Commutative Algebra

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Contents

0 Overview

This document is a recollection of things I have learned from the course in commutative algebra given by Professor Rankeya Datta at UIC. The course gives an overview of important concepts in commutative algebra, especially in dimension theory. I have tried not to include full proofs, instead I have included detailed hints for interested readers to work the proofs out themselves.

This note is not an accurate representation of the lectures, the official course notes are posted on https://rankeya.people.uic.edu/520f20.html. Please send comments and corrections to `tmzl dot sx at gmail`.

1 Dimension Theory

1.1 Filtered rings and modules

Definition 1.1 (Filtration)**.** Filtration on a ring R is a collection of ideals $\{I_n\}_{n\geq 0}$:

$$R = I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots, \quad I_n I_m \subseteq I_{n+m}$$

Filtration on an $(R, \{I_n\}_{n\geq 0})$-module M is a collection of submodules $\{M_n\}_{n\geq \mathbb{Z}}$:

$$\cdots \supseteq M_{n-1} \supseteq M_n \supseteq M_{n+1} \supseteq \cdots, \quad I_n M_m \subseteq M_{n+m}, \quad \bigcup_{n\in\mathbb{Z}} M_n = M.$$

***Example* 1.2** (*I*-adic filtration)**.** $(R, \{I^n\}_{n\geq 0})$, $(M, \{I^n M\}_{n\in\mathbb{Z}})$ ($I^n := R$ for $n < 0$).

Definition 1.3 (Induced filtration, quotient filtration)**.**

$$(M, \{M_n\}_{n\in\mathbb{Z}}) \rightsquigarrow (N, \{N \cap M_n\}_{n\in\mathbb{Z}}), \quad (M/N, \left\{\frac{N + M_n}{N}\right\}_{n\in\mathbb{Z}})$$

Definition 1.4 (Filtered homomorphism)**.** $\varphi(M_n) \subseteq N_n$.

Definition 1.5 (Topological abelian group)**.** The local topology at a point $a \in A$ is determined by the local topology at 0.

***Example* 1.6** (filtration topology)**.** If $(R, \{I_n\}_{n\geq 0})$, $(M, \{M_n\}_{n\in\mathbb{Z}})$ filtered, then $\{I_n\}_{n\geq 0}$ is a neighborhood basis of $0 \in R$ and $\{M_n\}_{n\in\mathbb{Z}}$ is a neighborhood basis of $0 \in M$.

Lemma 1.7.

$$(R, \{I_n\}_{n\geq 0}) \text{ Hausdorff in the filtration topology} \iff \bigcap_{n\geq 0} I_n = 0$$

$$(M, \{M_n\}_{n\geq 0}) \text{ Hausdorff in the filtration topology} \iff \bigcap_{n\in\mathbb{Z}} M_n = 0$$

Lemma 1.8. Two filtrations $\{M_n\}_{n\in\mathbb{Z}}, \{M'_n\}_{n\in\mathbb{Z}}$ on M gives the same topology if and only if

$$^\forall n_1, n_2 \in \mathbb{Z}, \ ^\exists m_1, m_2 \in \mathbb{Z} \text{ s.t. } M'_{m_1} \subseteq M_{n_1} \text{ and } M_{m_2} \subseteq M'_{n_2}$$

Corollary 1.9. Let $I, J \lhd R$.

$$I\text{-adic topology} = J\text{-adic topology} \overset{\star}{\iff} \sqrt{I} = \sqrt{J}$$

where $\Leftarrow$ holds when I, J are f.g.

***Problem* 1.10.** Let $N \subseteq M$, and $I \lhd R$, compare filtration topologies of $\{I^n N\}_{n\in\mathbb{Z}}$ and $\{N \cap I^n M\}_{n\in\mathbb{Z}}$ on N. Evidently $I^n N \subseteq N \cap I^n M$, so the first is finer than then second. The converse will be established by Artin-Rees Lemma 1.11.

Lemma 1.11 (Artin-Rees)**.** Let R be Noetherian, $I \lhd R$, M f.g. over R, $N \subseteq M$. Then

$$^\exists k \in \mathbb{Z}_{\geq 0} \text{ s.t. } \forall c \in \mathbb{Z}_{\geq 0}, (I^{k+c} M) \cap N = I^c (I^k M \cap N).$$

Hint. WTS: $\{I^n M \cap N\}_{n\geq 0}$ is I-stable, by 1.16 suffices to show $\oplus I^n M \cap N$ f.g. over $R[tI]$, but it is a $R[tI]$-submodule of $\oplus I^n M$ and $\oplus I^n M$ is f.g. as $\{I^n M\}_{n\geq 0}$ is I-stable, M f.g. and $R[tI]$ is noetherian.

Corollary 1.12 (Krull's Intersection Theorem)**.** Let R be Noetherian, $I \lhd R$ and M f.g. over R. If $I \subseteq Jac(R)$, then

$$\bigcap_{n\geq 0} I^n M = 0 \quad \text{i.e. } M \text{ is Hausdorff in } I\text{-adic topology.}$$

In particular, if $(R, \mathfrak{m})$ is Noetherian local, then $\cap_{n\geq 0}\mathfrak{m}^n = 0$.

Hint. Consider $N := \cap_{n\geq 0} I^n M$, use Artin-Rees, and NAK.

Definition 1.13 (I-stable module)**.** Let $(R, \{I^n\}_{n\geq 0})$ be filtered, a filtered R-module $(M, \{M_n\}_{n\in\mathbb{Z}})$ is I**-stable** (or I-good) if

$$^{\exists}k \in \mathbb{Z} \text{ s.t. } \forall n \geq 0, I^n M_k = M_{k+n}.$$

Definition 1.14 (Rees algebra)**.** Let $I \lhd R$, the **Rees algebra** (blowup algebra) of R wrt I is

$$R[tI] := \bigoplus_{n\geq 0} t^n I^n$$

Lemma 1.15. Let R be Noetherian, then $R[tI]$ is Noetherian as well.

Hint. The ideal I is f.g., so there is a surjection $R[x_1, \ldots, x_n] \twoheadrightarrow R[tI]$.

Proposition 1.16. Let R be Noetherian, M f.g. over R. Suppose $(M, \{M_n\}_{n\in\mathbb{Z}})$ is a filtered $(R, \{I^n\}_{n\geq 0})$-module, then

$$(M, \{M_n\}_{n\in\mathbb{Z}}) \text{ is } I\text{-stable} \iff \bigoplus_{n\geq 0} M_n \text{ is a f.g. } R[tI]\text{-module.}$$

Hint. I-stability gives $k \in \mathbb{Z}$ such that $I^n M_k = M_{k+n}$, then $\oplus_{n\geq 0} M_n$ is generated by $\oplus_{n=0}^k M_n$ over $R[tI]$, and $\oplus_{n=0}^k M_n$ is finitely generated because it is noetherian over R, and R is Noetherian. Conversely, pick $k \geq$ degrees of finitely generators of $\oplus_{n\geq 0} M_n$ suffices (by degree reason).

1.2 Graded rings and modules

Definition 1.17 (graded ring, graded module)**.** $R = \oplus_{n\geq 0} R_n$, $M = \oplus_{n\in\mathbb{Z}} M_n$.

***Example* 1.18.**

1. $R = k[x_1, \ldots, x_n]$.
2. Filtration $(R, \{I_n\}_{n\geq 0})$, $(M, \{M_n\}_{n\in\mathbb{Z}}) \rightsquigarrow$ gradation $R_\bullet := \oplus_{n\geq 0} I_n$, $M_\bullet := \oplus_{n\in\mathbb{Z}} M_n$.
3. Rees algebra $R[tI]$.

Lemma 1.19. Let $R = \oplus_{n\geq 0} R_n$ be a graded ring.

1. R_0 is a subring of R
2. $R_+ := \oplus_{n\geq 1} R_n$ is an ideal (irrelevant ideal) and $R_0 \cong R/R_+$.

Lemma 1.20. Let I be an ideal of a graded ring R, TFAE:

1. $\forall a \in I$, all homogeneous components of a are in I.

2. $I = \oplus_{n\geq 0} I \cap R_n$
3. I is generated by homogeneous elements.

Say I is **homogeneous**.

Lemma 1.21. Let $R = \oplus_{n\geq 0} R_n$ be a graded ring, TFAE:

1. R is Noetherian
2. R_0 is Noetherian and R_+ is f.g.

Hint. ($\Rightarrow$): R_0 is a quotient, R_+ is a submodule. ($\Leftarrow$): let $r_1, \ldots, r_n \in R^+$ be homogeneous generators, then show by induction on degree that $R = R' := R_0[r_1, \ldots, r_n]$.

Lemma 1.22. Let $R = \oplus_{n\geq 0} R_n$ be a graded ring and $M = \oplus_{n\in\mathbb{Z}} M_n$ a graded, Noetherian R-module. Then M_n is f.g. over R_0 ($\forall n \in \mathbb{Z}$).

Hint. $M_n = \oplus_{n\geq k} M_n / \oplus_{n\geq k+1} M_n$ as an $R_0 = R/R_+$-module.

Corollary 1.23. If $M = \oplus_{n\in\mathbb{Z}} M_n$ is a Noetherian $R = \oplus_{n\geq 0} R_n$-module and R_0 is Artinian, then $\ell_{R_0}(M_n) < \infty$ ($\forall n \in \mathbb{Z}$).

Hint. f.g. over Artin ring $\Rightarrow$ Artin + Noeth $\Rightarrow$ finite length.

Lemma 1.24 (Graded NAK)**.** Let $M = \oplus_{n\in\mathbb{Z}} M_n$ be a f.g. graded $R = \oplus_{n\geq 0} R_n$-module. If $M = R_+ M$, then $M = 0$.

Hint. Pick homogeneous generators $x_1, \ldots, x_r \in M$, and $d = \deg(x_1)$ has minimal degree. Then $x \in M_d \subseteq R_+ M = R_+ \oplus_{n\geq d} M_n = \oplus_{n\geq d+1} M_n$, a contradiction.

1.3 Numerical functions, polynomial like functions

Definition 1.25. Numerical function: $f : \mathbb{Z} \to \mathbb{Q}$.

Definition 1.26 (Difference operator)**.** $\Delta f(n) = f(n+1) - (fn)$.

***Example* 1.27** (Binomial polynomials)**.**

$$Q_k(x) = \frac{x(x-1)\cdots(x-k+1)}{k!}$$

Then:

1. $Q_k(n) = \binom{n}{k}$
2. $\Delta Q_{k+1} = Qk$
3. $\{Q_k(x) : k \in \mathbb{Z}_{\geq 0}\}$ is a $\mathbb{Q}$-basis of $\mathbb{Q}[x]$.

Definition 1.28. A numerical function $f : \mathbb{Z} \to \mathbb{Q}$ is **polynomial like** if

$$^{\exists} P_f(x) \in \mathbb{Q}[x] \text{ s.t. } f(n) = P_f(n) \quad \forall n \gg 0.$$

In this case say $P_f(x)$ is associated to f, and define $\deg f := \deg P_f$.
By Example 1.27, we can write

$$P_f(x) = a_d Q_d(x) + \cdots + a_1 Q_1(x) + a_0 Q_0(x) \quad a_i \in \mathbb{Q}$$

Define $\operatorname{mult} f := a_d$.

Lemma 1.29. Let $f : \mathbb{Z} \to \mathbb{Q}$ be a numerical function.

1. $$\Delta f(n) = 0 \ (\forall n \gg 0) \iff \begin{array}{l} f \text{ is polynomial like with associated} \\ \text{polynomial a constant} \end{array}$$

2. $$f \text{ is polynomial like} \iff \Delta f \text{ is polynomial like}$$

   In this case $\Delta P_f = P_{\Delta f}$. If $\deg \Delta f \geq 0$, then $\deg f = 1 + \deg \Delta f$ and $\operatorname{mult} f = \operatorname{mult} \Delta f$.

Hint. (1) is evident. For (2), ($\Rightarrow$) is evident, and for ($\Leftarrow$) write $P_{\Delta f}$ as a linear combination of Q_k's, and use Example 1.27 + (1).

Definition 1.30. A polynomial like function $f : \mathbb{Z} \to \mathbb{Q}$ is non-negative if $f(n) \geq 0$ ($\forall n \gg 0$); is integer-valued if $f(n) \in \mathbb{Z}$ ($\forall n \gg 0$).

Lemma 1.31. Let $f : \mathbb{Z} \to \mathbb{Q}$ be a polynomial like with associated polynomial

$$P_f(x) = a_d Q_d(x) + \cdots + a_1 Q_1(x) + a_0 Q_0(x) \quad a_i \in \mathbb{Q}\,.$$

TFAE:

1. f is integer-valued
2. $P_f(n) \in \mathbb{Z}$ ($\forall n \gg 0$)
3. $P_f(n) \in \mathbb{Z}$ ($\forall n \in \mathbb{Z}$)
4. $a_0, \ldots, a_n \in \mathbb{Z}$.

Hint. (1) $\Rightarrow$ (2) clear, (2) $\Rightarrow$ (1) because Q_k are integer valued. For (2) $\Rightarrow$ (4), induct on the degree of $P_f(x)$, establish the inductive step using Lemma 1.29.

Corollary 1.32. If f is an integer-valued polynomial like numerical function, then $\operatorname{mult} f \in \mathbb{Z}$. Moreover, if f is non-negative, then $\operatorname{mult} f \geq 0$, with $\operatorname{mult} f = 0 \Leftrightarrow f(n) = 0\, \forall n \gg 0$.

Lemma 1.33. If f is polynomial like, then

$$\operatorname{mult} f = (\deg f)! \times \text{leading coefficient of } P_f = \lim_{n\to\infty} \frac{f(n)}{n^{\deg f}/(\deg f)!} = \lim_{n\to\infty} \frac{f(n+c)}{n^{\deg f}/(\deg f)!}$$

for any fixed $c \in \mathbb{Z}$.

Lemma 1.34. Let f_1, f_2 be non-negative polynomial like numerical functions. Let $a, b, c, d, e, f \in \mathbb{Z}$ with $a, c > 0$. Then

1. $f_1(an + b) \geq f_2(cn + d)$ ($\forall n \gg 0$) $\implies$ $\deg f_1 \geq \deg f_2$.
2. $f_1(n + d) \geq f_2(n + d) \geq f_1(n + c)$ ($\forall n \gg 0$) $\implies$ $\deg f_1 = \deg f_2$, $\operatorname{mult} f_1 = \operatorname{mult} f_2$.

Hint. Lemma 1.33.

1.4 Hilbert-Samuel Function

In this section R is assumed to be noetherian.

Definition 1.35. Let $R = \oplus_{n\geq 0} R_n$ be a noetherian graded ring, R_0 Artinian. Let $M = \oplus_{n\in\mathbb{Z}} M_n$ be a f.g. graded R-module. Then $\ell_{R_0}(M_n) < \infty$ ($\forall n \in \mathbb{Z}$) (1.23). Define the **Hilbert function of** M:

$$H_M(n) := \ell_{R_0}(M_n), \quad n \in \mathbb{Z}\,.$$

***Example* 1.36.** $R = k[x_1, \dots, x_n]$, then

$$H_R(m) = \binom{m+n-1}{n-1}$$

is polynomial like of degree $n-1$.

Theorem 1.37. Let $R = \oplus_{n\geq 0} R_n$ be graded, R_0 Artinian. Suppose R_+ is f.g. by m elements $r_1, \dots, r_m$ in R_1. Then for any f.g. graded R-module M, the Hilbert function H_M is polynomial like of degree $\leq m-1$.

Hint. Induct on the number of generators m (zero polynomial has degree $-\infty$) . Establish the inductive step by considering exact sequence

$$0 \longrightarrow K_n \longrightarrow M_n \xrightarrow{r_m} M_{n+1} \longrightarrow C_{n+1} \longrightarrow 0$$

note K, C are graded $R/(r_m)$-module. By additivity of length, we have

$$\Delta H_M(n) = H_C(n+1) - H_K(n).$$

Definition 1.38 (Associated graded ring/module)**.** Let R be a Noetherian ring, $I \lhd R$, and M f.g. over R. Consider

$$\mathrm{gr}_I(R) := \bigoplus_{n\geq 0} I^n/I^{n+1}, \quad \mathrm{gr}_I(M) := \bigoplus_{n\geq 0} I^n M/I^{n+1}M.$$

Then by Lemma 1.21 $\mathrm{gr}_I(R)$ is Noetherian and $\mathrm{gr}_I(M)$ is f.g. over $\mathrm{gr}_I(R)$.

Definition 1.39 (Hilbert-Samuel function)**.** Let R be a Noetherian ring, $I \lhd R$ with R/I Artinian. Let M be f.g. over R, the **Hilbert-Samule function** of M wrt I is

$$H_{I,M}(n) := \ell_{R/I}(M/I^n M),$$

Another convenient form:

$$\widetilde{H}_{I,M}(n) := \ell_{R/I}(I^n M/I^{n+1}M).$$

By Theorem 1.37 $\widetilde{H}_{I,M}(m)$ is polynomial like with degree $\leq \mu_R(I) - 1$, where $\mu_R(I)$ equals the minimal number of generators of the R-module I.

Lemma 1.40. Let R be Noetherian, M f.g. over R, then

$$\ell_R(M) < \infty \iff R/\operatorname{Ann}_R(M) \text{ is Artinian.}$$

Hint. (⇐) is evident. For (⇒) note the composition series of M is a prime cyclic filtration of M, where each prime is maximal. So every associated prime of M is maximal, hence minimal elements of $\operatorname{Supp} M = \mathbb{V}(\operatorname{Ann}_R M)$ are maximal, hence $R/\operatorname{Ann}_R M$ is Artinian.

Corollary 1.41. Let $I \lhd R$ with R/I Artinian. For any f.g. R-module M, the Hilbert-Samuel function $H_{I,M}(n)$ is polynomial like of degree $\leq \mu_R(I)$. Call the associated polynomial $P_{I,M}(n)$ the **Hilbert-Samuel polynomial**, and call the multiplicity of $H_{I,M}$ the **Hilbert-Samuel multiplicity** of M wrt I, denoted $e_{I,M}$..

Hint. Use Lemma 1.40 to show $H_{I,M}(n) < \infty$ for all n. Then consider short exact sequence

$$0 \longrightarrow M/I^nM \longrightarrow M/I^{n+1}M \longrightarrow I^nM/I^{n+1}M \longrightarrow 0$$

Lemma 1.42. The degree of the Hilbert-Samuel function $H_{I,M}$ is impervious to taking radicals. Let $I, J \lhd R$ with $\sqrt{I} = \sqrt{J}$ and R/I Artinian (equivalently R/J Artinian), then $\deg H_{I,M} = \deg H_{J,M}$.

Hint. By finite generation of ideals, there exists $s, t > 0$ with $I^s \subseteq J$ and $J^t \subseteq I$. Then compare the two numerical function using Lemma 1.34.

Setup ⊛: $(R, \mathfrak{m})$ Noetherian local, M f.g. over R, $\mathfrak{q}$ an $\mathfrak{m}$-primary ideal of R (i.e. $\sqrt{\mathfrak{q}} = \mathfrak{m}$).

Definition 1.43 (Degree of module)**.** Under Setup ⊛, by Lemma 1.42,

$$\deg M := \deg H_{\mathfrak{m},M} = \deg H_{\mathfrak{q},M}.$$

Corollary 1.44. Under Setup ⊛, $\deg M \leq \mu_R(\mathfrak{m}) = \dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2)$.

Hint. NAK.

Proposition 1.45. Under Setup ⊛, if M' is a quotient of M, then $\deg M' \leq \deg M$.

Hint. If $0 \to N \to M \to M' \to 0$, then by right exactness of $\otimes R/\mathfrak{m}^n$, we have $M/\mathfrak{m}^nM \twoheadrightarrow M'/\mathfrak{m}^nM'$, hence $H_{\mathfrak{m},M'}(n) \leq H_{\mathfrak{m},M}(n)$ for all $n \geq 0$.

***Remark* 1.46.** The failure of left exactness of $\otimes R/\mathfrak{m}^n$ prevent us from conclude immediately $\deg N \leq \deg M$. To address this problem, we use Artin-Rees to compare filtrations $\{\mathfrak{m}^nN\}_{n\in\mathbb{Z}}$ and $\{N \cap \mathfrak{m}^nM\}_{n\in\mathbb{Z}}$. This motivates the following result.

Proposition 1.47. In the Setup ⊛, given SES $0 \to N \to M \to P \to 0$. Define

$$H := H_{\mathfrak{q},N} + H_{\mathfrak{q},P} - H_{\mathfrak{q},M}.$$

Then $H(n) \geq 0$ for all $n \geq 0$, and $\deg H < \deg H_{\mathfrak{q},M} = \deg M$.

Hint. Important ses's

$$0 \longrightarrow \frac{N}{\mathfrak{q}^nM \cap N} \longrightarrow \frac{M}{\mathfrak{q}^nM} \longrightarrow \frac{P}{\mathfrak{q}^nP} \longrightarrow 0$$

$$0 \longrightarrow \frac{\mathfrak{q}^nM \cap N}{\mathfrak{q}^nN} \longrightarrow \frac{N}{\mathfrak{q}^nN} \longrightarrow \frac{N}{\mathfrak{q}^nM \cap N} \longrightarrow 0$$

Let $G(n) := \ell_{R/\mathfrak{q}}(\frac{N}{\mathfrak{q}^n M \cap N})$. Then

$$H_{\mathfrak{q},M}(n) - H_{\mathfrak{q},P}(n) = G(n) = H_{\mathfrak{q},N} - \ell_{R/\mathfrak{q}}\left(\frac{\mathfrak{q}^n M \cap N}{\mathfrak{q}^n N}\right) \Rightarrow H(n) = \ell_{R/\mathfrak{q}}\left(\frac{\mathfrak{q}^n M \cap N}{\mathfrak{q}^n N}\right) \geq 0.$$

By Artin-Rees, $^\exists k \geq 0, ^\forall c \geq 0, I^{k+c} M \cap N = I^c(I^k M \cap N)$. Hence for $n \gg 0$, we have $\mathfrak{q}^n M \cap N = \mathfrak{q}^{n-k}(\mathfrak{q}^k M \cap N) \subseteq \mathfrak{q}^{n-k} N$, thus

$$\ell_{R/\mathfrak{q}}\left(\frac{\mathfrak{q}^n M \cap N}{\mathfrak{q}^n N}\right) \leq \ell_{R/\mathfrak{q}}\left(\frac{\mathfrak{q}^{n-k} N}{\mathfrak{q}^n N}\right) = H_{\mathfrak{q},N}(n-k) - H_{\mathfrak{q},N}(n).$$

Then we have

$$\deg H < \deg H_{\mathfrak{q},N} \implies \deg H < \deg G = \deg H_{\mathfrak{q},N} \leq \deg_{\mathfrak{q},M} = \deg M.$$

Corollary 1.48. For $0 \to N \to M \to P \to 0$, we have $\deg M = \max\{\deg N, \deg P\}$.

Hint. $H_{\mathfrak{q},M} = H_{\mathfrak{q},N} + H_{\mathfrak{q},P} - H$ with $\deg H < \deg M$.

Definition 1.49. For an R-module M, define

$$\dim_R M := \dim R/\operatorname{Ann}_R(M).$$

Proposition 1.50. Under the Setup $\circledast$,

$$\dim_R M \leq \deg M \leq \mu_R(M).$$

Hint. Induct on $\deg M$.

Corollary 1.51. $(R, \mathfrak{m})$ Noetherian local, then $\dim R \leq \dim_{R/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2$.

***Example* 1.52.** $(k = \overline{k})$. $R = k[x_1, \ldots, x_n]$ has dimension n. Show the non-trivial direction $\dim R \leq n$ by localizing at each maximal ideal $\mathfrak{m}$ and compute $\dim_{\kappa(\mathfrak{m})} \mathfrak{m}R_\mathfrak{m}/\mathfrak{m}^2 R_\mathfrak{m} = \dim_k \mathfrak{m}/\mathfrak{m}^2 \leq n$.

1.5 Krull's Hauptidealsatz, system of parameters

Definition 1.53 (Height)**.** The height of an ideal $\mathfrak{a} \lhd R$ is

$$\operatorname{ht} \alpha := \inf\left\{\dim R_\mathfrak{p} : \mathfrak{p} \text{ is a minimal prime of } \mathfrak{a}\right\}$$

***Remark* 1.54.** $\operatorname{ht} \mathfrak{a} \neq \sup\left\{\dim R_\mathfrak{p} : \mathfrak{p} \subseteq \mathfrak{a} \text{ prime}\right\}$. Consider $\mathfrak{a} = (x^2) \lhd k[x]$.

Lemma 1.55. Let R be a noetherian ring, the height of any ideal $\mathfrak{a} \lhd R$ is finite.

Hint. Pick a minimal prime $\mathfrak{p}$ of $\mathfrak{a}$, $\operatorname{ht} \alpha \leq \operatorname{ht} \mathfrak{p} = \dim R_\mathfrak{p} \leq \deg R_\mathfrak{p} \leq \mu(\mathfrak{p} R_\mathfrak{p}) \leq \mu(\mathfrak{p}) < \infty$.

Theorem 1.56 (Krull's Hauptidealsatz)**.** Let R be a noetherian ring, $\mathfrak{a} \lhd R$.

1. If $\mathfrak{a} = (a)$ is principal, then any minimal prime of $\mathfrak{a}$ has height ≤ 1. In particular, $\operatorname{ht} \mathfrak{a} \leq 1$.

2. Generally, any minimal prime of $\mathfrak{a}$ has height $\leq \mu_R(\alpha)$. In particular, $\operatorname{ht} \mathfrak{a} \leq \mu_R(\mathfrak{a})$.

Hint. If $\mathfrak{p}$ is a minimal prime of $\mathfrak{a}$, then $R_\mathfrak{p}/\mathfrak{a}R_\mathfrak{p}$ is Artinian, hence $\mathfrak{a}R_\mathfrak{p}$ is $\mathfrak{p}R_\mathfrak{p}$-primary, hence $\dim R_\mathfrak{p} \leq \deg R_\mathfrak{p} = \deg H_{\mathfrak{a}R_\mathfrak{p},R_\mathfrak{p}} \leq \mu_{R_\mathfrak{p}}(\mathfrak{a}R_\mathfrak{p}) \leq \mu_R(\mathfrak{a})$.

Proposition 1.57 (Partial converse to Hauptidealsatz)**.** Let $\mathfrak{a}$ be an ideal of a noetherian ring R with $\operatorname{ht}\mathfrak{a} = n$. Then

$$^\exists a_1, \dots, a_n \in \mathfrak{a} \text{ s.t. } \operatorname{ht}(a_1, \dots, a_i) = i \quad (1 \leq i \leq n).$$

Hint. Induct on $\operatorname{ht}\mathfrak{a}$.

Some consequences for noetherian local ring $(R, \mathfrak{m})$.

Corollary 1.58. Let $(R, \mathfrak{m})$ be a noetherian local ring of dimension d.

1. If $\mathfrak{q}$ is a $\mathfrak{m}$-primary ideal, then $d \leq \mu_R(\mathfrak{q})$.
2. $^\exists\mathfrak{m}$-primary ideal $\mathfrak{q}$ such that $\mu_R(\mathfrak{q}) = d$.
3. $\dim R = \deg R$.

Hint. (1) $\mathfrak{q}$ is $\mathfrak{m}$-primary $\Rightarrow$ $\mathfrak{m}$ is the only prime containing $\mathfrak{q}$, by Hauptidealsatz $d = \operatorname{ht}\mathfrak{m} \leq \mu_R(\mathfrak{q})$. (2) Use Proposition 1.57 and observe the height d ideal must be $\mathfrak{m}$-primary. (3) $d = \dim R \leq \deg R \leq \mu_R(\mathfrak{q}) = d$.

Definition 1.59 (System of parameters)**.** A **system of parameters** (s.o.p.) of a Noetherian local ring $(R, \mathfrak{m})$ of dimension d is a collection of elements $x_1, \dots, x_d \in \mathfrak{m}$ with $\operatorname{rad}(x_1, \dots, x_d) = \mathfrak{m}$. By Corollary 1.58 a s.o.p. exist for a Noetherian local ring.

Lemma 1.60. Let $\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n$ be a collection of prime ideals of a noetherian ring R. Then for $f \in \mathfrak{p}_n$,

$$^\exists\text{chain of primes } \mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_n \text{ s.t. } f \in \mathfrak{q}_1 \text{ and } \mathfrak{q}_n = \mathfrak{p}_n.$$

Hint. Induct on n.

Corollary 1.61. Let $(R, \mathfrak{m})$ be a Noetherian local ring and $a_1, \dots, a_n \in \mathfrak{m}$.

1. $\dim R/(a_1, \dots, a_n) \geq \dim R - n$.
2. $\dim R/(a_1, \dots, a_n) = \dim R - n \iff a_1, \dots, a_n$ can be extended to a s.o.p. of R.

Hint. (1) induct on n. (2) $\Rightarrow$ pull a s.o.p. of $R/(a_1, \dots, a_n)$ back to a s.o.p. of R. $\Leftarrow$ use Hauptidealsatz and (1).

***Exercise* 1.62.** Let R be noetherian ring and $a \in R$ not contained in any associated prime of R, then $\dim R/(a) = \dim R - 1$.

Hint. Minimal primes of R are associated, so every minimal prime of (a) has height 1.

1.6 Dimension of polynomial rings

Lemma 1.63. For an ideal $\mathfrak{a} \lhd R$, $R[x]/\mathfrak{a}[x] \cong (R/\mathfrak{a})[x]$. Moreover $\mathfrak{a}[x] \cap R = \mathfrak{a}$.

Corollary 1.64. If $\mathfrak{p} \in \operatorname{Spec} R$, then $\mathfrak{p}[x] \in \operatorname{Spec} R[x]$.

Lemma 1.65. Let $\mathfrak{p} \in \operatorname{Spec} R$ and $|fa$ an ideal of R. Then $\mathfrak{p}$ is minimal over $\mathfrak{a} \iff \mathfrak{p}[x]$ is minimal over $\mathfrak{a}[x]$.

Lemma 1.66. Let R be a noetherian ring. If $\mathfrak{p} \in \operatorname{Spec} R$, then $\operatorname{ht}\mathfrak{p} = \operatorname{ht}\mathfrak{p}[x]$.

Hint. $\operatorname{ht}\mathfrak{p} \leq \operatorname{ht}\mathfrak{p}[x]$ is clear. Conversely pick $a_1, \dots, a_n \in \mathfrak{p}$ with $\operatorname{ht}(a_1, \dots, a_n) = n$. Easy to see $\mathfrak{p}$ is minimal over $\mathfrak{a} = (a_1, \dots, a_n)$. Then $\mathfrak{p}[x]$ is minimal over $\mathfrak{a}[x]$, by Hauptidealsatz we conclude $\operatorname{ht}\mathfrak{p}[x] \leq \mu_{R[x]}(\mathfrak{a}[x]) \leq n = \operatorname{ht}\mathfrak{p}$.

Lemma 1.67. Let R be any ring. Let $\mathfrak{q}_1 \subsetneq \mathfrak{q}_2 \in \operatorname{Spec} R[x]$ s.t. $\mathfrak{q}_1 \cap R = \mathfrak{p} = \mathfrak{q}_2 \cap R$. Then $\mathfrak{q}_1 = \mathfrak{p}[x]$.

Hint. $\kappa(\mathfrak{p})[x]$ is a PID.

Theorem 1.68. Let R be a noetherian ring of finite Krull dimension. Then $\dim R[x_1, \dots, x_n] = \dim R + n$.

Hint. Suffices to show the case $n = 1$. The direction $\dim R[x] \geq \dim R + 1$ is evident. Conversely, pick a chain of primes $\mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_n$ of length $n = \dim R[x]$. Let j be the largest number with $0 \leq j \leq n-1$ such that $\mathfrak{q}_j \subseteq \mathfrak{q}_{j+1}$ contract to the same prime $\mathfrak{p}$ in R. Then by Lemma 1.66 Lemma 1.67 $\mathfrak{q}_j$ has height $\operatorname{ht}\mathfrak{p}$, moreover $\operatorname{ht}\mathfrak{p} = \operatorname{ht}\mathfrak{p}_j \geq j$. Note $\mathfrak{p} \subsetneq \mathfrak{q}_{j+2}^c \subsetneq \cdots \subsetneq \mathfrak{q}_n^c$ is a chain in R of length $n-j-1$. Hence $\dim R \geq n-j-1+\operatorname{ht}\mathfrak{p} \geq n-1$, hence $\dim R + 1 \geq n = \dim R[x]$.

1.7 Regular rings

Definition 1.69 (Regular rings)**.** A Noetherian local ring $(R, \mathfrak{m})$ is **regular** if $\dim R = \deg R = \mu_R(\mathfrak{m}) = \dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2)$. A Noetherian ring R is **regular** if for all maximal ideal $\mathfrak{m}$ of R, $R_\mathfrak{m}$ is a regular local ring. We know there is some $\mathfrak{m}$-primary ideal $\mathfrak{q}$ with $\mu_R(\mathfrak{q}) = \dim R$, now regularity requires generation of $\mathfrak{m}$.

Lemma 1.70. A Noetherian local ring $(R, \mathfrak{m})$ is regular $\iff$ $\exists$ s.o.p. of R generating $\mathfrak{m}$.

Hint. $\Rightarrow$ NAK. $\Leftarrow$ if $x_1, \dots, x_n$ is a s.o.p. generating $\mathfrak{m}$, then $n = \operatorname{ht}\mathfrak{m} = \dim R \leq \mu_R(\mathfrak{m}) \leq n$.

Definition 1.71. A system of parameters of a regular local ring $(R, \mathfrak{m})$ generating $\mathfrak{m}$ is called a **regular system of parameters** (r.s.o.p.).

Lemma 1.72. Let $(R, \mathfrak{m})$ be a regular local ring of dimension d, let $x_1, \dots, x_i \in \mathfrak{m}$, TFAE:

1. $x_1, \dots, x_i$ can be extended to a rsop of R.
2. $\overline{x_1}, \dots, \overline{x_i} \in \mathfrak{m}/\mathfrak{m}^2$ are linearly independent over $R/\mathfrak{m}$.
3. $R/(x_1, \dots, x_i)$ is a regular local ring of dimension $d - i$.

Hint. (1) $\Rightarrow$ (2): the images of extended rsop is a basis of $\mathfrak{m}/\mathfrak{m}^2$. (2) $\Rightarrow$ (3): extend to a basis, look at preimages, use NAK, and Corollary 1.61. (3) $\Rightarrow$ (1): evident.

Proposition 1.73. Let k be any field, $n \geq 0$, $\mathfrak{m} \lhd k[x_1, \dots, x_n]$ any maximal ideal.

1. $\mathfrak{m}$ is generated by n elements
2. $k[x_1, \dots, x_n]$ is regular.

Hint. (1) Induct on n, establish the inductive step by looking at contraction $\mathfrak{n} = \mathfrak{m} \cap R = k[x_1, \dots, x_{n-1}]$ – it is maximal by Nullstellensatz and Zariski's lemma. Use the fact $R[x_n]/\mathfrak{n}[x_n] \cong (R/\mathfrak{n})[x_n]$ is a PID to construct generators of $\mathfrak{m}$. (2) By going-down every maximal ideal has height n, hence all localization $k[\underline{x}]_\mathfrak{m}$ has dimension n, and its maximal ideal may be generated by n elements.

***Example* 1.74.** The cuspidal cubic $k[x,y]/(y^2-x^3)$ is not regular.

Hint. Look at the origin (x,y), by Exercise 1.62 the local ring has dimension 1, but the image of y^2-x^3 in $\mathfrak{m}/\mathfrak{m}^2$ is zero. (Use Lemma 1.72).

Proposition 1.75. Let $(R,\mathfrak{m})$ be a Noetherian local ring of dimension $d \geq 1$. TFAE:

1. $(R,\mathfrak{m})$ is regular local.
2. There is a degree-preserving isomorphism of graded $R/\mathfrak{m}$-algebras $(R/\mathfrak{m})[x_1,\dots,x_d] \to \operatorname{gr}_{\mathfrak{m}} R$.

Hint. Note: $\Delta H_{\mathfrak{m},R}(n) = \ell_{R/\mathfrak{m}}(R/\mathfrak{m}^{n+1}) - \ell_{R/\mathfrak{m}}(R/\mathfrak{m}^n) = H_{\operatorname{gr}_{\mathfrak{m}} R}(n)$, so $\deg H_{\operatorname{gr}_{\mathfrak{m}} R} = d-1$.

(1) $\Rightarrow$ (2): mapping $x_i \to \overline{y_i}$ where $y_1,\dots,y_d$ is a rsop of R. If f is in the kernel with degree $r \geq 1$, then $(R/\mathfrak{m})[\underline{x}]/f \twoheadrightarrow (R/\mathfrak{m})[\underline{x}]/\ker \cong \operatorname{gr}_{\mathfrak{m}} R$, but the Hilbert polynomial of $(R/\mathfrak{m})[\underline{x}]/f$ has degree $\leq d-2$, a contradiction.

(2) $\Rightarrow$ (1): the isomorphism in degree 1 establishes $\dim_{R/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 = d = \dim R$.

Corollary 1.76. A regular local ring $(R,\mathfrak{m})$ is a domain.

Hint. KIT + Proposition 1.75.

Corollary 1.77. Let $(R,\mathfrak{m})$ be a regular local ring of dimension $d \geq 1$, suppose $x_1,\dots,x_i \in \mathfrak{m}$ can be extended to a rsop of R. Then $(x_1,\dots,x_i)$ is prime of height i.

Hint. By Lemma 1.72 + Corollary 1.76, every $(x_1,\dots,x_j)$ is prime $(1 \leq j \leq i)$, and gives a strict chain as $\overline{x_i}$ are linearly independent in $\mathfrak{m}/\mathfrak{m}^2$.

***Exercise* 1.78.** $(R,\mathfrak{m})$ regular local, $f \neq 0$, then $R/(f)$ is regular $\iff$ $f \in \mathfrak{m} - \mathfrak{m}^2$.

Hint. $f \notin R^\times$ as $R/(f)$ is a domain, use Lemma 1.72.

Definition 1.79 (Regular sequence)**.** Let R be a ring, M an R-module Elements $x_1,\dots,x_n \in R$ is a **regular sequence on** M or a M**-regular sequence** if

1. x_1 is a nzd on M
2. $\forall 2 \leq i \leq n$, x_i is a nzd on $M/(x_1,\dots,x_{i-1})$
3. $(x_1,\dots,x_n)M \subsetneq M$.

Without property (3), it is called a **weak** M**-regular sequence**.

***Remark* 1.80.** Let R Noetherian, $M \neq 0$, then

$$x \in R \text{ is a } M\text{-regular sequence} \iff x \notin \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R(M)} \mathfrak{p} \text{ and } xM \neq M.$$

***Example* 1.81.** Let $(R,\mathfrak{m})$ be regular local of dimension $d \geq 1$, suppose $\mathfrak{m} = (x_1,\dots,x_d)$, then $x_1,\dots,x_d$ is a regular sequence.

Hint. Corollary 1.76 + Lemma 1.72.

***Remark* 1.82.** Permutations of a regular sequence may not be a regular sequence. Consider $R = k[x,y,z]$, $a_1 = x(y_1)$, $a_2 = y$, $a_3 = z(y-1)$. Then a_1,a_2,a_3 is a regular sequence, but a_1,a_3,a_2 is not. However the statement is true for regular local rings.

Proposition 1.83. Let $(R,\mathfrak{m})$ be regular local, M f.g. over R, then any permutation of a regular sequence on M is a still regular sequence.

Hint. Suffices to show for $\underline{x} = x_1, x_2$. Show $\ker(M \xrightarrow{x_2} M) = x_1 \ker(M \xrightarrow{x_2} M)$ and apply NAK. Show directly $\ker(M/x_2M \xrightarrow{x_1} M/x_2M) = 0$.

Proposition 1.84. Let R be a ring, M an R-module and $\underline{x}$ a weak M-regular sequence. Then exact sequence $N_2 \to N_1 \to N_0 \to M \to 0$ induces exact sequence $N_2/\underline{x}N_2 \to N_1/\underline{x}N_2 \to N_0/\underline{x}N_0 \to M/\underline{x}M$.

Hint. Suffices to show the case $\underline{x} = x$. Direct computation.

Corollary 1.85. Let $N_\bullet : \cdots \to N_m \to N_{m-1} \to \cdots \to N_0 \to N_{-1} \to 0$ be an exact sequence of R-modules. Suppose $\underline{x}$ is a weakly N_i-regular sequence for all i, then $N_\bullet \otimes R/\underline{x}R$ is also exact.

Hint. Break up the long exact sequence and use Proposition 1.84.

1.8 Koszul complex

Recall some homological algebra:

1. chain complex
2. homology functors $H_n : Comp(R) \to Mod(R)$
3. SES in $Comp(R)$ gives LES in homology
4. tensor product of complexes

$$\begin{aligned} T_\bullet &:= M_\bullet^{(1)} \otimes_R \cdots \otimes_R M_\bullet^{(k)} \\ T_n &= \bigoplus_{i_1+\cdots+i_k=n} M_{i_1}^{(1)} \otimes_R \cdots \otimes_R M_{i_k}^{(k)} \\ d_n^{T_\bullet}(x_{i_1} \otimes \cdots \otimes x_{i_k}) &= \sum_{j=1}^{k} (-1)^{i_1+\cdots+i_{j-1}} x_{i_1} \otimes \cdots \otimes \widehat{x_{i_j}} \otimes \cdots \otimes x_{i_k} \end{aligned}$$

***Example* 1.86.** For an R-module P, let $P[0]$ denote the complex with single nonzero entry P in degree 0. Then

$$M_\bullet \otimes_R P[0] = \left\{ \cdots \to M_{n+1} \otimes P \xrightarrow{d_{n+1} \otimes \mathrm{id}_P} M_n \otimes P \xrightarrow{d_n \otimes \mathrm{id}_P} M_{n-1} \otimes P \to \cdots \right\}$$

Definition 1.87 (Koszul complex)**.** The **Koszul complex on** R of a sequence $x_1, \ldots, x_k \in R$ is $K_\bullet(x_1, \ldots, x_k; R)$:

$$\begin{aligned} K_\bullet(x_1; R) &:= \left\{ \cdots \to 0 \to \underset{1}{R} \xrightarrow{x_1} \underset{0}{R} \to 0 \to \cdots \right\} \\ K_\bullet(x_1, \ldots, x_k; R) &:= K_\bullet(x_1; R) \otimes_R \cdots \otimes_R K_\bullet(x_k; R) \end{aligned}$$

For an R-module M, the **Koszul complex on** M of a sequence $x_1, \ldots, x_k \in R$ is

$$K_\bullet(x_1, \ldots, x_k; M) := K_\bullet(x_1, \ldots, x_k; R) \otimes_R M[0]$$

***Example* 1.88.** For $x \in R$ and $M_\bullet \in Comp(R)$, define $T_\bullet := M_\bullet \otimes_R K_\bullet(x;R)$, then

$$T_\bullet = \left\{ \cdots \longrightarrow \underset{n}{M_n \oplus M_{n-1}} \xrightarrow{\begin{bmatrix} d_n & (-1)^{n-1}x \\ & d_{n-1} \end{bmatrix}} \underset{n-1}{M_{n-1} \oplus M_{n-2}} \longrightarrow \cdots \right\}$$

Proposition 1.89. Let $M_\bullet \in Comp(R)$, $x \in R$, then there is a long exact sequence

$$\cdots \to H_n(M_\bullet) \xrightarrow{(-1)^n x} H_n(M_\bullet) \to H_n(M_\bullet \otimes K(x;R)) \to H_{n-1}(M_\bullet) \xrightarrow{(-1)^{n-1}x} H_{n-1}(M_\bullet) \to \cdots$$

Hint. Consider the complex $T_\bullet$ in Example 1.88, we have split ses $0 \to M_\bullet \to T_\bullet \to M_\bullet[-1] \to 0$, then trace the boundary map in associated les using snake's lemma.

Lemma 1.90. Let F denote the free R-module $F = \oplus_{i=1}^k Re_i$, then there is a canonical isomorphism $K_n(x_1,\dots,x_k;R) \xrightarrow{\sim} \bigwedge^n F$. In particular for an R-module M, we have $H_p(x_1,\dots,x_k;M) = 0$ for $p<0$ and $p>k$, and

$$H_0(x_1,\dots,x_k;M) = M/(x_1,\dots,x_k)M, \quad H_k(x_1,\dots,x_k;M) = (0 :_M (x_1,\dots,x_k)).$$

***Example* 1.91.**

$$K_\bullet(x_1,x_2;R) = \left\{ 0 \longrightarrow \underset{2}{R(e_1\wedge e_2)} \xrightarrow{\begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}} \underset{1}{Re_1 \oplus Re_2} \xrightarrow{\begin{bmatrix} x_1 & x_2 \end{bmatrix}} \underset{0}{R} \longrightarrow 0 \right\}$$

$$K_\bullet(x_1,x_2,x_3;R) = \left\{ 0 \longrightarrow \underset{3}{R} \xrightarrow{\begin{bmatrix} x_3 \\ -x_2 \\ x_1 \end{bmatrix}} \underset{2}{R^{\oplus 3}} \xrightarrow{\begin{bmatrix} -x_2 & -x_3 & 0 \\ x_1 & 0 & -x_3 \\ 0 & x_1 & x_2 \end{bmatrix}} \underset{1}{R^{\oplus 3}} \xrightarrow{\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}} \underset{0}{R} \longrightarrow 0 \right\}$$

Theorem 1.92. Let M be an R-module, $x_1,\dots,x_n$ a M-regular sequence, then

$$H_i(x_1,\dots,x_n;M) = \begin{cases} 0, & i \neq 0 \\ M/(x_1,\dots,x_n)M, & i = 0 \end{cases}$$

Hint. Induct on n, establish the inductive step using Proposition 1.89.

Recall

1. Projective module: $\mathrm{Hom}_R(P,-)$ exact
2. Injective module: $\mathrm{Hom}_R(-,I)$ exact
3. Flat module: $M \otimes_R -$ exact
4. free $\Rightarrow$ projective $\Rightarrow$ flat
5. Projective resolution

6. Tor_i^R functors: $\mathrm{Tor}_i^R(M,N) = L(M \otimes_R -)_i(N) = H_i(P_\bullet)$, if $P_\bullet \to N \to 0$ is a projective resolution.

Lemma 1.93. For an R-module M, TFAE:

1. M is projective.
2. Every se $0 \to N \to P \to M \to 0$ splits.
3. ${}^\exists N \subseteq M$ submodule, $N \oplus M$ is free.

Hint. (3) $\Rightarrow$ (1): $\mathrm{Hom}(N \oplus M, -) \cong \mathrm{Hom}(N,-) \oplus \mathrm{Hom}(M,-)$ is exact iff $\mathrm{Hom}(M,-)$ and $\mathrm{Hom}(N,-)$ are both exact.

Some properties of Tor_i^R functors:

1. $\mathrm{Tor}_i^R(M,N)$ is independent of the projective resolution.
2. $\mathrm{Tor}_0^R(M,N) = M \otimes_R N$.
3. $\mathrm{Tor}_i^R(M,N) \cong \mathrm{Tor}_i^R(N,M)$.
4. SES $0 \to N_1 \to N_2 \to N_3 \to 0$ gives LES

$$\cdots \to \mathrm{Tor}_{i+1}^R(M,N_3) \xrightarrow{\delta_{i+1}} \mathrm{Tor}_i^R(M,N_1) \to \mathrm{Tor}_i^R(M,N_2) \to \mathrm{Tor}_i^R(M,N_3) \xrightarrow{\delta_i} \mathrm{Tor}_i^R(M,N_1) \to \cdots$$

5. Lifting of linear maps

$$\begin{tikzcd}
\cdots \arrow[r] & P_2^{(1)} \arrow[r] \arrow[d, dashed, "\varphi_0"] & P_1^{(1)} \arrow[r] \arrow[d, dashed, "\varphi_1"] & P_0^{(1)} \arrow[r] \arrow[d, dashed, "\varphi_0"] & N_1 \arrow[r] \arrow[d, "\varphi"] & 0 \\
\cdots \arrow[r] & P_2^{(2)} \arrow[r] & P_1^{(2)} \arrow[r] & P_0^{(2)} \arrow[r] & N_2 \arrow[r] & 0
\end{tikzcd}$$

Hence get induced maps $\mathrm{Tor}_i^R(M,N_1) \to \mathrm{Tor}_i^R(M,N_2)$. If $\widetilde{\varphi} : P_\bullet^{(1)} \to P_\bullet^{(2)}$ is another lifting, then $\varphi \sim \widetilde{\varphi}$, hence they induces the same map on Tor's.

6. Finite free resolution. Let R be Noetherian, M f.g. over R, then there is a finite free resolution of M:

$$\begin{tikzcd}
\cdots \arrow[r] & R^{\oplus n_2} \arrow[rr, "d_2"] \arrow[dr, two heads] & & R^{\oplus n_1} \arrow[rr, "d_1"] \arrow[dr, two heads] & & R^{\oplus n_0} \arrow[r, two heads, "d_0"] & M \arrow[r] & 0 \\
& & \ker d_1 \arrow[ur, hook] & & \ker d_0 \arrow[ur, hook] & & &
\end{tikzcd}$$

***Remark* 1.94.** By Theorem 1.92 $K_\bullet(x_1,\ldots,x_n;R) \to R/(x_1,\ldots,x_n)R \to 0$ is a free resolution. Hence for R-module M,

$$\mathrm{Tor}_i^R(M, R/(x_1,\ldots,x_n)) \cong H_i(x_1,\ldots,x_n;M) \quad \forall i.$$

If $(R,\mathfrak{m})$ is a regular local ring, and $x_1,\ldots,x_n \in \mathfrak{m}$ a rsop, then for any R-module M,

$$\mathrm{Tor}_i^R(M, R/\mathfrak{m}) \cong H_i(x_1,\ldots,x_n;M) \quad \forall i.$$

Lemma 1.95. For R-module M, TFAE

1. M is flat
2. $\mathrm{Tor}_i^R(M,N)$ for all R-module N and $i \geq 1$
3. $\mathrm{Tor}_1^R(M,N)$ for all R-module N.

Hint. (3) $\Rightarrow$ (1): consider the LES of Tors.

Proposition 1.96. Let $(R,\mathfrak{m})$ be a Noetherian local ring, M f.g. over R, TFAE

1. M free
2. M projective
3. M flat

Hint. (3) $\Rightarrow$ (1): let $n := \mu_R(M) = \dim_{R/\mathfrak{m}}(M/\mathfrak{m}M)$, there is a surjection $R^n \twoheadrightarrow M$. Consider Tor sequence associated to $R/\mathfrak{m} \otimes -$, conclude by NAK that $R^n \twoheadrightarrow M$ is an iso.

***Example* 1.97.** Finite generation in Proposition 1.96 is necessarily. Consider a Noetherian local $(R,\mathfrak{m})$ with fraction field $R \neq K$. Since localization is exact, K is flat over R, however K is not projective over R. It suffices to show $\mathrm{Hom}_R(K,R) = 0$. Given $\varphi \in \mathrm{Hom}_R(K,R)$, then we have inclusion $K/\ker\varphi \hookrightarrow R$, hence the quotient is f.g. over R. Pick any $a \in \mathfrak{m} - 0$, then $a(K/\ker\varphi) = K/\ker\varphi$, hence $\ker\varphi = 0$ by NAK.

Proposition 1.98. For R-module M, TFAE:

1. $N_1 \xrightarrow{f} N_2 \xrightarrow{g} N_3$ exact $\iff$ $M \otimes N_1 \xrightarrow{1\otimes f} M \otimes N_2 \xrightarrow{1\otimes g} M \otimes n_3$ exact.
2. M flat + $M \otimes N = 0 \implies N = 0$ for all R-module N.
3. M flat + $M \neq \mathfrak{m}M$ for all maximal ideal $\mathfrak{m} \lhd R$.

M satisfying this is said to be **faithfully flat**.

Hint. (1) $\Rightarrow$ (2): consider $0 \to N \to 0$. (2) $\Rightarrow$ (1): flat modules commutes with image and quotient. (2) $\Rightarrow$ (3): $R/\mathfrak{m} \neq 0$. (3) $\Rightarrow$ (2): suppose there is some N with $M \otimes N = 0, N \neq 0$, then there is some injection $R/\mathfrak{m} \hookrightarrow N$ where $\mathfrak{m}$ maximal, tensor with M gives a contradiction.

Lemma 1.99. Let $\varphi : R \to S$ be a flat ring map, TFAE:

1. $\varphi : R \to S$ is faithfully flat
2. $^\forall M \in Mod(R)$, $\varphi \otimes \mathrm{id}_M : M \to S \otimes_R M$ is injective
3. $^\forall I \lhd R$, $\overline{\varphi} : R/I \to S/IS$ is injective.

Hint. (1) $\Rightarrow$ (2): let $K := \ker(\varphi \otimes \mathrm{id}_M) \subseteq M$, then $(K \xrightarrow{\varphi\otimes\mathrm{id}_K} S \otimes_R K \xrightarrow{\mathrm{id}_S \otimes \iota} S \otimes_R M) = 0$, since S is R-flat, $(K \xrightarrow{\varphi\otimes\mathrm{id}_K} S \otimes_R K) = 0$, hence $S \otimes_R K = 0$. Since S is R-faithfully flat, $K = 0$. (2) $\Rightarrow$ (3): $M := R/I$. (3) $\Rightarrow$ (1): for $\mathfrak{m} \lhd_{max} R$, we have $0 \neq R/\mathfrak{m} \hookrightarrow S/\mathfrak{m}S$, hence $\mathfrak{m}S \neq S$.

Lemma 1.100. Let $\varphi : R \to S$ be a faithfully flat ring map, then φ is injective.

Hint. Lemma 1.99

Theorem 1.101. Let $\varphi : R \to S$ be a ring map,

$$\varphi \text{ is faithfully flat} \iff \varphi \text{ is flat} + \operatorname{Spec}\varphi \text{ surjective.}$$

Hint. $\Rightarrow$: for $\mathfrak{p} \in \operatorname{Spec} R$, recall the fiber of $\mathfrak{p}$ are in one-to-one correspondence with $\operatorname{Spec}(R/\mathfrak{p} - 0)^{-1}(S/\mathfrak{p}S)$, by Lemma 1.99 we know the spectrum is nonempty. $\Leftarrow$: for $\mathfrak{m} \lhd_{max} R$, pick some fiber $\mathfrak{n} \in \operatorname{Spec} S$, then $\mathfrak{m}S \subseteq \mathfrak{n} \neq S$, hence φ is faithfully flat by Proposition 1.98.

***Example* 1.102.** The following ring maps are faithfully flat, using Theorem 1.101 and Proposition 1.98:

1. $R \hookrightarrow R[x]$
2. $R \hookrightarrow S$ integral extension
3. $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ flat, local.

Lemma 1.103. Let $\varphi : R \to S$ be a flat ring map, $\mathfrak{q} \in \operatorname{Spec} S$, $\mathfrak{p} := \varphi^{-1}(\mathfrak{q}) \in \operatorname{Spec} R$, then $R_{\mathfrak{p}} \to S_{\mathfrak{q}}$ is faithfully flat.

Hint. $R_{\mathfrak{p}} \to S_{\mathfrak{p}}$ is flat, $\mathfrak{q}S_{\mathfrak{p}} \in \operatorname{Spec} S_{\mathfrak{p}}$ and $(S_{\mathfrak{p}})_{\mathfrak{q}S_{\mathfrak{p}}} = S_{\mathfrak{q}}$, hence by composition $R_{\mathfrak{p}} \to S_{\mathfrak{q}}$ is flat. Faithful because $\mathfrak{p}S_{\mathfrak{q}} \subseteq \mathfrak{q}S_{\mathfrak{q}} \neq S_{\mathfrak{q}}$.

Corollary 1.104. Let $\varphi : R \to S$ be a faithfully flat ring map, then it satisfies the going down property.

$$\begin{array}{ccc} {}^{\exists}\mathfrak{q}_1 & \subsetneq & \mathfrak{q}_2 \\ | & & | \\ \mathfrak{p}_1 & \subsetneq & \mathfrak{p}_2 \end{array}$$

Hint. By Lemma 1.103 $R_{\mathfrak{p}_2} \to S_{\mathfrak{q}_2}$ is faithfully flat, hence the induced map on spectrum is surjective.

1.9 Projective dimension, global dimension, minimal free resolution

Let $\{\cdots \to P_1 \to P_0\} \to M \to 0$ be a projective resolution of R-modules, then the length of $P_\bullet$ is defined to be $\inf\{n : P_n \neq 0\}$. The **projective dimension** of an R-module M is the infimum of lengths of possible projective resolutions of M:

$$\operatorname{pd}_R(M) := \inf\left\{n : {}^{\exists} \text{ projective resolution of } M \text{ with length } n\right\}.$$

The **global dimension** of a ring R is

$$\operatorname{gl.dim} R := \sup\{\operatorname{pd}_R M : M \text{ is a f.g.} R\text{-module}\}$$

***Example* 1.105.** Let $x_1, \ldots, x_n$ be a regular sequence on R, then $\operatorname{pd}_R(R/(x_1, \ldots, x_n)) = n$.
Hint. The Koszul resolution is a projective resolution of length n, so $\operatorname{pd} \leq n$. Let $\mathfrak{m}$ be a maximal ideal containing $(x_1, \ldots, x_n)$, then all differentials of $R/\mathfrak{m} \otimes_R K_\bullet(x_1, \ldots, x_n; R)$ are zero, hence $\operatorname{Tor}_n^R(R/\mathfrak{m}, R/(x_1, \ldots, x_n)) \neq 0$, showing $\operatorname{pd} \geq n$.

1.9.1 Projective dimension of f.g. module over noethearin local ring

Definition 1.106 (Minimal free resolution)**.** Let $(R, \mathfrak{m})$ be a noetherian local ring and M f.g. over R. We know there is a finite free resolution $F_\bullet \to M \to 0$. Suppose $F_i \cong R^{\oplus n_i}$. Then differential $d_i : F_i \to F_{i-1}$ are given by a $(n_{i-1}) \times n_i$ matrix A_{d_i} with entries in R. We say $F_\bullet$ is a minimal free resolution of M if $^\forall i$ all entries of A_{d_i} are in $\mathfrak{m}$. Equivalently, $^\forall i$, $\operatorname{im} d_i \subseteq \mathfrak{m}F_{i-1}$.

Lemma 1.107. Let $(R, \mathfrak{m})$ be noetherian local, M f.g. over R. Then

1. If $n = \mu_R(M)$ and $f : R^{\oplus n} \twoheadrightarrow M$, then $\ker f \subseteq \mathfrak{m}R^{\oplus n}$.

2. M has a minimal free resolution.

Hint. (1) tensor with $R/\mathfrak{m}$, since $(R/\mathfrak{m})^{\oplus n} \to M/\mathfrak{m}M$ is an iso, $\ker f/\mathfrak{m}(\ker f) \to R^{\oplus n}/\mathfrak{m}R^{\oplus n}$ is zero, so $\ker f \subseteq \mathfrak{m}R^{\oplus n}$. (2) inductively build $F_\bullet$ using (1):

$$\begin{tikzcd}
\cdots \arrow[r] & R^{\oplus \mu_R(\ker d_1)} \arrow[rr, "d_2"] \arrow[dr, two heads, "g_2"'] & & R^{\oplus \mu_R(\ker g_0)} \arrow[rr, "d_1"] \arrow[dr, two heads, "g_1"'] & & R^{\oplus \mu_R(M)} \arrow[r, two heads, "d_0 = g_0"] & M \arrow[r] & 0 \\
& & \ker d_1 \arrow[ur, hook] & & \ker g_0 \arrow[ur, hook] & & &
\end{tikzcd}$$

Proposition 1.108. Let $(R, \mathfrak{m})$ be noetherain local, M f.g. over R. Let $F_\bullet$ be a minimal free resolution of M. Then

1. $^\forall i \geq 0$, $\operatorname{rank} F_i = \dim_{R/\mathfrak{m}} \operatorname{Tor}_i^R(R/\mathfrak{m}, M)$.

2. $\operatorname{pd}_R(M) =$ length of $F_\bullet$.

Hint. (1): all differentials of $R/\mathfrak{m} \otimes F_\bullet$ vanish. Suppose $\operatorname{pd}_R(M) = n < \infty$, then $\operatorname{Tor}_i^R(R/\mathfrak{m}, M) = 0$ for $i > n$, hence $\operatorname{rank} F_i = 0$ for $i > n$ by (1), hence length of $F_\bullet \leq n$.

Corollary 1.109. Let $(R, \mathfrak{m})$ be noetherian local, then $\operatorname{gl.dim} R = \operatorname{pd}_R(R/\mathfrak{m})$.

Hint. Know $\operatorname{gl.dim} R \geq \operatorname{pd}_R(R/\mathfrak{m})$. Let M be f.g. over R, and $F_\bullet \to M \to 0$ a minimal free resolution. Then differentials of $R/\mathfrak{m} \otimes F_\bullet$ vanish, hence $\operatorname{pd}_R(M) =$ length of $F_\bullet$ which equals $\sup \left\{ n : \operatorname{Tor}_n^R(R/\mathfrak{m}, M) \neq 0 \right\}$, which $\leq \operatorname{pd}_R(R/\mathfrak{m})$.

Corollary 1.110. Let $(R, \mathfrak{m})$ be a regular local ring, then

$$\operatorname{gl.dim} R = \operatorname{pd}_R(R/\mathfrak{m}) = \dim R.$$

Hint. Let $x_1, \ldots, x_n \in \mathfrak{m}$ be an rsop, then $K_\bullet(x_1, \ldots, x_n; R)$ is a minimal resolution of $R/\mathfrak{m}$, hence $\dim R = n = \operatorname{pd}_R(R/\mathfrak{m}) = \operatorname{gl.dim} R$ by Corollary 1.109 + Proposition 1.108.

Proposition 1.111. Let $(R, \mathfrak{m})$ be noetherian local, M f.g. over R. Let $x \in \mathfrak{m}$ be a nzd on R and on M. Then $\operatorname{pd}_R M = \operatorname{pd}_{R/(x)} M/xM$.

Hint. Let $F_\bullet \to M \to 0$ be a minimal free resolution of M over R, then $F_\bullet \otimes R/(x) \to M/xM \to 0$ is a minimal free resolution (Corollary 1.85) of M/xM over R/x of the same length (NAK) as $F_\bullet$. Another way to see $F_\bullet \otimes R/(x) \to M/(x) \to 0$ is a minimal free resolution is to observe $H_i(F_\bullet \otimes R/x) = \operatorname{Tor}_i^R(R/x, M) \cong \operatorname{Tor}_i^R(M, R/x)$, and $0 \to R \xrightarrow{x} R \twoheadrightarrow R/x \to 0$ is a projective resolution.

1.9.2 Auslander-Buchsbaum-Serre

Lemma 1.112. Let $(R, \mathfrak{m})$ be noetherian local, $\mathfrak{m} = \operatorname{ann}_R(r) \in \operatorname{Ass}_R(R)$, then for M f.g. over R, M is projective ($\operatorname{pd}_R M = 0$) or $\operatorname{pd}_R M = \infty$.

Hint. If $0 < \operatorname{pd}_R M = n < \infty$, then a minimal free resolution of M starts with $0 \to F_n \to F_{n-1}$. Choose basis element $e \in F_n$, then $re \neq 0$, but $d_n(re) \subseteq r\mathfrak{m}F_{n-1} = 0$, a contradiction.

Lemma 1.113. Let $(R, \mathfrak{m})$ be noetherian local, $x \in \mathfrak{m}$, $x \notin$ any minimal prime of R. If R/x is regular, then so is R.

Hint. Geometrically, $x \notin$ any minimal prime means the hypersurface defined by x is not contained in any irreducible component. In this case $\dim R/x = \dim R - 1 = d - 1$. Let $y_1, \ldots, y_{d-1} \in \mathfrak{m}$ such that $\mathfrak{m}/(x) = (y_1 + (x), \ldots, y_{d-1} + (x))$, then $\mathfrak{m} = (y_1, \ldots, y_{d-1}, x)$.

Lemma 1.114. Let $(R, \mathfrak{m})$ be noetherian local, then

$$\operatorname{pd}_R R/\mathfrak{m} < \infty \iff \operatorname{pd}_R \mathfrak{m} < \infty$$

Hint. Let $F_\bullet \to \mathfrak{m} \to 0$ be a minimal free resolution, then

$$\begin{tikzcd}
\cdots \arrow[r] & F_1 \arrow[r, "d_1"] & F_0 \arrow[rr, dashed, "\widetilde{d_0}"] \arrow[dr, two heads, "d_0"'] & & R \arrow[r, two heads] & R/\mathfrak{m} \arrow[r] & 0 \\
& & & \mathfrak{m} \arrow[ur] & & &
\end{tikzcd}$$

is a minimal free resolution of $R/\mathfrak{m}$, length $++$.

Lemma 1.115. Let $(R, \mathfrak{m})$ be noetherian local, M, N f.g. over R. Then

$$\operatorname{pd}_R(M \oplus N) = \max\{\operatorname{pd}_R(M), \operatorname{pd}_R(N)\}.$$

Hint. If $F_\bullet^M \to M \to 0$, $F_\bullet^N \to N \to 0$ be minimal free resolutions, then $F_\bullet^M \oplus F_\bullet^N \to M \oplus N \to 0$ is a minimal free resolution of $M \oplus N$.

Lemma 1.116 (General prime avoidence)**.** Let $J, I_1, \ldots, I_n$ be ideals of R such that $J \not\subseteq I_i$ for all i, and I_i are prime ($3 \leq i \leq n$). Then $J \not\subseteq \bigcup_{i=1}^n I_i$.

Hint. Induct on n, $n = 1$ is evident. $n = 2$: take $x \in J - I_1$, $y \in J - I_2$, then $x + y \in J - I_1 \cup I_2$. If $n > 2$, take $x \in J - \cup_{j=1}^{n-1} I_i$, may assume $x \in J_n$. May assume $I_i \not\subseteq I_n$ $(1 \leq i \leq n-1)$, then $JI_1 \cdots I_{n_1} \not\subset I_n$ since I_n is prime. Pick $y \in JI_1 \cdots I_{n_1} - I_n$, then $x + y$ suffices.

Lemma 1.117. Let $(R, \mathfrak{m})$ be noetherian local, $\mathfrak{m} \notin \operatorname{Ass}_R R$, then

$$^\exists x \in \mathfrak{m}, x \notin \mathfrak{m}^2, x \text{ is a nzd on R.}$$

Hint. Easy to see $\mathfrak{m} \neq 0$, then by $\mathfrak{m} \not\subseteq \mathfrak{m}^2$ by NAK. Since R is noetherian, $\operatorname{Ass}_R R = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_k\}$, by maximality of $\mathfrak{m}$, $\mathfrak{m} \not\subseteq \mathfrak{p}_i$ for all i. Finish by prime avoidence Lemma 1.116.

Theorem 1.118 (Auslander-Buchsbaum-Serre)**.** Let $(R, \mathfrak{m})$ be noetherian local, TFAE:

1. R is regular
2. $\operatorname{gl.dim} R < \infty$
3. $\operatorname{pd}_R R/\mathfrak{m} < \infty$

Hint. (1) $\Rightarrow$ (2) because $\dim R < \infty$, (2) $\Rightarrow$ (3) by definition, so suffices to show (3) $\Rightarrow$ (1). Induct on $\mu_R(\mathfrak{m})$, when $\mu_R(\mathfrak{m}) = 0$, then $\mathfrak{m} = (0)$ and R is a field. Assume $\mu_R(\mathfrak{m}) \geq 1$.

- $\operatorname{pd}_R(\mathfrak{m}) \geq 1$ because otherwise it is projective hence free by Proposition 1.96, but then $\mathfrak{m} = \operatorname{Ann}_R(R/\mathfrak{m}) = 0$, contradiction.
- Lemma 1.112 + Lemma 1.114 $\Rightarrow$ $\mathfrak{m}$ not associated, so by Lemma 1.117 there is $x \in \mathfrak{m} - \mathfrak{m}^2$, a nzd on R. $x \notin$ any associated prime $\Rightarrow$ $\dim R/(x) = \dim R - 1$. Lemma 1.113 $\Rightarrow$ STS $R/(x)$ regular.
- $\mu_{R/x}(\mathfrak{m}/(x)) = \mu_R(\mathfrak{m}) - 1$: $\mu_{R/x}(\mathfrak{m}/(x)) = \dim_{R/\mathfrak{m}} \frac{\mathfrak{m}/(x)}{\mathfrak{m}^2+(x)/(x)} = \dim_{R/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2 + x$, finish using SES over $R/\mathfrak{m}$:

$$0 \longrightarrow \frac{\mathfrak{m}^2+(x)}{\mathfrak{m}^2} \longrightarrow \frac{\mathfrak{m}}{\mathfrak{m}^2} \longrightarrow \frac{\mathfrak{m}}{\mathfrak{m}^2+x} \longrightarrow 0$$

  IH $\Rightarrow$ STS $\operatorname{pd}_{R/(x)} R/\mathfrak{m} < \infty$.
- $\operatorname{pd}_R R/\mathfrak{m} < \infty$ + Lemma 1.114 $\Rightarrow$ $\operatorname{pd}_R \mathfrak{m} < \infty$. Proposition 1.111 $\Rightarrow$ $\operatorname{pd}_{R/x} \mathfrak{m}/x\mathfrak{m} < \infty$. Have SES

$$0 \longrightarrow \frac{xR}{x\mathfrak{m}} \longrightarrow \frac{\mathfrak{m}}{x\mathfrak{m}} \longrightarrow \frac{\mathfrak{m}}{(x)} \longrightarrow 0$$

  Lemma 1.115 $\Rightarrow$ STS it splits.
- $xR/x\mathfrak{m} \cong R/\mathfrak{m}$ over R/x: iso induced by $R \xrightarrow{x} xR$. $\Rightarrow$ $xR/x\mathfrak{m}$ simple over R/x.
- Let $n := \mu_R(\mathfrak{m})$, then by NAK $^\exists y_1, \ldots, y_{n-1}$ with $\mathfrak{m} = (x, y_1, \ldots, y_{n-1})$. Consider

$$\eta : \frac{(y_1, \ldots, y_{n-1}) + x\mathfrak{m}}{x\mathfrak{m}} \hookrightarrow \frac{\mathfrak{m}}{x\mathfrak{m}} \twoheadrightarrow \frac{\mathfrak{m}}{(x)}$$

  Then η surjective. $\operatorname{Ker} \eta = \operatorname{Ker}(\mathfrak{m}/x\mathfrak{m} \twoheadrightarrow \mathfrak{m}/(x)) \cap \frac{(y_1,\ldots,y_{n-1})+x\mathfrak{m}}{x\mathfrak{m}} = \frac{xR}{x\mathfrak{m}} \cap \frac{(y_1,\ldots,y_{n-1})}{x\mathfrak{m}}$. Note $x \notin (y_1, \ldots, y_{n-1}) + x\mathfrak{m}$ as $\{x + \mathfrak{m}^2, y_1 + \mathfrak{m}^2, \ldots, y_{n-1} + \mathfrak{m}^2\}$ is a $R/\mathfrak{m}$ basis of $\mathfrak{m}/\mathfrak{m}^2$. Together with simplicity of $xR/x\mathfrak{m}$, η is injective.
- $\phi := \frac{\mathfrak{m}}{(x)} \xrightarrow{\eta^{-1}} \frac{(y_1,\ldots,y_{n-1}+x\mathfrak{m})}{x\mathfrak{m}} \hookrightarrow \frac{\mathfrak{m}}{x\mathfrak{m}}$ gives a splitting.

Corollary 1.119. Let $(R, \mathfrak{m})$ be regular local, then $R_\mathfrak{p}$ is a regular local for all $\mathfrak{p} \in \operatorname{Spec} R$.

Hint. By Auslander-Buchsbaum-Serre suffices to show $\operatorname{pd}_{R_\mathfrak{p}} \kappa(\mathfrak{p}) < \infty$. $(R, \mathfrak{m})$ regular local $\Rightarrow$ $\operatorname{gl.dim} R = \dim R < \infty \Rightarrow \operatorname{pd}_R R/\mathfrak{p} < \infty$. Let $F_\bullet \to R/\mathfrak{p} \to 0$ be a minimal free resolution over R, then $F_\bullet \otimes R_\mathfrak{p} \to k(\mathfrak{p}) \to 0$ is a free resolution over $R_\mathfrak{p}$ of finite length, hence $\operatorname{pd}_{R_\mathfrak{p}} \kappa(\mathfrak{p}) < \infty$.

1.10 Normal rings

Recall a regular local ring of dimension 0 is a field as it is a domain of dimension 0. We first give a characterization of dimension 1 regular local rings.

Theorem 1.120. Let $(R, \mathfrak{m})$ be noetherian local, TFAE:

1. R is regular, $\dim R = 1$

2. R is a PID

3. R is a domain, integrally closed in $K(R)$, and $\dim R = 1$

4. R is a doamin, integrally closed in $K(R)$, and $\mathfrak{m} \in \operatorname{Ass}_R(R/a)$ for some $a \neq 0 \in R$

5. R is a domain, and $\mathfrak{m} \neq (0)$ is principal.

Hint. (1) $\Rightarrow$ (2): $\mu_R(\mathfrak{m}) = 1$, so $\mathfrak{m} = (t)$ for some t, then show any element of R can be written as ut^n for $u \in R^\times$ and $n \geq 0$ using Krull's intersection theorem.

(2) $\Rightarrow$ (3): PID $\Rightarrow$ UFD.

(3) $\Rightarrow$ (4): pick any $a \neq 0 \in \mathfrak{m}$, then $R/a \neq 0$, hence $\operatorname{Ass}_R R/a \neq \varnothing$, it must equal $\{\mathfrak{m}\}$.

(4) $\Rightarrow$ (5): suppose $\mathfrak{m} = \operatorname{ann}_R(x + (a))$ then $a \in \mathfrak{m}$ and $x\mathfrak{m} \subseteq (a)$, then $\mathfrak{m} \neq (0)$, and $(x/a)\mathfrak{m} \subseteq R$. Show $(x/a)\mathfrak{m} = R$ by deriving a contradiction from $(x/a)\mathfrak{m} \subseteq \mathfrak{m}$ using Cayley-Hamilton and the fact that R is integrally closed in $K(R)$. Then $\mathfrak{m} = (a/x)$.

(5) $\Rightarrow$ (1): $\mu_R(\mathfrak{m}) = 1$, then $\dim R \leq \mu_R(\mathfrak{m}) = 1$, since R is a domain, not a field, it must have dimension 1.

Definition 1.121 (Normal ring)**.** A ring R (not necessarily a domain) is **normal** if for all $\mathfrak{p} \in \operatorname{Spec} R$:

1. $R_\mathfrak{p}$ is a domain

2. $R_\mathfrak{p}$ is integrally closed in $K(R_\mathfrak{p})$.

Lemma 1.122. Suppose R is a domain, then R is normal $\iff$ R is integrally closed in $K(R)$.

Hint. (nonzero) localization of a domain is a domain, hence $\Leftarrow$ is evident. $\Rightarrow$: let $f \in K(R)$ be integral over R, then show the ideal of denominators $I_f = \{x \in R : xf \in R\}$ is not contained in any maximal ideal.

***Remark* 1.123.** A normal ring need not be a domain itself: consider $k \times k$.

Lemma 1.124. Let R be a domain, $(0) \neq I \lhd R$ an ideal. Define $B^I := \big\{f \in K(R) : fI \subseteq I\big\}$. Then

1. B^I is a subring of $K(R)$ containing R

2. if R is noetherian then B^I is finite over R.

Hint. (1) is evident. (2): there is an injection $B^I \hookrightarrow \operatorname{Hom}_R(I, I)$ sending $f \mapsto \phi_f$, multiplication by f. R noeth $\Rightarrow$ $^\exists R^{\oplus n} \twoheadrightarrow I \to 0 \Rightarrow \operatorname{Hom}_R(I, I) \hookrightarrow \operatorname{Hom}_R(I, R^{\oplus n}) \cong I^{\oplus n}$. $I^{\oplus n}$ noeth $\Rightarrow$ $\operatorname{Hom}_R(I, I)$ noeth $\Rightarrow B^I$ finite over R.

Proposition 1.125. Let R be a noetherian normal domain, then

$$R = \bigcap_{\operatorname{ht}\mathfrak{p}=1} R_\mathfrak{p} \subseteq K(R).$$

Hint. Let I_f denote the ideal of denominators of $f \in K(R)$. Consider collection $\Sigma = \big\{I_f : f \in \text{RHS} - R\big\}$. Suppose $\Sigma \neq \varnothing$ then R noeth $\Rightarrow$ $^\exists I_g \in \Sigma$ maximal. Then $\mathfrak{q} := I_g$ is a prime, $g\mathfrak{q} \subseteq R$. Consider the ideal $g\mathfrak{q}R_\mathfrak{q} \lhd R_\mathfrak{q}$. Show $g\mathfrak{q}R_\mathfrak{q} \neq R_\mathfrak{q}$: if not $\mathfrak{q}R_\mathfrak{q} = g^{-1}R_\mathfrak{q}$, and $\operatorname{ht}\mathfrak{q} = \operatorname{ht}\mathfrak{q}R_\mathfrak{q} = 1$, so $g^{-1}, g \in R_\mathfrak{q}$, and $\mathfrak{q}R_\mathfrak{q} = (1)$ a contradiction. Then $g\mathfrak{q}R_\mathfrak{q} \subseteq \mathfrak{q}R_\mathfrak{q}$, hence $g\mathfrak{q} \subseteq \mathfrak{q}$, then $g \in B^\mathfrak{q}$ is integral over R by Lemma 1.124, a contradiction.

Lemma 1.126. Let R be noetherian, satisfying (S2), then for any nzd $a \in R$, if $\mathfrak{p} \in \operatorname{Ass}_R(R/a)$, then $\operatorname{ht}\mathfrak{p} = 1$.

Hint. $aR \subseteq \mathfrak{p} + a$ not contained in any minimal prime $\Rightarrow \operatorname{ht}\mathfrak{p} \geq 1$. Suppose ht ≥ 2, replace R by $R_\mathfrak{p}$, we may assume $(R, \mathfrak{m})$ is a noeth local, $a \in R$ a nzd, $\mathfrak{m} \in \operatorname{Ass}_R(R/a)$, and by (S2) ${}^\exists y_1, y_2 \in \mathfrak{m}$ form a regular sequence on R. Since y_2 is a nzdon R/y_1, $\mathfrak{m} \notin \operatorname{Ass}_R(R/y_1)$.

Suppose $\mathfrak{m} = \operatorname{ann}_R(x + (a))$ then $x \notin aR$, $\mathfrak{m}x \subseteq aR \Rightarrow {}^\exists z \in R, xy_1 = az$, $z \notin (y_1)$. Then $z + (y_1) \neq \overline{0} \in R/y_1$, hence $\operatorname{ann}_R(z + (y_1)) \subseteq \mathfrak{m}$. Show $z\mathfrak{m} \subseteq (y_1)$, so it is in fact an equality, so we get a contradiction to $\mathfrak{m} \notin \operatorname{Ass}_R(R/y_1)$. Hence $\operatorname{ht}\mathfrak{p} = 1$.

Theorem 1.127. Let R be a noetherian domain. Then R is normal if and only if it satisfies (R1) + (S2), where

(R1) ${}^\forall\mathfrak{p} \in \operatorname{Spec} R, \operatorname{ht}\mathfrak{p} = 1 : R_\mathfrak{p}$ is regular

(S2) ${}^\forall\mathfrak{p} \in \operatorname{Spec} R, \operatorname{ht}\mathfrak{p} \geq 2 : {}^\exists$ regular sequence on $R_\mathfrak{p}$ as an $R_\mathfrak{p}$-module of length ≥ 2.

Hint. Normal $\Rightarrow$ (R1) by Theorem 1.120. Suppose $\operatorname{ht}\mathfrak{p} \geq 2$, since $R_\mathfrak{p}$ is a domain any $a \neq 0 \in \mathfrak{p}R_\mathfrak{p}$ is a nzd. $\dim R_\mathfrak{p} \geq 2$ + Theorem 1.120 $\Rightarrow \mathfrak{p}R_\mathfrak{p} \notin \operatorname{Ass}_{R_\mathfrak{p}}(R_\mathfrak{p}/aR_\mathfrak{p})$, then by prive avoidance $\mathfrak{p}R_\mathfrak{p} \nsubseteq \cup_{\mathfrak{q}\in\operatorname{Ass}}\mathfrak{q}$, pick any $b \in LHS - RHS$, then a, b is a regular sequence on $R_\mathfrak{p}$.

Assume (R1) + (S2): suppose $x \in K(R)$ is integral over R, $x \notin R$, then $R \hookrightarrow R[x]$ is finite, so ${}^\exists\mathfrak{p} \in \operatorname{Ass}_R(R[x]/R)$, so $\mathfrak{p}R_\mathfrak{p} = \operatorname{ann}_{R_\mathfrak{p}}(y + R_\mathfrak{p}) \in \operatorname{Ass}_{R_\mathfrak{p}}(R_\mathfrak{p}[x]/R_\mathfrak{p})$. Then $y\mathfrak{p}R_\mathfrak{p} \subseteq R_\mathfrak{p}$ and $y \notin R_\mathfrak{p}$. Writing $y = r/s$, we have $r\mathfrak{p}R_\mathfrak{p} \subseteq sR_\mathfrak{p}$, and $r \notin sR_\mathfrak{p}$. Then $\mathfrak{p}R_\mathfrak{p} = \operatorname{ann}_{R_\mathfrak{p}}(r + sR_\mathfrak{p}) \in \operatorname{Ass}_{R_\mathfrak{p}}(R_\mathfrak{p}/sR_\mathfrak{p})$. By correspondence $\mathfrak{p} \in \operatorname{Ass}_R(R/s)$. By Lemma 1.126, $\operatorname{ht}\mathfrak{p} = 1$, then by (R1) $x \in R_\mathfrak{p}$, then $(R[x]/R)_\mathfrak{p} = R_\mathfrak{p}[x]/R_\mathfrak{p} = 0$, a contradiction.

Corollary 1.128. A regular local ring is normal.

Hint. Check (R1) + (S2), follows from the localization problem Corollary 1.119.

1.11 Completion

We first give an ideal theoretic characterization of flatness, and how Hom changes under base change.

Proposition 1.129. Let R be any ring, M an R-module, TFAE:

1. M is R-flat
2. $\operatorname{Tor}_1^R(R/I, M) = 0$ for all finitely generated ideal $I \lhd R$
3. The induced map $I \otimes_R M \to M$ is injective for all finitely generated ideals $I \lhd R$.

Hint. (1) $\Rightarrow$ (2): definition of flatness. (2) $\Rightarrow$ (3): consider the Tor sequence associated to the ses $0 \to I \to R \to R/I \to 0$. (3) $\Rightarrow$ (1): first show $\operatorname{Tor}_1^R(N, M) = 0$ for all N f.g. over R by induction on the number n of generaters. The case $n = 0$ is evident, the case $n = 1$ is established by (3). Let $N = Rx_1 + \cdots + Rx_n$, then consider submodule $N' = Rx_1 + \cdots + Rx_{n-1}$, consider the Tor sequence associated to the ses $0 \to N' \to N \to N/N' \to 0$. This shows $(P \hookrightarrow Q) \otimes_R M$ is injective if Q/P is f.g.

Now pass to arbitrary module by the observation that any R-module Q can be identified with the filtered colimit $\varinjlim_\alpha Q_\alpha$, where $\{Q_\alpha\}$ is the collection of f.g. submodules of Q. Explicitly, if $\varphi : P \hookrightarrow Q$ in an injection, let φ_α be the restriction $\varphi|_{\varphi^{-1}(Q_\alpha)}$. Then $\varphi_\alpha \otimes \mathrm{id}_M : \varphi^{-1}(Q_\alpha) \otimes M \to Q_\alpha \otimes M$ is injective for all α. Since tensor product commutes with filtered colimit (RAPL) and universal property of colim, $\varphi \otimes \mathrm{id}_M = \varinjlim_\alpha \varphi_\alpha \otimes \mathrm{id}_M$ is injective.

Let S be an R-algebra, and M, N be R-modules, then there is a natural map

$$\begin{aligned} S \otimes_R \operatorname{Hom}_R(M,N) &\xrightarrow{f_{M,N}} \operatorname{Hom}_S(S \otimes_R M, S \otimes_R N) \\ s \otimes \varphi &\longmapsto \ell_s \otimes \varphi \end{aligned}$$

***Exercise* 1.130.** If $f_{M,N}$ and $f_{P,N}$ are isomorphisms, so is $f_{M\otimes P,N}$.

Hint. Finite dirct sum commutes with tensor product and Hom:

$$\begin{tikzcd} \left(S \otimes_R \operatorname{Hom}_R(M,N)\right) \oplus \left(S \otimes \operatorname{Hom}_R(P,N)\right) \arrow[r, "f_{M,N}\oplus f_{P,N}"] \arrow[d, "\wr"] & \operatorname{Hom}_S(S \otimes_R M, S \otimes_R N) \oplus \operatorname{Hom}_S(S \otimes_R P, S \otimes_R N) \arrow[d, "\wr"] \\ S \otimes_R (\operatorname{Hom}_R(M \oplus P, N)) \arrow[r, "f_{M\oplus P,N}"] & \operatorname{Hom}_S(S \otimes_R (M \oplus P), S \otimes_R N) \end{tikzcd}$$

Proposition 1.131. Let $\varphi : R \to S$ be a flat ring map, $M, N \in Mod_R$, then

1. M f.g. $\Rightarrow f_{M,N}$ injective
2. M f.p. $\Rightarrow f_{M,N}$ isomorphism.

Hint. Explicitly work out the case $M = R$, the finite free case is then established by the exercise. If M f.g., there is some ses $0 \to K \to R^{\oplus n} \to M \to 0$. Left-exactness of Hom and exactness of $- \otimes_R S$ gives commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & S \otimes_R \operatorname{Hom}_R(M,N) \arrow[r] \arrow[d, "f_{M,N}"] & S \otimes_R \operatorname{Hom}_R(R^{\oplus n},N) \arrow[r] \arrow[d, "f_{R^{\oplus n},N}", "\wr"'] & S \otimes_R \operatorname{Hom}_R(K,N) \arrow[d, "f_{K,N}"] \\ 0 \arrow[r] & \operatorname{Hom}_S(S \otimes_R M, S \otimes_R N) \arrow[r] & \operatorname{Hom}_S(S \otimes_R R^{\oplus n}, S \otimes_R N) \arrow[r] & \operatorname{Hom}_S(S \otimes_R K, S \otimes_R N) \end{tikzcd}$$

The middle isomorphism shows the left map is injective. If M is f.p., the right map is also injective, in this case a diagram chase shows the left map is also surjective.

Definition 1.132 (Algebraic completion)**.** Let R be a ring, $I \lhd R$ an ideal, define the algebraic completion of R with respect to I:

$$\widehat{R}^I := \varprojlim_n (R/I^n, \pi_{m>n})_{n\in\mathbb{Z}_{\geq 1}}$$

where $\pi_{m>n} : R/I^m \to R/I^n$ are natural projection maps. Note equivalently,

$$\widehat{R}^I = \varprojlim_n \left\{ R/I \twoheadleftarrow R/I^2 \twoheadleftarrow R/I^3 \twoheadleftarrow \cdots \right\}$$

Definition 1.133 (Topological completion)**.** Let R be a ring, $I \lhd R$ an ideal, recall the filtration $\{I^n\}_{n\geq 0}$ defines the I-adic topology on R, where the neighborhood basis of $0 \in R$ is given by the family $\{I^n\}_{n\geq 0}$. Define **Cauchy sequence** and **null sequence** in R (wrt I) by

$$C_I(R) := \left\{ (a_n)_n \in \prod_{n\geq 1} R : {}^\forall j, a_n - a_m \in I^j \text{ for } m,n \gg 0 \right\}$$

$$N_I(R) := \left\{ (a_n)_n \in \prod_{n\geq 1} R : {}^\forall j, a_n \in I^j \text{ for } n \gg 0 \right\}.$$

Then $N_I(R)$ is an ideal in $\prod_{n\geq 1} R$, hence also an ideal in $C_I(R)$. The **topological completion of R wrt I** is the quotient $C_I(R)/N_I(R)$.

Proposition 1.134. There is a natural isomorphism $C_I(R)/N_I(R) \to \widehat{R}^I$.

Hint. For each $j \geq 1$, given $(x_n)_n \in C_I(R)$, the sequence $(x_n + I^j)_n$ stabilizes, defining a natural map $\phi_j : C_I(R) \to R/I^j$. It factors through the quotient $C_I(R)/N_I(R)$, hence we have a map $\phi : C_I(R)/N_I(R) \to \widehat{R}^I$. A sequence $(a_n)_n$ with $a_j \equiv a_{j+1} \bmod I^j$ for all $j \geq 1$ is evidently Cauchy, so we have surjectivity. Injectivity is evident.

Definition 1.135 (I-adic completion of module)**.** For $I \lhd R$, and $M \in Mod_R$, the I-adic completion of M wrt I is

$$\widehat{M}^I := \varprojlim_{n\geq 1} M/I^n M.$$

The quotient maps $M \to M/I^n M$ induces a natural map $M \to \widehat{M}^I$. Say M is **I-adically complete** if the module map $M \to \widehat{M}^I$ is an iso, R is I-adically complete if the ring map $R \to \widehat{R}^I$ is an iso.

Lemma 1.136. For $I \lhd R$, given $i \in I$, then $((1-i) + I^n)_{n\geq 1} \in \left(\widehat{R}^I\right)^\times$.

Hint.

$$\frac{1}{1-i} = 1 + i + i^2 + i^3 + \cdots.$$

Lemma 1.137. Given $I \lhd R$, $M \in Mod_R$.

1. $\ker(M \to \widehat{M}^I) = \cap_{n\geq 1} I^n M$
2. I-adic topology on M is Hausdorff $\iff$ $M \to \widehat{M}^I$ injective
3. If R is I-adically complete, then $^\forall i \in I$, $(1-i) \in R^\times$
4. If R is I-adically complete, then $I \subseteq Jac(R)$.

Hint. (4): $1 + ai = 1 - (-ai)$.

Corollary 1.138. Let R be a noetherian ring, $I \lhd R$ a proper ideal, M f.g. over R, then

1. R is a domain $\Rightarrow R \to \widehat{R}^I$ injective.
2. R is local $\Rightarrow R \to \widehat{R}^I$ injective.

Hint. (1): use Artin-Rees and Cayley-Himailton to show $\cap_{n\geq 1} I^n = 0$. (2): Krull's intersection.

Lemma 1.139. $M \mapsto \widehat{M}^I$ is a covariant functor,the natural map $M \to \widehat{M}^I$ factors as

$$\begin{tikzcd} M \arrow[rr] \arrow[dr] & & \widehat{M}^I \\ & M \otimes_R \widehat{R}^I \arrow[ur] & \end{tikzcd}$$

For $\varphi : M \to N$ linear over R, there is a commutative diagram

$$\begin{tikzcd} M \otimes_R \widehat{R}^I \arrow[r, "\varphi \otimes \mathrm{id}"] \arrow[d] & N \otimes_R \widehat{R}^I \arrow[d] \\ \widehat{M}^I \arrow[r, "\widehat{\varphi}^I"] & \widehat{N}^I \end{tikzcd}$$

Hence there is a natural transformation $- \otimes_R \widehat{R}^I \Rightarrow \widehat{(-)}^I$.

Lemma 1.140.

1. $P \xrightarrow{f} Q \to 0$ over $R \implies \widehat{P}^I \twoheadrightarrow \widehat{Q}^I$ over $\widehat{R}^I$
2. If M is f.g over R, then $M \otimes_R \widehat{R}^I \twoheadrightarrow \widehat{M}^I$ over $\widehat{R}^I$.

Hint. (1) build lifting inductively through diagram chase:

$$t_n + I^n(\ker f) \longmapsto (p_n - p_{n+1}) + I^n P \longmapsto 0$$

$$\begin{tikzcd}
\frac{\ker f}{I^n(\ker f)} \arrow[r] & \frac{P}{I^n P} \arrow[r] & \frac{Q}{I^n Q} \arrow[r] & 0 \\
& p_n + I^n P \arrow[r, mapsto] & q_n + I^n Q & \\
\frac{\ker f}{I^{n+1}(\ker f)} \arrow[r] \arrow[uu] & \frac{P}{I^{n+1} P} \arrow[r] \arrow[uu] & \frac{Q}{I^{n+1} Q} \arrow[r] \arrow[uu] & 0 \\
& (p_{n+1} + t_n) + I^{n+1} P \arrow[uu, mapsto] & & \\
& p_{n+1} + I^{n+1} P \arrow[r, mapsto] & q_{n+1} + I^{n+1} Q \arrow[uuu, mapsto] &
\end{tikzcd}$$

(2) Given $R^{\oplus n} \to M \to 0$, get

$$\begin{tikzcd}
R^{\oplus n} \times_R \widehat{R}^I \arrow[r] \arrow[d, "\wr"] & M \otimes_R \widehat{R}^I \arrow[r] \arrow[d] & 0 \\
\widehat{R^{\oplus n}}^I \arrow[r] & \widehat{M}^I \arrow[r] & 0
\end{tikzcd}$$

***Question* 1.141.** Is $\widehat{M}^I$ I-adically complete?

***Remark* 1.142.** The projection maps $\pi_j : \widehat{M}^I \to M/I^jM$ are surjective, with kernel

$$\begin{aligned}
\ker \pi_j &= \left\{ (m_n + I^n M)_{n\geq 1} : {}^\forall n, m_n \equiv 0 \bmod I^j M \forall n \right\} \\
&= \left\{ (m_n + I^n M)_{n\geq 1} : {}^\forall n \leq j, m_n \in I^n M, {}^\forall n > j, m_n \in I^j M \right\} \\
&\cong \varprojlim_{n \geq 1} \frac{I^j M}{I^{j+n} M} = \widehat{I^j M}^I.
\end{aligned}$$

Proposition 1.143. Let $I \lhd R$ be finitely generated, then

1. ${}^\forall j \geq 1$, $\ker \pi_j = I^j \widehat{M}^I$, hence $\widehat{M}^I / I^j \widehat{M}^I \cong M/I^jM$ for all $j \geq 1$
2. the map $M/I^jM \to \widehat{M}^I / I^j \widehat{M}^I$ induced by the canonical map $M \to \widehat{M}^I$ are isomorphisms
3. $\widehat{M}^I$ is I-adically complete.

Hint. (1): suppose $I^j = (i_1, \dots, i_k)$, then have $M^{\oplus k} \twoheadrightarrow I^j M$ by $(m_1, \dots, m_k) \mapsto i_1 m_1 + \cdots + i_k m_k$. Taking completion preserves surjectivity, and the resulting map is still given by $\left(\widehat{M}^I\right)^{\oplus k} \twoheadrightarrow \widehat{I^j M}^I$ by $(\alpha_1, \dots, \alpha_k) \mapsto i_1 \alpha_1 + \cdots + i_k \alpha_k$, hence $I^j \widehat{M}^I = \widehat{I^j M}^I$.

(2): $M/I^jM \to \widehat{M}^I / I^j \widehat{M}^I \underset{\sim}{\xrightarrow{\pi_j}} M/I^jM$

(3):

$$\begin{tikzcd}
\cdots \arrow[r] & M/I^{j+1}M \arrow[r] \arrow[d, "\wr"] & M/I^{j}M \arrow[r] \arrow[d, "\wr"] & M/I^{j-1}M \arrow[r] \arrow[d, "\wr"] & \cdots \\
\cdots \arrow[r] & \widehat{M}^I/I^{j+1}\widehat{M}^I \arrow[r] & \widehat{M}^I/I^{j}\widehat{M}^I \arrow[r] & \widehat{M}^I/I^{j-1}\widehat{M}^I \arrow[r] & \cdots
\end{tikzcd}$$

induces isomorphism, which is the canonical map:

$$\begin{aligned}
\widehat{M}^I &\longrightarrow \widehat{\widehat{M}^I}^I \\
(m_n+I^nM)_n &\longmapsto (m_n+I^jM)_{j\geq 1}+I^n\widehat{M}^I=(m_j+I^jM)_{j\geq 1}+I^n\widehat{M}^I
\end{aligned}$$

***Remark* 1.144.** Given ring map $R\to S$, and $I\lhd R$, $J:=IS\lhd S$. Viewing S as an R-module, we have $S/I^nS=S/(IS)^nS$, hence $\widehat{S}^I=\widehat{S}^J$.

Corollary 1.145. Let $I\lhd R$ f.g.,

1. $\widehat{R}^I$ is $I\widehat{R}^I$-adically complete
2. $I\widehat{R}^I\subseteq Jac(\widehat{R}^I)$
3. the canonical projection $\pi:\widehat{R}^I\twoheadrightarrow R/I$ induces bijection $mSpec(R/I)\to mSpec(\widehat{R}^I)$
4. if $(R,\mathfrak{m})$ local, $\mathfrak{m}$ f.g., then $(\widehat{R}^{\mathfrak{m}},\mathfrak{m}\widehat{R}^{\mathfrak{m}})$ is local.

Hint. (1): Remark 1.144. (2) Lemma 1.137.
(3): $mSpec(R/I)\leftrightarrow mSpec(\widehat{R}^I/I\widehat{R}^I)\leftrightarrow mSpec(\widehat{R}^I)$.
(4): $(R/\mathfrak{m},(0))$ is local, contraction of (0) is $\ker(R\to\widehat{R}^{\mathfrak{m}})$.

***Exercise* 1.146.** Let $I\lhd R$ f.g., then $I^n/I^m\cong I^n\widehat{R}^I/I^m\widehat{R}^I$ for $m\geq n>0$.

Hint.

$$\begin{tikzcd}
0 \arrow[r] & I^n/I^m \arrow[r] \arrow[d] & R/I^m \arrow[r] \arrow[d, "\wr"] & R/I^n \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & I^n\widehat{R}^I/I^m\widehat{R}^I \arrow[r] & \widehat{R}^I/I^m\widehat{R}^I \arrow[r] & \widehat{R}^I/I^n\widehat{R}^I \arrow[r] & 0
\end{tikzcd}$$

***Example* 1.147.** The p-adic integers: $\mathbb{Z}_p=\widehat{\mathbb{Z}}^{(p)}$.

***Example* 1.148.** $\widehat{R[x_1,\ldots,x_k]}^{(\underline{x})}\cong R[\![x_1,\ldots,x_k]\!]$.

Hint.

$$\begin{tikzcd}
R[x_1,\ldots,x_k]/(\underline{x})^j \arrow[r] \arrow[d, "\pi_{j>k}"] \arrow[dr, phantom, "\square"] & \{f\in R[x_1,\ldots,x_n]:\deg f\leq j-1\} \arrow[d, "\pi"] \\
R[x_1,\ldots,x_k]/(\underline{x})^k \arrow[r] & \{f\in R[x_1,\ldots,x_n]:\deg f\leq k-1\}
\end{tikzcd}$$

Proposition 1.149. Let R be noetherian, then $\widehat{R}^I$ is noetherian for any $I\lhd R$.

Hint. Let $I=(i_1,\ldots,i_k)$, then $R[x_1,\ldots,x_k]\to R\to 0$, $x_j\mapsto i_j$. Taking completion realizes $\widehat{R}^I$ as a quotient of $R[\![x_1,\ldots,x_k]\!]$.

Corollary 1.150. Let R be noetherian, $I \lhd R$ and M f.g. over R, then

1. $\widehat{M}^I$ is a noetherian $\widehat{R}^I$-module
2. If $I = \mathfrak{m}$ is maximal, $(\widehat{R}^{\mathfrak{m}}, \mathfrak{m}\widehat{R}^{\mathfrak{m}})$ is noetherian local.

Hint. (1): I f.g. $\Rightarrow M \otimes_R \widehat{R}^I \twoheadrightarrow \widehat{M}^I \Rightarrow \widehat{M}^I$ f.g. over $\widehat{R}^I$. (2): Corollary 1.145.

***Example* 1.151.** Completion does not commute with arbitrary direct sum.

Take $R = k[\![x]\!]$, $M = \oplus_{n\geq 1} k[\![x]\!]$, $I = (x)k[\![x]\!]$. Since R is I-adically complete, $\oplus_{n\geq 1}\widehat{k[\![x]\!]}^I = \oplus_{n\geq 1} k[\![x]\!]$. Consider $\alpha = (\alpha_n + I^n M)_{n\geq 1} \in \widehat{M}^I$ where $\alpha_n = (1, x, \ldots, x^{n-1}, 0, \ldots)$. Since $(1, x, x^2, \ldots) \notin \oplus_{n\geq 1} k[\![x]\!]$, $\alpha \notin \oplus_{n\geq 1} k[\![x]\!]$.

Lemma 1.152. Let $I \lhd R$ be f.g. then,

1. $I^{n-1}/I^n \cong I^{n-1}\widehat{R}^I/I^n\widehat{R}^I$ for all $n \geq 1$
2. $\mathrm{gr}_I(R) \cong \mathrm{gr}_{I\widehat{R}^I}(\widehat{R}^I)$.

Hint. Exercise 1.146.

Theorem 1.153. Let $(R, \mathfrak{m})$ be noetherian local,

1. $R/\mathfrak{m} \cong \widehat{R}^{\mathfrak{m}}/\mathfrak{m}\widehat{R}^{\mathfrak{m}}$ as fields
2. $\mu_R(\mathfrak{m}) = \mu_{\widehat{R}^{\mathfrak{m}}}(\mathfrak{m}\widehat{R}^{\mathfrak{m}})$
3. $\dim R = \dim \widehat{R}^{\mathfrak{m}}$
4. R is regular if and only if $\widehat{R}^{\mathfrak{m}}$ is regular.

Hint. (1): Lemma 1.152 (2): NAK + Lemma 1.152 (3): Lemma 1.152 $\Rightarrow H_{\mathfrak{m},R} = H_{\mathfrak{m}\widehat{R}^{\mathfrak{m}},\widehat{R}^{\mathfrak{m}}}$
(4): (3) + (2).

Proposition 1.154.

1. There is a bijection

$$mSpec(R[\![x_1, \ldots, x_n]\!]) \longleftrightarrow mSpec(R)$$

$$(\mathfrak{m}, x_1, \ldots, x_n) \longleftarrow\!\!\shortmid\ \mathfrak{m}$$

2. For $f \in R[\![x_1, \ldots, x_n]\!]$, $f \in R[\![x_1, \ldots, x_n]\!]^\times \iff f(0) \in R^\times$.

Hint. (1) is induced by the projection $R[\![x_1, \ldots, x_n]\!] \twoheadrightarrow R[x_1, \ldots, x_n]/(x_1, \ldots, x_n) \cong R$.
(2): the projection from (1) is a ring map, so have $\Rightarrow$. Conversely if $f \in (\mathfrak{m}, x_1, \ldots, x_n)$, then $f(0) \in \mathfrak{m}$, so have $\Leftarrow$.

1.11.1 $\widehat{R}^I$ is R-flat

Lemma 1.155. Let R be a ring, $I \lhd R$, and $0 \to N \to M \to P \to 0$ ses, then

$$0 \longrightarrow \varprojlim_{n\geq 1} \frac{N}{I^n M \cap N} \longrightarrow \widehat{M}^I \longrightarrow \widehat{P}^I \longrightarrow 0$$

is exact as well.

Hint. Have ses's

$$0 \longrightarrow \tfrac{N}{I^n M\cap N} \longrightarrow \tfrac{M}{I^n M} \longrightarrow \tfrac{P}{I^n P} \longrightarrow 0 \qquad n \geq 1$$

Inverse limit functor is left exact + Lemma 1.140.

Proposition 1.156. Let R be noetherian, $I \lhd R$, and $0 \to N \to M \to P \to 0$ ses of f.g. R-modules, then

$$0 \longrightarrow \widehat{N}^I \longrightarrow \widehat{M}^I \longrightarrow \widehat{P}^I \longrightarrow 0$$

is exact as well.

Hint. Artin-Rees (Lemma 1.11): $\widehat{N}^I \xrightarrow{\sim} \varprojlim_{n\geq 1} \dfrac{N}{I^n M \cap N}$ + Lemma 1.155.

Theorem 1.157. Let R be noetherian, $I \lhd R$, then

1. M f.g. over $R \Rightarrow M \otimes_R \widehat{R}^I \to \widehat{M}^I$ is an iso
2. $\widehat{R}^I$ is R-flat.

Hint. (1) first consider the case M is finite free, then f.g. over noetherian ring $\Rightarrow$ f.p. $\Rightarrow$ exact sequence $R^{\oplus m} \to R^{\oplus n} \to M \to 0$. Conclude using diagram (using Proposition 1.156)

$$\begin{tikzcd}
 & R^{\oplus m} \otimes_R \widehat{R}^I \arrow[r] \arrow[d, "\wr"] & R^{\oplus n} \otimes_R \widehat{R}^I \arrow[r] \arrow[d, "\wr"] & M \otimes_R \widehat{R}^I \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & \widehat{R^{\oplus m}}^I \arrow[r] & \widehat{R^{\oplus n}}^I \arrow[r] & \widehat{M}^I \arrow[r] & 0
\end{tikzcd}$$

(2) for $\mathfrak{a} \lhd R$: by (1) and Proposition 1.156

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{a} \otimes_R \widehat{R}^I \arrow[r] \arrow[d, "\wr"] & R \otimes_R \widehat{R}^I \arrow[d, "\wr"] \\
0 \arrow[r] & \widehat{\mathfrak{a}}^I \arrow[r] & \widehat{R}^I
\end{tikzcd}$$

Corollary 1.158. Let R be noetherian, $I \lhd R$,

$$\widehat{R}^I \text{ is faithfully flat over } R \iff I \subseteq Jac(R).$$

Hint. ($\Leftarrow$): for $\mathfrak{m} \in mSpec(R)$, extension of $\mathfrak{m}\widehat{R}^I$ along projection $\widehat{R}^I \twoheadrightarrow R/I$ is $\mathfrak{m}/I \subsetneq R/I$, hence $\mathfrak{m}\widehat{R}^I \neq \widehat{R}^I$, conclude using Proposition 1.98.

($\Rightarrow$): Corollary 1.145 $\Rightarrow I\widehat{R}^I \subseteq Jac(\widehat{R}^I)$. For $i \in I$, $x \in R$, $R/(1+xi) \otimes_R \widehat{R}^I = \widehat{R}^I/(1+xi) = 0$. Conclude using $\widehat{R}^I$ is faithfully flat.

Index