Algebraic Geometry

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These notes, taken by Markus Himmel, will at times differ significantly from what was lectured. In particular, all errors are almost certainly my own.

Contents

Introduction

Definition 0.1. Let A be a ring. Then $\operatorname{Spec} A \coloneqq \{p \subseteq A \mid p \text{ a prime ideal}\}$. For $I \subseteq A$ an ideal, define

$$V(I) \coloneqq \{p \subseteq A \mid p \text{ prime}, p \supseteq I\}.$$

Proposition 0.2. The sets $V(I)$ form the closed sets of a topology on $\operatorname{Spec} A$, called the Zariski topology.

Proof. (1) $V(A) = \varnothing$
(2) $V(0) = \operatorname{Spec} A$
(3) If $\{I_i\}_{i \in J}$ is a collection of ideals, then $V(\sum_{i \in J} I_j) = \bigcap V(I_i)$.
(4) We claim: $V(I_1 \cap I_2) = V(I_1) \cup V(I_2)$.
"$\supseteq$" is obvious.
"$\subseteq$": Follows from the fact that $p \supseteq I_1 \cap I_2$ is prime, then $p \supseteq I_1$ or $p \supseteq I_2$.

□

Example 0.3. Let $A = k[X_1, \ldots, X_n]$ with k algebraically closed. Let $I \subseteq A$ be an ideal. Then the maximal ideals m of A containing I are in one-to-one correspondence with $V(I)$ in $\mathbb{A}^n(k)$: by Nulstellensatz, every maximal ideal is of the form $(X_1 - a_1, \ldots, X_n - a_n)$, which corresponds to $(a_1, \ldots, a_n)$ in the old $V(I)$.

The new $V(I)$ now extends this notion of zero set by including other prime ideals.

Example 0.4. If k is a field, then $\operatorname{Spec} k = \{0\}$, so the topological space cannot see the field. We fix this by also thinking about what functions are on these spaces.

CHAPTER 1

Sheaves

Remark. Fix a topological space X.

Definition 1.1. A presheaf $\mathcal{F}$ on X consists of

(1) For every open set $U \subseteq X$ an abelian group $\mathcal{F}U$,
(2) for every inclusion $V \subseteq U \subseteq X$ a restriction map $\rho_{UV} \colon \mathcal{F}U \to \mathcal{F}V$ such that $\rho_{UU} = \mathrm{id}_{\mathcal{F}U}$ and $p_{UW} = \rho_{VW} \circ \rho_{UV}$.

Remark 1.2. A presheaf is just a contravariant functor from the poset category of open sets of X to the catgegory of abelian groups.

We can generalize this to any contravariant functor $X^{\mathrm{op}} \to \mathcal{C}$ for some category $\mathcal{C}$.

Definition 1.3. A morphism of presheaves $f \colon \mathcal{F} \to \mathcal{G}$ on X is a collection of morphisms $f_U \colon \mathcal{F}U \to \mathcal{G}U$ such that for all $V \subseteq U$ the diagram

$$\begin{tikzcd} \mathcal{F}U \arrow[r, "f_U"] \arrow[d, "\rho_{UV}"] & \mathcal{G}U \arrow[d, "\rho_{UV}"] \\ \mathcal{F}V \arrow[r, "f_V"] & \mathcal{G}V \end{tikzcd}$$

commutes.

Definition 1.4. A presheaf $\mathcal{F}$ is called a sheaf if it satisfies additional axioms:

(S1) If $U \subseteq X$ is covered by an open cover $\{U_i\}$ and $s \in \mathcal{F}U$ satsfies $s|_{U_i} := \rho_{UU_i}(s) = 0$ for all i, then $s = 0$
(S2) If U, and U_i are as before, and if $s_i \in \mathcal{F}U_i$ such that for all i and j we have $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, then there is some $s \in \mathcal{F}U$ such that $s|_{U_i} = s_i$ for all i.

Remark 1.5. (1) If $\mathcal{F}$ is a sheaf, then $\varnothing \subseteq X$ is covered by the empty covering; hence $\mathcal{F}(\varnothing) = 0$.
(2) The two sheaf axioms can be described as saying that given $U, \{U_i\}$,

$$0 \longrightarrow \mathcal{F}U \xrightarrow{\alpha} \prod_i \mathcal{F}U_i \underset{\beta_2}{\overset{\beta_1}{\rightrightarrows}} \prod_{i,j} \mathcal{F}(U_i \cap U_j)$$

is exact, where $\alpha(s) = (s|_{U_i})_{i \in I}$, $\beta_1((s_i)_{i \in I}) = (s_i|_{U_i \cap U_j})$, $\beta_2((s_i)_{i \in I}) = (s_j|_{U_i \cap U_j})_{i,j}$.

Exactness means that α is injective, $\beta_1 \circ \alpha = \beta_2 \circ \alpha$, and for any $(s_i) \in \prod_{i \in I} \mathcal{F}U_i$, with $\beta_1((s_i)) = \beta_2((s_i))$, there exists $s \in \mathcal{F}U$ with $\alpha(s) = (s_i)$.

This is all subsumed by saying that α is the equalizer of β_1 and β_2.

Example. (1) Let X be any topological space, $\mathcal{F}U$ the continuous functions $U \to \mathbb{R}$.

This is a sheaf: $\rho_{UV} \colon \mathcal{F}U \to \mathcal{F}V$ is just the restriction.

The first sheaf axiom says that a contiunous function is zero if it is zero on every open set of cover.

The second sheaf axiom says that continuous functions can be glued.

(2) Let $X = \mathbb{C}$ with the Euclidean topology.

Define $\mathcal{F}U$ to be the set of bounded analytic functions $f: U \to \mathbb{C}$.

This is a presheaf, since the restriction of bounded analytic functions is bounded analytic. It also satisfies the first sheaf axiom. However, it does not satisfy the second sheaf axiom.

For example, consider the cover $\{U_i\}_{i\in\mathbb{N}}$ of $\mathbb{C}$ given by $U_i = \{z \in \mathbb{C} \mid |z| < i\}$. Define $s_i: U_i \to \mathbb{C}$ by $z \mapsto z$. Note that if $i < j$, then $U_i \cap U_j = U_i$ and $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$. However, gluing yields the identity function on $\mathbb{C}$, which is not bounded (note that complex analysis tells us that $\mathcal{F}\mathbb{C} = \mathbb{C}$.

The underlying problem is that sheafs can only track properties that can be tested locally.

(3) Let G be a group and set $\mathcal{F}U \coloneqq G$ for any open set U. This is called the constant presheaf. This is in general not a sheaf (unless G is trivial).

Take U to be an disjoint union of open sets $U_1 \cup U_2$. If $\mathcal{F}U_1 = G$ and $\mathcal{F}U_2 = G$, then we need $\mathcal{F}(U_1 \cap U_2) = 0$.

If the second sheaf axiom was to be satisfied, we would want $s_1 \in \mathcal{F}U_1$ and $s_2 \in \mathcal{F}U_2$ to glue, so we should have $\mathcal{F}U = G \times G$.

Now give G the discrete topology, and define instead $\mathcal{F}U$ to be the set of continuous maps $f: U \to G$. By our choice of topology, this means that f is locally constant, i.e., for every $x \in U$ we have a neighborhood $V \subseteq U$ of x such that $f|_V$ is constant.

This is called the constant sheaf and if U is nonempty and connected then $\mathcal{F}U = G$.

(4) If X is an algebraic variety, $U \subseteq X$ a Zariski open subset, then define $\mathcal{O}_X(U)$ to be the regular functions $f: U \to k$.

Roughly, f regular means that every point of U has an open neighborhood on which f is expressed as a ratio of polynomials g/h with h nonvanishing on the neighborhood.

$\mathcal{O}_X$ is a sheaf, called the structure sheaf of X.

Definition 1.6. Let $\mathcal{F}$ be a presheaf on X and let $x \in X$. Then the stalk of $\mathcal{F}$ at x is $\mathcal{F}_x \coloneqq \{(U, s) \mid U \subseteq X \text{ open neighborhood at } x, s \in \mathcal{F}U\}/\sim$, w here $(U, s) \sim (V, s')$ if there is a neighborhood $W \subseteq U \cap V$ of x such that $s|_W = s'|_W$. An equivalence class of a pair (U, s) is called a germ.

Remark. $\mathcal{F}_x$ is just the colimit of $\mathcal{F}U$ where U ranges over the open neighborhoods of x.

Note that a morphism $f: \mathcal{F} \to \mathcal{G}$ of presheaves induces a morphism $f_p: \mathcal{F}_p \to \mathcal{G}_p$ via $f_p(U, s) \coloneqq (U, f_U(s))$.

Proposition 1.7. Let $f: \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then f is an isomorphism if and only if f_p is an isomorphism for every $p \in X$.

Proof. "$\Longrightarrow$" is obvious.

"$\Longleftarrow$": Assume that f_p is an isomorphism for all $p \in X$. Need to show that $f_U: \mathcal{F}U \to \mathcal{G}U$ is an isomorphism for all $U \subseteq X$, as then we can define $(f^{-1})_U = (f_U)^{-1}$. This defines a morphism of sheaves, as

$$\begin{aligned}\rho^{\mathcal{F}}_{UV} \circ f_U^{-1} &= f_V^{-1} \circ f_V \circ \rho^{\mathcal{F}}_{UV} \circ f_U^{-1}\\ &= f_V^{-1} \circ \rho^{\mathcal{G}}_{UV} \circ f_U \circ f_U^{-1}\\ &= f_V^{-1} \circ \rho^{\mathcal{G}}_{UV}.\end{aligned}$$

We will first check that f_U is injective. Suppose $s \in \mathcal{F}U$ and $f_U(s) = 0$. Then for all $p \in U$, we have $f_p(U, s) = (U, f_U(s)) = (U, 0) = 0 \in \mathcal{G}_p$. Since f_p is injective, this means that $(U, s) = 0$ in $\mathcal{F}_p$. This means that there is an open neighborhood V_p of p in U such that $s|_{V_p} = 0$. Since the sets $\{V_p\}_{p\in U}$ cover U, we see by sheaf axiom 1 that we have $s = 0$.

Next, we will show that f_U is surjective. Let $t \in \mathcal{G}U$ and write $t_p := (U, t) \in \mathcal{G}_p$. Since f_p is surjective, we find $s_p \in \mathcal{F}_p$ with $f_p(s_p) = t_p$. This means that we find an open neighborhood $V_p \subseteq U$ of p and a germ (V_p, s_p) such that $(V_p, f_{V_p}(s_p)) \sim (U, t)$. By shrinking V_p if necessary wen can assume that $t|_{V_p} = f_{V_p}(s_p)$.

Now on $V_p \cap V_q$, $f_{V_p\cap V_q}(s_p|_{V_p\cap V_q} - s_q|_{V_p\cap V_q}) = t_{V_p\cap V_q} - t|_{V_p\cap V_q} = 0$ and hence by injectivity of $f_{V_p\cap V_q}$ already proved, we have $s_p|_{V_p\cap V_q} = s_q|_{V_p\cap V_q}$. By the second sheaf axiom, the s_p glue to give an element $s \in \mathcal{F}U$ with $s|_{V_p} = s_p$ for every $p \in U$.

Now $f_U(s)|_{V_p} = f_{V_p}(s|_{V_p}) = f_{V_p}(s_p) = t|_{V_p}$. By the first sheaf axiom applied to $f_U(s) - t$ we get $f_U(s) = t$. This shows surjectivity of f_U, completing the proof. □

Theorem 1.8. Given a presheaf $\mathcal{F}$ there is a sheaf $\mathcal{F}^+$ and a morphism $\theta\colon \mathcal{F} \to \mathcal{F}^+$ satisfying the following universal propert:

For any sheaf $\mathcal{G}$ and morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ there is a unique morphism $\varphi^+\colon \mathcal{F}^+ \to \mathcal{G}$ such that $\varphi^+ \circ \theta = \varphi$.

The pair $(\mathcal{F}^+, \theta)$ is unique up to unique isomorphism and is called the sheafification of $\mathcal{F}$.

Proof. See exercises. □

Definition. Let $f\colon \mathcal{F} \to \mathcal{G}$ be a morphism of presheaves on a space X. We define

(1) The presheaf kernel of f, $\ker f$, is the presheaf given by
$$(\ker f)(U) := \ker f_U.$$
One should check that this is a presheaf.

(2) The presheaf cokernel of f, $\operatorname{coker} f$, is the presheaf given by
$$(\operatorname{coker} f)(U) := \operatorname{coker} f_U.$$

(3) The presheaf image $\operatorname{im} f$ is the presheaf given by
$$(\operatorname{im} f)(U) = \operatorname{im} f_U.$$

Remark. If $f\colon \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, then $\ker f$ is also a sheaf. The identity axiom is certainly satisfied: If $s \in (\ker f)(U) \subseteq \mathcal{F}U$ satisfies $s|_{U_i} = 0$ for all U_i in a cover of U, then we use the identity axiom for $\mathcal{F}$ to find that $s = 0$.

Given $s_i \in (\ker f)(U_i)$ with $\{U_i\}$ an open cover of U, and with $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$, then we find $s \in \mathcal{F}U$ with $s|_{U_i} = s_i$. But $f_U(s) = 0$ since
$$f_U(s)|_{U_i} = f_{U_i}(s|_{U_i}) = f_{U_i}(s_i) = 0,$$
and we can use the identity axiom to conclude that $f_U(s) = 0$.

Example. Let $X = \mathbb{P}^1$ (or think of the Riemann sphere). Let $P, Q \in X$ be distinct points. Let $\mathcal{G}$ be the sheaf of regular functions on X (alternatively, think of holomorphic functions on the Riemann sphere). Next, let $\mathcal{F}$ be the sheaf of regular functions which vanish on P and Q. Notice that $\mathcal{F}U = \mathcal{G}U$ if $U \cap \{P, Q\} = \varnothing$.

Let $U := \mathbb{P}^1 \setminus \{P\}$, $V = \mathbb{P}^1 \setminus \{Q\}$.

Note that $\mathcal{F}(\mathbb{P}^1) = 0$, $\mathcal{G}(\mathbb{P}^1) = k$, because regular functions on $\mathbb{P}^1$ are constants. Let $f\colon \mathcal{F} \to \mathcal{G}$ be the inclusion.

Then $(\operatorname{coker} f)(\mathbb{P}^1) \cong k$, $(\operatorname{coker} f)(U) = \mathcal{G}U/\mathcal{F}U = k[X]/(X) \cong k$a, $(\operatorname{coker} f)(V) \cong k$. However, $(\operatorname{coker} f)(U \cap V) = \mathcal{G}(U \cap V)/\mathcal{F}(U \cap V) \cong 0$.

Therefore, if the gluing axiom held, then we could need to have
$$(\operatorname{coker} f)(\mathbb{P}^1) \cong k \oplus k.$$

Note that this failure to be a sheaf is not a bug, but a feature!

DEFINITION. Let $f\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. The sheaf kernel of f is just the presheaf kernel.

The sheaf cokernel is the sheaf associated to the presheaf cokernel of f.

The sheaf image is the sheaf associated to the presheaf image of f.

We can check that these notions give kernels, cokernels and images in the category of sheaves.

EXERCISE. The sheaf image $\operatorname{im} f$ is a subsheaf of $\mathcal{G}$, where $\mathcal{F}$ is called a subsheaf of $\mathcal{G}$ if we have a morphism $f\colon \mathcal{F} \to \mathcal{G}$ such that f_U is a monomorphism for every open set U.

SOLUTION. See exercises. □

DEFINITION. We say that f is injective if $\ker f = 0$. We say that f is surjective if $\operatorname{im} f = \mathcal{G}$.

Note that surjectivity does not imply that f_U is surjective for every U.

We say that a sequence of morphisms of sheaves

$$\dots \longrightarrow \mathcal{F}^{i-1} \xrightarrow{f^i} \mathcal{F}^i \xrightarrow{f^{i+1}} \mathcal{F}^{i+1} \longrightarrow \dots$$

is exact if $\ker f^{i+1} = \operatorname{im} f^i$ for all i.

If $\mathcal{F}' \subseteq \mathcal{F}$ is a subsheaf, then we write $\mathcal{F}/\mathcal{F}'$ for the sheaf associated to the presheaf $U \mapsto \mathcal{F}U/\mathcal{F}'U$, so $\mathcal{F}/\mathcal{F}'$ is the cokernel of the inclusion $\mathcal{F}' \to \mathcal{F}$.

LEMMA 1.9. Let $f\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then for all $p \in X$ we have

$$(\ker f)_p = \ker(f_p\colon \mathcal{F}_p \to \mathcal{G}_p)$$
$$(\operatorname{im} f)_p = \operatorname{im} f_p$$

PROOF. We first define a map $(\ker f)_p \to \ker f_p$. If $(U,s) \in (\ker f)_p$, then $(U,s) \in \mathcal{F}_p$ and

$$f_p(U,s) = (U, f_U(s)) = (U,0) = 0 \in \mathcal{G}_p.$$

Therefore, $(U,s) \in \ker f_p$.

We will check injectivity and surjectivity of this map.

For injectivity, assume that $(U,s) = 0$ in $\mathcal{F}_p$, then there is $V \subseteq U$ of p such that $s|_V = 0$. Then we also have the equality

$$(U,s) = (V, s|_V) = (V,0) = 0$$

in $(\ker f)_p$.

For surjectivity, assume that $(U,s) \in \ker f_p$. This means that $(U, f_U(s)) = 0$ in $\mathcal{G}_p$, so there is $p \in V \subseteq U$ such that $0 = f_U(s)|_V = f_V(s|_V)$. Thus, $s|_V \in (\ker f)(V)$, and $(V, s|_V) \in (\ker f)_p$, and $(V, s|_V)$ maps to the element in $\ker f_p$ represented by (U,s).

For images: Let $\operatorname{im}' f$ be the presheaf image.

From the exercises we know that $\theta_p\colon \mathcal{F}_p \to \mathcal{F}_p^+$ is an isomorphism for every p.

Therefore $(\operatorname{im} f)_p \cong (\operatorname{im}' f)_p$, so we need to show that $(\operatorname{im}' f)_p \cong \operatorname{im} f_p$. Define a map $(\operatorname{im}' f)_p \to \operatorname{im} f_p$ by

$$(U,s) \in (\operatorname{im}' f)_p \mapsto (U,s) \in \operatorname{im} f_p.$$

Once again, we will check that this is injective and surjective.

For injectivity: if $(U,s) = 0$ in $\mathcal{G}_p$ then there is a neighborhood $V \subseteq U$ of p such that $s|_V = 0$. Then $(U,s) = (V,0)$ in $(\operatorname{im}' f)_p$.

For surjectivity: if $(U,s) \in \operatorname{im} f_p$, then there is $(V,t) \in \mathcal{F}_p$ with $(V, f_V(t)) = f_p(V,t) = (U,s)$, so after shrinking U and V if necessary, then we can take $U = V$ and $f_U(t) = s$. Then $(U,s) \in (\operatorname{im}' f)_p$. □

PROPOSITION. Let $f\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then f is injective if and only if for every $p \in X$ the map $f_p\colon \mathcal{F}_p \to \mathcal{G}_p$ is injective and f is surjective if and only if for every $p \in X$ the map $f_p\colon \mathcal{F}_p \to \mathcal{G}_p$ is surjective.

PROOF. f_p is injective for every p if and only if $\ker f_p = 0$ for every p if and only if $(\ker f)_p = 0$ for every p.

In the exercises, we show that for any sheaf $\mathcal{F}$, the map

$$\mathcal{F}U \to \prod_{p \in U} \mathcal{F}_p$$

is injective. Now if all of the $\mathcal{F}_p$ are trivial, then so is $\mathcal{F}U$.

Therefore $(\ker f)_p = 0$ for every p if and only if $\ker f = 0$.

Similarly, f_p is surjective for every p iff $\operatorname{im} f_p = \mathcal{G}_p$ for every p. Now consider the diagram

$$\begin{tikzcd}
\operatorname{im} f_p \arrow[r] \arrow[d, "\cong"] & \mathcal{G}_p \\
(\operatorname{im}' f)_p \arrow[r, "\cong"] \arrow[ur] & (\operatorname{im} f)_p, \arrow[u]
\end{tikzcd}$$

where

- the top arrow is the inclusion,
- the left arrow is the isomorphism defined in Lemma 1.9,
- the bottom arrow is the isomorphism on stalks induced by the inclusion into the associated sheaf,
- the diagonal arrow is the morphism on stalks induced by the inclusion of the presheaf image, and
- the right arrow is induced by the arrow making the sheaf image into a subsheaf.

The upper triangle commutes trivially, and the lower triangle commutes because by construction the right arrow is induced by the unique arrow making the non-stalk version of the triangle commute. Thus, since the bottom and left arrows are isomorphisms and the diagram commutes, we have that $\operatorname{im} f_p \to \mathcal{G}_p$ is an isomorphism (which just means that $\operatorname{im} f_p = \mathcal{G}_p$) if and only if $(\operatorname{im} f)_p \to \mathcal{G}_p$ is an isomorphism.

Now, the arrow $(\operatorname{im} f)_p \to \mathcal{G}_p$ is an isomorphism for every p if and only if $\operatorname{im} f \to \mathcal{G}$ is an isomorphism (Proposition 1.7), and this is the definition of surjectivity. $\square$

EXERCISE. Given $f\colon \mathcal{F} \to \mathcal{G}$, then we have $\mathcal{G}/\operatorname{im} f \cong \operatorname{coker} f$.

1. Passing between spaces

REMARK. Let $f\colon X \to G$ be a continuous map between topological spaces, $\mathcal{F}$ a sheaf on X, $\mathcal{G}$ a sheaf on Y.

DEFINITION. Define $f_*\mathcal{F}$ by setting

$$(f_*\mathcal{F}) \coloneqq \mathcal{F}(f^{-1}(U))$$

for $U \subseteq Y$ open.

EXERCISE. $f_*\mathcal{F}$ is a sheaf on Y.

SOLUTION. TODO $\square$

DEFINITION. Define $f^{-1}\mathcal{G}$ to be the sheaf associated to the presheaf

$$U \mapsto \{(V, s) \mid f(U) \subseteq V, s \in \mathcal{G}V\}/\sim$$

where $(V, s) \sim (v', s')$ if there is some $W \subseteq V \cap V'$ such that $f(U) \subseteq W$ and $s|_W = s'|_W$.

Example. If $f\colon \{p\} \to X$ is an inclusion of a point, then $f^{-1}\mathcal{G}$ is the sheaf on the one-point space given by $\mathcal{G}_p$.

More generally, if $i\colon Z \to X$ is the inclusion of a subspace, we often write $\mathcal{F}|_Z \coloneqq i^{-1}\mathcal{F}$. If Z is open in X, then we have

$$F|_Z(U) \cong \mathcal{F}U.$$

Notation 1.10. If $s \in \mathcal{F}U$, then we say that s is a section of $\mathcal{F}$ over U.

We often write $\mathcal{F}U = \Gamma(U, \mathcal{F})$. This allows us to think of $\Gamma(U, \cdot)$ as a functor from the category of presheaves on X to the category of abelian groups.

CHAPTER 2

Affine schemes

REMARK. Our goal is to construct a sheaf $\mathcal{O}$ on $\operatorname{Spec} A$, analogous to the sheaf of regular functions on a variety.

$\mathcal{O}$ will be a sheaf of rings, i.e., $\mathcal{O}U$ will be a ring for each open set U and restriction maps will be ring homomorphisms.

REMARK. Let A be a ring and let $S \subseteq A$ be a multiplicative subset (i.e., $1 \in S$ and S is closed under multiplication). We define a ring

$$S^{-1}A = \{(a, s) \mid a \in A, s \in S\}/\sim,$$

where $(a, s) \sim (a', s')$ iff there is $s'' \in S$ such that $s''(as' - a's) = 0$.

We write a/s for the equivalence class of (a, s).

Observe that the usual equivalence relation on fractions suggests that we should have $a/s = a'/s' \iff as' = a's$. We need the extra possibility of killing $as' - a's$ with s'' if A is not an integral domain.

The ring $S^{-1}A$ is called the localization of A at S.

EXAMPLE. (1) Take $f \in A$ and $S \coloneqq \{f^n \mid n \in \mathbb{N}_0\}$. Then we write $A_f \coloneqq S^{-1}A$.

This example will correspond to open subsets.

(2) Let $p \subseteq A$ be a prime ideal of A. Then $S \coloneqq A \setminus p$ satisfies $1 \in S$ and is closed under multiplication since p is prime. We define $A_p \coloneqq S^{-1}A$. This is the localization of A at (or rather, away from?) p.

This example will correspond to taking stalks.

DEFINITION. $\mathcal{O}$ should satisfy $\mathcal{O}_p = A_p$.

Define

$$\mathcal{O}U \coloneqq \{s \colon U \to \coprod_{p \in U} A_p \mid (\star)\},$$

where $(\star)$ means that

(1) $\forall p \in U \colon s(p) \in A_p$,
(2) for each $p \in U$ there is some $p \in V \subseteq U$ with V open and $a, f \in A$ such that for all $q \in V \colon f \notin q \wedge s(q) = a/f$.

LEMMA. For any $p \in \operatorname{Spec} A$, we have $\mathcal{O}_p \cong A_p$.

PROOF. We define a map

$$\begin{aligned} \mathcal{O}_p &\to A_p \\ (U, s) &\mapsto s(p) \end{aligned}$$

and will show that it is injective and surjective.

For surjectivity, notice that every element of A_p can be written as a/f for some $a \in A$, $f \notin p$. Then

$$D(f) \coloneqq \operatorname{Spec} A \setminus V(f) = \{p \in \operatorname{Spec} A \mid f \notin p\}$$

is an open set (in fact it is called a standard open). Now a/f defines an element of $s \in \mathcal{O}(D(f))$ given by $q \mapsto a/f \in A_q$. In particular, $s(p) = a/f \in A_p$.

For injectivity, let $p \in U \subseteq \operatorname{Spec} A$, $s \in \mathcal{O}U$ with $s(p) = 0$ in A_p. We need to show that $(U, s) = 0$ in $\mathcal{O}_p$. By shrinking U we can assume that s is given by $a, f \in A$ with $s(q) = a/f$ for all $q \in U$. In particular $f \notin q$ for every $q \in U$.

Thus, $a/f = 0/1$ in A_p. By definition of localization, this means that there is $h \in A \setminus p$ such that $h \cdot (a \cdot 1 + f \cdot 0) = 0$ in A, so we have $ah = 0$.

Now let $V = D(f) \cap D(h)$. Then $(V, s|_V) = 0$ in $\mathcal{O}_p$, since for $q \in V$, $s|_v(q) = s(q) = a/f \in A_q$ and $ha = 0$, $h \notin A \setminus q$, so $ha = 0$ implies $a/f = 0/1$ in A_q. Thus $(U, s) = 0$ in $\mathcal{O}_p$. □

Lemma. For any $f \in A$, we have $\mathcal{O}(D(f)) \cong A_f$.
In particular, since $\operatorname{Spec} A = D(1)$, we have $\mathcal{O}(\operatorname{Spec} A) \cong A_1 \cong A$.

Proof. Define

$$\begin{aligned} \Psi\colon A_f &\to \mathcal{O}(D(f)) \\ a/f^n &\mapsto (p \mapsto a/f^n). \end{aligned}$$

This makes sense since if $f \notin p$, then $f^n \notin p$. As usual, we will verify injectivity and surjectivity.

For injectivity, assume that $\Psi(a/f^n) = 0$. Then for all $p \in D(f)$, we have $a/f^n = 0$ in A_p, i.e., there is $h \in A \setminus p$ such that $ha = 0$ in A.

Let $I = \{q \in A \mid q \cdot a = 0\}$ (the annihilator of a). So $h \in I$, but $h \notin p$, so $I \nsubseteq p$. This is true for all $p \in D(f)$, so $V(I) \cap D(f) = \varnothing$. Thus $f \in \bigcap_{p \in V(I)} p = \sqrt{I}$, as we know from commutative algebra. This means that $f^n \in I$ for some $n > 0$. Thus $f^n \cdot a = 0$, so $a/f^n = 0$ in A_f, so Ψ is injective.

Next, we will prove surjectivity. Let $s \in \mathcal{O}(D(f))$. Cover $D(f)$ with open sets V_i on which s is represented by as a_i/g_i with $a_i, g_i \in A$, $g_i \notin p$ whenever $p \in V_i$. Thus $V_i \subseteq D(g_i)$. By question 1 on the first example sheet, the sets of the form $D(h)$ form a base for the Zariski topology on $\operatorname{Spec} A$. Thus we can assume $V_i = D(h_i)$ for some $h_i \in A$. Since $D(h_i) \subseteq D(g_i)$, we have $V(h_i) \supseteq V(g_i)$, so $\sqrt{(h_i)} \subseteq \sqrt{(g_i)}$, since the radical is the intersection of all the primes of $V(\cdot)$. Hence, $h_i^n \in (g_i)$ for some n, say $h_i^n = c_i g_i$, so we have $\frac{a_i}{g_i} = \frac{c_i a_i}{h_i^n}$. Now replace h_i by h_i^n. This does not change the open sets because in general $D(h_i) = D(h_i^n)$ and replace a_i by $c_i a_i$.

The situation so far is that we may assume that $D(f)$ is covered by sets $D(h_i)$ such that s is represented by a_i/h_i on $D(h_i)$.

We now claim that $D(f)$ can be covered by a finite number of the $D(h_i)$, i.e., $D(f)$ is quasicompact. Indeed, $D(f) \subseteq \bigcup_i D(h_i)$, which is equivalent to $V(f) \supseteq \bigcap_i V(h_i) = V(\sum (h_i))$. This in turn is equivalent to $f \in \sqrt{\sum_i (h_i)}$ (because it just says that f is in every prime ideal containing $\sum_i (h_i)$), which is equivalent to there being some n such that $f^n \in \sum_i (h_i)$. Hence, we can write $f^n = \sum_{i \in I} b_i h_i$ for some finite set I.

Reversing this argument yields that $D(f) \subseteq \bigcup_{i \in I} D(h_i)$ as required, completing the proof of the claim.

We now pass to this finite subcover $\{D(h_i)\}_{i \in I}$. On $D(h_i) \cap D(h_j) = D(h_i h_j)$, note a_i/h_i and a_j/h_j both represent s. Since we have already shown injectivity, this means that $a_i h_j / h_i h_j = a_j h_i / h_i h_j$ in $A_{h_i h_j}$.

Thus, for some n, $(h_i h_j)^n (h_j a_i - h_i a_j) = 0$ in A. We can pick an n sufficiently large to work for all pairs i, j (since there are only finitely many such pairs).

We rewrite this equality as $h_j^{n+1}(h_i^n a_i) - h_i^{n+1}(h_j^n a_j) = 0$. Now replace h_i by h_i^{n+1}, and a_i by $h_i^n a_i$ (this is allowed because $\frac{a_i}{h_i} = \frac{a_i h_i^n}{h_i^{n+1}}$. Thus we can assume that s is still represented on $D(h_i)$ by a_i/h_i but also for each i, j we have $h_i a_j = h_j a_i$.

Since $D(f) \subseteq \bigcup_{i \in I} D(h_i)$, we have $V(\sum (h_i)) = \bigcap_{i \in I} V(h_i) \subseteq V(f)$, hence $f^n = \sum b_i h_i$ for some h_i. Define $a \coloneqq b_i a_i$.

Then for any j, we have

$$h_j a = \sum_i b_i a_i h_j = \sum_i b_i a_j h_i = f^n a_j.$$

This means that $a/f^n = a_j/h_j$ on $D(h_j)$. Hence $\Psi(a/f^n) = s$, completing the proof of surjectivity. $\square$

Remark. We now have a topological space $\operatorname{Spec} A$ equipped with a sheaf of rings $\mathcal{O}$.

Definition. A ringed space is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X.

A morphism of ringed spaces $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ consists of a continuous map $X \to Y$ and a morphism of shaves of rings $f^{\#}\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$, i.e., for every open $O \subseteq Y$, a homomorphism of rings $f_U^{\#}\colon \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$.

Example. (1) Let X, Y be topological spaces and $\mathcal{O}_X$ and $\mathcal{O}_Y$ the sheaf of continuous $\mathbb{R}$-valued functions. Given $f\colon X \to Y$, we get $f^{\#}\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ defined by $f_U^{\#}(\varphi) = \varphi \circ f$.

(2) Let X be a variety and $\mathcal{O}_X$ the sheaf of regular functions on X. A morphism of varieties $f\colon X \to Y$ is a continuous map inducing

$$\mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U)),$$
$$\varphi \mapsto \varphi \circ f.$$

Definition. A locally ringed space $(X, \mathcal{O}_X)$ is a ringed space such that $\mathcal{O}_{X,p}$ is a local ring (i.e., has a unique maximal ideal) for every $p \in X$.

A morphism $f\colon (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of locally ringed spaces is a morphism of ringed spaces such that such that the induced map $f_p^{\#}\colon \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$ is a local homomorphism. Here,

- the map $f_p^{\#}$ is defined by $(U, s) \mapsto (f^{-1}(U), f_U^{\#}(s))$ for a section $s \in \mathcal{O}_Y(U)$, and
- a local homomorphism $\varphi\colon (A, m_A) \to (B, m_B)$ is a ring homomorphism between local rings such that $\varphi^{-1}(m_B) = m_A$. Note that $\varphi(A \setminus m_A) = \varphi(A^\times) \subseteq B^\times = B \setminus m_B$. Hence, $\varphi^{-1}(m_B) \subseteq m_A$ is always true, and the opposite inclusion is what makes a ring homomorphism local.

Remark. In the case of varieties, $\mathcal{O}_{X,p}$ has a unique maximal ideal $\{(U, f) \in \mathcal{O}_X(U) \mid f(p) = 0\}/\sim$, i.e., if $f(p) \neq 0$, then f is nowhere vanishing on some neighborhood of p, so after shrinking U, we can invert f.

The local homomorphism condition just follows from the pullback of a function φ vanishing at $f(p)$ vanishes at p.

Example. $(\operatorname{Spec} A, \mathcal{O})$ is a locally ringed space; which we call an affine scheme.

Theorem. The category of affine schemes with locally ringed morphisms is equivalent to the opposite of the category of rings.

Proof. We need to show the following things.

(1) If $\varphi\colon A \to B$ is a ring homomorphism, we obtain an induced morphism

$$(f, f^{\#})\colon (\operatorname{Spec} B, \mathcal{O}_B) \to (\operatorname{Spec} A, \mathcal{O}_A).$$

(2) Any morphism of affine schemes as locally ringed spaces arises in this way.

For the first part, let $\varphi\colon A \to B$ be a ring homomorphism and define

$$f\colon \operatorname{Spec} B \to \operatorname{Spec} A$$
$$p \mapsto \varphi^{-1}(p),$$

where we use that $\varphi^{-1}(p)$ is prime: if $ab \in \varphi^{-1}(p)$, then $\varphi(ab) = \varphi(a)\varphi(b) \in p$. Hence $\varphi(a) \in p$ or $\varphi(b) \in p$, hence $a \in \varphi^{-1}(p)$ or $b \in \varphi^{-1}(p)$.

We also need to show that f is continuous. Any closed set is of the form $V(I)$. We calculate

$$\begin{aligned} f^{-1}(V(I)) &= f^{-1}(\{p \in \operatorname{Spec} A \mid p \supseteq I\}) \\ &= \{q \in \operatorname{Spec} B \mid f(q) \supseteq I\} \\ &= \{q \in \operatorname{Spec} B \mid \varphi^{-1}(q) \supseteq I\} \\ &= \{q \in \operatorname{Spec} B \mid q \supseteq \varphi(I)\} \\ &= V(\varphi(I)). \end{aligned}$$

Hence the preimage of a closed set is closed, so f is continuous.

We need to construct a morphism of sheaves

$$f_{\#} \colon \mathcal{O}_{\operatorname{Spec} A} \to f_* \mathcal{O}_{\operatorname{Spec} B}.$$

For $p \in \operatorname{Spec} B$, we obtain a natural homomorphism

$$\begin{aligned} \varphi_p \colon A_{\varphi^{-1}(p)} &\to B_p \\ \frac{a}{s} &\mapsto \frac{\varphi(a)}{\varphi(s)}, \end{aligned}$$

where $a \in A$, $s \notin \varphi^{-1}(p)$. This makes sense since $\varphi(a) \in B$ and $\varphi(s) \notin p$.

The maximal ideal pB_p of B_p is generated by the image of p under the map $B \to B_p$. The maximal ideal $\varphi^{-1}(p)A_{\varphi^{-1}(p)}$ of $A_{\varphi^{-1}(p)}$ is generated by the image of $\varphi^{-1}(p)$ under the map $A \to A_p$.

Given $V \subseteq \operatorname{Spec} A$ open, we may define

$$\begin{aligned} f_V^{\#} \colon \mathcal{O}_{\operatorname{Spec} A}(V) &\to \mathcal{O}_{\operatorname{Spec} B}(f^{-1}(V)) \\ s &\mapsto (q \mapsto \varphi_q(s(f(q)))). \end{aligned}$$

We now have to check the local coherence condition of $\mathcal{O}$, i.e., if s is locally given by a/h, then $f_V^{\#}(s)$ is locally given by $\frac{\varphi(a)}{\varphi(h)}$. Indeed, let $s \in \mathcal{O}_{\operatorname{Spec} A}(V)$ and $p \in f^{-1}(V)$. Then we find $f(p) \in W \subseteq V$, $a, h \in A$ such that for all $q \in W$, $h \notin q$, $s(q) = a/h$. Define $U := f^{-1}(W)$. We have $p \in U$, since $f(p) \in W$. If $q \in U$, we have $\varphi(h) \notin q$, since $h \notin \varphi^{-1}(q) = f(q) \in W$. Hence,

$$f_V^{\#}(s)(q) = \varphi_q(s(f(q))) = \varphi_q(a/h) = \varphi(a)/\varphi(h)$$

as required.

This gives the desired map $f^{\#} \colon \mathcal{O}_{\operatorname{Spec} A} \to f_* \mathcal{O}_{\operatorname{Spec} B}$ and the induced map on stalks $f_p^{\#} \colon \mathcal{O}_{\operatorname{Spec} A, f(p)} \to \mathcal{O}_{\operatorname{Spec} B, p}$ agrees with $\varphi_p \colon A_{\varphi^{-1}(p)} \to B_p$ by construction. To be precise, we claim that we have a commutative diagram

$$\begin{tikzcd} \mathcal{O}_{\operatorname{Spec} A, f(p)} \arrow[r, "f_p^{\#}"] \arrow[d, "\alpha"] & \mathcal{O}_{\operatorname{Spec} B, p} \arrow[d, "\beta"] \\ A_{f(p)} \arrow[r, "\varphi_p"] & B_p \end{tikzcd}$$

for every $p \in X$, where α and β are the canonical isomorphisms defined in a previous result. Indeed, if $(U, s) \in \mathcal{O}_{\operatorname{Spec} A, f(p)}$, we have

$$\beta(f_p^{\#}(U, s)) = \beta(f^{-1}(U), f_U^{\#}(s)) = f_U^{\#}(s)(p) = \varphi_p(s(f(p))) = \varphi_p(\alpha(U, s)).$$

Hence, the pair $(f, f^{\#})$ is a morphism of locally ringed spaces.

Now suppose given a morphism $(f, f^\#)\colon \operatorname{Spec} B \to \operatorname{Spec} A$ of locally ringed spaces. We have

$$f^\#_{\operatorname{Spec} A}\colon \Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \to \Gamma(\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}),$$

but since the glocal sections of $\operatorname{Spec} R$ are just R, we get $\varphi\colon A \to B$.

We need to show that φ gives rise to $(f, f^\#)$. We have a local homomorphism

$$f^\#_p\colon A_{f(p)} \cong \mathcal{O}_{\operatorname{Spec} A, f(p)} \to \mathcal{O}_{\operatorname{Spec} B, p} \cong B_p.$$

This is compatible with the corresponding map on glocal sections in the sense that

$$\begin{tikzcd} \Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \arrow[r, "f^\#_{\operatorname{Spec} A}"] \arrow[d] & \Gamma(\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \arrow[d] \\ \mathcal{O}_{\operatorname{Spec} A, f(p)} \arrow[r, "f^\#_p"] & \mathcal{O}_{\operatorname{Spec} B, p} \end{tikzcd}$$

is a commutative diagram. By applying our calculations, this yields a diagram

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d] & B \arrow[d] \\ A_{f(p)} \arrow[r, "f^\#_p"] & B_p. \end{tikzcd}$$

Recall that $f^\#_p$ is a local homomorphism. Thus $(f^\#_p)^{-1}(pB_p) = f(p)A_{f(p)}$. Along the lower left path, the maximal ideal pB_p is pulled back to $f(p)A_{f(p)}$ and then to $f(p)$. Along the upper right path, it gets pulled back to p and then to $\varphi^{-1}(p)$. By commutativity, we conclude that $f(p) = \varphi^{-1}(p)$.

Thus f is induced by φ and by commutativity, $f^\#_p = \varphi_p$. Then $f^\#$ is as constructed previously (this needs to be checked). □

Remark. Note that demanding that $(f, f^\#)$ is a morphism of locally ringed spaces rather than merely ringed spaces was crucial to make the proof work.

Definition 2.1. An affine scheme is a locally ringed space that is isomorphic as a locally ringed space to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ for some ring A.

A scheme is a locally ringed space $(X, \mathcal{O}_X)$ with an open cover $\{(U_i, \mathcal{O}_X|_{U_i})\}$ such that each $(U_i, \mathcal{O}_X|_{U_i})$ is an affine scheme. Recall that we have $\mathcal{O}_X|_{U_i}(V) = \mathcal{O}_X(V)$ for $V \subseteq U$ open.

Example.

(1) Let k be a field. Then $\operatorname{Spec} k = (\{0\}, k)$.

What does giving a morphism $f\colon \operatorname{Spec} k \to X$ a scheme mean?

First, we need to choose a point $x \in X$, the image of f. Second, we get a local ring homomorphism

$$f^\#_0\colon \mathcal{O}_{X,x} \to \mathcal{O}_{\operatorname{Spec} k, 0} \cong k,$$

i.e., $(f^\#_0)^{-1}(0) = \mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$, the maximal ideal of $\mathcal{O}_{X,x}$. Thus we get a factorization $f^\#_0\colon \mathcal{O}_{X,x} \to \mathcal{O}_{X,x}/\mathfrak{m}_x \to k$. The middle quotient is a field denoted as $\kappa(x)$, the residue field of X at x.

Thus f induces an inclusion $\kappa(x) \to k$.

Conversely, given an inclusion $\iota\colon \kappa(x) \to k$ we get a morphism of schemes $\operatorname{Spec} k \to X$ by defining $f(0) = x$ and $f^\#\colon \mathcal{O}_X \to f_*k$ by defining $s \mapsto \iota(s(x)) \in k$, where $s(x)$ means taking the stalk of s at x.

Moral: Giving a morphism $f\colon \operatorname{Spec} k \to X$ is equivalent to giving a point $x \in X$ and an inclusion $\kappa(x) \to k$.

Note: If $X = \operatorname{Spec} A$, then giving a morphism $\operatorname{Spec} k \to \operatorname{Spec} A$ is equivalent to giving a homomorphism $A \to k$, which we viewed at the beginning of the course as a "k-valued point" on $\operatorname{Spec} A$.

(2) What does giving a morphism $X \to \operatorname{Spec} k$ mean? The continuous map $X \to \operatorname{Spec} k$ does not carry any information, since $\operatorname{Spec} k$ is a singleton space. We also have a map

$$f^{\#} \colon k \cong \mathcal{O}_{\operatorname{Spec} k} \to f_* \mathcal{O}_X,$$

i.e., a map $k \to \Gamma(\operatorname{Spec} k, f_*\mathcal{O}_X) = \Gamma(X, \mathcal{O}_X)$, i.e., $\Gamma(X, \mathcal{O}_X)$ carries a k-algebra structure.

Notice that this induces k-algebra structures on $\mathcal{O}_X(U)$ for all open sets U via the composite

$$k \longrightarrow \mathcal{O}_X(X) \xrightarrow{\rho} \mathcal{O}_X(U),$$

and similarly all stalks $\mathcal{O}_{X,p}$ are also k-algebras.

In this situation we say that X is a scheme (defined) over k.

(3) Consider $A = k[X_1, \dots, X_n]/I$ with $I = \sqrt{I}$. Then $\operatorname{Spec} A$ is a replacement for $V(I) \subseteq \mathbb{A}^n_k$, viewing $\operatorname{Spec} A$ as a scheme over k.

If $k \subseteq k'$ is a field extension, a k'-valued point of X/k is a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} k' \arrow[r] \arrow[dr] & X \arrow[d] \\ & \operatorname{Spec} k, \end{tikzcd}$$

which has the dual

$$\begin{tikzcd} k' & A \arrow[l] \\ & k, \arrow[ul] \arrow[u] \end{tikzcd}$$

so the top arrow is a homomorphism of k-algebras.

We write $X(k')$ for the set of such morphisms.

Remark. It is rare in algebraic geometry to work with schemes alone. Rather, we always work over a base scheme.

Fix a base scheme S. Define the category of schemes over S to be the category whose objects are morphisms $T \to S$ and morphisms are commutative triangles. This is just the normal comma construction.

We will frequently work with schemes over $\operatorname{Spec} k$, which we will also refer to as schemes over k.

Given two schemes over S, $T \to S$ and $X \to S$, we define a T-valued point of $X \to S$ as a morphism $T \to X$ over S. We write $X(T)$ for the set of T-valued points.

By Yoneda, the collection of $X(T)$ for every T determines X up to isomorphism.

Example. Fix a field k, and let $D = \operatorname{Spec} k[t]/(t^2) = (\{(t)\}, k[t]/(t^2))$, where (t) is the unique prime ideal. t doesn't make sense as a k-valued function any more, as $t^2 = 0$.

Let X be any scheme over k. What is $X(D)$? Given a morphism $f \colon D \to X$ of schemes over k, we get a point $x \in X$ as the image of f and a local homomorphism

$$f^{\#}_x \colon \mathcal{O}_{X,x} \to k[t]/(t^2),$$

such that $(f^{\#}_x)^{-1}((t)) = \mathfrak{m}_x$. Note thate $\mathfrak{m}_x^2$ maps to 0, hence we get a k-linear map

$$\mathfrak{m}_x/\mathfrak{m}_x^2 \to (t) \cong k,$$

where the isomorphism is as a k-vector space. We also have a composed k-algebra homomorphism

$$\mathcal{O}_{X,x} \to k[t]/(t^2) \to k[t]/(t) \cong k$$

with kernel $\mathfrak{m}_x$, and hence we have $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x \cong k$. To see this, we must use that this is a homomorphism of k-algebras, so the k sitting inside $\mathcal{O}_{X,x}$ maps to k on the right, i.e., the composite is surjective.

Se we get:

(1) a k-valued point with residue field k,
(2) a morphism of k-vector spaces $\mathfrak{m}_x/\mathfrak{m}_x^2 \to k$, i.e., an element of $(\mathfrak{m}_x/\mathfrak{m}_x^2)^*$, the dual vector space.

The space $(\mathfrak{m}_x/\mathfrak{m}_x^2)^*$ is called the Zariski tangent space to X at x. It can be thought of as a kind of "differentiation rule".

Think of D as a point plus an arrow: mapping D into a scheme X carries as data a point of X and a tangent vector[1].

Example (Glued Schemes). This is a special case of a question of Example Sheet 1.

Suppose we are given to schemes X_1, X_2 and open subsets $U_i \subseteq X_i$.

Recall U_i is also a locally ringed space $(U_i, \mathcal{O}_{X_i}|_{U_i})$ and in fact U_i is then a scheme (this is not obvious and will be discussed later).

Given an isomorphism $f\colon U_1 \to U_2$, we can glue X_1 and X_2 along U_1 and U_2 to get a scheme X with an open cover $\{X_1, X_2\}$.

As a topological space, X is just the topological gluing of X_1 and X_2. Refer to the example sheet for the construction of $\mathcal{O}_X$.

Now take $\mathbb{A}_k^n \coloneqq \operatorname{Spec} k[X_1, \dots, X_n]$. Hence, $\mathbb{A}_k^1 = \operatorname{Spec} k[X]$. Take $X_1 = X_2 = \mathbb{A}_k^1$. Glue $U_1 \coloneqq \mathbb{A}^1 \setminus \{0\} = D(X) \subseteq A_k^1 = X$, where 0 is the point corresponding to the prime ideal (X) and $U_2 \coloneqq \mathbb{A}^1 \setminus \{0\} = D(X) \subseteq X_2$ via the identity map. As a topological space, X is just the line with two origins. The resulting scheme is called the affine line with doubled origin. It is not a variety.

Note that $U_i = \operatorname{Spec} k[X]_X$ (localization). Hence, we could also glue U_1 and U_2 via the map given by $X \mapsto X^{-1}$.

When we glue this way, we get the projective line over k, $\mathbb{P}_k^1$.

[1]This is just a vague intuition.

CHAPTER 3

Projective schemes

Remark. Let S be a graded ring, i.e., $S = \bigoplus_{d \geq 0} S_d$ with S_d an abelian group, and we have the product law $S_d \cdot S_{d'} \subseteq S_{d+d'}$.

For example, if $S = k[X_0, \dots, X_n]$, then we get a grading such that S_d is the space of homogeneous polynomials of degree d.

We write $S_+ \coloneqq \bigoplus_{d \geq 1} S_d$, which we call the irrelevant ideal.

Definition. $I \subseteq S$ is called a homogeneous ideal if I is generated by its homogeneous elements, i.e., elements in S_d for various d.

Definition. $\operatorname{Proj} S \coloneqq \{p \in \operatorname{Spec} S \mid p \text{ homogeneous}, p \not\supseteq S_+\}$.

For $I \subseteq S$ a homogeneous ideal, set $V(I) \coloneqq \{p \in \operatorname{Proj} S \mid p \supseteq I\}$.

Exercise. Check the $V(I)$ form the closed sets of a topology on $\operatorname{Proj} S$.

Solution. TODO. $\square$

Remark 3.1. For $p \in \operatorname{Proj} S$, let

$$T = \{f \in S \setminus p \mid f \text{ is homogeneous}\}.$$

Then T is a multiplicateively closed subset of S and let $S_{(p)} \subseteq T^{-1}S$ be the subring of elements of degree 0, i.e., written in the form s/s' with $s \in S$ homogeneous, $s' \in T$ with $\deg s = \deg s'$.

For $f \in S$ homogeneous, we write $S_(f) \subseteq S_f$ for the subset of elements of degree 0.

Definition. For $U \subseteq \operatorname{Proj} S$ open, set

$$\mathcal{O}(U) \coloneqq \{s \colon U \to \coprod_{p \in U} S_(p) \mid (\star)\},$$

where $(\star)$ means that

(1) for all $p \in U$, $s(p) \in S_(p)$, and
(2) for all $p \in U$ there is $p \in V \subseteq U$ and $a, f \in S$ homogeneous of the same degree such that for all $q \in V$ we have $f \notin q$ and $\forall q \in V \colon s(q) = a/f$.

As before, we can calculate $\mathcal{O}_p \cong S_{(p)}$.

Important question: is the locally ringed space $(\operatorname{Proj} S, \mathcal{O})$ a scheme?

If $f \in S$ is homogeneous, then we write $D_+(f) = \{p \in \operatorname{Proj} S \mid f \notin p\}$. This is the open set, as we have $D_+(f) = \operatorname{Proj} S \setminus V(f)$.

Proposition. We have $(D_+(f), \mathcal{O}|_{D_+(f)}) \cong \operatorname{Spec} S_{(f)}$ as locally ringed spaces. Further, the open sets $D_+(f)$ for $f \in S_+$ cover $\operatorname{Proj} S$. Hence $(\operatorname{Proj} S, \mathcal{O})$ is a scheme.

Proof. This appears on the second example sheet. $\square$

Definition 3.2. If A is a ring, define

$$\mathbb{P}^n_A \coloneqq \operatorname{Proj} A[X_0, \dots, X_n].$$

Example. Let k be an algebraically closed field, consider $\mathbb{P}^1_k = \operatorname{Proj} k[X_0, X_1]$.

The closed points, i.e., points p such that $\{p\}$ is closed, correspond to maximal elements of $\operatorname{Proj} S$ (TODO: exercise!). These maximal elements are ideals of the form $(aX_0 - bX_1)$:

Note that the only maximal homogeneous ideal of $k[X_0, X_1]$ is $(X_0, X_1) = S_+$, which is the irrelevant ideal, hence not part of $\operatorname{Proj} S$ (TODO), since any maximal ideal is of the form $(X_0 - a_0, X_1 - a_1)$ by the Nullstellensatz.

The other prime ideals of $k[X_0, X_1]$ are principal, i.e. of the form (f) with f irreducible or zero.

For (f) to be homogeneous, f must be homogeneous. Any such polynomial splits into linear factors, all homogeneous, so in order for f to be irreducible, it must be linear.

Note that we hve a bijective correspondence between the collection of ideals $(aX_0 - bX_1)$ with $a, b \in k$, a, b not both zero and $(k^2 \setminus \{0, 0\})/k^\times$ given by $(aX_0 - bX_1) \mapsto (b : a)$.

Conclusion: The closed points of $\mathbb{P}^1_k = \operatorname{Proj} k[X_0, X_1]$ are in one-to-one correspondence with points of $(k^2 \setminus \{0\})/k^\times$.

More generally, the closed points of P^n_k are in one-to-one correspondence with points of $(k^{n+1} \setminus \{0\})/k^\times$. This is harder (but a good exercise), but it can be seen by using the open cover $(D_+(X_i))$ (note that if $p \nsubseteq D_+(X_i)$ for any i, then $X_i \in p$ for any i, hence $S_+ \subseteq p$, so $p \notin \operatorname{Proj} S$).

Example. Let $S = k[X_0, \dots, X_n]$, but grade by $\deg X_i = w_i$, where $w_0, \dots, w_n$ are positive integers. Define weighted projective space via $W\mathbb{P}^n(w_0, \dots, w_n) = \operatorname{Proj} S$.

For example, consider $W\mathbb{P}^2(1, 1, 2)$. This has an open cover $\{D_+(X_i)\}$. We have $D_+(X_2) \cong \operatorname{Spec} S_{(X_2)}$. Note

$$S_{(X_2)} = k[u := \frac{X_0^2}{X_2}, v := \frac{X_0 X_1}{X_2}, w := \frac{X_1^2}{X^2}] \subseteq S_{X_2} \cong k[u, v, w]/(uw - v^2)$$

$\operatorname{Spec} S_{(X_2)}$ then is a quadric cone (an image is missing here).

$D_+(X_0)$ and $D_+(X_1)$ are both isomorphic to $\mathbb{A}^2_k$.

Example. Let $M = \mathbb{Z}^n$, $M_\mathbb{R} = M \otimes_\mathbb{Z} \mathbb{R} = \mathbb{R}^n$. Let $\Delta \subseteq M_\mathbb{R}$ be a compact convex lattice polytope, i.e., there is some finite set $V \subseteq M$ such that Δ is the convex hull of V, i.e., the smallest convex set containing V.

(there is a picture missing here)

Let $C(\Delta) = \{(m, r) \in M_\mathbb{R} \oplus \mathbb{R} \mid m \in r\Delta, r \geq 0\}$. Here $r\Delta = \{r \cdot m \mid m \in \Delta\}$. This is the cone over Δ.

Let

$$S = k[C(\Delta) \cap (M \oplus \mathbb{Z})] = \bigoplus_{p \in C(\Delta) \cap (M \oplus \mathbb{Z})} k \cdot z^p.$$

The thing in the square bracket is a monoid (use convexity to prove this). We have a multiplication given by $z^p \cdot z^{p'} = z^{p+p'}$ making S into a ring, and it is graded by $\deg z^{m,r} = r$.

Define $\mathbb{P}_\Delta = \operatorname{Proj} S$. This is called a projective toric variety.

Examples: If Δ is a standard n-simplex, i.e., the convex hull of $\{0, e_1, \dots, e_n\}$, then it is possible to check that $S \cong k[X_0, \dots, X_n]$ with the standard grading such that $X_0 \leftrightarrow z^{(0,1)}$, $X_0 \leftrightarrow z^{(e_i, i)}$. Hence $\mathbb{P}_\Delta = P^n_k$.

Let $n = 2$ and Δ be the convex hull of $\{(0, 0), (1, 0), (0, 1), (1, 1)\}$, i.e., the unit square. In S, the degree d monomials are $\{z^{(a,b,d)} \mid 0 \leq a, b \leq d\}$. Any of these monomials can be written as a product of monomials of degree 1, i.e., $x := z^{(0,0,1)}, y := z^{(1,0,1)}, w := z^{(0,1,1)}, t := z^{(1,1,1)}$. Thus $S = k[x, y, w, t]/(xt - yw)$ (it is possible but nontrivial to verify that this is the only relation).

Hence, $\operatorname{Proj} S$ can be thought of as a quadric surface in $\mathbb{P}^3_k$.

CHAPTER 4

Properties of schemes

1. Open and closed subschemes

Definition. An open subschemes of a schemes X is a scheme $(U, \mathcal{O}_X|_U)$ for $U \subseteq X$ an open subset.

This is indeed a scheme, since from questions 1 and 11 on the first example sheet, we know that some open affine subsets of X form a basis for the topology of X. In particular, we can cover U by affine schemees.

An open immersion is a morphism $f\colon X \to Y$ which induces an isomorphism of X with an open subscheme of Y.

A closed immersion $f\colon X \to Y$ is a morphism which is a homeomorphism onto a closed subset of Y and the induced morhpism $f^{\#}\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective.

A closed subscheme of Y is an equivalence class of closed immersions, where two closed immersions are considered equivalent if there is an isomorphism $i\colon X \to X'$ making the diagram

$$\begin{tikzcd} X \arrow[r, "Y"] \arrow[d, "i"] & Y \\ X' \arrow[ur] & \end{tikzcd}$$

commute.

Example. (1) Let $Y = \operatorname{Spec} A$, let $I \subseteq A$ be an ideal and take $X = \operatorname{Spec} A/I$. Note that the map of schemes induced by the quotient map $A \to A/I$ identifies $\operatorname{Spec} A/I$ with $V(I) \subseteq \operatorname{Spec} A$. Thus the map $f\colon X \to Y$ induced by $A \to A/I$ satisfies the first condition of being a closed immersion.

Note that $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective on stalks. Indeed, for $p \in V(I)$, $\mathcal{O}_{Y,p} \cong A_p$, and furthermore $(f_*\mathcal{O}_X)_p \cong \mathcal{O}_{X,p}$, since all open sets in X are of the form $U \cap X$ for $U \subseteq Y$ open. We have $\mathcal{O}_{X,p} \cong (A/I)_(p/I)$. The induced map $A_p \to (A/I)_{(p/I)}$ is surjective (and we should convince ourselves that this map is indeed the one we get).

(2)
$$\operatorname{Spec} k[X,Y]/(X) \to \operatorname{Spec} k[X,Y] = \mathbb{A}^2$$
can be thought of as the y-axis. This gives "a closed subscheme structure" to the set $V(X)$.

Observe that $V(X^2, XY) = V(X)$. Hence this also gives a closed immersion
$$\operatorname{Spec} k[X,Y](X^2, XY) \to \mathbb{A}^2,$$
but we obtain a different closed subscheme structure on $V(X)$ (for example, in one we have nilpotents, in the other we do not).

If we were to draw a picture, we would think of the first subscheme as the y-axis, and the second subscheme as the y-axis where the origin is a special point.

Note that the subschemes are isomorphic away from the origin, which we can see by looking at $D(Y) \subseteq \operatorname{Spec} k[X,Y]/(X)$. Here $D(Y) \cong\cong$

$\operatorname{Spec}(k[X,Y]/(X))_Y \cong \operatorname{Spec} k[Y]_Y$. If we instead consider $D(Y) \subseteq \operatorname{Spec} k[X,Y]/(X^2, XY)$. Here $D(Y) \cong \operatorname{Spec}(k[X,Y]/(X^2,XY))_Y \cong \operatorname{Spec}(k[X,Y]_Y/(X)) \cong \operatorname{Spec} k[Y]_Y$. The second isomorphism is a good exercise in localizations.

1.1. Fibre products.

Definition. Let $\mathcal{C}$ be a category and

$$\begin{tikzcd} & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

be a diagram in $\mathcal{C}$.

Then a fibre product, if it exists, is an object W equipped with morphisms $p\colon W \to X$, $q\colon W \to Y$ such that $f \circ p = g \circ q$ satisfying the following universal property.

$$\begin{tikzcd} W' \arrow[dr, dotted, "h"] \arrow[drr, bend left, "q'"] \arrow[ddr, bend right, "p'"'] & & \\ & W \arrow[r, "q"] \arrow[d, "p"] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"] & Z \end{tikzcd}$$

For any W' equipped with maps $p'\colon W' \to X$, $q'\colon W' \to Y$ such that $f \circ p' = g \circ q'$, there exists a unique moprhism $h\colon W' \to W$ making the diagram commute, i.e., $p \circ h = p'$, $q \circ h = q'$.

If the fibre product exists, it is unique up to unique isomorphism.

As a key exmple, if $\mathcal{C}$ is the category of sets, then

$$X \times_Z Y = \{(x,y) \in X \times Y \mid f(x) = g(y)\}.$$

Remark. It will be helpful to think about the fibre product and more generally other universal properties via the Yoneda lemma.

Let $\mathcal{C}$ be a category. Write h_X for the contravariant functor

$$\begin{aligned} h_X\colon \mathcal{C} &\to \mathsf{Set} \\ Y &\mapsto \operatorname{Hom}(Y,X) \\ h_X(f\colon Y \to Z)\colon \operatorname{Hom}(Z,X) &\to \operatorname{Hom}(Y,Z) \\ \varphi &\mapsto \varphi \circ f. \end{aligned}$$

Recall that a natural transformation of contravariant functors $F, G\colon \mathcal{C} \to \mathcal{D}$ written as $T\colon F \to G$, consists of data $T_(X)\colon F(X) \to G(X)$ for every object X of $\mathcal{C}$ such that for all $f\colon X \to Y$ in $\mathcal{C}$ the diagram

$$\begin{tikzcd} F(X) \arrow[d, "T(X)"] & F(Y) \arrow[l, "F(f)"'] \arrow[d, "T(Y)"] \\ G(X) & G(Y) \arrow[l, "G(f)"'] \end{tikzcd}$$

commutes.

The Yoneda lemma sats that the set of natural transformations $h_X \to G$ for any functor $G\colon \mathcal{C} \to \mathsf{Set}$ is in natural bijection with $G(X)$.

A sketch of the proof is as follows: given $\eta \in G(X)$, we need to define a map $h_X(Y) \to G(Y)$ for all objects Y in $\mathcal{C}$. We do this by sending $f\colon Y \to X$ to $G(f)(\eta)$. One can check that this is indeed a natural transformation.

In the converse direction, if $T\colon h_X \to F$ is a natural transformation, we obtain an element $\eta\colon T(X)(1_X) \in F(X)$.

One can check that these two maps are inverse to each other.

The corollary we are interested in is the following: the set of natural transformations $h_X \to h_Y$ is in natural bijection with $h_Y(X) = \mathrm{Hom}(X, Y)$.

We call a contravariant functor $F\colon \mathcal{C} \to \mathsf{Set}$ representable if F is naturally isomorphic to h_X for some object X of $\mathcal{C}$.

Lots of questions in algebraic geometry boil down to whether some functor is representable.

In this light, we can redefine fibre products: a fibre product in a category $\mathcal{C}$ is an object which represents the functor $T \mapsto \mathrm{Hom}(T, X) \times_{\mathrm{Hom}(T,Z)} \mathrm{Hom}(T, Y)$.

The advantage of putting it this way is that we can check identities involving fibre products using identities of fibre products of sets. For example, consider the identity $(A \times_B C) \times_C D \cong A \times_B D$. On sets, we find that $((a, c), d) \mapsto (a, d)$ has the inverse $((a, f(d)), d)$, where $f\colon D \to C$ is the map we pulled back.

Then we have functors $T \mapsto (h_A(T) \times_{h_B(T)} h_C(T)) \times_{h_C(T)} h_D(T)$ and $T \mapsto h_A(T) \times_{h_B(T)} h_D(T)$. The bijection of sets above yields a natural isomorphism between the functors, which hence represent isomorphic objects by Yoneda.

Theorem 4.1. The category of schemes has fibre products.

Proof. We will construct $X \times_S Y$ for various cases, bootstrapping up to the general case.

For the first step, $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, $S = \operatorname{Spec} R$. Since we have an equivalence with the opposite category of the category of rings, we are looking for pushouts in the category of rings. But the pushout of A and B via R is just $A \otimes_R B$. Thus $\operatorname{Spec}(A \otimes_R B)$ is $\operatorname{Spec} A \times_{\operatorname{Spec} R} \operatorname{Spec} B$ in the category of affine schemes.

If T is an arbitrary scheme, then giving a morphism $T \to \operatorname{Spec} A$ (for any ring A) is the same as giving a morphism $A \to \Gamma(T, \mathcal{O}_T)$ by the first example sheet. Thus giving a diagram

$$\begin{tikzcd}
T \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& & \operatorname{Spec} B \arrow[d] \\
& \operatorname{Spec} A \arrow[r] & \operatorname{Spec} R
\end{tikzcd}$$

is equivalent to

$$\begin{tikzcd}
\Gamma(T, \mathcal{O}_T) & & \\
& A \otimes_R B \arrow[ul, dotted, "\exists! h"'] & B \arrow[l] \arrow[ull, bend right] \\
& A] \arrow[u] \arrow[uul, bend left] & R \arrow[l, "f"] \arrow[u, "g"']
\end{tikzcd}$$

and $h\colon A \otimes_R B \to \Gamma(T, \mathcal{O}_T)$ induces a map $T \to \operatorname{Spec} A \otimes_R B$. Thus $\operatorname{Spec} A \otimes_R B$ is the fibre product $\operatorname{Spec} A \times_{\operatorname{Spec} R} \operatorname{Spec} B$ in the category of schemes.

For the second step, the idea is to construct more general fibre products by gluing of schemes as seen on the first example sheet. We also glue morphisms: if X, Y are schemes, $\{U_i\}$ an open cover of X and we are given morhpsisms $f_i\colon U_i \to Y$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$, then we obtain $f\colon X \to Y$ such that $f|_{U_i} = f_i$ (this was proved in the examples class).

For the third step, if $X, Y \to S$ are given and $U \subseteq X$ is open, suppose that $X \times_S Y$ exists, with projectives $p_1 \colon X \times_S Y \to X$, $p_2 \colon X \times_S Y \to Y$. Then we claim $p_1^{-1}(U)$ has the universal property of $U \times_S Y$.

$$\begin{tikzcd}
T \arrow[dr, dotted, "\exists ! h"] \arrow[drr, dotted, "\exists ! h'"] \arrow[drrr, bend left] \arrow[ddr, bend right] & & & \\
& p_1^{-1}(U) \arrow[r, hook] \arrow[d] & X \times_S Y \arrow[r] \arrow[d] & Y \arrow[d] \\
& U \arrow[r, hook] & X \arrow[r] & S
\end{tikzcd}$$

Note that by commutativity, the image of $p_1 \circ h'$ is contained in U, hence the image of h' is contained in $p_1^{-1}(U)$. Hence we get a unique factoring h as required. Therefore, the universal property holds for $p^{-1}(U)$.

For the fourth step, suppose $\{X_i\}$ is an open cvover of X and $X_i \times_S Y$ exists for each i. Then we claim that $X \times_S Y$ exists.

Indeed, Define $X_{ij} \coloneqq X_i \cap X_j$, $U_{ij} = p_1^{-1}(X_{ij}) \subseteq X_i \times_S Y$. By the third step, we find $U_i j \cong X_{ij} \times_S Y$. By the universal propertu of fibre products, there is a unique isomorphism $\varphi_{ij} \colon U_{ij} \to U_{ji}$ compatible with the projections.

As an exercise: check that these gluing maps φ_{ij} satisfy the requirements of question 14 on the first example sheet.

Thus we can glue that $X_i \times_S Y$ via the φ_{ij} to get a scheme $X \times_S Y$, and we need to check that this is indeed a fibre product. So suppose we have a scheme T, $p_1' \colon T \to X$ and $p_2' \colon T \to Y$ satisfying the commutativity condition. Let $T_i \coloneqq (p_1')^{-1}(X_i)$, so we get a morphism $\theta_i \colon T_i \to X_i \times_S Y \hookrightarrow X \times_S Y$, where the latter is an open immersion by construction. On $T_i \cap T_j$, these maps agree, since they factor through $X_{ij} \times SY \subseteq X_i \times_S Y$ and $X_{ji} \times_S Y \subseteq X_j \times_S Y$ and by the universal property they agree.

Thus using step 2 we can glue the θ_i to get $\theta \colon T \to X \times_S Y$. It is an exercise to check that this is the unique map making the diagram commute.

For step 5, using step 4 and 1, we may construct $X \times_S Y$ when S and Y are affine. Repeating the process for Y, we obtain fibre products when S is affine and X and Y are arbitrary schemes.

Now, for step 6, if X, Y and S are all arbitrary, take an open affine cover $\{S_i\}$ of S, $f \colon X \to S : Y \to S$, $X_i \coloneqq f^{-1}(S_i)$, $Y_i \coloneqq g^{-1}(S_i)$. Then $X_i \times_{S_i} Y_i$ exists, and it is an exercise to verify using the universal property that $X_i \times_{S_i} Y_i \cong X_i \times_S Y_i$ (since the maps $X_i \to S$ and $Y_i \to S$ factor through S_i). Use the same gluing argument as above to glue the $X_i \times_S Y_i$ to $X \times_S Y$. □

2. Fibres of morphisms

Remark. In the category of sets, if $f \colon X \to Y$ and $y \in Y$, then the fibre product $\{y\} \times_Y X$ is just $f^{-1}(y)$.

Definition 4.2. Given a morphism of schemes $f \colon X \to Y$ and $y \in Y$, let $\kappa(y) \coloneqq \mathcal{O}_{Y,y}/\mathfrak{m}_y$ be the residue field of y. We get a morphism $\operatorname{Spec} \kappa(y) \to Y$ with image y. Then we define $X_y \coloneqq \operatorname{Spec} \kappa(y) \times_Y X$ to be the scheme-theoretic fibre of f at y.

Example. Let $f \colon X = \operatorname{Spec} k[X] \to Y = \operatorname{Spec} k[t]$ be induced by the map $k[t] \to k[X]$ given by $t \mapsto X^2$.

For $y = (t - a) \subseteq k[t]$, $a \in k$, then notice that $\kappa(y) \cong k[t]/(t-a) \cong k$ (exercise: verify the first isomorphism) $X_y \cong \operatorname{Spec} \kappa(y) \otimes_{k[t]} k[X]$. From commutative algebra, we know that for an A-algebra B we have $A/I \otimes_A B \cong B/IB$. Hence $X_y \cong \operatorname{Spec} k[X]/(X^2 - a)$. If $a \neq 0$ and $\operatorname{char} k \neq 2$, then we find that either X_y consists of two distinct points or a single point of $\sqrt{a} \notin k$. If $a = 0$, then we get $\operatorname{Spec} k[X]/(X^2)$.

This can be thought of like collapsing a rotated parabola to a line. There is an image missing here.

REMARK. (1) In general, it is hard to calculate fibre products. In general, $X \times_S Y$ will not be the set-theoretic fibre product, for example $\mathbb{A}^1_k \times_{\operatorname{Spec} k} \mathbb{A}^1_k \cong \operatorname{Spec} k[X] \otimes_k k[X] \cong \operatorname{Spec} k[X, Y] \cong \mathbb{A}^2_k$.
(2) If we are interested only in varieties, i.e., schemes over a field k, the usual product of varieties $X \times Y$ will correspond to $X \times_{\operatorname{Spec} k} Y$. More generally, if we are working in the category of schemes over S, the categorical product is given by $X \times_S Y$.
(3) Given schemes S, T with a morphism $T \to S$ we get a functor from schemes over S to schemes over T given by $X \to S \mapsto X \times_S T \to T$. This functor is called base change.

EXAMPLE. Consider a scheme X over $\operatorname{Spec} \mathbb{Z}$, for example $X = \operatorname{Proj} \mathbb{Z}[X, Y, Z]/(X^n + Y^n - Z^n) \to \operatorname{Spec} \mathbb{Z}$.

We may consider the base change

$$\operatorname{Spec} \mathbb{F}_p \to \operatorname{Spec} \mathbb{Z}$$

induced by $\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \cong \mathbb{F}_p$. Here we have

$$X \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \mathbb{F}_p \cong \operatorname{Proj} \mathbb{F}_p[X, Y, Z]/(X^n + Y^n - Z^n).$$

Another possible base change is $\operatorname{Spec} \mathbb{Q} \to \operatorname{Spec} \mathbb{Z}$ induced by $\mathbb{Z} \to \mathbb{Q}$. In this case we have

$$X \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \mathbb{Q} \cong \operatorname{Proj} \mathbb{Q}[X, Y, Z]/(X^n, Y^n - Z^n).$$

Yet another example is given by $\operatorname{Spec} \mathbb{C} \to \operatorname{Spec} \mathbb{Z}$ induced by $\mathbb{Z} \to \mathbb{C}$. This gives

$$X \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \mathbb{C} \cong \operatorname{Proj} \mathbb{C}[X, Y, Z]/(X^n + Y^n - Z^n) \subseteq \mathbb{P}^2_{\mathbb{C}}.$$

This illustrates the power of scheme-theoretic language to translate questions between different areas (e.g., number theory or geometry)

DEFINITION 4.3. A scheme X is called intergral if for every $U \subseteq X$ open $\mathcal{O}_X(U)$ is an integral domain.

DEFINITION 4.4. A scheme X is called reduced if for every $U \subseteq X$ open $\mathcal{O}_X(U)$ has no nilpotent elements.

DEFINITION 4.5. A scheme X is called irreducible if the underlying topological space X is irreducible, i.e., if $X = X_1 \cup X_2$ with $X_1, X_2 \subseteq X$ closed, then $X_1 = X$ or $X_2 = X$.

EXAMPLE. Take $X = \operatorname{Spec} k[X, Y]/(XY)$.

- X is not integral because $\Gamma(X, \mathcal{O}_X) \cong k[X, Y]/(XY)$ is not an integral domain, as $X \times Y = 0$.
- We can show that X is reduced, and this follows from the fact that the global sections to not have nilpotents (TODO: how?).
- X is not irreducible, since $X = V(X) \cup V(Y)$. We can think of X as the union of the x-axis and the y-axis.

THEOREM 4.6. A scheme X is integral if and only if X is reduced and irreducible.

PROOF. On the second example sheet. □

DEFINITION 4.7. Let X be a scheme. We call X locally Noetherian if there is a cover $\{U_i\}$ of X with $U_i \cong \operatorname{Spec} A_i$ affine and A_i Noetherian. We call X Noetherian if the cover may be taken to be finite.

As an example, the scheme $\operatorname{Spec} k[X_1, X_2, \ldots]$ is not locally Noetherian, and this is not obvious.

However, it is possible to show that X is locally Noetherian if and only if for every affine subset $U \subseteq X$, $U \cong \operatorname{Spec} A$ we have that A is Noetherian.

DEFINITION 4.8. A morphism $f\colon X \to Y$ of schemes is called locally of finite type if there is a covering of Y by affine open sets $\{V_i \cong \operatorname{Spec} B_i\}$ such that for each i, $f^{-1}(V_i)$ can be covered by affine open sets $\{U_{ij} \cong \operatorname{Spec} A_{ij}\}$, where each A_{ij} is a finitely generated B_i-algebra. This makes sense:

$$\begin{tikzcd} U_{ij} \arrow[d] & A_{ij} \\ V_i & B_i. \arrow[u] \end{tikzcd}$$

We say that f is of finite type if for each i the cover U_{ij} may be taken to be finite.

DEFINITION 4.9. Let k be an algebraically closed field. A variety over k is a scheme X over $\operatorname{Spec} k$ which is integral and $X \to \operatorname{Spec} k$ is of finite type.

Note that since $\operatorname{Spec} k$ only has one point, being of finite type just means that X can be covered by a finite number of open affines $U_i = \operatorname{Spec} A_i$ with A_i a finitely generated k-algebra. The A_i must be integral domains. In particular, this forces $A_i = k[X_1, \ldots, X_n]/I$, where I is a prime ideal.

Note this this still allows a "non-Hausdorff" scheme $\mathbb{A}^1 \cup \mathbb{A}^1$ obtained by gluing $D(X)$ to $D(X)$ (this is the line with two origins).

EXAMPLE 4.10. Let $X_i = \operatorname{Spec} k[X_i, Y_i]/(X_iY_i)$, $i \in \mathbb{Z}$. Glue X_i to X_{i+1} along the open subsets $U_{i,i+1} \subseteq X_i$ given by $D(X_i)$ and $U_{i+1,i} \subseteq X_{i+1}$ given by $D(Y_{i+1})$ via the map

$$U_{i,i+1} \cong \operatorname{Spec}\left(\frac{k[X_i, Y_i]}{(X_iY_i)}\right)_{X_i} \cong \operatorname{Spec} k[X_i]_{X_i} \to \operatorname{Spec} k[Y_{i+1}]_{Y_{i+1}} \cong U_{i+1,i}$$

induced by the ring homomorphism sending $Y_{i+1} \mapsto X_i^{-1}$.

Do this for all i to get an infinite chain of $\mathbb{P}^1$s.

There is an image missing here.

The $\{X_i\}$ form an open cover of X that has no finite subcover, so X is not quasicompact and only locally of finite type over $\operatorname{Spec} k$.

3. Separated and proper morphisms

REMARK. A topological space X is Hausdorff if and only if the diagonal $\Delta \subseteq X \times X$ is closed.

For example, if X is the Euclidean line with two origins is not Hausdorff, since every open set containing one origin also contains the other. The space $X \times X$ is $\mathbb{R}^2$ with doubled axes and four origin. The diagonal Δ only contains two of the origins. But the other two origins are still contained in the closure of Δ. There is a picture missing here.

DEFINITION 4.11. If $f\colon X \to Y$ is a morphism of schemes. Define $\Delta\colon X \to X \times_Y X$ to be the unique morphism making the diagram

$$\begin{tikzcd}
X \arrow[dr, "\Delta"] \arrow[drr, bend left, "\mathrm{id}_X"] \arrow[ddr, bend right, "\mathrm{id}_X"'] & & \\
& X \times_Y X \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}$$

commute.

Way say that f is separated if Δ is a closed immersion.

THEOREM 4.12. Let $f\colon X \to Y$ be a morphism of schemes, where X is Noetherian. Then f is separated if and only if the following condition holds.

For any field K and any valuation ring $R \subseteq K$ (i.e., for any $0 \neq x \in K$, $x \in R \vee x^{-1} \in R$), let $T = \operatorname{Spec} R$, $U = \operatorname{Spec} K$, and let $U \to T$ be the morphism induced by the inclusion $R \to K$. Given a commutative diagram

$$\begin{tikzcd}
U \arrow[r] \arrow[d, "i"] & X \arrow[d, "f"] \\
T \arrow[r] & Y
\end{tikzcd}$$

there exists at most one morphism $i'\colon T \to X$ making the diagram commute.

NOT PROVED IN THIS COURSE. It is recommended to read a proof in II.4 of Hartshorne. □

REMARK. If R is a valuation ring, it has 0 as a prime ideal and a unique maximal ideal. The point $\{0\}$ is dense (i.e., its closure is all of $\operatorname{Spec} R$). The closed point $\mathfrak{m}$ is in some sense a limit point. Very roughtly, the "at most one" condition in the valuative criterion correesponds to the fact that a sequence has at most one limit in a Hausdorff space.

REMARK. We may now define a variety over a field k as a scheme X which is integral and finite type and separated over $\operatorname{Spec} k$.

DEFINITION 4.13. A morphism $f\colon X \to Y$ is called proper if it is separated, of finite type and universally closed, i.e., for any morphism $Y' \to Y$ the induced projection $X \times_Y Y' \to Y'$ is a closed map, i.e., sends closed sets to closed sets.

EXAMPLE.
(1) $\mathbb{P}^n_k = \operatorname{Proj} k[X_0, \ldots, X_n] \to \operatorname{Spec} k$ is proper.
(2) $\mathbb{A}^1_k \to \operatorname{Spec} k$ is not proper. It is separated (we have not proved this) and of finite type (TODO: this should be clear). Consider the base change by $\mathbb{A}^1_k \to \operatorname{Spec} k$. We have the map

$$KP_2\colon \mathbb{A}^1_k \times_{\operatorname{Spec} k} \mathbb{A}^1_k = \mathbb{A}^2_k = \operatorname{Spec} k[X] \otimes_k k[Y] \to \operatorname{Spec} k[t] = \mathbb{A}^1_k$$
$$Y \leftarrow\!\shortmid t$$
$$(X, Y) \mapsto Y.$$

This is not a closed map, e.g., $P_2(V(XY-1)) = D(t)$, which is open, not closed. We are essentially projecting a hyperbola to the x axis.

THEOREM 4.14. Let $f\colon X \to Y$ be a morphism of finite type with X noetherian. Then f is proper if as in the criterion for separatedness, whenever given a diagram

$$\begin{tikzcd} \operatorname{Spec} k = U \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} R = T \arrow[r] \arrow[ur, dotted, "\exists! g"] & Y, \end{tikzcd}$$

where R is a valuation ring, there exists a unique morphism $g\colon T \to X$ making the diagram commute.

EXAMPLE. Projective varieties, i.e., closed subvariatites in $\mathbb{P}_n^k$ are proper over $\operatorname{Spec} k$.

CHAPTER 5

Sheaves of $\mathcal{O}_X$-modules

REMARK. We want to go from the notion of an A-module M to the notion of an $\mathcal{O}_X$-module F.

DEFINITION 5.1. Let $(X, \mathcal{O}_X)$ be a ringed space. A sheaf of $\mathcal{O}_X$-modules is a sheaf of abelian groups on X such that for each $U \subseteq X$, $\mathcal{F}(U)$ has the structure of an $\mathcal{O}_X(U)$-module, compatible with restriction, i.e., if $s \in \mathcal{O}_X(U)$, $m \in F(U)$ and $V \subseteq U$, then $s|_V \cdot m|_V = (s \cdot m)|_V$.

A morphism of sheaves of $\mathcal{O}_X$-modules $\varphi\colon \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves of abelian groups such that for all $U \subseteq X$ the map $\varphi_U\colon \mathcal{F}(U) \to \mathcal{G}(U)$ is $\mathcal{O}_X(U)$-linear.

Kernels, cokernals and images of morphisms of sheaves of $\mathcal{O}_X$-modules are sheaves of $\mathcal{O}_X$-modules (TODO: show that sheafification retains the $\mathcal{O}_X$-module structure).

We write $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ for the group of $\mathcal{O}_X$-module homomorphisms from $\mathcal{F}$ to $\mathcal{G}$. This is an $\mathcal{O}_X(X)$-module.

Then $U \mapsto \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ is a sheaf of $\mathcal{O}_X$-modules, written as $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ and pronounced "sheaf hom".

If $\mathcal{F}$, $\mathcal{G}$ are $\mathcal{O}_X$-modules, we denote by $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ with the trivial Lie bracket.

the sheaf associated to the presheaf $U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$.

Pushforwards and pullbacks: We will start with some motivation. Consider a map of rings $\varphi A \to B$ and let M be a B-module and N and A-module. Then M is also an A-module via restriction of scalars. Also, $B \otimes_A N$ is a B-module via extension of scalars. We want to find analogues of this for sheaves of $\mathcal{O}_X$-modules.

Given $f\colon X \to Y$ a morphism of ringed spaces (recall that this includes a map $f^\#\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$), a sheaf $\mathcal{F}$ of $\mathcal{O}_X$ modules and a sheaf $\mathcal{G}$ of $\mathcal{O}_Y$-modules, then

(1) $f_*\mathcal{F}$ is naturally a sheaf of $f_*\mathcal{O}_X$-modules since $(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$, which is a $\mathcal{O}_X(f^{-1}(U)) = (f_*\mathcal{O}_X)(U)$-module and hence $f_*\mathcal{F}$ becomes a $\mathcal{O}_Y$-module via restriction of scalars.

(2) $f^{-1}\mathcal{G}$ is naturally an $f^{-1}\mathcal{O}_Y$-module. But $f^\#$ induces the adjoint map $f^\#\colon f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ by Question 10 on the first example sheet. Define $f^*\mathcal{G} := f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$. This is a sheaf of $\mathcal{O}_X$-modules.

DEFINITION 5.2. We call a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules free if it is isomorphic to $\bigoplus_{i \in I} \mathcal{O}_X$ for some index set I. If I is finite of cardinality r, then we say that $\mathcal{F}$ has rank r.

A sheaf $\mathcal{F}$ is called locally free of rank r, if there exists an open cover $\{U_i\}$ of X such that $\mathcal{F}|_{U_i}$ is free (as a $\mathcal{O}_X|_{U_i}$-module!) of rank r for each i.

We say that $\mathcal{F}$ is a line bundle if it is rank 1.

Often, more generally, one might refer to a rank r locally free sheaf as a rank r vector bundle.

EXAMPLE. Let $X = \operatorname{Spec} A$ be an affine scheme and M an A-module. For $p \in \operatorname{Spec} A$, we have the localization M_p. Define a sheaf $\tilde{M}$ on $\operatorname{Spec} A$ by

$$\tilde{M}(U) = \{s\colon U \to \coprod_{p \in U} M_p \mid (\star)\},$$

where $(\star)$ means that

(1) $\forall p \in U \colon s(p) \in M_p$,
(2) for each $p \in U$ there is a neighborhood $p \in V \subseteq U$ and $m \in M$, $s \in A$ such that $s \notin q$ for all $q \in V$ and $s(q) = m/s$ for all $q \in V$.

We notice that $\tilde{A} = \mathcal{O}_{\operatorname{Spec} A}$. Observe:

(1) $(\tilde{M})_p \cong M_p$,
(2) $\tilde{M}(D(f)) \cong M_f$,
(3) $\Gamma(\operatorname{Spec} A, \tilde{M}) \cong M$.

The proofs are exactly the same as for $\mathcal{O}_{\operatorname{Spec} A}$.

Remark 5.3. One may define the notion of a vector bundle over a scheme X as another scheme E with a morphism $\pi \colon E \to X$ whose fibres are $\mathbb{A}^r$, and there is an open cover $\{U_i\}$ such that $\pi^{-1}(U_i) \cong U_i \times \mathbb{A}^r$ (and some other conditions).

We get a sheaf $\mathcal{E}(U) = \{s \colon U \to \pi^{-1}(U) \mid \pi \circ s = \mathrm{id}_U\}$. This gives a locally free sheaf on X. For details, see the exercises of Hartshorne, II.5.

As an example let $E = X \times \mathbb{A}^1$, then $\mathcal{E}(U) = \mathcal{O}_X(U)$. Giving a morphism $s \colon U \to U \times_{\operatorname{Spec} k} \mathbb{A}^1_k$ whose composition with $p_1 \colon U \times \mathbb{A}^1_k \to U$ is the identity is the same as giving a morphism $U \to \mathbb{A}^1_k$:

$$\begin{tikzcd}
U \arrow[drr, bend left, "f"] \arrow[dr, dotted, "s"] \arrow[ddr, bend right, "\mathrm{id}_U"'] & & \\
& U \times_{\operatorname{Spec} k} \mathbb{A}^1_k \arrow[r] \arrow[d] & \mathbb{A}^1_k \arrow[d] \\
& U \arrow[r] & \operatorname{Spec} k
\end{tikzcd}$$

But giving a morphism $f \colon U \to \mathbb{A}^1_k$ is the same thing as giving a homomorphism of k-algebras $k[X] \to \mathcal{O}_X(U)$ mapping $X \mapsto \varphi$ for some $\varphi \in \mathcal{O}_X(U)$. Hence the set of such homomorphisms is completely determined by the elements of $\mathcal{O}_X(U)$.

Definition 5.4. Let X be a scheme and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules on X. We say that $\mathcal{F}$ is quasi-coherent, if X can be covered with affines $U_i = \operatorname{Spec} A_i$ such that $\mathcal{F}|_{U_i} \cong \tilde{M}_i$ for some A_i-module M_i.

We say that $\mathcal{F}$ is coherent if each M_i can be taken to be finitely generated.

Example. A locally free sheaf is always quasi-coherent and it is coherent if it is of finite rank: if $U \subseteq X$ (where $U \cong \operatorname{Spec} A$) satisfies $\mathcal{F}|_U \cong \bigoplus_{i \in I} \mathcal{O}_U$, then $\mathcal{F}|_U = \widetilde{\bigoplus_{i \in I} A}$

Kernels, cokernels, images, tensor products and hom sheaves of quasi-coherent sheaves of $\mathcal{O}_X$-modules are themselves quasi-coherent.

This follows since these operations commute with $\tilde{\ }$, e.g., $\ker(\tilde{M}_1 \to \tilde{M}_2) = \ker(\widetilde{M_1 \to M_2})$, $\tilde{M}_1 \otimes_{\mathcal{O}_X} \tilde{M}_2 = \widetilde{M_1 \otimes_A M_2}$, $\operatorname{Hom}_{\mathcal{O}_X}(\tilde{M}_1, \tilde{M}_2) = \widetilde{\operatorname{Hom}_A(M_1, M_2)}$

Should check (TODO) that a morphism of sheaves is always induced by an underlying morphism of modules (look at global sections).

Remark. Note that if $\mathcal{L}$ is a line bundle, say with trivializing cover $\{U_i\}$, then we have on $U_i \cap U_j$:

$$\mathcal{O}_{U_i}|_{U_i \cap U_j} \xleftarrow[\cong]{} \mathcal{L}|_{U_i \cap U_j} \xrightarrow{\cong} \mathcal{O}_{U_j}|_{U_i \cap U_j},$$

where the first map comes from trivialization in U_i and the other using trivialization on U_j. Composing the second map with the inverse of the first yields a map φ_{ij} from left to right. Then φ_{ij} is an automorphism of $\mathcal{O}_{U_i \cap U_j}$ as an $\mathcal{O}_{U_i \cap U_j}$-module, and as such is given by multiplication by $g_{ij} \in \mathcal{O}_X^\times(U_i \cap U_j)$, where $\mathcal{O}_X^\times$ is the subsheaf of $\mathcal{O}_X$ consisting of invertible sections of $\mathcal{O}_X$.

Note on $U_i \cap U_j \cap U_k$ we have $g_{ij}g_{jk} = g_{ik}$.

Now suppose given $f\colon Y \to X$ a morphism and $\mathcal{L}$ still a line bundle on X. How do we think about $f^*\mathcal{L}$?

Let $Y_i \coloneqq f^{-1}(U_i)$. We get $f_i\colon Y_i \to U_i$. Then

$$f_i^*(\mathcal{L}|_{U_i}) \cong f_i^*(\mathcal{O}_U) \cong f_i^{-1}(O_{U_i}) \otimes_{f_i^{-1}\mathcal{O}_{U_i}} \mathcal{O}_{Y_i} \cong \mathcal{O}_{Y_i}.$$

Hence $(f^*\mathcal{L})|_{Y_i} \cong \mathcal{O}_{Y_i}$, so $\{U_i\}$ pulls back to a trivializing cover for $f^*\mathcal{L}$, i.e., the pullback of a line bundle is a line bundle.

Further, the transition maps are given by $f^\#(g_{ij})$ (this can be shown by tracing the previous chain of isomorphisms in a larger pulled back version of the above diagram).

Remark. The pushforward is not as well-behaved, e.g., $f_*\mathcal{L}'$ for $\mathcal{L}'$ a line bundle on Y need not be a line bundle. In fact, it will always be quasi-coherent, but not necessarily coherent.

If $\mathcal{L}_1, \mathcal{L}_2$ are line bundles on X, with a common trivializing cover $\{U_i\}$ (this is always possible) and with transition functions g_{ij}, h_{ij} respectively, then:

(1) The transition functions of $\mathcal{L}_1 \otimes_{\mathcal{O}_X} \mathcal{L}_2$ are $g_{ij}h_{ij}$. Note that if $(\cdot g)\colon A \to A$ and $(\cot h)\colon A \to A$ are given, then these two homomorphisms induce the homomorphism $(\cdot g) \otimes (\cdot h)\colon A \otimes_A A \to A \otimes_A A$, but $A \otimes_A A \cong A$, and applying this isomorphism on both sides yields the map $A \to A$ given by $(\cdot(g \cdot h))$.

(2) Set $\mathcal{L}_1^\vee \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}_1, \mathcal{O}_X)$. This is also a line bundle, because on U_i, $\mathcal{L}_1|_{U_i} \cong \mathcal{O}_{U_i}$, and $\mathcal{H}om_{\mathcal{O}_{U_i}}(\mathcal{O}_{U_i}, \mathcal{O}_{U_i}) = \mathcal{O}_{U_i}$, noting that $\operatorname{Hom}_A(A, A) \cong A$.

The transition maps are given by g_{ij}^{-1}:

$$\mathcal{O}_{U_i}|_{U_i \cap U_j} \xrightarrow{\cdot g_{ij}} \mathcal{O}_{U_j}|_{U_i \cap U_j},$$

and we need to take the transpose inverse of this map represented by a 1×1 matrix.

Note that $\mathcal{L}_1^\vee \otimes_{\mathcal{O}_X} \mathcal{L}_1$ has transition maps given by $g_{ij}^{-1}g_{ij} = 1$. Thus $\mathcal{L}_1^\vee \otimes_{\mathcal{O}_X} \mathcal{L}_1 \cong \mathcal{O}_X$.

Definition 5.5. Let X be a scheme. Define the Picard group $\operatorname{Pic}(X)$, the Picard group of X, to be the set of isomorphism classes of line bundles on X. This is a group with product law $\mathcal{L}_1 \cdot \mathcal{L}_2 \coloneqq \mathcal{L}_1 \otimes_{\mathcal{O}_X} \mathcal{L}_2$ and inverses given by $\mathcal{L}^{-1} \coloneqq \mathcal{L}^\vee = \mathcal{H}om(\mathcal{L}, \mathcal{O}_X)$.

Remark 5.6. Why are line bundles important? They tell you about morphisms to projective space.

Fix a base scheme $\operatorname{Spec} k$, $\mathbb{P}_k^n = \operatorname{Proj} k[X_0, \ldots, X_n]$. Denote by Sch/k the category of schemes over $\operatorname{Spec} k$. Let F be the functor taking a scheme T over k to the set of surjections $\mathcal{O}_T^{\oplus(n+1)} \to \mathcal{L}$ for $\mathcal{L}$ a line bundle on T modulo isomorphism, where $\varphi_i\colon \mathcal{O}_T^{\oplus(n+1)} \to \mathcal{L}_i$ for $i = 1, 2$ are considered isomorphic, if there exists an isomorphism $f\colon \mathcal{L}_1 \to \mathcal{L}_2$ of $\mathcal{O}_X$-modules making the diagram

$$\begin{tikzcd} \mathcal{O}_T^{\otimes(n+1)} \arrow[r, "\varphi_1"] \arrow[dr, "\varphi_2"'] & \mathcal{L}_1 \arrow[d, "f"] \\ & \mathcal{L}_2 \end{tikzcd}$$

commute.

Given a morphism $f\colon T_1 \to T_2$ over k, we get a map of sets $F(T_2) \to F(T_1)$ by mapping $\varphi\colon \mathcal{O}_{T_2}^{\oplus(n+1)} \to \mathcal{L}$ to $f^*\varphi\colon \mathcal{O}_{T_1}^{\oplus(n+1)} = f^*(\mathcal{O}_{T_2}^{\oplus(n+1)}) \to f^*\mathcal{L}$. We use that $f^*\varphi$ is still surjective, which follows from right exactness of the tensor product.

Theorem 5.7. The functor F is represented by $\mathbb{P}_k^n$, i.e., $F \cong h_{\mathbb{P}_n^k}$. In other words, giving a morphism to $\mathbb{P}_k^n$ is the same as giving a surjective morphism as above.

Proof. If the statement holds, then there is a "universal object", i.e., an lement of $F(\mathbb{P}_K^n)$ corresponding to the identity $1_{\mathbb{P}_k^n} \in h_{\mathbb{P}_k^n}(\mathbb{P}_k^n)$, i.e., a surjective map $\varphi\colon \mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)} \to \mathcal{L}$. Further, following the proof of Yoneda, given $f\colon X \to \mathbb{P}_k^n$ and $T\colon h_{\mathbb{P}^n} \to F$ the natural transformation giving the natural isomorphism of functors, we get a commutative diagram

$$\begin{array}{ccc} h_{\mathbb{P}^n}(\mathbb{P}^n) & \xrightarrow{T(\mathbb{P}^n)} & F(\mathbb{P}^n) \\ \big\downarrow{\scriptstyle h_{\mathbb{P}^n}(f)} & & \big\downarrow{\scriptstyle F(f)} \\ h_{\mathbb{P}^n}(X) & \xrightarrow{T(X)} & F(X) \end{array}$$

By commutativity of the diagram, we find that $f^*\varphi\colon \mathcal{O}_X^{\oplus(n+1)} \to f^*\mathcal{L}$ is the same as $T(X)(f)$.

So the representing scheme $\mathbb{P}^n$ comes with the universal object (surjective morphism) $\mathcal{O}_{\mathbb{P}^n}^{\otimes(n+1)} \to \mathcal{L}$.

Hence, our proof strategy will be to construct this universal object. The line bundle we construct has a name: $\mathcal{O}_{\mathbb{P}^n}(1)$.

Recall that $\mathbb{P}^n$ has an open cover $\{D_+(X_i) \mid i \leq i \leq n\}$, where $S = k[X_0, \ldots, X_n]$, $\mathbb{P}_k^n = \operatorname{Proj} S$, $D_+(X_i) = \{\mathfrak{p} \in \operatorname{Proj} S \mid X_i \in \mathfrak{p}\}$.

We will take U to be the trivializing cover for $\mathcal{O}_{\mathbb{P}^n}(1)$ with transition map $g_{ij} \in \mathcal{O}_{\mathbb{P}^n}^\times(D_+(X_i) \cap D_+(X_j)) = \mathcal{O}_{\mathbb{P}^n}^\times(D_+(X_iX_j)) \cong S_{(X_iX_j)}$ given by $g_{ij} = X_i/X_j = X_i^2/X_iX_j$. Note that $g_{ji} = X_j/X_i$, so g_{ij} is invertible and the compatibility condition $g_{ij} \cdot g_{jk} = g_{ik}$ is satisfied.

We have a morphism $\mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)} \to \mathcal{O}_{\mathbb{P}^n}(1)$ defined on $D_+(X_i)$ by $e_j \mapsto \frac{X_j}{X_i}$ using the trivialization of $\mathcal{O}_{\mathbb{P}^n}(1)$ on $D_+(X_i)$, i.e., we have an isomorphism $\mathcal{O}_{\mathbb{P}^n}(1)|_{U_i} \cong \mathcal{O}_{U_i}$.

We need to check that this is well-defined globally. We have maps

$$\begin{aligned} \mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)}|_{D_+(X_iX_k)} &\to \mathcal{O}_{D_+(X_i)}|_{D_+(X_jX_k)}, \\ e_j &\mapsto X_j/X_i, \\ \mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)}|_{D_+(X_iX_k)} &\to \mathcal{O}_{D_+(X_k)}|_{D_+(X_jX_k)}, \\ e_j &\mapsto X_j/X_k, \\ \mathcal{O}_{D_+(X_i)}|_{D_+(X_jX_k)} &\xrightarrow{\cdot g_{ik}} \mathcal{O}_{D_+(X_k)}|_{D_+(X_iX_k)} \end{aligned}$$

And the triangle they form commutes:

$$g_{ik} \cdot \frac{X_j}{X_i} = \frac{X_i}{X_k} \cdot \frac{X_j}{X_i} = \frac{X_i}{X_j}.$$

Note in particular, that each e_j maps to a global section of $\mathcal{O}_{\mathbb{P}^n}(1)$. (Remark: if instead we have chosen $g_{ij} = X_j/X_i$ we would have obtained the line bundle $\mathcal{O}_{\mathbb{P}^n}(1)^\vee =: \mathcal{O}_{\mathbb{P}^n}(-1)$, and $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-1)) = 0$).

We now have a morphism $\mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)} \to \mathcal{O}_{\mathbb{P}^n}(1)$, and we need to check that it is surjective. On $D_+(X_i)$, we have $e_i \mapsto X_i/X_i = 1 \in \Gamma(D_+(X_i), \mathcal{O}_{\mathbb{P}^n})$, so in particular, looking at sections over $D_+(X_i)$, we get a homomorphism of $S_{(X_i)}$-modules $S_{(X_i)}^{\oplus(n+1)} \to S_{(X_i)}$ sending $e_i \mapsto 1$, so it is clearly a surjective map of modules.

Thus, $(\psi\colon \mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)} \to \mathcal{O}_{\mathbb{P}^n}(1)) \in F(\mathbb{P}^n)$.

Ite remains to show that given $(X, \varphi\colon \mathcal{O}_X^{\oplus(n+1)} \to \mathcal{L})$, there exists a unique morphism $f\colon X \to \mathbb{P}^n$ such that $\varphi \cong (f^*\psi\colon \mathcal{O}_X^{\oplus(n+1)} \to f^*\mathcal{O}_{\mathbb{P}^n}(1))$. Indeed, this will give the natural transformation $F \to h_{\mathbb{P}^n}$ and the inverse natural transformation $h_{\mathbb{P}^n} \to F$ is given by pullback, $X \to \mathbb{P}^n$ gives $f^*\psi\colon \mathcal{O}_X^{\oplus(n+1)} \to f^*\mathcal{O}_{\mathbb{P}^n}(1)$.

To prove the existence of the unique morphism, let $\varphi(e_i) = s_i \in \Gamma(X, \mathcal{L})$. Define $Z_i \coloneqq \{x \in X \mid (s_i)_x \in \mathfrak{m}_x\mathcal{L}_X\}$, where $(s_i)_x$ is the germ of s_i at x and $\mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$ is the maximal ideal. We claim that Z_i is a closed set.

Observe that being a closed set can be checked on an open cover $\{U_i\}$, since $Z \subseteq X$ if and only if $Z \cap U_i$ is closed on U_i for all i.

Thus we may use a trivializing affine cover $\{U_i\}$ of X, so we reduce to the case that $X = \operatorname{Spec} A$, $\mathcal{L} \cong \mathcal{O}_{\operatorname{Spec} A}$, so $\Gamma(X, \mathcal{L}) \cong A$, so by abuse of notation we have $s_i \in A$. This induces $(s_i)_{\mathfrak{p}} = \frac{s_i}{1} \in A_{\mathfrak{p}}$.

Now $\frac{s_i}{1} \in \mathfrak{p}A_{\mathfrak{p}}$ if and only if s_i lies in the inverse image of $\mathfrak{m}_{\mathfrak{p}}A_{\mathfrak{p}}$ under the localization map $A \to A_{\mathfrak{p}}$. Hence, $Z_i = V(s_i)$ is closed.

Let $U_i \coloneqq X|_{Z_i}$ Then there is an isomorphism $\mathcal{O}_{U_i} \to \mathcal{L}|_{U_i}$ sencind $1 \mapsto s_i$, and the inverse sends $s \mapsto s/s_j$. We should interpret this as the element of $\mathcal{O}_{U_i}$ such that $\frac{s}{s_j} \cdot s_j = s$. Injectivity is obvious, surjectivity is by definition of Z_i.

(Remark: If we were working in the world of varieties, locally the section s_i is viewed as a function and Z_i is the locus where s_i vanishes. On U_i, we will define a morphism to projective space $U_i \to D_+(X_i) \subseteq \mathbb{P}^n$ via $p \mapsto (\frac{s_o(p)}{s_i(p)}, \dots, \frac{s_n(p)}{s_i(p)})$. Equivalently, on X, we can view this function as $X \to \mathbb{P}^n$, $p \mapsto (s_0(p), \dots, s_n(p))$.)

We may now define a morphism

$$f\colon U_i = X \setminus Z_i \to D_+(X_i) \cong \operatorname{Spec} S_{(X_i)}$$

by giving a homomorphism of rings $f_i^{\#}\colon S_{(X_i)} \to \Gamma(U_i, \mathcal{O}_X)$. But $S_{(X_i)}$ should be thought of as $k[\frac{X_0}{X_i}, \dots, \frac{X_n}{X_i}]$, so we define our map by sending $\frac{X_j}{X_i} \mapsto \frac{s_j}{s_i}$, which defines $f_i^{\#}$ as a homomorphism of k-algebras. To get a morphism $f\colon Z \to \mathbb{P}_k^n$ such that $f|_{U_i} = f_i$, we need to check $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$. We can now check that check that

$$(f_i^{\#})_{U_i \cap U_j}\colon S_{(X_iX_j)} \cong \Gamma(D_+(X_iX_j), \mathcal{O}_{\mathbb{P}^n}) = \Gamma(D_+(X_i) \cap D_+(X_j), \mathcal{O}_{\mathbb{P}^n}) \to \Gamma(U_i \cap U_j, \mathcal{O}_X)$$

maps $\frac{X_k}{X_i} \mapsto \frac{s_k}{s_i}$ and $\frac{X_k}{X_j} = \frac{X_k/X_i}{X_j/X_i} \mapsto \frac{s_k/s_i}{s_j/s_i} = \frac{s_k}{s_j}$ and similarly for $(f_j^{\#})_{U_i \cap U_j}$, and we find that they agree, i.e., $(f_i^{\#})_{U_i \cap U_j} = (f_j^{\#})_{U_i \cap U_j}$, so $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$, so the morphisms glue to give $f\colon X \to \mathbb{P}_k^n$.

This defines our morphism, and we have to show that $f^*\mathcal{O}_{\mathbb{P}^n}(1) \cong \mathcal{L}$. This is the case because the transition maps $g_{ij} = X_i/X_j$ of $\mathcal{O}_{\mathbb{P}^n}(1)$ pull back under $f^{\#}$ to s_i/s_j, which are the transition maps for $\mathcal{L}$ using the trivializations for $\mathcal{L}_{U_i}$ which we used above (this needs to be checked). We should also check that this isomorphism is suitably compatibe.

For uniqueness, suppose given a surjection $\varphi\colon \mathcal{O}_X^{\oplus(n+1)} \to \mathcal{L}$ and a morphism $g\colon X \to \mathbb{P}^n$ such that $g^*(\mathcal{O}_{\mathbb{P}^n}^{\oplus(n+1)} \to \mathcal{O}_{\mathbb{P}^n}(1)) \cong (\mathcal{O}_X^{\oplus(n+1)} \to \mathcal{L})$. We may think of φ as given by $n+1$ sections $s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$ with $s_i = \varphi(p_i)$. Similarly, the universal object on $\mathbb{P}^n$ is given by sections $X_i \in \Gamma(\imath^n, \mathcal{O}_{\mathbb{P}^n}(1))$.

Note that by the construction of the universal object, the section X_j is given on $D_+(X_i)$ by $X_j/X_i \in S_{(X_i)}$. In particular, the pullback of the section $X_i \in \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ is s_i.

A short remark on pullbacks: if $f\colon X \to Y$, and $\mathcal{F}$ is a sheaf of $\mathcal{O}_Y$-modules, then $s \in \Gamma(Y, \mathcal{F})$ induces a section (Y, s) in $\Gamma(X, f^{-1}\mathcal{F})$, and hence a section $f^*s = (Y, s) \otimes 1 \in \Gamma(X, f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X) = \Gamma(X, f^*\mathcal{F})$.

In particular, $(s_i)_x \in \mathfrak{m}_x\mathcal{L}_X$ for some $x \in X$ if and only if $(x_i)_{g(x)} \in \mathfrak{m}_{g(x)}\mathcal{O}_{\mathbb{P}^n}(1)_{g(x)}$. Thus, $U_i = \{x \in X \mid (s_i)_x \notin \mathfrak{m}_x\mathcal{L}_x\}$ satisfies $U_i = g^{-1}(D_+(X_i))$ (things need to be checked here).

So we have $g_i = g|_{U_i}\colon U_i \to D_+(X_i)$ and it is enough to show that $g_i = f_i$, where f_i was constructed previously from $\mathcal{O}_X^{\oplus(n+1)} \to \mathcal{L}$. For this, it suffices to show $g_i^\# = f_i^\#$. But we have $g_i^\#(X_j/X_i) = \frac{g^\#(X_j)}{g^\#(X_i)} = \frac{s_j}{s_i} = f_i^\#(\frac{X_j}{X_i})$ (the first equality needs to be checked), and uniqueness follows. $\square$

Remark. This is an example of a so-called Quot scheme, which is a scheme which represents a functor of the form $T \mapsto \{f\colon \mathcal{O}_T^{\oplus k} \to \mathcal{E}\}$, where f must be surjective and $\mathcal{E}$ is a coherent sheaf satisfying some properties.

CHAPTER 6

Divisors and the Picard Group

Remark. Very roughly, we will encounter two notions: Weil divisors, which can be thought of as "subvarieties of codimension one", and Cartier divisors, which can be thought of as "subschemes defined by a single equation".

Definition 6.1. The dimension of a topological space X is the length of the longest chain $Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$ of irreducible closed subsets of X.

For example $\dim \mathbb{A}^1_k = 1$, and the chains are $\{\star\} \subseteq \mathbb{A}^1_k$, where $\star$ is any closed point.

The Krull dimension of a ring A, $\dim A$, is the dimension of $\operatorname{Spec} A$ as a topological space. Equivalently, this is the length of the longest chain $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n$ of prime ideals.

If $Z \subseteq X$ is an irreducible closed subset, then $\operatorname{codim}(Z, X)$ is the length n of the longest chain $Z = Z_0 \subsetneq \cdots \subsetneq Z_n$ of irreducible closed subsets.

Note that the intuition on dimension may be faulty, even for Noetherian affine schemes.

However, if B is a domain and a finitely generated k-algebra for k a field, then for any prime ideal $\mathfrak{p} \subseteq B$, then $\operatorname{ht} \mathfrak{p} + \dim B/\mathfrak{p} = \dim B$. Here, $\operatorname{ht} \mathfrak{p}$ is the length of the longest chain of primes $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}$. Note $\dim B/\mathfrak{p} = \dim V(\mathfrak{p})$ and $\operatorname{ht} \mathfrak{p} = \operatorname{codim}(V(\mathfrak{p}), \operatorname{Spec} B)$, so we have from $(\star)$ that $\operatorname{codim}(V(\mathfrak{p}), \operatorname{Spec} B) + \dim V(\mathfrak{p}) = \dim \operatorname{Spec} B$. This implies that if X is a variety over k (integral and of finite type over k), and $Z \subseteq X$ is an irreducible closed subset, then $\dim Z + \operatorname{codim}(Z, X) = \dim X$.

Also, if $\eta \in Z \subseteq X$ is the generic point of Z, then $\dim \mathcal{O}_{X,\eta} = \operatorname{codim}(Z, X)$. This is shown on the third example sheet.

Proposition. If X is a Noetherian scheme, then X is a Noetherian topological space, i.e., every decreasing sequence of closed sets becomes stationary and every closed subset of X has a decomposition into a finite number of irreducible closed subsets.

Proof. Let $\{U_i\}$ be a finite cover such that $U_i \cong \operatorname{Spec} A_i$, where A_i is a Noetherian ring.

If $V_1 \supseteq V_2 \supseteq \cdots$ is a decreasing sequence of closed subsets of X, then for every i, $V_1 \cap U_i \supseteq V_2 \cap U_i \supseteq \cdots$ is a decreasing sequence of subsets which are closed in U_i. Hence there exist ideals $I_{i1} \subseteq I_{i2} \subseteq \cdots$ of A_i such that $V_j \cap U_i = V(I_{ij})$ for all j. Since A_i is Noetherian, this means that there is some K_i such that $V_j \cap U_i = V_{K_i} \cap U_i$ for all $j \geq K_i$. Since there are finitely many U_i and they cover X, we can define $K \coloneqq \max_i K_i$ and for all $j \geq K$ we have $V_j = V_K$, so X is a Noetherian topological space.

Now let V be any closed subset of X. If it is irreducible, we are done. Otherwise, decompose it as the union of two closed subsets. If possible, decompose these again, and so on. This process yields a rooted binary tree of closed sets in X, but the tree must be finite, since otherwise we would find an infinite descending sequence of closed sets in X (a binary tree of finite height is necessarily finite), which does not exist. Hence the tree has a finite number of leaves, and this is the desired decomposition into irreducible closed subsets. □

REMARK. In this chapter, we will use the symbol $(\star)$ for the following assumption: X is a Noetherian integral scheme over $\operatorname{Spec} k$ which is regular in codimension one, i.e., whenever a local ring $\mathcal{O}_{X,x}$ is of dimension one, it is regular, i.e., $\dim_{\mathcal{O}_{X,x}/\mathfrak{m}_x} \mathfrak{m}_x/\mathfrak{m}_x^2 = \dim \mathcal{O}_{X,x}$, i.e., the dimension of the Zarsiki tangent space to X at x coincides with $\dim \mathcal{O}_{X,x}$.

REMARK. Regularity measures nonsingularity, so we tend to say that a scheme X all of whose local rings are regular is a regular or nonsingular scheme.

For example, if X is a nonsingular curve in the classical sense, then X is regular in codimension 1, but the variety defined by $Y^2 = X^2(X-1)$ (there is an image missing here) is not regular at the origin. Indeed, the Zariski tangent space at the origin is 2-dimensional.

By a standard commutative algebra fact (which can be found in Atiyah-Macdonald), a regular Noetherian local domain A of dimension 1 is a discrete valuation ring, i.e., if K is the field of fractions of A, then there is a group homomorphism $\nu \colon K^\times \to \mathbb{Z}$ such that

$$A = \{x \in K^\times \mid \nu(x) \geq 0\} \cup \{0\}$$
$$\mathfrak{m} = \{x \in K^\times \mid \nu(x) > 0\} \cup \{0\}$$

Note that after rescaling ν so that $\nu(\mathfrak{m} \setminus \mathfrak{m}^2) = 1$, then $\nu(x) = k$ if and only if $x \in \mathfrak{m}^k \setminus \mathfrak{m}^{k+1}$.

DEFINITION. Assume $(\star)$ holds. Then a prime divisor on X is defined as a closed subvariety (i.e., an irreducible and reduced (equivalently integral) closed subscheme of X) of codimension 1.

Let $\operatorname{Div}(X)$ be the free abelian group generated by the prime divisors. Let $K(X)$ be the function field of X (cf. Sheet 2, Exercise 7). Notice that $K(X)$ is the field of fractions of A whenever $\operatorname{Spec} A \subseteq X$ is an open affine subset. Let $Y \subseteq X$ be a prime divisor and let $\eta \in Y$ be the generic point. Then $\dim \mathcal{O}_{X,\eta} = 1$ (this follows from the fact that $\operatorname{codim}(Y, X) = 1$) and hence we get a valuation $\nu_Y : K(X)^\times \to \mathbb{Z}$ (since $K(X)$ is the field of fractions of $\mathcal{O}_{X,\eta}$) such that

$$\mathcal{O}_{X,\eta} = \{f \in K(X)^\times \mid \nu(f) \geq 0\} \cup \{0\}.$$

Without loss of generality, we may assume that $\nu_Y(\mathfrak{m}_\mu \setminus \mathfrak{m}_\mu^2) = 1$.

EXAMPLE. Take $X = \mathbb{A}^1_k$ (i.e., $X = \operatorname{Spec} k[X]$), $\mathfrak{p} = (x-a) \subseteq k[X]$. Then $\mathcal{O}_{X,\mathfrak{p}} = k[X]_{(X-a)}$, $K(X) = k(X)$. Given $0 \neq \frac{f}{g} \in k(X)$, we may write $\frac{f}{g} = \frac{p}{q}(x-a)^k$ such that $\gcd(p, x-a) = \gcd(q, x-a) = 1$. Then we have $\nu_{\mathfrak{p}}(\frac{f}{g}) = k$ and

$$\mathcal{O}_{X,\mathfrak{p}} = \{\frac{f}{g} \in k(X)^\times \mid \nu_{\mathfrak{p}}(\frac{f}{g}) \geq 0\} \cup \{0\}.$$

We should think of ν as measuring the order of the zero or pole of $\frac{f}{g}$ at 0.

LEMMA 6.2. If X satisfies $(\star)$ and $f \in K(X)^\times$, then $\nu_Y(f) = 0$ for all but a finite number of prime divisors Y.

PROOF. We can find an open affine subset $U = \operatorname{Spec} A$ of X such that $f \in \Gamma(U, \mathcal{O}_X)$, e.g., first pass to an open affine $\operatorname{Spec} B$. On this, $f = \frac{a}{s}$ for some $a \in B$, $s \neq 0$, and the $f \in B_s$, wo we may take $U = D(s) \subseteq \operatorname{Spec} B$.

Then $Z = X \setminus U$ is a proper closed subset of X. Since X is Noetherian, so is Z as a topological space and hence decomposes into a finite number of irreducible closed subsets. Thus Z contains only a finite number of prime divisors.

Thus, it suffcies to check that statement on U, since any other prime divisor intersects U and its generic point η is contained in U (if $\eta \notin U$, then $\overline{\{\eta\}} \cap U = \varnothing$ as U is open).

Thus we may assume that $X = \operatorname{Spec} A$ is affine and $f \in A$. Thus $\nu_Y(f) \geq 0$ for all prime divisors Y in X and furthermore $\nu_Y(f) > 0$ if and only if $f \in \mathfrak{m}_\eta \subseteq \mathcal{O}_{\mathfrak{X},\eta}$ where η is the generic point of Y, which is the case if and only if $f/1 \in \mathfrak{p}$, where $\mathfrak{p} \subseteq A$ is the prime ideal corresponding to η if and only if $\mathfrak{p} \in V(f)$. Note that $V(f)$ is a proper closed subset of X since $f \neq 0$. Thus $V(f)$ decomposes into a finite number of irreducible components, none of which are equal to X, and hence there are at most a finite number of prime divisors contained in $V(f)$. $\square$

Definition 6.3. Let X satisfy $(\star)$, and $f \in K(X)^\times$. Then the divisor (of zeros and poles) of f, denoted as (f), is

$$(f) = \sum_{Y \subseteq X \text{ prime}} \nu_Y(f) \cdot Y \in \operatorname{Div} X.$$

By the previous lemma, this definition makes sense. Note that we have a map $K^\times \to \operatorname{Div} X$, $f \mapsto (f)$, and this map is in fact a homomorphism of groups since ν_Y is.

Definition 6.4. The class group of X, written as $\operatorname{Cl} X$, is the cokernel of the homomorphism $K^\times \to \operatorname{Div} X$, $f \mapsto (f)$.

Two divisors $D, D' \in \operatorname{Div} X$ are called linearly equivalent of there is some $f \in K(X)^\times$ such that $(f) = D - D'$. We write $D \sim D'$. If $D \sim 0$, i.e., $D = (f)$ for some f, we say D is a principal divisor. So $\operatorname{Cl} X$ is the group of divisors modulo linear equivalence.

Remark. If $X = \operatorname{Spec} \mathcal{O}_K$, where $\mathcal{O}_K$ is the ring of integers in a number field, then $\operatorname{Cl} \operatorname{Spec} \mathcal{O}_K = \operatorname{Cl} \mathcal{O}_K$ in algebraic number theory.

Theorem 6.5. Let A be a Noetherian integral domain. Then A is a unique factorization domain if and only if $X = \operatorname{Spec} A$ is normal (i.e., A is integrally closed in its field of fractions) and $\operatorname{Cl} X = 0$.

Proof. A UFD is integrally closed in its field of fractions (the argument is the same as for $\mathbb{Z} \subseteq \mathbb{Q}$). Also A is a UFD if and only if every prime ideal of height 1 of A is principal (exercise).

Thus, we need to show that if A is an integrally closed domain, we have the equivalence: Every height 1 prime of A is principal if and only if $\operatorname{Cl} \operatorname{Spec} A = 0$.

First, assume that every height 1 prime of A is principal. Given a prime divisor $Y \subseteq X$, Y corresponds to a height 1 prime $\mathfrak{p} \subseteq A$. By assumption, this means that there is some $f \in A \setminus \{0\}$ such that $\mathfrak{p} = (f)$, where (f) is the ideal generated by (f). Then $(f) = 1 \cdot Y$, where (f) is the divisor of f. Hence, every divisor in principal, i.e., $\operatorname{Cl} \operatorname{Spec} A = 0$.

Conversely, assume that every divisor in principal. Let $\mathfrak{p} \subseteq A$ be a prime of height 1. Let $Y = V(\mathfrak{p})$. Then we find $f \in K(X)^\times = (A_{(0)})^\times$ such that $(f) = Y$, where (f) is the divisor of f. Since $\nu_Y(f) = 1$, we have $f \in A_\mathfrak{p} = \mathcal{O}_{X,\eta}$ and f generates the maximal ideal $\mathfrak{p}A_\mathfrak{p}$. Here we use that in a discrete valuation rinh, every element of $\mathfrak{m} \setminus \mathfrak{m}^2$ generates $\mathfrak{m}$ (exercise).

Further, if $\mathfrak{p}' \subseteq A$ is any other height 1 prime, and $Y' = V(\mathfrak{p}')$, then $\nu_{Y'}(f) = 0$, so $f \in A_{\mathfrak{p}'}$ is a unit. Now apply the following result (e.g., from Matsumura, Commutative algebra, Theorem 38):

If A is an integrally closed Noetherian domain, then

$$A = \bigcap_{\operatorname{ht} \mathfrak{p}=1} A_\mathfrak{p} \subseteq A_{(0)}.$$

Thus, $f \in A$ and $f \in A \cap \mathfrak{p}A\mathfrak{p} = \mathfrak{p}$. If we can show that f generates $\mathfrak{p}$, we'll be done.

Let g be any other element of $\mathfrak{p}$. Then $\nu_Y(g) \geq 1$ and $\nu_{Y'}(g) \geq 0$ for all $Y' \neq Y$. In particular, $\nu_{Y'}(g/f) = \nu_{Y'}(g) - \nu_{Y'}(f) \geq 0$ for all Y'. Thus $g/f \in A$. Thus $g = \frac{g}{f} \cdot f \in (f)$, where (f) is the ideal generated by f, so $p = (f)$ as required. □

Proposition 6.6. Let X satisfy $(\star)$ and let $Z \subseteq X$ be a proper closed subset such that $U = X \setminus Z$ is an open subscheme of X.

Then

(1) there is a surjective homomorphism $\operatorname{Cl} X \to \operatorname{Cl} U$ given by $\sum n_i Y_i \mapsto \sum n_i (Y_i \cap U)$, where $Y_i \cap U$ is interpreted as -0 of $Y_i \cap U = \varnothing$.
(2) If $\operatorname{codim}(Z, X) \geq 2$, then this homomorphism is an isomorphism.
(3) If Z is irreducible of codimension 1, then we have an exact sequence

$$Z \to \operatorname{Cl} X \to \operatorname{Cl} U \to 0,$$

where the first map sends $1 \mapsto [Z]$.

Proof. For (1), observe that Y being a prime divisor of X implies $Y \cap U$ is either a prime divisor of U or it is empty. If $f \in K(X)^\times$ and $(f) = \sum n_i Y_i$, then the image of (f) is $\sum n_i (Y_i \cap U)$ and this coincides with $(f|_U)$. The main point here is that $K(X) = K(U)$ (since both are the field of fractions of any affine subscheme of either). Thus $\operatorname{Cl} X \to \operatorname{Cl} U$ is well-defined. It is surjective since if $Y \subseteq U$ is a prime divisor, then $\overline{Y} \subseteq X$ is a prime divisor of X with $Y = \overline{Y} \cap U$.

For (2), we note that $\operatorname{Div} X$ and $\operatorname{Cl} X$ only depend on subvarieties of codimension 1, so the claim follows (TODO).

For (3), we remark that the kernel of $\operatorname{Cl} X \to \operatorname{Cl} U$ consists only of divisors suppoerted on Z. If Z is irreducible of codimension 1, there is precisely one such prime divisor: Z itself. Hence, the claim follows. □

Proposition 6.7. We have $\operatorname{Cl} \mathbb{P}^n_k \cong \mathbb{Z}$, with a generator being the class of a hyperplane $H = V(X_i)$.

Proof. We have $\mathbb{P}^n \setminus H = D_+(X_i) \cong \mathbb{A}^n_k = \operatorname{Spec} k[X_1, \ldots, X_n]$ and $k[X_1, \ldots, X_n]$ is a UFD, so $\operatorname{Cl} \mathbb{A}^n_k = 0$. By the previous result, we have an exact sequence

$$\mathbb{Z} \to \operatorname{Cl} \mathbb{P}^n \to \operatorname{Cl} \mathbb{A}^n = 0,$$

and thus $\operatorname{Cl} \mathbb{P}^n$ is generated by $[H]$. It remains to show that the first map is also injective.

Now

$$K(\mathbb{P}^n) = k[X_0, \ldots, X_n]_{(0)} = \{f/g \mid f, g \in k[X_1, \ldots, X_n] \text{ homogeneous of the same degree, } g \neq 0\}/\sim.$$

Thus if $dH \sim 0$, then we would need to have a rational function f/g such that $(f/g) = dH$. This is only poassible if $d = 0$. To be more precise, $(f/g) = Y_1 - Y_2$, where Y_1, Y_2 are sums of hypersurfaces with the same total degree. This is not possible, hence injectivity follows. □

Remark. If X is a projective nonsingular curve, then $\operatorname{Cl} X$ was already defined in Part II.

Definition 6.8. Let X be a scheme. We define the sheaf of rational functions on X, $\mathcal{K}_X$, to be the sheaf associated with the presheaf

$$U \mapsto S(U)^{-1} \Gamma(U, \mathcal{O}_X),$$

where $S(U) \subseteq \Gamma(U, \mathcal{O}_X)$ is the subset of elements whose stalks in $\mathcal{O}_{X,x}$ for every $x \in U$ are non-zero divisors.

For example, if X is integral, then $S(U) \subseteq \Gamma(U, \mathcal{O}_X)$ is just the set of non-zero elements of $\Gamma(U, \mathcal{O}_X)$. Then $\mathcal{K}_X$ is the constant sheaf $U \mapsto K(X)$.

This is more subtle than it looks. Grothendieck got it wrong in EGA, and also the first few editions of Hartshorne copied the incorrect definition from Grothendieck.

Definition. Let $\mathcal{K}_X^\times \subseteq \mathcal{K}_X$ be the sheaf of invertible elements of $\mathcal{K}_X$. Then there is an inclusion $\mathcal{O}_X^\times \to \mathcal{K}_X^\times$.

This can be checked at presheaf level, i.e., check that $\Gamma(U, \mathcal{O}_X^\times) \to S(U)^{-1}\Gamma(U, \mathcal{O}_X)$ is injective.

Definition 6.9. A Cartier divisor on X is a global section of $\mathcal{K}_X^\times/\mathcal{O}_X^\times$.

A Cartier divisor is said to be principal if it is in the image of the natural map $\Gamma(X, \mathcal{K}_X \times) \to \Gamma(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times)$.

Two divisors are linearly equivalent if their difference is principal.

Note that we use additive language for divisors.

We write $\operatorname{CaCl} X$ (the Cartier-class group of X) to be the Cartier divisors modulo principal divisors, i.e., the cokernel of the map $\Gamma(X, \mathcal{K}_X^\times) \to \Gamma(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times)$.

Remark. Note that an element of $\Gamma(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times)$ can be represented by $\{(U_i, f_i)\}$, where $\{U_i\}$ is some open cover of X and $f_i \in \Gamma(U_i, \mathcal{K}_X^\times)$ such that on $U_i \cap U_j$, we have

$$\frac{f_i|_{U_i \cap U_j}}{f_j|_{U_i \cap U_j}} \in \Gamma(U_i \cap U_j, \mathcal{O}_X^\times).$$

Remark. Let X satisfy $(\star)$. Then there is a homomorphism $\Gamma(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times) \to \operatorname{Div} X$ descending to $\operatorname{CaCl} X \to \operatorname{Cl} X$.

Indeed, given $\{(U_i, f_i)\}$ as in the previous remark, and Y a prime divisor on X, associate a coefficient n_Y to Y by choosing some U_i such that $Y \cap U_i \neq \varnothing$ and setting $n_Y = \nu_Y(f_i)$.

We check that this is well-defined: if $Y \cap U_j \neq \varnothing$, then $Y \cap U_i \cap U_j \neq \varnothing$ (as $U_i \cap Y$ is dense in Y by irreducibility). Then $\nu_Y(f_j) = \nu_Y(f_i \cdot \frac{f_j}{f_i}) = \nu_Y(f_i) + \nu_Y(\frac{f_j}{f_i}) = \nu_Y(f_i)$ since (as we noted in the previous remark), f_j/f_i is invertible on $U_i \cap U_j$, hence has no zeros or poles. Now take the Cartier divisor $\{(U_i, f_i)\}$ to $\sum_Y n_Y Y$.

There are several things that should be checked here: it has to be independent of the choice of representative $\{(U_i, f_i)\}$. We should also convince ourselves that the cover $\{U_i\}$ may be assumed to be finite (this follows since X is Noetherian, so in particular quasi-compact).

Note also that a principal divisor coming form $f \in \Gamma(X, \mathcal{K}_X^\times)$ is represented by $\{(X, f)\}$. Then this is mapped to (f) by construction. Hence, we get an induced map on class groups as claimed.

Proposition 6.10. If X satisfies $(\star)$ and all local rings $\mathcal{O}_{X,x}$ are unique factorisation domains, then the above map $\Gamma(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times) \to \operatorname{Div} X$ is an isomorphism.

Proof. Our goal is to define an inverse map $\operatorname{Div}(C) \to \Gamma(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times)$. Let $x \in X$ be any point. Then we get a morphism $\operatorname{Spec} \mathcal{O}_{X,x} \to X$ (e.g., if $x \in \operatorname{Spec} A \subseteq X$ open affine, then $\mathcal{O}_{X,x} \cong A_\mathfrak{p}$ where $\mathfrak{p}$ corresponds to x, so $A \to A_\mathfrak{p}$ induces a morphism $\operatorname{Spec} \mathcal{O}_{X,x} \to \operatorname{Spec} A \to X$ (should check that this is independent of the choice of A).

A prime divisor on X pulls back to a prime divisor on $\operatorname{Spec} \mathcal{O}_{X,x}$ by taking inverse image. More precisely, given a prime divisor $Y \subseteq X$, if $x \notin Y$, then the pullback is empty. Otherise, $(\operatorname{Spec} A) \cap Y$ is nonempty and is of the form $V(q)$ for a prime ideal $\mathfrak{q} \subseteq A$ with $\mathfrak{q} \subseteq \mathfrak{p}$. Hence, $\mathfrak{q}$ corresponds to a prime ideal $\mathfrak{q}A_\mathfrak{p}$, hence a prime divisor $V(\mathfrak{q}A_\mathfrak{p})$ of $\operatorname{Spec} A_\mathfrak{p}$.

This gives a map $\operatorname{Div} X \to \operatorname{Div} \operatorname{Spec} \mathcal{O}_{X,x}$, $D \mapsto D_x$. Since $\mathcal{O}_{X,x}$ is a UFD, D_x must be a principal divisor on $\operatorname{Spec} \mathcal{O}_{X,x}$, i.e., $D_x = (f_x)$ (on $\operatorname{Spec} \mathcal{O}_{X,x}$ for some $f_x \in K(X)$. Thus, D and (f_x) in X differ only in divisors which don't contain x (since otherwise we could pull back to $\mathcal{O}_{X,x}$ and the order of vanishing, which is the same on X and $\operatorname{Spec} \mathcal{O}_{X,x}$, must be zero).

Thus, if U_x is the complement of the union of prime divisors of X at which D and (f_x) have different coefficient, then $D|_{U_x} = (f_x)|_{U_x}$.

Do this for every point x, and then represent a Cartier divisor by $\{(U_x, f_x)\}$. Need to check that this is really a Cartier divisor: on $U_x \cap U_y$, (f_x) and (f_y) agree, as both agree with $D|_{U_x \cap U_y}$. Hence $(f_x/f_y) = 0$, so f_x/f_y is invertible in $\mathcal{O}_{X,p}$ for all points $p \in U_x \cap U_y$ of height 1 (i.e., generic points of prime divisors).

If we cover $U_x \cap U_y$ with open affines $\operatorname{Spec} A$, this says that $f_x/f_y \in A_{\mathfrak{p}}^\times$ for all prime ideals $\mathfrak{p} \subseteq A$ of height 1. Now since all $A_{\mathfrak{q}}$ are UFDs, for all $\mathfrak{q} \subseteq A$ prime, $A_{\mathfrak{q}}$ must be integrally closed. Since being intrally closed is a local property (cf. Atiyah-Macdonald, Prop. 5.13), A is integrally closed, so by a result we cited previously, $A = \bigcap_{\mathfrak{p} \subseteq A, \operatorname{ht}(\mathfrak{p})=1} A_{\mathfrak{p}}$, so $f_x/f_y \in A^\times$, so $f_x/f_y \in \Gamma(U_i \cap U_j, \mathcal{O}_X^\times)$.

Thus $\{(U_x, f_x)\}$ represents a section of $\mathcal{K}_X^\times/\mathcal{O}_X^\times$, i.e., a Cartier divisor. This gives the inverse map. □

Remark. If X is a nonsingular variety, i.e., all local rings of X are regular, then the hypotheses of the previous result are satisfied, as all regular local rings are UFDs (this is a nontrivial result from commutative algebra).

Definition 6.11. If all local rings of X are UFDs, we say that X is locally factorial.

1. Cartier divisors and line bundles

Definition 6.12. Let D be a Cartier divisor on X represented by $\{(U_i, f_i)\}$. Define $\mathcal{O}_X(D)$ to be the subsheaf of $\mathcal{O}_X$-modules of $\mathcal{K}_X$ generated by f_i^{-1} on U_i. Note that as f_i/f_j is invertible on $U_i \cap U_j$, f_i^{-1} and f_j^{-1} generate the same $\mathcal{O}_{U_i \cap U_j}$-module.

This is a line bundle, and for the transition maps consider

$$\mathcal{O}_X|_{U_i \cap U_j} \xrightarrow{1 \mapsto f_i^{-1}} \mathcal{O}_X(D)|_{U_i \cap U_j} \xleftarrow[1 \mapsto f_j^{-1}]{} \mathcal{O}_X|_{U_i \cap U_j}$$

and the first map followed by the inverse of the second sends $1 \mapsto f_j/f_i$, so $g_{ij} = f_j/f_i$ are the transition maps.

Consequently, if D_1, D_2 are Cartier divisors represented by $\{(U_i, f_i\})$, $\{(U_i, g_i)\}$, then $D_1 - D_2$ is represented by $\{(U_i, f_i/g_i)\}$ and the transition maps for $\mathcal{O}_X(D_1 - D_2)$ are $\frac{f_j}{g_j}/\frac{f_i}{g_i} = \frac{f_j}{f_i}/\frac{g_j}{g_i}$, which are also the transition maps for $\mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)^\vee$. Thus $\mathcal{O}_X(D_1 - D_2) \cong \mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)^\vee$, so we obtain a group homomorphism

$$\Gamma(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times) \to \operatorname{Pic} X, \qquad D \mapsto \mathcal{O}_X(D).$$

Lemma 6.13. We have $D_1 \sim D_2$ if and only if $\mathcal{O}_X(D_1) \cong \mathcal{O}_X(D_2)$.

Proof. It will suffice to show that D is principal if and only if $\mathcal{O}_X(D) \cong \mathcal{O}_X$.

If D is principal, then D is represented by a rational function (X, f) with $f \in \Gamma(X, \mathcal{K}_X^\times)$. So $\mathcal{O}_X(D) = \mathcal{O}_X \cdot f^{-1} \cong \mathcal{O}_X$, where the middle term is interpreted as a subsheaf of $\mathcal{K}_X$.

Conversely, if $\mathcal{O}_X(D) \cong \mathcal{O}_X$ let

$$1 \in \Gamma(X, \mathcal{O}) \mapsto f \in \Gamma(X, \mathcal{O}_X(D)) \subseteq \Gamma(X, \mathcal{K}_X)$$

and note that we in fact have $f \in \Gamma(X, \mathcal{K}_X^\times)$. Then (X, f^{-1}) representes $D = \{(U_i, g_i)\}$ as f^{-1} and g_i only differ by a factor of an invertible function on U_i. Thus D is principal. □

Corollary 6.14. On any scheme X, $D \mapsto \mathcal{O}_X(D)$ defines an injective homomorphism $\operatorname{CaCl} X \to \operatorname{Pic} X$.

Proposition 6.15. If X is integral, then this homomorphism is an isomorphism.

Proof. We need to show that every line bundle on X is isomorphic to a subsheaf of $\mathcal{K}_X$, which, since X is integral, is the constant sheaf $U \mapsto K(X)$. Once this is shown, a trivialization on a cover U_i leads to rational functions given by $1 \in \mathcal{O}_{U_i} \to \mathcal{L}|_{U_i} \subseteq \mathcal{K}_X|_{U_i}$, where $1 \mapsto f_i$, and then $D = \{(U_i, f_i^{-1})\}$ satisfies $\mathcal{L} \cong \mathcal{O}_X(D)$.

So let $\mathcal{L}$ be any line bundle on X and consider $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{K}_X$. On any open U with $\mathcal{L}|_U \cong \mathcal{O}_U$, we have $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{K}_X|_U \cong \mathcal{O}_U \otimes_{\mathcal{O}_U} \mathcal{K}_X|_U \cong \mathcal{O}_X|_U$. This is the constant sheaf $V \mapsto K(X)$ for $V \subseteq U$.

We will shwo that $\mathcal{F} \coloneqq \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{K}_X$ is also the constant sheaf $V \mapsto \mathcal{K}(X)$. Indeed, if V is any nonempty open subset and $\{U_i\}$ is a trivializing cover of $\mathcal{L}$, then $\mathcal{F}(V \cap U_i)$ can be identified with $K(X)$ canonically, as we can identify $\mathcal{F}_\eta$ with $K(X)$ where η is the generlic point of X.

Then the sheaf gluing axioms tell us that $\mathcal{F}U \cong K(X)$. Thus, $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{K}_X \cong \mathcal{K}_X$, and we have a natural map $\mathcal{L} \to \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{K}_X$, $s \mapsto s \otimes 1$, thus exhibiting $\mathcal{L}$ as a subsheaf of $\mathcal{K}_X$. □

Definition 6.16. We say that a Weil divisor $\sum a_i Y_i$ is effective if $a_i \geq 0$ for all i.

We say that a Cartier divisor $\{(U_i, f_i)\}$ is effective if $f_i \in \mathcal{O}_X(U_i)$ for all i.

If $\mathcal{L}$ is a line bundle, $s \in \Gamma(X, \mathcal{L})$ and $\{U_i\}$ is a trivializing cover for $\mathcal{L}$ with trivializations $\varphi_i \colon L|_{U_i} \to \mathcal{O}_{U_i}$, we obtain a Cartier divisor $(s)_0$, the so-called divisor of zeros of s, by setting $(s)_0 = \{(U_i, \varphi(s))\}$. This is an effective Cartier divisors, since the right tuple elements are in $\mathcal{O}_X(U_i)$.

Theorem 6.17. Let $X \subseteq \mathbb{P}^n_k$ be a closed subscheme and $\mathcal{F}$ a coherent sheaf of $\mathcal{O}_X$-modules. Then $\Gamma(X, \mathcal{F})$ is a finite-dimensional k-vector space.

Not proved in this course. See Hartshorne, II.5.19. □

Remark. If $X = \mathbb{A}^1$, $\mathcal{F} = \mathcal{O}_X$, then $\Gamma(X, \mathcal{F}) = k[X]$ is not a finite-dimensional k-vector space.

Theorem 6.18. If $X \subseteq \mathbb{P}^n_k$ is an integral closed subscheme with k algebraically closed, then $\Gamma(X, \mathcal{O}_X) = k$.

Not proved in this course. See Hartshorne I.3.4. □

Remark. As an exercise, find an example where k is not algebraically closed and the previous result fails.

Theorem 6.19. Let X be an integral closed subscheme of $\mathbb{P}^n_k$ with k algebraically closed. Let D_0 be a Carier divisor on X, $\mathcal{L} = \mathcal{O}_X(D_0)$. Then

(1) For every $0 \neq s \in \Gamma(X, \mathcal{L})$, $(s)_0$ is an effective divisor linearly equivalent to D_0.
(2) Every effective divisor linearly equivalent to D_0 is $(s)_0$ for some section $s \in \Gamma(X, \mathcal{L})$.
(3) Two sections $s, s' \in \Gamma(X, \mathcal{L})$ have the same divisor of zeros if and only if $\exists \lambda \in k^\times : s = \lambda s'$.

Proof. For (1), we have $\mathcal{O}_X(D_0) \subseteq \mathcal{K}_X$, so $s \in \Gamma(X, \mathcal{L})$ corresponds to a rational function $f \in \Gamma(X, \mathcal{K}_X) = K(X)$. If D_0 is represented by $\{(U_i, f_i)\}$, then $\mathcal{O}_X(D_0)$ is locally generated as an $\mathcal{O}_{U_i}$-module by f_i^{-1}, giving trivializations

$$\begin{aligned} \varphi_i\colon \mathcal{O}(D_0)|_{U_i} &\to \mathcal{O}_{U_i} \\ t &\mapsto t \cdot f_i. \end{aligned}$$

So $D = (s)_0 = \{(U_i, f \cdot f_i)\} = D_0 + (f)$, since we had defined $(f) = \{(X, f)\}$. Hence, $D \sim D_0$.

For (2), if D is effective and $D = D_0 + (f)$, then if we write $D = \{(U_i, g_i)\}$, $D_0 = \{(U_i, f_i)\}$, then $g_i = f_i \cdot f$ and $g_i \in \mathcal{O}_X(U_i)$. Then $\varphi^{-1}(g_i) = g_i f_i^{-1} = f_i \cdot f \cdot f_i^{-1} = f$. So f in fact is a section of $\mathcal{O}_X(D_0) \cong \mathcal{L}$, and the $(s)_0 = D$.

If $(s)_0 = (s')_0$, then $(s)_0 = D_0 + (f)$ and $(s')_0 = D_0 + (f')$. But then $(f/f') = 0$, i.e., $f/f' \in \Gamma(X, \mathcal{O}_X^\times)$. Now we use the fact that $\Gamma(X, \mathcal{O}_X) = k$, so $f/f' \in k^\times$. $\square$

Example. Observe that $\mathbb{P}^n_k$ satisfies all of the hypotheses of this theorem. We have isomorphisms $\mathbb{Z} \cong \operatorname{Cl}\mathbb{P}^n \cong \operatorname{CaCl}\mathbb{P}^n \cong \operatorname{Pic}\mathbb{P}^n$, using that $\mathbb{P}^n_k$ is nonsingular, i.e., all local rings are regular. The generator of $\operatorname{Cl}\mathbb{P}^n$ is H, a hyperplane, and it is not hard to see that $\mathcal{O}_{\mathbb{P}^n}(H) = \mathcal{O}_{\mathbb{P}^n}(1)$ constructed previously (checking this is an very important exercise). Hence, $\operatorname{Pic}\mathbb{P}^n$ is generated by $\mathcal{O}_{\mathbb{P}^n}(1)$. Thus, it makes sense to define $\mathcal{O}_{\mathbb{P}^n}(d)$ as $\mathcal{O}_{\mathbb{P}^n}(1)^{\otimes d}$ for $d > 0$ and $\mathcal{O}_{\mathbb{P}^n}(-d)^\vee$ for $d < 0$. We have $\mathcal{O}_{\mathbb{P}^n}(d) \cong \mathcal{O}_{\mathbb{P}^n}(dH)$.

We will see that $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \cong S_d$, where $S = k[X_0, \dots, X_n] = \bigoplus_d S_d$, where S_d is the component of degree d.

As an exercise, show that if $f \in S_d$ is a homogeneous polynomial of degree d and $f = \prod_{i=1}^n f_i^{d_i}$ is its prime factorization, then $(f)_0 = \sum d_i V(f_i)$.

Remark. Let X an integral subscheme of $\mathbb{P}^n_k$, where k is algebraically closed.

Line bundles	Linear systems
A line bundle $\mathcal{L}$ $s \in \Gamma(X, \mathcal{L})$, $s \neq 0$ $\mathbb{P}(\Gamma(X, \mathcal{L})) = (\Gamma(X, \mathcal{L}) \setminus \{0\})/k^\times$	$D \in \operatorname{CaCl} X$ such that $\mathcal{L} \cong \mathcal{O}_X(D)$ $(s)_0 \sim D$ an effective divisor A complete linear system $\lvert D \rvert$, the set of all effective D' such that $D' \sim D$
$s_0, \dots, s_n \in \Gamma(X, \mathcal{L})$ defines a morphism $\mathcal{O}_X^{\oplus(n+1)} \to \mathcal{L}$ given by $e_i \mapsto s_i$. If this map is surjective, we say $\mathcal{L}$ is generated by global sections and we obtain a morphism $X \to \mathbb{P}^n_k$	A linear subspace $\mathcal{D} \subseteq \lvert D \rvert$ is called a linear system (think of this as the linear subspace of $\lvert D \rvert$ spanned by $(s_i)_0$ for $0 \leq i \leq n$. We say that $\mathcal{D}$ basepoint-free if for every $x \in X$, there is $D' \in \mathcal{D}$ such that $x \notin \operatorname{Supp} D'$ (if $D' = \sum a_i Y_i$ with $a_i > 0$, then $\operatorname{Supp} D' = \bigcup_i Y_i$). In this case, D gives a morphism $X \to \mathbb{P}^n$. Note: if $\mathcal{D}$ is determined by $s_0, \dots, s_n$, then $\mathcal{D}$ is basepoint-free if and only if $s_0, \dots, s_n$ generate $\mathcal{L} = \mathcal{O}_X(D)$. Also the pullbacks of hyperplanes in $\mathbb{P}^n$ give elements of $\mathcal{D}$.
If section of $\mathcal{L}$ induce a closed immersion in some $\mathbb{P}^n_k$, we say $\mathcal{L}$ is very ample	If (D) induces a closed immersion, we say that D is very ample.
$\mathcal{L}$ is ample if $\mathcal{L}^{\otimes n}$ is very ample for some $n > 0$	D is ample if nD is very ample for some $n > 0$.

Remark. There exists a good geometric criterion for very ampleness, see example sheets. There is also a “numerical” criterion for ampleness.

The central size of $\Gamma(X, \mathcal{L})$ is useful.

CHAPTER 7

Cohomology of sheaves

Remark. Given a short exact sequence of sheaves

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0,$$

we know that

$$0 \longrightarrow \Gamma(X, F') \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{F}'')$$

is exact. Can we extend this to a long exact sequence?

This is indeed possible using the right derived functors of $\Gamma(X, \bullet)$, which are written as $H^i(X, \bullet)$.

The idea of the construction is as follows. An abelian group I is called injective if given any diagram of abelian groups

$$\begin{tikzcd}
 & I & \\
0 \arrow[r] & A \arrow[u] \arrow[r] & B \arrow[ul, dotted]
\end{tikzcd}$$

admits a lift making the diagram commute.

Fact: Every abelian group A has an injection into an injective group. This allows us to construct injective resolutions: construct the diagram

$$\begin{tikzcd}
 & & & & 0 \arrow[dr] & & 0 & \\
 & & & & & C_1 \arrow[ur] \arrow[dr] & & \\
0 \arrow[r] & A \arrow[r] & I_0 \arrow[rr] \arrow[dr] & & I_1 \arrow[rr] \arrow[ur] & & I_2 \arrow[r] & \dots \\
 & & & C_0 \arrow[ur] \arrow[dr] & & & & \\
 & & 0 \arrow[ur] & & 0, & & &
\end{tikzcd}$$

where the C_i are cokernels. This gives a long exact sequence $0 \to A \to I_\bullet$.

We then get injective resolutions in the category of sheaves of abelian groups: if $\mathcal{F}$ is a sheaf on X, then for $x \in X$ we have an inclusion $0 \to \mathcal{F}_x \to I_x$ with $\mathcal{I}_x$ injective. Then define

$$\mathcal{I} = \prod_{x \in X} i_{x*}\mathcal{I}_x,$$

where $i_x \colon \{x\} \to X$, i.e., $\mathcal{I}(U) = \prod_{x \in U} \mathcal{I}_x$.

Then we have an inclusion $0 \to \mathcal{F} \to I$, where $s \in \mathcal{F}(U) \mapsto (f_x(U, s))_{x \in U}$ and it is possible to check that $\mathcal{I}$ is an injective object in the category of sheaves of abelian groups.

Hence we get injective resolutions

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \xrightarrow{d_0} \mathcal{I}^1 \xrightarrow{d_1} \dots$$

Then define

$$H^i(X,\mathcal{F}) \coloneqq \frac{\ker(d_i :- \Gamma(X,\mathcal{I}^i) \to \Gamma(X,\mathcal{I}^{i+1}))}{\operatorname{im}(d_{i-1}\colon \Gamma(X,\mathcal{I}^{i-1}) \to \Gamma(X,\mathcal{I}^i))},$$

i.e., take the cohomology of the chain complex of glocal sections of the $\mathcal{I}^i$.

There are some immediate properties:

(1) $H^i(X,\bullet)$ is a well-defined coaraiant functor (i.e., it is independent of the choice of resolution and $f\colon \mathcal{F} \to \mathcal{G}$ and we get induced maps $H^i(X,\mathcal{F}) \to H^(X,\mathcal{G})$.
(2) Whenever

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

is exact, then we obtain connecting homomorphisms $\delta\colon H^i(X,\mathcal{F}'') \to H^{i+1}(X,\mathcal{F}')$ the fit into a long exact sequence.
(3) Given a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}' \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F}'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{G}' \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{G}'' \arrow[r] & 0 \end{tikzcd}$$

with exact rows, we get a commutative square

$$\begin{tikzcd} H^i(X,\mathcal{F}'') \arrow[r, "\delta"] \arrow[d] & H^{i+1}(X,\mathcal{F}') \arrow[d] \\ H^i(X,\mathcal{G}'') \arrow[r, "\delta"] & H^{i+1}(X,\mathcal{G}'). \end{tikzcd}$$

(4) Whenever $\mathcal{F}$ is a flabby (flasque) sheaf, i.e., all restriction maps are surjective, then $H^i(X,\mathcal{F}) = 0$ for all $i > 0$.
(5) $H^0(X,\mathcal{F}) = \Gamma(X,F)$.

As a remark, we may also work on a ringed space $(X,\mathcal{O}_X)$ and consider only sheaves of $\mathcal{O}_X$-modules. Injective resolutions of $\mathcal{O}_X$-modules by injective $\mathcal{O}_X$-modules exist, so we could have defined cohomology in terms of such resolutions (but as remarked previously, we get the same cohomology either way).

Theorem 7.1 (Grothendieck). Let X be a Noetherian topological space of dimension n, $\mathcal{F}$ a sheaf of abelian groups on X. Then $H^i(X,\mathcal{F}) + 0$ for all $i > n$.

Not proved in this course. See Hartshorne III.2.7. □

1. Cech Cohomology

Remark. How do we calculate cohomology in practice?

Let X be a topological space, $\mathcal{F}$ a sheaf of abelian groups on X and $\mathcal{U} \coloneqq \{U_i\}_{i\in I}$ an open cover of X.

Choose a well-ordering on I and write $U_{i_0,\dots,i_p} = U_{i_0} \cap \dots \cap I_{i_p}$.

Define the group of Cech p-cochains to be

$$C^p(U,\mathcal{F}) \coloneqq \prod_{i_0<\dots<i_p} \mathcal{F}(U_{i_0,\dots,i_p}).$$

Write $\alpha \in C^p(\mathcal{U},\mathcal{F})$ as

$$\alpha = (\alpha_{i_0,\dots,i_p})_{i_0<\dots<i_p}.$$

Define the Cech coboundary $d\colon C^p(\mathcal{U},\mathcal{F}) \to C^{p+1}(\mathcal{U},\mathcal{F})$ by

$$(d\alpha)i_0,\dots,i_{p+1} = \sum_{k=0}^{p+1}(-1)^k \alpha_{i_0,\cdots,\hat{i_k},\cdots,i_{p+1}}|_{U_{i_0,\dots,i_{p+1}}}.$$

As an exercise, show that $d^2 = 0$ (this is the same as for singular cohomology).

Define $\check{H}^p(\mathcal{U},\mathcal{F}) := H^p(C^\bullet(\mathcal{U},\mathcal{F}))$.

This is nice, because if the cover $\mathcal{U}$ is small, this is actually computable.

Example. (1) Take $X = S^1$ with the usual topology and $\mathcal{F} = \underline{\mathbb{Z}}$ the constant sheaf, i.e., the sheaf associated to the presheaf of locally constant functions.

Consider the open cover consisting of two open sets, one, U, a slightly enlarged left hemisphere and the other, V, a slightly enlared right hemisphere. There is an image missing here. Then $C^0(\mathcal{U},\mathcal{F}) = \Gamma(U,\mathcal{F}) \times \Gamma(V,\mathcal{F}) = \mathbb{Z}\times\mathbb{Z}$. Also, $C^1(\mathcal{U},\mathcal{F}) = \Gamma(U\cap V,\mathcal{F}) = \mathbb{Z}^2$. Next, we have to check the boundary map $d\colon C^0(\mathcal{U},\mathcal{F}) \to C^1(\mathcal{U},\mathcal{F})$. It sends (a,b) to $(b-a,b-a)$. Hence,

$\check{H}^0(\mathcal{U},\mathcal{F}) = \ker d \cong \mathbb{Z}$, and $\check{H}^1(\mathcal{U},\mathcal{F}) = \operatorname{coker} d \cong \mathbb{Z}$.

Note that this agrees with the singular cohomology of S^1. This is not an accident. In this case, this also agrees with the sheaf cohomology $H^i(S^1,\mathcal{F})$.

(2) Next, take $\mathcal{F} = \mathcal{O}_{\mathbb{P}^1}(-2)$, $\mathbb{P}^1 = \operatorname{Proj} k[X_0,X_1]$. Recall that $\mathcal{O}_{\mathbb{P}^1}(1)$ has a transition map from $U_0 = D_+(X_0)$ to $U_1 = D_+(X_1)$ given by X_0/X_1. Thus $\mathcal{O}_{\mathbb{P}^1}(-2)$ has transition map X_1^2/X_0^2. Taking $\mathcal{U} = \{U_0,U_1\}$, we get

$$C^0(\mathcal{U},\mathcal{F}) = \Gamma(U_0,\mathcal{O}_{\mathbb{P}^1}(-2)) \times \Gamma(U_1,\mathcal{O}_{\mathbb{P}^1}(-2)) = k[X_1/X_0]\times k[X_0/X_1].$$

Furthermore, $C^1(\mathcal{U},\mathcal{F}) = \Gamma(U_0\cap U_1,\mathcal{O}_{\mathbb{P}^1}(-2)) = k[X_1/X_0]_{X_1/X_0}$, using the same trivialization on $U_0\cap U_1$ which we used on U_1. Then $d(f,g) = g - f\cdot\frac{X_1^2}{X_0^2}$, where we had to take into account that f uses a different trivialization, so we had to apply the transition map.

Then $\ker d = 0$ and $\operatorname{coker} d$ is one-dimensional and generated by the monomial X_1/X_0. Hence $\check{H}^0(\mathcal{U},\mathcal{O}_{\mathbb{P}^1}(-2)) = 0$ and $\check{H}^1(\mathcal{U},\mathcal{O}_{\mathbb{P}^1}(-2)) = k$.

Theorem 7.2. Let X be a Noetherian scheme with an open affine cover $\mathcal{U} = \{U_i\}_{i\in I}$ with the property that $U_{i_0,\dots,i_n}$ are affine for all $i_0 < \cdots < i_n$. Then if $\mathcal{F}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules, then $\check{H}^i(\mathcal{U},\mathcal{F}) \cong H^i(X,\mathcal{F})$.

Remark. If $X \to S$ is a separated morphism with S affine, then any open affine cover of X has the intersection property required by the comparison theorem is always satisfied.

Cohomology of projective space.

Remark. Fix a field k and let $X = \mathbb{P}^r_k$. We saw every line bundle on $\mathbb{P}^r_k$ is of the form $\mathcal{O}_{\mathbb{P}^r}(m) = \mathcal{O}_X(m) = \mathcal{O}_X(mH)$ for some $m\in\mathbb{Z}$.

Definition 7.3. A perfect pairing is a bilinear map $\langle\ ,\ \rangle\colon V\times W \to k$, where V,W are k-vector spaces, such that the map

$$V \to W^*, \qquad v \mapsto \langle v,\cdot\rangle$$

is an isomorphism.

Theorem 7.4. Let $S = k[X_0,\dots,X_r]$. Then

(1) there is an isomorphism of graded S-modules
$$S \cong \bigoplus_{n\in\mathbb{Z}} H^0(X, \mathcal{O}_X(n)),$$
i.e., $H^0(X, \mathcal{O}_X(n))$ is just the degree n homogeneous polynomials.
(2) For $0 < i < n$ we have $H^i(X, \mathcal{O}_X(n))$.
(3) We have $H^r(X, \mathcal{O}(-r-1)) \cong k$.
(4) There is a perfect pairing
$$H^0(X, \mathcal{O}_X(n)) \times H^r(X, \mathcal{O}_X(-n-r-1)) \to H^r(X, \mathcal{O}_X(-r-1)) \cong k$$
of finite-dimensional vector spaces for all $n \in \mathbb{Z}$.

Proof. We will calculate this using the comparison theorem and the standard affine cover $\mathcal{U} = \{U_i = D_+(X_i) \mid 0 \leq i \leq r\}$. Furthermore, we will be calculating the cohomology of $\mathcal{F} \coloneqq \bigoplus_{n\in\mathbb{Z}} \mathcal{O}_X(n)$ (this is not coherent, but it is quasicoherent) and then we will be done because cohomology commutes with coproducts (this must be checked).

The key point is the following: recall that the transition map for $\mathcal{O}_X(1)$ from U_i to U_j is X_i/X_j, and so the transition maps for $\mathcal{O}_X(m)$ are X_i^m/X_j^m. For $I \subseteq \{0, \ldots r\}$, we have $U_I = \cap_{i\in I} D_+(X_i) = D_+(X_I)$, where $X_i \coloneqq \prod_{i\in I} X_i$.

Thus $\Gamma(U_I, \mathcal{O}_{\mathbb{P}^r} \cong S_{(X_I)}$. We will identify $\Gamma(U_I, \mathcal{O}_X(m))$ with the k-subspace of S_{X_I} spanned by Laurent monomials of degree m, i.e., monomials of the form $X_0^{a_0} \cdots X_r^{a_r}$ with $\sum a_i = n$ and $a_i < 0 \implies i \in I$.

Given such a monomial M, using the trivialization on U_i, we will identify the section of $\mathcal{O}_X(m)$ defined by M with $M/X_i^m \in \Gamma(U_I, \mathcal{O}_{\mathbb{P}^r}$ (with $i \in I$).

If $i, j \in I$, then note
$$\frac{M}{X_i^m} \cdot \frac{X_i^m}{X_j^m} = \frac{M}{X_j^m}.$$
Thus we have a canonical identification of $\Gamma(U_I, \mathcal{O}_{\mathbb{P}^r}(m))$ with the space spanned by Laurent monomials of degree m.

Thus $\Gamma(U_I, \mathcal{F})$ can be identified with S_{X_I}.

Now we have a Cech complex $C^\bullet(\mathcal{U}, \mathcal{F})$ which looks like
$$\prod_{0\leq i_0\leq r} S_{X_{i_0}} \xrightarrow{d_0} \prod_{0\leq i_0\leq i_1\leq r} S_{X_{i_0}X_{i_1}} \longrightarrow \cdots \xrightarrow{d_{r-1}} S_{X_0\cdots X_r}.$$

Note that $H^0(X, \mathcal{F}) = \ker d_0$. Note also that all modules in the Cech complex are sub-S-modules of $S_{X_0\cdots X_r}$. We have
$$d_0((f_i)_{i\in\{0,\ldots,r\}}) = (f_j - f_i)_{0\leq i<j\leq r}.$$
Thus if $(f_i)_{0\leq i\leq r} \in \ker d_0$ we actually have $f_i = f_j$ for all i.

Thus $f_i, f_j \in S$ since otherwise f_i involves a negative power of X_i, which can't occur in f_j, or vice versa. Thus $_i = f$ for all i with $f \in S$, and hence $\ker d_0 \cong S$. Thus $H^0(X, \mathcal{F}) = S$ preserving degrees, so $H^0(X, Ocal_X(m)) = S_m$.

Now consider
$$d_{r-1} \colon \prod_{0\leq k\leq r} S_{X_0\cdots\widehat{X_k}\cdots X_r} \to S_{X_0\cdots X_r}.$$
Note $S_{X_0\cdots X_r}$ is the k-vector space with basis $\prod_{i=0}^r X_i^{a_i}$, where the $a_i \in \mathbb{Z}$ and the image of d_{r-1} is spanned by monomials of the form $\prod_{i=0}^r X_i^{a_i}$ with at least one $a_i \geq 0$. Thus a basis for $\operatorname{coker} d_{r-1}$ is given by
$$\{\prod_{i=0}^r X_i a_i \mid a_i \leq -1 \forall i\}.$$
In particular, $H^r(X, \mathcal{O}_X(-r-1))$ is generated by $X_0^{-1} \cdots X_r^{-1}$, and thus $H^r(X, \mathcal{O}_X(r-1)) \cong k$.

For the perfect pairing, note that $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n)) = 0$ for $n < 0$, as $S_n = 0$ for $n < 0$, and $H^r(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(-n-r-1)) = 0$ for $n < 0$ as there are no monomials with only negative exponents of degree greater than $-r-1$, and so there is nothing to check in this case.

If $n \geq 0$, we have a basis

$$\{\prod_i X_i^{m_i} \mid \sum m_i = n, m_i \geq 0\}$$

for $H^r(X, \mathcal{O}_X(n))$ and a basis

$$\{\prod_i X_i^{\ell_i} \mid \sum \ell_i = -n-r-1, \ell_i \leq -1\}$$

for $H^r(X, \mathcal{O}_X(-n-r-1))$. The perfect pairing is given by

$$(X_0^{m_0} \cdots X_r^{m_r}) \cdots (X_0^{\ell_0} \cdots X_r^{\ell_r}) = X_0^{m_0+\ell_0} \cdots X_r^{m_r+\ell_r},$$

where we interpret the result as 0 if $m_i + \ell_i \leq 0$ for any i. Indeed, this gives a pairing

$$H^0(X, \mathcal{O}_X(n)) \times H^r(X, \mathcal{O}_X(-n-r-1)) \to H^r(X, \mathcal{O}_X(-r-1)) = k(X_0 \cdots X_r)^{-1}$$

and it is easy to check that it is perfect.

It remains to show that $H^i(X, \mathcal{O}_X(m)) = 0$ for $0 < i < r$. The proof will be by induction on r.

In the base case $r = 1$ the statement is vacuous. For the induction step, if we localize $C^\bullet(\mathcal{U}, \mathcal{F})$ at X_r as graded S-modules, we get a Cech complex which calculates the cohomology groups $H^i(\mathcal{U}_r, \mathcal{F}|_{\mathcal{U}_r})$ (calculate using the Cech cover $\mathcal{U}' = \{U_i \cap U_r \mid 0 \leq i \leq r\}$). But $U_r \cong \mathbb{A}_k^r$, and Cech cohomology can also be calulated via the cover $\{U_r\}$, so $H^i(\mathcal{U}_r, \mathcal{F}|_{\mathcal{U}_r}) = 0$ for all i[1]

Now localizing at X_r is an exact functor, so

$$H^i(C^\bullet(\mathcal{U}, \mathcal{F})_{X_r}) = H^i(C^\bullet(\mathcal{U}, \mathcal{F}))_{X_r},$$

so thus, $H^i(X, \mathcal{F}_{X_r} = H^i(\mathcal{U}_r, \mathcal{F}|_{\mathcal{U}_r}) = 0$ for $i > 0$. For this to be the case, every element of $H^i(X, \mathcal{F})$ must be annihilated by some power of X_r.

Now let $H = V(X_r) \subseteq \mathbb{P}^r$. Thinking of this as a closed subscheme, we may write $H = \operatorname{Proj} S/(X_r) = \operatorname{Proj} k[X_0, \ldots, X_{r-1}] = \mathbb{P}^{r-1}$.

We have a surjective map $\mathcal{O}_{\mathbb{P}^r} \to i_*\mathcal{O}_H$, where $i\colon H \to \mathbb{P}^r$ is the inclusion. Because H is defined locally by a single equation, the kernel of $\mathcal{O}_{\mathbb{P}^r} \to i_*\mathcal{O}_H$ is a line bundle. Note that this kernel is the ideal sheaf corresponding to H. On $U_i = \operatorname{Spec} S_{(X_i)}$, this kernel is generated by X_r/X_i and hence the transition maps for the ideal sheaf $\mathcal{I}_{H/X}$ are given by

$$\mathcal{O}_{U_i}|_{U_i \cap U_j} \xrightarrow{X_r/X_i} \mathcal{I}_{H/X}|_{U_i \cap U_j} \xleftarrow[X_r/X_j]{} \mathcal{O}_{U_j}|_{U_i \cap U_j}$$

the first map followed by the inverse of the second is given by $\frac{X_r}{X_i} \cdot \frac{X_j}{X_r} = \frac{X_j}{X_i}$, i.e., the inverse of the transition map of $\mathcal{O}_{\mathbb{P}^r}(1)$, so we conclude that $\mathcal{I}_{H/X} \cong \mathcal{O}_{\mathbb{P}^r}(-1) = \mathcal{O}_{\mathbb{P}^r}(-H)$.

A remark: In general, given an effective Cartier divisor $D = \{(U_i, f_i)\}$, $f_i \in \mathcal{O}_X(U_i)$, D defines a closed subscheme of X whose ideal in U_i is generated by f_i. This coincides with the line bundles $\mathcal{O}_X(-D)$.

[1]Observe that this implies that if $\mathcal{F}$ is any quasicoherent sheaf on an affine scheme X, then $H^i(X, \mathcal{F}) = 0$ for all $i > 0$. But this is not really a proof, because the proof of the comparison theorem uses this fact. Actually, vanishing cohomology for any quasicoherent sheaf in positive degree characterises affine schemes.

The upshot of all of this is that we have an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^r}(-1) \xrightarrow{\cdot X_r} \mathcal{O}_{\mathbb{P}^r} \longrightarrow i_*\mathcal{O}_H \longrightarrow 0,$$

where the multiplication by X_r makes sense: on U_i, it means multiplying by X_r/X_i, recalling that X_r corresponds to the section X_r/X_i of $\mathcal{O}_{\mathbb{P}^r}(1)$ on U_i.

We can tensor the exact sequence with $\mathcal{O}_{\mathbb{P}^r}(n)$. This is still exact, so we get

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^r}(n-1) \longrightarrow \mathcal{O}_{\mathbb{P}^r}(n) \longrightarrow i_*\mathcal{O}_H(n) \longrightarrow 0.$$

Exactness on the left follows since $\mathcal{O}_{\mathbb{P}^n}(n)$ is locally free, haence flat (or, more simply, on U_i we have $\mathcal{O}_{\mathbb{P}^r}(n) \cong \mathcal{O}_{U_i}$, so tensoring with $\mathcal{O}_{U_i}$ doesn't do anything). Note also that $(i_*\mathcal{O}_H) \otimes_{\mathcal{O}_{\mathbb{P}^r}} \mathcal{O}_{\mathbb{P}^r}(n) \cong i_*(\mathcal{O}_H(n))$, which can be seen by considering transition maps. Frequently, we will drop the i_* when dealing with sheaves on a closed subscheme, i.e., if $\mathcal{F}$ is a sheaf on H we often write $\mathcal{F}$ for $i_*\mathcal{F}$ (where $(i_*\mathcal{F})(U) = \mathcal{F}(U \cap H)$).

Thus we have an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^r}(-n-1) \xrightarrow{\cdot X_r} \mathcal{O}^r_{\mathbb{P}}(n) * \mathcal{O}_H(n) \longrightarrow 0.$$

Summing over all n, we get

$$0 \longrightarrow \mathcal{F}(-1) \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_H \longrightarrow 0,$$

where $\mathcal{F}(-1) \coloneqq \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^r}(-1)$. and $\mathcal{F}_H = \bigoplus_{n\in\mathbb{Z}} \mathcal{O}_H(n)$.

By the induction hypothesis, we have $H^i(\mathbb{P}^r, \mathcal{F}_H) = 0$ for $0 < i < r-1$, where we have used that $H^i(\mathbb{P}^r, \mathcal{F}_H) = H^i(H, \mathcal{F}_H)$ because the Cech complexes calculating them are the same, i.e., if we use $\mathcal{U} = \{U_i\}$ or $\mathcal{U}_H = \{U_i \cap H\}$. This is a general fact: if $i\colon Y \to X$ is a closed immersion and $\mathcal{F}$ is a sheaf on Y, then $H^p(X, i_*\mathcal{F}) = H^p(Y, \mathcal{F})$.

So if $1 < i < r-1$, we get a piece of the long exact cohomology sequence

$$0 = H^{i-1}(\mathbb{P}^r, \mathcal{F}_H) \to H^i(\mathbb{P}^r, \mathcal{F}(-1)) \to H^i(\mathbb{P}^r, \mathcal{F}) \to H^i(\mathbb{P}^r, \mathcal{F}H) = 0,$$

and so $(\cdot X_r)\colon H^i(\mathbb{P}^r, \mathcal{F}(-1)) \to H^i(\mathbb{P}^r, \mathcal{F})$ is an isomorphism. But note that $H^i(\mathbb{P}^r, \mathcal{F}(-1)) = H^i(\mathbb{P}^r, \mathcal{F})$ as non-graded S-modules. But we know that every element of $H^i(\mathbb{P}^r, \mathcal{F})$ is annihilated by some power of X_r. Thus, we must have $H^i(\shortmid^r, \mathcal{F}) = 0$ for $1 < i < r-1$.

It remains to deal with the cases $i = 1$ and $i = r-1$

For $i = 1$ we have

$$0 \to H^0(\mathbb{P}^r, \mathcal{F}(-1)) \to H^0(\mathbb{P}^n, \mathcal{F}) \to H^0(\mathbb{P}^r, \mathcal{F}_H) \to H^1(\mathbb{P}^r, \mathcal{F}(-1)) \to H^1(\mathbb{P}^r, \mathcal{F}) \to 0.$$

This first two non-zero terms are just S, but with different gradings. The first one is the S-module $S(-1)$, i.e., we have $S(-1)_d = S_{d-1}$. The third term is $S/(X_r)$, and the map $S \to S/(X_r)$ is surjective. But then the map after that is zero, which in turn means that the map $(\cdot X_r)\colon H^1(\mathbb{P}^r, \mathcal{F}(-1)) \to H^1(\mathbb{P}^r, \mathcal{F})$ is injective, and hence, as before, $H^1(\mathbb{P}^r, \mathcal{F}) = 0$.

For $i = r-1$, we get

$$H^{r-1}(\mathbb{P}^r, \mathcal{F}_H) \to H^r(\mathbb{P}^r, \mathcal{F}(-1)) \to H^r(\mathbb{P}^r, \mathcal{F}) \to H^r(\mathbb{P}^r, \mathcal{F}_H) = 0,$$

where the second map is again multiplication by X_r.

By our calculation, the kernel of $(\cdot X_r)$ is generated by

$$\{X_0^{\ell_0} \cdots X_r^{\ell_r} \mid \forall i\colon \ell_i \le -1, \ell_r = -1\}.$$

This is identified with $H^{r-1}(\mathbb{P}^r, \mathcal{F}_H)$, and so the first map is injective (this should be checked, which involves understanding the Cech cohomology connecting maps) and we conclude that

$$H^{r-1}(\mathbb{P}^r, \mathcal{F}(-1)) \xrightarrow{\cdot X_r} H^{r-1}(\mathbb{P}^r, \mathcal{F})$$

is surjective. Thus multiplucation by X_r is an isomorphism, and we conclude as before that $H^{r-1}(\mathbb{P}^r, \mathcal{F}) = 0$. □

2. Normal and conormal bundles

REMARK. Let X be a scheme, $i\colon Z \to X$ a closed immersion. Then we have

$$\mathcal{I}_Z := \mathcal{I}_{Z/X} := \ker(i^*\colon \mathcal{O}_X \to i_*\mathcal{O}_Z).$$

We saw on the example sheet that $\mathcal{I}_Z$ is a coherent sheaf of $\mathcal{O}_X$-modules if X is Noetherian.

We define the conormal sheaf of Z in X to be

$$N^\vee_{Z/X} = \mathcal{I}_Z/\mathcal{I}_Z^2 \subseteq \mathcal{O}_X/\mathcal{I}_Z^2.$$

Here, $\mathcal{I}_Z^2$ is the sheaf associated to the presheaf $U \mapsto \mathcal{I}_Z(U)^2 \subseteq \mathcal{O}_X(U)$.

We will not prove the following fact: suppose X and Z are nonsingular (i.e., all local rings of X and Z are regular). Then $N^\vee_{Z/X}$ is a locally free sheaf of rank $\operatorname{codim}(Z, X)$. In this case, we define the normal bundle of Z in X to be

$$N_{Z/X} = \mathcal{H}om_{\mathcal{O}_Z}(N^\vee_{Z/X}, \mathcal{O}_Z).$$

Here we are using that $N^\vee_{Z/X}$ is a sheaf of $\mathcal{O}_Z = \mathcal{O}_X/\mathcal{I}_Z$-modules.

DEFINITION 7.5. Suppose $f\colon X \to Y$ is a separated morphism, so that $\Delta\colon X \to X \times_Y X$ is a closed immersion. Then the sheaf of differentials $\Omega_{X/Y}$ is the sheaf $\Delta^*(N^\vee_{X/X\times_Y X})$.

REMARK. We will give some motivation for this construction.

Let B be an algebra (think $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$) and M a B-module. An A-derivation $d\colon B \to M$ is a map such that for all $a \in A$, $b, b' \in B$ we have

(1) $d(b + b') = d(b) + d(b')$,
(2) $d(bb') = bd(b') + b'd(b)$, and
(3) $d(a) = 0$.

The module of relative differentials $\Omega_{B/A}$ is a B-module satisfying a universal property: there is an A-derivation $d\colon B \to \Omega_{B/A}$ such that for any other A-derivation $d'\colon B \to M$ we find a homomorphism of B-modules $g\colon \Omega_{B/A} \to M$ making the diagram

$$\begin{tikzcd} B \arrow[r, "d"] \arrow[dr, "d'"'] & \Omega_{B/A} \arrow[d, "g"] \\ & M \end{tikzcd}$$

commute.

As an example, take $B = k[X_1, \dots, X_n]$ and $A = k$. We may describe $\Omega_{B/A}$ as a direct sum

$$\Omega_{B/A} = \bigoplus_{i=1}^n B dX_i,$$

where dX_i is a generator of the summand. We set $d(X_i) = dX_i$, $d(f) = \sum_{i=1}^n \frac{\partial f}{\partial X_i} dX_i$. Given $d'\colon B \to M$ define $g\colon \Omega_{B/A} \to B$ by $g(dX_i) = d'(X_i)$.

More generally, we can construct $\Omega_{B/A}$ as follows. We have a homomorphism $\varphi\colon B \otimes_A B \to B$ sending $b \otimes b' \mapsto bb'$. Let $I := \ker \varphi$. Then I/I^2 is a B-module and we may define $d\colon B \to I/I^2$ via $d(b) = 1 \otimes b - b \otimes 1$ and we can check that $\Omega_{B/A} = I/I^2$ has the universal property (cf. Example Sheet 4).

Notive that if $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$, then φ induces the diagonal morphism $\Delta\colon X \to X \times_Y X$ and $\tilde{I} = \mathcal{I}_{X/X\times_Y X}$.

Then $\Delta^* \mathcal{I}_{X/X\times_Y X}/\mathcal{I}^2_{X/X\times_Y X}$ coincides with the sheafification of I/I^2 viewing I/I^2 as a B-module.

In this way, the sheaf of differentials is the geometric version of the module of relative differentials.

For example if $Y = \operatorname{Spec} k$ and X is a nonsingular connected variety, then so is $X \times_k X$ and $\operatorname{codim}(\Delta(X), X \times_k X) = \dim X$. So $\Omega_{X/\operatorname{Spec} k} = \Omega_X$ is a locally free sheaf of rank $\dim X$.

As an example within the example, if $X = \mathbb{A}^n_k$, then $\Omega_X = \bigoplus_{i=1}^n \mathcal{O}_X dX_i$. One can think of Ω_X of as the cotangent bundle. Then $\mathcal{T}_X = \mathcal{H}om_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X)$ is the tangent bundle.

Definition 7.6. If X is a as above, we define the canonical bundle of X to be

$$\omega_X \coloneqq \bigwedge^{\dim X} \Omega_X.$$

This is the sheaf associated with the presheaf

$$U \mapsto \bigwedge_{\mathcal{O}_X(U)}^{\dim X} \Omega_X(U).$$

Alternatively, if one takes a trivializing cover $\{U_i\}$ for Ω_X with transition matrices $g_{ij} \in \mathrm{GL}_n(\Gamma(U_i \cap U_j, \mathcal{O}_X))$, then the transition functors for ω_X are $\det g_{ij}$.

ω_X is a line bundle, and we write its corresponding Cartier divisor class as K_X. This is called the canonical divisor of X. Understanding this canonical divisor is one of the central aims of algebraic geometry.

There is an important result called Serre duality: let X be a non-singular projective variety over $\operatorname{Spec} k$ of dimension n. Then for any locally free sheaf $\mathcal{F}$ on X of finite rank, there is a natural isomorphism $H^i(X, \mathcal{F}^\vee \otimes \omega_X) \to H^{n-i}(X, \mathcal{F})^*$, where star in the right hand side is the dual vector space.

The proof of Serre duality is mostly homological algebra, but ultimately it reduces to the calculation of $H^i(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n))$

In fact, for $\mathbb{P}^r$ we have $\omega_{\mathbb{P}^r} \cong \mathcal{O}_{\mathbb{P}^r}(-r-1)$, so the perfect pairing we constructed during the calculation of the cohomology of projective space,

$$H^r(X, \mathcal{O}_X(n)) \times H^0(X, \mathcal{O}_X(-n-r-1)) \to k,$$

exhibits the isomorphism from Serre duality in this special case.

Definition 7.7. In general, if X is a projective scheme over k, then $H^i(X, \mathcal{F})$ is a finite-dimensional k-vector space (for $\mathcal{F}$ a coherent sheaf on X). Then we may define the Euler characteristic of $\mathcal{F}$ to be

$$\chi(\mathcal{F}) \coloneqq \sum_{i=0}^{\dim X} (-1)^i \dim H^i(X, \mathcal{F}).$$

This is additive in exact sequences, i.e., if

$$\cdots \longrightarrow \mathcal{F}_{i-1} \longrightarrow \mathcal{F}_i \longrightarrow F_{i+1} \longrightarrow \cdots$$

is exact, then $\sum (-1)^i \chi(\mathcal{F}_i) = 0$. In particular, if $\mathcal{F}', \mathcal{F}, \mathcal{F}''$ fit in a short exact sequence, then $\chi(\mathcal{F}) = \chi(\mathcal{F}') + \chi(\mathcal{F}'')$. These statements essentially follow from the corresponding statements about dimension of vector spaces.

Remark. The Riemann-Roch theorem roughly says that $\chi(\mathcal{F})$ is a topological invariant.

To make this more precise, we need to talk about curves. For now, let X be a projective nonsingular curve over an algebraically closed field k (algebraic closure is not needed, but will make things easier). If $P \in X$ is a closed point, we may think of it as a prime divisor defining a closed subscheme, and wehave an exact sequence

$$0 \longrightarrow \mathcal{I}_P \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_P \longrightarrow 0,$$

where the last term is the structure sheaf of the point P. We also know that $\mathcal{I}_P \cong O_X(-P)$. Now tensor with a line bundle $\mathcal{L}$. Then we get

$$0 \longrightarrow \mathcal{L}(-P) \longrightarrow \mathcal{L} \longrightarrow \mathcal{L} \otimes \mathcal{O}_P \longrightarrow 0,$$

since tensoring with line bundles is exact as remarked previously. Notice that $\mathcal{L} \otimes \mathcal{O}_P \cong \mathcal{O}_P$.

So $\chi(\mathcal{L}) = \chi(\mathcal{L}(-P)) + \chi(\mathcal{O}_P)$. Since k is algebraically closed, $\mathcal{O}_P$ has no higher cohomology and has cohomology k in degree zero, so $\chi(L) = \chi(\mathcal{L}(-P)) + 1$.

So if $D \in \operatorname{Div} X$, then $\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \deg D$, where $D = \sum a_i P_i$, and $\deg D = \sum a_i$.

Definition 7.8. The genus of a curve X is $g = \dim_K H^1(X, \mathcal{O}_X)$.

Theorem 7.9 (Riemann-Roch for curves). *For $D \in \operatorname{Div} X$, we have*

$$\dim H^0(X, \mathcal{O}_X(D)) - \dim H^0(X, \omega_X \otimes \mathcal{O}_X(-D)) = \deg D + 1 - g.$$

Proof. We have

$$\begin{aligned} \chi(\mathcal{O}_X(D)) &= \dim H^0(X, \mathcal{O}_X(D)) - \dim H^1(X, \mathcal{O}_X(D)) \\ &= \dim H^0(X, \mathcal{O}_X(D)) - \dim H^0(X, \omega_X \otimes \mathcal{O}_X(-D)). \end{aligned}$$

This is the left-hand side of the claim. But

$$\begin{aligned} \chi(\mathcal{O}_X)D)) &= \chi(\mathcal{O}_X) + \deg D \\ &= \dim H^0(X, \mathcal{O}_X) - \dim H^1(X, \mathcal{O}_X) + \deg D \\ &= 1 - g + \deg D, \end{aligned}$$

and this is the right-hand side. $\square$

Exercises

Example Sheet 1

Exercise 1.

Exercise. Let A be a ring. Show that the sets $D(f) \coloneqq \{\mathfrak{p} \in \operatorname{Spec} A \mid f \notin \mathfrak{p}\}$ with f ranging over elements of A form a basis of the topology on $\operatorname{Spec} A$.

Solution. We have $\operatorname{Spec} A = D(1)$ and for $f, g \in A$ we have

$$\begin{aligned} D(f) \cap D(g) &= \{\mathfrak{p} \in \operatorname{Spec} A \mid f \notin \mathfrak{p} \wedge g \notin \mathfrak{p}\} \\ &= \{\mathfrak{p} \in \operatorname{Spec} A \mid fg \notin \mathfrak{p}\} \\ &= D(fg), \end{aligned}$$

so the collection $\{D(f)\}$ forms the basis of *a* topology, and it remains to show that the topology generated by the $D(f)$ is the Zariski topology. Firstly, for any $f \in A$ we have

$$\begin{aligned} \operatorname{Spec} A \setminus D(f) &= \{\mathfrak{p} \in \operatorname{Spec} A \mid f \in p\} \\ &= \{\mathfrak{p} \in \operatorname{Spec} A \mid (f) \subseteq \mathfrak{p}\} \\ &= V((f)), \end{aligned}$$

so each $D(f)$ is open. It remains to show that every open set is the union of sets of the form $D(f)$. Indeed, if I is any ideal of A, then

$$\begin{aligned} \operatorname{Spec} A \setminus V(I) &= \{\mathfrak{p} \in \operatorname{Spec} A \mid I \nsubseteq \mathfrak{p}\} \\ &= \{\mathfrak{p} \in \operatorname{Spec} A \mid \exists f \in I \colon f \notin \mathfrak{p}\} \\ &= \bigcup_{f \in I} D(f) \end{aligned}$$

as required. □

Exercise. An element $f \in A$ is nilpotent if and only if $D(f) = \varnothing$.

Solution. If f is nilpotent, say $f^n = 0$, and $\mathfrak{p}$ is a prime ideal, then we have $f^n = 0 \in \mathfrak{p}$, so $f \in \mathfrak{p}$. Hence, $D(f) = \varnothing$.

If f is not nilpotent, then define $\mathcal{S}$ to be the collection of all ideals I such that $f^n \notin I$ for every $n > 0$. Since f is not nilpotent, $(0) \in \mathcal{S}$. The set $\mathcal{S}$ is partially ordered by inclusion and admits upper bounds, since the increasing union of ideals disjoint from $\{f^n\}$ is still an ideal disjoint from $\{f^n\}$. Hence $\mathcal{S}$ admits a maximal member I. We will show that I is prime.

Let $x, y \in A$ such that $xy \in I$ and suppose that $x \notin I$, $y \notin I$. Then $I + Ax$ and $I + Ay$ are not disjoint from $\{f^n\}$ so we find $n, m \in \mathbb{N}$, $i, j \in I$ and $a, b \in A$ such that $f^n = i + ax$, $f^m = j + by$. But then $f^{n+m} = ij + iby + jax + abxy \in I$, a contradiction, so $x \in I$ or $y \in I$ and I is prime. Hence, $I \in D(f)$, so $D(f) \neq \varnothing$. □

Exercise 4.

Notation. For $s \in \mathcal{F}U$ and $p \in U$ we will write $s_p \coloneqq (U, s) \in \mathcal{F}_p$.

Definition. Let $\mathcal{F}$ be a presheaf and $U \subseteq X$ an open set. Define

$$\mathcal{F}^+U \coloneqq \{s \colon U \to \coprod_{p \in U} \mathcal{F}_p \mid \forall p \in U \colon s(p) \in \mathcal{F}_p, (\star)\},$$

where $(\star)$ is the following statement: for every $p \in U$ there is an open $p \in V_p \subseteq U$ and a section $s_{V_p} \in \mathcal{F}U$ such that for ever $q \in V_p$ we have $(s_{V_p})_q = s(q)$.

Exercise. $\mathcal{F}^+$ together with the obvious restriction maps forms a sheaf.

Solution. $\mathcal{F}^+U$ is an abelian group with pointwise addition, as the sum of $s, t \in F^+U$ still satisfies $(\star)$ by taking the intersection of the V_p obtained from s and t.

It is obvious that $\mathcal{F}^+$ is a presheaf.

Next, let $s \in \mathcal{F}^+U$ and $\{U_i\}$ an open cover such that $\forall i, s|_{U_i} = 0$. Let $p \in U$. Then $p \in U_i$ for some i and we have $s(p) = (s|_{U_i})(p) = 0$, so $s = 0$, so the identity axiom is satisfied.

Next, let $\{U_i\}_{i \in I}$ be a cover, $s_i \in \mathcal{F}^+U_i$ such that $\forall i, j \colon s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$. Given $p \in U$, define $s(p) \coloneqq s_i(p)$ for $p \in U_i$. This is well-defined because of the compatibility condition. We need to show that $s \in \mathcal{F}^+U$. Indeed, let $p \in U$. Then $s(p) = s_i(p)$ for some i, and since $s_i \in \mathcal{F}^+U_i$ and taking stalks is compatible with restrictions, we get a neighborhood that satisfies the required condition. It remains to show that for all i, $s|_{U_i} = s_i$, but that is true by definition. □

Definition. For a presheaf $\mathcal{F}$ and an open set U, define

$$\theta_U \colon \mathcal{F}U \to \mathcal{F}^+U; \qquad s \mapsto (p \mapsto s_p).$$

This is obviously a homomorphism of groups. It also defines a morphism of shaves, because for $s \in \mathcal{F}U$, $V \subseteq U$ and $p \in V$ we have

$$\theta_U(s)|_V(p) = \theta_U(s)(p) = s_p = (s|_V)_p = \theta_V(s|V)(p).$$

Lemma. Let $\mathcal{F}$ be a sheaf and U an open set. Then the natural map

$$\mathcal{F}U \to \prod_{p \in U} \mathcal{F}_p$$

is injective.

Proof. Let $s, t \in \mathcal{F}U$ such that $s_p = t_p$ for every p. Let $p \in U$. By definition of a stalk, $s_p = t_p$ means that there is an open $p \in V_p \subseteq U$ such that $s|_{V_p} = t|_{V_p}$. These V_p cover U so by the identity axiom we have $s = t$. □

Lemma. Let $\mathcal{F}$ be a sheaf. Let U be an open set. Let $s \colon U \to \coprod_{p \in U} \mathcal{F}_p$ such that for every $p \in U$ we have $s(p) \in \mathcal{F}_p$ and there is an open $p \in V_p \subseteq U$ together with $s_{V_p} \in \mathcal{F}V_p$ such that for every $q \in V_p$ we have $(s_{V_p})_q = s(q)$. Then there is a unique $t \in \mathcal{F}U$ such that $t_q = s(q)$ for every $q \in U$.

Proof. Uniqueness follows from the previous lemma. For existence, notice that the V_p cover U. Let $p, q \in U$. The s_{V_p} are gluable because their stalks agree on the intersection, so the conditions of the gluing axiom are satisfied by the previous lemma. Since talking stalks is compatible with restrictions, the glued section has the correct stalks. □

Exercise. Let $\mathcal{F}$ be a presheaf, $\mathcal{G}$ a sheaf and $\varphi \colon \mathcal{F} \to \mathcal{G}$ a morphism of presheaves. Then there is a unique morphism of sheaves $\varphi^+ \colon \mathcal{F}^+ \to \mathcal{G}$ such that $\varphi = \varphi^+ \circ \theta$.

SOLUTION. Let U be an open and let $s \in \mathcal{F}^+U$. Cover U with the V_p from the definition of $\mathcal{F}^+$ and obtain the associated $s_{V_p} \in \mathcal{F}V_p$. Define $t_{V_p} \coloneqq \varphi_{V_p}(s_{V_p}) \in \mathcal{G}V_p$. We can calulate that for $q \in V_p$ we have

$$(t_{V_p})_q = (\varphi_{V_p}(s_{V_p}))_q = \varphi_q((s_{V_p})_q) = \varphi_q(s(q)).$$

Therefore, Lemma 2 gives us a unique $t_U \in \mathcal{G}_U$ such that

$$(\star) \qquad \forall q \in U\colon (t_U)_q = \varphi_q(s(q)).$$

We define $\varphi^+_U(s) = t_U$.

This is indeed a morphism of sheaves: if $V \subseteq U$ and $s \in \mathcal{F}^+U$, then

$$\varphi^+(s|_V) = \varphi^+(s)|_V$$

follows from the fact that, using $(\star)$, the germ of both sides at $p \in V$ is just $\varphi_p(s(p))$. By Lemma 1, the two sides are equal.

Similarly, if $s \in \mathcal{F}U$ and $p \in U$, then

$$(\varphi^+_U \theta_U(s))_p \overset{(\star)}{=} \varphi_q(\theta(s)(q)) = \varphi_q(s_q) = (\varphi_U(s))_q,$$

so $\varphi^+_U \circ \theta_U = \varphi_U$ by Lemma 1, so $\varphi^+ \circ \theta = \varphi$.

Finally, to see uniqueness, assume that $\varphi^\#$ satisfies $\varphi^\# \circ \theta = \varphi$. Let $s \in \mathcal{F}^+U$ and $p \in U$. By definition of $\mathcal{F}^+$ there is $p \in V_p \subseteq U$, $s_{V_p} \in \mathcal{F}V_p$ such that $\forall q \in V_p\colon (s_{V_p})_q = s(q)$. The condition can be reprased as $s|_{V_p} = \theta(s_{V_p})$ and we calculate

$$\begin{aligned}(\varphi^\#_U(s))_p &= (\varphi^\#_U(s)|_{V_p})_p = (\varphi^\#_{V_p}(s|_{V_p}))_p = (\varphi^\#_{V_p}(\theta(s_{V_p})))_p \\ &= (\varphi^+_{V_p}(\theta(s_{V_p})))_p = \ldots = (\varphi^+_U(s))_p,\end{aligned}$$

so by Lemma 1, we have $\varphi^+_U = \varphi^\#_U$, so $\varphi^+ = \varphi^\#$, completing the proof of uniqueness. $\square$

EXERCISE. We have $(\mathcal{F}^+)_p = \mathcal{F}_p$ for $p \in X$. Show that if $f\colon \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves, then there is an induced morphism $f^+\colon \mathcal{F}^+ \to \mathcal{G}^+$ with $(f^+)_p = f_p$.

SOLUTION. Let $p \in X$. Of course, $(\mathcal{F}^+)_p$ and $\mathcal{F}_p$ cannot be literally equal. Instead, we show the following more precise statement: The map $\theta_p\colon \mathcal{F}_p \to \mathcal{F}^+_p$ is an isomorphism.

Indeed, we define $g_p\colon \mathcal{F}^+_p \to \mathcal{F}_p$ as follows: for an open U and $s \in \mathcal{F}^+U$ we define $g_p(s_p) \coloneqq s(p)$. This is well-defined because sections $s \in \mathcal{F}^+U$, $t \in \mathcal{F}^+V$ that have the same germ at p must satisfy $s|_W = t|_W$ for some W that contains p, so $s(p) = s|_W(p) = t|_W(p) = t(p)$.

Next, let U be an open and $s \in \mathcal{F}^+_p$. By definition of $\mathcal{F}^+$, there is some $p \in V_p \subseteq U$ open, $s_{V_p} \in \mathcal{F}V_p$ such that for all $q \in V_p$ we have $(s_{V_p})_q = s(q)$. This is equivalent to saying that $s|_{V_p} = \theta_{V_p}(s_{V_p})$, so in particular, in $\mathcal{F}^+_p$, we have $s_p = (\theta_{V_p}(s_{V_p}))_p$. This lets us calculate

$$\theta_p(g_p(s_p)) = \theta_p(s(p)) = \theta_p((s_{V_p})_p) = (\theta_{V_p}(s_{V_p}))_p = s_p,$$

so we have $\theta_p \circ g_p = \mathrm{id}_{\mathcal{F}^+_p}$.

Next, let U be an open and $s \in \mathcal{F}U$. Then we have

$$g_p(\theta_p(s_p)) = g_p(\theta_U(s)_p) = g_p((q \mapsto s_q)_p) = (q \mapsto s_q)(p) = s_p,$$

so $g_p \circ \theta_p = \mathrm{id}_{\mathcal{F}_p}$, and θ_p is an isomorphism as required.

Next, let $\mathcal{F}$ and $\mathcal{G}$ be presheaves and let $\theta\colon \mathcal{F} \to \mathcal{F}^+$ and $\iota\colon \mathcal{G} \to \mathcal{G}^+$ denote the natural maps to the associated sheaf. If $f\colon \mathcal{F} \to \mathcal{G}$ is a map of presheaves, we can invoke the universal property of $\mathcal{F}^+$ on the composite $\iota \circ f$ and find a morphism $f^+ \colon \mathcal{F}^+ \to \mathcal{G}^+$ making the diagram

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\theta"] \arrow[d, "f"] & \mathcal{F}^+ \arrow[d, dotted, "f^+"] \\
\mathcal{G} \arrow[r, "\iota"] & \mathcal{G}^+
\end{tikzcd}$$

commute.

On stalks, we have

$$f_p^+ \circ \theta_p = (f^+ \circ \theta)_p = (\iota \circ f)_p = \iota_p \circ f_p,$$

and since θ_p is an isomorphism, we have

$$f_p^+ = \iota_p \circ f_p \circ \theta_p^{-1},$$

which is how we should interpret the "equality" $(f^+)_p = f_p$ under the natural identifications θ_p and ι_p. $\square$

Exercise 5.

Exercise. Show that if $f \colon \mathcal{F} \to \mathcal{G}$ is a morphism between sheaves, then the sheaf image $\operatorname{im} f$ can be naturally identified with a subsheaf of $\mathcal{G}$.

Solution. We will prove the following more general statement: if $\mathcal{F}$ is a presheaf satisfying the identity axiom, $\mathcal{G}$ is a sheaf and $f \colon \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves such that f_U is injective for every U, then the induced morphism $f_U^+ \colon \mathcal{F}^+ U \to \mathcal{G} U$ is injective for every U.

Indeed, the inclusion of the presheaf image into $\mathcal{G}$ satisfies these conditions. It satisfies sheaf axiom 1 for the same reason that the presheaf kernel does.

We will now prove the claim. Let U be an open set, $s \in \mathcal{F}^+ U$ such that $f_U^+(s) = 0$. From the construction of the associated sheaf we see that $f_U^+(s) = f_U(t)$ where t is the unique element of $\mathcal{F} U$ such that $\forall q \in U \colon t_q = f_q(s(q))$.

So we have $0 = f_U^+(s) = f_U(t)$, so since f_U is injective we have $t = 0$. Let $q \in U$. Then $f_q(s(q)) = t_q = 0_q = 0$. The element $s(q)$ of $\mathcal{F}_q$ is represented by some open set V and a section $u \in \mathcal{F} V$. Thus $0 = f_q(s(q)) = f_q(V, u) = (V, f_V(u))$. Thus, there is some open $W \subseteq V$ such that $0 = f_V(u)|_W = f_W(u|_W)$. Since f_W is injective, we conclude $u|_W = 0$, and $u|_W$ represents the same element in $\mathcal{F}_q$ as u, but that element is just $s(q)$, so $s(q) = 0$. Since q was arbitrary, we conclude $s = 0$. $\square$

Exercise 6.

Exercise. A sequence of sheaves is exact if and only if for every $p \in X$ the corresponding sequence of maps of abelian groups is exact.

Solution. Assume that $f \colon \mathcal{F} \to \mathcal{G}$ and $g \colon \mathcal{G} \to \mathcal{H}$ are morphisms of sheaves such that $g \circ f = 0$. Consider the diagram

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "f"] \arrow[d] & \mathcal{G} \arrow[r, "g"] & \mathcal{H} \\
\operatorname{im}' f \arrow[ur] \arrow[r, "\theta"] \arrow[rr, bend right, "\iota"'] & \operatorname{im} f \arrow[u] \arrow[r, "\varphi"] & \ker g, \arrow[ul]
\end{tikzcd}$$

where the map ι is an inclusion of subpresheaves of $\mathcal{G}$ and φ is induced by ι. We say that $\operatorname{im} f = \ker g$ if φ is an isomorphism. By a result of the lecture, this is the case if and only if forall $p \in X$, the induced map $\varphi_p\colon (\operatorname{im} f)_p \to (\ker g)_p$ is an isomorphism. Since θ induces isomorphisms on stalks and the bottom triangle commutes, this is the case if and only if $\iota_p\colon (\operatorname{im}' f)_p \to (\ker g)_p$ is an isomorphism for every $p \in X$. Now consider the diagram

$$\begin{tikzcd} (\operatorname{im}' f)_p \arrow[r, "\iota_p"] \arrow[d, "\cong"] & (\ker g)_p \arrow[d, "\cong"] \\ \operatorname{im} f_p \arrow[r, "i"] & \ker g_p, \end{tikzcd}$$

where the left and right maps are the isomorphisms defined in the proof of a result from the lecture and the bottom map is just the inclusion (this makes sense since $g \circ f = 0 \iff \forall p \in X\colon g_p \circ f_p = 0$ as stalks characterize morphisms). The diagram commutes since none of the maps actually does anything. Since the left and right maps are isomorphisms, we have that the top map is an isomorphism if and only if the bottom map is an isomophism.

But the bottom map is an isomorphism if and only if the sequence

$$\mathcal{F}_p \xrightarrow{f_p} \mathcal{G}_p \xrightarrow{g_p} \mathcal{H}_p$$

is exact, so putting everything together, we find that (f, g) is exact if and only if (f_p, g_p) is exact for every $p \in X$. $\square$

Exercise 7.

Exercise. Show that a morphism of sheaves is an isomorphism if and only if it is injective and surjective.

Solution. Let $f\colon \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. By a result from the lecture, f is an isomorphism if and only if $f_p\colon \mathcal{F}_p \to \mathcal{G}_p$ is an isomorphism for every p. Since f_p is a morphism of abelian groups, this is the case if and only if f_p is injective and surjective for every p. By another result from the lecture, this is the case if and only if f is injective and surjective. $\square$

Exercise 8.

Exercise. Let $\mathcal{F}'$ be a subsheaf of a sheaf $\mathcal{F}$. Then the natural map $\mathcal{F} \to \mathcal{F}/\mathcal{F}'$ is surjective and has kernel $\mathcal{F}'$ so that there is an exact sequence

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}/\mathcal{F}' \longrightarrow 0.$$

Solution. The natural map $e\colon \mathcal{F} \to \mathcal{F}/\mathcal{F}'$ is given as the composite $\theta \circ \hat{e}$, where $\hat{e}$ is the map $\mathcal{F} \to \operatorname{coker}' i$, where $i\colon \mathcal{F}' \to \mathcal{F}$ is the inclusion and $\operatorname{coker}' i$ is the presheaf cokernel of i, and θ is the natural map into the sheafification.

For every $p \in X$, θ_p is surjective because θ induces isomorphisms on stalks, and $\hat{e}_p$ is surjective, because $\hat{e}$ is surjective on open sets, which in particular implies surjectivity on stalks. Hence e_p is surjective as the composite of two open maps. By a result from the lecture, this implies that e is surjective.

Since for any open set U and $s \in \mathcal{F}'U$ we have $e_U(s) = \theta_U(\hat{e}_U(s)) = \theta_U(0) = 0$, we obtain a map $\varphi\colon \mathcal{F}' \to \ker e$. Let $p \in X$.

$$\begin{tikzcd} \mathcal{F}'_p \arrow[r, "i_p"] \arrow[d, "\varphi_p"] & \mathcal{F}_p \arrow[r, "e_p"] \arrow[drr, "\hat{e}_p"] & (\mathcal{F}/\mathcal{F}')_p & \\ (\ker e)_p \arrow[r, "\cong"] \arrow[ur] & \ker e_p \arrow[u] & & (\operatorname{coker}' i)_p \arrow[ul, "\theta_p"'] \end{tikzcd}$$

The map φ_p is injective because i_p is, and it is surjective, because $\ker e_p = \ker \hat{e}_p = i_p$. Hence $\ker e = \mathcal{F}'$ as subsheaves of $\mathcal{F}$.

Now we have $\operatorname{im} i = \mathcal{F}'$ as subsheaves of $\mathcal{F}$, since the map $\theta\colon \mathcal{F}' = \operatorname{im}' i \to \operatorname{im} i$ induces isomorphisms on stalks, but since the domain already is a sheaf this forces θ to be an isomorphism. Hence $\operatorname{im} i = \ker e$ as subsheaves of $\mathcal{F}$, so the sequence is exact at $\mathcal{F}$. Exactness at $\mathcal{F}'$ and $\mathcal{F}/\mathcal{F}'$ is trivially checked on stalks using Exercise 6. $\square$

Exercise. If

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

is an exact sequence, then $\mathcal{F}'$ is isomorphic to a subsheaf of $\mathcal{F}$ and $\mathcal{F}''$ is isomorphic to the quotient of $\mathcal{F}$ by this subsheaf.

Solution. We have a commutative diagram

$$\begin{tikzcd}
\mathcal{F}' \arrow[r, "i"] \arrow[d, "\hat{\imath}"] & \mathcal{F} \\
\operatorname{im}' i \arrow[r, "\theta"] \arrow[ur] & \operatorname{im} i, \arrow[u, "\iota"']
\end{tikzcd}$$

where the diagonal arrow is the inclusion, the bottom arrow is the natural map into the associated sheaf, and the right arrow is induced by the diagonal arrow. By Exercise 5, $\operatorname{im} i$ can be regarded as a subsheaf of $\mathcal{F}$. Since i is injective, for every $p \in X$, i_p is injective, so by commutativity, $\hat{\imath}_p$ is injective. Furthermore, for every $p \in X$, $\hat{\imath}$ is surjective, because it is surjective on open sets. Hence, the composite $\theta \circ \hat{\imath}$ is an isomorphism on stalks. Since it is a map between sheaves, this means that is is an isomorphism. Hence, $\mathcal{F}$ is isomorphic to the subsheaf $\operatorname{im} i$.

Next, consider the diagram

$$\begin{tikzcd}
 & \operatorname{coker}' \iota \arrow[r, "\eta"] \arrow[dr, "\hat{p}"] & \operatorname{coker} \iota \arrow[d, "\hat{p}^+"] \\
\operatorname{im} i \arrow[r, "\iota"] & \mathcal{F} \arrow[r, "p"] \arrow[u, "\pi"] & \mathcal{F}'',
\end{tikzcd}$$

where the map $\hat{p}$ is defined on open sets using the fact that $p \circ \iota = 0$, hence $p_U \circ \iota_U = 0$, hence $(\operatorname{im} i)(U) \subseteq (\ker p)(U)$. The map, η is the natural map into the associated sheaf and $\hat{p}^+$ is obtained from the universal property. Since p is surjective, it is surjective on stalks, hence by commutatity $\hat{p}^+$ must also be surjective on stalks.

I do not have a proof that $\hat{p}^+$ is injective on stalks. $\square$

Exercise 9.

Exercise. If $U \subseteq X$ is an open subset and

$$0 \longrightarrow \mathcal{F}' \xrightarrow{\ i\ } \mathcal{F} \xrightarrow{\ p\ } \mathcal{F}''$$

is exact, then

$$0 \longrightarrow \Gamma(U, \mathcal{F}') \xrightarrow{\ i_U\ } \Gamma(U, \mathcal{F}) \xrightarrow{\ p_U\ } \Gamma(U, \mathcal{F}'')$$

is exact.

Solution. Since $(0, i)$ is exact, $(0, i_x)$ is exact for every $x \in U$, hence i_x is injective for every $x \in U$, hence i is injective, hence $\ker i = 0$, hence $0 = (\ker i)(U) = \ker i_U$, hence i_U is injective, so the sequence is exact at $\Gamma(U, \mathcal{F}')$.

It remains to show exactness at $\Gamma(U, \mathcal{F})$. Since $p_U \circ i_U = (p \circ i)_U = 0_U = 0$, we have $\operatorname{im} i_U \subseteq \ker i_U$.

Conversely, let $s \in \ker p_U$, i.e., $p_U(s) = 0$. By Exercise 6 we know that (i_x, p_x) is exact for every $x \in U$. Since $p_U(s) = 0$, we have $p_x(U, s) = 0$ for every $x \in U$, hence $(U, s) \in \operatorname{im} i_x$ for all $x \in U$, i.e., we find $(V_x, t_x) \in \mathcal{F}'_x$ such that $i_x(V_x, t_x) = (U, s)$. If necessary, shrink V_x such that $i_{V_x}(t_x) = s|_{V_x}$.

For $x, y \in U$, we have

$$i_{V_x \cap V_y}(t_x|_{V_x \cap V_y} - t_y|_{V_x \cap V_y}) = s|_{V_x \cap V_y} - s|_{V_x \cap V_y} = 0.$$

Since i is injective, we conclude $t_x|_{V_x \cap V_y} = t_y|_{V_x \cap V_y}$, hence we can glue the t_x to a $t \in \mathcal{F}'U$. For any $x \in U$ we have

$$\mathcal{F}_x \ni (U, i_U(t)) = (V_x, i_{V_x}(t|_{V_x})) = (V_x, i_{V_x}(t_x)) = (V_x, s|_{V_x}) = (U, s),$$

and since stalks characterize sections, this implies that $i_U(t) = s$, hence $s \in \operatorname{im} i_U$ as required. □

Example sheet 2.

Exercise 7.

Exercise. Let X be an integral scheme. There is a unique point η such that the closure of $\{\eta\}$ is X; this is called the generlic point of X. The stalk of $\mathcal{O}_X$ at η is a field, called the function field of X, denote by $K(X)$. More precisely, if $U = \operatorname{Spec} A$ is any open affine subset of X, then $K(X)$ is a field of fractions of A.

Solution. If $U \cong \operatorname{Spec} A$ is an open affine subset of X, then $A \cong \mathcal{O}_X(U)$ is an integral domain. Hence we have a point $\eta_U \in X$ corresponding to the prime ideal (0).

We will show:

(i) η_U is the unique point of U that is contained in every nonempty open subset of U.
(ii) If U is an open affine subset of X, and V is an open affine subset of U, then $\eta_U = \eta_V$.
(iii) If U, V are open affine subsets of X, then $\eta_U = \eta_V$. Thus we may define $\eta \coloneqq \eta_U$, where U is any open affine subset of X.
(iv) The closure of $\{\eta\}$ is X and η is the unique point with this property.
(v) If $U \cong \operatorname{Spec} A$ is an open affine subset of X, then the stalk of η is a field of fractions of A.

For (i), notice that a point $\mathfrak{p} \in \operatorname{Spec} A$ is contained in every nonempty open subset of $\operatorname{Spec} A$ if and only if $\mathfrak{p} \in V(I)$ implies $V(I) = \operatorname{Spec} A$. Since A is an integral domain, then $V(I) = \operatorname{Spec} A$ is equivalent to $I = (0)$. Clearly, $\mathfrak{p} \in V(\mathfrak{p})$ for any $\mathfrak{p} \in \operatorname{Spec} A$, so $\mathfrak{p}$ is contained in every nonempty open subset of $\operatorname{Spec} A$ if and only if $\mathfrak{p} = (0)$.

For (ii), we observe that every open subset of V is also an open subset of U, hence η_U is a point contained in every nonempty open subset of V, so by uniqueness, $\eta_U = \eta_V$.

For (iii), notice that since X is integral, it is irreducible, so $U \cap V \neq \varnothing$. Combining Exercises 1 and 11 from the first example sheet, we know that the affine open subsets of X form a base for the topology of X. Hence we find an affine open subset $W \subseteq U \cap V$. Hence, by (i), we have $\eta_U = \eta_W = \eta_V$.

For (iv), we remark that the closure of $\{\eta\}$ being X is the same as saying that η is contained in every nonempty open subset of X. But since the open affine subsets form a base, it suffices to know that η is contained in every affine open subsets, which is clear from what we have shown previously. Uniqueness follows by applying the uniqueness statement from (i) for any open affine U.

For (v), note that $\mathcal{O}_{X,\eta} = \mathcal{O}_{X,\eta_U} \cong A_{(0)} = (A \setminus \{0\})^{-1}A$, which is precisely the field of fractions of A. □

Examples classes

Examples class 1

Exercise 12.

Exercise.

Solution. Want to construct an inverse β to α. Define for $\varphi\colon A \to \Gamma(X, \mathcal{O}_X)$. Define $\beta(\varphi)$ as follows. Cover X with affine schemes $\{U_i\}$, $U_i = \operatorname{Spec} B_i$. We have restriction maps $\Gamma(X, \mathcal{O}_X) \to \Gamma(U_i, \mathcal{O}_X) = \Gamma(U_i, \mathcal{O}_{\operatorname{Spec} B_i}) = B_i$.

This igves by composition with φ maps $\varphi_i\colon A \to B_i$. This induces a morphism $f_i\colon U_i = \operatorname{Spec} B_i \to \operatorname{Spec} A$. We want to show that we can glue the f_i, by first showing that they agree on $U_i \cap U_j$.

Using Exercises 1 and 11, we may cover $U_i \cap U_j$ with affine schemes $\{U_{ijk}\}$, where $U_{ijk} = \operatorname{Spec} B_{ijk}$. Then we have a commutative diagram

$$\begin{tikzcd} \Gamma(X, \mathcal{O}_X) \arrow[r] \arrow[d] & B_j \arrow[d] \\ B_i \arrow[r] & B_{ijk} \end{tikzcd}$$

of restriction maps for $\mathcal{O}_X$.

Thus the compositions

$$U_{ijk} \hookrightarrow U_i \xrightarrow{f_i} \operatorname{Spec} A$$

$$U_{ijk} \hookrightarrow U_j \xrightarrow{f_j} \operatorname{Spec} A$$

agree. Thus $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$. We can now glue these morphisms to get a morphism $f\colon X \to \operatorname{Spec} A$.

- Obtaining f as a continuous map is no problem.
- We need to construct $f^\#$.

Given $V \subseteq \operatorname{Spec} A$, we need a map $f_V^\#\colon \Gamma(V, \mathcal{O}_{\operatorname{Spec} A} \to \Gamma(f^{-1}(V), \mathcal{O}_X)$.

Note $f^{-1}(V)$ is covered by the sets $f_i^{-1}(V) = f^{-1}(V) \cap U_i$. So we have for $s \in \Gamma(V, \mathcal{O}_{\operatorname{Spec} A})$, $f_i^\#(s) \in \Gamma(f_i^{-1}(v), \mathcal{O}_X)$ and since $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ we have

$$f_i^\#(s)|_{f_i^{-1}(V) \cap f_j^{-1}(V)} = f_j^\#(s)|_{f_i^{-1}(V) \cap f_j^{-1}(V)}.$$

By the sheaf gluing axiom, we obtain $f^\#(s) \in \Gamma(f^{-1}(V), \mathcal{O}_X)$.

Note: This is a general fact: given $\{U_i\}$ a cover of X and $f_i\colon U_i \to Y$ morphisms such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$, then we obtain a glued morphism.

This gives $\beta\colon \operatorname{Hom}_{\mathrm{Ring}}(A, \Gamma(X, \mathcal{O})X)) \to \operatorname{Hom}_{\mathrm{Sch}}(X, \operatorname{Spec} A)$. Need to check:

- $\alpha \circ \beta$ is the identity: given φ, $f = \beta(\varphi)$ is contructed such that the composition

$$A \xrightarrow{f^\#} \Gamma(X, \mathcal{O}_X) \longrightarrow \Gamma(U_i, \mathcal{O}_X)$$

coincides with

$$A \xrightarrow{\varphi} \Gamma(X, \mathcal{O}_X) \longrightarrow \Gamma(U_i, \mathcal{O}_X),$$

so by the first sheaf axiom we must have $f^\# = \varphi = \alpha(f)$.

- $\beta \circ \alpha$ is the identity: given a morphism $f\colon X \to \operatorname{Spec} A$, this induces by restriction to U_i a morphism $U_i \to \operatorname{Spec} A$; necessarily induced by the composition

$$A \xrightarrow{f^\#} \Gamma(X, \mathcal{O}_X) \longrightarrow \Gamma(U_i, \mathcal{O}_X) = B_i.$$

SO f is induced by $\alpha(f)$ on open sets U_i. So $f|_{U_i} = \beta(\alpha(f))|_{U_i}$. Thus $= \beta(\alpha(f))$.

Note: not every details has been checked here, for example that $f^\#$ constructed above is indeed a morphism of locally ringed spaces (the reason is because locally, we already started with a morphism). Checking details is essential to understanding what is important and what isn't. During marking, not checking unimportant/easy things is usually not a problem, forgetting to check important things is.

$\square$

Exercise 11.

Exercise. If $f \in A$, then $(D(f), \mathcal{O}_{\operatorname{Spec} A}|_{D(f)}) \cong \operatorname{Spec} A_f$.

Solution. Note that $D(f)$ is the set of all prime ideals of A not containing f. There is a one-to-one correspondence between primes of A disjoint from S and primes of $S^{-1}A$ for S any multiplicatively closed subset.

Thus time primes of A_f are in one-to-one correspondence with primes of A disjoint from $\{1, f, f^2, \ldots\}$, i.e., primes in $D(f)$. This gives a bijection $\operatorname{Spec} A_f \to D(f)$ and the composition with the inclusion $D(f) \to \operatorname{Spec} A$ is induced by the localization map $\varphi\colon A \to A_f$.

Hence, the induced map $\operatorname{Spec} A_f \to \operatorname{Spec} A$ is continuous, so $\operatorname{Spec} A_f \to D(f)$ is continuous. Note that if $I \subseteq A_f$ is an ideal, then $i(V(I)) = V(\varphi^{-1}(I)) \cap D(f)$, so i is a homeomorphism.

To check that i is an isomorphism of schemes, we need to check that we get an isomorphism $i^\#\colon \mathcal{O}_{\operatorname{Spec} A}|_{D(f)} \to \mathcal{O}_{\operatorname{Spec} A_f}$. Note that it is enough to check that this is an isomorphism on stalks. Note that the stalk of $\mathcal{O}_{\operatorname{Spec} A}$ at $p \in D(f)$ is A_p and the stalk of $\mathcal{O}_{\operatorname{Spec} A_f}$ at pA_f is $(A_f)_{pA_f \xleftarrow{\cong}}$, by sending $a/s \mapsto a/s$. Since $f \notin p$, one can check algebraically that this is an isomorphism. $\square$

Exercise 10.

Exercise.

Solution. Given a continuous map $f\colon X \to Y$ and a sheaf $\mathcal{F}$ over X and a sheaf $\mathcal{G}$ over Y.

We will first give a morphism morphism $\Psi\colon f^{-1}f_*\mathcal{F} \to \mathcal{F}$. $s \in (f^{-1}f_*\mathcal{F})(U) = \{(V, s') \mid V \supseteq f(U)\}/\sim$, where $s' \in (f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$, i.e., $s = (V, s)$, where $V \supseteq f(U)$, $s \in \mathcal{F}(f^{-1}(V))$, but $f^{-1}(V) \supseteq U$. Then the natural Ψ map takes $(V, s) \mapsto s|_U$.

Next, we give a morphism $\Phi\colon \mathcal{G} \to f_*f^{-1}(\mathcal{G}$. Given $s \in \mathcal{G}(U)$, we have $f_*f^{-1}(\mathcal{G})(U) = (f^{-1}(\mathcal{G}))(f^{-1}(U))$. The natural map Φ sends $s \mapsto (U, s)$, which makes sense since $U \supseteq f(f^{-1}(U))$.

We want to construct a bijection

$$F\colon \operatorname{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}) \to \operatorname{Hom}_Y(\mathcal{G}, f_*\mathcal{F}).$$

Given a map $\varphi\colon f^{-1}\mathcal{G} \to \mathcal{F}$, apply f_* to get $f_*\varphi\colon f_*f^{-1}\mathcal{G} \to f_*\mathcal{F}$ and then $(f_*\varphi)\circ\Phi\colon \mathcal{G} \to f_*\mathcal{F}$, se define $F(\varphi) = (f_*\varphi)\circ\Phi$.

Similarly, we define

$$G\colon \operatorname{Hom}_Y(\mathcal{G}, f_*\mathcal{F}) \to \operatorname{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}).$$

Given $\varphi\colon \mathcal{G} \to f_*\mathcal{F}$, apply f^{-1} to get $f^{-1}\varphi\colon f^{-1}\mathcal{G} \to f^{-1}f_*\mathcal{F}$, so we can define $G(\varphi) \coloneqq \Psi\circ f^{-1}\varphi$.

Now we need to check that $F\circ G$ and $G\circ F$ are identities.

Given $\psi\colon \mathcal{G} \to f_*\mathcal{F}$, the map $FG(\psi)$ is given as a composition

$$\begin{array}{ccccccc} \mathcal{G} & \xrightarrow{\Phi} & f_*f^{-1}\mathcal{G} & \longrightarrow & f_*f^{-1}f_*\mathcal{F} & \longrightarrow & f_*\mathcal{F} \\ \\ s & \longmapsto & (U,s) & \longmapsto & (U,\psi_U(s)) & \longmapsto & \psi_U(s), \end{array}$$

so FG is the identity.

Similarly, given $\varphi\colon f^{-1}\mathcal{G} \to \mathcal{F}$ is given as a composition

$$f^{-1}\mathcal{G} \longrightarrow f^{-1}f_*f^{-1}\mathcal{G}, f^{-1}f_*\mathcal{F} \xrightarrow{\Psi} \mathcal{F}$$

A section of $(f^{-1}\mathcal{G})(U)$ is represented by (V,s) for $s \in \mathcal{G}(V)$ with $V \supseteq f(U)$.

The sequence of maps is given as

$$(V,s) \mapsto (V,s) \mapsto (V,\varphi(s)) \mapsto \varphi(s),$$

so GF is the identity.

Note that we are working with presheaves here, so we have to splice in the universal property of sheafification everywhere. $\square$

Exercise 14.

Exercise. Given a collection $\{X_i\}$ of schemes and open sets $U_{ij} \subseteq X_i$ together with isomorphisms $\varphi_{ij}\colon U_{ij} \to U_{ji}$ such that

(1) $\varphi_{ij} = \varphi_{ji}^{-1}$,
(2) for all i,j,k we have $\varphi_{ij}(U_{ij}\cap U_{ik}) = U_{ji}\cap U_{jk}$ and $\varphi_{ik} = \varphi_{jk}\circ\varphi_{ij}$ on $U_{ij}\cap U_{ik}$.

Solution. We can glue the X_i as topological spaces to get a space X and open subsets $X_i \subseteq X$ by usual gluing of topological spaces. We have sheaves $\mathcal{O}_{X_i}$ on X_i along with isomorphisms

$$\varphi_{ij}^{\#}\colon \mathcal{O}_{X_j}|_{X_i\cap X_j} \to \mathcal{O}_{X_i}|_{X_i\cap X_j},$$

and for each i,j,k, we have $\varphi_{ik}^{\#} = \varphi_{jk}^{\#}\circ\varphi_{ij}^{\$}$ on $X_i\cap X_j\cap X_k$.

Let's glue these sheaves. Define $\mathcal{O}_X$ be defining $\mathcal{O}_X(U)$ to be the set of tuples $(s_i)_{i\in I}$ with $s_i \in \mathcal{O}_{X_i}(U\cap X_i)$ subject to the contraint that on $X_i\cap X_j\cap U$, $\varphi_{ij}^{\#}(s_j) = s_i$.

One now checks that this is a sheaf of rings, i.e, we inherit a ring structure, the sheaf axioms are satisfied. This follows from the sheaf axioms for the $\mathcal{O}_{X_i}$.

Finally, we need to check that this is a scheme. For this it is enough to show that $\mathcal{O}_X|_{X_i} \cong \mathcal{O}_{X_i}$. This isomorphism is given by sending $s \in \mathcal{O}_{X_i}(U)$ to $(\varphi_{ij}^{\#}(s|_{U\cap X_j}))_{j\in I}$. To see that this is a section of $\mathcal{O}_X$, we use the compatibility condition for $\varphi_{ij}^{\#}$ from above. $\square$