Notes on Mathematical Logic By Ian Chiswell and Wilfrid Hodges

Sixuan Thomas Lou

April 4, 2016

Contents

1 Basic Notions

1.1 Statement:

Later formalized with first-order language

1.2 set of statements

1.3 Sequent

$$\Gamma \vdash \phi$$

where Γ is a set of statements (assumptions) ϕ is a statement (conclusion)

The sequent is correct if and only if there is a proof of ϕ with no undischarged assumptions

1.3.1 Sequent rules

- Axiom: If $\phi \in \Gamma$, then $\Gamma \vdash \phi$ is correct
- Transitivity : If $\Delta \vdash \phi$ and for all statements δ in Δ, the sequent $\Gamma \vdash \delta$ is correct, then $\Gamma \vdash \phi$ is correct

1.4 Natural Deduction

Natural Deduction rules tells how to build a <u>proof</u> for sequents, especially called <u>derivations</u>.

Derivations + rules of using them gives <u>natural deduction calculus</u>

1.4.1 Natural Deduction Rules

- Axiom:

Let ϕ be a statement, then

$$\phi$$

is a derivation whose conclusion is ϕ , with one undischarged assumption, namely ϕ

1.5 Simple logical connectives with introduction / elimination rules

1.5.1 And ($\wedge$)

1. Natural Deduction Rules

   - If $\begin{matrix} D \\ \phi \end{matrix}$ and $\begin{matrix} D' \\ \psi \end{matrix}$ are derivations of ϕ and ψ respectively, then

$$\dfrac{\begin{matrix} D \\ \phi \end{matrix} \qquad \begin{matrix} D' \\ \psi \end{matrix}}{\phi \wedge \psi} \, (\wedge I)$$

   is a derivation of $(\phi \wedge \psi)$.

   - If $\begin{matrix} D \\ \phi \wedge \psi \end{matrix}$ is a derivation of $(\phi \wedge \psi)$, then

$$\dfrac{\begin{matrix} D \\ \phi \wedge \psi \end{matrix}}{\phi} \, (\wedge E)$$

$$\dfrac{\begin{matrix} D \\ \phi \wedge \psi \end{matrix}}{\psi} \, (\wedge E)$$

   are derivations of ϕ and ψ respectively.

2. Sequent Rules

   - If $\Gamma \vdash \phi$ and $\Delta \vdash \psi$ are correct sequents, then $\Gamma \cup \Delta \vdash (\phi \wedge \psi)$ is a correct sequent.
   - If $\Gamma \vdash (\phi \wedge \psi)$ is a correct sequent, then $\Gamma \vdash \phi$ and $\Gamma \vdash \psi$ are correct sequents.

1.5.2 Implication ($\to$)

1. Natural Deduction Rules

   - If $\begin{array}{c} D \\ \phi \end{array}$ is a derivation of ψ , ϕ is a statement,

   then

$$\dfrac{\begin{array}{c} \cancel{\phi} \\ D \\ \psi \end{array}}{\phi \to \psi} \,(\to I)$$

   is a derivation of $(\phi \to \psi)$.

   - If $\begin{array}{c} D \\ \phi \end{array}$ and $\begin{array}{c} D' \\ \phi \to \psi \end{array}$ are derivations of ϕ and $\phi \to \psi$ respectively,

   then

$$\dfrac{\begin{array}{c} D \\ \phi \end{array} \qquad \begin{array}{c} D' \\ \phi \to \psi \end{array}}{\psi} \,(\to E)$$

   is a derivation of ψ.

2. Sequent Rules

   - If $\Gamma \cup \{\phi\} \vdash \psi$ is a correct sequent, then $\Gamma \vdash (\phi \to \psi)$ is also correct.
   - If $\Gamma \vdash (\phi \vdash \psi)$ and $\Delta \vdash \phi$ are correct sequents, then so is $\Gamma \cup \Delta \vdash \psi$.

1.5.3 Equivalence ($\leftrightarrow$)

1. Natural Deduction Rules

   - If $\begin{array}{c} D \\ \phi \to \psi \end{array}$ and $\begin{array}{c} D' \\ \psi \to \phi \end{array}$ are derivations of $\phi \to \psi$ and $\psi \to \phi$ respectively,

   then

$$\dfrac{\begin{array}{cc} D & D' \\ \phi \to \psi & \psi \to \phi \end{array}}{\phi \leftrightarrow \psi}\ (\leftrightarrow I)$$

is a derivation of $(\phi \leftrightarrow \psi)$.

- If $\begin{array}{c} D \\ \phi \leftrightarrow \psi \end{array}$ is a derivation of $(\phi \leftrightarrow \psi)$,

then

$$\dfrac{\begin{array}{c} D \\ \phi \leftrightarrow \psi \end{array}}{\phi \to \psi}\ (\leftrightarrow E)$$

$$\dfrac{\begin{array}{c} D \\ \phi \leftrightarrow \psi \end{array}}{\psi \to \phi}\ (\leftrightarrow E)$$

are derivations of $(\phi \to \psi)$ and $(\psi \to \phi)$ respectively.

2. Sequent Rules

- If $\Gamma \vdash \phi \to \psi$ and $\Delta \vdash \psi \to \phi$ are correct sequents, then so is $\Gamma \cup \Delta \vdash \phi \leftrightarrow \psi$
- If $\Gamma \vdash \phi \leftrightarrow \psi$ is a correct sequent, then so is $\Gamma \vdash \phi \to \psi$ and $\Gamma \vdash \psi \to \phi$

1.5.4 Negation ($\neg$)

1. Natural Deduction Rules

- If $\begin{array}{c} D \\ \perp \end{array}$ is a derivation if $\perp$,

then

$$\dfrac{\begin{array}{c} \cancel{\phi} \\ D \\ \perp \end{array}}{(\neg\phi)}\ (\neg I)$$

is a derivation of $(\neg\phi)$.

- If $\begin{matrix} D \\ \phi \end{matrix}$ and $\begin{matrix} D' \\ \neg\phi \end{matrix}$ are derivations of ϕ and $(\neg\phi)$ respectively,

then

$$\dfrac{\begin{matrix} D \\ \phi \end{matrix} \qquad \begin{matrix} D' \\ \neg\phi \end{matrix}}{\perp}\,(\neg E)$$

is a derivation of $\perp$.

- If $\begin{matrix} D \\ \perp \end{matrix}$ is a derivation of $\perp$,

then

$$\dfrac{\begin{matrix} \cancel{(\neg\phi)} \\ D \\ \perp \end{matrix}}{\phi}\,(\text{RAA})$$

is a derivation of ϕ.

2. Sequent Rules

- If $\Gamma \cup \{\phi\} \vdash \perp$ is a correct sequent, then $\Gamma \vdash (\neg\phi)$ is also a correct sequent.
- If $\Gamma \vdash \phi$ and $\Delta \vdash \neg\phi$ are correct sequents, then $\Gamma \cup \Delta \vdash \perp$ is also correct.
- If $\Gamma \cup \{(\neg\phi)\} \vdash \perp$ is a correct sequent, then $\Gamma \vdash \phi$ is also correct.

3. Example. Find natural deduction proofs of the following sequent.

$$\{(\neg(\phi \leftrightarrow \psi))\} \vdash (\neg\phi) \leftrightarrow \psi$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\cancel{\phi}\qquad\cancel{\neg\phi}}{\bot}}{\psi}}{\phi\to\psi}\qquad\dfrac{\dfrac{\dfrac{\cancel{\psi}\qquad\cancel{\neg\psi}}{\bot}}{\phi}}{\psi\to\phi}}{(\phi\to\psi)\wedge(\psi\to\phi)}}{\phi\leftrightarrow\psi}\qquad\neg(\phi\leftrightarrow\psi)}{\bot}}{\psi}}{(\neg\phi)\to\psi}\qquad\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\cancel{\psi}}{\phi\to\psi}\qquad\dfrac{\cancel{\phi}}{\psi\to\phi}}{(\phi\to\psi)\wedge(\psi\to\phi)}}{\phi\leftrightarrow\psi}\qquad\neg(\phi\leftrightarrow\psi)}{\bot}}{\neg\phi}}{\psi\to(\neg\phi)}}{((\neg\phi)\to\psi)\wedge(\psi\to(\neg\phi))}}{(\neg\phi)\leftrightarrow\psi}
$$

1.5.5 Or ($\vee$)

1. Natural Deduction Rules

   - If $\begin{matrix} D \\ \phi \end{matrix}$ is a derivation of ϕ and ψ is a statement, then

   $$\dfrac{\begin{matrix} D \\ \phi \end{matrix}}{\phi \vee \psi}\,(\vee I)$$

   - If $\begin{matrix} D \\ \psi \end{matrix}$ is a derivation of ψ and ϕ is a statement, then

   $$\dfrac{\begin{matrix} D \\ \psi \end{matrix}}{\phi \vee \psi}\,(\vee I)$$

   - If $\begin{matrix} D \\ (\phi \vee \psi) \end{matrix}$, $\begin{matrix} D' \\ \chi \end{matrix}$ and $\begin{matrix} D'' \\ \chi \end{matrix}$ are derivations of $(\phi \vee \psi)$ and χ respectively, then,

   $$\dfrac{\begin{matrix} D \\ (\phi \vee \psi) \end{matrix}\qquad\begin{matrix} \cancel{\phi} \\ D' \\ \chi \end{matrix}\qquad\begin{matrix} \cancel{\phi} \\ D'' \\ \chi \end{matrix}}{\chi}\,(\vee E)$$

2. Sequent Rules

- If $\Gamma \vdash \phi$ is a correct sequent, ψ is a statement, then $\Gamma \vdash (\phi \vee \psi)$ is also a correct sequent.
- If $\Gamma \vdash \psi$ is a correct sequent, ψ is a statement, then $\Gamma \vdash (\phi \vee \psi)$ is also a correct sequent.
- If $\Gamma \cup \{\phi\} \vdash \chi$, $\Delta \cup \{\psi\} \vdash \chi$ are correct sequents, then $\Gamma \cup \Delta \cup \{(\phi \vee \psi)\} \vdash \chi$ is correct.

1.6 First order language