

Category Theory

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These notes, taken by Markus Himmel, will at times differ significantly from what was lectured. In particular, all errors are almost certainly my own.

Contents

Chapter 3. Adjunctions	5
Exercises	7
Chapter 1	7
Chapter 2	9

CHAPTER 3

Adjunctions

THEOREM 3.3. If $g: FA \rightarrow B$, then consider the square

$$\begin{array}{ccc} (FA \rightarrow FA) & \longrightarrow & (A \rightarrow GFA) \\ \downarrow & & \downarrow \\ (FA \rightarrow B) & \longrightarrow & A \rightarrow GB. \end{array}$$

Along the top right 1_{FA} is mapped to η_A and then to $Gg \circ \eta_A$. Along the bottom left, 1_{FA} is mapped to g and then to the morphism corresponding to g . Hence we have the the morphism corresponding with g is precisely $Gg \circ \eta_A$, i.e., if $f = Gg \circ \eta_A$, then f must correspond to g .

COROLLARY 3.4. From the initial objects we obtain the components:

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & GFA \\ & \searrow \eta'_A & \uparrow G\theta_A^{-1} \\ & & GF'A \end{array}$$

It remains to show naturality. Let $f: A \rightarrow A'$ be a morphism. By initiality, there is a unique morphism $\alpha: FA \rightarrow FA'$ making the square

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & GFA \\ \downarrow f & & \downarrow G\alpha \\ A' & \xrightarrow{\eta_{A'}} & GFA' \end{array}$$

commute. Recall that in the proof of 3.3 we saw that the morphism corresponding to $GFf \circ \eta_A: A \rightarrow GFA'$ is $Ff: FA \rightarrow FA'$. On the other hand, consider the adjunction square

$$\begin{array}{ccc} (FA' \rightarrow FA') & \longrightarrow & (A' \rightarrow GFA') \\ \downarrow & & \downarrow \\ (FA \rightarrow FA') & \longrightarrow & (A \rightarrow GFA'). \end{array}$$

Along the top right path, $1_{FA'}$ is mapped to $\eta_{A'}$ and then to $\eta_{A'} \circ f$. Along the bottom left path $1_{FA'}$ is mapped to Ff and then to the morphism corresponding with Ff . Hence, Ff corresponds to $\eta_{A'} \circ f$. But Ff also corresponds to $GFf \circ \eta_A$, so we must have $\eta_{A'} \circ f = GFf \circ \eta_A$, which just means that η is a natural transformation, and in particular, $\alpha = Ff$.

On the other hand, we may calculate that

$$\begin{aligned} G\theta_{A'}^{-1} \circ GF'f \circ G\theta_A \circ \eta_A &= G\theta_{A'}^{-1} \circ GF'f \circ \eta'_A \\ &= G\theta_{A'}^{-1} \circ \eta'_{A'} \circ f \\ &= \eta_{A'} \circ f, \end{aligned}$$

where we use that η' is a natural transformation for the same reason as η and that the triangle at the start commutes. Therefore, we find that $\alpha = \theta_{A'}^{-1} \circ F'f \circ \theta_A$. Rearranging, this yields $\theta_{A'} \circ Ff = F'f \circ \theta_A$, so θ is natural, which is what we wanted to show.

THEOREM 3.7. Let A be an object of \mathcal{C} . The naturality in the first variable of an adjunction asserts that

$$\begin{array}{ccc} (GFA \rightarrow GFA) & \longrightarrow & (FGFA \rightarrow FA) \\ \downarrow & & \downarrow \\ (A \rightarrow GFA) & \longrightarrow & (FA \rightarrow FA) \end{array}$$

is a commutative diagram, where the horizontal arrows are the adjunction and the vertical arrows are given by precomposition with η_A resp. $F\eta_A$.

Starting with 1_{GFA} , along the top right way we map to ϵ_{FA} and then to $\epsilon_{FA} \circ F\eta_A$. Along the bottom left way we map to $\eta_A \circ 1_{GFA} = \eta_A$ and then to 1_{FA} , since this is how we defined η_A . Thus $\epsilon_{FA} \circ F\eta_A = 1_{FA}$, so the first triangular identity holds.

Exercises

Chapter 1

Exercise 17.

EXERCISE. A morphism $e: A \rightarrow A$ is called idempotent if $ee = e$. An idempotent e is said to split if it can be factored as fg where gf is an identity morphism.

- (i) Let \mathcal{E} be a collection of idempotents in a category \mathcal{C} : show that there is a category $\mathcal{C}[\check{\mathcal{E}}]$ whose objects are the members of \mathcal{E} , whose morphisms $e \rightarrow d$ are those morphisms $f: \text{dom } e \rightarrow \text{dom } d$ in \mathcal{C} for which $dfe = f$, and whose composition coincides with composition in \mathcal{C} . [Hint: first show that the single equation $dfe = f$ is equivalent to the two equations $df = f = fe$. Note that the identity morphism on an object e is not $1_{\text{dom } e}$ in general.]
- (ii) If \mathcal{E} contains all identity morphisms of \mathcal{C} , show that there is a full and faithful functor $I: \mathcal{C} \rightarrow \mathcal{C}[\check{\mathcal{E}}]$, and that an arbitrary functor $T: \mathcal{C} \rightarrow \mathcal{D}$ can be factored as $\hat{T}I$ for some \hat{T} iff it sends the members of \mathcal{E} to split idempotents in \mathcal{D} .
- (iii) Deduce that if all idempotents split in \mathcal{D} , then the functor categories $[\mathcal{C}, \mathcal{D}]$ and $[\hat{\mathcal{C}}, \mathcal{D}]$ are equivalent, where $\hat{\mathcal{C}} = \mathcal{C}[\check{\mathcal{E}}]$ for \mathcal{E} the class of all idempotents in \mathcal{C} .

SOLUTION. We will first show that if $f: C \rightarrow D$ is any morphism and $c: C \rightarrow C$ and $d: D \rightarrow D$ are idempotents, then $dfe = f \iff df = f = fe$.

Indeed, if $df = f = fe$, then $dfe = fe = f$. Conversely, if $dfe = f$, then $f = dfe = ddfe = df$ and $f = dfe = dfee = fe$.

To show that $\mathcal{C}[\check{\mathcal{E}}]$ is a category, we need to show that the composition of two morphisms is indeed a morphism and that there are identity morphism.

Assume that $c: C \rightarrow C$, $d: D \rightarrow D$, $e: E \rightarrow E$ are idempotents and that $f: C \rightarrow D$ and $g: D \rightarrow E$ satisfy $dfe = f$ and $egd = g$. We need to show that $egfc = gf$. Using the lemma, we have $egf = (eg)f = gf$ and $gfc = g(fc) = gf$, so, again by the lemma, the claim follows.

If $e: E \rightarrow E$ is an idempotent, define $1_e := e \xrightarrow{e} e$. By idempotency of e , this is indeed a morphism. If $f: d \rightarrow e$ is a morphism, then the morphism $f1_d$ is the morphism $fd = f$ (here we use the lemma again) in \mathcal{C} , so $f1_d = f$ as required. Similarly, $1_e f = f$. This completes part (i).

Next, assume that \mathcal{E} contains all identity morphisms of \mathcal{C} . Define the functor I via

$$\begin{aligned} I: \mathcal{C} &\rightarrow \mathcal{C}[\check{\mathcal{E}}] \\ A &\mapsto 1_A \\ (f: A \rightarrow B) &\mapsto (f: 1_A \rightarrow 1_B) \end{aligned}$$

This is indeed a functor and since the data of a morphism $A \rightarrow B$ in \mathcal{C} is precisely the same as the data of a morphism $1_A \rightarrow 1_B$ in $\mathcal{C}[\check{\mathcal{E}}]$, I is fully faithful.

Now let $T: \mathcal{C} \rightarrow \mathcal{D}$ be any functor.

First, assume that there is some functor $\widehat{T}: \mathcal{C}[\check{\mathcal{E}}] \rightarrow \mathcal{D}$ such that $T = \widehat{T}I$. Let $e: A \rightarrow A \in \mathcal{E}$ be an idempotent. Then we have

$$\begin{aligned} Te &= \widehat{T}(1_A \xrightarrow{e} 1_A) \\ &= \widehat{T}(1_A \xrightarrow{e} e \xrightarrow{e} 1_A) \\ &= \widehat{T}(e \xrightarrow{e} 1_A) \circ \widehat{T}(1_A \xrightarrow{e} e), \end{aligned}$$

and we also have

$$\begin{aligned} \widehat{T}(1_A \xrightarrow{e} e) \circ \widehat{T}(e \xrightarrow{e} 1_A) &= \widehat{T}(e \xrightarrow{e} 1_A \xrightarrow{e} e) \\ &= \widehat{T}(e \xrightarrow{ee} e) \\ &= \widehat{T}(e \xrightarrow{e} e) \\ &= \widehat{T}(1_e) \\ &= 1_{\widehat{T}e}, \end{aligned}$$

which shows that Te is split.

Next, assume that Te is split for any $e \in \mathcal{E}$. For any $e \in \mathcal{E}$, choose a splitting

$$TA \xrightleftharpoons[f_e]{g_e} B_e,$$

i.e., $f_e \circ g_e = Te$, $g_e \circ f_e = 1_{B_e}$. For identity morphisms 1_A (A an object of \mathcal{C}), choose the specific splitting given by $B_{1_A} := TA$, $f_{1_A} := 1_{TA}$, $g_{1_A} := 1_{TA}$.

Now define the functor \widehat{T} via

$$\begin{aligned} \widehat{T}: \mathcal{C}[\check{\mathcal{E}}] &\rightarrow \mathcal{D} \\ (e: A \rightarrow A) &\mapsto B_e \\ (f: d \rightarrow e) &\mapsto g_e \circ Tf \circ f_d. \end{aligned}$$

If $e \in \mathcal{E}$, then we have

$$\begin{aligned} \widehat{T}(1_e) &= g_e \circ Te \circ f_e \\ &= g_e \circ f_e \circ g_e \circ f_e \\ &= 1_{B_e} \circ 1_{B_e} = 1_{B_e} \end{aligned}$$

Furthermore, if $f: c \rightarrow d$ and $g: d \rightarrow e$, then we have

$$\begin{aligned} \widehat{T}(g \circ f) &= g_e \circ T(g \circ f) = f_c \\ &= g_e \circ Tg \circ Tf = f_c \\ &= g_e \circ Tg \circ T(d \circ f) \circ f_c \\ &= g_e \circ Tg \circ Td \circ Tf \circ f_c \\ &= g_e \circ Tg \circ f_d \circ g_d \circ Tf \circ f_c \\ &= \widehat{T}g \circ \widehat{T}f. \end{aligned}$$

So \widehat{T} is indeed a functor. If A is an object of \mathcal{C} , then

$$\widehat{T}IA = \widehat{T}1_A = B_{1_A} = TA$$

and if $f: C \rightarrow D$ is a morphism in \mathcal{C} , then

$$\widehat{T}If = \widehat{T}(1_C \xrightarrow{f} 1_D) = g_{1_D} \circ Tf \circ f_{1_C} = 1_{TD} \circ Tf \circ 1_{TC} = Tf,$$

so \widehat{T} is the required factorisation, completing part (ii).

Define a functor $\Phi: [\widehat{\mathcal{C}}, \mathcal{D}] \rightarrow [\mathcal{C}, \mathcal{D}]$ via $F \mapsto F \circ I$, $\eta \mapsto I\eta$, where $I\eta$ is defined via $I\eta_C := \eta_{IC} = \eta_{1_C}$. Naturality of $I\eta$ immediately follows from naturality of η . Functoriality is also clear.

We will show that this functor is full, faithful and essentially surjective.

Indeed, let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a functor. Then \widehat{F} as defined in the previous part satisfies $\Phi\widehat{F} = F$, so Φ is essentially surjective.

Next, let $F, G: \widehat{\mathcal{C}} \rightarrow \mathcal{D}$ be functors and $\eta: F \circ I \rightarrow G \circ I$ a natural transformation. For an idempotent $e: A \rightarrow A$ in \mathcal{C} , define $\hat{\eta}_e$ to be the composite

$$Fe \xrightarrow{F(e \xrightarrow{e} 1_A)} F1_A = (F \circ I)A \xrightarrow{\eta_A} (G \circ I)A = G1_A \xrightarrow{G(1_A \xrightarrow{e} e)} Ge.$$

We claim that this defines a natural transformation $\hat{\eta}: F \rightarrow G$. Indeed, if $f: d \rightarrow e$ is a morphism, then

$$\begin{aligned} \hat{\eta}_e \circ Ff &= G(1_A \xrightarrow{e} e) \circ \eta_E \circ F(e \xrightarrow{e} 1_E) \circ F(d \xrightarrow{f} e) \\ &= G(1_E \xrightarrow{e} e) \circ \eta_E \circ F(d \xrightarrow{f} e \xrightarrow{e} 1_E) \\ &= G(1_E \xrightarrow{e} e) \circ \eta_E \circ F(d \xrightarrow{d} 1_D \xrightarrow{d} d \xrightarrow{f} e \xrightarrow{e} 1_E) \\ &= G(1_E \xrightarrow{e} e) \circ \eta_E \circ F(d \xrightarrow{d} 1_D \xrightarrow{d} d \xrightarrow{f} e \xrightarrow{e} 1_E) \\ &= G(1_E \xrightarrow{e} e) \circ \eta_E \circ F(1_D \xrightarrow{efd} 1_E) \circ F(d \xrightarrow{d} 1_D) \\ &= G(1_E \xrightarrow{e} e) \circ \eta_E \circ F(efd) \circ F(d \xrightarrow{d} 1_D) \\ &= G(1_E \xrightarrow{e} e) \circ G(efd) \circ \eta_D \circ F(d \xrightarrow{d} 1_D), \end{aligned}$$

and doing the whole thing backwards we conclude that $\hat{\eta}_e \circ Ff = Gf \circ \hat{\eta}_d$, so $\hat{\eta}$ is indeed a natural transformation.

For any $A \in \mathcal{C}$ we have

$$\begin{aligned} (I\hat{\eta})_A &= \hat{\eta}_{IA} = \hat{\eta}_{1_A} = G(1_A \xrightarrow{1_A} 1_A) \circ \eta_A \circ F(1_A \xrightarrow{1_A} 1_A) \\ &= G(1_{1_A}) \circ \eta_A \circ F(1_{1_A}) = \eta_A, \end{aligned}$$

which means that $\Phi(\hat{\eta}) = \eta$, so Φ is full.

Finally, let $F, G: \widehat{\mathcal{C}} \rightarrow \mathcal{D}$ be functors and $\eta, \eta': F \rightarrow G$ be natural transformations such that $\Phi(\eta) = \Phi(\eta')$. To show that Φ is faithful, we need to prove that $\eta = \eta'$. The assumption $\Phi(\eta) = \Phi(\eta')$ means that for all $A \in \mathcal{C}$ we have $\eta_{IA} = \eta'_{IA}$, so $\eta_{1_A} = \eta'_{1_A}$.

Let $e: A \rightarrow A$ be any idempotent in \mathcal{C} . We need to show that $\eta_e = \eta'_e$. Indeed, we have

$$\begin{aligned} \eta_e &= G(1_e) \circ \eta_e \\ &= G(e \xrightarrow{e} e) \circ \eta_e \\ &= G(e \xrightarrow{ee} e) \circ \eta_e \\ &= G(e \xrightarrow{e} 1_A \xrightarrow{e} e) \circ \eta_e \\ &= G(1_A \xrightarrow{e} e) \circ G(e \xrightarrow{e} 1_A) \circ \eta_e \\ &= G(1_A \xrightarrow{e} e) \circ \eta_{1_A} \circ F(e \xrightarrow{e} 1_A) \\ &= G(1_A \xrightarrow{e} e) \circ \eta'_{1_A} \circ F(e \xrightarrow{e} 1_A), \end{aligned}$$

and the same argument in backwards direction shows that $\eta_e = \eta'_e$, completing the proof. \square

Chapter 2

Exercise 13.

EXERCISE. The inner automorphisms of \mathcal{C} form a normal subgroup of the group of all automorphisms of \mathcal{C} . [Don't worry about whether these groups are sets or proper classes!]

SOLUTION. Let $F, G: \mathcal{C} \rightarrow \mathcal{C}$ be automorphisms and let $\alpha: F \rightarrow 1_{\mathcal{C}}$ be a natural isomorphism.

Let $A \in \mathcal{C}$. Define $\beta: GFG^{-1} \rightarrow 1_A$ via $\beta_A := G(\alpha_{G^{-1}A})$ (so $\beta_A: GFG^{-1}A \rightarrow GG^{-1}A = A \rightarrow GG^{-1}A = 1_{\mathcal{C}}A$).

This is indeed a natural transformation: let $f: A \rightarrow B \in \mathcal{C}$, then we can write the naturality square in a funny way,

$$\begin{array}{ccc} GFG^{-1}A & \xrightarrow{G(\alpha_{G^{-1}A})} & G1_{\mathcal{C}}G^{-1}A \\ \downarrow GFG^{-1}(f) & & \downarrow G1_{\mathcal{C}}G^{-1}f \\ GFG^{-1}B & \xrightarrow{G(\alpha_{G^{-1}B})} & G1_{\mathcal{C}}G^{-1}B \end{array}$$

and we see that it is just the functor G applied to the naturality diagram for α and the morphism $G^{-1}f$.

Therefore, β is a natural transformation, and since functors map isomorphisms to isomorphisms, it is also a natural isomorphism. So GFG^{-1} is an inner automorphism as required. \square

LEMMA. Let $1 \in \mathcal{C}$ be a terminal object and $F: \mathcal{C} \rightarrow \mathcal{C}$ an automorphism. Then $F1$ is a terminal object.

PROOF. If $A \in \mathcal{C}$, the functor F , which is fully faithful, induces a bijection between the collection of morphisms $F^{-1}A \rightarrow 1$ and the collection of morphisms $A \rightarrow F1$. Since 1 is terminal, there is exactly one morphism $A \rightarrow F1$. \square

EXERCISE. If $F: \mathbf{Set} \rightarrow \mathbf{Set}$ is an automorphism, then there is a unique natural isomorphism $1_{\mathcal{C}} \rightarrow F$.

SOLUTION. Of course, the terminal object in the category of sets is just the one-element set $1 = \{\star\}$. Since $F1$ is also terminal, it is in bijection with 1 . We write $F1 = \{\star_{F1}\}$.

By the Yoneda lemma, the set of natural transformations $\mathbf{Set}(1, -) \rightarrow F$ is in bijection with $F1$, so there is a unique natural transformation $\eta: \mathbf{Set}(1, -) \rightarrow F$. Examining the proof, we see that the components of this natural transformation are given by

$$\begin{aligned} \eta_A: \mathbf{Set}(1, A) &\rightarrow FA \\ f &\mapsto Ff(\star_{F1}) \end{aligned}$$

for any object A of \mathcal{C} . Let A be an object of \mathcal{C} . We will show that η_A is an isomorphism, i.e., a bijection.

First, let $x \in FA$. Then $\eta_A(F^{-1}(\star_{F1} \mapsto x)) = x$, so η_A is surjective.

Additionally, let $f, g: 1 \rightarrow A$ such that $\eta_A(f) = \eta_A(g)$. Since a map $F1 \rightarrow FA$ is completely determined by its value at \star_{F1} , we must have $Ff = Fg$. But then $f = F^{-1}F(f) = F^{-1}F(g) = g$.

This means that η_A is an isomorphism, so η is in fact a natural isomorphism.

We define a natural transformation $\alpha: 1_{\mathbf{Set}} \rightarrow \mathbf{Set}(1, -)$ by setting

$$\alpha_A(a)(\star) := a.$$

The naturality square for $f: A \rightarrow B$ is

$$\begin{array}{ccc} A & \xrightarrow{\alpha_A} & \mathbf{Set}(1, A) \\ \downarrow f & & \downarrow g \mapsto f \circ g \\ B & \xrightarrow{\alpha_B} & \mathbf{Set}(1, B) \end{array}$$

Both paths are just $a \mapsto (\star \mapsto f(a))$, so α is natural. It is also clear that α_A is bijective, so α is a natural isomorphism. In other words, \star is a universal element of the identity functor.

In particular, this tells us that composition with α and its inverse exhibits a bijection between the collection of natural transformations

$$1_{\mathbf{Set}} \rightarrow F$$

and the collection of natural transformations

$$\mathbf{Set}(1, -) \rightarrow F.$$

This means that there is a unique natural transformation $1_{\mathbf{Set}} \rightarrow F$, and it is given by $\alpha \circ \eta$, and since α and η are both natural isomorphisms, so is $\alpha \circ \eta$, completing the proof. \square

EXERCISE. The Sierpiński space S is, up to isomorphism, the unique topological space which has precisely three endomorphisms.

SOLUTION. Let X be a topological space. Then for any $x \in X$, the constant map $c_x: X \rightarrow X$ sending $y \in X$ to x is continuous. Furthermore, the identity on X is continuous. This, if X is infinite, then X has infinitely many endomorphisms, and if X is finite, then X has at least $|X| + 1$ endomorphisms.

Now assume that X has precisely three endomorphisms. Then X is finite and has at most two points. Clearly, if X has zero or one point, then there is only one endomorphism. So X has two points, say $X = \{a, b\}$. There are four set-functions $\{a, b\} \rightarrow \{a, b\}$, three of which (the identity and the two constant maps) are continuous regardless of the topology. The final map interchanges a and b and is not continuous.

The empty set and all of X are open. If X had the trivial or the discrete topology, then the interchange would be continuous, a contradiction. Hence, precisely one of the sets $\{a\}$ and $\{b\}$ is open. Sending the member of that set to 1 and the other element to 0 describes a homeomorphism with S . \square

EXERCISE. Let \mathcal{C} be a full subcategory of **Top** containing the singleton space 1 and the Sierpiński space S and let F be an automorphism of \mathcal{C} . Then

- (a) we have $FS \cong S$,
- (b) there is a unique natural isomorphism $\alpha: U \rightarrow UF$, where $U: \mathcal{C} \rightarrow \mathbf{Set}$ is the forgetful functor,
- (c) if \mathcal{C} contains a space in which not every union of closed sets is closed, then α_S is continuous, and
- (d) F is uniquely naturally isomorphic to the identity functor.

SOLUTION. For (a), we just need to notice that F is fully faithful, so it induces a bijection between the sets of morphisms $S \rightarrow S$ and $FS \rightarrow FS$. Since S is determined up to isomorphism by having exactly three endomorphisms, the claim follows.

The proof of (b) is entirely analogous to the proof of 2.13(ii).

For (c), write $FS = \{\tilde{0}, \tilde{1}\}$ such that $\{\tilde{1}\}$ is open. Suppose that α_S is not continuous. Then α_S must send $0 \mapsto \tilde{1}$ and $1 \mapsto \tilde{0}$. Now let $U \subseteq X$ be an open set of some topological space in \mathcal{C} . Consider the map $g: X \rightarrow S$ which sends $x \in X$ to 1

if and only if $x \in U$. This map is continuous. Define $f := F^{-1}g$, then by naturality we have

$$(\alpha_{F^{-1}X})^{-1}(U) = \alpha_{F^{-1}X}^{-1}((UFf)^{-1}(\{\tilde{1}\})) = (Uf)^{-1}((\alpha_S)^{-1}(\tilde{1})) = (Uf)^{-1}(\{0\}).$$

Since f is continuous, the right hand side is closed. Hence the preimage under $\alpha_{F^{-1}X}$ of an open set is closed. In analogous fashion and using the fact that F^{-1} is also an automorphism (noting that α^{-1} must be the unique natural isomorphism $U \rightarrow UF^{-1}$), we find that for any space X in \mathcal{C} we have

- the preimage under α_X of an open set is closed,
- the preimage under α_X of a closed set is open,
- the image under α_X of an open set is closed,
- the image under α_X of a closed set is open.

Now let X be a space in \mathcal{C} and a collection U_i closed sets such that $\bigcup U_i$ is not closed. We have

$$\alpha_X^{-1}(\bigcup U_i) = \bigcup \alpha_X^{-1}(U_i),$$

where the left hand side is not open, since otherwise $\bigcup U_i$ would be closed, but the right hand side is open, since $\alpha_X^{-1}(U_i)$ is open for every i . This is a contradiction, so α_S is continuous.

For (d), we can now carry out the same calculation as above to find that α_X and α_X^{-1} are continuous for every X , so α lifts to a natural isomorphism $1_C \rightarrow F$, which must be unique since the forgetful functor $[\mathcal{C}, \mathcal{C}] \rightarrow [\mathbf{Set}, \mathbf{Set}]$ is faithful. \square

Exercise 14.

EXERCISE. Let $e: A \rightarrow A$ be an idempotent. Then the following are equivalent:

- (i) e is split,
- (ii) the pair $(e, 1_A)$ has an equaliser,
- (iii) the pair $(e, 1_A)$ has a coequaliser.

SOLUTION. We will show that (i) is equivalent to (ii). By duality, this implies that (i) is equivalent to (iii).

Assume that there are $f: B \rightarrow A$ and $g: A \rightarrow B$ such that $fg = e$ and $gf = 1_B$. We claim that f is an equaliser of e and 1_A . We must show that any $h: C \rightarrow A$ satisfying $he = h$ factors uniquely through f .

$$\begin{array}{ccccc} & & C & & \\ & \swarrow h' & \downarrow h & & \\ B & \xleftarrow{f} & A & \xrightarrow[e]{1_A} & A \end{array}$$

Indeed, given such h . Then $fgh = eh = h$, hence gh is one such factoring. If $h': C \rightarrow B$ is another factoring such that $fh' = h$, then $h' = gh' = gh$, so the factoring is unique.

Conversely, assume that the pair $(e, 1_A)$ admits an equaliser $f: B \rightarrow A$. Since $ee = e = 1_A e$, e factors through f via some $g: A \rightarrow B$. Hence, $fg = e$. On the other hand, $fgf = ef = f$, and by a result from the lecture, f is monic, so $gf = 1_A$, so e is split. \square

EXERCISE. A split monomorphism is regular.

SOLUTION. If $f: A \rightarrow B$ is a split monomorphism, then there is some $g: B \rightarrow A$ such that $gf = 1_A$. Then $fgfg = f1_A g = fg$, so fg is a split idempotent. By what we just saw, this means that f is an equaliser of $(fg, 1_A)$, hence f is a regular monomorphism. \square

Exercise 15.

EXERCISE. Every regular monomorphism is strong.

SOLUTION. Let f be the equaliser of u and v and take a commutative square as in the definition of strongness.

$$\begin{array}{ccccc}
 C & \xrightarrow{h} & A & & \\
 \downarrow g & \nearrow t & \downarrow f & & \\
 D & \xrightarrow{k} & B & \xrightleftharpoons[u]{u} & E
 \end{array}$$

We have $ukg = ufh = vfh = vkg$. Since g is epi, this means that $uk = vk$, and since f is the equaliser of u and v , we find $t: D \rightarrow A$ such that $ft = k$. Now $ftg = kg = fh$. Since f is mono, we conclude that $tg = h$, so t has the desired properties. Hence, f is a strong monomorphism. \square

EXERCISE. Let \mathcal{C} be the finite category whose non-identity morphisms are represented by the diagram

$$\begin{array}{ccccc}
 A & \xrightarrow{f} & B & \xleftarrow{g} & C \\
 & \searrow l & \downarrow h & \downarrow k & \swarrow m \\
 & & D & &
 \end{array}$$

The morphism f is strong monic but not regular monic.

SOLUTION. The strongness condition for f is actually vacuous: if we have a diagram

$$\begin{array}{ccc}
 \bullet & \xrightarrow{u} & A \\
 \downarrow v & & \downarrow f \\
 \bullet & \xrightarrow{w} & B,
 \end{array}$$

then we must have $u = 1_A$. The morphism f is not an epimorphism, as witnessed by the fact that $hf = kf$, but $h \neq k$, so we must have $v = l$. Then w is a morphism $D \rightarrow B$, but such a morphism does not exist. Hence, the square does not exist, so f is strong.

However, the only pairs of morphisms that f can be an equaliser of are $(1_B, 1_B)$, (k, k) , (h, h) and (h, k) . If f was the equaliser of any of these pairs, g would factor through f , but there is no morphism $C \rightarrow A$, hence that is not the case. So we conclude that f is not regular. \square

Exercise 16.

EXERCISE. Let $f: A \rightarrow B$, $g: B \rightarrow C$ be two morphisms.

- If f and g are monic, then gf is monic,
- If f and g are strong monic, then gf is strong monic,
- If f and g are split monic, then gf is split monic,
- If gf is monic, then f is monic,
- If gf is strong monic, then f is strong monic,
- If gf is split monic, then f is split monic.
- If gf is regular monic and g is monic, then f is regular monic.

SOLUTION. (a) If $gfu = gfv$, then $fu = fv$ since g is monic, and $u = v$, since f is monic.

(b) Consider the diagram

$$\begin{array}{ccc}
 D & \xrightarrow{h} & A \\
 \downarrow l & \nearrow u & \downarrow f \\
 & E & B \\
 & \searrow t & \downarrow g \\
 E & \xrightarrow{k} & C
 \end{array}$$

Since g is strong monic, using the square (fh, g, l, k) , we find $t: E \rightarrow B$ such that $gt = k$ and $tl = fh$. Since f is strong epic, using the square (h, f, l, t) , we find $u: E \rightarrow A$ such that $fu = t$ and $ul = h$. Then we have $gf u = gt = k$, so u is the required morphism.

- (c) If $u: B \rightarrow A$ satisfies $uf = 1_A$ and $v: C \rightarrow B$ satisfies $vg = 1_B$, then uv is the desired retraction, as $uv g f = u 1_B f = u f = 1_A$.
 (d) If $fu = fv$, then trivially, $gf u = gf v$, so $u = v$.
 (e) Consider the diagram

$$\begin{array}{ccc}
 D & \xrightarrow{h} & A \\
 \downarrow l & \nearrow t & \downarrow f \\
 E & \xrightarrow{k} & B \\
 & \searrow gk & \downarrow g \\
 & & C
 \end{array}$$

Since gf is strong monic, using the square (h, gf, l, gk) we find $t: E \rightarrow A$ such that $tl = h$ (and $gft = gk$, but that is not important). We have $ftl = fh = kl$, so since l is epi, we have $ft = k$, so t is indeed the required diagonal morphism, so f is strong monic.

- (f) If $u: C \rightarrow A$ satisfies $ugf = 1_A$, then $(ug)f = 1_A$, so f is split monic.
 (g) Say gf is an equalizer of u and v .

$$\begin{array}{ccccc}
 & & T & & \\
 & \swarrow \ell & \downarrow h & & \\
 A & \xrightarrow{f} & B & \xrightarrow{g} & C \xrightarrow[u]{v} D
 \end{array}$$

If $h: T \rightarrow B$ satisfies $ugh = vgh$, then since gf is an equaliser of u and v , we find a unique $\ell: T \rightarrow A$ such that $gf\ell = gh$. Since g is monic, we have $f\ell = h$. The morphism ℓ is the unique morphism satisfying $f\ell = h$, since if $\hat{\ell}$ also satisfies $f\hat{\ell} = h$, then certainly $gf\hat{\ell} = gh$, hence $\ell = \hat{\ell}$. \square

EXERCISE. Let \mathcal{C} be the full subcategory of \mathbf{Ab} whose objects are groups having no elements of order 4 (though they may have elements of order 2). Then

- (i) multiplication by 2 is a regular monomorphism $\mathbb{Z} \rightarrow \mathbb{Z}$,
- (ii) multiplication by 4 is not a regular monomorphism $\mathbb{Z} \rightarrow \mathbb{Z}$,
- (iii) there is a pair of morphisms (f, g) such that gf is regular monix but f is not.

SOLUTION. (i) We claim that multiplication by 2 is an equalizer in \mathcal{C} of the projection $\pi: \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$ and the zero map $0: \mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$.

$$\begin{array}{ccccc} & & G & & \\ & \swarrow & \downarrow f & & \\ \mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Z} & \xrightarrow[\quad 0 \quad]{\pi} & \mathbb{Z}/2\mathbb{Z} \end{array}$$

Indeed, if $f: G \rightarrow \mathbb{Z}$ equalizes π and 0, then its image is contained in $2\mathbb{Z}$, hence it factors uniquely through multiplication by 2 via the map $g \mapsto f(g)/2$.

(ii) Assume that multiplication by 4 is an equalizer in \mathcal{C} of f and g .

$$\begin{array}{ccccc} & & \ker(f - g) & & \\ & \swarrow & \downarrow \iota & & \\ \mathbb{Z} & \xrightarrow{\cdot 4} & \mathbb{Z} & \xrightarrow[\quad g \quad]{f} & G \end{array}$$

Clearly, the kernel of $f - g$ has no elements of order 4 and the inclusion equalizes f and g , hence it factors through multiplication by 4. Consider the element $\alpha := f(1) - g(1) \in G$. We know that $\alpha + \alpha + \alpha + \alpha = f(4) - g(4) = 0$, since multiplication by 4 equalises f and g . Since G is an object of \mathcal{C} , the order of α is 2 or 1. In either case, we have $2 \in \ker(f - g)$, which is not in the image of multiplication by 4, hence ι cannot factor through multiplication by 4, so multiplication by 4 is not an equalizer of f and g . \square

Exercise 17.

EXERCISE. The functor F is irreducible if and only if there is an epimorphism $\mathcal{C}(A, -) \rightarrow F$ for some object A of \mathcal{C} .

SOLUTION. If F is irreducible, then applying the irreducibility property to the epimorphism constructed in 2.12 gives the desired result.

Conversely, if A is an object of \mathcal{C} such that there is an epimorphism $\beta: \mathcal{C}(A, -) \rightarrow F$, then by 2.11 we get a factoring $\gamma: \mathcal{C}(A, -) \rightarrow \coprod_{i \in I} G_i$. Define $x := f_A(1_A) \in G_j(A)$ for some $j \in I$. By Yoneda, we know that for any object B and morphism $f: A \rightarrow B$ we have

$$\gamma_B(f) = \left(\prod_{i \in I} G_i \right) (f)(x) = G_j(f)(x),$$

i.e., the image of γ_B is completely contained in $G_j(B)$ for every B . Hence we have a commutative diagram

$$\begin{array}{ccc} G_j & \xrightarrow{\alpha|_{G_j}} & F \\ & \nwarrow \gamma & \uparrow \beta \\ & & \mathcal{C}(A, -), \end{array}$$

and by the dual of Exercise 2.16(ii), the natural transformation $\alpha|_{G_i}$ must be an epimorphism. \square

EXERCISE. A functor $F: \mathcal{C} \rightarrow \mathbf{Set}$ is irreducible and projective if and only if there is a split epimorphism $e: \mathcal{C}(A, -) \rightarrow F$ for some A .

SOLUTION. If F is irreducible and projective, then by part (i) we find an epimorphism $e: \mathcal{C}(A, -) \rightarrow F$ for some A . Applying the projectivity of F to the diagram

$$\begin{array}{ccc} & & F \\ & \swarrow s & \downarrow 1_F \\ \mathcal{C}(A, -) & \xrightarrow{e} & F \end{array}$$

yields $s: F \rightarrow \mathcal{C}(A, -)$ such that $es = 1$, so e is split.

Conversely, if $e: \mathcal{C}(A, -) \rightarrow F$ admits a section $s: F \rightarrow \mathcal{C}(A, -)$ such that $es = 1$, then F is irreducible by part (i). Suppose we have a morphism $f: F \rightarrow R$ and an epimorphism $g: Q \rightarrow R$.

$$\begin{array}{ccc} \mathcal{C}(A, -) & \xrightleftharpoons[s]{e} & F \\ \downarrow h & & \downarrow f \\ Q & \xrightarrow{g} & R \end{array}$$

Since $\mathcal{C}(A, -)$ is projective by 2.11, we find some $h: \mathcal{C}(A, -) \rightarrow Q$ such that $fe = gh$. But then $ghs = fes = f$, hence $hs: F \rightarrow Q$ solves the lifting problem, and F is projective. \square

EXERCISE. If all idempotents in \mathcal{C} split, then the irreducible projectives in $[\mathcal{C}, \text{Set}]$ are exactly the representable functors.

SOLUTION. If F is representable, then we have a natural isomorphism $\mathcal{C}(A, -) \rightarrow F$, which in particular is a split epimorphism, hence F is irreducible and projective by part (ii).

Conversely, if F is irreducible and projective, by (ii) we find an epimorphism $e: \mathcal{C}(A, -) \rightarrow F$ and a section $s: F \rightarrow \mathcal{C}(A, -)$ such that $es = 1$. se is a natural transformation $\mathcal{C}(A, -) \rightarrow \mathcal{C}(A, -)$. Define $f := (se)_A(1_A)$. By Yoneda, for any $u: A \rightarrow B$, we have

$$(se)_B(u) = \mathcal{C}(A, u)(f) = uf.$$

Since se is idempotent, in particular we get

$$f = (se)_A(1_A) = (sese)_A(1_A) = (se)_A((se)_A(1_A)) = (se)_A(f) = ff,$$

so f is idempotent. By assumption, f is split, so we find some object B , $g: B \rightarrow A$ and $h: A \rightarrow B$ such that $f = gh$, $hg = 1_B$. Defining

$$\begin{array}{ll} x: \mathcal{C}(A, -) \rightarrow \mathcal{C}(B, -) & y: \mathcal{C}(B, -) \rightarrow \mathcal{C}(A, -) \\ x_C: u \mapsto ug & y_C: u \mapsto uh, \end{array}$$

(these are natural, which we can see either using Yoneda or by noticing that the naturality squares are just associativity of composition), we find that $yx = se$ and $xy = 1$. But then we have $xsey = xyxy = 1$, $eyxs = eses = 1$, so $xs: F \rightarrow \mathcal{C}(B, -)$ and $ey: \mathcal{C}(B, -) \rightarrow F$ are two-sided inverses of each other, hence F is representable. \square

Exercise 18.

EXERCISE. Let \mathcal{D} be the full subcategory of the category \mathcal{C} in Exercise 2.15 with objects A , B and D , and let 2 be the category with objects 0 and 1 and one non-identity morphism $0 \rightarrow 1$. Find an example of a morphism in the functor category $[2, \mathcal{D}]$ which is epic but not pointwise epic.

SOLUTION. Let $G: 2 \rightarrow D$ be the functor that sends the morphism $0 \rightarrow 1$ to h . Consider any functor $H: 2 \rightarrow D$ and a natural transformation $\eta: G \rightarrow H$.

$$\begin{array}{ccc} B & \xrightarrow{\eta_0} & H0 \\ \downarrow h & & \downarrow H(0 \rightarrow 1) \\ D & \xrightarrow{\eta_1} & H1. \end{array}$$

Clearly, $H_1 = D$, $\eta_1 = 1_D$. H_0 is either B or D . If $H_0 = B$, then $\eta_0 = 1_B$ and $H(0 \rightarrow 1) = h$. If $H_0 = D$, then $\eta_0 = h$ and $H(0 \rightarrow 1) = 1_D$. In both cases, there is only one natural transformation $G \rightarrow H$. Hence, any natural transformation $\alpha: F \rightarrow G$ is automatically epic. Choose F to be the functor that sends $0 \rightarrow 1$ to f and set $\alpha_0 := f$, $\alpha_1 := h$. Then α is a natural transformation. By what we have just seen, it is epic, but $\alpha_0 = f$ is not an epimorphism, hence α is not pointwise epic. \square