DEFINING AND EXPLORING SECONDARY SCHOOL STUDENT USE OF STRUCTURE

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INTRODUCTION

Our study began as an inquiry into one of the Common Core State Standards (CCSS), specifically one of the mathematical practices (MP), CCSS-MP7, *Look for and make use of structure*. While many of the mathematical practices are easier to interpret, from conversations with numerous fellow educators, there is a lack of clarity and coherence when math teachers discuss MP7. Often, as in “good art,” the gist of MP7 is explained with the comment, “I know it when I see it,” but little is provided in terms of a clear definition or evidence. Consequently, our goal was to both define structure in a way that made sense to teachers and to understand what MP7 looked like in practice.

In order to understand the use of *structure*, we first had to get a grasp on what we understood to be the meaning of *structure* in a mathematical setting. We began by looking for existing definitions of structure which referred to mathematical *objects* that attach to a *set* to give operations on the set additional meaning or significance (Corry, 1992). We wanted our definition to be one to which both high school teachers and students could relate and understand.

While structure is not explicitly defined in the CCSS documents, there did seem to be an underlying understanding that the use of structure could be seen when students exhibited the following behaviors: they interpreted the parts of an expression and attempted to give each part a real world context; they interpreted a complex expression by viewing one or more of its parts as a single entity; they discerned patterns in expressions and used those patterns to simplify or rewrite the expression to a form easier to calculate or understand (Algebra >> Seeing Structure in Expressions (n.d.). Retrieved from http://www.corestandards.org/Math/Content/HSA/SSE/#CCSS.Math.Content.HSA.SSE.A.2).

With this in mind, we define “making use of structure” as follows: *Making use of mathematical structure is the ability to make the solving of a math problem more efficient and effective, in some manner, by attending to the components of the problem and their relationships.* We felt this definition conveyed much of the same mathematical meaning as other definitions, while making MP7 relevant to both teachers and students at the secondary school level. We then set out to determine how the *use of structure* manifested in our students’ mathematical engagement, by making connections with relevant literature, collecting data on the issue, and then reflecting on the process in order to plan our next steps.

RATIONALE

As secondary school teachers, our experience with students’ mathematical engagement served as a foundation for our inquiry, leading to specific questions. Does the way that a task is posed affect students’ tendency to invoke structure? What evidence is there of middle school and high–school–support students invoking structure? We set out to answer these questions by examining two classrooms: a middle school integrated pre-algebra class and a high school integrated algebra support class. In both cases, the term *integrated* refers to classes in which students are exposed to more than a single mathematical field (e.g. integrated algebra support includes some statistics and geometry).

Note that structure itself is not inherent in a task, but instead is in the mind of the person looking at the task. As such, the use of structure is influenced by the task as well as the student, and the relationship between how a student sees the task and the task itself varies based on the individual. Looking at how each task is approached by a variety of students allows us to better understand the task and its tendency to invoke structure. The classes in our study represent a diverse population from an international border community in southern California. We also made use of a first–year Advanced Placement (AP) calculus class as a pilot for our tasks, and as a means to evaluate the validity of our tasks, because by using the tasks with students deemed by the educational establishment to be mathematically successful, we hoped to find a ceiling effect on the potential for these tasks to invoke structure. We used the data from the pilot class to help guide our decisions regarding our written tasks, and to evaluate the tendency of students to make use of structure.

We started collecting data from the pilot class by administering a written worksheet to each student, asking them to show their logic progression, and judging which students seemed to be utilizing structure in their problem-solving methodology. The results of this

Table 1 – Determining Student Use of Structure.

No evidence of structure	Evidence of structure
$48 + 98 - 48$ $146 - 48 = 98$	$\underbrace{48 + 98 - 48}_{\text{cancle out}} = \boxed{98}$

written work (see Table 1 for an example) formed the basis for the creation of the written work to be administered to the other two classes. We decided to administer the written work to the classes, select a range of students for a follow up exercise, and then to utilize clinical interviews as our follow-up data source, because not only do interviews enable us to gain deeper insight into student understanding, but unlike with paper–and–pencil written tasks, clinical interviews enable us to adapt the questions we ask, thereby making it possible to develop and test theories about student understanding (Ginsberg, 1997).

METHODS

In designing our initial written problems, we chose tasks that we thought would invoke structure in our students' solution methods. To do so, we took a number of arithmetic expressions and short algebraic equations and wrote them in a way that we thought would *highlight* structure. Similar to the Blanton and Kaput (2003) technique of A*lgebrafying* problems to provide students with the opportunity to generalize their thinking, we designed and modified tasks in an attempt to create opportunities for the use of structure. For example, since the CCSS suggests making use of structure is exhibited by interpreting a complex expression by viewing one or more of its parts as a single entity, we designed the task $10 - \frac{x}{99} = 9$ in the hopes that students would view $\frac{x}{99}$ as a single entity, specifically the number 1.

Our first collection of data focused on students' written responses to these tasks. We examined the written responses and attempted to decide if, based on their solutions, students were invoking an efficient and/or effective use of structure. The data was collected from both classes, coded, and analyzed. We compared the results of the two classes' written data to the results of the pilot class data, and selected tasks that we felt had the highest probability for students to make use of structure during the interviews.

Based on our first data source, the paper–and–pencil assessment (Appendix A), we classified the students at three levels: students who used structure often, occasionally, or never. For our second data source, we interviewed six students, three from each class. During the interview, the tasks were not only posed visually, but we also engaged the students with questions written in a specific way and/or posed with intentional auditory emphasis. Our thought was that by posing a question in a different visual and/or auditory construction, we could *highlight* the structure and invoke its use by *focusing* students on the components and/or organization of the task (Lobato, Hohensee, & Rhodehamel, 2013).

Based on the results of the interviews, we decided that our third data set would consist of four interviews, two interviewees from each class selected on the basis of their having used little to no structure on their written work in the pen-and-pencil assessment, data set one. Interview tasks for data source three remained the same as data source two. The tasks utilized were those that seemed to invoke structure in both target classes and where structure appeared to be invoked regularly by students in the AP calculus (pilot) class. Specific strategies for *highlighting* structure were used to orient students' *focus* to the structure.

The interviewing strategies used to invoke structure were to read and/or write tasks in ways that *highlight* components of the problem. For instance, instead of writing 48 + 98 – 48, the instructor would write and say, "48   - 48", pause, hoping that the student will recognize the relationship, and then write-in "+ 98". Components such as number choice and organization of operations were designed and implemented for students to recognize relationships between components, prior to engaging with the entire task.

ANALYSIS AND RESULTS

Paper and Pencil Survey

When designing a task to invoke structure, it is important that the task be neither too trivial nor too complicated. For example, in designing a task with relational thinking for students to see *one more than doubles* (e.g. 10 + 11), the task 2 + 3 may be too trivial to invoke the desired structure, whereas 3,789 + 3,790 may be too difficult (Land, Drake, Sweeney, Franke, & Johnson, 2015).

Based on the results of data set one, we concluded that tasks #1 and #3 were too simple and provided little to no information concerning students' use of structure to solve the problem. In task #1, students were asked to fill in the blank for 8 + 4 = __ + 5, and students simply reasoned that because 8 + 4 = 12 and 12 – 5 = 7, the answer was 7. Only six percent of students provided written solutions that were coded as using structure to solve this problem (Table 2).

Table 2 – Percentage of students who used structure by task (*N* = 53)

Based on the results of data set one	Task #1	Task #2	Task #3	Task #4	Task #5	Task #6	Task #7	Task #8
Middle & Secondary Support	**6%**	40%	**92%***	2%	8%	8%	40%	40%
AP Calculus Students	**6%**	52%	**100% ***	42%	55%	55%	68%	84%
Task Selected for Interviews		✓			✓	✓		✓

*Unsure or difficult to determine if a student used structure or not.

In task #3, the students were asked to fill in the blank for 19 x 37 = 37 x __ and almost every student, ninety-two percent, recognized that the missing piece was 19. After more consideration, we concluded that the task might have been answered without attending to the multiplicative relationship at all, much the same way that one might reason when presented with the problem posed in Figure 1. We concluded that the task in Figure 1 is one whereby students may get the right answer without thinking about mathematical structure.

Example: Fill in the blank.

$$\alpha + \beta = \beta + \_\_\_$$

Figure 1

Task #4 asked students to evaluate the expression (6 x 77) + (4 x 77), a task designed with the intent for students to invoke the distributive property (e.g. (6 + 4) x 77). Only one of 53 students provided evidence of the use of structure in solving this problem. In hindsight, we concluded that our students are not used to seeing math problems that make use of the distributive property (e.g. $(a \cdot b) + (c \cdot b) = (a + c)(b)$); however, forty-two percent of the calculus students were able to recognize the structure. We posited that for many of our students, this task is ineffective for helping us understand how they see structure, although it might be useful in higher levels of math where students have more experience using the distributive property.

For almost every task, calculus students used structure more than the middle/support students, leading us to theorize that there is a link between structure and content knowledge. In tasks #4, #5, and #6, 2%, 8%, and 8%, respectively, of the middle/support students were coded as making use of structure to solve the problem. Likewise, on the same tasks, the percentages for thirty-one calculus students were 42%, 55%, and 55%, respectively.

In task #5, there are two reasons that we believe middle/support students didn't use structure. First, students have been given strict and often consistent instruction regarding arithmetic computation which they have been taught must be done left to right. If students had more flexibility with computation, they could have been comfortable subtracting prior to adding.

Second, students are generally taught order of operations using an acronym or mnemonic like, "*please excuse my dear Aunt Sally*" (PEMDAS). Some students had very fixed orientations about order of operations based on the way they were taught and believed that they must add before they subtract. Depending on how the acronym is presented can affect how students view order of operations (see Figure 2). The vertical representation focuses on teaching students that addition and subtraction are at the same

Order of Operations Acronyms	
(horizontal)	(vertical)
PEMDAS	P E MD AS

Figure 2

"level" and can occur in any order, while the horizontal representation can lead to the fixed view of requiring *addition before subtraction*.

In task #6 we discovered that our students were not proficient in adding and subtracting fractions. Of the students that used structure, 50% answered correctly. Of the students who did not use structure, 16% answered correctly. The responses to this task, once again, lead us to believe that content knowledge is a foundational necessity for students to make use of structure (Falkner, Levi, & Carpenter, 1999).

The Clinical Interviews

Three points stand out in our six clinical interviews. First, if students did not understand the content, the instructor could not invoke structure. This confirmed our conjecture that content knowledge is an important link to the use of structure. Second, when students invoked structure in one problem, they were more likely to look for it in another problem. Third, once structure was realized, students felt that it made math "quicker" and "easier", hence more effective and efficient.

The clinical interview was composed of four warm-up problems to test for content knowledge, and, depending upon the student's responses to the warm-up, between four and twelve interview questions (see Appendix B). Of the six clinical interviews, two students were not able to complete the fraction warm-ups $\frac{3}{7}+\frac{5}{7}$ and $1-\frac{2}{7}$. Since these students didn't have the content knowledge or foundation to complete fractional addition and subtraction problems, they were not able to solve or use structure on task #3 $\left(\frac{3}{7}-\frac{5}{9}+\frac{4}{7}\right)$ of the interview.

During the interviews, the students were more likely to see structure after it had been observed in a previous problem. There was one form of problem that seemed to switch students into a mode where they were deliberately looking for structure, i.e. 132 + 55 – 32 (see Figure 3).

Student #1: 132 - 32 to equal 100 and then plus 55 would be 155. Oh Snap.

Instructor: What does oh snap mean?

Student #1: Um, you still switch the operations and you switch the numbers around but it still equals the same answer as you would even if you'd like kept it the same.

Instructor: Ok, would that strategy help you on either these problems?

Student #1: I think it'd be the same outcome. So, like the first one, 3573 plus 5131 subtracted by 2573, I think if you subtract 3573 from 2573 (sic) and then added 5131 you would get the same answer…

Figure 3

In one situation, Figure 4, a student used the same strategy invoked here and applied it to a fraction problem. As students completed each interview question, there was a major turning point in their thinking when structure was invoked. After discovering it

Task $\frac{3}{7} - \frac{5}{9} + \frac{4}{7}$

Student #5: Okay, then I'm going to try the same thing, cause look you got common denominators, and you get, and that's, that equals 1, so then you subtract that and, uh, I think it's four ninths, I think I'm wrong though, but yeah... yeah

Figure 4

initially, they would actively look for it in each problem, because they felt that it was "quicker" and "easier."

Follow-up Interviews

We conducted four additional interviews of students who used little to no structure in our first data source (the paper–and–pencil survey). Our interviews seemed to validate our hypothesis regarding content knowledge and its relation to how we can invoke structure.

Students' content knowledge of each of the problems affected their ability to see structure. If a student did not have the content knowledge necessary to solve the problem, then we were unsuccessful in invoking structure. Similarly, if a student had the necessary

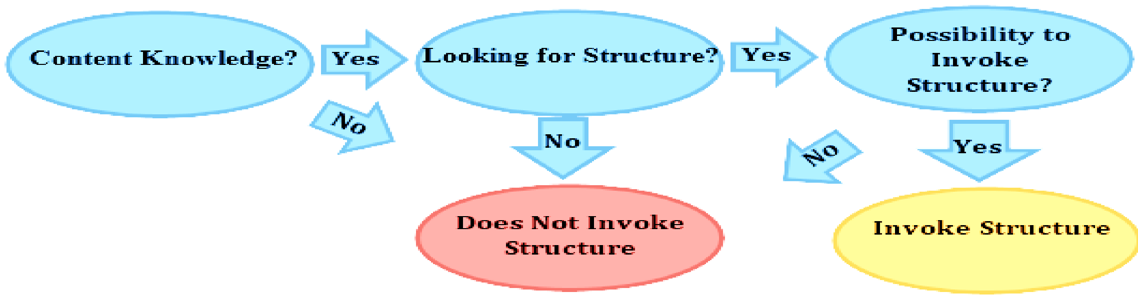

Figure 5

content knowledge, then this opened the door to *looking for structure* (Figure 5). Once a student identified one instance of *structure,* they continued to *look for* and attempted to *make use of structure* in other tasks.

There were instances where alternate conceptions were hindering a student's ability to see structure. Alternate conceptions are concepts about a curriculum that inhibit student learning (Isabelle, Millham, & Cunha, 2014). In our study, the most common alternate conceptions were in regard to *order of operations and* especially the *computation of fractions*, e.g. the idea that each fraction *must* have the same common denominator. Four out of ten students were not able to see structure in fraction division because of an alternate conception.

Task #3: $\frac{5}{8}+\frac{1}{6}+\frac{3}{8}$

Instructor: They have the same denominator, so you can add 5 + 3?

Student #10: Yeah.

Instructor: So with that 1/6 there can you still do that, or do you need to do it this way (make the denominators 24)?

Student #10: I have to do it that way (make the denominators 24).

Instructor: The other way, and why do you think that way?

Student #10: Cause in order to solve it you have to have the same denominator.

Figure 6

Figure 6 serves as an example of a typical dialogue showing an example of an alternate conception. When the instructor covered up the $\left(\frac{1}{6}\right)$ component of the problem, the student was able to compute the fractions because they had a common denominator, aided by the instructor's suggestion on the possibility of adding 5 + 3. After the instructor moved his thumb, the denominators were not equal, and the student was unable to calculate the answer. For this student, having equal denominators was an absolute necessity when calculating with fractions.

While the interviews did not seek to teach students' content knowledge, they did seek to determine if students could be *oriented* to *look for structure*. Instructors were successful in *highlighting* structure if the student had the content knowledge necessary, void of *alternate conceptions,* by using specific presentation techniques in each interview.

Ten out of ten students were eventually able to see structure in interview task #4 (Appendix B). This led us to believe that there is a necessary set of criteria for teachers and students to implement MP7 in their classrooms.

IMPLICATIONS

While this study was an initial exploration with a small sample size, and our findings are in a no way a road map for teaching structure, common factors did present themselves. *Content knowledge* seems to be a foundational component when attempting to show students structure. This implies that structure may be taught at any time if the teacher maintains an awareness of the students' level of content knowledge, and tailors the instruction to that perceived level.

The presentation of a task also effects how students *orient* themselves towards solving a problem. In at least one task posed during our clinical interviews, every student was able to see structure, in some form, when the *presentation* of the problem was *oriented* to *highlight* the intended structure. All of the students interviewed were able to see *structure* in some form when the presentation of a problem was tailored to *highlight* the intended structure. Once students are *oriented* to *look for structure*, they tend to look for and see structure in other problems. This could indicate that when attempting to teach structure, it is important to first *condition* students to look for structure by *presenting* tasks using a deliberate method with which they have a strong base *content knowledge*.

> "So too the child's thinking. Like the river, it does have some general features. But it is always changing in response to its environment, fluid, swirling, shifting, meandering. If you dip into it in different places, you seem to place yourself in different rivers, as in a sense you do." – Ginsburg, 1997, p. 63

Appendix A

Data Source 1: Survey Written Task

Task #1:

Fill in the blank. $8 + 4 = \_\_\_ + 5$

Task #2:

Solve for x. $10 - \frac{x}{99} = 9$

Task #3:

Fill in the blank. $19 \times 37 = 37 \times \_\_\_$

Task #4:

Evaluate the following expression. $(6 \times 77) + (4 \times 77)$

Task #5:

Evaluate the following expression. $3476 + 5000 - 1476$

Task #6:

Evaluate the following expression. $\frac{3}{7} - \frac{1}{9} + \frac{4}{7}$

Task #7:

Fill in the blank to make the equation true. $101 + \_\_\_ = 100 + 43$

Task #8:

Evaluate the following expression. $48 + 98 - 48$

Appendix B

Data Source 2/3: Interview Tasks – Script

#1-#4 are warm-up problems.

1. $8 + 4 = \_\_ + 5$ **2.** $\frac{3}{7} + \frac{5}{7}$ **3.** $125 - x = 100$ **4.** $1 - \frac{2}{7}$

If #2 and #3 correct,
Interview Task #1 & Interview Task #3 ok.
Otherwise only use Interview Task #2 and Interview Task #4

Interview Task #1:

$23 - \frac{x}{11} = 22 \rightarrow$ If procedure applied…Write 37 = 36, then minus x/33.

a. $37 - \frac{x}{33} = 36$ → Thumb over missing subtrahend. Explicit highlighting.

b. $53 - \frac{x}{577} = 52$

Interview Task #2:

$3573 + 5131 - 2573$ →132 pause + come back -132 add the 2 and 1 for the thousands (Vocal)

a. $2132 + 3777 - 1132$ → 32 pause + skip – 32 add the 1 for hundreds then the + 55 (Vocal)

b. $132 + 55 - 32$

Interview Task #3:

$\frac{3}{7} - \frac{5}{9} + \frac{4}{7}$

a. $\frac{5}{8} - \frac{1}{6} + \frac{3}{8}$

b. $\frac{1}{2} - \frac{1}{5} + \frac{1}{2}$

Interview Task #4:

$37 + 113 - 37$

a. $117 + 237 - 117$

b. $125 + 50 - 125$

Remember to look for and discuss that "AHA!" moment.

If no structure on any tasks.
Push towards/invoke structure in interview task #4.
Then ask if there is another way?
And rewind to a potential use of structure in a prior task.

If they have used structure in a previous problem, then check to see if they continue to use structure…

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