4. Vector Space

1) Vector spaces and Subspaces
2) Linear Transformation
3) Null Spaces, Column spaces, Row spaces
4) Basis
5) Dimension
6) Rank
7) Coordinate systems

1. vector spaces and subspaces

1) Vector spaces

(1) Definition : vector spaces

A vector space is a nonempty set V of objects, called vectors, on which are defined two operations, called addition and multiplication by scalars.

① V is nonempty set
② if $\boldsymbol{v}$, $\boldsymbol{w}$ are in V, then $\boldsymbol{v}+\boldsymbol{w}$ is in V
③ if $\boldsymbol{v}$ are in V, then $k\boldsymbol{v}$ is in V (k is scalar)

and vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ in V, and for scalars c, d satisfies these axioms

④ $\boldsymbol{v}+\boldsymbol{w} = \boldsymbol{w}+\boldsymbol{v}$
⑤ $\boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w}) = (\boldsymbol{u}+\boldsymbol{w})+\boldsymbol{v}$
⑥ There is a zero vector $\mathbf{0}$ in V such that $\boldsymbol{u}+\mathbf{0} = \boldsymbol{u}$
⑦ For each $\boldsymbol{u}$ in V, there is a vector $-\boldsymbol{u}$ such that $\boldsymbol{u}+(-\boldsymbol{u}) = \mathbf{0}$
⑧ $c(\boldsymbol{u}+\boldsymbol{v}) = c\boldsymbol{u}+d\boldsymbol{v}$
⑨ $(c+d)\boldsymbol{u} = c\boldsymbol{u}+d\boldsymbol{u}$
⑩ $c(d\boldsymbol{u}) = (cd)\boldsymbol{u}$
⑪ $1\boldsymbol{u} = \boldsymbol{u}$

ex) The spaces $\mathrm{R}^{\mathrm{n}}(n \geqq 1)$: vector spaces

2) Subspaces

(1) Definition : Subspaces

A subspace of a vector space V is a subset H of V that has three properties

① The zero vector of V is in H
② H is closed under vector addition. For each $\boldsymbol{u}$, $\boldsymbol{v}$ in H, the sum $\boldsymbol{u}+\boldsymbol{v}$ is in H
③ H is closed under multiplication by scalars. For each $\boldsymbol{u}$ in H and each scalar c, the vector $c\boldsymbol{u}$ is in H

ex) The set consisting of only the zero vector in a vector space V : zero subspace, $\{0\}$

ex) The vector space R^2 and R^3

ex) $H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} : s \text{ and } t \; are \, real \right\}$

ex) A plane in R^3 not through the origin

(2) A subspace Spanned by a set

Theorem

If v_1 , $\cdots$, v_p are in a vector space V, then $Span\{v_1, \cdots, v_p\}$ is a subspace of V

Given any subspace H of V, a spanning set for H is a set $\{v_1, \cdots, v_p\}$ in H such that

$H = Span\{v_1, \cdots, v_p\}$

2. Linear Transformation

1) Linear Transformation

(1) Definition : Transformation

A function whose inputs and outputs are vectors is called a Transformation

$T: \mathrm{R}^{\mathrm{n}} \rightarrow \mathrm{R}^{\mathrm{m}}$

A tranformation $T: \mathrm{R}^{\mathrm{n}} \rightarrow \mathrm{R}^{\mathrm{n}}$ is called operator on R^{n}

ex) T : transformation that maps a vector $\boldsymbol{x} = (x_1, x_2)$ in R^2 into the vector $2\boldsymbol{x} = (2x_1, 2x_2)$ in R^2

(2) Definition : Linear Transformation

A function $T: \mathrm{R}^{\mathrm{n}} \to \mathrm{R}^{\mathrm{m}}$ is called a linear transformation from R^{n} to R^{m} if following two properties hold for all vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ in R^{n} and for all scalars c

① $T(c\boldsymbol{u}) = c\,T(\boldsymbol{u})$
② $T(\boldsymbol{u}+\boldsymbol{v}) = T(\boldsymbol{u}) + T(\boldsymbol{v})$

in the special case where $m=n$, the linear transformation T is called a linear operator on R^{n}

ex) T : transformation that maps a vector $\boldsymbol{x} = (x_1, x_2)$ in R^2 into the vector $2\boldsymbol{x} = (2x_1, 2x_2)$ in R^2

(3) Definition : Matrix Transformation

If A is an $m \times n$ matrix, and if $\boldsymbol{x}$ is a column vector in R^{n}, then the product $A\boldsymbol{x}$ is a vector in R^{m}. So multiplying $\boldsymbol{x}$ by A creates a transformation from R^{n} to R^{m} and this transformation is called the multiplication by A or the transforamtion A, and is denoted by T_A to emphasize the matrix A.

ex) $A = \begin{bmatrix} 1 & -1 \\ 2 & 5 \\ 3 & 4 \end{bmatrix}$ $\boldsymbol{b} = \begin{bmatrix} 7 \\ 0 \\ 7 \end{bmatrix}$

ex) Zero transformation T_0

ex) Identity transformation T_I

(4) Theorem

All linear transformations are matrix transformations

(5) Theorem

Let $T : \mathrm{R}^{\mathrm{n}} \rightarrow \mathrm{R}^{\mathrm{m}}$ be a linear transformation. If $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\cdots$, $\boldsymbol{e}_n$ are standard unit vectors in R^{n}, and if $\boldsymbol{x}$ is any vector in R^n, then $T(\boldsymbol{x})$ can be expressed as
$T(\boldsymbol{x}) = A\boldsymbol{x}$

$A = \left[\, T(\boldsymbol{e}_1)\ T(\boldsymbol{e}_2)\ \cdots\ T(\boldsymbol{e}_n) \,\right]$

A : standard matrix for T and denoted by $A = [T]$

2) Kernel of transformation

(1) Definition : Kernel of transformation

If $T: \mathrm{R}^{\mathrm{n}} \rightarrow \mathrm{R}^{\mathrm{m}}$ is a linear transformation, then the set of vectors in R^{n} that T maps into $\mathbf{0}$ is called kernel of T and is denoted by $\ker(T)$

$$\ker(T) = \{\boldsymbol{x} : \ T(\boldsymbol{x}) = \mathbf{0}\}$$

ex) $A = \begin{bmatrix} 1 & -1 \\ 2 & 5 \\ 3 & 4 \end{bmatrix}$

ex) The zero operator

ex) The identity operator

(2) Theorem

① The kernel of a linear transformation always contains the zero vector.

② If $T: \mathrm{R}^{\mathrm{n}} \rightarrow \mathrm{R}^{\mathrm{m}}$ is a linear transformation, then the kernel of T is a subspace of R^{n}

(3) one to one

A linear transformation $T: \mathrm{R}^{\mathrm{n}} \rightarrow \mathrm{R}^{\mathrm{m}}$ is one to one if T maps distinct vectors in R^{n} into distinct vectors in R^{m}

(4) Theorem

If $T: \mathrm{R}^{\mathrm{n}} \rightarrow \mathrm{R}^{\mathrm{m}}$ is a linear transformation, then the followings are equivalent

① T is one to one
② $\ker(T) = \{\mathbf{0}\}$

(5) Theorem

If A is an $m \times n$ matrix, then the corresponding linear transformation $T_A : \mathrm{R}^{\mathrm{n}} \rightarrow \mathrm{R}^{\mathrm{m}}$ is one-to-one if and only if the linear system $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution.

3) Range of transformation

(1) Definition : Range of transformation

If $T: \mathrm{R}^{\mathrm{n}} \rightarrow \mathrm{R}^{\mathrm{m}}$ is a linear transformation, then the range(image) of T, denoted by $ran(T)$, is the set of all vectors in R^{m} that are images of at least one vector in R^{n}

$$ran(T) = \{\boldsymbol{b} : \boldsymbol{b} = T(\boldsymbol{x}) \text{ for } all\ \boldsymbol{x} \in \mathrm{R}^{\mathrm{m}}\}$$

ex) $A = \begin{bmatrix} 1 & -1 \\ 2 & 5 \\ 3 & 4 \end{bmatrix}$

ex) The zero operator

ex) The identity operator

(2) Theorem

If $T: \mathrm{R}^{\mathrm{n}} \to \mathrm{R}^{\mathrm{m}}$ is a linear transformation, then $ran(T)$ is a subspace of R^{m}

(3) Onto

A linear transformation $T: \mathrm{R}^{\mathrm{n}} \to \mathrm{R}^{\mathrm{m}}$ is said to be onto if the range of T is the entire codomain R^{m}

Theorem

If A is an $m \times n$ matrix, then the corresponding linear transforamtion $T_A: \mathrm{R}^{\mathrm{n}} \to \mathrm{R}^{\mathrm{m}}$ is onto if and only if the linear system $A\boldsymbol{x} = \boldsymbol{b}$ is consistent for every $\boldsymbol{b}$ in R^{n}

4) Linear operator

(1) Theorem

If $T: \mathrm{R}^n \rightarrow \mathrm{R}^n$ is a linear operator on R^n, then T is one-to-one if and only if it is onto.

(2) Unifying theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix
b. A is row equivalent to the $n \times n$ identity matrix
c. A has n pivot positions
d. The equation $A\boldsymbol{x} = \mathbf{0}$ has only the trivial solution
e. The columns of A form a linearly independent set
f. The columns of A span R^n
g. There is an $n \times n$ matrix C such that $CA = I$
h. There is an $n \times n$ matrix D such that $AD = I$
j. The equation $A\boldsymbol{x} = \boldsymbol{b}$ has at least one solution for each $\boldsymbol{b}$ in R^n
k. A^T is an invertible matrix.
l. $\det A \neq 0$
m. T_A is one to one
n. T_A is onto

3. Column space, Null space, Row space

1) Null space

(1) Definition : Null space

The null space of an $m \times n$ matrix A, written as $NulA$, is the set of all solutions of the homogeneous equation $A\boldsymbol{x} = \mathbf{0}$.

$$Nul\,A = \{\boldsymbol{x} : A\boldsymbol{x} = \mathbf{0},\ \boldsymbol{x} \in \mathrm{R}^{\mathrm{n}}\}$$

(2) Theorem

The null space of an $m \times n$ matrix A is a subspace of R^{n}

ex) $A = \begin{bmatrix} 1 & -1 \\ 2 & 5 \\ 3 & 4 \end{bmatrix}$

2) Column space

(1) Definition : Column space

The column space of an $m \times n$ matrix A, written as $ColA$, is the set of all linear combinations of the columns of A, If $A = [\, \boldsymbol{a}_1 \;\; \boldsymbol{a}_2 \;\; \cdots \;\; \boldsymbol{a}_n \,]$, then

$$Col\,A = Span\{\, \boldsymbol{a}_1,\, \boldsymbol{a}_2,\, \cdots,\, \boldsymbol{a}_n \}$$

(2) Theorem

The column space of an $m \times n$ is a subspace of R^{m}

A typical vector in $ColA$ can be written as $A\boldsymbol{x}$ for some $\boldsymbol{x}$ because the notation $A\boldsymbol{x}$ stands for a linear combination of the columns of A

$$ColA = \{\boldsymbol{b} : \boldsymbol{b} = A\boldsymbol{x} \;\text{ for } some \;\; \boldsymbol{x} \in \mathrm{R}^{\mathrm{n}}\}$$

ex) $A = \begin{bmatrix} 1 & -1 \\ 2 & 5 \\ 3 & 4 \end{bmatrix}$

3) The contrast between $NulA$ and $ColA$

ex) $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$, $\boldsymbol{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}$, $\boldsymbol{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$

(1) Table : The contrast between $NulA$ and $ColA$ for an $m \times n$ matrix A

$NulA$	$ColA$
① $NulA$ is a subspace of $\mathbb{R}^n$ ② $NulA$ is implicitly defined (you are given only condition $(A\boldsymbol{x} = \boldsymbol{0})$ that vectors in $NulA$ must satisfy. ③ It takes time to find vectors in $NulA$ Row operations on $[A\,\boldsymbol{0}]$ are required ④ There is no obvious relation between $NulA$ and the entries in A ⑤ A typical vector $\boldsymbol{v}$ in $NulA$ has the property that $A\boldsymbol{v} = \boldsymbol{0}$ ⑥ Given a specific vector $\boldsymbol{v}$, it is easy to tell if $\boldsymbol{v}$ is in $NulA$. Just compute $A\boldsymbol{v}$ ⑦ $NulA = \{\boldsymbol{0}\}$ if and only if the equation $A\boldsymbol{x} = \boldsymbol{0}$ has only the trivial solution. ⑧ $NulA = \{\boldsymbol{0}\}$ if and only if the linear transformation $\boldsymbol{x} \to A\boldsymbol{x}$ is one-to-one	① $ColA$ is a subspace of $\mathbb{R}^m$ ② $ColA$ is explicitly defined (you are told how to bulid the vectors in $ColA$) ③ It is easy to find $ColA$. The columns of A are displayed; others are formed from them. ④ There is obvious relation between $ColA$ and the entries of A, since each columns of A is in $ColA$ ⑤ A typical vector $\boldsymbol{v}$ in $ColA$ has the property that the equation $A\boldsymbol{x} = \boldsymbol{v}$ is consistent ⑥ Given a specific vector $\boldsymbol{v}$, it may take time to tell if $\boldsymbol{v}$ is in $ColA$ Row operations on $[A\,\boldsymbol{v}]$ are required ⑦ $ColA = \mathbb{R}^m$ if and only if the equation $A\boldsymbol{x} = \boldsymbol{b}$ has a solution for every $\boldsymbol{b}$ in $\mathbb{R}^m$ ⑧ $ColA = \mathbb{R}^m$ if and only if the linear transformation $\boldsymbol{x} \to A\boldsymbol{x}$ is onto

4) Row space

(1) Definition : Row space

The row space of an $m \times n$ matrix A, written as $Row A$, is the set of all linear combinations of the rows of A.

Each row has n entries, so $Row A$ is a subspace of R^{n}

$$Row A = Col A^T$$

(2) Theorem

If two matrices A and B are row equivalent, then their row spaces are the same.

ex) $A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$

4. Linear independent sets ; Bases

1) Review of linear independence

(1) Linear independence

An indexed set of vectors $\{v_1, v_2, \cdots, v_p\}$ in V is linearly independent if the vector equation
$c_1 v_1 + \cdots + c_p v_p = 0$
has only the trivial solution. $c_1 = c_2 = \cdots = c_p = 0$

The set $\{v_1, v_2, \cdots, v_p\}$ is linearly dependent if the vector equation has non trivial solution.

(2) Theorem

An indexed set $\{v_1, v_2, \cdots, v_p\}$ of two or more vectors with $v_1 \neq \mathbf{0}$, is linearly dependent if and only if some v_j (with $j > 1$) is linear combination of the preceding vectors $v_1, \cdots, v_p$

2) Basis

(1) Definition : Basis

Let H be a subspace of a vector space V. An indexed set of vectors $B = \{b_1, b_2, \cdots, b_p\}$ in V is a basis for H if

① B is a linearly independent set
② The subspace spanned by B is H. That is

$H = Span\{b_1, b_2, \cdots, b_p\}$

The definition of a basis applies to the case when $H = V$, because any vector space is a subspace of itself

Thus a basis of V is a linearly independent set that spans V

ex) Let A be an invertible $n \times n$ matrix, say $A = [\, a_1 \ a_2 \ \cdots \ a_n \,]$. Then the columns of A form a basis for R^n because they are linearly independent and they span R^n, by invertible matrix theorem

ex) The nonzero row vectors of a matrix in row echelon form are linearly independent.

ex) Let e_1, e_2, $\cdots$, e_n be the columns of the $n \times n$ matrix I_n. The set $\{e_1, e_2, \cdots, e_n\}$ is called the standard basis of R^{n}

ex) $v_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}$, $v_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}$, $v_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$

(2) Theorem

Let S be a finite set of vectors in a nonzero subspace V.
If S spans V, but is not a basis for V, then a basis for V can be obtained by removing appropriate vectors from S

Let S be a finite set of vectors in a nonzero subspace V. If S is linearly independent, but is not a basis for V, then a basis for V can be obtained by adding appropriate vectors from V to S

5. Dimension

1) Dimension

(1) Definition : Dimension

If V is spanned by a finite set, then V is said to be finite-dimensional, and the dimension of V, written as $\dim V$, is the number of vectors in a basis for V.

The dimension of the zero vector space $\{0\}$ is defined to be zero.

If V is not spanned by a finite set, then V is said to be infinite-dimensional.

ex) $\dim \mathrm{R}^{\mathrm{n}}$

ex) $H = \left\{ \begin{bmatrix} a-3b+6c \\ 5a+4d \\ b-2c-d \\ 5d \end{bmatrix} : a,\ b,\ c,\ d,\ in\ \mathrm{R}^{\mathrm{n}} \right\}$

ex) Solution space of the linear system

$$\begin{aligned} 2x_1+4x_2-2x_3+x_4&=0 \\ -2x_1-5x_2+7x_3+3x_4&=0 \\ 3x_1+7x_2-8x_3+6x_4&=0 \end{aligned}$$

(2) Theorem

① If V and W are subspaces of R^{n}, and if V is a subspace of W, then

$0 \leqq \dim V \leqq \dim W \leqq n$

$V = W$ if and only if $\dim V = \dim W$

② Let S be a nonempty set of vectors in a vector space V, and let S' be a set that results by adding additional vectors in V to S

a) If the additional vectors are in $Span S$, then $Span S' = Span S$

b) If $Span S' = Span S$, then the additional vectors are in $Span S$.

c) If $Span S$ and $Span S'$ have the same dimension, then the additional vectors are in $Span S$ and $Span S' = Span S$

③ Let V be a p-dimensional vector space, $p \geqq 1$.

Any linearly independent set of exactly p elements in V is automatically a basis for V

Any set of exactly p elements that spans V is automatically a basis for V

(3) Finding dimension of $Nul\,A$ and $Col\,A$

ex) $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$

6. Rank

1) Rank

(1) Definition : Rank

The rank of A is the dimension of the column space of A

Since $Row A$ is the same as $Col A^T$, the dimension of the row space of A is the rank of A^T

(2) Definition : Nullity

The nullity of A is the dimension of the null space of A

ex) $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$

2) Rank Theorem

(1) Rank Theorem

The dimensions of the column space and row space of an $m \times n$ matrix A are equal. This common dimension, the rank of A, also equals the number of pivot positions in A and satisfies the equation

$rankA + \dim NulA = n$

(2) The pivot Theorem

The pivot columns of a matrix A form a basis for $Col\,A$

ex) $A = \begin{bmatrix} 2 & -1 & 1 & -6 & 8 \\ 1 & -2 & -4 & 3 & -2 \\ -7 & 8 & 10 & 3 & -10 \\ 4 & -5 & -7 & 0 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -2 & -4 & -3 & -2 \\ 0 & 3 & 9 & -12 & 12 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Find $rankA$ and $nullityA$, Find bases for $Col\,A$ and $Row\,A$, and $Nul\,A$

ex) Could a 6×9 matrix have a two-dimensional null space?

7. Coordinate System

1) Coordinate System

(1) Theorem : The uniqueness representation theorem

Let $B=\{\boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_n\}$ be a basis for a vector space V. Then for each $\boldsymbol{x}$ in V, there exists a unique set of scalars $c_1, c_2, \cdots, c_n$ such that

$$\boldsymbol{x} = c_1\boldsymbol{b}_1 + \cdots + c_n\boldsymbol{b}_n$$

(2) Definition : coordinate

Suppose $B=\{\boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_n\}$ is a basis for V and $\boldsymbol{x}$ is in V. The coordinates of $\boldsymbol{x}$ relative to the basis B (or the B-coordinate of $\boldsymbol{x}$) are the weights $c_1, \cdots, c_n$ such that $\boldsymbol{x} = c_1\boldsymbol{b}_1 + \cdots + c_n\boldsymbol{b}_n$

If $c_1, \cdots, c_n$ are the B-coordinate of $\boldsymbol{x}$, then the vector in R^{n}

$$[\boldsymbol{x}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \cdots \\ c_n \end{bmatrix}$$

is the coordinate vector of $\boldsymbol{x}$, or the B-coordinate vector of $\boldsymbol{x}$

The mapping $\boldsymbol{x} \rightarrow [\boldsymbol{x}]_B$ is the coordinate mapping(determined by B)

ex) Consider a basis $B = \{\boldsymbol{b}_1, \boldsymbol{b}_2\}$ for R^2, where $\boldsymbol{b}_1 = \begin{bmatrix}1\\0\end{bmatrix}$, $\boldsymbol{b}_2 = \begin{bmatrix}1\\2\end{bmatrix}$

Suppose an $\boldsymbol{x}$ in R^2 has the coordinate vector $[\boldsymbol{x}]_B = \begin{bmatrix}-2\\3\end{bmatrix}$

ex) The entries in the vector $\boldsymbol{x} = \begin{bmatrix}1\\6\end{bmatrix}$ are the coordinates of $\boldsymbol{x}$ relative to the standard basis $\epsilon = \{\boldsymbol{e}_1, \boldsymbol{e}_2\}$

$$\boldsymbol{x} = \begin{bmatrix}1\\6\end{bmatrix} = 1\boldsymbol{e}_1 + 6\boldsymbol{e}_2$$

$$[\boldsymbol{x}]_\epsilon = \boldsymbol{x}$$

(3) Coordinates in R^n

When a basis B for R^n is fixed, the B-coordinate vector of a specified $\boldsymbol{x}$ is easily found

ex) $\boldsymbol{b}_1 = \begin{bmatrix}1\\0\end{bmatrix}$, $\boldsymbol{b}_2 = \begin{bmatrix}1\\2\end{bmatrix}$, $B = \{\boldsymbol{b}_1, \boldsymbol{b}_2\}$, $\boldsymbol{x} = \begin{bmatrix}3\\-2\end{bmatrix}$, find the coordinate vector $[\boldsymbol{x}]_B$ of $\boldsymbol{x}$ relative to B

The matrix in $\begin{bmatrix}1 & 1\\0 & 2\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}3\\-2\end{bmatrix}$ changes the B-coordinates of a vector $\boldsymbol{x}$ into the standard coordinates for $\boldsymbol{x}$

An analogous change of coordinates can be carried out in R^{n} for a basis $B=\{\boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_n\}$

Let

$P_B = [\boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_n]$

Then the vector equation
$\boldsymbol{x} = c_1 \boldsymbol{b}_1 + \cdots + c_n \boldsymbol{b}_n$
is equivalent to $\boldsymbol{x} = P_B [\boldsymbol{x}]_B$

P_B : change of coordinates matrix from B to the standard basis in R^{n}

Left multiplication by P_B transforms the coordinate vector $[\boldsymbol{x}]_B$ into $\boldsymbol{x}$

Since the columns of P_B forms a basis for R^{n}, P_B is invertible

$[\boldsymbol{x}]_B = {P_B}^{-1} \boldsymbol{x}$

The correspondence $\boldsymbol{x} \rightarrow [\boldsymbol{x}]_B$, produced by ${P_B}^{-1}$, is the coordinate mapping.

Since ${P_B}^{-1}$ is an invertible matrix, the coordinate mapping is a one-to-one linear operator.

(4) The Coordinate Mapping

Theorem

Let $B=\{\boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_n\}$ be basis for a vector space V. Then the coordinate mapping $\boldsymbol{x} \rightarrow [\boldsymbol{x}]_B$ is a one-to-one linear transformation from V onto R^{n}

The linearity of the coordinate mapping extends to linear combinations.

If $\boldsymbol{u}_1, \cdots, \boldsymbol{u}_p$ are in V and if $c_1, c_2, \cdots, c_p$ are scalars, then

$$[c_1\boldsymbol{u}_1 + \cdots + c_p\boldsymbol{u}_p]_B = c_1[\boldsymbol{u}_1]_B + \cdots + c_p[\boldsymbol{u}_p]_B$$

2) Change of basis

(1) change of basis

ex) $x = 3b_1 + b_2$, $x = 6c_1 + 4c_2$

$[x]_B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $[x]_C = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$

if $b_1 = 4c_1 + c_2$ and $b_2 = -6c_1 + c_2$

$[x]_B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $[x]_C$?

Theorem

Let $B = \{b_1, b_2, \cdots, b_n\}$ and $C = \{c_1, \cdots, c_n\}$ be bases of a vector space V. Then there is a unique $n \times n$ matrix $P_{C \leftarrow B}$ such that

$[x]_C = P_{C \leftarrow B}[x]_B$

The columns of $P_{C \leftarrow B}$ are the C - coordinate vectors of the vectors in the basis B

$P_{C \leftarrow B} = [\ [b_1]_C \ \ [b_2]_C \ \ \cdots \ \ [b_n]_C\ ]$

The matrix $P_{C \leftarrow B}$ is called the change-of-coordinates matrix from B to C.
Multiplication by $P_{C \leftarrow B}$ converts B-coordinates into C-coordinates.

The columns of $P_{C\leftarrow B}$ are linearly independent because they are the coordinate vectors of the linearly independent set B

Since $P_{C\leftarrow B}$ is square, it must be invertible, by the invertible matrix theorem

$[\boldsymbol{x}]_B = {P_{C\leftarrow B}}^{-1}[\boldsymbol{x}]_C$

${P_{C\leftarrow B}}^{-1}$ is the matrix that converts C-coordinates into B-coordinates

${P_{C\leftarrow B}}^{-1} = P_{B\leftarrow C}$

If $C = \epsilon = \{e_1, \cdots, e_n\}$

$P_{\epsilon\leftarrow B} = P_B = [\boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_n]$

ex) $\mathbf{b}_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$, $\mathbf{c}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$,

$B = \{\mathbf{b}_1, \mathbf{b}_2\}$, $C = \{\mathbf{c}_1, \mathbf{c}_2\}$ be the basis for $\mathbb{R}^2$.

Find the change-of-coordinates matrix from B to C

Find the change-of-coordinates matrix from C to B