1강 : System of Linear Equation and Linear Independence

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1. System of Linear Equation

1) System of Linear Equation

(1) Definition : Linear Equation

A linear equation in the n variables x_1, x_2, ..., x_n is one that can be expressed in the form

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b$$

where a_1, a_2, ... , a_n and b are constants and the $a's$ are not all zero

ex) $x + 2y = 5$

$2x - y + 3z = 10$

$x_1 + x_2 + \cdots + x_n = 1$

(2) Definition : System of Linear Equation

A collection of one or more linear equations involving the same variables.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots \\ a_{p1}x_1 + a_{p2}x_2 + \cdots + a_{pn}x_n &= b_p \end{aligned}$$

ex)
$$\begin{aligned} 2x_1 - x_2 + 5x_3 &= 1 \\ x_1 \qquad + 3x_3 &= 0 \end{aligned}$$

(3) Solution of system

A list of numbers ($s_1, s_2, \cdots, s_n$) that makes each every equation true

Solution set : The set of all possible solutions.

Two linear systems are equivalent : Two linear systems have the same solution set

ex)
$$\begin{aligned} 2x_1 - x_2 &= 0 \\ x_1 + x_2 &= 3 \end{aligned}$$

2) Solution set and linear system

A system of linear equation has

① No solution
② exactly one solution
③ infinitely many solution

ex)
$$\begin{aligned} x + y &= 1 \\ x + 2y &= 3 \end{aligned} \qquad \begin{aligned} x + 2y &= 1 \\ 2x + 4y &= 3 \end{aligned} \qquad \begin{aligned} x + 2y &= 1 \\ 2x + 4y &= 2 \end{aligned}$$

3) Matrix Notation

A general linear system can be abbreviated by writing only the rectangular array of numbers

$$
\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
&a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
&\cdots \\
&a_{p1}x_1 + a_{p2}x_2 + \cdots + a_{pn}x_n = b_p
\end{aligned}
$$

$$
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\
a_{31} & a_{32} & \cdots & a_{3n} & b_3 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
a_{p1} & a_{p2} & \cdots & a_{pn} & b_n
\end{bmatrix}
$$

- augmented matrix for the system

A general linear system can be abbreviated by writing only the rectangular array of numbers

$$
\begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
a_{31} & a_{32} & \cdots & a_{3n} \\
\cdots & \cdots & \cdots & \cdots \\
a_{p1} & a_{p2} & \cdots & a_{pn}
\end{bmatrix}
$$

- coefficient matrix for the system

ex) $$
\begin{aligned}
2x_1 - x_2 + 5x_3 &= 1 \\
x_1 + 2x_2 + x_3 &= 3 \\
-x_1 - x_2 + x_3 &= 2
\end{aligned}
$$

2. Solving a Linear system

1) Elementary Row Operation

(1) Operations on equations

① Replacement : Add a multiple of one equation to another
② Interchange : Interchange two equations
③ Scaling : Multiple an equation through by a nonzero number

(2) Operations on matrix

① Replacement : Add a multiple of one row to another
② Interchange : Interchange two rows
③ Scaling : Multiple a row through by a nonzero number

ex) $$\begin{aligned} 2x_1 - x_2 + 5x_3 &= 1 \\ x_1 + 2x_2 + x_3 &= 3 \\ -x_1 - x_2 + x_3 &= 2 \end{aligned}$$

Definition : Row Equivalent

Two matrices are row equivalent if there is a sequence of elementary row operations that transforms one matrix into another

Theorem

If augmented matrices of two linear systems are row equivalent, then two systems have the same solution set

2) Row Echelon Form

Definition

A rectangular matrix is in (row) echelon form if it has the following three properties

① All nonzero rows are above any rows of all zeros
② Each leading entry of a row is in a column to the right of the leading entry of the row above it.
③ All entries in a column below the leading entry are zeros

If a matrix in echelon form satisfies the following additional conditions, then it is in reduced echelon form (or reduced row echelon form)

④ The leading entry in each nonzero row is 1.
⑤ Each leading 1 is the only nonzero entry in its column.

*leading entry : The leftmost nonzero entry in a row of a matrix

ex) $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ $\qquad$ $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

Theorem : Uniqueness of the Reduced Echelon form

Each matrix is row equivalent to one and only one reduced echelon matrix
If a matrix A is row equivalent to an echelon matrix U, we call U an (row) echelon form of A
If a matrix U is in reduced echelon form, we call U the reduced echelon form of A

ex) $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix}$ $\qquad$ $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

3) Pivot Position

Definition

A pivot position in a matrix A is the location in A that corresponds to a leading 1 in the reduced echelon form of A

A pivot column is a column of A that contains a pivot positions

ex) $\begin{aligned} 2x_1 - x_2 + 5x_3 &= 1 \\ x_1 + 2x_2 + x_3 &= 3 \\ -x_3 - x_2 + x_3 &= 2 \end{aligned}$

Definition : Free variables

basic variable : the variables corresponding to pivot columns in the matrix

free variable : the variables except the pivot columns in the matrix

ex) $\begin{aligned} 2x_1 - x_2 + 5x_3 &= 1 \\ x_1 + 2x_2 + x_3 &= 3 \\ -x_1 - x_2 + x_3 &= 2 \end{aligned}$

4) Row Reduction Algorithm

Step 1. Locate the leftmost column that does not consist entirely of zero.

Step 2. Interchange the top row with another row, if necessary, to bring a nonzero entry to the top of the column found in Step 1.

Step 3. If the top entry of the column in Step 2 is a, multiply the first row by 1=a to introduce a leading 1.

Step 4. Add suitable multiple of the top row to the rows below so that all entries below the leading 1 become zeros.

Step 5. Now cover the top row and begin again with Step 1 to the submatrix that remains. Continue in this way until the entire matrix is in row echelon form.

Step 6. Beginning with the last nonzero row and working upward, add suitable multiple of each row to the rows above to introduce zeros above the leading 1's.

ex)
$$\begin{aligned} 2x_3 + 4x_4 + 4x_5 &= 0 \\ 2x_1 - 4x_2 - 2x_3 + 2x_4 + 2x_5 &= 0 \\ 2x_1 - 4x_2 \qquad + 9x_4 + 6x_5 &= 0 \\ 3x_1 - 6x_2 \qquad + 9x_4 + 9x_5 &= 0 \end{aligned}$$

4) Solution of Linear System

ex) augmented matrix of system

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

ex) augmented matrix of system

$$\begin{bmatrix} 1 & 3 & -5 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Theorem : Existence and Uniqueness Theorem

A linear system is consistent if and only if the rightmost column of the augmented matrix is not a pivot column.
that is, if and only if an echelon form of the augmented matrix has no row of the form

$[0\ 0\ \ldots\ 0\ b\ ]$

If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least one free variable.

3. Vector Equation

1) Vector and Scalar

Definition

1. Engineers and physicists distinguish between two types of physical quantities-scalars, which are quantities that can be described by a numerical value alone, and vectors, which require both a numerical value and a direction for their complete description.

2. If n is a positive integer, then an ordered n-tuple is a sequence of n real numbers $(v_1, v_2, \cdots, v_n)$. The set of all ordered n-tuples is called n-space and is denoted by R^{n}.

3. We will denote n-tuples using the vector notation $(v_1, v_2, \cdots, v_n)$, and we will write $0 = (0, 0, \cdots, 0)$ for the n-tuple whose components are all zero. We will call this the zero vector or sometimes the origin of R^{n}.

2) Vector notation

$$\boldsymbol{v} = (v_1, v_2, \cdots, v_n)$$

$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ \cdots \\ v_n \end{bmatrix}$$

$$\boldsymbol{v} = [\ v_1\ v_2\ \ldots\ v_n\ ]$$

3) Equality of vectors

Vectors $\boldsymbol{v} = (v_1, v_2, \cdots, v_n)$ and $\boldsymbol{w} = (w_1, w_2, \cdots, w_n)$ in R^{n} are said to be equivalent(equal) if

$v_1 = w_1$, $v_2 = w_2$,⋯, $v_n = w_n$

$\boldsymbol{v} = \boldsymbol{w}$

4) Operation of vectors

$\boldsymbol{v} = (v_1, v_2, \cdots, v_n)$, $\boldsymbol{w} = (w_1, w_2, \cdots, w_n)$ are vectors in R^{n}, and scalar k

① $\boldsymbol{v}+\boldsymbol{w} = (v_1+w_1, v_2+w_2, \cdots, v_n+w_n)$

② $k\boldsymbol{v} = (kv_1, kv_2, \cdots, kv_n)$

③ $-\boldsymbol{v} = (-v_1, -v_2, \cdots, -v_n)$

④ $\boldsymbol{v}-\boldsymbol{w} = (v_1-w_1, v_2-w_2, \cdots, v_n-w_n)$

ex) $\boldsymbol{v} = (3, 1, -1, 4)$, $\boldsymbol{w} = (1, -2, -3, 5)$

① $\boldsymbol{v}+\boldsymbol{w}$

② $3\boldsymbol{v}$

③ $-\boldsymbol{v}$

④ $\boldsymbol{v}-\boldsymbol{w}$

5) Linear Combination

Definition Linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_p$

Given vectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_p$ in $\mathbb{R}^n$ and given $c_1, c_2, \cdots, c_p$, the vector $\boldsymbol{y}$ defined by

$$\boldsymbol{y} = c_1\boldsymbol{v}_1 + c_2\boldsymbol{v}_2 + \cdots + c_p\boldsymbol{v}_p$$

is called a linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \cdots, \boldsymbol{v}_p$ with weights $c_1, c_2, \cdots, c_p$

ex) $\boldsymbol{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$, $\boldsymbol{a}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$

6) Vector equation

A vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_p\mathbf{a}_p = \mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_p & \mathbf{b} \end{bmatrix}$$

In particular, $\mathbf{b}$ can be generated by a linear combination of $\mathbf{a}_1, \cdots, \mathbf{a}_p$ if and only if there exists a solution to the linear system corresponding to the augmented matrix

ex) $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$

Solution set of $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_p\mathbf{a}_p = \mathbf{b}$

7) Spanning set

Definition : Spanning set

If $v_1, v_2, \cdots, v_p$ are in R^n, then the set of all linear combinations of $v_1, v_2, \cdots, v_p$ is denoted by $\mathrm{Span}\{\mathrm{v}_1, \mathrm{v}_2, \cdots, \mathrm{v}_p\}$ and is called the subset of R^n spanned(generated) by $v_1, v_2, \cdots, v_p$

Asking whether a vector b is in $\mathrm{Span}\{\mathrm{v}_1, \mathrm{v}_2, \cdots, \mathrm{v}_p\}$ amounts to asking whether the vector equation $x_1v_1+x_2v_2+\cdots+x_pv_p=b$ has a solution, or equivalently, asking whether the linear system with augmented matrix $\begin{bmatrix} v_1 & v_2 & \cdots & v_p & b \end{bmatrix}$ has a solution.

8) parametric vector form

ex) $10x_1 - 3x_2 - 2x_3 = 0$

ex) $$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0 \\ -3x_1 - 2x_2 + 4x_3 &= 0 \\ 6x_1 + x_2 - 8x_3 &= 0 \end{aligned}$$

4. The Matrix Equation $Ax=b$

1) Matrix Equation

Definition : If A is an $m\times n$ matrix, with columns $a_1, a_2, \cdots, a_n$, and if x is in R^n, then the product of A and x, denoted by Ax, is the linear combination of the columns of A using the corresponding entries in x as weights ; that is,

$$Ax = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$$

Ax is defined only if the number of columns of A equals the number of entries in x

ex) $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 5 & -3 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}$

$\begin{bmatrix} 2 & -3 \\ 8 & 0 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 7 \end{bmatrix}$

ex) for v_1, v_2, v_3 in R^m, write the linear combination $3v_1 - 5v_2 + 7v_3$ as a matrix times a vector

2) Matrix Equation, Vector Equation, System of Linear Equation

Theorem : If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$, and if $\mathbf{b}$ is in R^m, then the matrix equation

$$A\mathbf{x} = \mathbf{b}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}$$

5. Linear Independence

1) homogeneous linear systems

Definition

A system of linear equations is said to be homogeneous if it can be written in the form $Ax = 0$, where A is $m \times n$ matrix and $\mathbf{0}$ is the zero vector in R^{m}.

Such a system $Ax = 0$ always has at least one solution, namely, $\boldsymbol{x} = \mathbf{0}$

The zero solution is usually called the trivial solution

For a given equation $Ax = 0$, The important question is whether there exists a non trivial solutions, i.e, a nonzero vector $\boldsymbol{x}$ that satisfies $Ax = 0$

A system of linear equations is said to be non homogeneous if it can be written in the form $Ax = \boldsymbol{b}$, where A is $m \times n$ matrix and $\boldsymbol{b}$ is not a zero vector.

2) Linear Independence

Definition

An indexed set of vectors $\{v_1, v_2, \cdots, v_p\}$ in R^n is said to be linearly independent if the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = 0$$

has only trivial solution.

The set $\{v_1, v_2, \cdots, v_p\}$ is said to be linearly dependent if there exist weights $c_1, c_2, \cdots, c_p$, not all zero, such that

$$c_1 v_1 + c_2 v_2 + \cdots + c_p v_p = 0$$

The indexed set is linearly dependent if and only if it is not linearly independent

ex) $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, $v_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

3) Linear Independence of Matrix Columns

Suppose that we begin with a matrix $A = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_n \end{bmatrix}$ instead of a set of vectors. The matrix equation $A\boldsymbol{x} = 0$ can be written as

$$x_1\boldsymbol{a}_1 + x_2\boldsymbol{a}_2 + \cdots + x_p\boldsymbol{a}_p = 0$$

The columns of matrix A are linearly independent if and only if the equation $A\boldsymbol{x} = 0$ has only the trivial solution.

ex) $A = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix}$

4) Vectors and linearly independence

(1) A set containing only zero vector

(2) A set containing one vector

(3) A set containing two vector

(4) Theorem : Characterization of Linearly Dependent Sets

An indexed set $S = \{v_1, v_2, \cdots, v_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others

In fact, if S is linearly dependent and $v_1 \neq \mathbf{0}$, then some v_j (with $j > 1$) is a linear combination of the preceding vectors, $v_1, \cdots, v_{j-1}$

5) Theorem

(1) If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\{v_1, v_2, \cdots, v_p\}$ in R^n is linearly dependent if $p > n$

(2) If a set $S = \{v_1, v_2, \cdots, v_p\}$ in R^n contains the zero vector, then the set is linearly dependent

ex) determine by inspection if the given set is linearly dependent

① $\begin{bmatrix}1\\7\\6\end{bmatrix}$, $\begin{bmatrix}2\\0\\9\end{bmatrix}$, $\begin{bmatrix}3\\1\\5\end{bmatrix}$, $\begin{bmatrix}4\\1\\8\end{bmatrix}$

② $\begin{bmatrix}2\\3\\5\end{bmatrix}$, $\begin{bmatrix}0\\0\\0\end{bmatrix}$, $\begin{bmatrix}1\\1\\8\end{bmatrix}$

③ $\begin{bmatrix}-2\\4\\6\\10\end{bmatrix}$, $\begin{bmatrix}3\\-6\\-9\\-15\end{bmatrix}$

ex) determine if the vectors are linearly independent

① $\begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}, \begin{bmatrix} 9 \\ 4 \\ -8 \end{bmatrix}$

② $\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ -8 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$

ex) determine if the columns of the matrix form a linearly independent set.

$$\begin{bmatrix} 0 & -8 & 5 \\ 3 & -7 & 4 \\ -1 & 5 & -4 \\ 1 & -3 & 2 \end{bmatrix}$$

ex) Let $u = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$, $v = \begin{bmatrix} 6 \\ 1 \\ 7 \end{bmatrix}$, $w = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix}$, $z = \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix}$

① Are the sets $\{u, v\}$, $\{u, w\}$, $\{u, z\}$, $\{v, w\}$, and $\{w, z\}$ each linearly independent? Why or why not?

② Does the answer to Part ① imply that $\{u, v, w, z\}$ is linearly independent, is it wise to check if, say w is a linear combination of u, v, and z respectively?

③ To determine if $\{u, v, w, z\}$ is linearly dependent, is it wise to check if, say w is a linear combination of u, v, and z?

④ Is $\{u, v, w, z\}$ is linearly independent?

ex) Suppose that $\{v_1, v_2, v_3\}$ is a linearly dependent set of vectors in $\mathbb{R}^n$ and v_4 is a vector in $\mathbb{R}^n$. Show that $\{v_1, v_2, v_3, v_4\}$ is also a linearly dependent set.