Last name(s) **Name** **ID**

Midterm EDA exam   Length: 2.5 hours   19/04/2018

- *The exam has 4 sheets, 8 sides and 4 problems.*
- *Write your full name and ID on every sheet.*
- *Write your answers to all problems in the exam sheets within the reserved space.*
- *Unless otherwise indicated, all your answers must be justified.*

Problem 1 (1.5 points)

(a) (0.5 pts.) Consider the functions $f(n) = n^{n^2}$ and $g(n) = 2^{2^n}$.

Which of the two functions grows asymptotically faster?

Justification:

Note: Read n^{n^2} as $n^{(n^2)}$, and similarly 2^{2^n} as $2^{(2^n)}$.

(b) (0.5 pts.) Assume that the inputs of size n of a certain algorithm are of the following types:

- Type 1: for each input of this type, the algorithm takes time $\Theta(n^4)$. Moreover, the probability that the input is of this type is $\frac{1}{n^3}$.
- Type 2: for each input of this type, the algorithm takes time $\Theta(n^3)$. Moreover, the probability that the input is of this type is $\frac{1}{n}$.
- Type 3: for each input of this type, the algorithm takes time $\Theta(n)$. Moreover, the probability that the input is of this type is $1 - \frac{1}{n^3} - \frac{1}{n}$.

Then the cost of the algorithm in the average case is $\boxed{}$.

Justification:

(c) (0.5 pts.) Solve the recurrence $T(n) = 2T(n/4) + \Theta(\sqrt{n})$.

Answer: $T(n) = \Theta(\boxed{})$.

Justification:

Last name(s) **Name** **ID**

Problem 2 **(2 points)**

In this problem we take n as a constant. Given a vector $x_0 \in \mathbb{Z}^n$ and a square matrix $A \in \mathbb{Z}^{n\times n}$, we define the sequence $x(0), x(1), x(2), \ldots$ as:

$$x(k) = \begin{cases} x_0 & \text{if } k = 0 \\ A \cdot x(k-1) & \text{if } k > 0 \end{cases}$$

For example, for $n = 3$, if

$$x_0 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad \text{and} \qquad A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

we have that

$$x(0) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \qquad x(1) = \begin{pmatrix} 6 \\ 1 \\ 2 \end{pmatrix}, \qquad x(2) = \begin{pmatrix} 9 \\ 6 \\ 1 \end{pmatrix}, \qquad \ldots$$

(a) (1 pt.) Show by induction that $x(k) = A^k \cdot x_0$ for all $k \geq 0$.

(b) (1 pt.) Explain at a high level how you would implement a function

$$\textit{vector}<\textbf{int}>\ \textit{k\_th}(\textbf{const}\ \textit{vector}<\textit{vector}<\textbf{int}\gg\&\ A,\ \textbf{const}\ \textit{vector}<\textbf{int}>\&\ \textit{x0},\ \textbf{int}\ k);$$

for computing the k-th term $x(k)$ of the sequence defined by the matrix A and the vector *x0*. Analyse also the cost in time as a function of k. The cost must be better than $\Theta(k)$.

Last name(s) **Name** **ID**

Problem 3 **(3.5 points)**

Given a natural number x, let us consider its representation in base 3. Assume that the number of digits n in this representation is a power of 3. We split the sequence of digits in three parts of the same size x_2, x_1 and x_0, corresponding to the highest, middle and lowest digits of x.

For example, if $x = 102111212_3$, then $x_2 = 102_3$, $x_1 = 111_3$ and $x_0 = 212_3$.

(a) (0.5 pts.) Express x in terms of x_2, x_1 and x_0. No justification is needed.

$x =$ ______

(b) (1 pt.) Let y be another natural number with n digits in its representation in base 3, and let us define y_2, y_1 and y_0 analogously. Express $x \cdot y$ in terms of x_2, x_1, x_0, y_2, y_1 and y_0. No justification is needed.

$$
\begin{aligned}
x \cdot y \quad = \quad & \boxed{} \cdot 3^{4n/3} \\
+ \; & \boxed{} \cdot 3^{3n/3} \\
+ \; & \boxed{} \cdot 3^{2n/3} \\
+ \; & \boxed{} \cdot 3^{n/3} \\
+ \; & \boxed{}
\end{aligned}
$$

(c) (1 pt.) Express $x \cdot y$ in terms of x_2, x_1, x_0, y_2, y_1 and y_0 in such a way that strictly less than 9 products are introduced. No justification is needed.

$$
\begin{aligned}
x \cdot y \quad = \quad & \boxed{} \cdot 3^{4n/3} \\
+ \; & \boxed{} \cdot 3^{3n/3} \\
+ \; & \boxed{} \cdot 3^{2n/3} \\
+ \; & \boxed{} \cdot 3^{n/3} \\
+ \; & \boxed{}
\end{aligned}
$$

Hint: Consider $(x_0 + x_1 + x_2) \cdot (y_0 + y_1 + y_2)$.

(d) (1 pt.) Describe at a high level an algorithm for computing the product of two given natural numbers represented in base 3 with a number of digits that is a power of 3. Analyse the asymptotic cost in time.

Last name(s) **Name** **ID**

Problem 4 **(3 points)**

(a) (1 pt.) Consider a recurrence of the form

$$T(n) = T(n/b) + \Theta(\log^k n)$$

where $b > 1$ and $k \geq 0$. Show that its solution is $T(n) = \Theta(\log^{k+1} n)$.

Justification:

Note: $\log^k n$ is short for $(\log(n))^k$.

Hint: do a change of variable.

(b) (1 pt.) Given an $n \geq 1$, a sequence of n integers $a_0, \ldots, a_{n-1}$ is *unimodal* if there exists t with $0 \leq t < n$ such that $a_0 < \ldots < a_{t-1} < a_t$ and $a_t > a_{t+1} > \ldots > a_{n-1}$. The element a_t is called the *top* of the sequence. For example, the sequence $1, 3, 5, 9, 4, 1$ is unimodal, and its top is 9 (take $t = 3$).

Fill the gaps in the following code so that the function

bool *search* (**const** *vector* <**int**>& *a*, **int** *x*),

given a non-empty vector a that contains a unimodal sequence and an integer x, returns whether x appears in the sequence or not. No justification is needed.

```
bool search(const vector<int>& a, int x, int l, int r) {
  if ( ______ ) return x == a[l];
  int m = (l+r)/2;
  auto beg = a.begin();
  if (a[m] < ______ )
    return search(a, x, m+1, r) or binary_search(beg + l, beg + ______ , x);
  else
    return search(a, x, l, m) or binary_search(beg + ______ , beg + r + 1, x);
}

bool search(const vector<int>& a, int x) { return search(a, x, 0, ______ ); }
```

Note: Given an element x and iterators *first*, *last* such that the interval $[first, last)$ is sorted (increasingly or decreasingly), the function $binary\_search(first, last, x)$ returns **true** if x appears in the interval $[first, last)$ and **false** otherwise, in logarithmic time in the size of the interval in the worst case.

(c) (1 pt.) Analyse the cost in time in the worst case of a call *search*(a, x), where a is a vector of size $n > 0$ that contains a unimodal sequence and x is an integer. Describe a situation in which this worst case can take place.