n]$  is noetherian, by Hilbert's basis theorem and induction.
- $\mathbb{Z}[X_1, \dots, X_n]$  is noetherian: every finitely generated commutative ring is noetherian, since if  $R$  is generated by  $r_1, \dots, r_n$ , we have a surjective map  $\mathbb{Z}[X_1, \dots, X_n] \rightarrow R$  given by  $X_i \mapsto r_i$ .
- Group algebras of free abelian groups of finite rank: if  $A$  is an abelian group, the group algebra of  $A$  is the free  $\mathbb{Z}$ -module with basis  $A$ . It is an  $A$ -algebra with the multiplication defined as the  $\mathbb{Z}$ -bilinear continuation of  $(a, b) \mapsto ab$ . If  $A$  is generated by  $g_1, \dots, g_n$ , then  $\mathbb{Z}A$  is generated as a ring by  $g_1, g_1^{-1}, \dots, g_n, g_n^{-1}$ .

- The ring of formal power series  $k[[X]]$  is noetherian if  $k$  is noetherian, see below.

Here are some non-commutative rings which are left and right noetherian:

- The enveloping algebra of a finite dimensional Lie algebra.
- The Iwasawa algebras of compact  $p$ -adic groups.

**THEOREM 1.8.** If  $R$  is a noetherian ring, then the ring  $R[[X]]$  of formal power series over  $R$  is noetherian.

**PROOF 1.** Adapt the proof of Hilbert's basis theorem, but use trailing coefficients rather than leading coefficients. See the first exercise sheet.  $\square$

**THEOREM 1.9 (Cohen's theorem).** If every prime ideal in a ring  $R$  is finitely generated, then  $R$  is noetherian.

**PROOF.** Assume that  $R$  is not noetherian. Let  $\mathcal{S}$  be the collection of non-finitely generated ideals of  $R$ .  $\mathcal{S}$  is nonempty by assumption and partially ordered by inclusion. Furthermore, every chain of ideals in  $\mathcal{S}$  has an upper bound (indeed, the union of an increasing chain of ideals in  $\mathcal{S}$  is an ideal and not finitely generated, since otherwise all generators would lie in some member of the chain, which would then be finitely generated), so by Zorn's lemma there is a maximal member  $I \in \mathcal{S}$ .  $I$  has the property that it is not finitely generated, but every ideal  $J$  such that  $I \subsetneq J$  is finitely generated.

We will now show that  $I$  is a prime ideal. Suppose  $a$  and  $b$  are such that  $ab \in I$ ,  $a \notin I$ ,  $b \notin I$ . Since  $I$  is maximally non-finitely-generated,  $I + Ra$  is finitely generated, say by  $i_1 + r_1a, \dots, i_n + r_na$ . Define

$$J := \{s \in R \mid sa \in I\}.$$

$J$  is an ideal, and it satisfies  $I \subsetneq I + Ra \subseteq J$  (here we use that  $ab \in I$ ). Again by maximality of  $I$ ,  $J$  is finitely generated. Therefore, if we can show that  $I = Ri_1 + \dots + Ri_n + Ja$ , then  $I$  is finitely generated, a contradiction.

The inclusion " $\supseteq$ " follows by definition of  $J$ , so let  $t \in I \subseteq I + Ra$ , so

$$t = u_1(i_1 + r_1a) + \dots + u_n(i_n + r_na)$$

for suitable  $u_i \in R$ . We may rewrite this as

$$t = u_1i_1 + \dots + u_ni_n + (u_1r_1 + \dots + u_nr_n)a.$$

Since the whole right hand side is in  $I$  and everything but the last summand is also in  $I$ , the last summand is in  $I$ , so  $u_1r_1 + \dots + u_nr_n \in J$  by definition of  $J$ , so indeed  $t \in Ri_1 + \dots + Ri_n + Ja$  and we are done.  $\square$

**LEMMA 1.10.** Let  $p$  be a prime ideal of  $R[[X]]$  and  $\theta: R[[X]] \rightarrow R$  given by  $X \mapsto 0$ . The  $p$  is a finitely generated ideal of  $R[[X]]$  if and only if  $\theta(p)$  is a finitely generated ideal of  $R$ .

**PROOF.** We already know that images of finitely generated ideals are finitely generated.

Conversely, suppose that  $\theta(p) = Ra_1 + \dots + Ra_n$ .

If  $X \in p$ , then  $p$  is generated by  $a_1, \dots, a_n, X$ : given any  $f \in p$ , we can find  $g$  such that  $f - Xg \in R$  and so indeed  $a_i \in p$  (!) and  $f \in Ra_1 + \dots + Ra_n + X$ .

On the other hand, if  $X \notin p$ , let  $f_1, \dots, f_n \in p$  have constant terms  $a_1, \dots, a_n$  (these exist by definition of  $\theta$ ). We will show that  $p$  is generated by  $f_1, \dots, f_n$ . Let  $g_0 \in p$  and let  $b = \sum_{i=1}^n b_i a_i$  be the constant term of  $g$ , so there is  $g_1$  such that  $g_0 - \sum_{i=1}^n r_{0,i} f_i = g_1 X$ . We have  $g_1 X \in p$ , but since  $p$  is prime and  $X \notin p$ , we have  $g_1 \in p$ . Continuing inductively, we find  $r_{j,i} \in R$  and  $g_{j+1} \in p$  such that  $g_j - \sum_{i=1}^n r_{j,i} f_i = g_{j+1} X$ .

Define  $h_j := \sum_{i=0}^{\infty} r_{i,j} X^i$ . We can calculate

$$\begin{aligned}
 \sum_{i=1}^n h_i f_i &= \sum_{i=1}^n \left( \sum_{j=0}^{\infty} r_{j,i} X^j \right) f_i \\
 &= \sum_{i=1}^n \sum_{j=0}^{\infty} r_{j,i} f_i X^j \\
 &= \sum_{j=0}^{\infty} \sum_{i=1}^n r_{j,i} f_i X^j \\
 &= \sum_{j=0}^{\infty} X^j \sum_{i=1}^n r_{j,i} f_i \\
 &= \sum_{j=0}^{\infty} X^j (g_j - g_{j+1} X) \\
 &= g_0,
 \end{aligned}$$

so  $g_0$  is in the span of  $f_1, \dots, f_n$  as required.  $\square$

LEMMA 1.11. The set  $N(R)$  of all nilpotent elements of  $R$  is an ideal and  $R/N(R)$  has no nonzero nilpotent elements.

PROOF. If  $x \in N(R)$ , then there is  $m \in \mathbb{N}$  such that  $x^m = 0$ , which implies  $(rx)^m = 0$ , so  $rx \in N(R)$ . If  $x, y \in N(R)$ , there are  $n, m \in \mathbb{N}$ ,  $x^n = y^m = 0$ . Then  $(x+y)^{m+n-1}$  is a linear combination of terms  $\lambda x^s y^t$  with  $s+t = m+n-1$ . In particular,  $s \geq n \vee t \leq m$ , and so  $(x+y)^{m+n-1} = 0$  and  $x+y \in N(R)$ .

Furthermore, if  $s \in R/N(R)$ , then  $s = x + N(R)$ . If  $s$  is nilpotent, i.e.,  $s^n = 0$ , then  $0 = s^n = (x + N(R))^n = x^n + N(R)$ , i.e.,  $x^n \in N(R)$ . That means that for some  $m$  we have  $x^{nm} = 0$ , so  $x \in N(R)$ , so  $s = 0$ .  $\square$

DEFINITION 1.12. The ideal  $N(R)$  is called the nilradical of  $R$ .

THEOREM 1.13. The nilradical  $N(R)$  is the intersection of all prime ideals of  $R$ .

PROOF. Define  $I := \bigcap_{p \text{ prime}} p$ .

If  $x \in N(R)$ , i.e.,  $x^n = 0$ , and  $p$  is prime, then  $x^n = 0 \in p$ , so  $x \in p$ . Hence,  $N(R) \subseteq I$ .

To show that  $I \subseteq N(R)$ , we will show that  $x \notin N(R)$  implies  $x \notin I$ . Indeed, if  $x \notin N(R)$ , define  $\mathcal{S}$  to be the collection of all ideals  $J$  that are disjoint from the set  $\{x^n \mid n > 0\}$ . We have  $(0) \in \mathcal{S}$ , so  $\mathcal{S}$  is nonempty, and as usual, upper bounds of chains exist, so Zorn's lemma gives us a maximal member  $J_1$  of  $\mathcal{S}$ . We have  $x \notin J_1$ , so if we can show that  $J_1$  is prime, we are done.

Suppose  $yz \in J_1$ ,  $y, z \notin J_1$ . Then  $J_1 + Ry$  and  $J_1 + Rz$  are strictly larger than  $J_1$ , so we find  $n, m$  such that  $x^n \in J_1 + Ry$ ,  $x^m \in J_1 + Rz$ . This implies  $x^{n+m} \in J_1 + Ryz$  (write  $x^n = j_1 + r_1 y$ ,  $x^m = j_2 + r_2 z$ ), but then  $x^{n+m} \in J_1 + Ryz = J_1$ , which is a contradiction because  $J_1 \in \mathcal{S}$ .  $\square$

DEFINITION 1.14. The radical  $\sqrt{I}$  of an ideal  $I$  is defined as

$$\sqrt{I} := \{r \in R \mid \exists n \in \mathbb{N}: r^n \in I\}$$

We call an ideal radical if  $I = \sqrt{I}$ .

REMARK. It is unsubstantial whether 0 is allowed as an exponent or not: if  $r^0 = 1 \in I$ , then  $I = R$ , so  $r^1 \in I$ .

We have an equality  $\sqrt{I} + I = N(R/I)$  of ideals of  $R/I$ .

$\sqrt{I}$  is the intersection of all prime ideals that contain  $I$ :  $\sqrt{I}/I$  is the intersection of all prime ideals of  $R/I$ , then use the correspondence between prime ideals of  $R/I$  and prime ideals of  $R$  that contain  $I$ .

DEFINITION 1.15. The Jacobson radical  $J(R)$  of  $R$  is the intersection of all maximal ideals of  $R$ .

REMARK. We have  $N(R) \subseteq J(R)$ .

THEOREM 1.16 (Nakayama's lemma). If  $M$  is a finitely generated  $R$ -module such that  $J(R)M = M$ , then  $M = 0$ .

PROOF. Suppose that  $M \neq 0$ . Define  $\mathcal{S}$  to be the collection of proper submodules of  $M$ . Then  $(0) \in \mathcal{S}$ , and if we have an ascending chain of proper submodules, then the union is also a proper submodule (otherwise all generators would already lie in one of the proper submodules). So by Zorn, there is a maximal proper submodule  $M_1$ .

The quotient  $M/M_1$  is a simple module, as we can pullback any submodule of  $M/M_1$  to a submodule of  $M$  lying between  $M_1$  and  $M$ . If  $0 \neq m \in M/M_1$ , the submodule generated by  $m$  is all of  $M/M_1$ .

The homomorphism  $R \rightarrow M/M_1$  of  $R$ -modules given by  $r \mapsto rm + M_1$  is surjective. If  $I$  is the kernel of this map, then there is an isomorphism of  $R$ -modules  $M/M_1 \cong R/I$ , but since the former is a simple  $R$ -module, so is the latter. Now if  $J$  is an ideal of  $R/I$ , then it is also an  $R$ -submodule of  $R/I$ , which shows that  $R/I$  has only two ideals, so it is a field. This means that  $I$  is a maximal ideal.

Let  $n \in M$ . Since  $m$  generates  $M/M_1$ , we can write  $n = rm + m'$  for some  $r \in R$ ,  $m' \in M_1$ . If  $i \in I$ , then  $in = rim + im' \in M'$ , since  $im \in M'$  by definition of  $I$ . This means that  $IM \subseteq M_1$ .

Since  $I$  is maximal, we have  $J(R) \subseteq I$ , and so

$$J(R)M \subseteq IM \subseteq M_1 \subsetneq M,$$

contrary to our assumption.  $\square$

REMARK. In a commutative ring,  $N(R) \leq J(R)$ . They are in general not equal, take for example  $R_p = \{\frac{m}{n} \in \mathbb{Q} \mid p \nmid n\}$  for some prime  $p$ . This has a unique maximal ideal  $p = \{\frac{m}{n} \in \mathbb{Q} \mid p \mid n, p \nmid m\}$ , but it is an integral domain, so  $N(R) = (0)$  while  $J(R) = p$ .

On the other hand, for  $R = k[X_1, \dots, X_n]/I$ , where  $k$  is algebraically closed and  $I$  is any ideal, then we do indeed have  $N(R) = J(R)$ . This is Hilbert's Nullstellensatz.

EXAMPLE. A commutative ring is called artinian if it does not contain an infinite, strictly decreasing chain of ideals (equivalently, if every nonempty set of ideals has a minimal member). An  $R$ -module is called artinian if it satisfies that analogous property for submodules.

Examples of artinian rings:  $\mathbb{Z}/p\mathbb{Z}$ ,  $k[X]/(f)$ , where  $k$  is a field and  $f \neq 0$ .  $k[X]$  is not artinian: we have the chain  $(X) \supseteq (X^2) \supseteq \dots$ .

Recall that an ideal  $I$  is prime if and only iff  $R/I$  is an integral domain if and only if  $I_1, I_2 \subseteq I$  implies that  $I_1 \subseteq I \vee I_2 \subseteq I$ .

We will now show that if  $R$  is artinian, then prime ideals are maximal, which in particular means that  $N(R) = J(R)$ . Indeed, let  $p$  be a prime ideal and  $x \in R$  such that  $x \notin p$ . By the descending chain condition,  $(x) \supseteq (x^2) \supseteq \dots$  becomes stationary, so there is a number  $n$  and some  $y \in R$  such that  $x^n = yx^{n+1}$ . Rearranging, we have  $x^n(1 - xy) = 0 \in p$ . Since  $p$  is prime and  $x \notin p$ ,  $x^n \notin p$ , so we must have  $1 - xy \in p$ , so  $x + p$  has the inverse  $y + p$  in  $R/p$ . Since  $x$  was arbitrary,  $R/p$  is a field, so  $p$  is maximal.

**THEOREM 1.17** (Artin-Tate lemma). Let  $R \subseteq S \subseteq T$  be commutative rings. Suppose that  $R$  is noetherian,  $T$  is finitely generated as an  $R$ -algebra and  $T$  is a finitely generated  $S$ -module. Then  $S$  is a finitely generated  $R$ -algebra.

**PROOF.** Suppose  $T$  is generated as an  $R$ -algebra by  $t_1 = 1, \dots, t_n \in T$ . By assumption, we have  $x_1 = 1, \dots, x_m \in T$  such that  $T = Sx_1 + \dots + Sx_m$ . Therefore, if  $1 \leq i \leq n$ , we may write

$$(1) \quad t_i = \sum_{j=1}^m s_{ij}x_j$$

for some  $s_{ij} \in S$ . Furthermore,  $1 \leq i, j \leq m$ , we find  $s_{ijk} \in S$  satisfying

$$(2) \quad x_i x_j = \sum_{k=1}^m s_{ijk} x_k.$$

Define  $S_0$  as the  $R$ -subalgebra of  $S$  generated by the  $s_{ij}$  and the  $s_{ijk}$ . We have  $R \subseteq S_0 \subseteq S$ . If  $t \in T$ , we may write  $t$  as a polynomial in the  $t_i$ . Since  $t_1 = 1$ , we may assume that this polynomial does not have a constant term. Substituting (1) and then repeatedly substituting (2), we find that  $T$  is finitely generated by the  $x_i$  as a  $S_0$  module.

Next, we note that  $S_0$  is a noetherian ring. Since  $S_0$  is finitely generated as an  $R$ -algebra, we have a surjective homomorphism of rings  $\varphi: R[X_1, \dots, X_k] \rightarrow S_0$ . Then  $S_0$  is isomorphic to a quotient of  $R[X_1, \dots, X_k]$ , which is noetherian by the Basissatz. Quotients of noetherian rings are noetherian rings: indeed,  $R[X_1, \dots, X_n]/\ker \varphi$  is a noetherian  $R[X_1, \dots, X_n]$ -module, which implies that it is a  $R[X_1, \dots, X_n]/\ker \varphi$ -module.

As a finitely generated module over a noetherian ring, we find that  $T$  is a noetherian  $S_0$ -module. Since  $S$  is an  $S_0$ -submodule of  $T$ , we find that  $S$  is finitely generated as a  $S_0$ -module.

This allows us to write every element of  $S$  as a polynomial in the generators of  $S$  as an  $S_0$ -module and the  $s_{ij}$  and  $s_{ijk}$ , so  $S$  is a finitely generated  $R$ -algebra.  $\square$

**LEMMA 1.18** (Zariski's lemma). If  $k$  is a field, and  $R$  is a finitely generated  $k$ -algebra which is a field, then  $R$  is a finite-dimensional  $k$ -vector space (i.e., a finite algebraic extension of  $k$ ).

**PROOF.** Denote the generators of  $R$  as a  $k$ -algebra by  $x_1, \dots, x_n \in R$ . Suppose that  $R$  is not a finite algebraic extension of  $k$ . Then we may reorder the  $x_i$  such that there is an  $1 \leq m \leq n$  such that  $x_1, \dots, x_m$  is a transcendence basis, i.e.,  $x_1, \dots, x_m$  are all transcendental, but  $k(x_1, \dots, x_m) \subseteq R$  is finite algebraic.

Therefore we have  $k \subseteq k(x_1, \dots, x_m) \subseteq R$ , and Artin-Tate tells us that  $k(x_1, \dots, x_m)$  is a finitely generated  $k$ -algebra, say with generators  $q_1, \dots, q_k$ , where  $q_i = f_i/g_i$  for some  $f_i, g_i \in k[x_1, \dots, x_n]$  and  $g_i \neq 0$ . This means that we can write every element  $q \in k(x_1, \dots, x_m)$  as

$$q = \frac{f}{q_1^{e_1} \cdots q_k^{e_k}}.$$

However, since  $k[x_1, \dots, x_n]$  is a UFD, we can see that

$$\frac{1}{q_1 \cdots q_k + 1}$$

is not of this form, a contradiction.  $\square$

**THEOREM 1.19** (Hilbert's Nullstellensatz (weak version)). Let  $k$  be a field,  $T$  a finitely generated  $k$ -algebra, and  $m$  a maximal ideal of  $T$ . Then  $T/m$  is a finite

algebraic extension of  $k$ . In particular, if  $k$  is algebraically closed, and  $T$  is the polynomial algebra, then maximal ideals  $m$  are of the form  $(X_1 - a_1, \dots, X_n - a_n)$ .

PROOF. Let  $m$  be a maximal ideal of  $T$ . Define  $R := T/m$ . This is a field. By Zariski's lemma,  $k \subseteq T/m$  is a finite algebraic extension. If  $k$  is algebraically closed and  $T = k[X_1, \dots, X_n]$ , then this means that the map natural map  $\Phi: k \rightarrow k[X_1, \dots, X_n] \rightarrow k[X_1, \dots, X_n]/m$  is an isomorphism. Let  $a_i := \Phi^{-1}(X_i)$ . Then we have that  $I := (X_1 - a_1, \dots, X_n - a_n) \subseteq \ker \Phi = m$ .

On the other hand the natural map  $k \rightarrow k[X_1, \dots, X_n]/I$  is injective, because the kernel is not trivial and  $k$  is a field, and it is surjective, because every polynomial in the quotient by  $I$  "reduces" to an element of  $k$ , so  $I$  is maximal, so  $I = m$  since  $m \supseteq I$  is a proper ideal.  $\square$

THEOREM 1.20. Let  $k$  be an algebraically closed field, and  $R$  a finitely generated  $k$ -algebra. Then  $N(R) = J(R)$ . Thus if  $I$  is a radical ideal of  $k[X_1, \dots, X_n]$  and  $R = k[X_1, \dots, X_n]/I$  then the intersection of the maximal ideals of  $R$  is 0.

Furthermore, any radical ideal is the intersection of the maximal ideals containing it.

### Minimal and associated primes

LEMMA 1.21. If  $R$  is a noetherian ring, then any ideal  $I$  contains a power of its radical  $\sqrt{I}$ .

For  $I = (0)$ , this means that  $N(R)$  is nilpotent.

PROOF. Since  $R$  is noetherian,  $\sqrt{I}$  is finitely generated, say by  $x_1, \dots, x_n$ . Then we find natural numbers  $m_i$  such that  $x_i^{m_i} \in I$ . If we define  $m := 1 + \sum_{i=1}^n (m_i - 1)$ , then the binomial theorem tells us that elements of the form  $x_1^{r_1} \cdots x_n^{r_n}$  with  $\sum_{i=1}^n r_i = m$  generate the ideal  $\sqrt{I}^m$ . By our choice of  $m$ , for some  $i$  we must have  $r_i \geq m_i$ , so every generator lies in  $I$ , so  $\sqrt{I}^m \subseteq I$ .  $\square$

LEMMA 1.22. If  $R$  is noetherian, then every radical ideal of  $I$  is the intersection of finitely many primes.

PROOF. Let  $\mathcal{S}$  be the set of radical ideals that are not the intersection of finitely many prime ideals. Suppose that  $\mathcal{S}$  is nonempty. Since  $R$  is noetherian,  $\mathcal{S}$  has a maximal member  $I$ . We will show that  $I$  is prime (a contradiction, since  $I$  is not the intersection of finitely many prime ideals).

Indeed, if  $I$  is not prime, then there are ideals  $J'_1, J'_2 \not\subseteq I$  such that  $J'_1 J'_2 \subseteq I$  (indeed we can find principal ideals that work). Defining  $J_1 := J'_1 + I$ ,  $J_2 := J'_2 + I$ , we find that  $I \subsetneq J_i$ , but  $J_1 J_2 \subseteq I$ . Since  $I$  was maximal, we can write

$$\sqrt{J_1} = Q_1 \cap \cdots \cap Q_n, \quad \sqrt{J_2} = Q'_1 \cap \cdots \cap Q'_m,$$

where all  $Q_i, Q'_i$  are prime.

Now define

$$J := \sqrt{J_1} \cap \sqrt{J_2} = Q_1 \cap \cdots \cap Q_n \cap Q'_1 \cap \cdots \cap Q'_m.$$

From the preceding lemma, we obtain  $n_1$  and  $n_2$  such that  $J^{n_1} \subseteq J_1^{n_1} \subseteq J_1$  and  $J^{n_2} \subseteq J_2^{n_2} \subseteq J_2$ . Then we have  $J^{n_1+n_2} \subseteq J_1 J_2 \subseteq I$ . Since  $I \in \mathcal{S}$ ,  $I$  is a radical ideal, which means that  $J \subseteq I$ .

On the other hand,  $I \subseteq J_i \subseteq \sqrt{J_i}$ , so  $I \subseteq J$ .

This means that  $I = J$  is the intersection of finitely many prime ideals, which is a contradiction to  $I \in \mathcal{S}$ .  $\square$

REMARK. If we have written  $\sqrt{I} = p_1 \cap \cdots \cap p_m$  with  $p_i$  prime (as we have just seen is always possible), then we can remove any  $p_i$  from the list if it is a superset



of one of the others. Therefore, we may assume that  $p_i \not\subseteq p_j$  for all pairs  $i \neq j$ . Now if  $p$  is another prime ideal and  $\sqrt{I} \subseteq p$ , then  $p_1 \cdots p_m \subseteq \bigcap p_i = \sqrt{I} \subseteq p$ , some since  $p$  is prime, one of the  $p_i$  must be fully contained in  $p$ .

**DEFINITION 1.23.** The minimal primes  $p$  over an ideal  $I$  of a noetherian ring are those prime ideals such that if  $p'$  is a prime ideal and  $I \subseteq p' \subseteq p$ , then  $p = p'$ .

If  $I$  is radical and we choose  $p_i$  as in the previous remark, then  $p_i$  is a minimal prime: indeed, if  $p'$  is prime such that  $I \subseteq p' \subseteq p_i$ , then by the remark some  $p_j$  satisfies  $p_j \subseteq p' \subseteq p_i$ , but due to the way we chose the  $p_i$  this means that  $i = j$  and  $p' = p_i$ .

**LEMMA 1.24.** Let  $I$  be an ideal of a noetherian ring. Then  $\sqrt{I}$  is the intersection of the minimal primes over  $I$ . Furthermore, there is a finite product of minimal primes over  $I$  that is contained in  $I$ .

**PROOF.** If  $p$  is a prime over  $I$ , then  $\sqrt{I} \subseteq p$  as  $p$  is prime. This implies that the minimal primes over  $I$  are exactly the minimal primes over  $\sqrt{I}$ , so the intersection of the minimal primes over  $I$  is the intersection of the minimal primes over  $\sqrt{I}$ , which is  $\sqrt{I}$  itself.

By a previous remark, we can find minimal primes  $p_1, \dots, p_n$  such that  $p_1 \cdots p_n \subseteq \sqrt{I}$ . Since there is some  $m$  such that  $\sqrt{I}^m \subseteq I$ , we have that  $p_1^m \cdots p_n^m \subseteq I$  as required.  $\square$

**EXAMPLE.** Recall that the Nullstellensatz gives a bijection between radical ideals  $\mathbb{C}[X_1, \dots, X_n]$  and algebraic subsets of  $\mathbb{C}^n$ .

If  $I$  is a radical ideal of  $\mathbb{C}[X_1, \dots, X_n]$ , then  $(a_1, \dots, a_n)$  is a common zero of all  $f \in I$  if and only if  $I \subseteq (X_1 - a_1, \dots, X_n - a_n)$ <sup>1</sup>. Consider the ideal

$$J := \bigcap_{(a_1, \dots, a_n) \in V(I)} (X_1 - a_1, \dots, X_n - a_n),$$

This is a radical ideal (TODO: why?). The bijection in the Nullstellensatz tells us that  $I = J$ . Therefore, we may write any radical ideal as the intersection of maximal ideals it is contained in, which are all of the form  $(X_1 - a_1, \dots, X_n - a_n)$  (as we already know).

Furthermore, Hilbert's Nullstellensatz tells us that if  $J \subseteq \mathbb{C}[X_1, \dots, X_n]$  is an ideal, then  $N(\mathbb{C}[X_1, \dots, X_n]/J) = J(\mathbb{C}[X_1, \dots, X_n]/J)$ .

**DEFINITION 1.25.** Let  $R$  be a noetherian ring and let  $M$  be a finitely generated  $R$ -module. We call a prime ideal  $p$  an associated prime of  $M$  if it is the annihilator of an element of  $M$ , i.e., there is  $m \in M$  such that  $p = \text{ann}(m) = \{r \in R \mid rm = 0\}$ .

We further define

$$\text{Ass}(M) := \{p \mid p \text{ prime}, \exists m \in M : p = \text{ann}(m)\}.$$

**EXAMPLE.** If  $p$  is a prime ideal of  $R$ , then  $\text{Ass}(R/p) = \{p\}$ . Indeed, if  $r \in R$ , then there are two cases. If  $r \in p$ , then  $\text{ann}(r+p) = \text{ann}(0) = R$ , which is not prime. Otherwise, if  $r \notin p$ , then if  $0+p = (s+p)(r+p)$ , we have  $rs \in p$ , and since  $p$  is prime and  $r \notin p$ , we have  $s \in p$ . Conversely,  $p$  is trivially contained in the annihilator, and we conclude that  $\text{ann}(r) = p$ .

**DEFINITION 1.26.** If  $M$  is an  $R$ -module, then we call a submodule  $N$  of  $M$   $p$ -primary (or just primary) if  $\text{Ass}(M/N) = \{p\}$  for a prime ideal  $p$ . Since ideals are just submodules, the definition extends to ideals.

<sup>1</sup>Indeed, if  $\{(a_1, \dots, a_n)\} \subseteq V(I)$ , then  $I = \sqrt{I} = I(V(I)) \subseteq I(\{(a_1, \dots, a_n)\}) = (X_1 - a_1, \dots, X_n - a_n)$ . Conversely, if  $I \subseteq (X_1 - a_1, \dots, X_n - a_n)$ , then  $\{(a_1, \dots, a_n)\} \subseteq V(I)$ . To see that  $I(\{(a_1, \dots, a_n)\}) = (X_1 - a_1, \dots, X_n - a_n)$ , note that " $\supseteq$ " is clear, but the latter is maximal as we have seen before.

LEMMA 1.27. If  $\text{ann}(M) := \bigcap_{m \in M} \text{ann}(m) = p$  for some prime ideal  $p$ , then we have  $p \in \text{Ass}(M)$ .

PROOF. Suppose  $M$  is generated by  $m_1, \dots, m_k$ . Define  $I_j := \text{ann}(m_j)$ . Then

$$\prod I_j \subseteq \bigcap I_j = \bigcap \text{ann}(m_j) = \text{ann}(M) = p.$$

Since  $p$  is prime, this forces  $I_j \subseteq p$ , but  $p = \text{ann}(M) \subseteq \text{ann}(m_j) = I_j$ , so  $p = I_j$ , hence  $p \in \text{Ass}(M)$ .  $\square$

LEMMA 1.28. Let  $Q$  be maximal amongst the annihilators of nonzero elements of  $M$ . Then  $Q$  is prime, hence  $Q \in \text{Ass}(M)$ .

PROOF. Let  $Q \in \text{ann}(m)$  and  $r_1 \cdot r_2 \in Q$ , but  $q_2 \notin Q$ . We will show that  $r_1 \in Q$ . Since  $r_1 r_2 \in Q$  we have  $r_1 r_2 m = 0$ . This means that  $r_1 \in \text{ann}(r_2 m)$ . Since,  $r_2 \notin Q$ , we have that  $r_2 m \neq 0$ .

We have  $Q = \text{ann}(m) \subseteq \text{ann}(r_2 m)$ , and since  $r_2 m$  is nonzero as we have just seen, by maximality of  $Q$ , we have  $Q = \text{ann}(r_2 m)$ . Hence,  $r_1 \in \text{ann}(r_2 m) = Q$  as required.  $\square$

LEMMA 1.29. Let  $M$  be a nonzero finitely generated module over a noetherian ring  $R$ . Then there is a chain

$$0 \subsetneq M_1 \subsetneq M_2 \subsetneq \dots \subsetneq M_t = M$$

of submodules with  $M_i/M_{i-1} \cong R/p_i$  for some prime ideal  $p_i$ .

PROOF. By the previous lemma we find  $0 \neq m_1 \in M$  such that  $\text{ann}(m_1)$  is a prime ideal. Set  $M_1 = Rm_1$ . Then the kernel of the map  $R \rightarrow M_1$  given by  $r \mapsto rm_1$  is precisely  $\text{ann}(m_1)$ , so  $M_1 \cong R/p_1$  (as  $R$ -modules).

Similarly, if  $M_i$  is a proper submodule of  $M$ , then we find  $m_{i+1} + M_i \in M/M_i$  such that  $\text{ann}(m_{i+1} + M_i)$  is a prime ideal. Set  $M_{i+1} := M_i + Rm_{i+1}$ . Then the map  $R \rightarrow M_{i+1}/M_i$  given by  $r \mapsto rm_{i+1} + M_i$  is surjective and has kernel  $\text{ann}(m_{i+1} + M_i)$ . Furthermore,  $m_{i+1} \notin M_i$ , since otherwise the annihilator of  $m_{i+1} + M_i$  would be all of  $R$ . Therefore,  $M_i$  is a proper submodule of  $M_{i+1}$ .

By the ascending chain condition, this process terminates.  $\square$

LEMMA 1.30. If  $N$  is a submodule of a finitely generated module  $M$  over a noetherian ring  $R$ , then  $\text{Ass}(M) \subseteq \text{Ass}(N) \cup \text{Ass}(M/N)$ .

PROOF. Let  $\text{ann}(m) \in \text{Ass}(M)$  for some  $m \in M$ . Define  $M_1 := Rm \cong R/\text{ann}(m)$ .

Let  $rm \in M_1$ . It is trivial that  $\text{ann}(m) \subseteq \text{ann}(rm)$ . Conversely, if  $s \in \text{ann}(rm)$ , then  $srm = 0$ , but  $\text{ann}(m)$  is prime and  $rm \neq 0$ , so we must have  $s \in \text{ann}(m)$ . Hence  $\text{ann}(rm) = \text{ann}(m)$ .

Now if  $M_1 \cap N \neq 0$ , then by what we just saw there is  $rm \in M_1 \cap N$  with  $\text{ann}(rm) = \text{ann}(m)$ , so  $\text{ann}(m) \in \text{Ass}(N)$ .

On the other hand, if  $M_1 \cap N = 0$ , then  $r \in \text{ann}(m + N)$  iff  $r \cdot m \in N$  iff  $r \cdot m = 0$ , so  $\text{ann}(m) = \text{ann}(m + N) \in \text{Ass}(M/N)$ .  $\square$

LEMMA 1.31. If  $R$  is a noetherian ring and  $M$  is finitely generated, then  $\text{Ass}(M)$  is finite.

PROOF. Take a chain

$$M_0 = 0 \subsetneq M_1 \subsetneq M_2 \subsetneq \dots \subsetneq M_t = M.$$

such that  $M_{i+1}/M_i \cong R/p_i$  for  $i \geq 0$ .

We will show inductively that  $M_{i+1} = 0$  for  $i \geq 0$ . Indeed, if  $i = 0$ , then  $M_i \cong R/p_0$ , and we have previously calculated that  $\text{Ass}(R/p_0) = \{p_0\}$ .

If  $i > 0$ , then  $M_i$  is a submodule of  $M_{i+1}$ . By the previous lemma, we have  $\text{Ass}(M_{i+1}) \subseteq \text{Ass}(M_i) \cup \text{Ass}(M_{i+1}/M_i)$ . The former is finite by the inductive hypothesis, while the latter is a one-element set.  $\square$

PROPOSITION 1.32. Each minimal prime over an ideal  $I$  is an associated prime of  $R/I$ .

PROOF. By (1.24), we find minimal primes  $p_1, \dots, p_n$  and natural numbers  $s_1, \dots, s_n$  such that  $p_1^{s_1} \cdots p_n^{s_n} \subseteq I$ . Additionally, we may assume that  $i \neq j$  implies  $p_i \neq p_j$ .

Define

$$M := (p_2^{s_2} \cdots p_n^{s_n} + I)/I$$

and let  $J := \text{ann}(M)$ . Clearly, every element of  $p_1^{s_1}$  annihilates  $M$ , so  $p_1^{s_1} \subseteq J$ . Furthermore, we have

$$Jp_2^{s_2} \cdots p_n^{s_n} \subseteq I \subseteq p_1,$$

but  $p_1$  is prime and we cannot have  $p_i^{s_i} \subseteq p_1$  for  $i \neq 1$  as the  $p_i$  are minimal primes, so we must have  $J \subseteq p_1$ . In particular,  $J \neq R$ , so  $M \neq 0$ .

Invoke (1.29) to obtain a chain

$$0 \subsetneq M_1 \subsetneq M_2 \subsetneq \cdots \subsetneq M_t = M$$

of submodules with  $M_i/M_{i-1} \cong R/q_i$  for some prime ideal  $q_i$ .

Since  $p_1^{s_1}$  annihilates  $M$ , in particular it annihilates  $M_j/M_{j-1}$  for every  $j$ . So we have  $p_1^{s_1} \subseteq \text{ann}(M_j/M_{j-1}) = \text{ann}(R/q_j) = q_j$  for every  $j$ . Since  $q_j$  is prime, we conclude  $p_1 \subseteq q_j$  for every  $j$ .

On the other hand,  $\prod q_j \subseteq J$ : by induction on  $j$  assume that  $\prod_{k=1}^j q_k$  annihilates  $M_j$ . If  $x \in M_{j+1}$  and  $r \in \prod_{k=1}^j q_k$ , and  $s \in q_{j+1}$  then  $sx \in M_j$ , since  $q_{j+1}$  annihilates  $M_{j+1}/M_j$ . By the inductive hypothesis,  $rsx = 0$ , so  $\prod_{k=1}^{j+1} q_k$  annihilates  $M_{j+1}$ .

Hence  $\prod q_j \subseteq J \subseteq p_1$ , so there is some  $j$  such that  $q_j \subseteq p_1$ , but we have seen that  $p_1 \subseteq q_j$ , so there is  $j$  such that  $q_j = p_1$ . Let  $j$  be the least such  $j$ . In particular,  $\prod_{k < j} q_k \subsetneq p_1$ .

We will now show that  $p_1 \in \text{Ass}(M)$ . For this, take  $x \in M_j \setminus M_{j-1}$ . If  $j = 1$ , then  $\text{ann}(x) = p_1$  (since  $M_1 \cong R/p_1$ ), but  $x \in M \subseteq R/I$ , so  $p_1 \in \text{Ass}(R/I)$ .

On the other hand, if  $j > 1$ , choose some  $r \in (\prod_{k < j} q_k) \setminus p_1$  (this is indeed nonempty, since otherwise one of the  $q_k$  would be contained in  $p_1$ ). Note that if  $s \in p_1 = q_j$ , then  $r(sx) = 0$  (this is just the induction we did earlier). So we have  $s(rx) = 0$ , which means that we have  $p_1 \subseteq \text{ann}(rx)$ .

Note that  $\text{ann}(rx + M_{j-1}) = p_1$  since  $rx + M_{j-1} \neq 0$ , but  $M_j/M_{j-1} \cong R/q_j = R/p_1$ . Since  $r \notin p_1$ , we conclude that  $rx \notin M_{j-1}$ . Now if  $s \in \text{ann}(rx)$ , then certainly  $s \in \text{ann}(rx + M_{j-1}) = p_1$ , so  $\text{ann}(rx) \subseteq p_1$ .

Putting the last two paragraphs together, we have  $\text{ann}(rx) = p_1$ , so  $p_1 \in \text{Ass}(M) \subseteq \text{Ass}(R/I)$ .

By changing the order of the  $p_i$ , we see that  $p_j \in \text{Ass}(R/I)$  for every  $j$ , completing the proof.  $\square$

EXAMPLE 1.33. The converse of the previous theorem fails in general. For example, take  $R = k[X, Y]$ ,  $p = (X, Y) > q = (X)$  and  $I = pq = (X^2, XY)$ .

We have  $\sqrt{I} = q$ . Since this is a prime, (1.24) tells us that  $q$  is the only minimal prime over  $q$ . It is possible to show that  $\text{Ass}(R/I) = \{p, q\}$ . In particular,  $I$  is not primary, but we can write

$$I = (X^2, XY, Y^2) \cap (X),$$

where  $(X^2, XY, Y^2) = (X, Y)^2$  is  $p$ -primary and  $(X)$  is  $q$ -primary. This is an example of a primary decomposition.

DEFINITION 1.34. If  $R$  is a noetherian ring,  $M$  is a finitely generated  $R$ -module and  $N \subseteq M$  is a submodule, then a primary decomposition of  $N$  consists of submodules  $N_1, \dots, N_s$  of  $M$  containing  $N$  such that  $N_i$  is  $p_i$ -primary, where the  $p_i$  are pairwise distinct, such that  $N = \bigcup_{i=1}^n N_i$  (in particular, this means that there is an embedding  $M/N \rightarrow \bigoplus M/N_i$ ).

REMARK. This primary decomposition exists (which we will not show) and is not necessarily unique. However, Atiyah-Macdonald Chapter 4 contains two uniqueness theorems for finitely generated modules over noetherian rings:

- (1) the  $p_i$  occurring in a primary decomposition are unique and are precisely  $\text{Ass}(M/N)$ ;
- (2) the  $N_j$  belonging to  $p_j$  which are minimal elements of the set  $\{p_i\}$  are unique. The  $N_j$  belonging to the rest of the  $p_j$  (which are called embedded), are not necessarily unique.

In the previous example,  $q$  is minimal and  $p$  is embedded, Hence, the ideal  $(X)$  is unique and the decomposition shows that  $\text{Ass}(R/I) = \{p, q\}$ , which is rather tricky to prove from first principles.

## CHAPTER 2

### Localisation

REMARK. As always, all rings  $R$  are commutative with unity.

Let  $S$  be a multiplicatively closed subset of  $R$  (i.e.,  $S$  is closed under multiplication and  $1 \in S$ ). We define a relation  $\equiv$  on  $R \times S$  by saying that  $(r_1, s_1) \equiv (r_2, s_2) \iff \exists x \in S: (r_1 s_2 - r_2 s_1)x = 0$ . Reflexivity and symmetry are immediate, for transitivity, assume that

$$(r_1 s_2 - r_2 s_1)x = 0 = (r_2 s_3 - r_3 s_2)y.$$

Multiplying the left hand side with  $s_3 y$  and the right hand side with  $s_1 x$  and the subtracting the two yields the desired identity

$$(r_1 s_3 - r_3 s_1)s_2 xy = 0,$$

since  $s_2 xy \in S$ .

This shows that  $\equiv$  is an equivalence relation, and we will denote equivalence classes of  $(r_1, s_1)$  by  $\frac{r_1}{s_1}$  and the quotient by  $S^{-1}R$ . We make  $S^{-1}R$  into a ring by setting

$$\begin{aligned} \frac{r_1}{s_1} + \frac{r_2}{s_2} &:= \frac{r_1 s_2 + r_2 s_1}{s_1 s_2}, \\ \frac{r_1}{s_1} \cdot \frac{r_2}{s_2} &:= \frac{r_1 r_2}{s_1 s_2}. \end{aligned}$$

Furthermore, we have a ring homomorphism  $R \rightarrow S^{-1}R$  given by  $r \mapsto \frac{r}{1}$ .

LEMMA 2.1. Let  $\varphi: R \rightarrow T$  be a ring homomorphism with  $\varphi(s)$  a unit in  $T$  for all  $s \in S$ . Then there is a unique homomorphism of rings  $\alpha: S^{-1}R \rightarrow T$  such that  $\varphi = \alpha \circ \theta$ , i.e., the diagram

$$\begin{array}{ccc} R & \xrightarrow{\theta} & S^{-1}R \\ & \searrow \varphi & \downarrow \exists! \alpha \\ & & T \end{array}$$

is commutative.

PROOF. We will first show uniqueness. Suppose we have  $\alpha: S^{-1}R \rightarrow T$  satisfying  $\alpha \circ \theta = \varphi$ .

Then we have

$$\forall r \in R: \alpha\left(\frac{r}{1}\right) = \alpha(\theta(r)) = \varphi(r),$$

$$\forall s \in S: \alpha\left(\left(\frac{s}{1}\right)^{-1}\right) = \alpha\left(\frac{s}{1}\right)^{-1} = \alpha(\theta(s))^{-1} = \varphi(s)^{-1}.$$

Thus,  $\alpha\left(\frac{r}{s}\right) = \alpha\left(\frac{r}{1}\right)\alpha\left(\frac{1}{s}\right) = \varphi(r)\varphi(s)^{-1}$  is uniquely determined by  $\varphi$ .

For existence, we define  $\alpha\left(\frac{r}{s}\right) := \varphi(r)\varphi(s)^{-1}$ . We need to show that this is well-defined. If  $\frac{r_1}{s_1} = \frac{r_2}{s_2}$ , then we find  $x \in S$  such that  $(r_1 s_2 - r_2 s_1)x = 0$ . Applying  $\varphi$ , we find  $(\varphi(r_1)\varphi(s_2) - \varphi(r_2)\varphi(s_1))\varphi(x) = 0$ . Since  $\varphi(x)$  is a unit, we can cancel it and since the  $\varphi(s_i)$  are units, we can rewrite this two the required relation  $\varphi(r_1)\varphi(s_1)^{-1} = \varphi(r_2)\varphi(s_2)^{-1}$ .

It is also possible to check that  $\alpha$  is indeed a homomorphism of rings.  $\square$

- EXAMPLE. (1) If  $R$  is an integral domain and  $S = R \setminus \{0\}$ , then  $S^{-1}R$  is just the field of fractions of  $R$ .
- (2) We have that  $S^{-1}R$  is the zero ring if and only if  $0 \in S$ .
- (3) If  $I$  is an ideal of  $R$ , then  $S = 1 + I$  is multiplicatively closed.
- (4) Let  $p$  be a prime ideal. Then  $S = R \setminus p$  is multiplicatively closed (indeed, if  $x, y \in S$ , then if  $xy \in R \setminus S = p$ , then  $x \in p = R \setminus S$  or  $y \in p = R \setminus S$ , which is not possible). We write  $R_p$  for  $S^{-1}R$ , and the process of passing from  $R$  to  $R_p$  is called localisation at  $p$ . The elements  $\frac{r}{s}$  with  $r \in p$  form an ideal of  $R_p$ . This is a unique maximal ideal in  $R_p$ : if  $\frac{r}{s}$  satisfies  $r \notin p$ , then  $r \in S$ , so  $\frac{r}{s}$  has an inverse in  $R_p$  and is not part of any maximal ideal.

DEFINITION 2.2. A ring with a unique maximal ideal is called local.

REMARK. Some authors require a local ring to also be noetherian. We do not.

- EXAMPLE. (1) Let  $R = \mathbb{Z}$ , and  $p$  prime number. Then  $(p)$  is a prime ideal, and we have  $R_{(p)} = \{\frac{m}{n} \mid p \nmid n\} \subseteq \mathbb{Q}$ .  
The maximal ideal is given by  $\{\frac{m}{n} \mid p \mid m, p \nmid n\}$ .
- (2) Let  $R = k[X_1, \dots, X_n]$ ,  $p = (X_1 - \alpha_1, \dots, X_n - \alpha_n)$ . Then we can interpret  $R_p$  as a subring of  $k(X_1, \dots, X_n)$  consisting of those rational functions that are defined at  $(\alpha_1, \dots, \alpha_n) \in k^n$ , and the unique maximal ideal consists of those rational functions which are zero at  $(\alpha_1, \dots, \alpha_n)$ .

### 1. Localization of modules

DEFINITION. Given a left  $R$ -module  $M$ , define a relation  $\equiv$  on  $M \times S$ , where  $S$  is a multiplicatively closed subset  $S \subseteq R$  by

$$(m_1, s_1) \equiv (m_2, s_2) \iff \exists x \in S: x(m_1 s_2 - m_2 s_1) = 0.$$

This is again an equivalence relation with  $\frac{m}{s}$  denoting the equivalence class of  $(m, s)$ . The quotient is denoted by  $S^{-1}M$ .  $S^{-1}M$  has the structure of an  $S^{-1}R$ -module via

$$\begin{aligned} \frac{m_1}{s_1} + \frac{m_2}{s_2} &:= \frac{s_2 m_1 + s_1 m_2}{s_1 s_2} \\ \frac{r_1}{s_1} \frac{m_2}{s_2} &= \frac{r_1 m_2}{s_1 s_2}. \end{aligned}$$

Again, we write  $M_p$  in the case  $S = R \setminus p$  for a prime ideal  $p$ .

If  $\theta: M_1 \rightarrow M_2$  is an  $R$ -linear map, then an  $S^{-1}R$ -linear map  $S^{-1}\theta: S^{-1}M_1 \rightarrow S^{-1}M_2$  is given by  $\frac{m_1}{s} \mapsto \frac{\theta(m_1)}{s}$ . This is functorial in the sense that if  $\varphi: M_2 \rightarrow M_3$  is another  $R$ -linear map then  $S^{-1}(\varphi \circ \theta) = S^{-1}\varphi \circ S^{-1}\theta$ .

LEMMA 2.3. If

$$M_1 \xrightarrow{\theta} M \xrightarrow{\varphi} M_2$$

is exact at  $M$ , then

$$S^{-1}M_1 \xrightarrow{S^{-1}\theta} S^{-1}M \xrightarrow{S^{-1}\varphi} S^{-1}M_2$$

is exact at  $S^{-1}M$ .

PROOF. By functoriality, we have

$$(S^{-1}\varphi) \circ (S^{-1}\theta) = S^{-1}(\varphi \circ \theta) = S^{-1}0 = 0,$$

hence  $\text{im}(S^{-1}\theta) \subseteq \ker(S^{-1}\varphi)$ .

Now suppose  $\frac{m}{s} \in \ker(S^{-1}\varphi) \subseteq S^{-1}M$ . This means that  $\frac{\varphi(m)}{s} = 0$  in  $S^{-1}M_2$ . By definition of localization, this means that there is  $t \in S$  such that  $t\varphi(m) = 0$  in

$M_2$ . By linearity,  $0 = t\varphi(m) = \varphi(tm)$ , hence  $tm \in \ker \varphi = \text{im } \theta$ , so we find  $m_1 \in M_1$  such that  $\theta(m_1) = tm$ . Then we can calculate in  $S^{-1}M$  that

$$\frac{m}{s} = \frac{tm}{ts} = \frac{\theta(m_1)}{ts} = (S^{-1}\theta)\left(\frac{m_1}{ts}\right),$$

hence  $\frac{m}{s} \in \text{im } S^{-1}\theta$ , and we conclude that  $\ker S^{-1}\varphi = \text{im } S^{-1}\theta$  as claimed.  $\square$

REMARK. If  $N \subseteq M$  is a submodule, then  $S^{-1}N \subseteq S^{-1}M$  is a submodule in the natural way. In particular, if  $I \subseteq R$  is an ideal, then  $S^{-1}I$  is an ideal of  $S^{-1}R$ .

LEMMA 2.4. Let  $N \subseteq M$  be a submodule. Then  $S^{-1}(M/N) \cong S^{-1}M/S^{-1}N$ .

PROOF. Applying the previous lemma to the short exact sequence

$$0 \longrightarrow N \xrightarrow{\iota} M \xrightarrow{\varepsilon} M/N \longrightarrow 0$$

yields exactness of

$$0 \longrightarrow S^{-1}N \xrightarrow{S^{-1}\iota} S^{-1}M \xrightarrow{S^{-1}\varepsilon} S^{-1}(M/N) \longrightarrow 0.$$

Since  $S^{-1}\iota$  is just the inclusion  $S^{-1}N \subseteq S^{-1}M$ , we find that  $S^{-1}(M/N) \cong S^{-1}M/S^{-1}N$ .  $\square$

LEMMA 2.5. (i) Every ideal in  $S^{-1}R$  is of the form  $S^{-1}I$  for some ideal  $I$  of  $R$ .

(ii) The prime ideals of  $S^{-1}R$  are in one-to-one correspondence with the prime ideals of  $R$  that do not meet  $S$ .

PROOF. For the first part, let  $J$  be an ideal of  $S^{-1}R$  and define  $I := \{r \in R \mid \frac{r}{1} \in J\}$ . This is clearly an ideal. Now if  $\frac{r}{s} \in J$ , then  $\frac{r}{1} = \frac{s}{1} \frac{r}{s} \in J$ , hence  $r \in I$ , so  $\frac{r}{s} \in S^{-1}I$  and  $J \subseteq S^{-1}I$ .

Conversely, if  $\frac{r}{s} \in S^{-1}I$ , i.e.,  $r \in I$  and  $s \in S$ , then  $\frac{r}{1} \in J$ , so  $\frac{r}{s} = \frac{1}{s} \frac{r}{1} \in J$ .

Hence,  $S^{-1}I = J$ , completing the first part.

Let  $q$  be a prime ideal of  $S^{-1}R$  and set  $p := \{r \in R \mid \frac{r}{1} \in q\}$ . By the previous part,  $p$  is an ideal and  $q = S^{-1}p$ .

The ideal  $p$  is prime, since if  $xy \in p$ , then  $\frac{xy}{1} = \frac{x}{1} \frac{y}{1} \in q$ , so either  $\frac{x}{1}$  or  $\frac{y}{1}$  is in  $q$ , hence,  $x \in p$  or  $y \in p$ .

Furthermore, we have  $p \cap S = \emptyset$ , since if  $r \in S \cap p$ , then  $\frac{r}{1} \in q$  by definition of  $p$  and  $\frac{1}{r}$  is valid element of  $S^{-1}R$ , so  $1 = \frac{1}{r} \frac{r}{1} \in q$  since  $q$  is an ideal, but  $q$  is prime, so  $1 \notin q$ , a contradiction.

Conversely, let  $p$  be a prime ideal of  $R$  that does not meet  $S$ . If  $\frac{r}{s}, \frac{x}{y} \in S^{-1}R$  such that  $\frac{rx}{sy} \in S^{-1}p$ , then by definition of localisation we have  $zrx \in p$  for some  $z \in S$ . Since  $z \in S$ , we have  $z \notin p$ , so since  $p$  is prime, we must have  $rx \in p$ . Again since  $p$  is prime, we find that  $r \in p$  or  $x \in p$ , so  $\frac{r}{s} \in S^{-1}p$  or  $\frac{x}{y} \in S^{-1}p$ , so  $S^{-1}p$  is prime.

Hence, the mappings  $p \mapsto S^{-1}p$  and  $q \mapsto \{r \in R \mid \frac{r}{1} \in q\}$  are inverse bijections (one half is given by the first part of the proof, the other half is obvious) that preserve primality in both directions.  $\square$

LEMMA 2.6. If  $R$  is noetherian, then  $S^{-1}R$  is noetherian.

PROOF. Using the previous lemma, a chain  $J_1 \subseteq J_2 \subseteq \dots$  in  $S^{-1}R$  lifts to a chain  $I_1 \subseteq I_2 \subseteq \dots$  in  $R$  such that  $J_i = S^{-1}I_i$  for each  $i$ . Since  $R$  is noetherian, the chain  $\{I_i\}$  terminates, so the chain  $\{J_i\} = \{S^{-1}I_i\}$  must terminate as well.  $\square$

DEFINITION 2.7. A property  $P$  of a ring  $R$  or  $R$ -module  $M$  is called local if  $R$  or  $M$  has the property  $P$  if and only if  $R_p$  (resp.  $M_p$ ) has property  $P$  for each prime ideal  $p$  of  $R$ .

LEMMA 2.8. The following are equivalent for an  $R$ -module  $M$ .

- (i)  $M = 0$ ,
- (ii) for all prime ideals  $p$ , we have  $M_p = 0$ ,
- (iii) for all maximal ideals  $q$ , we have  $M_q = 0$ .

PROOF. It is obvious that (i) implies (ii) and (ii) implies (iii), so it will suffice to show that (iii) implies (i). Indeed, suppose that  $M_q = 0$  for every maximal ideal  $q$ , but  $M \neq 0$ .

Let  $0 \neq m \in M$ . The annihilator  $\{r \in R \mid rm = 0\}$  of  $m$  is a proper ideal of  $R$ , hence it is contained in a maximal ideal  $q$  of  $R$ . Since  $M_q$  is trivial, we have  $\frac{m}{1} = 0$  in  $M_q$ , so there is some  $s \in R \setminus q$  such that  $sm = 0$  in  $R$ . But since  $s \notin q$ , we have  $s \notin \text{ann}(m)$ , i.e.,  $sm \neq 0$ , a contradiction.  $\square$

LEMMA 2.9. Let  $\varphi: M \rightarrow N$  be a homomorphism of  $R$ -modules. The following are equivalent.

- (i)  $\varphi$  is injective,
- (ii)  $\varphi_p: M_p \rightarrow N_p$  is injective for all primes  $p$  of  $R$ ,
- (iii)  $\varphi_q: M_q \rightarrow N_q$  is injective for all maximal ideals  $q$  of  $R$ .

PROOF. (i) implies (ii) by exactness of localization. It is obvious that (ii) implies (iii).

Now assume that  $\varphi_q$  is injective for all maximal ideals  $q$ . The sequence

$$0 \longrightarrow \ker \varphi \xrightarrow{\iota} M \xrightarrow{\varphi} N$$

is exact. Hence

$$0 \longrightarrow (\ker \varphi)_q \xrightarrow{\iota_q} M_q \xrightarrow{\varphi_q} N_q$$

is exact for every maximal ideal  $q$ . By exactness,  $(\ker \varphi)_q$  is isomorphic to  $\ker \varphi_q$ , which is trivial by assumption. Hence  $(\ker \varphi)_q$  is trivial for every maximal ideal  $q$ , so by the previous result, we have  $\ker \varphi = 0$  as required.  $\square$

LEMMA 2.10. Let  $p$  be a prime ideal of  $R$  and  $S$  a multiplicatively closed subset of  $R$  such that  $S \cap p = \emptyset$ . By (2.5),  $S^{-1}p$  is a prime ideal of  $S^{-1}R$ . Then  $(S^{-1}R)_{S^{-1}p} \cong R_p$ . In particular, if  $q$  is a prime ideal of  $R$  with  $p \subseteq q$ , then  $(R_q)_{p_q} \cong R_p$ , by taking  $S = R \setminus q$ .

PROOF. On the second exercise sheet.  $\square$

## 2. A proof of the Nullstellensatz

REMARK. Let  $k$  be a field and  $R$  a  $k$ -algebra which is also an integral domain. If  $R$  is a finite-dimensional  $k$ -vector space, then  $R$  is a field: indeed, if  $0 \neq r \in R$ , then multiplication by  $r$  is a  $k$ -linear map. Since  $R$  is an integral domain, this map is injective, and since  $R$  is finite-dimensional, every injective map is surjective. Hence we find an inverse of  $r$ .

THEOREM. Let  $k$  be an algebraically closed field, and  $R$  a finitely generated  $k$ -algebra. Then  $N(R) = J(R)$ . Thus if  $I$  is a radical ideal of  $k[X_1, \dots, X_n]$  and  $R = k[X_1, \dots, X_n]/I$  then the intersection of the maximal ideals of  $R$  is 0.

Furthermore, any radical ideal is the intersection of the maximal ideals containing it.

PROOF. Let  $p$  be any prime ideal of  $R$  and let  $s \in R \setminus p$ . The set  $S := \{1, s, s^2, \dots\}$  is multiplicative, so we get a localization  $S^{-1}R$  and a map  $\theta: R \rightarrow S^{-1}R$ .  $R$  is a finitely generated  $k$ -algebra and  $S^{-1}R$  generated as a  $k$ -algebra by  $\theta(R)$  and  $1/s$ . Hence  $S^{-1}R$  is a finitely generated  $k$ -algebra. Let  $q$  be a maximal



ideal of  $S^{-1}R$  containing  $S^{-1}p$ . By the weak Nullstellensatz,  $S^{-1}R/q$  is a finite field extension of  $k$ .

The ideal  $p_1 := \theta^{-1}(q)$  is a prime ideal containing  $p$ , and by the correspondence of prime ideals we know that  $p_1$  does not meet  $S$ . Hence  $\theta$  induces an injective  $k$ -vector space homomorphism  $R/p_1 \rightarrow S^{-1}R/q$ . Since  $S^{-1}R/q$  is finite-dimensional, this implies that  $R/p_1$  is also finite-dimensional.

By the remark, this implies that  $R/p_1$  (which is an integral domain since  $p_1$  is prime) is a field, hence  $p_1$  is a maximal ideal. Hence, for any  $s \notin p$ , we find a maximal ideal containing  $p$  but not containing  $s$ , i.e.,

$$R \setminus p \subseteq \bigcup \{\text{complements of maximal ideals containing } p\}.$$

By elementary set theory, this means that

$$\bigcap \{\text{maximal ideals containing } p\} \subseteq p.$$

Since the converse inclusion is trivial, we have that  $p$  is the intersection of maximal ideals containing  $p$ . Hence the intersection of all primes is the same as the intersection of all maximals, which is what we wanted to show.  $\square$



## CHAPTER 3

### Tensor products

DEFINITION 3.1. If  $L, M, N$  are  $R$ -modules, then a function  $\varphi: M \times N \rightarrow L$  is called  $R$ -bilinear if

$$\begin{aligned}\varphi(r_1 m_1 + r_2 m_2, n) &= r_1 \varphi(m_1, n) + r_2 \varphi(m_2, n), \\ \varphi(m, r_1 n_1 + r_2 n_2) &= r_1 \varphi(m, n_1) + r_2 \varphi(m, n_2).\end{aligned}$$

REMARK. The idea is to reduce the study of bilinear maps to the of linear (i.e,  $R$ -module) maps.

If  $\varphi: M \times N \rightarrow T$  is bilinear and  $\theta: T \rightarrow L$  is linear, then  $\theta \circ \varphi$  is bilinear. Composition with  $\varphi$  gives a well defined function  $\varphi^*$  from  $R$ -linear maps  $T \rightarrow L$  to bilinear maps  $M \times N \rightarrow L$ .

$$\begin{array}{ccc} M \times N & \xrightarrow{\varphi} & T \\ & \searrow \varphi^*(\theta) & \downarrow \theta \\ & & L \end{array}$$

We say that  $\varphi$  is universal if  $\varphi^*$  is a bijection for every  $L$ . If this happens that study of bilinear maps  $M \times N \rightarrow L$  is reduced to the study if linear maps  $T \rightarrow L$ .

- LEMMA 3.2. (i) Given  $R$ -modules  $M, N$ , there is an  $R$ -module  $T$  and a universal map  $\varphi: M \times N \rightarrow T$ .  
(ii) Given two universal maps  $\varphi_1: M \times N \rightarrow T_1$ ,  $\varphi_2: M \times N \rightarrow T_2$ , there is a unique isomorphism  $\beta: T_1 \rightarrow T_2$  such that  $\varphi_2 = \beta \circ \varphi_1$ .

PROOF. Let  $F$  be the free  $R$ -module on the generators  $e_{(m,n)}$  indexed by pairs  $(m, n) \in M \times N$ . Let  $X$  be the  $R$ -submodule generated by all elements of the forms

$$e_{(r_1 m_1 + r_2 m_2, n)} - r_1 e_{(m_1, n)} - r_2 e_{(m_2, n)}, \quad e_{(m, r_1 n_1 + r_2 n_2)} - r_1 e_{(m, n_1)} - r_2 e_{(m, n_2)}.$$

Define  $T := F/X$  and write  $m \otimes n$  for the image of the basis element  $e_{(m,n)}$  in  $T$ .  $T$  is generated by elements of the form  $m \otimes n$ , and we have the relations

$$\begin{aligned}(r_1 m_1 + r_2 m_2) \otimes n &= r_1 (m_1 \otimes n) + r_2 (m_2 \otimes n) \\ m \otimes (r_1 n_1 + r_2 n_2) &= r_1 (m \otimes n_1) + r_2 (m \otimes n_2).\end{aligned}$$

Define  $\varphi: M \times N \rightarrow T$  via  $(m, n) \mapsto m \otimes n$  and note that  $\varphi$  is bilinear. Any map  $\alpha: M \times N \rightarrow L$  extends to a map of  $R$ -modules  $\bar{\alpha}: F \rightarrow L$  by sending  $e_{(m,n)} \mapsto \alpha(m, n)$ . If  $\alpha$  is bilinear then  $\bar{\alpha}$  vanishes on the generators of  $X$ , hence it induces a map of  $R$ -modules  $\alpha': T \rightarrow L$  such that  $\alpha'(m \otimes n) = \alpha(m, n)$ , and  $\alpha'$  is uniquely determined by these relations. Hence  $\varphi$  is universal.

The proof of uniqueness is just the usual dance with universal properties.  $\square$

DEFINITION 3.3. The module  $T$  is usually denoted  $M \otimes_R N$  and is called the tensor product of  $M$  and  $N$  over  $R$ .

REMARK.  $\bullet$  We often drop the subscript  $R$  if it is clear what ring we are using.

- Not all elements of  $M \otimes_R N$  are of the form  $m \otimes n$ . A general element is of the form  $\sum_{i=1}^r m_i \otimes n_i$ .
- If  $R = k$  is a field and  $k^s, k^t$  are finite-dimensional vector spaces over  $k$ , then the map  $M \times N \rightarrow k^{st}$  given by numbering basis elements of  $k^{st}$  by pairs  $(i, j)$ ,  $1 \leq i \leq s, 1 \leq j \leq t$  and sending  $(a_i, b_j) \mapsto e_{(i,j)}$  is universal, hence  $M \otimes N \cong k^{st}$ .
- It is possible to define tensor products over non-commutative rings, where  $M$  is a right  $R$ -module and  $N$  is a left  $R$ -module. In this situation,  $M \otimes N$  is only an abelian group, not necessarily an  $R$ -module. The construction is analogous, but you take the free abelian group instead of the free  $R$ -module and use the relations

$$\begin{aligned} e_{(m_1+m_2, n)} &= e_{(m_1, n)} + e_{(m_2, n)} \\ e_{(m, n_1+n_2)} &= e_{(m, n_1)} + e_{(m, n_2)} \\ e_{(mr, n)} &= e_{(m, rn)}. \end{aligned}$$

If  $M$  is an  $(R, S)$ -bimodule and  $N$  is an  $(S, T)$ -bimodule, then  $M \otimes N$  becomes a  $(R, T)$ -bimodule.

- On the exercise sheet we will see that  $\mathbb{Z}/r\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/s\mathbb{Z} \cong \mathbb{Z}/\gcd(r, s)\mathbb{Z}$ .
- one can construct a universal trilinear map  $L \times M \times N \rightarrow T$ , unique up to isomorphism, denoted by  $L \otimes M \otimes N$ .

LEMMA 3.4. The following maps exist and are isomorphisms.

(i)

$$\begin{aligned} M \otimes N &\rightarrow N \otimes M \\ m \otimes n &\mapsto n \otimes m, \end{aligned}$$

(ii)

$$\begin{aligned} (M \otimes N) \otimes L &\rightarrow M \otimes (N \otimes L) \rightarrow M \otimes N \otimes L \\ (m \otimes n) \otimes \ell &\mapsto m \otimes (n \otimes \ell) \mapsto m \otimes n \otimes \ell, \end{aligned}$$

(iii)

$$\begin{aligned} (M \oplus N) \otimes L &\rightarrow (M \otimes L) \oplus (N \otimes L) \\ (m, n) \otimes \ell &\mapsto (m \otimes \ell, n \otimes \ell), \end{aligned}$$

(iv)

$$\begin{aligned} R \otimes_R M &\rightarrow M \\ r \otimes m &\mapsto rm. \end{aligned}$$

PROOF. (i) The map  $M \times N \rightarrow N \otimes M$  given by  $(m, n) \mapsto n \otimes m$  is bilinear, hence it induces the map  $M \otimes N \rightarrow N \otimes M$  given by  $m \otimes n \mapsto n \otimes m$ . Swapping the roles of  $M$  and  $N$  yields an inverse.

(ii) Exercise (appears on the second example sheet).

(iii) We have a bilinear map

$$\begin{aligned} (M \oplus N) \times L &\rightarrow (M \otimes L) \oplus (N \otimes L) \\ ((m, n), \ell) &\mapsto (m \otimes \ell, n \otimes \ell). \end{aligned}$$

This map induces a linear map

$$\begin{aligned} (M \oplus N) \otimes L &\rightarrow (M \otimes L) \oplus (N \otimes L) \\ (m, n) \otimes \ell &\mapsto (m \otimes \ell, n \otimes \ell), \end{aligned}$$

and we will find an inverse. Indeed, the maps

$$\begin{aligned} M \times L &\rightarrow (M \oplus N) \otimes L & N \times L &\rightarrow (M \oplus N) \otimes L \\ (m, \ell) &\mapsto (m, 0) \otimes \ell & (n, \ell) &\mapsto (0, n) \otimes \ell \end{aligned}$$

are bilinear, and by the universal property of the tensor product and the universal property of the direct sum we obtain a linear map

$$\begin{aligned} \Psi: (M \otimes L) \oplus (N \otimes L) &\rightarrow (M \oplus N) \otimes L \\ (m \otimes \ell_1, n \otimes \ell_2) &\mapsto (m, 0) \otimes \ell_1 + (0, n) \otimes \ell_2. \end{aligned}$$

We trivially calculate that this is the required inverse.

(iv) Another exercise, cf. Proposition 2.14 in Atiyah-Macdonald.

□

EXAMPLE. We have a natural bijection

$$\mathrm{Hom}(M \otimes N, L) \cong \mathrm{Hom}(M, \mathrm{Hom}(N, L))$$

Indeed, if  $\varphi: M \times N \rightarrow L$  is a bilinear map, we get  $\theta: M \rightarrow \mathrm{Hom}(N, L)$  as  $m \mapsto (n \mapsto \varphi(m, n))$ . Conversely, if  $\theta: M \rightarrow \mathrm{Hom}(N, L)$  is linear, then we obtain a bilinear map  $M \times N \rightarrow L$  by setting  $(m, n) \mapsto \theta(m)(n)$ .

### 1. Restriction and extension of scalars

DEFINITION 3.5. If  $\varphi: R \rightarrow T$  is a homomorphism of rings, and  $N$  is a  $T$ -module, then  $N$  may be regarded as an  $R$ -module via  $rm := \varphi(r)m$ . In particular,  $T$  itself is an  $R$ -module. This process is called restriction of scalars.

If  $M$  is an  $R$ -module, then  $T \otimes_R M$  is an  $R$ -module. It is also a  $T$ -module via  $t_1(t_2 \otimes m) := (t_1 t_2) \otimes m$ .

EXAMPLE. Localisation of a module is just extension of scalars using the map  $R \rightarrow S^{-1}R$ . Indeed, given an  $R$ -module  $M$  and a multiplicatively closed set  $S$ , we find an isomorphism of  $R$ -modules  $f: S^{-1}R \otimes_R M \rightarrow S^{-1}M$  given by  $\frac{r}{s} \otimes m \mapsto \frac{rm}{s}$ .

Indeed, the map  $S^{-1}R \times M \rightarrow S^{-1}M$ ,  $(r/s, m) \mapsto (rm)/s$  is bilinear, so it induces  $f$  as above. It is obviously surjective. For injectivity, recall that a general element of the left hand side is of the form  $\sum_{i=1}^n r_i/s_i \otimes m_i$ . Let  $s = s_1 \cdots s_n$  and  $t_i = \prod_{j \neq i} s_j$ . Then we may calculate

$$\sum \frac{r_i}{s_i} \otimes m_i = \sum \frac{r_i t_i}{s} \otimes m_i = \sum \frac{1}{s} \otimes r_i t_i m_i = \frac{1}{s} \otimes \sum r_i t_i m_i.$$

Hence, every element of the left hand side is of the form  $1/s \otimes m$ .

Suppose that  $f(1/s \otimes m) = 0$ . Then  $m/s = 0$  in  $S^{-1}M$ , i.e., we find  $x \in S$  such that  $xm = 0$ . But then

$$\frac{1}{s} \otimes m = \frac{x}{sx} \otimes m = \frac{1}{sx} \otimes xm = \frac{1}{sx} \otimes 0 = 0,$$

and so  $f$  is injective.

DEFINITION 3.6. Given  $R$ -linear maps  $\theta: M_1 \rightarrow M_2$  and  $\varphi: N_1 \rightarrow N_2$ , the tensor product of  $\theta$  and  $\varphi$  is the map

$$\begin{aligned} \theta \otimes \varphi: M_1 \otimes N_1 &\rightarrow M_2 \otimes N_2 \\ m_1 \otimes n_1 &\mapsto \theta(m_1) \otimes \varphi(n_1), \end{aligned}$$

which exists because  $M_1 \times N_1 \rightarrow M_2 \otimes N_2$ ,  $(m, n) \mapsto \theta(m) \otimes \varphi(n)$  is bilinear.

DEFINITION 3.7. Given a ring homomorphism  $\varphi_1: R \rightarrow T_1$  (which in particular makes  $T_1$  into an  $R$ -module), we say that  $T_1$  together with  $\varphi_1$  is an  $R$ -algebra. Given another ring homomorphism  $\varphi_2: R \rightarrow T_2$ , we can take the tensor product of the  $R$ -modules  $T_1$  and  $T_2$  to give  $T_1 \otimes_R T_2$ . We can declare a product on  $T_1 \otimes T_2$  by

$$(T_1 \otimes T_2) \times (T_1 \otimes T_2) \rightarrow T_1 \otimes T_2$$

$$(t_1 \otimes t_2, t'_1 \otimes t'_2) \mapsto t_1 t'_1 \otimes t_2 t'_2.$$

As usual, it needs to be checked that this map actually exists: first, we notice that multiplication is bilinear, hence it induces a map  $T_1 \times T_i \rightarrow T_i$ . The composite

$$(T_1 \otimes T_1) \times (T_2 \otimes T_2) \rightarrow T_1 \times T_2 \rightarrow T_1 \otimes T_2$$

is again bilinear, hence it induces a map

$$(T_1 \otimes T_1) \otimes (T_2 \otimes T_2) \rightarrow T_1 \otimes T_2$$

$$(t_1 \otimes t'_1) \otimes (t_2 \otimes t'_2) \mapsto t_1 t'_1 \otimes t_2 t'_2.$$

By (3.4), we can reassociate and permute this to a map

$$(T_1 \otimes T_2) \otimes (T_1 \otimes T_2) \rightarrow T_1 \otimes T_2,$$

and we see that composition with the tensoring map gives exactly the map we postulated above. Hence the product exists, and  $1 \otimes 1$  is the multiplicative identity. This makes  $T_1 \otimes T_2$  into a ring and we have an  $R$ -algebra structure via  $r \mapsto \varphi_1(r) \otimes 1 = 1 \otimes \varphi_2(r)$ .

- EXAMPLE. (i) If  $k$  is a field then  $k[X]$  is a  $k$ -algebra. We have an isomorphism  $k[X] \otimes_k k[X] \cong k[X, Y]$ .  
(ii) We have an isomorphism  $\mathbb{Q}[X]/(X^2 + 1) \otimes_{\mathbb{Q}} \mathbb{C} \cong \mathbb{C}[X]/(X^2 + 1)$ .  
(iii) We have  $k[X]/(f) \otimes_k k[X]/(g) \cong k[X, Y]/(f(X), g(Y))$ .

DEFINITION 3.8. If  $R$  is a  $k$ -algebra and  $M$  and  $N$  are  $R$ -modules, then  $M \otimes_k N$  is an abelian group, and we can declare an  $R$ -module structure on it by defining  $r(m \otimes n) := rm \otimes rn$ . This is called the diagonal action. If  $M$  and  $N$  are finitely generated as  $R$ -modules, then so is  $M \otimes N$ .

LEMMA 3.9. If

$$M_1 \xrightarrow{\theta} M \xrightarrow{\varphi} M_2 \longrightarrow 0$$

is a sequence of  $R$ -modules, then it is exact if and only if for all  $R$ -modules  $N$ , the sequence

$$0 \longrightarrow \text{Hom}(M_2, N) \xrightarrow{\alpha} \text{Hom}(M, N) \xrightarrow{\beta} \text{Hom}(M_1, N)$$

is exact.

PROOF. First, assume that the first sequence is exact. Let  $N$  be an  $R$ -module. If  $f \in \text{Hom}(M_2, N)$ , then  $f \circ \varphi \in \text{Hom}(M, N)$ . This is the map  $\alpha: \text{Hom}(M_2, N) \rightarrow \text{Hom}(M, N)$ , and it is injective, since if  $f \neq 0$ , there is  $m_2 \in M_2$  with  $f(m_2) \neq 0$ , but  $m_2 = \varphi(m)$  for some  $m \in M$ , so  $f(\varphi(m)) \neq 0$ , hence  $\alpha$  is injective.

If  $g \in \text{Hom}(M, N)$  then  $\beta$  sends it to  $g \circ \theta \in \text{Hom}(M_1, N)$ . If  $g \circ \theta = 0$ , then  $\theta(M_1) \subseteq \ker g$ . By exactness, we have  $\theta(M_1) = \ker \varphi$ , so we conclude  $\ker \varphi \subseteq \ker g$ . Now  $\varphi$  factors as  $M \rightarrow M/\ker \varphi \rightarrow M_2$ , with the latter map being an isomorphism. Hence,  $g$  factors as  $M \rightarrow M_2 \rightarrow M/\ker \varphi \rightarrow M/\ker g \rightarrow N$ , where the first map is  $\varphi$  and the second map is the inverse of the isomorphism. Taking  $f$  to be the composite of the latter maps, we have written  $g = f \circ \varphi$  for  $f \in \text{Hom}(M_2, N)$ , hence  $\ker \beta \subseteq \text{im } \alpha$ . Since the reverse inclusion is trivial, we have exactness at  $\text{Hom}(M, N)$ .

Conversely, suppose that the sequence of Homsets is exact. If the map  $M_2 \rightarrow M_2/\text{im } \varphi$  was nonzero, then by injectivity of  $\alpha$ , so would be the composite  $M \rightarrow M_2 \rightarrow M_2/\text{im } \varphi$ , which is obviously not the case. Hence  $\varphi$  is surjective.

We have  $0 = \beta(\alpha(\text{id}_{M_2})) = \varphi \circ \theta$ , hence  $\text{im } \theta \subseteq \ker \varphi$ . Let  $p: M \rightarrow M/\text{im } \theta$  be the projection. Then  $p \circ \theta = 0$ , i.e.,  $p \in \ker \beta = \text{im } \alpha$ , so we find  $g: M_2 \rightarrow N$   $p = g \circ \varphi$ . Then  $\text{im } \theta = \ker p \supseteq \ker \varphi$ . Putting things together, we have exactness at  $M$ .  $\square$

REMARK. • The functor  $\text{Hom}(-, N)$  is not exact. The failure of exactness is studied using cohomology.

- We do have an analogous statement for the other Hom functor: The sequence

$$0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0$$

is exact if and only if

$$0 \longrightarrow \text{Hom}(N, M_1) \longrightarrow \text{Hom}(N, M) \longrightarrow \text{Hom}(N, M_2)$$

is exact for all modules  $N$ .

LEMMA 3.10. If

$$M_1 \xrightarrow{\theta} M \xrightarrow{\varphi} M_2 \longrightarrow 0$$

is an exact sequence and  $N$  is an  $R$ -module, then

$$\begin{aligned} M_1 \otimes N &\xrightarrow{\theta \otimes \text{id}} M \otimes N \xrightarrow{\varphi \otimes \text{id}} M_2 \otimes N \longrightarrow 0 \\ N \otimes M_1 &\xrightarrow{\text{id} \otimes \theta} N \otimes M \xrightarrow{\text{id} \otimes \varphi} N \otimes M_2 \longrightarrow 0 \end{aligned}$$

are exact.

PROOF. The second statement follows from the first statement by commutativity of tensor products.

Let  $N'$  be any  $R$ -module. By the previous lemma, we have an exact sequence

$$0 \rightarrow \text{Hom}(M_2, \text{Hom}(N, N')) \rightarrow \text{Hom}(M, \text{Hom}(N, N')) \rightarrow \text{Hom}(M_1, \text{Hom}(N, N')).$$

By the tensor-hom adjunction, we have  $\text{Hom}(M, \text{Hom}(N, N')) \cong \text{Hom}(M \otimes N, N')$ , so we get an exact sequence

$$0 \longrightarrow \text{Hom}(M_2 \otimes N, N') \longrightarrow \text{Hom}(M \otimes N, N') \longrightarrow \text{Hom}(M_1 \otimes N, N').$$

The previous lemma yields the desired exact sequence (after we have verified that the maps are indeed what we expect).  $\square$

EXAMPLE. Again, observe that these are not short exact sequences. Applying  $- \otimes N$  does not in general preserve injectivity of the left hand map. For example, consider the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

Tensoring with  $\mathbb{Z}/2\mathbb{Z}$ , we obtain a sequence

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0,$$

which fails to be exact. Hence exactness is not preserved.

DEFINITION 3.11. An  $R$ -module  $N$  is called flat if given any short exact sequence

$$0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0,$$

the sequence

$$0 \longrightarrow M_1 \otimes N \longrightarrow M \otimes N \longrightarrow M_2 \otimes N \longrightarrow 0$$

is exact.

EXAMPLE.           •  $R$  is a flat  $R$ -module.

- $R^n$  is a flat  $R$ -module.
- If  $R = \mathbb{Z}$ ,  $\mathbb{Q}$  is a flat  $\mathbb{Z}$ -module. In fact, it can be shown that every torsion-free abelian group is a flat  $\mathbb{Z}$ -module.

Homology measures the failure of a module to be flat.



## CHAPTER 4

### Integrality and dimension

DEFINITION 4.1. The spectrum of  $R$ , denoted  $\text{Spec } R$ , is  $\{\mathfrak{p} \mid \mathfrak{p} \text{ prime ideal of } R\}$ .

DEFINITION 4.2. The length of a chain of prime ideals  $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n$  is  $n$ .

DEFINITION 4.3. The (Krull) dimension of  $R$ , denoted  $\dim R$ , is

$$\dim R = \begin{cases} \sup\{n \mid \exists \text{ chain of prime ideals of length } n\}, & \text{if it exists,} \\ \infty, & \text{otherwise.} \end{cases}$$

DEFINITION 4.4. The height of  $\mathfrak{p} \in \text{Spec } R$ , denoted  $\text{ht}(\mathfrak{p})$  or  $\text{ht}_R(\mathfrak{p})$ , is

$$\sup\{n \mid \exists \text{ chain of prime ideals } \mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}\}.$$

REMARK. By the one-to-one correspondence between primes with empty intersection with  $R \setminus \mathfrak{p}$  and the primes of  $R_{\mathfrak{p}}$ , we have  $\text{ht}(\mathfrak{p}) = \dim R_{\mathfrak{p}}$ .

EXAMPLE. 

- An artinian ring has dimension 0 (since all prime ideals are maximal). Conversely, it is possible to show that any noetherian ring of dimension 0 is artinian.
- We have  $\dim \mathbb{Z} = 1$ . Indeed, a chain of maximal length must be of the form  $0 \subsetneq (p)$ , where  $p$  is a prime number. Also,  $\dim k[X] = 1$  if  $k$  is a field. These are examples of Dedekind rings, i.e., integrally closed domains of dimension 1. Dedekind rings are essential ingredients of the theory of algebraic curves and number theory.
- The dimension of  $k[X_1, \dots, X_n]$  is at least  $n$ , because we have the chain of prime ideals

$$0 \subsetneq (X_1) \subsetneq (X_2) \subsetneq \cdots \subsetneq (X_1, \dots, X_n).$$

Indeed, if  $k$  is a field, it is possible to show that the dimension is exactly  $n$ . To prove this, we need some results about the relationship between chains of prime ideals in subrings and chains in the whole ring under some condition relating the subring to the larger ring.

LEMMA 4.5. The height 1 primes of  $k[X_1, \dots, X_n]$  are precisely those of the form  $(f)$  where  $f$  is irreducible.

PROOF. Certainly such an ideal is prime, since  $k[X_1, \dots, X_n]$  is a unique factorization domain.

If  $\mathfrak{p}$  is a nonzero prime ideal, it contains such an  $(f)$ , since if  $g \in \mathfrak{p}$ , then by primality, it contains at least one of its irreducible factors. This means that if a prime has height 1, then it must be of the form  $(f)$  with  $f$  irreducible.

Conversely, suppose that  $f$  is irreducible, and  $0 \subsetneq \mathfrak{p} \subseteq (f)$  for some prime ideal  $\mathfrak{p}$ . Then by what we just saw, we find an irreducible  $h$  with  $(h) \subseteq \mathfrak{p}$ . In particular,  $h \in (f)$ . Since  $h$  is irreducible, we must have  $(h) = (f)$ , so  $\mathfrak{p} = (f)$ , so  $\text{ht}((f)) = 1$  as claimed.  $\square$

### 1. Integral extensions

DEFINITION 4.6. If  $R \subseteq S$  are rings, then  $x \in S$  is called integral over  $R$  if there is some monic polynomial  $f \in R[X]$  such that  $f(x) = 0$ .

EXAMPLE. Elements of  $\mathbb{Q}$  which are integral over  $\mathbb{Z}$  are precisely the elements of  $\mathbb{Z}$ : if  $x = r/s$  is integral over  $\mathbb{Z}$  with  $\gcd(r, s) = 1$ , then

$$0 = s^n f(x) = r^n + r_{n-1}s + \cdots + r_0s^n = 0$$

for integers  $r_i$ . Thus  $s \mid r^n$ , but since  $\gcd(r, s) = 1$ , this means that  $s = \pm 1$ , hence  $x = \pm r \in \mathbb{Z}$ .

LEMMA 4.7. The following statements are equivalent for an element  $x \in S$ .

- (i)  $x$  is integral over  $R$ ,
- (ii) the ring  $R[x]$  (i.e., the subring of  $S$  generated by  $R$  and  $x$ ) is finitely generated as an  $R$ -module,
- (iii)  $R[x]$  is contained in a subring  $T$  of  $S$  such that  $T$  is a finitely generated  $R$ -module.

PROOF. If  $x$  is integral, i.e.,  $f(x) = 0$  for monic  $f \in R[X]$ , we can write  $x^{n+j} = -(r_{n-1}x^{n+j-1} + \cdots + r_0x^j)$  for all  $j \geq 0$ , where  $\deg f = n$ . Hence,  $R[X]$  is generated as an  $R$ -module by  $1, x, \dots, x^{n-1}$ . This shows that (i) implies (ii).

Trivially, (ii) implies (iii).

Finally, let  $y_1, \dots, y_n$  generate  $T$  as an  $R$ -module. Then we find  $r_{ij}$  such that  $xy_i = \sum_j r_{ij}y_j$ . Then

$$\sum_j (x\delta_{ij} - r_{ij})y_j = 0,$$

where  $\delta$  is the Kronecker delta. Define a matrix  $A$  with coefficients in  $S$  via  $a_{ij} = x\delta_{ij} - r_{ij}$ . Then what we have just seen means that  $Ay_i = 0$  for all  $i$ , so in particular  $0 = \operatorname{adj}(A)Ay = (\det A)Iy = \det A y$ , where  $y = (y_1, \dots, y_n)^\top$ . Since the  $y_i$  generate  $T$ ,  $1$  is a linear combination of the  $y_i$ , so we find some row vector  $v$  such that  $0 = \det A y v = \det(A)1 = \det A$ . But  $\det A_{ij}$  is a monic polynomial expression in  $x$  with coefficients in  $R$ , so we are done.  $\square$

DEFINITION. Some authors use the phrase “ $S$  is finite over  $R$ ” to say that  $S$  is finitely generated as an  $R$ -module, and the phrase “ $R$  is of finite type” if  $R$  is finitely generated as a  $k$ -algebra.

REMARK. This proof is very similar to a proof of Nakayama’s lemma appearing in Atiyah-Macdonald.

Some authors say  $S$  is of finite type over  $R$  if  $S$  is generated as a ring by  $R$  and a finite set.

LEMMA 4.8. If  $x_1, \dots, x_m \in S$  are integral over  $R$ , then  $R[x_1, \dots, x_m]$  is a finitely generated  $R$ -module.

PROOF. We do induction on  $m$ . The case  $m = 1$  is just 4.7. If the claim is true for  $m$ , since  $x_{m+1}$  is integral over  $R$ , it is definitely integral over  $R[x_1, \dots, x_m]$ . Hence  $R[x_1, \dots, x_{m+1}]$  is a finitely generated  $R[x_1, \dots, x_m]$ -module, say with generators  $z_1, \dots, z_t$ . By the inductive hypothesis,  $R[x_1, \dots, x_m]$  is finitely generated as an  $R$ -module, say by  $y_1, \dots, y_\ell$ . Then  $R[x_1, \dots, x_{m+1}]$  is generated as an  $R$ -module by elements of the form  $z_i y_j$  with  $1 \leq i \leq t$  and  $1 \leq j \leq \ell$ .  $\square$

LEMMA 4.9. The set  $T \subseteq S$  of elements integral over  $R$  forms a subring of  $S$ .

PROOF. Every element of  $R$  is integral over  $R$ . Furthermore, if  $x, y \in T$ , then by 4.8 we have that  $R[x, y]$  is finitely generated. Since  $x \pm y, xy \in R[x, y]$ , these are integral by 4.7(iii).  $\square$

DEFINITION 4.10. The subring of integral elements over  $R$  in  $S$  is called the integral closure of  $R$  in  $S$ . If the integral closure of  $R$  in  $S$  is just  $R$ , then we say that  $R$  is integrally closed in  $S$ . If the integral closure is the entirety of  $S$ , we say that  $S$  is integral over  $R$ .

If  $R$  is an integral domain, we say that  $R$  is integrally closed if it is integrally closed in its field of fractions.

- EXAMPLE. • As we say,  $\mathbb{Z}$  is integrally closed.
- $k[X_1, \dots, X_n]$  is integrally closed (in  $k(X_1, \dots, X_n)$ ).
  - If  $K$  is an algebraic number field with  $[K : \mathbb{Q}] < \infty$ , then the integral closure of  $\mathbb{Z}$  in  $K$  is the ring of integers  $\mathcal{O}_K$ .
  - Being integrally closed is a local property. This will be proved later.

LEMMA 4.11. If  $R \subseteq T \subseteq S$  are rings,  $T$  is integral over  $R$  and  $S$  is integral over  $T$ , then  $S$  is integral over  $R$ .

PROOF. Take  $x \in S$ . By assumption, we have a relation  $x^n + t_{n-1}x^{n-1} + \dots + t_0 = 0$  with  $t_i \in T$ . By (4.8), the subring  $R[t_0, \dots, t_{n-1}]$  is a finitely generated  $R$ -module, and  $x$  is integral over it, so  $R[t_0, \dots, t_{n-1}, x]$  is a finitely generated  $R[t_0, \dots, t_{n-1}]$ -module. As seen in the proof of (4.8), this implies that  $R[t_0, \dots, t_{n-1}, x]$  is a finitely generated  $R$ -module, so by 4.7(iii),  $x$  is integral over  $R$  as required.  $\square$

LEMMA 4.12. Let  $R \subseteq T$  be rings such that  $T$  is integral over  $R$ .

- (i) If  $J \subseteq T$  is an ideal, then  $T/J$  is integral over  $R/(J \cap R)$ , where we identify  $R/(J \cap R)$  with the subring  $(R + J)/J$  of  $T/J$ .
- (ii) If  $S$  is a multiplicatively closed subset of  $R$ , then  $S^{-1}T$  is integral over  $S^{-1}R$ .

PROOF. For (i), if  $x \in T$ , then we have an expression  $x^n = r_{n-1}x^{n-1} + \dots + r_0 = 0$  for some  $r_i \in R$ . Projecting onto  $T/J$ , this yields an equation  $\bar{x}^n + \bar{r}_{n-1}\bar{x}^{n-1} + \dots + \bar{r}_0 = \bar{0}$  in  $T/J$ , such that  $\bar{r}_i \in (R + J)/J$ , hence  $\bar{x}$  is integral over  $(R + J)/J$  as required.

For (ii), let  $x/s \in S^{-1}T$ . Again we have  $x^n = r_{n-1}x^{n-1} + \dots + r_0 = 0$  for some  $r_i \in R$ . In particular, this implies

$$\left(\frac{x}{s}\right)^n + \frac{r_{n-1}}{s} \left(\frac{x}{s}\right)^{n-1} + \dots + \frac{r_0}{s^n} = 0$$

in  $S^{-1}T$ , so  $x/s$  is integral over  $S^{-1}R$ .  $\square$

LEMMA 4.13. Let  $R \subseteq T$  both be integral domains such that  $T$  is integral over  $R$ . Then  $T$  is a field if and only if  $R$  is a field.

PROOF. First assume that  $R$  is a field. Let  $0 \neq t \in T$ . Let  $t^n + r_{n-1}t^{n-1} + \dots + r_0 = 0$  be such that  $n$  is minimal among all monic expressions. We have  $r_0 \neq 0$ , otherwise we could factor out a  $t$ , so since  $T$  is a domain,  $t^{n-1} + \dots + r_1 = 0$  would be a shorter expression, contradicting minimality. Then  $s := -r_0^{-1}(t^{n-1} + \dots + r_1)$  satisfies  $st = 1$ , hence  $t$  has an inverse, so  $T$  is a field.

Conversely suppose that  $T$  is a field. Let  $0 \neq x \in R$ . Then  $x$  has an inverse  $x^{-1}$  in  $T$ . We find  $r'_i \in R$  such that  $x^{-m} + r'_m x^{-m+1} + \dots + r'_0 = 0$ . Multiply by  $x^{m-1}$  and rearrange to find  $x^{-1} = -(r'_m + r'_{m-1}x + \dots + r'_0 x^{m-1}) \in R$ , so  $R$  is a field.  $\square$

LEMMA 4.14. Let  $R \subseteq T$  be rings and  $T$  integral over  $R$ . Let  $\mathfrak{q}$  be a prime ideal in  $T$  and set  $\mathfrak{p} := R \cap \mathfrak{q}$ . Then  $\mathfrak{q}$  is maximal if and only if  $\mathfrak{p}$  is maximal.

PROOF. By 4.12(i),  $T/\mathfrak{q}$  is integral over  $R/\mathfrak{p}$ , and since  $\mathfrak{p}$  and  $\mathfrak{q}$  are prime,  $T/\mathfrak{q}$  and  $R/\mathfrak{p}$  are integral domains. The result now follows from 4.13.  $\square$

**THEOREM 4.15** (Incomparability theorem). Let  $R \subseteq T$  be rings with  $T$  integral over  $R$ . Let  $\mathfrak{q} \subseteq \mathfrak{q}_1$  be prime ideals of  $T$ . Suppose  $\mathfrak{q} \cap R = \mathfrak{p} = \mathfrak{q}_1 \cap R$ . Then  $\mathfrak{q} = \mathfrak{q}_1$ .

In particular, every strict chain of primes in  $T$  will induce a strict chain of primes in  $R$ , hence  $\dim R \geq \dim T$ .

**PROOF.** Set  $S := R \setminus \mathfrak{p}$ . By 4.12(ii),  $S^{-1}T$  is integral over  $R_{\mathfrak{p}}$ .  $R_{\mathfrak{p}}$  is local with the unique maximal ideal  $S^{-1}\mathfrak{p}$ . As seen in Chapter 2, the ideals  $S^{-1}\mathfrak{q}$  and  $S^{-1}\mathfrak{q}_1$  are prime ideals in  $S^{-1}T$ . But the assumption implies that  $S^{-1}\mathfrak{q} \cap S^{-1}R = S^{-1}\mathfrak{p} = S^{-1}\mathfrak{q}_1 \cap S^{-1}R$ .

By 4.14,  $S^{-1}\mathfrak{q}$  and  $S^{-1}\mathfrak{q}_1$  are both maximal, but  $S^{-1}\mathfrak{q} \subseteq S^{-1}\mathfrak{q}_1$ , so we have equality. Using the correspondence between prime ideals of  $S^{-1}T$  and prime ideals of  $T$  that do not meet  $S$ , we conclude  $\mathfrak{q} = \mathfrak{q}_1$ .  $\square$

**THEOREM 4.16** (Lying over theorem). Let  $R \subseteq T$  be rings,  $T$  integral over  $R$ . Let  $\mathfrak{p}$  be a prime ideal of  $R$ . Then there is some prime ideal  $\mathfrak{q}$  of  $T$  with  $\mathfrak{q} \cap R = \mathfrak{p}$ . In this situation, we say that  $\mathfrak{q}$  lies over  $\mathfrak{p}$ . In other words, the map  $\text{Spec } T \rightarrow \text{Spec } R$  is surjective.

**PROOF.** Again, let  $S := R \setminus \mathfrak{p}$ . Then  $S^{-1}T$  is integral over  $R_{\mathfrak{p}}$ . By the correspondence, a maximal ideal of  $S^{-1}T$  is of the form  $S^{-1}\mathfrak{q}$ , where  $\mathfrak{q}$  is a prime ideal of  $T$ .

Then  $S^{-1}\mathfrak{q} \cap S^{-1}R$  is maximal by 4.14, but then it must be the unique maximal ideal  $S^{-1}\mathfrak{p}$  of  $R_{\mathfrak{p}}$ . So  $S^{-1}\mathfrak{q} \cap S^{-1}R = S^{-1}\mathfrak{p}$ , and by the correspondence we conclude  $\mathfrak{q} \cap R = \mathfrak{p}$ .  $\square$

**REMARK.** The next two theorems are due to Cohen and Seidelberg (1946) and are called the “going up” and “going down” theorems. They allow us to move from chains of prime ideals of  $R$  to such chains in  $T$ , where  $R \subseteq T$  is an integral extension. The second theorem requires stronger conditions.

**THEOREM 4.17** (Going-up theorem). Let  $R \subseteq T$  be an integral ring extension. Let  $\mathfrak{p}_1 \subseteq \cdots \subseteq \mathfrak{p}_n$  be a chain of prime ideals of  $R$ , and  $\mathfrak{q}_1 \subseteq \cdots \subseteq \mathfrak{q}_m$ , where  $m < n$  a chain of prime ideals of  $T$  such that  $\mathfrak{q}_i \cap R = \mathfrak{p}_i$  ( $1 \leq i \leq m$ ).

Then the chain of  $\mathfrak{q}$ s extends to a chain  $\mathfrak{q}_1 \subseteq \cdots \subseteq \mathfrak{q}_n$  such that  $\mathfrak{q}_i \cap R = \mathfrak{p}_i$  for  $1 \leq i \leq n$ .

**PROOF.** By induction, it will be enough to consider the case  $n = 2$ ,  $m = 1$ . Write  $\bar{R}$  for  $R/\mathfrak{p}_1$  and  $\bar{T}$  for  $T/\mathfrak{q}_1$ . Then we have an integral extension  $\bar{R} \subseteq \bar{T}$ , using the fact that  $\mathfrak{q}_1 \cap R = \mathfrak{p}_1$  and 4.12(i). By the Lying over theorem, we find a prime ideal  $\bar{\mathfrak{q}}_2$  of  $\bar{T}$  such that  $\bar{\mathfrak{q}}_2 \cap \bar{R} = \bar{\mathfrak{p}}_2$ . Lifting out of the quotient yields a prime ideal  $\mathfrak{q}_2$  of  $T$  with  $\mathfrak{q}_1 \subseteq \mathfrak{q}_2$  and  $\mathfrak{q}_2 \cap R = \mathfrak{p}_2$ .  $\square$

**THEOREM 4.18** (Going-down theorem). Let  $R \subseteq T$  be an integral extension of integral domains, such that  $R$  is integrally closed. Let  $\mathfrak{p}_1 \supseteq \cdots \supseteq \mathfrak{p}_n$  be a chain of prime ideals of  $R$  and  $\mathfrak{q}_1 \supseteq \cdots \supseteq \mathfrak{q}_m$  ( $m < n$ ) be a chain of prime ideals of  $T$  such that  $\mathfrak{q}_i \cap R = \mathfrak{p}_i$  for  $1 \leq i \leq m$ .

Then the chain of  $\mathfrak{q}$ s extends to a chain  $\mathfrak{q}_1 \supseteq \cdots \supseteq \mathfrak{q}_n$  with  $\mathfrak{q}_i \cap R = \mathfrak{p}_i$  ( $1 \leq i \leq n$ ).

**PROOF.** By induction, it will be sufficient to consider the case  $m = 1$ ,  $n = 2$ , i.e., we are given prime ideals  $\mathfrak{p}_1 \supsetneq \mathfrak{p}_2$  of  $R$  and a prime ideal  $\mathfrak{q}_1$  of  $T$  such that  $\mathfrak{q}_1 \cap R = \mathfrak{p}_1$ . We need to produce a prime ideal  $\mathfrak{q}_2 \subseteq \mathfrak{q}_1$  such that  $\mathfrak{q}_2 \cap R = \mathfrak{p}_2$ .

Let  $S_1 = T \setminus \mathfrak{q}_1$  and  $S_2 = R \setminus \mathfrak{p}_2$ .  $S := S_1 S_2$  is a multiplicatively closed subset of  $T$  satisfying  $S_1 \subseteq S$  and  $S_2 \subseteq S$ .

Assume for now that  $T\mathfrak{p}_2 \cap S = \emptyset$  (we will prove this later).  $T\mathfrak{p}_2$  is an ideal of  $T$ , so we have  $S^{-1}(T\mathfrak{p}_2) \subseteq S^{-1}T$  is an ideal, and it is a proper ideal, since otherwise we would have  $x \in T\mathfrak{p}_2$  and  $s, y \in S$  such that  $s(x - y) = 0$ , but then

$sx = sy \in T\mathfrak{p}_2 \cap S = \emptyset$  (since  $T\mathfrak{p}_2$  is an ideal and  $S$  is multiplicatively closed), a contradiction.

Hence  $S^{-1}T\mathfrak{p}_2$  is contained in some maximal ideal  $S^{-1}T$ , which by 2.5, is of the form  $S^{-1}\mathfrak{q}_2$  for some prime ideal  $\mathfrak{q}_2$  of  $T$  satisfying  $\mathfrak{q}_2 \cap S = \emptyset$ . Furthermore, if  $x \in T\mathfrak{p}_2$ , then  $x/1 \in S^{-1}(T\mathfrak{p}_2)$ , i.e., we find  $y \in \mathfrak{q}_2$ ,  $s_1, s_2 \in S$  such that  $s_2(y - xs_1) = 0$ . Hence,  $x(s_1s_2) \in \mathfrak{q}_2$ , but  $\mathfrak{q}_2$  is prime,  $s_1s_2 \in S$  and  $\mathfrak{q}_2 \cap S = \emptyset$ , so  $x \in \mathfrak{q}_2$ . We conclude that  $T\mathfrak{p}_2 \subseteq \mathfrak{q}_2$  and hence  $\mathfrak{p}_2 \subseteq T\mathfrak{p}_2 \cap R \subseteq \mathfrak{q}_2 \cap R$ . On the other hand, if  $\mathfrak{q}_2 \cap R \not\subseteq \mathfrak{p}_2$ , then we find  $x \in \mathfrak{q}_2 \cap R$  such that  $x \notin \mathfrak{p}_2$ . The latter means that  $x \in R \setminus \mathfrak{p}_2 = S_2 \subseteq S$ , but then  $x \in \mathfrak{q}_2 \cap S = \emptyset$ , a contradiction. We conclude  $\mathfrak{p}_2 = \mathfrak{q}_2 \cap R$ .

Similarly, if  $\mathfrak{q}_2 \not\subseteq \mathfrak{q}_1$ , then we find  $x \in \mathfrak{q}_2$  such that  $x \notin \mathfrak{q}_1$ . Then  $x \in T \setminus \mathfrak{q}_1 = S_1 \subseteq S$ , so  $x \in \mathfrak{q}_2 \cap S = \emptyset$ . This shows,  $\mathfrak{q}_2 \subseteq \mathfrak{q}_1$ , so  $\mathfrak{q}_2$  has the desired properties.

It remains to show that  $T\mathfrak{p}_2 \cap S = \emptyset$ . Suppose  $x \in T\mathfrak{p}_2 \cap S$ . Using the definition of  $S$  and the fact that  $T$  is an integral domain, we see that  $x \neq 0$ . Lemma 4.22 with  $I = \mathfrak{p}_2$  tells us that the integral closure of  $\mathfrak{p}_2$  in  $T$  is  $\sqrt{T\mathfrak{p}_2}$ . In particular,  $x$  is in the integral closure of  $\mathfrak{p}_2$ . Lemma 4.23 then tells us that it is algebraic over  $K$ , the field of fractions of  $R$ , and the coefficients of the minimal polynomial  $f$  of  $x$  over  $K$  are contained in  $\sqrt{\mathfrak{p}_2} = \mathfrak{p}_2$ .

Since  $x \in S$ , it is of the form  $x = rt$  with  $t \in S_1$ ,  $r \in S_2$ . If  $r_i$  are the coefficients of  $f$ , then the minimal polynomial of  $t = x/r$  over  $K$  has coefficients  $r'_i := r_i/r^{n-i} \in K$ . Since  $t$  is integral over  $R$  by assumption, we may apply 4.23 with  $I = R$  and find that  $r'_i$  is in fact contained in  $R$ .

Now  $r_i = r'_i r^{n-i} \in \mathfrak{p}_2$  and  $r \notin \mathfrak{p}_2$  since  $r \in S_2$ , so we must have  $r'_i \in \mathfrak{p}_2$ . Since these coefficients belong to a monic polynomial killing  $t$ , this just means that  $t$  is integral over  $\mathfrak{p}_2$ . Hence, 4.22 tells us that  $t \in \sqrt{T\mathfrak{p}_2}$ . We have  $\mathfrak{p}_2 \subseteq \mathfrak{p}_1 \subseteq \mathfrak{q}_1$ , which implies  $T\mathfrak{p}_2 \subseteq T\mathfrak{q}_1 = \mathfrak{q}_1$ . Since  $\mathfrak{q}_1$  is prime, this implies  $\sqrt{T\mathfrak{p}_2} \subseteq \mathfrak{q}_1$ , so we conclude  $t \in \mathfrak{q}_1$ . On the other hand,  $t \in S_1 = T \setminus \mathfrak{q}_1$ , so we have arrived at the desired contradiction.  $\square$

REMARK. Later we apply these to finitely generated  $k$ -algebras to prove the Noether Normalisation theorem. If  $T$  is a domain then  $T$  is integral over a subalgebra  $R$  isomorphic to a polynomial algebra.

COROLLARY 4.19. Let  $R \subseteq T$  be an integral extension. Then  $\dim R = \dim T$ .

PROOF. Take a chain  $\mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_n$  of primes in  $T$ . Intersection with  $R$  yields a chain  $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n$  of primes in  $R$ , with  $\mathfrak{q}_i \cap R = \mathfrak{p}_i$ , and this chain is strict by 4.15. Thus  $\dim R \geq \dim T$ .

Conversely, if  $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n$  is a chain of primes in  $R$ . Then 4.16 yields  $\mathfrak{q}_0$  lying over  $\mathfrak{p}_0$ , which we can complete using the going-up theorem. This chain is obviously strictly increasing, hence  $\dim R \leq \dim T$ .  $\square$

COROLLARY 4.20. Let  $R \subseteq T$  be an integral extension, where  $T$  is an integral domain and  $R$  is integrally closed. Let  $\mathfrak{q}$  be a prime ideal of  $T$ . Then  $\text{ht}(\mathfrak{q} \cap R) = \text{ht}(\mathfrak{q})$ .

PROOF. Take a chain  $\mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_n = \mathfrak{q}$  in  $\text{Spec}(T)$ . As above this yields a strict chain of prime ideals in  $R$  by defining  $\mathfrak{p}_i := \mathfrak{q}_i \cap R$ . Hence,  $\text{ht}(\mathfrak{q} \cap R) \geq \text{ht}(\mathfrak{q})$ .

Conversely, if we have a chain of prime ideals ending in  $\mathfrak{q} \cap R$ , then  $\mathfrak{q}$  is the beginning of a lift which can be completed using 4.18. Hence  $\text{ht}(\mathfrak{q} \cap R) \leq \text{ht}(\mathfrak{q})$ .  $\square$

DEFINITION 4.21. If  $I \subseteq R$  is an ideal and  $R \subseteq T$ , then  $x \in T$  is called integral over  $I$  if it satisfies a monic equation  $x^n + r_{n-1}x^{n-1} + \cdots + r_0 = 0$  with  $r_i \in I$ . The integral closure of  $I$  in  $T$  is the set of such  $x$ .

LEMMA 4.22. Let  $R \subseteq T$  be an integral extension. Let  $I \subseteq R$  be an ideal. Then the integral closure of  $I$  in  $T$  is the radical  $\sqrt{TI}$  (note that  $TI$  is an ideal of  $T$ ) and

this is closed under addition and multiplication. In particular, if  $R = T$ , then the integral closure of  $I$  in  $R$  is  $\sqrt{I}$ .

PROOF. If  $x$  is integral over  $I$ , then by definition we get a description  $x^n = -(a_1x^{n-1} + \dots + a_n) \in TI$ . Hence,  $x \in \sqrt{TI}$ .

Conversely, if  $x \in \sqrt{TI}$ , then we have  $x^n = \sum_i t_i r_i$  for some  $n \in \mathbb{N}$ ,  $r_i \in I$  and  $t_i \in T$ . Since  $R \subseteq T$  is integral, using 4.8 we find that  $M := R[t_1, \dots, t_n]$  is finitely generated as an  $R$ -module. Observe that  $x^n M - \sum_i r_i (t_i M) \subseteq IM$ . Let  $y_1, \dots, y_s$  be a generating set for  $M$ . We may write  $x^n y_j = \sum_\ell r_{j\ell} y_\ell$  for suitable  $r_{j\ell} \in I$  (first write  $t_i y_j$  as an  $R$ -linear combination of the  $y_j$  and then multiply with  $r_i$  to obtain an  $I$ -linear combination. Summing up, we get  $x^n y_j$  as required). Rearrange to obtain

$$\sum_\ell (x^n \delta_{j\ell} - r_{j\ell}) y_\ell = 0,$$

and an argument identical to that in the proof of 4.7 yields that  $x^n$ , and hence  $x$ , is integral over  $I$ .  $\square$

LEMMA 4.23. Let  $R \subseteq T$  be integral domains,  $R$  integrally closed, and let  $x \in T$  be integral over an ideal  $I$  of  $R$ . Then  $x$  is algebraic over the field of fractions  $K$  of  $R$ , and if  $f := X^n + r_{n-1}X^{n-1} + \dots + r_0$  is the minimal polynomial of  $x$  over  $K$ , then  $r_i \in \sqrt{I}$  for all  $i$ .

PROOF. Certainly  $x$  is algebraic over  $K$ , since  $x$  is integral over  $I$ . It remains to show that the coefficients  $r_i$  are in  $\sqrt{I}$ . By the previous result, it will be sufficient to show that they are integral over  $I$ , since in that case they will be contained in  $R$ , since  $R$  is integrally closed, and 4.22 with  $T = R$  yields the desired result.

To show that the  $r_i$  are integral over  $I$ , consider the extension  $K \subseteq L$ , where  $L$  is a splitting field of  $f$ . If  $y$  is a root of  $f$ , then there is a  $K$ -automorphism of  $L$  sending  $x \mapsto y$  (cf. Galois theory). Since  $x$  is integral over  $R$ , it satisfies some monic equation  $x^m + s_{m-1}x^{m-1} + \dots + s_0 = 0$  with  $s_i \in I$ . Applying the automorphism (which fixes  $K$ , so in particular  $I$ ), yields  $y^m + s_{m-1}y^{m-1} + \dots + s_0 = 0$ , so  $y$  is integral over  $I$ .

Since  $f = \prod_y (X - y)$  for the roots  $y$ , the coefficients  $r_i$  are expressible as sums and products of the  $y$ . By 4.22, sums and products of elements integral over an ideal are again integral over the ideal, so we are done.  $\square$

## 2. Transcendence Degree

DEFINITION. If  $k$  is a field, then an affine algebra over  $k$  is just a  $k$ -algebra which is finitely generated as a  $k$ -algebra (i.e., there is a surjective map  $k[X_1, \dots, X_n] \rightarrow A$  for some  $n$ ). Our main result will be:

THEOREM 4.24. Let  $T$  be an affine algebra which is also an integral domain with fraction field  $K$ . Then  $\dim T = \text{tr deg}_k K$ , where  $\text{tr deg}_k K$  is the so-called transcendence degree of  $K$  over  $k$ .

PROOF. We proceed by induction over  $r := \text{tr deg}_k K$ . For  $r = 0$ , there is nothing to do.

By Noether Normalisation, we find algebraically independent elements  $x_1, \dots, x_r \in T$  such that  $T$  is integral over  $R = k[x_1, \dots, x_r]$ . This implies that  $K$  is algebraic over  $k(x_1, \dots, x_r)$ , which in turn implies that  $\text{tr deg}_k(K) = \text{tr deg}_k k(x_1, \dots, x_r) = r$ . By 4.19,  $\dim T = \dim R = \dim k[x_1, \dots, x_r]$ .

Thus, it remains to show that  $\dim k[x_1, \dots, x_r] = r$ . We have already seen that  $\dim k[x_1, \dots, x_r] \geq r$ . Take a chain  $\mathfrak{p}_0 \subsetneq \dots \subsetneq \mathfrak{p}_s$  of primes in  $R$ .

Since  $R$  is an integral domain, we may assume that  $\mathfrak{p}_0 = 0$ . Furthermore, we may assume using Lemma 4.5 that  $\mathfrak{p}_1 = (f)$ .

As shown in an example below, we have  $\text{tr deg}_k(K_f) = r - 1$ , where  $K_f$  is the field of fractions of  $R/(f)$ . By induction, we find  $\dim R/(f) = r - 1$ .

Next, consider the chain

$$\mathfrak{p}_1/\mathfrak{p}_1 \subsetneq \mathfrak{p}_2/\mathfrak{p}_1 \subsetneq \cdots \subseteq \mathfrak{p}_s/\mathfrak{p}_1.$$

This is again a chain of prime ideals (since  $\mathfrak{p}_1 \subseteq \mathfrak{p}_i$  and the quotient map is surjective) in  $R/(f)$  of length  $s - 1$ . Hence  $s - 1 \leq r - 1$  and so  $s \leq r$ , so we conclude  $\dim R = r$  as required.  $\square$

**DEFINITION.** We say that  $x_1, \dots, x_n$  are algebraically independent over  $k$  if the ring map  $k[X_1, \dots, X_n] \rightarrow k[x_1, \dots, x_n]$  which sends  $X_i \mapsto x_i$  is an isomorphism. In this situation,  $k[x_1, \dots, x_n]$  may be regarded as a polynomial algebra.

As in linear algebra, we consider maximal algebraically independent sets: they all have the same size (we will not prove this here). Such a set is called a transcendence basis over  $k$  and the transcendence degree is the cardinality.

There are some concepts that carry over from linear algebra: an algebraically independent set can be thought of like a linearly independent set, the algebraic closure of a set  $S$  is like the span of  $S$  and transcendence degree is like dimension.

**EXAMPLE.** Let  $L = k(X_1, \dots, X_n)$  be the fraction field of  $k[X_1, \dots, X_n]$  and  $f \in k[X_1, \dots, X_n]$  an irreducible polynomial. Define  $K$  to be the field of fractions of  $k[X_1, \dots, X_n]/(f)$ . Then we have  $\text{tr deg}_k L = n$ , since  $X_1, \dots, X_n$  is a maximal algebraically independent set, and  $\text{tr deg } K = n - 1$ , since  $K$  is algebraic over  $k(X_1, \dots, X_{i-1}, X_{i+1}, X_n)$ , where  $X_i$  is a variable that appears in some term in  $f$ .

**THEOREM 4.25 (Noether Normalisation).** Let  $T$  be an affine algebra. Then  $T$  is integral over a subalgebra of the form  $R = k[x_1, \dots, x_r]$ , where  $x_1, \dots, x_r \in T$  are algebraically independent.

**PROOF.** Since  $T$  is affine, we have  $T = k[a_1, \dots, a_n]$  for some  $n \in \mathbb{N}$ ,  $a_i \in T$ . We proceed by induction on  $n$ . Let  $r$  denote the maximal number of algebraically independent elements of  $\{a_i\}$ . Without loss of generality,  $r \geq 1$ , since otherwise all elements of  $T$  are integral over  $k$  (the kernel of the map  $k[X] \rightarrow k[a]$  contains a monic polynomial for every  $a$ ), so  $R = k$  will do the trick.

If  $r = n$ , there is nothing to do. Reorder the  $a_i$  in such a way that  $a_1, \dots, a_r$  are algebraically independent and  $a_{r+1}, \dots, a_n$  are algebraically dependent on  $a_1, \dots, a_r$  over  $k$ .

In particular, we find  $0 \neq f \in k[X_1, \dots, X_r, X_n]$  such that  $f(a_1, \dots, a_r, a_n) = 0$  (this exists because  $a_1, \dots, a_r, a_n$  are algebraically dependent). Then  $f$  is a sum of terms of the form  $\lambda_\ell X_1^{\ell_1} \cdots X_r^{\ell_r} X_n^{\ell_n}$ , where  $\ell = (\ell_1, \dots, \ell_r, \ell_n)$  and  $\ell_i \in \mathbb{N}_0$ .

We claim that there are positive integers  $m_1, \dots, m_r$  such that  $\varphi: \ell \mapsto m_1 \ell_1 + \cdots + m_r \ell_r + \ell_n$  is injective for those  $\ell$  with  $\lambda_\ell \neq 0$ .

Since there are only finitely many  $\ell$  such that  $\lambda_\ell = 0$ , there are only finitely many  $d = \ell - \ell'$  with  $\lambda_\ell \neq 0$  and  $\lambda_{\ell'} \neq 0$ . Writing  $d = (d_1, \dots, d_r, d_n)$ , consider the finitely many  $d = (d_1, \dots, d_r) \neq 0$  obtained in this way (observe that we have dropped the final component). Vectors in  $\mathbb{Q}^n$  that are orthogonal to one of these  $r$ -tuples lie in finitely many  $(r - 1)$ -dimensional subspaces. Hence it is possible to pick  $(q_1, \dots, q_r)$  such that each  $q_i$  satisfies  $q_i > 0$  and  $\sum q_i d_i \neq 0$  for all of the finitely many  $(d_1, \dots, d_r) \neq 0$ . Multiplying by a sufficiently large positive integer, we obtain  $(m_1, \dots, m_r) \in \mathbb{Z}_{\geq 0}^r$  satisfying  $|\sum m_i d_i| > |d_n|$  for all of the  $(d_1, \dots, d_r) \neq 0$ .

Now if  $\ell$  and  $\ell'$  are such that  $\varphi(\ell) = \varphi(\ell')$ , define  $d = \ell - \ell'$  such that  $\varphi(d) = 0$ . Notice that this implies  $d_1 = \dots = d_r = 0$ , since otherwise we would have  $\varphi(d) \neq 0$  by the inequality of absolute values. But then  $0 = \varphi(d) = d_n$ , hence  $\ell = \ell'$ . This completes the proof of the claim.

Pick  $m_1, \dots, m_r$  as in the claim and set

$$g(X_1, \dots, X_n) := f(X_1 + X_n^{m_1}, \dots, X_r + X_n^{m_r}, X_n).$$

Thus,  $g$  is a sum of the form

$$\sum_{\ell: \lambda_\ell \neq 0} \lambda_\ell (X_1 + X_n^{m_1})^{\ell_1} \cdots (X_r + X_n^{m_r})^{\ell_r} X_n^{\ell_n}.$$

By our choice of  $m_i$ , different terms will have different powers of  $x_n$ . Hence, there will be a single term with the highest power of  $X_n$ . Viewing  $g$  as a polynomial in  $X_n$ , the leading coefficient is one of the  $\lambda_\ell$ , so in particular it is an element of  $k$ .

Next, define  $b_i := a_i - a_n^{m_i}$  ( $1 \leq i \leq r$ ) and  $h(X_n) := g(b_1, \dots, b_r, X_n)$ .

Then the leading coefficient of  $h$  is once again in  $k$  and all coefficients are elements of  $k[b_1, \dots, b_r]$ . Moreover, we may calculate

$$h(a_n) = g(b_1, \dots, b_r, a_n) = f(a_1, \dots, a_r, a_n) = 0,$$

using the definition of  $g$  and the defining property of  $f$ .

Now divide  $h$  by its leading coefficient to find that  $a_n$  is integral over  $k[b_1, \dots, b_r]$ . Additionally, for each  $i \leq r$ ,  $a_i = b_i + a_n^{m_i}$  is also integral over  $k[b_1, \dots, b_r]$  (recall that sums and products of integral elements are integral).

This means that  $T$  is integral over  $S := k[b_1, \dots, b_r, a_{r+1}, \dots, a_{n-1}]$ . Since that is one generator less than before, we find that  $S$  is integral over some polynomial algebra  $R$ , so  $T$  is also integral over  $R$ .  $\square$

**COROLLARY 4.26.** If  $\mathfrak{q}$  is a prime of an affine domain  $T$ , then we have

$$\text{ht}(\mathfrak{q}) + \dim(T/\mathfrak{q}) = \dim T.$$

**PROOF.** Define  $m := \text{ht}(\mathfrak{q})$  and pick a chain

$$\mathfrak{q}_0 \subsetneq \cdots \subseteq \mathfrak{q}_m = \mathfrak{q}$$

of primes of maximal length. By Noether normalization, we find some polynomial subalgebra  $R$  of  $T$  such that  $T$  is integral over  $R$ . We have  $\dim T = \dim R$  by 4.19 and  $\dim T = \dim R = \text{tr deg } K$ , where  $K$  is the field of fractions of  $T$ , by 4.24. Furthermore,  $\dim R$  is the number of indeterminates of  $R$ .

Write  $\mathfrak{p}_i := \mathfrak{q}_i \cap R$ . By maximality of the chain, we must have  $\text{ht}(\mathfrak{q}_1) = 1$ . Since  $R$  is a UFD, it is integrally closed (the proof for  $R = \mathbb{Z}$  generalized to any UFD), hence 4.20 tells us that  $\text{ht}(\mathfrak{p}_1) = 1$ . But by 4.5 this implies  $\mathfrak{p}_1 = (f)$  for an irreducible polynomial  $f$ . By a previous calculation, we conclude that the transcendence degree of the field of fractions of  $R/\mathfrak{p}_1$  is  $\dim R - 1$ .

Now  $\text{ht}(\mathfrak{q}/\mathfrak{q}_1) = m - 1$  (TODO: why? Maybe you can make some argument comparing chains in  $T$  and  $T/\mathfrak{q}_1$  work, but doesn't that just prove the entire lemma?), and  $R/\mathfrak{p}_1 \subseteq T/\mathfrak{q}_1$  is an integral extension, so  $\dim(T/\mathfrak{q}_1) = \dim(R/\mathfrak{p}_1) = \dim T - 1$  (4.19 and 4.24). Finally, the rings  $(T/\mathfrak{q}_1)/(\mathfrak{q}/\mathfrak{q}_1)$  and  $T/\mathfrak{q}$  are isomorphic, so putting things together and applying the inductive hypothesis, we find

$$(m - 1) + \dim(T/\mathfrak{q}) = \dim T - 1,$$

and adding 1 on both sides yields the claim.  $\square$

**THEOREM 4.27.** Let  $R$  be a noetherian integral domain that is integrally closed in its field of fractions  $K$ . Let  $L$  be a separable extension over  $K$ , and let  $T_1$  be the integral closure of  $R$  in  $L$ . Then  $T_1$  is a finitely generated  $R$ -module.

Recall that separability always holds in characteristic 0.

**PROOF.** We will make use of the trace function. There are many ways to define it (cf. Galois theory). We will use it as a black box with the following property (cf.



Reid 8.13): if  $K \subseteq L$  is separable, then the map  $L \times L \rightarrow K$  given by  $(x, y) \mapsto \text{Tr}(xy)$  is a non-degenerate symmetric  $K$ -bilinear form.

Pick a  $K$ -basis  $y_1, \dots, y_n$  of  $L$ . □

COROLLARY 4.28. If  $L$  is any number field, then the integral closure of  $\mathbb{Z}$  in  $L$  is a finitely generated abelian group.



# Exercises

## Example Sheet 1

### Exercise 1.

LEMMA. Let  $R$  and  $S$  be (commutative unital) rings. Denote by  $\mathcal{I}_R$  the set of ideals of  $R$ . Then there is a bijective correspondence

$$\begin{aligned}\mathcal{I}_{R \times S} &\leftrightarrow \mathcal{I}_R \times \mathcal{I}_S, \\ I &\mapsto (\pi_1(I), \pi_2(I)), \\ I_1 \times I_2 &\leftrightarrow (I_1, I_2).\end{aligned}$$

PROOF. We need to show the following.

- (i) If  $I$  is an ideal of  $R \times S$ , then  $\pi_1(I)$  is an ideal of  $R$  and  $\pi_2(I)$  is an ideal of  $S$ ,
- (ii) if  $I_1$  is an ideal of  $R$ ,  $I_2$  is an ideal of  $S$ , then  $I_1 \times I_2$  is an ideal of  $R \times S$ ,
- (iii) if  $I$  is an ideal of  $R \times S$ , then  $I = \pi_1(I) \times \pi_2(I)$  and
- (iv) if  $I_1$  is an ideal of  $R$ ,  $I_2$  is an ideal of  $S$ , then  $I_1 = \pi_1(I_1 \times I_2)$  and  $I_2 = \pi_2(I_1 \times I_2)$ .

Indeed (i) follows from surjectivity of the projection and (ii) and (iv) are obvious. It remains to show (iii).

If  $(r, s) \in I$ , then  $r = \pi_1((r, s)) \in \pi_1(I)$  and  $s = \pi_2((r, s)) \in \pi_2(I)$ , so  $(r, s) \in \pi_1(I) \times \pi_2(I)$ .

Conversely, if  $(r, s) \in \pi_1(I) \times \pi_2(I)$ , then there are  $r', s'$  such that  $(r, s') \in I$  and  $(r', s) \in I$ . We conclude that  $(r, s) = (r, s') \cdot (1, 0) + (r', s) \cdot (0, 1) \in I$ .  $\square$

EXERCISE. The direct product of finitely many noetherian rings is noetherian.

SOLUTION. Since the terminal object in the category of rings is the zero ring, which is noetherian, by induction it suffices to show that if  $R$  and  $S$  are noetherian, then  $R \times S$  is noetherian.

Let  $I$  be an ideal of  $R \times S$ . We have to show that  $I$  is finitely generated. By the Lemma,  $I = I_1 \times I_2$  for an ideal  $I_1$  of  $R$  and an ideal  $I_2$  of  $S$ . Since  $R$  and  $S$  are noetherian,  $I_1$  is finitely generated, say by  $r_1, \dots, r_n$  and so is  $I_2$ , say by  $s_1, \dots, s_m$ . Then if  $(r, s) \in I_1 \times I_2$ , we have

$$(r, s) = \left( \sum_{i=1}^n \lambda_i r_i, \sum_{i=1}^m \lambda'_i s_i \right) = \sum_{i=1}^n (\lambda_i, 0)(r_i, 0) + \sum_{i=1}^m (0, \lambda'_i)(0, s_i),$$

so  $I_1 \times I_2$  is finitely generated by  $(r_1, 0), \dots, (r_n, 0), (0, s_1), \dots, (0, s_m)$ .  $\square$

### Exercise 3.

EXERCISE. The set of prime ideals in a non-zero rings possesses a minimal member with respect to inclusion.

SOLUTION. Denote the set of prime ideals of  $A$  by  $\mathcal{S}$ . Since  $A$  is nonzero,  $(0)$  is a proper ideal, which is contained in a maximal ideal, hence  $\mathcal{S}$  is nonempty.

The set  $\mathcal{S}$  is partially ordered using the relation “ $\supseteq$ ”. Let  $\mathcal{S}' \subseteq \mathcal{S}$  denote a totally ordered subset of  $\mathcal{S}$ . We will show that  $\mathcal{S}'$  admits an upper bound. Indeed, define  $S := \bigcap_{P \in \mathcal{S}'} P$ .  $S$  is obviously an ideal, and we will show that it is prime. Assume that  $x, y \in A$  such that  $xy \in S$ . Since every  $P \in \mathcal{S}'$  is prime, we may write  $\mathcal{S}' = \mathcal{S}_x \cup \mathcal{S}_y$ , where  $\mathcal{S}_x := \{P \in \mathcal{S}' \mid x \in P\}$  and  $\mathcal{S}_y := \{P \in \mathcal{S}' \mid y \in P\}$ . We claim that it is true that

$$(\star) \quad (\forall P \in \mathcal{S}' \exists P' \in \mathcal{S}_x: P' \subseteq P) \vee (\forall P \in \mathcal{S}' \exists P' \in \mathcal{S}_y: P' \subseteq P).$$

Indeed, the negation of this statement is

$$(\exists P \in \mathcal{S}' \forall P' \in \mathcal{S}_x: P' \not\subseteq P) \wedge (\exists Q \in \mathcal{S}' \forall Q' \in \mathcal{S}_y: Q' \not\subseteq Q),$$

but then  $P \cap Q$ , which is either  $P$  or  $Q$ , since  $\mathcal{S}'$  is totally ordered, is part of neither  $\mathcal{S}_x$  nor  $\mathcal{S}_y$ , which is a contradiction.

Therefore, without loss of generality, we may assume that the left hand side of  $(\star)$  is true (the case where the right hand side is true works exactly the same). Since  $P' \in \mathcal{S}_x$  and  $P' \subseteq P$  implies  $P \in \mathcal{S}_x$ , we have that  $\mathcal{S}' = \mathcal{S}_x$ , so  $x \in S$ , and  $S$  is indeed a prime ideal, and therefore every chain in  $\mathcal{S}$  admits an upper bound.

Applying Zorn's lemma gives a maximal element of  $\mathcal{S}$ , which is precisely a minimal prime ideal of  $A$ .  $\square$

### Exercise 7.

EXERCISE. Let  $M$  be a noetherian  $A$ -module and  $\theta$  be an endomorphism.

- (i) If  $\theta$  is surjective, then it is an isomorphism.
- (ii) If  $M$  is artinian and  $\theta$  is injective, then it is an isomorphism.

[Hint: in (i) consider the submodules  $\ker \theta^n$ ; in (ii), consider the quotient modules  $\text{coker } \theta^n$ .]

SOLUTION. For (i), assume that  $\theta$  is not injective. Then there is some  $x \in \ker \theta \setminus \{0\}$ . Let  $n \in \mathbb{N}$ . Since  $\theta$  is surjective, so is  $\theta^n$ , so there is some  $y \in M$  such that  $\theta^n(y) = x$ . Therefore,  $y \in \ker \theta^{n+1} \setminus \ker \theta^n$  and we have an infinite strictly increasing chain

$$\ker \theta \subsetneq \ker \theta^2 \subsetneq \ker \theta^3 \subsetneq \cdots.$$

For (ii), assume that  $\theta$  is not surjective. This means that there is some  $x \notin \text{im } \theta$ . Let  $n \in \mathbb{N}$ . Then we have  $\theta^n(x) \in \text{im } \theta^n$ . Suppose that  $\theta^n(x) \in \text{im } \theta^{n+1}$ . Then there would be  $y \in M$  such that  $\theta^{n+1}(y) = \theta^n(x)$ . By injectivity of  $\theta$ , this means that  $\theta(y) = x$ , a contradiction. Therefore,  $\theta^n(x) \in \text{im } \theta^n \setminus \text{im } \theta^{n+1}$  and we have an infinite strictly decreasing chain

$$\text{im } \theta \supsetneq \text{im } \theta^2 \supsetneq \text{im } \theta^3 \supsetneq \cdots. \quad \square$$

### Exercise 8.

EXERCISE. Let  $A$  be a Noetherian ring and  $f \in A[[X]]$ . Then  $f$  is nilpotent if and only if all of its coefficients are nilpotent.

SOLUTION. First assume that  $f$  is nilpotent. Write  $f = \sum_{i=0}^{\infty} a_i X^i$  for some  $a_i \in A$ . We will argue by induction. Since  $f$  is nilpotent, there is some  $k \in \mathbb{N}$  such that  $f^k = 0$ . The constant term of  $f^k$  is  $a_0^k$ , hence  $a_0$  is nilpotent.

Next, assume that  $a_0, \dots, a_n$  are nilpotent for some  $n \in \mathbb{N}$ . Then they are also nilpotent as elements of  $A[[X]]$ . Since the set of nilpotent elements forms an ideal, we have that  $g := \sum_{i=0}^n a_i X^i$  is nilpotent, so  $f - g = \sum_{i=n+1}^{\infty} a_i X^i$  is nilpotent, i.e., there is some  $k$  such that  $(f - g)^k = 0$ . But the  $X^{k(n+1)}$ -coefficient of  $(f - g)^k$  is just  $a_{n+1}^k$ , hence  $a_{n+1}$  is nilpotent.

Next, assume that  $f = \sum_{i=0}^{\infty} a_i X^i$  and every  $a_i$  is nilpotent. Denote by  $I$  the ideal of  $A$  generated by all  $a_i$ . Then  $I \subseteq N(A)$ . Since  $N(A) = \sqrt{0}$ , by Lemma 1.21,

there is some natural number  $n$  such that  $N(A)^n \subseteq (0)$ . Since  $I^n \subseteq N(A)^n$ , this implies that  $I^n = (0)$ . Since the coefficients of  $f^n$  are elements of  $I^n$ , we conclude that  $f^n = 0$ , so  $f$  is nilpotent.  $\square$

### Exercise 9.

EXERCISE. Let  $A$  be a ring and  $M$  an  $R$ -module.

- (i)  $M[X]$  is an  $A[X]$ -module,
- (ii) If  $P$  is a prime ideal in  $A$ , then  $P[X]$  is a prime ideal in  $A[X]$ . If  $Q$  is a maximal ideal of  $A$ , is  $Q[X]$  a maximal ideal of  $A[X]$ ?
- (iii) Let  $M$  be a noetherian  $A$ -module. Then  $M[X]$  is a noetherian  $A[X]$ -module.

SOLUTION. For the second part, let  $P$  be a prime ideal in  $A$ .  $P[X]$  is obviously an ideal of  $A[X]$ . Let  $f, g \in A[X]$  such that  $fg \in P[X]$ . We can write

$$f = \sum_{i=0}^n a_i X^i, \quad g = \sum_{i=0}^m b_i X^i,$$

and also define  $a_i = 0$  for  $i > n$  and  $b_i = 0$  for  $i > m$ . Suppose that  $f \notin P[X]$  and  $g \notin P[X]$ . Then we find  $i$  and  $j$  such that  $a_i \notin P$ ,  $b_j \notin P$ . Choose  $i$  and  $j$  to be minimal among the possible  $i$  and  $j$ . Then the coefficient of  $fg$  for  $X^{i+j}$  is given by

$$\left( \sum_{k=0}^{i-1} a_k b_{i+j-k} \right) + a_i b_j + \left( \sum_{k=i+1}^{i+j} a_k b_{i+j-k} \right).$$

The coefficient is in  $P$ , and so are the sums on the left and the right, by minimality of  $i$  and  $j$ . But then  $a_i b_j \in P$ , so  $a_i \in P$  or  $b_j \in P$ , a contradiction. Hence  $f \in P[X]$  or  $g \in P[X]$ .

Let  $Q$  be a maximal ideal of  $A$ . Then  $1 \notin Q$ , hence  $1 \notin Q[X]$  and  $X \notin Q[X]$ , hence  $1 \notin (Q[X], X)$ , but  $X \in (Q[X], X)$ . We conclude that  $Q[X] \subsetneq (Q[X], X) \subsetneq A[X]$ , so  $Q[X]$  is not a maximal ideal.

The proof of the third part is almost identical to the proof of Hilbert's basis theorem. We will show that every submodule of  $M[X]$  is finitely generated. Let  $N$  be an  $A[X]$ -submodule of  $M[X]$  and define  $N_n := \{f \in N \mid \deg f \leq n\}$ . We have  $0 \in N_n$  and  $N_0 \subseteq N_1 \subseteq \dots$  form an ascending chain.

Define  $M_n$  to be the set of coefficients of  $X^n$  appearing in elements of  $N_n$ . If  $m + n \in M_n$  and  $a \in A$ , then  $m + n \in M_n$  and  $am \in M_n$ . Therefore  $M_n$  is an  $A$ -submodule of  $M$ .

Furthermore, if  $m \in M_n$ , then  $m \in M_{n+1}$  by multiplying the corresponding polynomial by  $X$ .

Since  $M$  is noetherian, the chain  $M_0 \subseteq M_1 \subseteq \dots$  terminates, so we have  $k$  such that  $\forall n \geq k: M_n = M_k$ . Each of  $M_0, \dots, M_k$  is a finitely generated submodule of  $M$ , say  $M_j$  is generated by  $m_{j1}, \dots, m_{j\ell_j}$ . There are polynomials  $f_{j1}, \dots, f_{j\ell_j} \in N$  such that  $\deg f_{ji} = j$  and the leading coefficient of  $f_{ji}$  is  $m_{ji}$ .

We will show that the finite set  $\{f_{ji} \mid 0 \leq j \leq N, 1 \leq i \leq \ell_j\}$  generated  $N$ .

We will use induction on  $\deg f$ , where  $f \in N$ . If  $\deg f = 0$ , then  $f = m$  for some  $m \in M$ . By definition of  $M_0$ ,  $m \in M_0$ , and  $m$  is in the submodule generated by the  $f_{0i}$ .

Assume next that  $0 < \deg f \leq k$  and that the claim is true for smaller degrees. Let  $m$  be the leading coefficient of  $f$ . Then  $m \in M_n$  so we may write

$$m = \sum_j a_{nj} m_{nj}.$$

Then

$$f - \sum_j a_{nj} f_{nj}$$

is in  $N$  and of smaller degree, so is expressible as a linear combination of the  $f_{ij}$ , so  $f$  is expressible as a linear combination as well.

Finally, assume that  $N < \deg f$  and that the claim is true for smaller degrees. If  $m$  is the leading coefficient of  $f$ , then  $m \in M_n = M_k$  so we may write

$$m = \sum_j a_{kj} m_{kj}.$$

Then

$$f - X^{n-k} \sum_j a_{kj} f_{kj}$$

is in  $N$  and of smaller degree, so is expressible as a linear combination of the  $f_{ij}$ , so  $f$  is expressible as a linear combination as well.  $\square$

### Exercise 10.

EXERCISE. An element  $r$  lies in the Jacobson radical of  $A$  iff  $1 - rs$  is a unit for all  $s$  in  $A$ .

SOLUTION. Let  $r \in J(A)$  and  $s \in A$ . Then  $rs \in J(A)$ , so  $rs$  is contained in every maximal ideal of  $A$ . If  $1 - rs$  were contained in a maximal ideal  $M$ , then we would have  $1 \in M$ , a contradiction. So  $1 - rs$  is not contained in any maximal ideal, so  $(1 - rs)$  is not contained in any maximal ideal, so we must have  $(1 - rs) = (1)$ , hence  $1 - rs$  is a unit.

Conversely, assume that  $1 - rs$  is a unit for every  $s$ , and let  $M$  be a maximal ideal of  $A$ . Suppose that  $r \notin M$ . Then  $A = M + Ar$ , so we find  $m \in M$  and  $s \in A$  such that  $1 = m + rs$ , but then  $m = 1 - rs$  is a unit, a contradiction. Hence  $r \in M$  and therefore  $r \in J(R)$ .  $\square$

### Exercise 11.

EXERCISE. Any field  $K$  which is finitely generated as a ring is a finite field.

SOLUTION. Suppose that  $K$  has characteristic zero. Then we can identify  $\mathbb{Q}$  with a subfield of  $K$ . Since  $K$  is finitely generated as a  $\mathbb{Z}$ -algebra (this is just a different way of saying that  $K$  is finitely generated as a ring), it is certainly finitely generated as a  $\mathbb{Q}$ -algebra. By Zariski's lemma,  $K$  is a finite-dimensional  $\mathbb{Q}$ -vector space.

Hence, all assumptions for the Artin-Tate lemma for the chain  $\mathbb{Z} \subseteq \mathbb{Q} \subseteq K$  are satisfied, so we find that  $\mathbb{Q}$  is a finitely generated  $\mathbb{Z}$ -algebra. This is of course nonsense: if we had finitely many generators, then only finitely many primes could appear as divisors of denominators in  $\mathbb{Q}$ .

Therefore,  $K$  has characteristic  $p > 0$  and is a finitely generated  $\mathbb{Z}$ -algebra, so  $K$  is also a finitely generated  $\mathbb{Z}/p\mathbb{Z}$ -algebra. Hence, by Zariski's lemma,  $K$  is a finite-dimensional  $\mathbb{Z}/p\mathbb{Z}$ -vector space, hence  $K$  is finite.  $\square$

### Exercise 12.

EXERCISE. Let  $I$  be an ideal contained in the Jacobson radical of  $A$ , and let  $M$  be an  $A$ -module and  $N$  be a finitely generated  $A$ -module. Let  $\theta: M \rightarrow N$  be a homomorphism of  $A$ -modules. If the induced map  $M/IM \rightarrow N/IN$  is surjective, then  $\theta$  is surjective.

SOLUTION. Let  $n \in N$ . By surjectivity of the induced map, we find  $m \in M$  such that  $\theta(m) + IN = n + IN$ . Hence we find  $i \in I$  and  $n_1 \in N$  such that  $\theta(m) = n + in_1$ , hence  $n + \theta(M) = i(-n_1) + \theta(M)$ . Since  $n$  was arbitrary, we conclude

$$\frac{N}{\theta(M)} \subseteq I \frac{N}{\theta(M)} \subseteq J(A) \frac{N}{\theta(M)} \subseteq \frac{N}{\theta(M)}.$$

Since  $N$  is finitely generated, so is  $N/\theta(M)$ , and by Nakayama's lemma, we must have  $N/\theta(M) = 0$ , so  $\theta$  is surjective.  $\square$

### Exercise 13.

EXERCISE. In the ring  $A$ , let  $\Sigma$  be the set of all ideals in which every element is a zero-divisor. Show that the set  $\Sigma$  has maximal elements and that every maximal element of  $\Sigma$  is a prime ideal. Hence show that the set of zero-divisors in  $A$  is a union of prime ideals.

SOLUTION. For a zero divisor  $a \in A$  denote by  $\Sigma_a$  the set of ideals containing  $a$  in which every element is a zero divisor (notice that  $\Sigma = \Sigma_0$ ). Since  $(a) \in \Sigma_a$ , we know that  $\Sigma_a$  is nonempty. Furthermore, the union of a chain of ideals in  $\Sigma_a$  is once again an element of  $\Sigma_a$ , hence  $\Sigma_a$  admits a maximal element  $I_a$  by Zorn's lemma.

Let  $x, y \in A$  such that  $xy \in I_a$ ,  $x \notin I_a$  and  $y \notin I_a$ . Then The ideals  $I_a + Ax$  and  $I_a + Ay$  contain non-zero-divisors  $u$  and  $v$ . Write  $u = i + u_1x$  and  $v = j + u_2y$  with  $i, j \in I_a$ ,  $u_1, u_2 \in A$ . Then  $uv = ij + iu_2y + ju_1x + u_1u_2xy \in I_a$ , hence  $uv$  is a zero divisor, but then  $u$  and  $v$  are also zero divisors, a contradiction.

Hence  $I_a$  is prime and if  $Z$  is the set of zero divisors, then we find that

$$Z = \bigcup_{a \in Z} I_a$$

as required.  $\square$

### Exercise 15.

LEMMA. Let  $q_1, \dots, q_n$  be pairwise distinct maximal ideals of a ring  $A$ . Then we have

$$\bigcap_{i=1}^n q_i = \prod_{i=1}^n q_i.$$

PROOF. We will proceed by induction on  $n$ . The claim is obviously true for  $n = 1$ . Suppose that

$$Q := \bigcap_{i=1}^n q_i = \prod_{i=1}^n q_i$$

and  $q_{n+1}$  is a maximal ideal distinct from the  $q_i$ . We have  $Q \not\subseteq q_{n+1}$ , because otherwise there would be some  $i \leq n$  such that  $q_i \subseteq q_{n+1}$ , since  $q_{n+1}$  is prime. But then we would have  $q_i = q_{n+1}$ , a contradiction. Hence  $Q + q_{n+1} = A$  by maximality of  $q_{n+1}$ , so we find  $u \in Q$  and  $v \in q_{n+1}$  such that  $u + v = 1$ . It is obvious that  $Qq_{n+1} \subseteq Q \cap q_{n+1}$ . Conversely, let  $x \in Q \cap q_{n+1}$ . Then  $x = x(u + v) = xu + xv \in Qq_{n+1}$ , so the claim follows.  $\square$

LEMMA. Let  $A$  be an artinian ring. Then  $A$  has finitely many maximal ideals.

PROOF. Otherwise, let  $q_1, q_2, \dots$  denote pairwise distinct maximal ideals of  $A$ . Define  $Q_n := \bigcap_{i=1}^n q_i$ . Then  $Q_n \supsetneq Q_{n+1}$ , since otherwise we would have  $Q_n \subseteq q_{n+1}$ , but by the preceding lemma and primality of  $q_{n+1}$ , this would imply that  $q_i \subseteq q_{n+1}$  for some  $i \leq n$ , hence  $q_i = q_{n+1}$ , which is not the case. Therefore, the  $Q_i$  form a strictly descending chain, which cannot exist since  $A$  is artinian.  $\square$

EXERCISE. Let  $A$  be an artinian ring. Then  $A$  is noetherian.

SOLUTION. By the second lemma,  $A$  has finitely many maximal ideals  $q_1, \dots, q_n$ . By a result from the lecture, since  $A$  is artinian, we have  $N(A) = J(A)$ , hence  $\sqrt{0} = \bigcap_{i=1}^n q_i = \prod_{i=1}^n q_i$ , where we have used the first lemma in the second step. Define  $Q_m := \prod_{i=1}^n q_i^m = \sqrt{0}^m$ . The  $Q_i$  form a decreasing chain of ideals. Since  $A$  is artinian, this chain terminates, say at  $Q_k$ . We claim that  $Q_k = 0$ . Indeed, assume that there is some  $0 \neq a \in Q_k$ . Since  $a \in \sqrt{0}$ ,  $a_\ell = 0$  for some  $\ell$ . But then...?  $\square$

**Exercise 4.**

EXERCISE. If  $A$  is a noetherian ring, then  $A[[X]]$  is a noetherian ring.

SOLUTION. Let  $I \subseteq A[[X]]$  be an ideal. For a natural number  $n$  define  $R(n)$  to be the set of trailing coefficients of elements of the form  $a_n X^n +$  higher order terms in  $I \cap (X^n)$ . As in the proof of Hilbert's basis theorem, we have  $R(0) \subseteq \dots$ . Since  $A$  is noetherian, we find  $N$  such that  $R(n) = R(N)$  for all  $n \geq N$ . For  $0 \leq i \leq N$ ,  $R(i)$  is finitely generated, say by  $r_{ij}$ ,  $0 \leq i \leq N$ ,  $1 \leq j \leq k_i$ . We find  $f_{ij} \in I$  such that  $f_{ij} = r_{ij} X^i +$  higher order terms. We claim that  $I$  is generated by the  $f_{ij}$ . Indeed, if  $f \in I$ , we can choose  $c_{ij}$  for  $1 \leq i \leq N$ ,  $1 \leq j \leq k_i$  such that  $f' := f - \sum_{i,j} c_{ij} f_{ij} \in (X^{N+1})$ .

Now let  $g_i \in A[[X]]$ ,  $1 \leq i \leq k_N$ . Write  $f' = \sum_{j=N+1}^{\infty} a_j X^j$ ,  $f_{Ni} = \sum_{j=N}^{\infty} b_{ij} X^j$ ,  $g_i = \sum_{j=0}^{\infty} c_{ij} X^j$ . For any  $k$ , the  $k$ -th coefficient of  $\sum_{i=1}^{k_N} f_{Ni} g_i$  is given by

$$\begin{aligned} \sum_{t=1}^{k_N} \sum_{i+j=k} b_{ti} c_{tj} &= \sum_{t=1}^{k_N} \sum_{i=1}^k b_{ti} c_{t(k-i)} \\ &= \left( \sum_{t=1}^{k_N} \sum_{i=N+1}^k b_{ti} c_{t(k-i)} \right) + \sum_{t=1}^{k_N} r_{Ni} c_{t(k-N)}. \end{aligned}$$

Hence, we can define the  $g_i$  inductively in such a way that the  $k$ -th coefficient of  $\sum f_{Ni} g_i$  is precisely  $a_k$ : since  $R(k) = R(N)$ , there is a choice of  $c_{t(k-N)}$  that works. Therefore,  $f' = \sum f_{Ni} g_i$ , so  $f$  is indeed in the span of the  $f_{ij}$ , hence  $I$  is finitely generated.  $\square$

LEMMA. If  $\varphi: R \rightarrow S$  is a surjective homomorphism of rings and  $R$  is noetherian, then  $S$  is noetherian.

PROOF. Any chain of ideals  $I_i$  of  $R$  can be pulled back to a chain  $\varphi^{-1}(I_i)$  of ideals in  $R$ . Since  $R$  is noetherian, this chain terminates, but since  $I_i = \varphi(\varphi^{-1}(I_i))$  by surjectivity of  $\varphi$ , the chain  $I_i$  terminates as well.  $\square$

EXERCISE. If  $A$  is a ring and  $A[X]$  or  $A[[X]]$  is noetherian, then so is  $A$ .

SOLUTION. There are surjective maps  $A[X] \rightarrow A$  and  $A[[X]] \rightarrow A$  sending a polynomial or formal power series to its constant term, hence the claim follows using the previous lemma.  $\square$

**Exercise 14.**

EXERCISE. If  $M, M', M''$  have finite length and we have a short exact sequence

$$0 \longrightarrow M' \xrightarrow{\iota} M \xrightarrow{\varphi} M'' \longrightarrow 0,$$

then  $\ell(M') - \ell(M) + \ell(M'') = 0$

SOLUTION. If

$$M' = M'_0 \supset M'_1 \supset \dots \supset M'_n = 0$$

and

$$M'' = M''_0 \supset M''_1 \supset \dots \supset M''_m = 0$$



are composition series, then

$$M = \varphi^{-1}(M''_0) \supset \cdots \supset \varphi^{-1}(M''_m) = \iota(M'_0) \supset \cdots \supset \iota(M'_n) = 0$$

is a composition series, since  $\varphi$  induces an isomorphism

$$\frac{\varphi^{-1}(M'_i)}{\varphi^{-1}(M'_{i+1})} \rightarrow \frac{M'_i}{M'_{i+1}}.$$

Hence  $\ell(M) = \ell(M') + \ell(M'')$ .  $\square$

EXERCISE. If  $V$  is a  $k$ -vector space, the following are equivalent:

- (1)  $V$  has finite dimension,
- (2)  $V$  has finite length,
- (3)  $V$  satisfies the ascending chain condition,
- (4)  $V$  satisfies the descending chain condition.

SOLUTION. If  $V$  has a finite basis  $v_1, \dots, v_n$ , then defining  $V_i := \langle v_1, \dots, v_{n-i} \rangle$  gives a composition series, hence (1) implies (2).

(2) implies (3) and (2) implies (4) by part (i) of the exercise.

If  $V$  is not finite-dimensional, then choose a basis  $B$  and let  $v_1, v_2, \dots \in B$  pairwise distinct. Then

$$\langle v_1 \rangle \subsetneq \langle v_1, v_2 \rangle \subsetneq \cdots$$

is an infinite strictly ascending chain and

$$\langle B \rangle \supsetneq \langle B \setminus \{v_1\} \rangle \supsetneq \cdots$$

is an infinite strictly descending chain. Hence (3) implies (1) and (4) implies (1).  $\square$

EXERCISE. If  $A$  is a ring in which the zero ideal is a product  $P_1 \cdots P_n$  of not necessarily distinct maximal ideals, then  $A$  is noetherian iff  $A$  is artinian

SOLUTION. Consider the chain

$$A \supseteq P_1 \supseteq P_1 P_2 \supseteq \cdots \supseteq P_1 \cdots P_n = 0.$$

The  $A$ -module  $A_i := P_1 \cdots P_i / P_1 \cdots P_{i+1}$  is an  $A/P_{i+1}$ -vector space in the obvious way. If  $A$  is noetherian or artinian, then so is  $A_i$  (since it is a quotient of a submodule of  $A$ ), and by part (iii) we obtain a composition series for  $A_i$ , using that an  $A$ -submodule is the same thing as a  $A/P_{i+1}$ -submodule.

Pulling back the composition series along the projection for every  $i$  and stitching together the results, we obtain a composition series for  $A$ . Again by part (i), we find that  $A$  is both noetherian and artinian.  $\square$

## Example Sheet 2

### Exercise 1.

EXERCISE. If  $S$  is a multiplicatively closed subset of a ring  $R$ , and  $M$  is a finitely generated  $R$ -module, then  $S^{-1}M = 0$  if and only if there is some  $s \in S$  such that  $sM = 0$ .

SOLUTION. If  $S^{-1}M = 0$ , then for all  $m \in M$  we have  $(m, 1) \sim (0, 1)$ , hence we find  $s \in S$  such that  $sm = 0$ . In particular, if  $M$  is generated by  $m_1, \dots, m_n$ , we find  $s_i$  such that  $s_i m_i = 0$ . Define  $s := \prod s_i$ , then for any  $m \in M$ , we find  $r_i \in R$  such that  $sm = s(r_1 m_1 + \cdots + r_n m_n) = 0$ , so  $sM = 0$ .

The converse direction is trivial.  $\square$

**Exercise 2.**

EXERCISE. Let  $I$  be an ideal of  $R$ , and define  $S := 1 + I$ . Then  $S^{-1}I \subseteq J(S^{-1}R)$ .

SOLUTION. Let  $i/s \in S^{-1}I$  and let  $r/t \in S^{-1}R$ . Then

$$\alpha := 1 - \frac{i}{s} \frac{r}{t} = 1 - \frac{ri}{st} = \frac{st - ri}{st}.$$

We have  $st \in 1 + I$  and  $ri \in I$ , hence  $st - ri \in 1 + I$ , so  $\alpha$  is a unit. By Exercise 10 on Example Sheet 1 we have  $i/s \in J(S^{-1}R)$ .  $\square$

**Exercise 3.**

EXERCISE. A multiplicatively closed set is saturated if and only if  $R \setminus S$  is a union of prime ideals.

SOLUTION. If  $S$  is saturated and  $x \in R \setminus S$ , let  $\Sigma$  denote the set of ideals  $I$  such that  $I \subseteq R \setminus S$  and  $x \in I$ .

If  $y \in R$ , then  $xy \in R \setminus S$ , since otherwise we would have  $x \in S$  by saturation of  $S$ . Hence  $(x) \in \Sigma$ .

The set  $\Sigma$  admits upper bounds, as the union of a chain of ideals once again is an ideal in  $\Sigma$ .

Hence we have a maximal element  $I \in \Sigma$ , which is prime, since if  $ab \in I$  and  $a \notin I$ ,  $b \notin I$ , then  $I + Ra$  and  $I + Rb$  both intersect nontrivially with  $S$ , so for  $s_1 \in S \cap I + Ra$ ,  $s_2 \in S \cap I + Rb$  we have  $s_1 s_2 \in S \cap I = \emptyset$ , a contradiction.

Hence every element of  $R \setminus S$  is contained in a prime ideal which is fully contained in  $R \setminus S$ , so  $R \setminus S$  is the union of these prime ideals.

Conversely, if  $R \setminus S$  is the union of prime ideals and  $xy \in S$ , then if  $x \notin S$ , then  $x$  was contained in one of the ideals, and by the ideal property, so would be  $xy$ , a contradiction. Hence  $x \in S$  and symmetrically  $y \in S$ .  $\square$

EXERCISE. If  $S$  is a multiplicatively closed subset of  $R$ , there is a unique smallest saturated multiplicatively closed subset  $S'$  containing  $S$ , and it is given as the complement in  $R$  of the union of the prime ideals which do not meet  $S$ .

SOLUTION. Define  $S'$  as the complement of the unions of the prime ideals which do not meet  $S$ . The set  $S'$  is multiplicatively closed (since the ideals are prime) and saturated by (i). Furthermore, by definition have  $R \setminus S' \subseteq R \setminus S$ , hence  $S \subseteq S'$ .

Let  $S''$  be a saturated multiplicatively closed subset satisfying  $S \subseteq S''$ . By (i),  $R \setminus S''$  is a union of prime ideals  $p_i$ . Let  $p_i \subseteq R \setminus S'' \subseteq R \setminus S$  be one of these prime ideals. Then  $p_i \cap S = \emptyset$ , so  $p_i \subseteq R \setminus S'$ . Hence we have  $R \setminus S'' \subseteq R \setminus S'$ , so  $S' \subseteq S''$ , completing the proof that  $S'$  is minimal.  $\square$

**Exercise 4.**

EXERCISE. Let  $S, T$  be two multiplicatively closed subsets of  $R$ , and let  $U$  be the image of  $T$  in  $S^{-1}R$ . Then  $(ST)^{-1}R$  and  $U^{-1}S^{-1}R$  are isomorphic as rings.

SOLUTION. Consider the following commutative diagram.

$$\begin{array}{ccccc} R & \xrightarrow{\alpha} & S^{-1}R & \xrightarrow{\delta} & U^{-1}S^{-1}R \\ & \searrow \beta & \downarrow \gamma & \nearrow \Phi & \uparrow \Psi \\ & & (ST)^{-1}R & & \end{array}$$

The maps  $\alpha, \beta$  and  $\delta$  are localization maps. Since  $S \subseteq ST$ ,  $\beta(s)$  is a unit for every  $s \in S$ , hence from the universal property of localization we have  $\gamma: S^{-1}R \rightarrow (ST)^{-1}R$  satisfying  $\gamma \circ \alpha = \beta$ . An element of  $U$  is of the form  $\alpha(t)$  for some  $t \in T$ .

We have  $\gamma(\alpha(t)) = \beta(t)$ , which is invertible, hence again from the universal property we have a map  $\Phi: U^{-1}S^{-1}R \rightarrow (ST)^{-1}R$  such that  $\Phi \circ \delta = \gamma$ . We know how this map is defined: if  $r \in R$ ,  $s \in S$ ,  $t \in T$ , we have

$$\Phi\left(\frac{r/s}{\alpha(t)}\right) = \gamma(r/s)\gamma(\alpha(t))^{-1} = \beta(r)\beta(s)^{-1}\beta(t)^{-1} = \frac{r}{1} \frac{1}{s} \frac{1}{t} = \frac{r}{st}.$$

Next, let  $st \in ST$ . We have

$$\delta(\alpha(st)) = \delta(\alpha(s))\delta(\alpha(t)) = \frac{s/1}{1} \frac{\alpha(t)}{1}.$$

This has the inverse

$$\frac{1/s}{1} \frac{1}{\alpha(t)},$$

so it is a unit, and the universal property yields  $\Psi: (ST)^{-1}R \rightarrow U^{-1}S^{-1}R$  satisfying  $\Psi \circ \beta = \delta \circ \alpha$ . Again, if  $r \in R$ ,  $s \in S$  and  $t \in T$ , we have

$$\Psi\left(\frac{r}{st}\right) = \delta(\alpha(r))\delta(\alpha(st))^{-1} = \frac{r/1}{1} \frac{1/s}{1} \frac{1}{\alpha(t)} = \frac{r/s}{\alpha(t)},$$

where we have used our inverse calculation from above.

Hence,  $\Phi$  and  $\Psi$  are two-sided inverses of each other, finishing the proof.  $\square$

### Exercise 5.

EXERCISE. Let  $R$  be a ring. Suppose that for each prime ideal  $P$  the local ring  $R_P$  has no non-zero nilpotent element. Then  $R$  has no nonzero nilpotent element.

SOLUTION. Let  $x \in R$  be a nilpotent element. Consider the ideal  $\text{ann}(x) = \{r \in R \mid rx = 0\}$ . If  $\text{ann}(x) \neq R$ , then  $\text{ann}(x) \subseteq \mathfrak{m}$  for some maximal ideal  $\mathfrak{m}$ . Then  $\mathfrak{m}$  is prime. Let  $\varphi: R \rightarrow R_{\mathfrak{m}}$ . Since  $x$  is nilpotent, we find  $n$  such that  $x^n = 0$ . Then  $\varphi(x)^n = \varphi(x^n) = 0$ , hence  $\varphi(x) = 0$ . By definition of localization, this means that there is some  $s \in R \setminus \mathfrak{m}$  such that  $sx = 0$ . But then  $s \in \text{ann}(x)$ , which is a contradiction. Hence we must have  $\text{ann}(x) = R$ , in particular  $x = 1 \cdot x = 0$ .  $\square$

EXERCISE. There is a ring  $R$  such that  $R$  is not an integral domain, but for every prime ideal  $P$  of  $R$ ,  $R_P$  is an integral domain.

SOLUTION. Define  $R := \mathbb{Z}/6\mathbb{Z}$ . The prime ideals of  $R$  are (2) and (3). By writing down all elements and checking the relations between them, we can check that the localizations at both of them are fields, hence integral domains.  $\square$

### Exercise 7.

EXERCISE. Suppose  $R \neq 0$  and let  $\Sigma$  be the set of all multiplicatively closed subsets  $S$  of  $R$  such that  $0 \notin S$ . Then  $\Sigma$  has maximal elements, and  $S \in \Sigma$  is maximal if and only if  $R \setminus S$  is a minimal prime ideal of  $R$ .

SOLUTION. The union of a chain in  $\Sigma$  is again an element of  $\Sigma$ , and the singleton set  $\{1\}$  is an element of  $\Sigma$ . Hence,  $\Sigma$  admits maximal elements by Zorn's lemma.

If  $S \in \Sigma$  is maximal, we claim that  $I := R \setminus S$  is a prime ideal. If  $r, s \in I$ , then  $SM_r$  and  $SM_s$ , where  $M_r := \{1, r, r^2, \dots\}$ , are multiplicatively closed subsets. Since  $r, s \notin S$ , these are strictly larger than  $S$ , hence must contain 0, i.e., we find natural numbers  $n, m$  and  $x, y \in S$  such that  $xr^n = 0 = ys^m$ . Then  $xy(r+s)^{n+m} = 0$  by the binomial theorem, so we must have  $r+s \in I$ , since otherwise we would have  $0 \in S$ , a contradiction.

If  $r \in R$ ,  $t \in I$ , then again we find  $n \in \mathbb{N}$  and  $x \in S$  such that  $xt^n = 0$ . Then  $r^n xt^n = 0$ , so if we have  $rt \in S$ , then  $0 \in S$ , hence  $rt \in I$ . This makes  $I$  into an ideal.

Next, let  $r, s \in R$  such that  $rs \in I$ . Again, this means that we find  $n \in \mathbb{N}$  and  $t \in S$  such that  $(rs)^n t = 0$ . If  $r$  and  $s$  were both in  $S$ , this would again lead to the contradiction  $0 \in S$ , hence  $r \in I \vee s \in I$ , making  $I$  into a prime ideal.

It remains to show that  $I$  is minimal. If  $\mathfrak{p} \subseteq I$  is a prime ideal, then  $R \setminus \mathfrak{p}$  is multiplicative, does not contain 0, and satisfies  $S \subseteq R \setminus \mathfrak{p}$ . By maximality of  $S$ , we find  $S = R \setminus \mathfrak{p}$ , so  $\mathfrak{p} = I$ .

Conversely, assume that  $R \setminus S$  is a minimal prime ideal. Then  $S$  is multiplicative, because  $R \setminus S$  is prime. Suppose  $S$  is not maximal. Then we have  $S \subsetneq S'$  for some maximal element  $S'$  of  $\Sigma$ . Then by what we have just shown,  $R \setminus S'$  is a minimal prime ideal, but then  $R \setminus S$  cannot be a minimal prime, since it is a strict superset of  $R \setminus S'$ .  $\square$

**EXERCISE.** Every minimal prime ideal of  $R$  is contained in  $D$ , the set of zero divisors of  $R$ .

**SOLUTION.** Let  $a \in S_0$  be a non-zero-divisor and let  $S$  be a maximal element of  $\Sigma$ . Then  $SM_a$  cannot contain 0, since  $S$  does not contain 0 and  $M_a$  does not contain zero divisors. Hence  $S \subseteq SM_a \in \Sigma$ , which implies  $SM_a = S$  by maximality. In particular,  $a \in S$ , so  $S_0 \subseteq S$  for every maximal element  $S$  of  $\Sigma$ .

Now if  $\mathfrak{p}$  is a minimal prime ideal, then  $R \setminus \mathfrak{p}$  is a maximal element of  $\Sigma$ . Hence we have  $S_0 \subseteq R \setminus \mathfrak{p}$ . Taking complements, we obtain  $\mathfrak{p} \subseteq R \setminus S_0 = D$  as required.  $\square$

- EXERCISE.**
- (i)  $S_0$  is the largest multiplicatively closed subset of  $R$  for which the homomorphism  $R \rightarrow S_0^{-1}R$  is injective.
  - (ii) Every element in  $S_0^{-1}R$  is either a zero-divisor or a unit.
  - (iii) Every ring in which every non-unit is a zero-divisor is equal to its total ring of fractions satisfies that  $R \rightarrow S_0^{-1}R$  is bijective.

**SOLUTION.** For (i), let  $S$  be any multiplicatively closed set. We claim that  $\varphi: R \rightarrow S^{-1}R$  is injective if and only if  $S$  contains no zero divisors.

Indeed,  $\varphi(r) = 0$  if and only if  $r/1 = 0/1 \in S^{-1}R$ , i.e., if and only if there exists  $s \in S$  such that  $rs = 0$ . So there is some nonzero  $r$  satisfying  $\varphi(r) = 0$  if and only if  $S$  contains a zero divisor.

Since  $S_0$  is the largest multiplicatively closed subset without zero divisors, the claim follows.

For (ii), let  $r/s \in S_0^{-1}R$ . If  $r \in S_0$ , then  $r/s$  is a unit, since  $r/s \cdot s/r = 1$ . Conversely, if  $r \notin S_0$ , then  $r$  is a zero divisor, so we find  $0 \neq q \in R$  such that  $rq = 0$ . Since  $S_0$  does not contain zero divisors, we have  $q/1 \neq 0 \in S^{-1}R$ . Then  $r/s \cdot q/1 = 0/s = 0 \in S^{-1}R$ , so  $r/s$  is a zero divisor.

For (iii), observe that if every non-unit is a zero divisor, every non-zero-divisor is a unit. Hence the universal property of localisation yields a map  $\theta$  making the diagram

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & S_0^{-1}R \\ & \searrow \text{id} & \downarrow \theta \\ & & R \end{array}$$

commute. It remains to verify that  $\varphi \circ \theta = \text{id}_{S_0^{-1}R}$ . Indeed, if  $r \in R$  and  $s \in S_0$ , then  $\varphi(\theta(r/s)) = \varphi(rs^{-1}) = (rs^{-1})/1$ . But since  $1(rs^{-1}s - r) = 0 \in R$ , we have  $r/s = (rs^{-1})/1 \in S_0^{-1}R$ , completing the proof.  $\square$

### Exercise 8.

**EXERCISE.** If  $m, n \in \mathbb{Z}$  are coprime, then  $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$  is trivial.

SOLUTION. Bézout's lemma yields  $a, b \in \mathbb{Z}$  such that  $am + bn = 1$ . The module  $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$  is generated by elements of the form  $x \otimes y$ ,  $x \in \mathbb{Z}/m\mathbb{Z}$ ,  $y \in \mathbb{Z}/n\mathbb{Z}$ . For any such element, we have

$$\begin{aligned} x \otimes y &= (am + bn)(x \otimes y) = am(x \otimes y) + bn(x \otimes y) = a(mx \otimes y) + b(x \otimes ny) \\ &= a(0 \otimes y) + b(x \otimes 0) = 0. \end{aligned} \quad \square$$

**Exercise 11.**

EXERCISE. Let  $M_i$  ( $i \in I$ ) be a family of  $R$ -modules and let  $M$  be their direct sum. Then  $M$  is flat if and only if each  $M_i$  is flat.

SOLUTION. The proof of the distributive property for tensor products generalizes without changes to yield an isomorphism

$$\begin{aligned} A \otimes \bigoplus_i M_i &\rightarrow \bigoplus_i A \otimes M_i \\ a \otimes m_i &\mapsto a \otimes m_i, \end{aligned}$$

where  $a \in A$  and  $m_i \in M_i$ .

Since a sequence of direct sums of maps is exact if and only if the corresponding sequences are exact, the claim follows.  $\square$

EXERCISE.  $R[X]$  is a flat  $R$ -algebra.

SOLUTION.  $R[X]$  is an  $R$ -algebra. Flatness is a property of  $R$ -modules, and as an  $R$ -module, we have  $R[X] \cong \bigoplus_{n \in \mathbb{N}} R$ . Since  $R$  is flat, the claim follows.  $\square$

**Exercise 14.**

EXERCISE. The torsion elements of  $M$  form a submodule  $T(M)$  of  $M$ .

SOLUTION. If  $m \in T(M)$ , i.e., we find  $0 \neq r \in A$  such that  $rm = 0$  and  $s \in A$ , then  $sm \in T(M)$ , since  $rs m = srm = s0 = 0$ .

If  $m, n \in T(M)$ , i.e., we find  $0 \neq r, s \in A$  such that  $rm = 0 = sn$ , then  $rs \neq 0$  since  $A$  is an integral domain, so  $m+n \in T(M)$ , since  $rs(m+n) = srm + rs n = 0$ .  $\square$

EXERCISE. If  $M$  is an  $A$ -module, then  $M/T(M)$  is torsion-free.

SOLUTION. If  $m + T(M) \in M/T(M)$  satisfies  $r(m + T(M)) = 0$  for  $r \neq 0$ , then  $rm \in T(M)$ , hence we find  $s \neq 0$  such that  $srm = 0$ . Since  $A$  is an integral domain,  $sr \neq 0$ , hence  $m \in T(M)$ , so  $m + T(M) = 0$ , so  $T(M/T(M)) = 0$  as claimed.  $\square$

EXERCISE. If  $f: M \rightarrow N$  is a homomorphism of  $A$ -modules, then  $f(T(M)) \subseteq T(N)$ .

SOLUTION. Let  $m \in T(M)$ , i.e., we find  $0 \neq r \in A$  such that  $rm = 0$ . Then  $rf(m) = f(rm) = f(0) = 0$ , hence  $f(m) \in T(N)$ .  $\square$

EXERCISE. If

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M''$$

is exact, then

$$0 \longrightarrow T(M') \xrightarrow{f|_{T(M')}} T(M) \xrightarrow{g|_{T(M)}} T(M'')$$

is exact.

SOLUTION. It is obvious that  $f|_{T(M')}$  is injective and that  $\text{im } f|_{T(M')}$  is contained in  $\ker g|_{T(M)}$ .

Let  $m \in \ker g|_{T(M)}$ . By exactness of the original sequence, we find  $m' \in M'$  such that  $f(m') = m$ . Since  $m \in T(M)$ , we find  $0 \neq r \in A$  such that  $0 = rm = rf(m') = f(rm')$ . By injectivity of  $f$ , we conclude  $rm' = 0$ , so  $m' \in T(M')$ , so  $m = f|_{T(M')}(m')$ , so  $m \in \text{im } f|_{T(M')}$ .  $\square$

**Exercise 15.**

EXERCISE. If  $S$  is a multiplicatively closed subset of an integral domain  $A$ , then  $T(S^{-1}M) = S^{-1}T(M)$  as  $S^{-1}R$ -submodules of  $S^{-1}M$ .

SOLUTION. The claim is trivial if  $0 \in S$ . Hence, in the remainder, we will assume that  $0 \notin S$ .

Let  $m/s \in T(S^{-1}M)$ , i.e., we find  $0 \neq r/t \in S^{-1}R$  such that  $rm/st = 0/1$ , i.e., there is some  $u \in S$  such that  $urm = 0$ . Observe that  $r \neq 0$ , otherwise we would have  $r/t = 0 \in S^{-1}R$ , and  $u \neq 0$ , since  $u \in S$ . Since  $A$  is an integral domain, this implies that  $ru \neq 0$ , hence  $m \in T(M)$ , so  $m/s \in S^{-1}T(M)$ .

Conversely, let  $m/s \in S^{-1}(T(M))$ . This means that we find  $0 \neq r \in A$  such that  $rm = 0$ . Then  $r/1 \cdot m/s = 0/s = 0 \in S^{-1}M$ , i.e.,  $m/s \in T(S^{-1}M)$ , completing the proof.  $\square$

EXERCISE. The following are equivalent for a module  $M$  over an integral domain  $A$ .

- (a)  $M$  is torsion-free,
- (b)  $M_{\mathfrak{p}}$  is torsion-free for all prime ideals  $\mathfrak{p}$ ,
- (c)  $M_{\mathfrak{m}}$  is torsion-free for all maximal ideals  $\mathfrak{m}$ .

SOLUTION. To show that (a) implies (b), notice that from the previous result we have  $T(M_{\mathfrak{p}}) = T(M)_{\mathfrak{p}}$  as  $S^{-1}R$ -submodules of  $M_{\mathfrak{p}}$ . But the right hand side is trivial as  $T(M) = 0$ .

The implication from (b) to (c) is trivial.

Finally, assume that  $T(M_{\mathfrak{m}})$  is trivial for all maximal ideals  $\mathfrak{m}$ . Let  $m \in T(M)$ . Consider the annihilator  $\text{ann}(m) = \{r \in R \mid rm = 0\}$ . Suppose  $\text{ann}(m)$  is a proper ideal. Then  $\text{ann}(m) \subseteq \mathfrak{m}$  for some maximal ideal  $\mathfrak{m}$ . By Exercise 14(ii) and our assumption,  $m$  is in the kernel of the map  $M \rightarrow M_{\mathfrak{m}}$ , so we find  $s \in R \setminus \mathfrak{m}$  such that  $sm = 0$ . But then  $s \in \text{ann}(m) \cap R \setminus \mathfrak{m} = \emptyset$ , a contradiction. We conclude  $\text{ann}(m) = R$ , so in particular  $m = 1 \cdot m = 0$ , i.e.,  $M$  is torsion-free.  $\square$