3강. Determinant

1. Introduction to Determinants

1) Determinants

(1) Definition : minor, cofactor

A : $n \times n$ matrix

minor of entry a_{ij} : determinant of submatrix that remains when ith row and jth columns of A are deleted - M_{ij}

cofactor of entry a_{ij} : $C_{ij} = (-1)^{i+j} M_{ij}$

ex) $A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{bmatrix}$

(2) Determinant of A

The determinant of an $n \times n$ matrix A can be computed by multiplying the entries in any row (or column) by their cofactors and adding the resulting products

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

ex) $A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{bmatrix}$ $\qquad B = \begin{bmatrix} 2 & 0 & 0 & 5 \\ -1 & 2 & 4 & 1 \\ 3 & 0 & 0 & 3 \\ 8 & 6 & 0 & 0 \end{bmatrix}$

(3) Theorem : determinant of triangular matrix

If A is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of A

2. Properties of Determinants

1) Effect of elementary row operation

Let A be $n \times n$ matrix

① If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$

② If two rows of A are interchanged to produce B, then $\det B = -\det A$

③ If one row of A is multiplied by k to produce B, then $\det B = k\det A$

E is $n \times n$ elementary matrix, then

ex) $A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$

2) Theorems of Determinant and Invertible matrix

(1) A square matrix A is invertible if and only if $\det A \neq 0$

(2) If A has two identical rows or columns, then $\det A = 0$

(3) If A has two identical rows or columns, then $\det A = 0$

(4) $\det(kA) = k^n \det(A)$

(5) $\det(AB) = \det A \det B$

(6) If A is a square matrix, $\det A = \det A^T$

(7) If A is invertible, then $\det A^{-1} = \dfrac{1}{\det A}$

3) Unifying Theorems

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix
b. A is row equivalent to the $n \times n$ identity matrix
c. A has n pivot positions
d. The equation $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution
e. The columns of A form a linearly independent set
f. The columns of A span R^n
g. There is an $n \times n$ matrix C such that $CA = I$
h. There is an $n \times n$ matrix D such that $AD = I$
j. The equation $A\boldsymbol{x} = \boldsymbol{b}$ has at least one solution for each $\boldsymbol{b}$ in R^n
k. A^T is an invertible matrix.
l. $\det A \neq 0$

3. Cramer's Rule

1) Cramer's Rule

(1) Definition

For any $n \times n$ matrix A and any $\boldsymbol{b}$ in R^{n}, let $A_i(\boldsymbol{b})$ be the matrix obtained from A by replacing column i by the vector $\boldsymbol{b}$

$$A_i(\boldsymbol{b}) = [\, \boldsymbol{a}_1 \ \ \cdots \ \ \boldsymbol{a}_{i-1} \ \ \boldsymbol{b} \ \boldsymbol{a}_{i+1} \ \ \cdots \ \ \boldsymbol{a}_n]$$

(2) Cramer's Rule

Let A be an invertible $n \times n$ matrix. For any $\boldsymbol{b}$ in R^{n}, the unique solution $\boldsymbol{x}$ of $A\boldsymbol{x} = \boldsymbol{b}$ has entries given by

$$x_i = \frac{\det A_i(\boldsymbol{b})}{\det A} \quad i = 1, 2, \cdots, n$$

ex) $\begin{aligned} 3x_1 - 2x_2 &= 6 \\ -5x_1 + 4x_2 &= 8 \end{aligned}$

2) A Formula for A^{-1}

(1) Finding A^{-1}

Let A be an invertible $n \times n$ matrix. Then, the jth column of A^{-1} is a vector $\boldsymbol{x}$ that satisfies

$$A\boldsymbol{x} = \boldsymbol{e}_j$$

By using Cramer's rule,

$\boldsymbol{x}_{ij} = \dfrac{\det A_i(\boldsymbol{b})}{\det A}$ - (i, j) entry of A^{-1}

$$\det A_i(\boldsymbol{e}_j) = (-1)^{j+i} M_{ji} = C_{ji}$$

using this fact, we can get

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \cdots & \cdots & & \cdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

adjugate of A

$$adj\,A = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \cdots & \cdots & & \cdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

Theorem : An inverse formula

Let A be $n \times n$ matrix. Then

$$A^{-1} = \frac{1}{\det A} adj A$$

ex) $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & 2 \end{bmatrix}$