2강. Matrix Algebra

1. Matrix Operation
2. The Inverse of a Matrix
3. Characterizations of Invertible Matrices

1. Matrix Operation

1) Matrix

(1) Definition : Matrix

$m \times n$ matrix A : a matrix with m rows and n columns

The scalar entry in the ith rows and jth columns of A is denoted by a_{ij}
: (i, j)-entry of A

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \cdots & & \cdots & & \cdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{ij} \\ \cdots & & \cdots & & \cdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

Notation

Each column of A is a list of m real numbers, which identifies a vector in R^{m}

These columns are denoted by $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\cdots$, $\boldsymbol{a}_n$

$A \ = [\boldsymbol{a}_1 \ \boldsymbol{a}_2 \ \cdots \ \boldsymbol{a}_n]$

The number a_{ij} is the ith entry (from the top) of the jth column vector $\boldsymbol{a}_j$

The diagonal entries in an $m \times n$ matrix A : a_{11}, a_{22}, $\cdots$ – they form the main diagonal of A

(2) Basic Matrix

① Zero matrix

A matrix whose entries are all zero

ex) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

② Square matrix

A matrix which the number of rows and columns are same ($n \times n$ matrix)

ex) $\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$

③ Identity matrix

A matrix whose diagonal entries are all 1, and the non-diagonal entries are all 0

ex) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

④ Triangular matrix

Upper triangular matrix : a square matrix in which all entries below the main diagonal are zero
Lower triangular matrix : a square matrix in which all entries above the main diagonal are zero

ex) $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 3 & 2 & 0 \end{bmatrix}$

⑤ Diagonal matrix

A square matrix in which all entries off the main diagonal are zero

ex) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

2) Operation of the matrix

(1) Sum and Scalar multiple

① Equality of matrices

The two matrices A, B are equal

- They have the same size(i.e the same number of rows and columns)
- Their corresponding columns are equal

② Sum of matrices

If A, B are $m \times n$ matrices, then the sum $A+B$ is the $m \times n$ matrix whose columns are the sums of the corresponding columns in A and B

Each entry in $A+B$ is the sum of the corresponding columns in A and B

The sum $A+B$ is defined only when A and B are the same size

ex) $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 5 & 3 \\ 2 & 1 & -4 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$

Find $A+B$, $A+C$

③ Scalar multiple

If r is a scalar and A is a matrix, the scalar multiple rA is the matrix whose columns are r times the corresponding columns in A

ex) $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 5 & 3 \\ 2 & 1 & -4 \end{bmatrix}$

Find $A-2B$

(2) Properties of sum and scalar multiples of matrices

Theorem Let A, B, and C be matrices of the same size, and let r and s be scalars.

① $A+B=B+A$
② $(A+B)+C=A+(B+C)$
③ $A+0=A$
④ $r(A+B)=rA+rB$
⑤ $(r+s)A=rA+sA$
⑥ $r(sA)=(rs)A$

(2) Matrix multiplication

① Definition : Matrix multiplication

If A is an $m \times n$ matrix, and if B is an $n \times p$ matrix with columns $\mathbf{b}_1$, $\mathbf{b}_2$, $\cdots$, $\mathbf{b}_p$
The product AB is the $m \times p$ matrix whose columns are $A\mathbf{b}_1$, $A\mathbf{b}_2$, $\cdots$, $A\mathbf{b}_p$

$$AB = [\ A\mathbf{b}_1 \quad A\mathbf{b}_2 \quad \cdots \quad A\mathbf{b}_p\ ]$$

ex) $B = \begin{bmatrix} -1 & 5 & 3 \\ 2 & 1 & -4 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 3 & 2 \\ 2 & -1 \end{bmatrix}$, find BC

Since the first column of AB is $A\mathbf{b}_1$, this column is a linear combination of the columns of A using the entries in $\mathbf{b}_1$ as weights. A similar statement is true for each column of AB

Each column of AB is a linear combination of the columns of A using weights from the corresponding column of B

Obviously, the number of columns of A must match the number of rows in B in order for a linear combination such as $A\mathbf{b}_1$ to be defined.

The definition of AB shows that AB has the same number of rows as A and the same number of columns as B

ex) If A is a 3×5 matrix and B is a 5×2 matrix, what are the sizes of AB and BA, if they are defined

② Row Column rule for computing AB

If a product AB is defined, then the entry in row i and column j of AB is the sum of the products of corresponding entries from row i of A and column j of B

If $(AB)_{ij}$ denotes the (i, j) entry in AB, and if A is $m \times n$ matrix, then

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

ex) $B = \begin{bmatrix} -1 & 5 & 3 \\ 2 & 1 & -4 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \\ 3 & 2 \\ 2 & -1 \end{bmatrix}$, find BC

ex) find the second row of AB where

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & -1 & 2 & 3 \\ 0 & 1 & 3 & 4 \\ 2 & 5 & 1 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 5 & 3 \\ -3 & 2 \\ 1 & 2 \end{bmatrix}$$

③ Properties of multiplication

Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined.

① $A(BC) = A(BC)$
② $A(B+C) = AB + AC$
③ $(B+C)A = BA + CA$
④ $r(AB) = (rA)B = A(rB)$ for any scalar r
⑤ $I_m A = A = I_n$

주의

① $AB \neq BA$
② $AB = AC \Rightarrow B = C$: False
③ $AB = 0 \Rightarrow A = 0$ or $B = 0$: False

④ Powers of matrix

If A is $n \times n$ matrix and if k is a positive integer, then

$A^k = AA \cdots A$

$A^0 = I_n$

(3) Transpose of matrix

Definition : Transpose of matrix

Given an $m \times n$ matrix A, the transpose of A is the $n \times m$ matrix whose columns are formed from the corresponding rows of A (denotes A^{T}

ex) $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 2 & 4 \end{bmatrix}$

Properties of Transpose of matrix

① $(A^T)^T = A$
② $(A+B)^T = A^T + B^T$
③ For any scalar r, $(rA)^T = rA^T$
④ $(AB)^T = B^T A^T$

◎ Definition : Symmetric Matrix

A square matrix is called symmetric if $A^T = A$

ex) $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 2 \\ 3 & 2 & 1 \end{bmatrix}$

Theorem

If A and B are symmetric matrices with the same size and k is any scalar , then

① A^T is symmetric
② $A+B$ and $A-B$ are symmetric
③ kA is symmetric

(4) Trace of matrix

Definition : Trace of matrix

If A is a square matrix, then the trace of A, denoted by $tr(A)$ is defined to be the sume of the entries on the main diagonal of A

ex) $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & -1 & 2 & 3 \\ 0 & 1 & 3 & 4 \\ 2 & 5 & 1 & 9 \end{bmatrix}$

2. The Inverse of a Matrix

1) Invertible Matrix

(1) invertible matrix

Definition : invertible matrix, Inverse of a matrix

An $n \times n$ matrix A is said to be invertible if there is an $n \times n$ matrix C such that

$CA = AC = I$

where $I = I_n$, the $n \times n$ identity matrix

C : the inverse of A

In fact, C is uniquely determined by A, because if B were another inverse of A

then, $B = BI = B(AC) = (BA)C = IC = C$

The unique inverse is denoted by A^{-1}, so that

$A^{-1}A = AA^{-1} = I$

Definition : singular, nonsingular matrix

Singular matrix : A matrix that is not invertible
Nonsingular matrix : A matrix that is invertible

ex) $A = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}$, $B = \begin{bmatrix} -7 & 5 \\ 3 & -2 \end{bmatrix}$

(2) Invertible matrix : 2×2 matrix

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $ad-bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$ad-bc$: determinant of A

$\det A = ad-bc$

if $ad-bc=0$, A is not invertible

ex) $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$

(2) Theorem

① If A is $n \times n$ matrix, then for each $\boldsymbol{b}$ in R^{n}, the equation $A\boldsymbol{x} = \boldsymbol{b}$ has the unique solution

$\boldsymbol{x} = A^{-1}\boldsymbol{b}$

ex) $\begin{aligned} 3x+4y&=0 \\ 5x+6y&=1 \end{aligned}$

② If A is an invertible matrix, then A^{-1} is invertible and
$(A^{-1})^{-1} = A$
③ If A, B are $n \times n$ invertible matrices, then so is AB, and the inverse of AB is
$(AB)^{-1} = B^{-1}A^{-1}$
④ If A is an invertible matrix, then so is A^T, and the inverse of A^T is
$(A^T)^{-1} = (A^{-1})^T$

Generalization of ③

The product of $n \times n$ invertible matrices is invertible, and the inverse is the product of their inverses in the reverse order.

An inverse matrix A is row equivalent to and identity matrix, and we can find A^{-1} by watching the row reduction of A to I

2) Elementary matrix

(1) Definition : Elementary matrix

The matrix that is obtained by performing single elementary row operation on an identity matrix

ex) $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

(2) Elementary matrix and invertible matrix

If an elementary row operation is performed on an $m \times n$ matrix A, the resulting matrix can be written as EA, where $m \times m$ matrix E is created by performing the same row operation on I_m

Since row operations are reversible, elementary matrices are invertible.

There is an elementary matrix F such that $EF = FE = I$

Each elementary matrix E is invertible, and the inverse of E is the elementary matrix of the same type that transforms E back to I

ex) $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$

(3) Theorem

An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}.

3) An Algorithm for Finding A^{-1}

(1) Algorithm for finding A^{-1}

Row reduce the augmented matrix $[\, A \;\; I \,]$. If A is row equivalent to I, then $[\, A \;\; I \,]$ is row equivalent to $[\, I \;\; A^{-1} \,]$. Otherwise, A does not have an inverse.

ex) $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{bmatrix}$

(2) Another view of matrix inversion

columns of I_n = $e_1, \; e_2, \; \cdots \;, \; e_n$

Then row reduction of $[\, A \;\; I \,]$ to $[\, I \;\; A^{-1} \,]$ can be viewed as the simultaneous solution of the n systems

$$Ax = e_1 , \; Ax = e_2 , \; \cdots , \; Ax = e_n \quad \cdots \text{①}$$

where the ‘augmented columns’ of these systems have all been placed next to A to form $[\, A \;\; e_1 \;\; e_2 \cdots \; e_n \,] = [\, A \;\; I \,]$

The equation $AA^{-1} = I$ and the definition of matrix multiplication show that the columns of A^{-1} are precisely the solutions of the systems in ①

3. Characterization of Invertible Matrices

1) Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false

a. A is an invertible matrix
b. A is row equivalent to the $n \times n$ identity matrix
c. A has n pivot positions
d. The equation $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution
e. The columns of A form a linearly independent set
f. The columns of A span R^{n}
g. There is an $n \times n$ matrix C such that $CA = I$
h. There is an $n \times n$ matrix D such that $AD = I$
j. The equation $A\boldsymbol{x} = \boldsymbol{b}$ has at least one solution for each $\boldsymbol{b}$ in R^{n}
k. A^T is an invertible matrix.

proof)

a. $\Rightarrow$ g. $\Rightarrow$ d. $\Rightarrow$ c. $\Rightarrow$ b. $\Rightarrow$ a

h. $\Rightarrow$ j. $\Rightarrow$ a. $\Rightarrow$ h.

d. $\Leftrightarrow$ e.

j. $\Leftrightarrow$ f.

a. $\Leftrightarrow$ l.

Practice Problems

ex) Suppose A is an $m \times n$ matrix, all of whose rows are identical. Suppose B is an $n \times p$ matrix, all of whose columns are identical. What can be said about the entries in AB?

ex) Determine if $A = \begin{bmatrix} 2 & 3 & 4 \\ 2 & 3 & 4 \\ 2 & 3 & 4 \end{bmatrix}$ is invertible

ex) Determine A is invertible

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}$$

ex) Suppose that A and B are $n \times n$ matrices and the equation $AB\boldsymbol{x} = \boldsymbol{b}$ has a nontrivial solution. What can you say about the matrix AB?

ex) Show that if AB is invertible, so is B