WHAT HAVE I LEARNED TODAY?

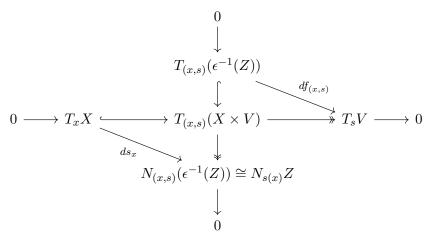
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1. 2019 May

05-23: Today I finished writing up the proof of the Riemann-Roch Theorem (finally!). The theorem states: if X is a compact Riemann surface, $L \in \operatorname{Pic}(X)$, then $\chi(X, L) = \deg(L) + \chi(X, \mathcal{O}_X)$. The proof uses (1) there's a positive line bundle \mathcal{O}_X on X, (2) the ability to make $L \otimes \mathcal{O}_X(n)$ and $\mathcal{O}_X(n)$ bpf for all large n (this uses Kodaira vanishing and Hodge Theorem) (3) Bertini's Theorem to find transverse sections of bpf line bundles (which depends on Sard's Theorem), (4) ideal sheaf sequence associated to some smooth hypersurface, (5) the fact that we can compute cohomolgies of $i_*i^{-1}\mathcal{L}$ and $j_*\mathcal{O}_X$ easily by taking open cover that covers each point of the discrete set of points (Leray's Theorem, flasque sheaves are acyclic), and (6) $c_1(L)$ is Poincaré dual to the zero locus of some transverse section of L.

If we interpret $c_1(\omega_X^{\vee})$ as the Gauss curvature on X, then the Gauss-Bonnet Theorem $\int_X c_1(\omega_X^{\vee}) = 2 - 2g$ becomes a easy consequence of the Riemann-Roch Theorem and Serre Duality.

05-22: Today I learned the proof of the Bertini's Theorem. The theorem says, if $L \in \operatorname{Pic}(X)$, and $V \subseteq \Gamma(X, L)$ a bpf linear system, then the set of elements of V that are not transverse to the zero section has measure zero. The main tool is the Sard's Theorem. We consider the evaluation map $\epsilon: X \times V \to L$, show it's a submersion, and then $\epsilon^{-1}(Z) \subseteq X \times V$ is an embedded submanifold. Then apply Sard's Theorem to $f: \epsilon^{-1}(Z) \hookrightarrow X \times V \xrightarrow{\operatorname{pr}_2} V$. A section s is a regular value of f if and only if it's transverse to the zero section. To prove this fact we consider the following diagram, by a diagram chase we show ds_x is surjective iff $df_{(x,s)}$ is surjective.



I have also learned an important technique of showing certain maps are surjective via sheaf cohomology. One realization of this idea is: if X is a compact Riemann surface, $\mathcal{O}_X(1)$ is a positive line budle, then for any $L \in \operatorname{Pic}(X)$, there exists $N \in \mathbb{N}$ such that $L \otimes \mathcal{O}_X(n)$ is bpf (n > N). We reduce the problem of showing $L \otimes \mathcal{O}_X(n)$ is bpf to showing the s.e.s of sheaves of \mathcal{O}_X -modules

$$0 \longrightarrow \mathcal{L} \otimes \mathcal{O}_X(n) \mathcal{I}_x \longrightarrow \mathcal{L} \otimes \mathcal{O}_X(n) \longrightarrow i_*((L \otimes \mathcal{O}_X(n))_x) \longrightarrow 0$$

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is exact on global sections. Prove exactness by showing $H^1(X, L \otimes \mathcal{O}_X(n) \otimes \mathcal{I}_x) = 0$ using Kodaira vanishing, because we can make $\omega_X^{\vee} \otimes L \otimes \mathcal{I}_x \otimes \mathcal{O}_X(n)$ positive for large n.

05-21: Today I learned the degree theory of smooth maps between compact, oriented n-manifolds. We first investigate the case of proper maps between \mathbb{R}^n . Let $f:\mathbb{R}^n\to\mathbb{R}^n$ be a proper map, the pullback $f^*:H^n_{dR}(\mathbb{R}^n)\to H^n_{dR}(\mathbb{R}^n)$ maps compactly supported n-forms to compactly supported forms, hence $f^*:H^n_{c}(\mathbb{R}^n)\to H^n_{c}(\mathbb{R}^n)$. Let ω be a generator of $H^n_{c}(\mathbb{R}^n)$ (this means $\int_{\mathbb{R}^n}\omega=1$, this is possible by the Poincaré Lemma that $H^n_{c}(\mathbb{R}^n)\cong\mathbb{R}$), we define the degree $\deg(f):=\int_M f^*\omega$. To prove this is an integer, pick a regular value $q\in\mathbb{R}^n$ of f by Sard's Theorem. Since f is proper, $f^{-1}(q)$ is a finite set of points $\{p_1,\ldots,p_K\}$. Since f is a regular value, the map f is locally a diffeomorphism when restricted to small neighborhoods f0 of each f1 such that f2 are disjoint and f3 and f4 fractional f5. By partition of unity we may pick a generator f6 of f7, with Supp f7 and f8 are f8 and each f9 fractional f9 fractional f9. Then f9 fractional f9 fractio

References