Hypersurfaces in $\mathbb{P}^n$

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In this note, we would like to consider hypersurfaces C in $\mathbb{P}^n$, defined by a single homogeneous polynomial $F \in S^d\langle x_0, \dots, x_n\rangle$. We would in particular aim to derive the *degree-genus-formula*, that is,

$$g(C) = \binom{d-1}{n}.$$

In the case of plane curves, that is $C \subseteq \mathbb{P}^2$, this gives us

$$g(C) = \frac{1}{2}(d-1)(d-2).$$

1 The Setup

Before diving in, we fix some notations. Let $L \subset \mathbb{P}^n \times \mathbb{C}^{n+1}$ denote the tautological line bundle $L = \{[\ell] \times v : [\ell] \in \mathbb{P}^n, v \in \ell\}$, let $\mathcal{O}(-1)$ denote the associated sheaf of holomorphic sections of L. Let $-L$ denote the dual bundle $-L := L^\vee$. For $k \in \mathbb{Z}$, define

$$kL := \begin{cases} L^{\otimes k}, & k > 0 \\ \mathbb{P}^n \times \mathbb{C}, & k = 0 \\ (-L)^{\otimes(-k)}, & k < 0. \end{cases}$$

Let $\mathcal{O}(-k)$ denote the sheaf of sections of kL.

If $E \xrightarrow{\pi} X$ is a holomorphic vector bundle, $Y \xrightarrow{f} X$ is a holomorphic map that is transverso to π. Then we may form the pullback bundle f^*E that fits into the pullback diagram:

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d, "f^*\pi"'] & E \arrow[d, "\pi"] \\ Y \arrow[r, "f"] & X \end{tikzcd}.$$

Lemma 1.1. For $d \in \mathbb{Z}_{>0}$, there is a canonical isomorphism $\Gamma(\mathbb{P}^n, \mathcal{O}(d)) \cong S^d\langle x_0, \dots, x_n\rangle$.

Lemma 1.2. Let $\pi : E \to X$ be a holomorphic vector bundle of rank r, let $s \in \Gamma(X, E)$ be a global section that is transverse to the zero section $z \in \Gamma(X, E)$, which were defined by $z(x) := (x, 0)$ (sending x to the zero element in vector space E_x). Define $Y \subseteq X$ by $Y = \{x \in X : s(x) = (x, 0)\}$, then

$$N_{Y \subseteq X} \cong E|_Y.$$

Let $C \subseteq \mathbb{P}^n$ be a smooth hypersurface defined as the zeros of a homogeneous polynomial $F \in S^d\langle x_0, \ldots, x_n\rangle \cong \Gamma(\mathbb{P}^n, \mathcal{O}(d))$ of degree d that's transverse to the zero section. By Lemma 1.2 $N_{C\subseteq\mathbb{P}^n} \cong (-dL)|_C$. We have a short exact sequence of vector bundles over C:

$$0 \longrightarrow T_C \longrightarrow T_{\mathbb{P}^n}|_C \longrightarrow (-dL)|_C \longrightarrow 0.$$

Taking the duals we have a short exact sequence:

$$0 \longrightarrow (dL)|_C \longrightarrow T^*_{\mathbb{P}^n}|_C \longrightarrow T^*_C \longrightarrow 0.$$

By adjunction formula,

$$\det(T^*_{\mathbb{P}^n}|_C) \cong \det(dL|_C) \otimes \det(T^*_C) \cong dL|_C \otimes \det(T^*_C).$$

Tensoring with $(-dL)|_C$ we get,

$$[(-dL) \otimes \det(T^*_{\mathbb{P}^n})]|_C = (-dL)|_C \otimes \det(T^*_{\mathbb{P}^n})|_C \cong \det(T^*_C),$$

Consider the associated sheaf of holomorphic sections, we have isomorphism of $\mathcal{O}_C$-modules:

$$(\mathcal{O}(d) \otimes \omega_{\mathbb{P}^n})|_C \cong \omega_C.$$

Recall the geometric genus of a complex manifold, $g(C) = \dim_{\mathbb{C}}(\Gamma(C, \omega_C)) = \Gamma(C, (\mathcal{O}(d) \otimes \omega_{\mathbb{P}^n})|_C)$. We will first compute the canonical bundle $\omega_{\mathbb{P}^n}$ of the projective space.

2 Computation of $\omega_{\mathbb{P}^n}$

Let $\Delta : \mathbb{P}^n \to \mathbb{P}^n \times \mathbb{P}^n$ denote the diagonal map, let $\mathrm{pr}_1, \mathrm{pr}_2 : \mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^n$ denote the projection to the first and the second factors respectively. The projections are holomorphic submersions, hence we may define pullback vector bundles $\mathrm{pr}_1^* L$ and $\mathrm{pr}_2^* L$ of the tautological line bundle to $\mathbb{P}^n \times \mathbb{P}^n$. By construction, fibers of $\mathrm{pr}_1^* L$ and $\mathrm{pr}_2^* L$ are,

$$(\mathrm{pr}_1^* L)_{[\ell]\times[m]} = L_{[\ell]} = \ell, \quad (\mathrm{pr}_2^* L)_{[\ell]\times[m]} = L_{[m]} = m.$$

Let $\iota : L \hookrightarrow \mathbb{P}^n \times \mathbb{C}^{n+1}$ denote the inclusion of bundle and let Q denote the quotient bundle $(\mathbb{P}^n \times \mathbb{C}^{n+1})/L$. We have short exact sequence of holomorphic bundle:

$$0 \longrightarrow L \xrightarrow{\iota} \mathbb{P}^n \times \mathbb{C}^{n+1} \xrightarrow{q} Q \longrightarrow 0.$$

Then we have short exact sequences of the pullbacks:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{pr}_1^* L & \xrightarrow{\mathrm{pr}_1^* \iota} & \mathbb{P}^n \times \mathbb{P}^n \times \mathbb{C}^{n+1} & \xrightarrow{\mathrm{pr}_1^* q} & \mathrm{pr}_1^* Q & \longrightarrow & 0 \\ & & & & \| & & & & \\ 0 & \longrightarrow & \mathrm{pr}_2^* L & \xrightarrow{\mathrm{pr}_2^* \iota} & \mathbb{P}^n \times \mathbb{P}^n \times \mathbb{C}^{n+1} & \xrightarrow{\mathrm{pr}_2^* q} & \mathrm{pr}_2^* Q & \longrightarrow & 0. \end{array}$$

Lemma 2.1.

1. For all $[\ell] \times [m] \in \mathbb{P}^n \times \mathbb{P}^n$,

$$(\mathrm{pr}_1^* L)_{[\ell]\times[m]} = (\mathrm{pr}_2^* L)_{[\ell]\times[m]} \iff (\mathrm{pr}_2^* q \circ \mathrm{pr}_1^* \iota)_{[\ell]\times[m]} = 0.$$

2. Consider the image of the diagonal,

$$\Delta(\mathbb{P}^n) = \left\{[\ell] \times [m] \in \mathbb{P}^n \times \mathbb{P}^n : l = m \subseteq \mathbb{C}^{n+1}\right\}.$$

   Then

$$\Delta(\mathbb{P}^n) = \left\{[\ell] \times [m] \in \mathbb{P}^n \times \mathbb{P}^n : (\mathrm{pr}_2^* q \circ \mathrm{pr}_1^* \iota)_{[\ell]\times[m]} = 0\right\}.$$

Proof. Suppose $\ell = m \subseteq \mathbb{C}^{n+1}$, then on the fiber over $[\ell] \times [m]$ of the exact sequences of bundles we have short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \ell \arrow[r, "(\mathrm{pr}_1^* \iota)_{[\ell]\times[m]}"] \arrow[d, equal] & \mathbb{C}^{n+1} \arrow[r, "(\mathrm{pr}_1^* q)_{[\ell]\times[m]}"] \arrow[d, equal] & Q_{[\ell]} \arrow[r] & 0 \\
0 \arrow[r] & m \arrow[r, "(\mathrm{pr}_2^* \iota)_{[\ell]\times[m]}"] & \mathbb{C}^{n+1} \arrow[r, "(\mathrm{pr}_2^* q)_{[\ell]\times[m]}"] & Q_{[m]} \arrow[r] & 0.
\end{tikzcd}.$$

Observe $(\mathrm{pr}_1^* \iota)_{[\ell]\times[m]} = \ell \hookrightarrow \mathbb{C}^{n+1}$ and $(\mathrm{pr}_2^* \iota)_{[\ell]\times[m]} = m \hookrightarrow \mathbb{C}^{n+1}$. Then the square in the diagram above commutes. In particular, $(\mathrm{pr}_2^* q \circ \mathrm{pr}_1^* \iota)_{[\ell]\times[m]} = 0$. Conversely, if $(\mathrm{pr}_2^* q \circ \mathrm{pr}_1^* \iota)_{[\ell]\times[m]} = 0$, then by universal property of kernel, $\ell \subseteq m$. But $\dim_{\mathbb{C}} \ell = \dim_{\mathbb{C}} m = 1$, it must be the case that $m = \ell$. □

Lemma 2.2. Let F and E be holomorphic vector bundles over complex manifold X of rank f and e respectively. Then there's a bijection

$$\{\text{vector bundle map } F \to E\} \longleftrightarrow \Gamma(X, F^\vee \otimes E).$$

Proof. Suppose $\alpha : F \to E$ is a holomorphic vector bundle map, define $\tilde{\alpha} \in \Gamma(X, F^\vee \otimes E)$ by $\tilde{\alpha}(x) := \alpha_x \in \mathrm{Hom}_{\mathbb{C}}(F_x, E_x)$. Conversely given $s \in \Gamma(X, F^\vee \otimes E)$ define vector bundle map $\hat{s} : F \to E$ by defining it on each fiber with $\hat{s}_x := s(x)$. To show $\tilde{\alpha}$ and $\hat{s}$ are holomorphic, it suffice to show they are locally holomorphic, hence we may assume the E and F are trivial, $E \cong X \times \mathbb{C}^e$, $F \cong X \times \mathbb{C}^f$. Observe a holomorphic map $\alpha : X \times \mathbb{C}^f \to X \times \mathbb{C}^e$ is equivalent to a holomorphic map $\alpha : X \to M_{f\times e}(\mathbb{C})$. This proves the lemma. □

By the lemma above we may view the holomorphic map $(\mathrm{pr}_2^* q \circ \mathrm{pr}_1^* \iota)$ as a smooth section of $\mathrm{pr}_1^* L^\vee \otimes \mathrm{pr}_2^* Q$.

Lemma 2.3. The section $(\mathrm{pr}_2^* q \circ \mathrm{pr}_1^* \iota) \in \Gamma(\mathbb{P}^n \times \mathbb{P}^n, \mathrm{pr}_1^* L^\vee \otimes \mathrm{pr}_2^* Q)$ is a holomorphic submersion, hence in particular is transverse to the zero section.

Proof. The strategy is to compute what the map $(\mathrm{pr}_2^* q \circ \mathrm{pr}_1^* \iota)$ looks like locally (it should be a matrix of holomorphic functions, after we fixing the coordinates) and check its differential is surjective. Since we are only interested in points $[\ell] \times [m] \in \mathbb{P}^n \times \mathbb{P}^n$ such that $(\mathrm{pr}_2^* q \circ \mathrm{pr}_1^* \iota)_{[\ell]\times[m]} = 0$, we only need to look locally on the diagonal.

Consider holomorphic coordinate neighborhoods $U_i \times U_i \subseteq \Delta(\mathbb{P}^n) \subseteq \mathbb{P}^n \times \mathbb{P}^n$, where U_i consists of points $[z_0 : \cdots : z_n]$ with $z_i \neq 0$. Let $x_0, \ldots, x_n$ (resp. $y_0, \ldots, y_n$) denote the homogeneous coordinates on left (resp. right) copy of $\mathbb{P}$ is $\mathbb{P}^n \times \mathbb{P}^n$. Then we have affine coordinates $(\frac{x_1}{x_i}, \ldots, \widehat{\frac{x_i}{x_i}}, \ldots, \frac{x_n}{x_i})$ and $(\frac{y_1}{y_i}, \ldots, \widehat{\frac{y_i}{y_i}}, \ldots, \frac{y_n}{y_i})$ on the two copies of $\mathbb{P}^n$. Let $t_j := x_j/x_i$ and $s_j := y_j/y_i$.

Consider the dual of the associated sequence of sheaves of $0 \to L \xrightarrow{\iota} \mathbb{P}^n \times \mathbb{C}^{n+1} \xrightarrow{q} Q \to 0$,

$$0 \to \mathcal{Q} \xrightarrow{q^\vee} \mathcal{O}_{\mathbb{P}^n}^{\oplus n+1} \xrightarrow{\iota^\vee} \mathcal{O}(1) \to 0.$$

Claim: locally we have isomorphism:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{Q}|_{U_i} \arrow[r] & \mathcal{O}_{\mathbb{P}^n}^{\oplus n+1}|_{U_i} \arrow[r] & \mathcal{O}(1)|_{U_i} \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O}_{U_i}^{\oplus n} \arrow[r, "A"] \arrow[u, dashed] & \mathcal{O}_{U_i}^{\oplus n+1} \arrow[r, "B"] \arrow[u, equal] & \mathcal{O}_{U_i} \arrow[r] \arrow[u, "\cdot x_i"] & 0
\end{tikzcd} \tag{2.1}$$

where

$$A = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ -t_0 & \cdots & -t_{i-1} & -t_{i+1} & \cdots & -t_n \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} t_0 & \cdots & t_{i-1} & 1 & t_{i+1} & \cdots & t_n \end{pmatrix}.$$

1. The second row is exact. Clearly B is surjective and A is injective, we want to show it's exact at the middle term. Direct computation shows $BA = 0$, hence $\operatorname{Im} A \subseteq \operatorname{Ker} B$. Furthermore if $(f_0, \ldots, f_n)^t \in \operatorname{Ker} B$, then we have relation

$$t_0 f_0 + \cdots + t_{i-1} f_{i-1} + f_i + t_{i+1} f_{i+1} + \cdots + t_n f_n = 0.$$

   Then

$$\begin{pmatrix} f_0 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ -t_0 & \cdots & -t_{i-1} & -t_{i+1} & \cdots & -t_n \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \begin{pmatrix} f_0 \\ \vdots \\ f_{i-1} \\ f_{i+1} \\ \vdots \\ f_n \end{pmatrix}.$$

   It follows $(f_0, \ldots, f_n)^t \in \operatorname{Im} A$.

2. The following square commutes:

$$\begin{tikzcd} \mathcal{O}_{U_i}^{\oplus n+1} \arrow[r] & \mathcal{O}(1)|_{U_i} \arrow[r] & 0 \\ \mathcal{O}_{U_i}^{\oplus n+1} \arrow[u] \arrow[r, "B"] & \mathcal{O}_{U_i} \arrow[u, "\cdot x_i"] \arrow[r] & 0 \end{tikzcd}$$

   Here the map $x_i : \mathcal{O}_{U_i} \to \mathcal{O}(1)|_{U_i}$ denotes multiplication by the section $x_i \in \Gamma(U_i, L^\vee)$. This map is invertible with inverse sending a section $\sigma \in \Gamma(U_i, L^\vee)$ to the function $\sigma / x_i \in \mathcal{O}_{U_i}$, the inverse is well-defined since the section x_i is nonzero on U_i. This shows the square commutes and all vertical maps are isomorphisms. Therefore two rows share the same kernel, this proves the claim.

Take dual of Equation (2.1), we see ι_{U_i} and $q|_{U_i}$ are locally of the form B^t and A^t respectively. Therefore, locally (on U_i), the section $(\mathrm{pr}_2^* q \circ \mathrm{pr}_1^* \iota)|_{U_i}$ is of the form

$$(\mathrm{pr}_2^* q \circ \mathrm{pr}_1^* \iota)|_{U_i} = \begin{pmatrix} 1 & & -s_0 & & & \\ & \ddots & \vdots & & & \\ & & -s_{i-1} & & & \\ & & -s_{i+1} & 1 & & \\ & & \vdots & & \ddots & \\ & & -s_n & & & 1 \end{pmatrix} \begin{pmatrix} t_0 \\ \vdots \\ t_{i-1} \\ 1 \\ t_{i+1} \\ \vdots \\ t_n \end{pmatrix} = \begin{pmatrix} t_0 - s_0 \\ \vdots \\ t_{i-1} - s_{i-1} \\ t_{i+1} - s_{i+1} \\ \vdots \\ t_n - s_n \end{pmatrix}.$$

Whose differential is

$$\begin{pmatrix} 1 & & & 0 & & & & -1 & & & 0 & & & \\ & \ddots & & \vdots & & & & & \ddots & & \vdots & & & \\ & & 1 & 0 & & & & & & -1 & 0 & & & \\ & & & 0 & 1 & & & & & & 0 & -1 & & \\ & & & \vdots & & \ddots & & & & & \vdots & & \ddots & \\ & & & 0 & & & 1 & & & & 0 & & & -1 \end{pmatrix}.$$

The differential is clearly surjective. □

By Lemma 2.3 we may conclude that $\Delta(\mathbb{P}^n)$ is an embedded submanifold of $\mathbb{P}^n \times \mathbb{P}^n$, and

$$N_{\Delta(\mathbb{P}^n)\subseteq \mathbb{P}^n\times\mathbb{P}^n} \cong (\mathrm{pr}_1^* L^\vee \otimes \mathrm{pr}_2^* Q)\Big|_{\Delta(\mathbb{P}^n)}.$$

Pull these two vector bundles back along the diagonal Δ, we get

$$\Delta^* N_{\Delta(\mathbb{P}^n)} \cong \Delta^*(\mathrm{pr}_1^* L^\vee \otimes \mathrm{pr}_2^* Q) \cong (\mathrm{pr}_1 \circ \Delta)^* L^\vee \otimes (\mathrm{pr}_2 \circ \Delta)^* Q \cong L^\vee \otimes Q.$$

Also note by definition of the normal bundle, we have exact sequence

$$0 \longrightarrow T_{\Delta(\mathbb{P}^n)} \longrightarrow T_{\mathbb{P}^n\times\mathbb{P}^n}\Big|_{\Delta(\mathbb{P}^n)} \longrightarrow N_{\Delta(\mathbb{P}^n)} \longrightarrow 0$$

Pulling back along Δ we get exact sequence

$$0 \longrightarrow \Delta^*(T_{\Delta(\mathbb{P}^n)}) = T_{\mathbb{P}^n} \longrightarrow \Delta^*(T_{\mathbb{P}^n\times\mathbb{P}^n}\Big|_{\Delta(\mathbb{P}^n)}) = T_{\mathbb{P}^n} \oplus T_{\mathbb{P}^n} \longrightarrow \Delta^*(N_{\Delta(\mathbb{P}^n)}) \longrightarrow 0$$

Therefore,

$$L^\vee \otimes Q \cong \Delta^*(N_{\Delta(\mathbb{P}^n)}) \cong T_{\mathbb{P}^n}.$$

Tensoring the exact sequence

$$0 \longrightarrow L \xrightarrow{\iota} \mathbb{P}^n \times \mathbb{C}^{n+1} \xrightarrow{q} Q \longrightarrow 0$$

by $L^\vee$ we get

$$0 \longrightarrow \mathbb{P}^n \times \mathbb{C} \longrightarrow L^\vee \otimes (\mathbb{P}^n \times \mathbb{C}^{n+1}) \cong (L^\vee)^{\oplus(n+1)} \longrightarrow L^\vee \otimes Q \cong T_{\mathbb{P}^n} \longrightarrow 0.$$

Take the dual short exact sequence, we have

$$0 \longrightarrow T^*_{\mathbb{P}^n} \longrightarrow L^{\oplus(n+1)} \longrightarrow \mathbb{P}^n \times \mathbb{C} \longrightarrow 0.$$

By the adjunction formula,

$$(n+1)L \cong \det(L^{\oplus(n+1)}) \cong \det(T^*_{\mathbb{P}^n}) \otimes \det(\mathbb{P}^n \times \mathbb{C}) \cong \det(T^*_{\mathbb{P}^n}).$$

Consider the associated sheaves of holomorphic sections, we have

$$\mathcal{O}(-n-1) \cong \omega_{\mathbb{P}^n}.$$

Therefore,

$$\omega_C \cong (\mathcal{O}(d) \otimes \omega_{\mathbb{P}^n})|_C \cong (\mathcal{O}(d) \otimes \mathcal{O}(-n-1))|_C \cong \mathcal{O}(d-n-1)|_C.$$

On the other hand, via the correspondence between ideal sheaves and line bundles (see for example Lemma 2.3.22 in [2]), we have exact sequence

$$0 \longrightarrow \mathcal{O}(-d) \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow i_*\mathcal{O}_C \longrightarrow 0.$$

Tensoring with $\mathcal{O}(d-n-1)$, we get exact sequence

$$0 \longrightarrow \mathcal{O}(-n-1) \longrightarrow \mathcal{O}(d-n-1) \longrightarrow (i_*\mathcal{O}_C) \otimes \mathcal{O}(d-n-1) = i_*(\mathcal{O}(d-n-1)|_C) \longrightarrow 0$$

The short exact sequence of sheaves induces a long exact sequence of global sections:

$$\begin{tikzcd} 0 \arrow[r] & \Gamma(\mathbb{P}^n, \mathcal{O}(-n-1)) = 0 \arrow[r] & \Gamma(\mathbb{P}^n, \mathcal{O}(d-n-1)) \arrow[r] & \Gamma(\mathbb{P}^n, i_*(\mathcal{O}(d-n-1)|_C)) \arrow[dll, "\delta"] \\ & H^1(\mathbb{P}^n, \mathcal{O}(-n-1)) \arrow[r] & \cdots \end{tikzcd}$$

Observe

$$\Gamma(\mathbb{P}^n, i_*(\mathcal{O}(d-n-1)|_C)) \cong \Gamma(C, \mathcal{O}(d-n-1)|_C) \cong \Gamma(C, \omega_C).$$

And by Theorem 5.1 of [1], $H^1(\mathbb{P}^n, \mathcal{O}(-n-1)) = 0$, we conclude that

$$\Gamma(C, \omega_C) \cong \Gamma(\mathbb{P}^n, \mathcal{O}(d-n-1)).$$

Hence we may compute the geometric genus of C:

$$g(C) = \dim_{\mathbb{C}} \Gamma(C, ((n+1-d)L)|_C) = \dim_{\mathbb{C}} \Gamma(\mathbb{P}^n, \mathcal{O}(d-n-1)) = \dim_{\mathbb{C}} S^{(d-n-1)}\langle x_0, \ldots, x_n\rangle = \binom{d-1}{n}.$$

References

[1] Robin Hartshorne, *Algebraic geometry*, Vol. 52, Springer Science & Business Media, 2013.

[2] Daniel Huybrechts, *Complex geometry: an introduction*, Springer Science & Business Media, 2006.