WHAT HAVE I LEARNED TODAY?

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05-23: Today I finished writing up the proof of the Riemann-Roch Theorem (finally!). The theorem states: if X is a compact Riemann surface, $L \in \mathrm{Pic}(X)$, then $\chi(X, L) = \deg(L) + \chi(X, \mathcal{O}_X)$. The proof uses (1) there's a positive line bundle $\mathcal{O}_X$ on X, (2) the ability to make $L \otimes \mathcal{O}_X(n)$ and $\mathcal{O}_X(n)$ bpf for all large n (this uses Kodaira vanishing and Hodge Theorem) (3) Bertini's Theorem to find transverse sections of bpf line bundles (which depends on Sard's Theorem), (4) ideal sheaf sequence associated to some smooth hypersurface, (5) the fact that we can compute cohomolgies of $i_* i^{-1} \mathcal{L}$ and $j_* \mathcal{O}_X$ easily by taking open cover that covers each point of the discrete set of points (Leray's Theorem, flasque sheaves are acyclic), and (6) $c_1(L)$ is Poincaré dual to the zero locus of some transverse section of L.

If we interpret $c_1(\omega_X^\vee)$ as the Gauss curvature on X, then the Gauss-Bonnet Theorem $\int_X c_1(\omega_X^\vee) = 2 - 2g$ becomes a easy consequence of the Riemann-Roch Theorem and Serre Duality.

05-22: Today I learned the proof of the Bertini's Theorem. The theorem says, if $L \in \mathrm{Pic}(X)$, and $V \subseteq \Gamma(X, L)$ a bpf linear system, then the set of elements of V that are not transverse to the zero section has measure zero. The main tool is the Sard's Theorem. We consider the evaluation map $\epsilon : X \times V \to L$, show it's a submersion, and then $\epsilon^{-1}(Z) \subseteq X \times V$ is an embedded submanifold. Then apply Sard's Theorem to $f : \epsilon^{-1}(Z) \hookrightarrow X \times V \xrightarrow{\mathrm{pr}_2} V$. A section s is a regular value of f if and only if it's transverse to the zero section. To prove this fact we consider the following diagram, by a diagram chase we show ds_x is surjective iff $df_{(x,s)}$ is surjective.

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & T_{(x,s)}(\epsilon^{-1}(Z)) \arrow[d, hook] \arrow[dr, "df_{(x,s)}"] & & \\
0 \arrow[r] & T_x X \arrow[r, hook] \arrow[dr, "ds_x"'] & T_{(x,s)}(X \times V) \arrow[r, two heads] \arrow[d, two heads] & T_s V \arrow[r] & 0 \\
 & & N_{(x,s)}(\epsilon^{-1}(Z)) \cong N_{s(x)} Z \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

I have also learned an important technique of showing certain maps are surjective via sheaf cohomology. One realization of this idea is: if X is a compact Riemann surface, $\mathcal{O}_X(1)$ is a positive line budle, then for any $L \in \mathrm{Pic}(X)$, there exists $N \in \mathbb{N}$ such that $L \otimes \mathcal{O}_X(n)$ is bpf ($n > N$). We reduce the problem of showing $L \otimes \mathcal{O}_X(n)$ is bpf to showing the s.e.s of sheaves of $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathcal{L} \otimes \mathcal{O}_X(n) \mathcal{I}_x \longrightarrow \mathcal{L} \otimes \mathcal{O}_X(n) \longrightarrow i_*((L \otimes \mathcal{O}_X(n))_x) \longrightarrow 0$$

is exact on global sections. Prove exactness by showing $H^1(X, L\otimes\mathcal{O}_X(n)\otimes\mathcal{I}_x) = 0$ using Kodaira vanishing, because we can make $\omega_X^\vee\otimes L\otimes\mathcal{I}_x\otimes\mathcal{O}_X(n)$ positive for large n.

05-21: Today I learned the degree theory of smooth maps between compact, oriented n-manifolds. We first investigate the case of proper maps between $\mathbb{R}^n$. Let $f:\mathbb{R}^n\to\mathbb{R}^n$ be a proper map, the pullback $f^*: H^n_{dR}(\mathbb{R}^n)\to H^n_{dR}(\mathbb{R}^n)$ maps compactly supported n-forms to compactly supported forms, hence $f^*: H^n_c(\mathbb{R}^n)\to H^n_c(\mathbb{R}^n)$. Let ω be a generator of $H^n_c(\mathbb{R}^n)$ (this means $\int_{\mathbb{R}^n}\omega = 1$, this is possible by the Poincaré Lemma that $H^n_c(\mathbb{R}^n)\cong\mathbb{R}$), we define the degree $\deg(f) := \int_M f^*\omega$. To prove this is an integer, pick a regular value $q\in\mathbb{R}^n$ of f by Sard's Theorem. Since f is proper, $f^{-1}(q)$ is a finite set of points $\{p_1,\dots,p_K\}$. Since q is a regular value, the map f is locally a diffeomorphism when restricted to small neighborhoods V_i of each p_i such that V_i are disjoint and $f(V_i) = W$ for all i. By partition of unity we may pick a generator ω of $H^n_c(\mathbb{R}^n)$, with $\operatorname{Supp}\omega\subseteq W$. Then $\int_M f^*\omega = \sum_i\int_{V_i} f^*\omega$ and each $\int_{V_i} f^*\omega$ is $+1$ if df_{p_i} is orientation preserving and -1 if df_{p_i} is orientation reversing. To prove the case of smooth maps between compact, oriented n-manifolds, we assume the fact that $H^n(M)\cong\mathbb{R}$ as well. This fact could be proved from Poincaré Duality.

References