WHAT HAVE I LEARNED TODAY?

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05-22: Today I learned the proof of the Bertini's Theorem. The theorem says, if $L \in \mathrm{Pic}(X)$, and $V \subseteq \Gamma(X, L)$ a bpf linear system, then the set of elements of V that are not transverse to the zero section has measure zero. The main tool is the Sard's Theorem. We consider the evaluation map $\epsilon : X \times V \to L$, show it's a submersion, and then $\epsilon^{-1}(Z) \subseteq X \times V$ is an embedded submanifold. Then apply Sard's Theorem to $f : \epsilon^{-1}(Z) \hookrightarrow X \times V \xrightarrow{\mathrm{pr}_2} V$. A section s is a regular value of f if and only if it's transverse to the zero section. To prove this fact we consider the following diagram, by a diagram chase we show ds_x is surjective iff $df_{(x,s)}$ is surjective.

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & T_{(x,s)}(\epsilon^{-1}(Z)) \arrow[d, hook] \arrow[dr, "df_{(x,s)}"] & & \\
0 \arrow[r] & T_xX \arrow[r, hook] \arrow[dr, "ds_x"'] & T_{(x,s)}(X \times V) \arrow[r, two heads] \arrow[d, two heads] & T_sV \arrow[r] & 0 \\
 & & N_{(x,s)}(\epsilon^{-1}(Z)) \cong N_{s(x)}Z \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

I have also learned an important technique of showing certain maps are surjective via sheaf cohomology. One realization of this idea is: if X is a compact Riemann surface, $\mathcal{O}_X(1)$ is a positive line budle, then for any $L \in \mathrm{Pic}(X)$, there exists $N \in \mathbb{N}$ such that $L \otimes \mathcal{O}_X(n)$ is bpf ($n > N$). We reduce the problem of showing $L \otimes \mathcal{O}_X(n)$ is bpf to showing the s.e.s of sheaves of $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathcal{L} \otimes \mathcal{O}_X(n)\,\mathcal{I}_x \longrightarrow \mathcal{L} \otimes \mathcal{O}_X(n) \longrightarrow i_*((L \otimes \mathcal{O}_X(n))_x) \longrightarrow 0$$

is exact on global sections. Prove exactness by showing $H^1(X, L \otimes \mathcal{O}_X(n) \otimes \mathcal{I}_x) = 0$ using Kodaira vanishing, because we can make $\omega_X^\vee \otimes L \otimes \mathcal{I}_x \otimes \mathcal{O}_X(n)$ positive for large n.

05-21: Today I learned the degree theory of smooth maps between compact, oriented n-manifolds. We first investigate the case of proper maps between $\mathbb{R}^n$. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a proper map, the pullback $f^* : H^n_{dR}(\mathbb{R}^n) \to H^n_{dR}(\mathbb{R}^n)$ maps compactly supported n-forms to compactly supported forms, hence $f^* : H^n_c(\mathbb{R}^n) \to H^n_c(\mathbb{R}^n)$. Let ω be a generator of $H^n_c(\mathbb{R}^n)$ (this means $\int_{\mathbb{R}^n} \omega = 1$, this is possible by the Poincaré Lemma that $H^n_c(\mathbb{R}^n) \cong \mathbb{R}$), we define the degree $\deg(f) := \int_M f^*\omega$. To prove this is an integer, pick a regular value $q \in \mathbb{R}^n$ of f by Sard's Theorem. Since f is proper, $f^{-1}(q)$ is a finite set of points $\{p_1, \ldots, p_K\}$. Since q is a regular value, the map f is locally a diffeomorphism when restricted to small neighborhoods V_i of each p_i such that V_i are disjoint and

$f(V_i) = W$ for all i. By partition of unity we may pick a generator ω of $H_c^n(\mathbb{R}^n)$, with $\operatorname{Supp}\omega \subseteq W$. Then $\int_M f^*\omega = \sum_i \int_{V_i} f^*\omega$ and each $\int_{V_i} f^*\omega$ is $+1$ if df_{p_i} is orientation preserving and -1 if df_{p_i} is orientation reversing. To prove the case of smooth maps between compact, oriented n-manifolds, we assume the fact that $H^n(M) \cong \mathbb{R}$ as well. This fact could be proved from Poincaré Duality.

References