Algebraic Geometry

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Introduction

Definition 0.1. Let A be a ring. Then $\operatorname{Spec} A := \{p \subseteq A \mid p \text{ a prime ideal}\}$. For $I \subseteq A$ an ideal, define

$$V(I) := \{p \subseteq A \mid p \text{ prime}, p \supseteq I\}.$$

Proposition 0.2. The sets $V(I)$ form the closed sets of a topology on $\operatorname{Spec} A$, called the Zariski topology.

Proof.

(1) $V(A) = \varnothing$
(2) $V(0) = \operatorname{Spec} A$
(3) If $\{I_i\}_{i \in J}$ is a collection of ideals, then $V(\sum_{i \in J} I_j) = \bigcap V(I_i)$.
(4) We claim: $V(I_1 \cap I_2) = V(I_1) \cup V(I_2)$.
"$\supseteq$" is obvious.
"$\subseteq$": Follows from the fact that $p \supseteq I_1 \cap I_2$ is prime, then $p \supseteq I_1$ or $p \supseteq I_2$.

$\square$

Example 0.3. Let $A = k[X_1, \ldots, X_n]$ with k algebraically closed. Let $I \subseteq A$ be an ideal. Then the maximal ideals m of A containing I are in one-to-one correspondence with $V(I)$ in $\mathbb{A}^n(k)$: by Nulstellensatz, every maximal ideal is of the form $(X_1 - a_1, \ldots, X_n - a_n)$, which corresponds to $(a_1, \ldots, a_n)$ in the old $V(I)$.

The new $V(I)$ now extends this notion of zero set by including other prime ideals.

Example 0.4. If k is a field, then $\operatorname{Spec} k = \{0\}$, so the topological space cannot see the field. We fix this by also thinking about what functions are on these spaces.

CHAPTER 1

Sheaves

REMARK. Fix a topological space X.

DEFINITION 1.1. A presheaf $\mathcal{F}$ on X consists of

(1) For every open set $U \subseteq X$ an abelian group $\mathcal{F}U$,
(2) for every inclusion $V \subseteq U \subseteq X$ a restriction map $\rho_{UV} : \mathcal{F}U \to \mathcal{F}V$ such that $\rho_{UU} = \mathrm{id}_{\mathcal{F}U}$ and $p_{UW} = \rho_{VW} \circ \rho_{UV}$.

REMARK 1.2. A presheaf is just a contravariant functor from the poset category of open sets of X to the catgegory of abelian groups.

We can generalize this to any contravariant functor $X^{\mathrm{op}} \to \mathcal{C}$ for some category $\mathcal{C}$.

DEFINITION 1.3. A morphism of presheaves $f : \mathcal{F} \to \mathcal{G}$ on X is a collection of morphisms $f_U : \mathcal{F}U \to \mathcal{G}U$ such that for all $V \subseteq U$ the diagram

$$\begin{tikzcd} \mathcal{F}U \arrow[r, "f_U"] \arrow[d, "\rho_{UV}"] & \mathcal{G}U \arrow[d, "\rho_{UV}"] \\ \mathcal{F}V \arrow[r, "f_V"] & \mathcal{G}V \end{tikzcd}$$

commutes.

DEFINITION 1.4. A presheaf $\mathcal{F}$ is called a sheaf if it satisfies additional axioms:

(S1) If $U \subseteq X$ is covered by an open cover $\{U_i\}$ and $s \in \mathcal{F}U$ satsfies $s|_{U_i} := \rho_{UU_i}(s) = 0$ for all i, then $s = 0$
(S2) If U, and U_i are as before, and if $s_i \in \mathcal{F}U_i$ such that for all i and j we have $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, then there is some $s \in \mathcal{F}U$ such that $s|_{U_i} = s_i$ for all i.

REMARK 1.5. (1) If $\mathcal{F}$ is a sheaf, then $\varnothing \subseteq X$ is covered by the empty covering; hence $\mathcal{F}(\varnothing) = 0$.
(2) The two sheaf axioms can be described as saying that given $U, \{U_i\}$,

$$0 \longrightarrow \mathcal{F}U \xrightarrow{\alpha} \prod_i \mathcal{F}U_i \underset{\beta_2}{\overset{\beta_1}{\rightrightarrows}} \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

is exact, where $\alpha(s) = (s|_{U_i})_{i \in I}$, $\beta_1((s_i)_{i\in I}) = (s_i|_{U_i \cap U_j})$, $\beta_2((s_i)_{i \in I}) = (s_j|_{U_i \cap U_j})_{i,j}$.

Exactness means that α is injective, $\beta_1 \circ \alpha = \beta_2 \circ \alpha$, and for any $(s_i) \in \prod_{i \in I} \mathcal{F}U_i$, with $\beta_1((s_i)) = \beta_2((s_i))$, there exists $s \in \mathcal{F}U$ with $\alpha(s) = (s_i)$.

This is all subsumed by saying that α is the equalizer of β_1 and β_2.

EXAMPLE. (1) Let X be any topological space, $\mathcal{F}U$ the continuous functions $U \to \mathbb{R}$.

This is a sheaf: $\rho_{UV} : \mathcal{F}U \to \mathcal{F}V$ is just the restriction.

The first sheaf axiom says that a contiunous function is zero if it is zero on every open set of cover.

The second sheaf axiom says that continuous functions can be glued.

(2) Let $X = \mathbb{C}$ with the Euclidean topology.

Define $\mathcal{F}U$ to be the set of bounded analytic functions $f\colon U \to \mathbb{C}$.

This is a presheaf, since the restriction of bounded analytic functions is bounded analytic. It also satisfies the first sheaf axiom. However, it does not satisfy the second sheaf axiom.

For example, consider the cover $\{U_i\}_{i\in\mathbb{N}}$ of $\mathbb{C}$ given by $U_i = \{z \in \mathbb{C} \mid |z| < i\}$. Define $s_i\colon U_i \to \mathbb{C}$ by $z \mapsto z$. Note that if $i < j$, then $U_i \cap U_j = U_i$ and $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$. However, gluing yields the identity function on $\mathbb{C}$, which is not bounded (note that complex analysis tells us that $\mathcal{F}\mathbb{C} = \mathbb{C}$.

The underlying problem is that sheafs can only track properties that can be tested locally.

(3) Let G be a group and set $\mathcal{F}U \coloneqq G$ for any open set U. This is called the constant presheaf. This is in general not a sheaf (unless G is trivial).

Take U to be an disjoint union of open sets $U_1 \cup U_2$. If $\mathcal{F}U_1 = G$ and $\mathcal{F}U_2 = G$, then we need $\mathcal{F}(U_1 \cap U_2) = 0$.

If the second sheaf axiom was to be satisfied, we would want $s_1 \in \mathcal{F}U_1$ and $s_2 \in \mathcal{F}U_2$ to glue, so we should have $\mathcal{F}U = G \times G$.

Now give G the discrete topology, and define instead $\mathcal{F}U$ to be the set of continuous maps $f\colon U \to G$. By our choice of topology, this means that f is locally constant, i.e., for every $x \in U$ we have a neighborhood $V \subseteq U$ of x such that $f|_V$ is constant.

This is called the constant sheaf and if U is nonempty and connected then $\mathcal{F}U = G$.

(4) If X is an algebraic variety, $U \subseteq X$ a Zariski open subset, then define $\mathcal{O}_X(U)$ to be the regular functions $f\colon U \to k$.

Roughly, f regular means that every point of U has an open neighborhood on which f is expressed as a ratio of polynomials g/h with h nonvanishing on the neighborhood.

$\mathcal{O}_X$ is a sheaf, called the structure sheaf of X.

Definition 1.6. Let $\mathcal{F}$ be a presheaf on X and let $x \in X$. Then the stalk of $\mathcal{F}$ at x is $\mathcal{F}_x \coloneqq \{(U,s) \mid U \subseteq X \text{ open neighborhood at } x, s \in \mathcal{F}U\}/\sim$, w here $(U,s) \sim (V,s')$ if there is a neighborhood $W \subseteq U \cap V$ of x such that $s|_W = s'|_W$. An equivalence class of a pair (U,s) is called a germ.

Remark. $\mathcal{F}_x$ is just the colimit of $\mathcal{F}U$ where U ranges over the open neighborhoods of x.

Note that a morphism $f\colon \mathcal{F} \to \mathcal{G}$ of presheaves induces a morphism $f_p\colon \mathcal{F}_p \to \mathcal{G}_p$ via $f_p(U,s) \coloneqq (U, f_U(s))$.

Proposition 1.7. Let $f\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then f is an isomorphism if and only if f_p is an isomorphism for every $p \in X$.

Proof. "$\Longrightarrow$" is obvious.

"$\Longleftarrow$": Assume that f_p is an isomorphism for all $p \in X$. Need to show that $f_U\colon \mathcal{F}U \to \mathcal{G}U$ is an isomorphism for all $U \subseteq X$, as then we can define $(f^{-1})_U = (f_U)^{-1}$. This defines a morphism of sheaves, as

$$\begin{aligned}\rho^{\mathcal{F}}_{UV} \circ f_U^{-1} &= f_V^{-1} \circ f_V \circ \rho^{\mathcal{F}}_{UV} \circ f_U^{-1}\\ &= f_V^{-1} \circ \rho^{\mathcal{G}}_{UV} \circ f_U \circ f_U^{-1}\\ &= f_V^{-1} \circ \rho^{\mathcal{G}}_{UV}.\end{aligned}$$

We will first check that f_U is injective. Suppose $s \in \mathcal{F}U$ and $f_U(s) = 0$. Then for all $p \in U$, we have $f_p(U,s) = (U, f_U(s)) = (U,0) = 0 \in \mathcal{F}_p$. Since f_p is injective, this means that $(U,s)=0$ in $\mathcal{F}_p$. This means that there is an open neighborhood V_p of p in U such that $s|_{V_p}=0$. Since the sets $\{V_p\}_{p\in U}$ cover U, we see by sheaf axiom 1 that we have $s=0$.

Next, we will show that f_U is surjective. Let $t\in\mathcal{G}U$ and write $t_p := (U,t)\in\mathcal{G}_p$. Since f_p is surjective, we find $s_p\in\mathcal{F}_p$ with $f_p(s_p)=t_p$. This means that we find an open neighborhood $V_p\subseteq U$ of p and a germ (V_p,s_p) such that $(V_p,f_{V_p}(s_p))\sim(U,t)$. By shrinking V_p if necessary wen can assume that $t|_{V_p}=f_{V_p}(s_p)$.

Now on $V_p\cap V_q$, $f_{V_p\cap V_q}(s_p|_{V_p\cap V_q}-s_q|_{V_p\cap V_q})=t_{V_p\cap V_q}-t|_{V_p\cap V_q}=0$ and hence by injectivity of $f_{V_p\cap V_q}$ already proved, we have $s_p|_{V_p\cap V_q}=s_q|_{V_p\cap V_q}$. By the second sheaf axiom, the s_p glue to give an element $s\in\mathcal{F}U$ with $s|_{V_p}=s_p$ for every $p\in U$.

Now $f_U(s)|_{V_p}=f_{V_p}(s|_{V_p})=f_{V_p}(s_p)=t|_{V_p}$. By the first sheaf axiom applied to $f_U(s)-t$ we get $f_U(s)=t$. This shows surjectivity of f_U, completing the proof. □

Theorem 1.8. Given a presheaf $\mathcal{F}$ there is a sheaf $\mathcal{F}^+$ and a morphism $\theta\colon\mathcal{F}\to\mathcal{F}^+$ satisfying the following universal propert:

For any sheaf $\mathcal{G}$ and morphism $\varphi\colon\mathcal{F}\to\mathcal{G}$ there is a unique morphism $\varphi^+\colon\mathcal{F}^+\to\mathcal{G}$ such that $\varphi^+\circ\theta=\varphi$.

The pair $(\mathcal{F}^+,\theta)$ is unique up to unique isomorphism and is called the sheafification of $\mathcal{F}$.

Proof. See exercises. □

Exercises

Example Sheet 1

Exercise 4.

NOTATION. For $s \in \mathcal{F}U$ and $p \in U$ we will write $s_p := (U, s) \in \mathcal{F}_p$.

DEFINITION. Let $\mathcal{F}$ be a presheaf and $U \subseteq X$ an open set. Define

$$\mathcal{F}^+U := \{s \colon U \to \coprod_{p \in U} \mathcal{F}_p \mid \forall p \in U \colon s(p) \in \mathcal{F}_p, (\star)\},$$

where $(\star)$ is the following statement: for every $p \in U$ there is an open $p \in V_p \subseteq U$ and a section $s_{V_p} \in \mathcal{F}U$ such that for ever $q \in V_p$ we have $(s_{V_p})_q = s(q)$.

EXERCISE. $\mathcal{F}^+$ together with the obvious restriction maps forms a sheaf.

SOLUTION. $\mathcal{F}^+U$ is an abelian group with pointwise addition, as the sum of $s, t \in F^+U$ still satisfies $(\star)$ by taking the intersection of the V_p obtained from s and t.

It is obvious that $\mathcal{F}^+$ is a presheaf.

Next, let $s \in \mathcal{F}^+U$ and $\{U_i\}$ an open cover such that $\forall i, s|_{U_i} = 0$. Let $p \in U$. Then $p \in U_i$ for some i and we have $s(p) = (s|_{U_i})(p) = 0$, so $s = 0$, so the identity axiom is satisfied.

Next, let $\{U_i\}_{i \in I}$ be a cover, $s_i \in \mathcal{F}^+U_i$ such that $\forall i, j \colon s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$. Given $p \in U$, define $s(p) := s_i(p)$ for $p \in U_i$. This is well-defined because of the compatibility condition. We need to show that $s \in \mathcal{F}^+U$. Indeed, let $p \in U$. Then $s(p) = s_i(p)$ for some i, and since $s_i \in \mathcal{F}^+U_i$ and taking stalks is compatible with restrictions, we get a neighborhood that satisfies the required condition. It remains to show that for all i, $s|_{U_i} = s_i$, but that is true by definition. $\square$

DEFINITION. For a presheaf $\mathcal{F}$ and an open set U, define

$$\theta_U \colon \mathcal{F}U \to \mathcal{F}^+U; \qquad s \mapsto (p \mapsto s_p).$$

This is obviously a homomorphism of groups. It also defines a morphism of shaves, because for $s \in \mathcal{F}U$, $V \subseteq U$ and $p \in V$ we have

$$\theta_U(s)|_V(p) = \theta_U(s)(p) = s_p = (s|_V)_p = \theta_V(s|V)(p).$$

LEMMA. Let $\mathcal{F}$ be a sheaf and U an open set. Then the natural map

$$\mathcal{F}U \to \prod_{p \in U} \mathcal{F}_p$$

is injective.

PROOF. Let $s, t \in \mathcal{F}U$ such that $s_p = t_p$ for every p. Let $p \in U$. By definition of a stalk, $s_p = t_p$ means that there is an open $p \in V_p \subseteq U$ such that $s|_{V_p} = t|_{V_p}$. These V_p cover U so by the identity axiom we have $s = t$. $\square$

Lemma. Let $\mathcal{F}$ be a sheaf. Let U be an open set. Let $s\colon U \to \coprod_{p\in U} \mathcal{F}_p$ such that for every $p \in U$ we have $s(p) \in \mathcal{F}_p$ and there is an open $p \in V_p \subseteq U$ together with $s_{V_p} \in \mathcal{F}V_p$ such that for every $q \in V_p$ we have $(s_{V_p})_q = s(q)$. Then there is a unique $t \in \mathcal{F}U$ such that $t_q = s(q)$ for every $q \in U$.

Proof. Uniqueness follows from the previous lemma. For existence, notice that the V_p cover U. Let $p, q \in U$. The s_{V_p} are gluable because their stalks agree on the intersection, so the conditions of the gluing axiom are satisfied by the previous lemma. Since talking stalks is compatible with restrictions, the glued section has the correct stalks. □

Exercise. Let $\mathcal{F}$ be a presheaf, $\mathcal{G}$ a sheaf and $\varphi\colon \mathcal{F} \to \mathcal{G}$ a morphism of presheaves. Then there is a unique morphism of sheaves $\varphi^+\colon \mathcal{F}^+ \to \mathcal{G}$ such that $\varphi = \varphi^+ \circ \theta$.

Solution. Let U be an open and let $s \in \mathcal{F}^+U$. Cover U with the V_p from the definition of $\mathcal{F}^+$ and obtain the associated $s_{V_p} \in \mathcal{F}V_p$. Define $t_{V_p} := \varphi_{V_p}(s_{V_p}) in \mathcal{G}V_p$. We can calulate that for $q \in V_p$ we have

$$(t_{V_p})_q = (\varphi_{V_p}(s_{V_p}))_q = \varphi_q((s_{V_p})_q) = \varphi_q(s(q)).$$

Therefore, Lemma 2 gives us a unique $t_U \in \mathcal{G}_U$ such that

$$(\star) \qquad \forall q \in U\colon (t_U)_q = \varphi_q(s(q)).$$

We define $\varphi^+_U(s) = t_U$.

This is indeed a morphism of sheaves: if $V \subseteq U$ and $s \in \mathcal{F}^+U$, then

$$\varphi^+(s|_V) = \varphi^+(s)|_V$$

follows from the fact that, using $(\star)$, the germ of both sides at $p \in V$ is just $\varphi_p(s(p))$. By Lemma 1, the two sides are equal.

Similarly, if $s \in \mathcal{F}U$ and $p \in U$, then

$$(\varphi^+_U \theta_U(s))_p \overset{(\star)}{=} \varphi_q(\theta(s)(q)) = \varphi_q(s_q) = (\varphi_U(s))_q,$$

so $\varphi^+_U \circ \theta_U = \varphi_U$ by Lemma 1, so $\varphi^+ \circ \theta = \varphi$.

Finally, to see uniqueness, assume that $\varphi^\#$ satisfies $\varphi^\# \circ \theta = \varphi$. Let $s \in \mathcal{F}^+U$ and $p \in U$. By definition of $\mathcal{F}^+$ there is $p \in V_p \subseteq U$, $s_{V_p} \in \mathcal{F}V_p$ such that $\forall q \in V_p\colon (s_{V_p})_q = s(q)$. The condition can be reprased as $s|_{V_p} = \theta(s_{V_p})$ and we calculate

$$\begin{aligned}(\varphi^\#_U(s))_p &= (\varphi^\#_U(s)|_{V_p})_p = (\varphi^\#_{V_p}(s|_{V_p}))_p = (\varphi^\#_{V_p}(\theta(s_{V_p})))_p \\ &= (\varphi^+_{V_p}(\theta(s_{V_p})))_p = \ldots = (\varphi^+_U(s))_p,\end{aligned}$$

so by Lemma 1, we have $\varphi^+_U = \varphi^\#_U$, so $\varphi^+ = \varphi^\#$, completing the proof of uniqueness. □

Exercise. We have $(\mathcal{F}^+)_p = \mathcal{F}_p$ for $p \in X$. Show that if $f\colon \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves, then there is an induced morphism $f^+\colon \mathcal{F}^+ \to \mathcal{G}^+$ with $(f^+)_p = f_p$.

Solution. Let $p \in X$. Of course, $(\mathcal{F}^+)_p$ and $\mathcal{F}_p$ cannot be literally equal. Instead, we show the following more precise statement: The map $\theta_p\colon \mathcal{F}_p \to \mathcal{F}^+_p$ is an isomorphism.

Indeed, we define $g_p\colon \mathcal{F}^+_p \to \mathcal{F}_p$ as follows: for an open U and $s \in \mathcal{F}^+U$ we define $g_p(s_p) := s(p)$. This is well-defined because sections $s \in \mathcal{F}^+U$, $t \in \mathcal{F}^+V$ that have the same germ at p must satisfy $s|_W = t|_W$ for some W that contains p, so $s(p) = s|_W(p) = t|_W(p) = t(p)$.

Next, let U be an open and $s \in \mathcal{F}_p^+$. By definition of $\mathcal{F}^+$, there is some $p \in V_p \subseteq U$ open, $s_{V_p} \in \mathcal{F}V_p$ such that for all $q \in V_p$ we have $(s_{V_p})_q = s(q)$. This is equivalent to saying that $s|_{V_p} = \theta_{V_p}(s_{V_p})$, so in particular, in $\mathcal{F}_p^+$, we have $s_p = (\theta_{V_p}(s_{V_p}))_p$. This lets us calculate

$$\theta_p(g_p(s_p)) = \theta_p(s(p)) = \theta_p((s_{V_p})_p) = (\theta_{V_p}(s_{V_p}))_p = s_p,$$

so we have $\theta_p \circ g_p = \mathrm{id}_{\mathcal{F}_p^+}$.

Next, let U be an open and $s \in \mathcal{F}U$. Then we have

$$g_p(\theta_p(s_p)) = g_p(\theta_U(s)_p) = g_p((q \mapsto s_q)_p) = (q \mapsto s_q)(p) = s_p,$$

so $g_p \circ \theta_p = \mathrm{id}_{\mathcal{F}_p}$, and θ_p is an isomorphism as required.

Next, let $\mathcal{F}$ and $\mathcal{G}$ be presheaves and let $\theta\colon \mathcal{F} \to \mathcal{F}^+$ and $\iota\colon \mathcal{G} \to \mathcal{G}^+$ denote the natural maps to the associated sheaf. If $f\colon \mathcal{F} \to \mathcal{G}$ is a map of presheaves, we can invoke the universal property of $\mathcal{F}^+$ on the composite $\iota \circ f$ and find a morphism $f^+\colon \mathcal{F}^+ \to \mathcal{G}^+$ making the diagram

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\theta"] \arrow[d, "f"] & \mathcal{F}^+ \arrow[d, dotted, "f^+"] \\
\mathcal{G} \arrow[r, "\iota"] & \mathcal{G}^+
\end{tikzcd}$$

commute.

On stalks, we have

$$f_p^+ \circ \theta_p = (f^+ \circ \theta)_p = (\iota \circ f)_p = \iota_p \circ f_p,$$

and since θ_p is an isomorphism, we have

$$f_p^+ = \iota_p \circ f_p \circ \theta_p^{-1},$$

which is how we should interpret the "equality" $(f^+)_p = f_p$ under the natural identifications θ_p and ι_p. $\square$