5. Eigenvector, Eigenvalue, and Diagonalization

1) Eigenvectors and Eigenvalues
2) Characteristic Equation
3) Diagonalization

1. Eigenvectors and Eigenvalues

1) Eigenvectors and Eigenvalues

(1) Definition : Eigenvectors and Eigenvalues

A : $n \times n$ matrix

eigenvector of A : nonzero vector $\boldsymbol{x}$ such that $A\boldsymbol{x} = \lambda\boldsymbol{x}$ for some scalar λ.
λ : eigenvalue of A

λ exists if there is nontrivial solution $\boldsymbol{x}$ of $A\boldsymbol{x} = \lambda\boldsymbol{x}$
$\boldsymbol{x}$: an eigenvector corresponding to λ

λ is an eigenvalue of A if and only if the equation

$(A - \lambda I)\boldsymbol{x} = 0$

has nontrivial solution.

The set of solutions of $(A - \lambda I)\boldsymbol{x} = 0$: null space of the matrix $A - \lambda I$
: eigenspace of A corresponding to λ
The eigenspace consists of the zero vector and all the eigenvectors corresponding to λ

ex) $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $\boldsymbol{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$, $\boldsymbol{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

ex) $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, $\lambda = 2$

ex) $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$, an eigenvalue of A is 2. Find a basis for the corresponding eigenspace.

2) Properties of eigenvalue and eigenvectors

(1) Theorem

The eigenvalues of a triangular matrix are the entries on its main diagonal.

(2) Theorem

If $\mathbf{v}_1$, $\mathbf{v}_2$, $\cdots$, , $\mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues λ_1, λ_2, $\cdots$, λ_r of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_r\}$ is linearly independent.

(3) Properties

① Eigenvalues and invertible matrix

A is not invertible if and only if one of eigenvalues of A is 0

② solution of $x_{k+1} = Ax_k$

$x_{k+1} = Ax_k$

2. The Characteristic Equation

1) The Characteristic Equation

(1) Definition : The Characteristic Equation

A : $n \times n$ matrix

$\det(A - \lambda I) = 0$: characteristic equation of A

$\det(A - \lambda I)$: characteristic polynomial of A

solution of characteristic equation of A : Eigenvalues of A

(If the solution has multiplicity : multiplicity of an eigenvalue λ is its multiplicity as a root of the characteristic equation.)

ex) $A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

ex) The characteristic polynomial of a 6×6 matrix is $\lambda^6 - 4\lambda^5 - 12\lambda^4$. Find the eigenvalues and their multiplicities.

2) Similarity

(1) Definition : Similarity

A, B : $n \times n$ matrices
A is similar to B if there is an invertible matrix P such that $P^{-1}AP = B$, or $A = PBP^{-1}$
B is also similar to A - A and B are similar
Changing A into $P^{-1}AP = B$: similarity transformation.

(2) Theorem

① Similar matrices have the same determinant.
② Similar matrices have the same rank.
③ Similar matrices have the same nullity.
④ Similar matrices have the same trace.
⑤ Similar matrices have the same characteristic equation and have the same eigenvalues

(3) Caution

① The matrices are not similar even though they have the same eigenvalues
② Similarity is not the same as row equivalence

3. Diagonalization

1) Diagonalization

(1) Efficiency of diagonal matrix

ex) $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

ex) $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$, $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$

(2) Definition : Diagonalization

A square matrix A is diagonalizable if A is similar to a diagonal matrix

$A = PDP^{-1}$ for some invertible matrix P and some diagonal matrix D

(3) Theorem : The Diagonalization Theorem

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.
In fact, $A = PDP^{-1}$, with D a diagonal matrix, columns of P are n linearly independent eigenvectors of A, and diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P

In other words, A is diagonalizable if and only if there are enough eigenvectors to form a basis of R^n. - eigenvector basis of R^n

ex) $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$

ex) $A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$

(4) Theorem

An $n \times n$ matrix with n distinct eigenvalues is diagonalizable

(5) Theorem

Let A be an $n \times n$ matrix whose distinct eigenvalues are λ_1, λ_2, $\cdots$, λ_p

① For $1 \leqq k \leqq p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of the eigenvalue λ_k.

② The matrix A is diagonalizable if and only if the sum of the dimensions of the eigenspaces equals n

This happens if and only if

a. The characteristic polynomial factors completely into linear factors

b. the dimension of the eigenspace for each λ_k equals the multiplicity of λ_k

③ If A is diagonalizable and B_k is a basis for the eigenspace corresponding to λ_k for each k, then the total collection of vectors in the sets B_1, $\cdots$, B_p forms an eigenvector basis for R^{n}

ex) $A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & -3 \end{bmatrix}$

ex) $A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$, A^8