WHAT HAVE I LEARNED TODAY?

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05-21: Today I learned the degree theory of smooth maps between compact, oriented n-manifolds. We first investigate the case of proper maps between $\mathbb{R}^n$. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a proper map, the pullback $f^* : H^n_{dR}(\mathbb{R}^n) \to H^n_{dR}(\mathbb{R}^n)$ maps compactly supported n-forms to compactly supported forms, hence $f^* : H^n_c(\mathbb{R}^n) \to H^n_c(\mathbb{R}^n)$. Let ω be a generator of $H^n_c(\mathbb{R}^n)$ (this means $\int_{\mathbb{R}^n} \omega = 1$, this is possible by the Poincaré Lemma that $H^n_c(\mathbb{R}^n) \cong \mathbb{R}$), we define the degree $\deg(f) := \int_M f^*\omega$. To prove this is an integer, pick a regular value $q \in \mathbb{R}^n$ of f by Sard's Theorem. Since f is proper, $f^{-1}(q)$ is a finite set of points $\{p_1, \ldots, p_K\}$. Since q is a regular value, the map f is locally a diffeomorphism when restricted to small neighborhoods V_i of each p_i such that V_i are disjoint and $f(V_i) = W$ for all i. By partition of unity we may pick a generator ω of $H^n_c(\mathbb{R}^n)$, with $\operatorname{Supp}\omega \subseteq W$. Then $\int_M f^*\omega = \sum_i \int_{V_i} f^*\omega$ and each $\int_{V_i} f^*\omega$ is $+1$ if df_{p_i} is orientation preserving and -1 if df_{p_i} is orientation reversing. To prove the case of smooth maps between compact, oriented n-manifolds, we assume the fact that $H^n(M) \cong \mathbb{R}$ as well. This fact could be proved from Poincaré Duality.

References