THE LIFTABLE MAPPING CLASS GROUP OF BALANCED SUPERELLIPTIC COVERS

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Abstract. The hyperelliptic mapping class group has been studied in various contexts within topology and algebraic geometry. What makes this study tractable is that there is a surjective map from the hyperelliptic mapping class group to a mapping class group of a punctured sphere. The more general family of superelliptic mapping class groups does not, in general, surject on to a mapping class group of a punctured sphere, but on to a finite index subgroup. We call this finite index subgroup the liftable mapping class group. In order to initiate the generalization of results on the hyperelliptic mapping class group to the broader family of superelliptic mapping class groups, we study an intermediate family called the balanced superelliptic mapping class group. We compute the index of the liftable mapping class group in the full mapping class group of the sphere and show that the liftable mapping class group is independent of the degree of the cover. We also build a presentation for the liftable mapping class group, compute its abelianization, and show that the balanced superelliptic mapping class group has finite abelianization. Although our calculations focus on the subfamily of balanced superellipitic mapping class groups, our techniques can be extended to any superelliptic mapping class group, even those not within the balanced family.

1. Introduction

Let Σ_g be a surface of genus g, and let ζ be a finite order homeomorphism of Σ_g such that $\Sigma_g/\langle\zeta\rangle$ is homeomorphic to the sphere Σ_0. The quotient map is a branched covering map $p : \Sigma_g \to \Sigma_0$ with the deck group D generated by ζ. The points on Σ_g which are fixed by a non-trivial power of ζ map to the branch points $\mathcal{B} \subset \Sigma_0$.

The mapping class group of Σ_0 relative to $\mathcal{B}$, denoted $\mathrm{Mod}(\Sigma_0, \mathcal{B})$, consists of homotopy classes of orientation preserving homeomorphisms of Σ_0 where both homotopies and homeomorphisms preserve $\mathcal{B}$. On the other hand, homeomorphisms of Σ_g need not preserve the points fixed by ζ.

Let $\hat{D}$ be the image of the deck group D in $\mathrm{Mod}(\Sigma_g)$. Let $\mathrm{SMod}_p(\Sigma_g)$ be the subgroup of $\mathrm{Mod}(\Sigma_g)$ consisting of isotopy classes of fiber preserving homeomorphisms. Then $\mathrm{SMod}_p(\Sigma_g)$ is equal to the normalizer of $\hat{D}$ in $\mathrm{Mod}(\Sigma_g)$ [8, Theorem 4].

Due to work of Birman and Hilden [7, 8], it is known that $\mathrm{SMod}(\Sigma_g)/\hat{D}$ is isomorphic to a finite index subgroup of $\mathrm{Mod}(\Sigma_0, \mathcal{B})$ provided $g > 1$.

We will call the finite index subgroup of $\mathrm{Mod}(\Sigma_0, \mathcal{B})$ the *liftable mapping class group*, denoted $\mathrm{LMod}_p(\Sigma_0, \mathcal{B})$. The liftable mapping class group is exactly comprised of isotopy classes of homeomorphisms of Σ_0 that lift to homeomorphisms of Σ_g.

The hyperelliptic involution: The isomorphism $\mathrm{SMod}(\Sigma_g)/\hat{D} \cong \mathrm{LMod}_p(\Sigma_0, \mathcal{B})$ has been successfully expoited, most notably in the case where ζ is a hyperelliptic involution eg. A'Campo [1], Arnol'd [2], Brendle-Margalit-Putman [10], Gries [16], Hain [17], Magnus-Peluso [19], Morifuji [22], Stukow [23]. Here $\mathrm{SMod}_p(\Sigma_g)$ is called the *hyperelliptic mapping class group*. When $g = 2$, the hyperelliptic mapping class group is equal to $\mathrm{Mod}(\Sigma_2)$. Birman and Hilden used this fact to find the first presentation for $\mathrm{Mod}(\Sigma_2)$. Bigelow and Budney proved that $\mathrm{SMod}_p(\Sigma_g)$ is linear [4] when ζ is a hyperelliptic involution.

One of the reasons the covering space induced by a hyperelliptic involution has been fertile ground for research is that the liftable mapping class group $\mathrm{LMod}_p(\Sigma_0, \mathcal{B})$ equals $\mathrm{Mod}(\Sigma_0, \mathcal{B})$. In general $\mathrm{LMod}_p(\Sigma_0, \mathcal{B})$ is only finite index in $\mathrm{Mod}(\Sigma_0, \mathcal{B})$. Although the finite index implies that $\mathrm{LMod}_p(\Sigma_0, \mathcal{B})$ enjoys many properties of $\mathrm{Mod}(\Sigma_0, \mathcal{B})$ such as finite presentability and linearity, it must be better understood in order to use the relationship $\mathrm{LMod}_p(\Sigma_0, \mathcal{B}) \cong \mathrm{SMod}(\Sigma_g)/\hat{D}$ for explicit calculations.

Cyclic branched covers of a sphere: Every finite cyclic branched covering space of a sphere can be modeled by a *superelliptic curve*, a plane curve with equation of the form $y^k = f(x)$ for some $f(x) \in \mathbb{C}[x]$, $k \in \mathbb{N}$. Indeed, choose distinct points $a_1, \ldots, a_t \in \mathbb{C}$. Then a cyclic branched cover of the sphere can be modeled by an irreducible plane curve C defined by

$$y^k = (x - a_1)^{d_1} \cdots (x - a_t)^{d_t}$$

where $1 \leqslant d_i \leqslant k-1$ for all i. Let $\widetilde{C}$ be the normalization of the plane curve C. Projection onto the x axis gives a k-sheeted cyclic branched covering $\widetilde{C} \to \mathbb{P}^1$ branched at the roots of $f(x)$ and possibly at infinity.

Removing the branch points $\mathcal{B} \subset \mathbb{P}^1$ and their preimages in $\widetilde{C}$, we obtain a cyclic (unbranched) covering space of $\mathbb{P}^1 \backslash \mathcal{B}$. By the Galois correspondence for covering spaces, this covering is determined by the kernel of a surjective homomorphism $\phi : \pi_1(\mathbb{P}^1 \backslash \mathcal{B}, x) \to \mathbb{Z}/k\mathbb{Z}$ for some point $x \in \mathbb{P}^1 \backslash \mathcal{B}$. Let γ_i be a loop based at x that runs counterclockwise around the branch point a_i. Then $\phi(\gamma_i) \equiv d_i \mod k$. Note that the irreducibility of C implies the surjectivity of ϕ.

The family of balanced superelliptic covers: In this paper we study a specific family of superelliptic curves, where

$$y^k = (x - a_1)(x - a_2)^{k-1} \cdots (x - a_{2n+1})(x - a_{2n+2})^{k-1}. \tag{1}$$

There is no branching at infinity. As k and n vary, we call the family of normalized curves *balanced superelliptic curves*.

Topologically, the balanced superelliptic curves describe a covering space as follows. Fix integers $g, k \geqslant 2$ such that $k-1$ divides g. Let $p_{g,k} : \Sigma_g \to \Sigma_0$ be a cyclic branched covering map of degree k branched at $2n+2$ points, where $n = g/(k-1)$. In this case, we will denote $\mathrm{LMod}_p(\Sigma_0, \mathcal{B})$ by $\mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B})$. We will refer to the surface Σ_g and the covering map $p_{g,k}$ together as a *balanced superelliptic cover.* When $k = 2$ we recover the case where the deck group is generated by a hyperelliptic involution. The example where $g = 4$ and $k = 3$ is shown in figure 1.

Goals: The goals of this paper are to intiate the study of $\mathrm{LMod}_p(\Sigma_0, \mathcal{B})$ and $\mathrm{SMod}_p(\Sigma_g)$ in general, and to remove the restriction that $\mathrm{LMod}_p(\Sigma, \mathcal{B})$ is equal to $\mathrm{Mod}(\Sigma, \mathcal{B})$ in programs such as Brendle–Margalit–Putman's [10] and McMullen's [21]. In the case where $\Sigma_g \to \Sigma_0$ is a degree k balanced superelliptic cover, we call $\mathrm{SMod}_p(\Sigma_g)$ the *balanced superelliptic mapping class group* and denote it $\mathrm{SMod}_{g,k}(\Sigma_g)$.

We focus on the family of balanced superelliptic covers for a number of reasons. First, when $k > 2$ it is no longer the case that $\mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B}) = \mathrm{Mod}(\Sigma_0, \mathcal{B})$. Therefore the balanced superelliptic covers provide a family of counterexamples to Lemma 5.1 of [8], which is in error (see [5] and [14] for a correction). Second, the covers can be embedded in $\mathbb{R}^3$ so that the deck group is generated by a rotation about the z-axis. This picture should provide insight into the study of the balanced superelliptic mapping class group. Third, as we prove in theorem 1.2 below, the first betti number of $\mathrm{SMod}_{g,k}(\Sigma_g)$ is always 0.

McMullen [21], Venkataramana [24], Chen [11], and others have studied a family of cyclic branched covering spaces of the sphere $p : \Sigma_g \to \Sigma_0$ that also generalize the cover induced by a hyperelliptic involution. Their family arises from curves of the form

$$y^k = (x - a_1)(x - a_2) \cdots (x - a_n).$$

Note that there may be branching at infinity. In their family, every homeomorphism of Σ_0 that fixes the point at infinity lifts to a homeomorphism of Σ_g. When $k = 2$ we recover the cover induced by a hyperelliptic involution.

General cyclic covers: Although our focus is on the balanced superelliptic covers, the results in this paper could be generalized to any cyclic branched cover over the sphere. Indeed, let $p : \Sigma_g \to \Sigma_0$ be a cyclic branched cover with branch points $\mathcal{B} \subset \Sigma_0$. Let $\widehat{\Psi} : \mathrm{LMod}_p(\Sigma_0, \mathcal{B}) \to GL_{|\mathcal{B}|-1}(\mathbb{Z})$ be the homomorphism given by the action of $\mathrm{LMod}_p(\Sigma_0, \mathcal{B})$ on $H_1(\Sigma_0 \backslash \mathcal{B}; \mathbb{Z})$. Then $\widehat{\Psi}(\mathrm{LMod}_p(\Sigma_0, \mathcal{B}))$ is isomorphic to a subgroup of the symmetric group $S_{|\mathcal{B}|}$. While calculating this subgroup of $S_{|\mathcal{B}|}$ is feasible in practice for a single cover or family of covers, we do not see an explicit general form. If one were able to find a presentation for $\Psi(\mathrm{LMod}_p(\Sigma_0, \mathcal{B}))$ in general, then the results of this paper could be generalized to all cyclic branched covers using the techniques developed within.

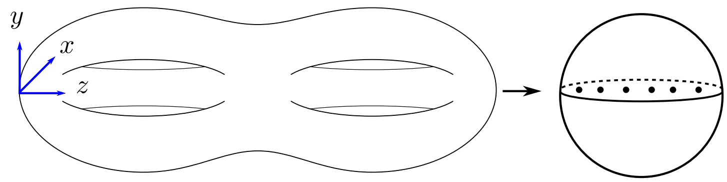

FIGURE 1. The 3-fold cyclic branched covering space of Σ_4 and $\Sigma_4 \to \Sigma_0$.

1.1. **Results.** Let $p : \Sigma_g \to \Sigma_0$ be the k-fold superelliptic covering space branched at $2n+2$ points. For $k > 2$ we compute the index $|\mathrm{Mod}(\Sigma_0, \mathcal{B}) : \mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B})| = \frac{(2n+2)!}{2((n+1)!)^2}$ in scholium 3.7. In fact, for a fixed number of branch points, the liftable mapping class is independent of the degree of the cover. That is, for any integers g_1, g_2 and $k_1, k_2 > 2$ such that $k_i - 1$ divides g_i and $g_1/(k_1 - 1) = g_2/(k_2 - 1)$ for $i = 1, 2$, $\mathrm{LMod}_{g_1,k_1}(\Sigma_0, \mathcal{B}) = \mathrm{LMod}_{g_2,k_2}(\Sigma_0, \mathcal{B})$.

The main technical result in the paper is an explicit presentation for $\mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B})$ in theorem 5.7. This allows us to prove our main theorems.

Theorem 1.1. *Let $k \geqslant 3$. Then*

$$H_1(\mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B}); \mathbb{Z}) \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/(n(n-1)^2)\mathbb{Z} & \textit{if } n \textit{ is odd} \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/(2n(n-1)^2)\mathbb{Z} & \textit{if } n \textit{ is even.} \end{cases}$$

Theorem 1.2. *The abelianization of the balanced superelliptic mapping class group $H_1(\mathrm{SMod}_{g,k}(\Sigma_g); \mathbb{Z})$ is a finite non-cyclic abelian group. In particular, the first betti number of $\mathrm{SMod}_{g,k}(\Sigma_g)$ is 0.*

Kevin Kordek pointed out that since the first betti number of $\mathrm{SMod}_{g,k}(\Sigma_g)$ is 0, there is an isomorphism between $H_1(\mathrm{SMod}_{g,k}(\Sigma_g); \mathbb{Z})$ and the torsion subgroup of the orbifold Picard group of the orbifold $T_g(\hat{D})/\mathrm{SMod}_{g,k}(\Sigma_g)$. Here $T_g(\hat{D})$ is the sublocus of Teichmüller space consisting of the points fixed by the deck group $\hat{D}$.

1.2. **Applications and Future Work.** In the family of covers where $p : \widetilde{\Sigma} \to \Sigma$ is a 3-fold, simple branched cover of the disk, Birman and Wajnryb found a presentation for $\mathrm{LMod}_p(\Sigma, \mathcal{B})$ [9]. However, a 3-fold simple cover does not induce an isomorphism between $\mathrm{LMod}_p(\Sigma, \mathcal{B})$ and $\mathrm{SMod}_p(\widetilde{\Sigma})$ [3, 25]. In contrast, the balanced superelliptic covers we study do induce the Birman–Hilden isomorphism. Therefore the presentation of $\mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B})$ can be used to find a presentation for $\mathrm{SMod}_{g,k}(\Sigma_g)$.

In particular, the generators of $\mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B})$ give us the generators of $\mathrm{SMod}_{g,k}(\Sigma_g)$, which is an infinite-index subgroup of $\mathrm{Mod}(\Sigma_g)$.

Corollary 1.3. *Let Σ_g be a surface of genus $g \geqslant 2$. Let $p_{g,k} : \Sigma_g \to \Sigma_0$ be a balanced superelliptic cover of degree $k \geqslant 3$ with set of branch points $\mathcal{B} = \mathcal{B}(2n + 2)$. Choose lifts of each of the generators of* $\mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B})$.

The subgroup $\mathrm{SMod}_{g,k}(\Sigma_g, \mathcal{B})$ *is generated by these lifts and a generator of the deck group of* $p_{g,k}$.

Generation by torsion elements: Stukow proved that the hyperelliptic mapping class $\mathrm{SMod}_{g,2}(\Sigma_g)$ is generated by two torsion elements [23]. We ask if there is an analogue for general $\mathrm{SMod}_{g,k}(\Sigma_g)$.

Question. Can $\mathrm{SMod}_{g,k}(\Sigma_g)$ be generated by a small number of torsion elements?

Monodromy representation: Let Σ_g be a genus g surface and the map $\Sigma_g \to \Sigma_0$ be a cyclic branched cover. Let D be the deck group of the covering space and $\hat{D}$ the image of D in $\mathrm{Mod}(\Sigma_g)$. The mapping class group $\mathrm{Mod}(\Sigma_g)$ acts on $H_1(\Sigma_g, \mathbb{Z})$ and the action preserves the interesection form on $H_1(\Sigma_g, \mathbb{Z})$. Thus the action induces a surjective representation $\rho : \mathrm{Mod}(\Sigma_g) \to \mathrm{Sp}(2g, \mathbb{Z})$. Let G be a group. Let $C_G(H)$ denote the centralizer of a subgroup H in G. McMullen asks when $\rho(C_{\mathrm{Mod}(\Sigma_g)}(D))$ is finite index in $C_{\mathrm{Sp}(2g,\mathbb{Z})}(\rho(D))$ [21]. While McMullen looks at a different family of covering spaces than we do, our work could be used to extend his program to the family of balanced superelliptic curves.

Question. What is the image $\rho(\mathrm{SMod}_{g,k}(\Sigma_g, \mathcal{B}))$ in $\mathrm{Sp}(2g, \mathbb{Z})$?

Since $\mathrm{SMod}_{g,k}(\Sigma_g)$ is the normalizer of $\hat{D}$ in $\mathrm{Mod}(\Sigma_g)$, we have an analogue of McMullen's question [21]:

Question. Let $p : \Sigma_g \to \Sigma_0$ be any cyclic branched cover of the sphere. When is $\rho(\mathrm{SMod}_p(\Sigma_g))$ finite index in the normalizer of $\rho(\hat{D})$?

The generators for $\mathrm{SMod}_{g,k}(\Sigma_g)$ in corollary 1.3 may be useful in answering this question for balanced superelliptic covers, and as noted above, it is possible to extend our techniques to other superelliptic covers.

Outline of paper: In section 2, we review the necessary combinatorial group theory and lifting properties for constructing our presentation. In section 3, we explicitly construct the family of balanced superelliptic covers, and we prove that $\mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B})$ is an extension of a subgroup W_{2n+2} of the symmetric group S_{2n+2} by the pure mapping class group $\mathrm{PMod}(\Sigma_0, \mathcal{B})$. In section 4 we find presentations for $\mathrm{PMod}(\Sigma_0, \mathcal{B})$ and W_{2n+2} in the group extension. We build the presentation for $\mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B})$ in section 5. Finally, we prove theorems 1.1 and 1.2 in section 6.

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2. Preliminary definitions and lemmas

In this section, we survey the combinatorial group theory and algebraic topology results used later in the paper. We first find a presentation of a group when given a short exact sequence of groups in section 2.1. We then use homological arguments to characterize the mapping classes that lift.

2.1. Group Presentations and Short Exact Sequences.

To obtain the presentation in section 5, we use two well-known results concerning short exact sequences and group presentations.

Lemma 2.1. *Let*

$$1 \longrightarrow K \xrightarrow{\alpha} G \xrightarrow{\pi} H \longrightarrow 1$$

be a short exact sequence of groups. Let $\langle S \mid R\rangle$ be a presentation for G where each symbol $s \in S$ denotes a generator $g_s \in G$. Let K be normally generated by $\{k_\beta\} \subset K$ and for each β, let w_β be a word in the symbols S denoting $\alpha(k_\beta)$. Then H admits the presentation $\langle S \mid R \cup \{w_\beta\}\rangle$ where $s \in S$ denotes $\pi(g_s)$. ■

A proof of lemma 2.1 can be found in [18, Section 2.1]

For lemma 2.2, let

$$1 \longrightarrow K \xrightarrow{\alpha} G \xrightarrow{\pi} H \longrightarrow 1$$

be a short exact sequence of groups. Let $K \cong \langle S_K \mid R_K\rangle$. Let $t \in S_K$. Assign the generator $k_t \in K$ to t. Similarly, let $H \cong \langle S_H \mid R_H\rangle$. Let $s \in S_H$. Assign the generator $h_s \in H$ to s.

For each generator h_s of H, choose an element $g_s \in G$ such that $\pi(g_s) = h_s$. Then let $s \in S_H$, and let $\tilde{s}$ denote g_s. Let $\widetilde{S}_H = \{\tilde{s} : s \in S_H\}$. For each $k_t \in K$, let $\tilde{t} \in \widetilde{S}_K$ denote $\alpha(k_t) \in G$. Let $\widetilde{S}_K = \{\tilde{t} : t \in S_K\}$.

Each word in R_H can be written in the form $s_1^{\epsilon_1} \cdots s_m^{\epsilon_m}$ with $s_i \in S_H$ and $\epsilon_i \in \{\pm 1\}$. Let $r \in R_H$ be $s_1^{\epsilon_1} \cdots s_m^{\epsilon_m}$. Denote the word $\widetilde{s_1}^{\epsilon_1} \cdots \widetilde{s_m}^{\epsilon_m}$ in $\widetilde{S}_H$ by $\tilde{r}$. Then $\tilde{r}$ is a word in $\widetilde{S}_H$ denoting some $g \in G$. The element g is such that $\pi(g) = \mathrm{id}_H$. Since the sequence is exact, this means that $g \in \alpha(K)$. Let w_r be a word in $\widetilde{S}_K$ denoting g and define the set of words

$$R_1 := \{\tilde{r} w_r^{-1} : r \in R_H\}.$$

Since $\alpha(K)$ is normal in G, for every $k_t \in K$ and $g_s \in G$, the element $g_s \alpha(k_t) g_s^{-1} \in \alpha(K)$. Let $v_{s,t}$ be a word in $\widetilde{S}_K$ that denotes $g_s \alpha(k_t) g_s^{-1}$. Define the set of words

$$R_2 := \{\tilde{s}\tilde{t}\tilde{s}^{-1} v_{s,t}^{-1} : \tilde{t} \in \widetilde{S}_K, \tilde{s} \in \widetilde{S}_H\}.$$

Finally, let $\widetilde{R}_K := \{\tilde{r} : r \in R_K\}$ where $\tilde{r}$ is the word in $\widetilde{S}_K$ obtained by replacing every symbol t by $\tilde{t}$ in the same way as in the definition of R_1.

Lemma 2.2. *Let*

$$1 \longrightarrow K \xrightarrow{\alpha} G \xrightarrow{\pi} H \longrightarrow 1$$

be a short exact sequence of groups. Then G admits the presentation

$$G \cong \langle \widetilde{S}_K \cup \widetilde{S}_H \mid R_1 \cup R_2 \cup \widetilde{R}_K \rangle.$$

where $\widetilde{S}_K, \widetilde{S}_H, R_1, R_2$, and $\widetilde{R}_K$ are defined as above. ■

A proof is left to the reader.

2.2. **Lifting mapping classes.** Our goal is to characterize which mapping classes in $\mathrm{Mod}(\Sigma_0, \mathcal{B})$ belong to $\mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B})$. Because all homotopies of Σ_0 lift to homotopies of Σ_g, it is sufficient to determine which homeomorphisms of Σ_0 lift to homeomorphisms of Σ_g. In 2.2.1 we characterize curves in Σ_0 that lift to closed curves in Σ_g. In 2.2.2 we characterize homeomorphisms of Σ_0 that lift to homeomorphisms of Σ_g.

2.2.1. *Lifting Curves.* Throughout this section we will work in generality. Let $\widetilde{X}$ be a path connected topological space.

Let $p : \widetilde{X} \to X$ be an unbranched covering space. Let $c : S^1 \to X$ be a curve in X. Recall that c *lifts* if there exists $\tilde{c} : S^1 \to \widetilde{X}$ such that $p\tilde{c} = c$.

Let $p : \widetilde{X} \to X$ be an abelian covering space with deck group D. Fix a base point $x_0 \in X$. There is a one-to-one correspondence between regular covering spaces of X and normal subgroups of $\pi_1(X, x_0)$. The covering space $p : \widetilde{X} \to X$ corresponds to the kernel of a surjective homomorphism $\varphi : \pi_1(X, x_0) \to D$. Let $\Phi : \pi_1(X, x_0) \to H_1(X; \mathbb{Z})$ be the Hurewicz homomorphism. Since D is abelian, there exists a homomorphism $\overline{\varphi} : H_1(X; \mathbb{Z}) \to D$ such that $\varphi = \overline{\varphi}\Phi$.

Conversely, given a homomorphism $\overline{\varphi} : H_1(X; \mathbb{Z}) \to D$, we can define a homomorphism $\varphi : \pi_1(X, x_0) \to D$ by setting $\varphi = \overline{\varphi}\Phi$. Since $\ker(\varphi)$ is a normal subgroup of $\pi_1(X, x_0)$, it determines a regular cover. So, for a regular abelian cover we will call the homomorphism $\overline{\varphi} : H_1(X; \mathbb{Z}) \to D$ the *defining homomorphism* of the cover. Note that this homomorphism is well defined up to an automorphism of D.

Unwrapping these definitions we get the following lemma.

Lemma 2.3. *Let $p : \widetilde{X} \to X$ be a regular abelian cover with deck group D, and let $\overline{\varphi} : H_1(X; \mathbb{Z}) \to D$ be the defining homomorphism. A curve $c : S^1 \to X$ lifts if and only if $[c] \in \ker \overline{\varphi} < H_1(X; \mathbb{Z})$.* ■

2.2.2. *Lifting homeomorphisms.* Let $\widetilde{X}$ be a path connected topological space. Let $p : \widetilde{X} \to X$ be a finite-sheeted, covering map.

A homeomorphism $f : X \to X$ *lifts* if there exists a homeomorphism $\tilde{f} : \widetilde{X} \to \widetilde{X}$ such that $p\tilde{f} = fp$.

Let f be a homeomorphism of X and let f_* be the induced map on $H_1(X, \mathbb{Z})$.

Lemma 2.4. *Let $p : \widetilde{X} \to X$ be a regular abelian cover, with $X, \widetilde{X}$ path connected. Let D be the deck group and let $\overline{\varphi} : H_1(X) \to D$ be the defining homomorphism. Then a homeomorphism $f : X \to X$ lifts if and only if the induced map on homology f_* satisfies $f_*(\ker \overline{\varphi}) = \ker \overline{\varphi}$.* ■

A well known corollary follows immediately:

Corollary 2.5. *Let $\widetilde{X}$ be a path connected topological space. Let $p : \widetilde{X} \to X$ be an abelian cover. A homeomorphism $f : X \to X$ lifts if and only if for all curves c that lift, $f(c)$ also lifts.* ■

Surfaces: Let Σ_g be a closed surface and let $\mathcal{B}(m) \subset \Sigma_g$ be a set of m marked points in Σ_g. Let $\Sigma_{g,m}^\circ = \Sigma_g \backslash \mathcal{B}(m)$. If the number of punctures m is either clear from context or irrelevent, we will write Σ_g° to denote a surface with punctures.

2.3. **The group extension.** Let $\mathrm{PMod}(\Sigma_{0,m}^\circ)$ denote the pure mapping class group of $\Sigma_{0,m}^\circ$. The pure mapping class group is the subgroup of $\mathrm{Mod}(\Sigma_{0,m}^\circ)$ that fixes the punctures. Let S_m be the symmetric group on m elements. There is an exact sequence:

$$1 \to \mathrm{PMod}(\Sigma_{0,m}^\circ) \to \mathrm{Mod}(\Sigma_{0,m}^\circ) \to S_m \to 1. \tag{2}$$

Let $p : \Sigma_g \to \Sigma_0$ be a finite branched covering space with set of m branch points $\mathcal{B}(m)$. Our goal is to find an sequence analogous to (2) for $\mathrm{LMod}_p(\Sigma_0, \mathcal{B}(m))$.

Action on homology: The first homology group $H_1(\Sigma_{0,m}^\circ; \mathbb{Z})$ is isomorphic to $\mathbb{Z}^{\oplus m-1}$, and a basis can be chosen as follows. Number the punctures $1, \ldots, m$, and let x_i be the homology class of curve on $\Sigma_{0,m}^\circ$ surrounding the ith puncture, oriented counterclockwise around the puncture. Then $\{x_1, \ldots, x_{m-1}\} \subset H_1(\Sigma_{0,m}^\circ)$ forms a basis.

Let $\Psi_m : \mathrm{Mod}(\Sigma_{0,m}^\circ) \to \mathrm{GL}_{m-1}(\mathbb{Z})$ be the homomorphism given by the action of $\mathrm{Mod}(\Sigma_{0,m}^\circ)$ on $H_1(\Sigma_{0,m}^\circ; \mathbb{Z})$. Since each basis element is support on a neighborhood of a puncture, any element of the pure mapping class group will act trivially on homology. Conversely, any homeomorphism which induces a non-trivial permutation on the punctures will permute homology classes of loops surrounding the punctures.

From this discussion we see that the kernel of Ψ_m is equal to the pure mapping class group $\mathrm{PMod}(\Sigma_{0,m}^\circ)$, and the image of Ψ_m is isomorphic to the symmetric group S_m. Indeed, if f is a homeomorphism of $\Sigma_{0,m}^\circ$, $\Psi_m([f])$ is the permutation induced on the m punctures. We can now conclude that the short exact sequence (2) above is obtained from the action of $\mathrm{Mod}(\Sigma_{0,m}^\circ)$ on $H_1(\Sigma_{0,m}^\circ; \mathbb{Z})$.

Punctures and marked points: Our lifting criteria above can only be applied to unbranched covering spaces. However we ultimately want a presentation for $\mathrm{LMod}_p(\Sigma_0, \mathcal{B})$, where Σ_0 is a surface with branch points $\mathcal{B}$.

Let $p : \widetilde{\Sigma} \to \Sigma$ be a branched covering space of surfaces with set of branch points $\mathcal{B} \subset \Sigma$. As above, it may be necessary to remove the branch points in Σ to obtain the punctured surface $\Sigma^\circ = \Sigma \backslash \mathcal{B}$. We then must also remove the preimages of the branch points in $\widetilde{\Sigma}$ to obtain the punctured surface $\widetilde{\Sigma}^\circ = \widetilde{\Sigma} \backslash p^{-1}(\mathcal{B})$. Let $p|_{\widetilde{\Sigma}^\circ} : \widetilde{\Sigma}^\circ \to \Sigma^\circ$ be an unbranched covering map. We use the Σ° notation specifically when we work with a surface with branch points removed (or $\Sigma^\circ_{0,m}$ when we need to specify the number of punctures). There is an inclusion map $\iota : \Sigma^\circ \to \Sigma$ where the punctures of Σ° are filled in with marked points. These marked points exactly comprise the set of branch points $\mathcal{B}$ of the cover. Then $\mathrm{Mod}(\Sigma^\circ)$ is isomorphic to $\mathrm{Mod}(\Sigma, \mathcal{B})$ because homeomorphisms and homotopies of Σ°_0 must fix the set of punctures and $\mathrm{Mod}(\Sigma, \mathcal{B})$ must fix the set of branch points. On the other hand, in the inclusion map $\widetilde{\iota} : \widetilde{\Sigma}^\circ \to \widetilde{\Sigma}$ the punctures are filled in with non-marked points. Then $\mathrm{Mod}(\widetilde{\Sigma})$ and $\mathrm{Mod}(\widetilde{\Sigma}^\circ)$ are not isomorphic because the set of points $p^{-1}(\mathcal{B})$ are not treated as marked points in $\mathrm{Mod}(\widetilde{\Sigma})$.

Work of Birman and Hilden [7, 8] gives an isomorphism between $\mathrm{LMod}_p(\Sigma, \mathcal{B})$ and a subgroup of $\mathrm{Mod}(\widetilde{\Sigma})$ modulo the homotopy classes of the deck transformations. The group $\mathrm{Mod}(\widetilde{\Sigma})$ need not stabilize the set $p^{-1}(\mathcal{B})$ in their work.

3. The balanced superelliptic covers

3.1. **The construction.** Choose a pair of integers $g, k \geqslant 2$ such that $k-1$ divides g, and let $n = g/(k-1)$. Embed Σ_g, a surface of genus g, in $\mathbb{R}^3$ so it is invariant under a rotation by $2\pi/k$ about the z axis as we describe below.

The intersection of Σ_g with the plane $z = a$ is:

- Empty for $a < 0$ and $a > 2n+1$
- A point at the origin for $a = 0$ and $a = 2n+1$
- Homeomorphic to a circle for $2m < a < 2m+1$ with $m \in \{0, \dots, n\}$
- A rose with k petals for $a \in \{1, \dots, 2n\}$
- k disjoint simple closed curves invariant under a rotation of $2\pi/k$ about the z axis for $2m-1 < a < 2m$ with $m \in \{1, \dots, n\}$. In the special case $a = 2m - 1/2$, put polar coordinates (r, θ) on the plane $z = 2m - 1/2$. Then we have k disjoint circles with centers on the rays $\theta = 2\pi d/k$, $d \in \{0, \dots, k-1\}$.

See figure 1 for the embedding when $g = 4$ and $k = 3$.

Consider a homeomorphism $\zeta : \Sigma_g \to \Sigma_g$ of order k given by rotation about the z axis by $2\pi/k$. The homeomorphism ζ fixes $2n+2$ points, which are the points of intersection of Σ_g with the z axis. Define an equivalence relation on Σ_g given by $x \sim y$ if and only if $\zeta^q(x) = y$ for some q. The resulting surface $\Sigma_g/\sim$ is homeomorphic to a closed sphere Σ_0. The quotient map $p_{g,k} : \Sigma_g \to \Sigma_0$ is a k-fold cyclic branched covering map with $2n+2$ branch points, which are the images of the points fixed by ζ. The deck group of $p_{g,k}$ is a cyclic group of order k generated by ζ. When $k = 2$, ζ is a hyperelliptic involution.

An important collection of arcs: Fix a pair of integers $g, k \geqslant 2$ such that $k - 1 \mid g$, and consider the surface Σ_g embedded in $\mathbb{R}^3$ as described above. Using cylindrical coordinates in $\mathbb{R}^3$, let $P_{\theta_0} = \{(r, \theta_0, z) \in \mathbb{R}^3 : r \geqslant 0\}$ be a closed half plane. The intersection of Σ_g and $P_{\pi/k}$ is a collection of

$n+1$ arcs where $n = g/(k-1)$. Call these arcs $\beta_1, \ldots, \beta_{n+1}$. For each arc $\beta_i : [0,1] \to \Sigma_g$, orient it so that $\beta_i(0) = (0,0,2i-2)$ and $\beta_i(1) = (0,0,2i-1)$ in $\mathbb{R}^3$. Number the endpoints $1, \cdots, 2n+2$ in order of increasing z value and fix the numbering for the remainder of the paper.

Consider the balanced superelliptic covering map $p_{g,k}$ as defined above. For each i with $1 \leqslant i \leqslant n+1$, let $\alpha_i = p_{g,k}\beta_i$. Each α_i is an arc $\alpha_i : [0,1] \to \Sigma_0$. The endpoints of α_i are in the set of branch points $\mathcal{B}(2n+2) \subset \Sigma_0$. Let α be the union of the arcs α_i in Σ_0. Let $[\alpha]$ denote the relative homology class of α in $H_1(\Sigma_0, \mathcal{B}; \mathbb{Z})$. The class $[\alpha]$ is calculated $\sum_{i=1}^{n+1} [\alpha_i]$. Figure 2 shows the embeddings of the arcs $\beta_1, \beta_2, \beta_3 \in \Sigma_4$ and $\alpha_1, \alpha_2, \alpha_3 \in \Sigma_0$ for the 3-fold balanced superelliptic cover of Σ_4 over Σ_0.

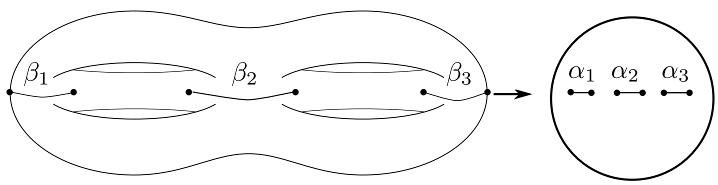


FIGURE 2. The arcs $\beta_1, \beta_2, \beta_3 \in \Sigma_4$ and the arcs $\alpha_1, \alpha_2, \alpha_3 \in \Sigma_0$.

3.2. **A lifting criterion for superellipitic covers.** The goal of this section is to prove lemma 3.4.

An intersection form for punctured surfaces: In lemma 3.1, we abuse notation and identify curves in $\Sigma_{0,m}^{\circ}$ with their image in Σ_0 under the inclusion $\Sigma_{0,m}^{\circ} \to \Sigma_0$.

Lemma 3.1. *Let Σ_g be a closed surface surface and $\mathcal{B}(m)$ a set of m points in Σ_g. There exists a homomorphism*

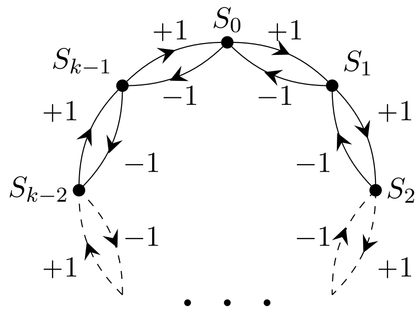
$$\bullet : H_1(\Sigma_{g,m}^{\circ}; \mathbb{Z}) \times H_1(\Sigma_g, \mathcal{B}(m); \mathbb{Z}) \to \mathbb{Z}$$

given by $c \bullet a = \hat{i}(c,a)$ where c is a homotopy class of curves in $\Sigma_{g,m}$, a is a homotopy class curves or arcs in $(\Sigma_g, \mathcal{B})$, and $\hat{i}(c,a)$ is the algebraic intersection of c and a. ∎

This is a well-known result and a proof can be found in the appendix of [15], for example.

We need the following combinatorial lemma.

Lemma 3.2. *Let G be the weighted digraph*



and let Γ *be a finite walk on* G *beginning at* S_0*. Define the weight of* Γ*, which we denote* $w(\Gamma)$*, as the sum of the weights of the edges traversed in the walk. Then* Γ *terminates at* S_i*, if and only if* $w(\Gamma) \equiv i \mod k$*.*

Proof. Induct on the length of the walk. ∎

In order to apply this lemma, we use the union of arcs α in Σ_0 defined in section 3.1 and their preimages in Σ_g.

The full preimage $p_{g,k}^{-1}(\alpha)$ is a collection of $k(n+1)$ oriented arcs in Σ_g and we will denote the union by $\widetilde{\alpha}$. The union of arcs $\widetilde{\alpha}$ consists of the orbits β_i under the action of the deck group of $p_{g,k}$.

The surface $\Sigma_g \backslash \{\widetilde{\alpha}\}$ is a union of k components of Σ_g. The components are cyclically permuted by the action of the deck group. Label one of these connected components R_0, and for each $\ell \in \{1, \ldots, k-1\}$ label $\zeta^\ell(R_0)$ by R_ℓ. We will refer to the embedding of each R_ℓ in Σ_g as a *region* in Σ_g.

Consider a curve $\widetilde{\gamma}$ in Σ_g that does not contain any of the $2n+2$ points on the z-axes and that intersects $\widetilde{\alpha}$ transversely. Choose an orientation for $\widetilde{\gamma}$.

Denote the algebraic intersection of $\widetilde{\gamma}$ and $\widetilde{\alpha}$ by $\hat{i}(\widetilde{\gamma}, \widetilde{\alpha})$. Homotopy preserves algebraic intersection, so the algebraic intersection is well defined for all representatives within the class. The orientation of $\widetilde{\alpha}$ is consistent with respect to the covering map $p_{g,k}$.

Consider a parameterization $\widetilde{\gamma} : [0,1] \to \Sigma_g$ with $\widetilde{\gamma}(0) = \widetilde{\gamma}(1)$. Let $t_0 \in [0,1]$ be a value such that $\widetilde{\gamma}(t_0) \in \widetilde{\gamma} \cap \widetilde{\alpha}$. Fix $\epsilon > 0$ such that $\widetilde{\gamma}(t_0 - \epsilon, t_0 + \epsilon) \cap \widetilde{\alpha} = t_0$. Then either $\widetilde{\gamma}(t_0 - \epsilon) \in R_{(i \mod k)}$ and $\widetilde{\gamma}(t_0 + \epsilon) \in R_{(i+1 \mod k)}$ for some $0 \leqslant i \leqslant k-1$ or $\widetilde{\gamma}(t_0 - \epsilon) \in R_{(i \mod k)}$ and $\widetilde{\gamma}(t_0 + \epsilon) \in R_{(i-1 \mod k)}$ for some $1 \leqslant i \leqslant k$. All intersections of $\widetilde{\gamma}$ and $\widetilde{\alpha}$ where the index of R_i increases modulo k will have the same sign of intersection. All intersections of $\widetilde{\gamma}$ and $\widetilde{\alpha}$ where the index of R_i decreases modulo k will have sign of intersection opposite to those where the index of R_i increases.

Lemma 3.3. *Let* $\overline{p}_{g,k} : \Sigma_g^\circ \to \Sigma_0^\circ$ *be an unbranched balanced superelliptic covering map of degree* k*. If a curve* γ *in* Σ_0° *lifts, then* $\hat{i}(\gamma, \alpha) \equiv 0 \mod k$*.*

Proof. Consider the regions $R_0, \cdots, R_{k-1} \subset (\Sigma_g \backslash \widetilde{\alpha})$ as above. For each $R_j \in \Sigma_g$, there is a corresponding embedding of $R_j \backslash p_{g,k}^{-1}(\mathcal{B})$ in Σ_g°. We will also denote the embeddings of $R_j \backslash p_{g,k}^{-1}(\mathcal{B})$ in Σ_g° by R_j as it will be clear from context when we are referring to punctured regions.

Let γ be a curve in Σ_0° that lifts to a multicurve $\widetilde{\gamma}$ in Σ_g°. The multicurve $\widetilde{\gamma}$ has k components in Σ_g°. Each component of $\widetilde{\gamma}$ is a map $[0,1]/\{0,1\} \to \Sigma_g^\circ$. Let $\widetilde{\gamma}_i$ denote the component of $\widetilde{\gamma}$ such that $\widetilde{\gamma}_i(0) = \widetilde{\gamma}_i(1) \in R_i$.

By compactness, $|\sigma \cap \widetilde{\alpha}| < \infty$. We may assume that $\widetilde{\gamma}$ is transverse to $\widetilde{\alpha}$. Let $x \in \widetilde{\gamma} \cap \widetilde{\alpha}$. Since the action of the deck group is transitive, the orbit of $\widetilde{x}$ is of order k. Indeed, the orbit of $\widetilde{x}$ is exactly $\bar{p}_{g,k}^{-1}(\bar{p}_{g,k}(\widetilde{x}))$ where $x \in \gamma \cap \alpha$. The signs of intersection of all points in the orbit of $\widetilde{x}$ are equal. Thus all components of $\widetilde{\gamma}$ have the same algebraic and geometric intersections with $\widetilde{\alpha}$. Therefore $\hat{i}(\gamma, \alpha) = \hat{i}(\widetilde{\gamma}_i, \widetilde{\alpha})$ for any component $\widetilde{\gamma}_i$ of $\widetilde{\gamma}$.

Let G be the weighted digraph as in lemma 3.2. Let S_i be the vertex in G corresponding to the region R_i in Σ_g.

We now construct a walk $\Gamma_{\widetilde{\gamma}_i}$ in G corresponding to each $\widetilde{\gamma}_i$. The walk $\Gamma_{\widetilde{\gamma}_i}$ begins at the vertex S_i. If $\widetilde{\gamma}_i \cap \widetilde{\alpha}$ is empty, the point S_i is the entire path G.

If $\widetilde{\gamma}_i \cap \widetilde{\alpha} \neq \varnothing$, let $\{t_j\} \subset [0,1]$ be the set of values such that $\widetilde{\gamma}_i(t_j) \in \widetilde{\gamma}_i \cap \widetilde{\alpha}$ and $t_j < t_{j+1}$. Choose a value $\epsilon > 0$ so that $\widetilde{\gamma}_i(t_j - \epsilon, t_j + \epsilon) \cap \widetilde{\alpha} = t_j$ for each t_j. We construct the walk $\Gamma_{\widetilde{\gamma}_i}$ by adding an edge and a vertex for each t_j, in the order of increasing j. The vertices will be those corresponding to the regions containing the elements $\widetilde{\gamma}_i(t_j + \epsilon)$ for each j. Add the edge corresponding to t_ℓ, which connects the vertex corresponding to the region containing $\widetilde{\gamma}_i(t_\ell - \epsilon)$ to the vertex corresponding to the region containing $\widetilde{\gamma}_i(t_\ell + \epsilon)$.

For each component $\widetilde{\gamma}_i$, the walk $\Gamma_{\widetilde{\gamma}_i}$ begins and terminates at S_i. By lemma 3.2, $\hat{i}(\widetilde{\gamma}_i, \widetilde{\alpha}) \equiv 0 \mod k$. Then by the discussion above, $\hat{i}(\gamma, \alpha) \equiv 0 \mod k$ as well. ∎

We are now ready to prove lemma 3.4, which is lemma 3.3 and its converse.

Lemma 3.4 (A lifting criterion for curves). *Let $\overline{p}_{g,k} : \Sigma_g^\circ \to \Sigma_0^\circ$ be the unbranched balanced superelliptic covering space. Let γ be a curve on Σ_0°. Then $[\gamma] \in \ker(\overline{\varphi})$ if and only if $\hat{i}(\gamma, \alpha) \equiv 0 \mod k$.*

We note that an analogue of lemma 3.4 is true for all cyclic branched covers of the sphere, but the collection of arcs α is specific to the balanced superellipitic covers.

Proof. Let $\hat{i}(-, \alpha) : H_1(\Sigma_0^\circ; \mathbb{Z}) \to \mathbb{Z}$ be the homomorphism from lemma 3.1, and let $\pi : \mathbb{Z} \to \mathbb{Z}/k\mathbb{Z}$ be the natural projection map. Let $\phi = \pi \circ \hat{i}(-, \alpha) : H_1(\Sigma_0^\circ; \mathbb{Z}) \to \mathbb{Z}/k\mathbb{Z}$. The homomorphism ϕ is surjective since there is a curve γ such that $\hat{i}(\gamma, \alpha) = 1$.

Let $\overline{\varphi} : H_1(\Sigma_0^\circ; \mathbb{Z}) \to \mathbb{Z}/k\mathbb{Z}$ be the defining homomorphism for the unbranched balanced superelliptic cover. By lemma 2.3, $\ker(\overline{\varphi}) = \{[\gamma] \in H_1(\Sigma_0^\circ; \mathbb{Z}) : \gamma \text{ lifts}\}$. Lemma 3.3 shows that $\ker(\phi) \subset \ker(\overline{\varphi})$. However, these are both index k subgroups of $H_1(\Sigma_0^\circ; \mathbb{Z})$, so $\ker(\phi) = \ker(\overline{\varphi})$.

By the definition of ϕ, $\ker(\phi) = \{[\gamma] \in H_1(\Sigma_0^\circ;\mathbb{Z}) : \hat{i}(\gamma,\alpha) \equiv 0 \mod k\}$. This completes the proof. ∎

Recall from lemma 2.3 that a curve γ lifts if and only if $[\gamma] \in \ker(\overline{\varphi})$. This together with lemma 3.4 allows us to decide whether or not a curve lifts simply by computing its algebraic intersection number with the collection of arcs α.

Lemma 3.5. *Let $\overline{p}_{g,k}$ be an unbranched balanced superelliptic covering map of degree k. Number the punctures of Σ_0° from 1 to $2n+2$ as in section 3.1. Let x_j be the homology class of curves surrounding the jth puncture of Σ_0° for $1 \leqslant j \leqslant 2n+1$ and oriented counterclockwise. The set $\{x_1, \ldots, x_{2n+1}\}$ forms a basis for $H_1(\Sigma_0^\circ;\mathbb{Z})$. Let γ be a curve in Σ_0° with $[\gamma] = (\gamma_1, \ldots, \gamma_{2n+1}) \in H_1(\Sigma_0^\circ;\mathbb{Z})$ with respect to this basis. Then γ lifts if and only if*

$$\sum_{i=1}^{2n+1} (-1)^{i+1}\gamma_i \equiv 0 \mod k.$$

Note that lemma 3.5 is the key lemma that distinguishes the balanced superelliptic covers from other superelliptic covers. In order to use our methods for other families of superelliptic covers, one must characterize the curves that lift, as we do here for the balanced superelliptic covers.

Proof. Let α be the collection of arcs defined above and observe that

$$\hat{i}(x_j,\alpha) = \begin{cases} 1 & \text{if } j \text{ is odd} \\ -1 & \text{if } j \text{ is even.} \end{cases}$$

Then $\hat{i}(\gamma,\alpha) = \sum_{i=1}^{2n+1}(-1)^{i+1}\gamma_i$. Combining this with lemma 3.4 completes the proof. ∎

This lemma shows that the family of balanced superelliptic covers are modeled by plane curves defined by equation (1).

3.3. **The exact sequence for** $\mathrm{LMod}_p(\Sigma_0,\mathcal{B})$**.** Let $p : \widetilde{\Sigma} \to \Sigma_0$ be a finite cyclic branched cover of the sphere. Let $\mathcal{B}(m)$ be the set of m branch points of the covering space. Recall that $\mathrm{Mod}(\Sigma_0,\mathcal{B}(m))$ and $\mathrm{Mod}(\Sigma_{0,m}^\circ)$ are isomorphic. Recall that the action of $\mathrm{Mod}(\Sigma_{0,m}^\circ)$ on $H_1(\Sigma_{0,m}^\circ,\mathbb{Z})$ induces a homomorphism $\Psi_m : \mathrm{Mod}(\Sigma_{0,m}^\circ) \to S_m$ on the short exact sequence (2). We will also consider the map $\widehat{\Psi}_m : \mathrm{Mod}(\Sigma_0,\mathcal{B}(m)) \to S_m$ by precomposing Ψ_m with the isomorphism $\mathrm{Mod}(\Sigma_0,\mathcal{B}(m)) \cong \mathrm{Mod}(\Sigma_{0,m}^\circ)$. By lemma 2.4 and the short exact sequence (2), $\mathrm{PMod}(\Sigma_0,\mathcal{B}(m)) = \ker\widehat{\Psi}_m$ is contained in $\mathrm{LMod}_p(\Sigma_0,\mathcal{B}(m))$. This gives us the short exact sequence

$$1 \to \mathrm{PMod}(\Sigma_0,\mathcal{B}(m)) \to \mathrm{LMod}_p(\Sigma_0,\mathcal{B}(m)) \to \widehat{\Psi}_m(\mathrm{LMod}_p(\Sigma_0,\mathcal{B}(m))) \to 1.$$

Since $\widehat{\Psi}_m(\mathrm{Mod}(\Sigma_0,\mathcal{B}(m))) \cong S_m$, the group $\widehat{\Psi}_m(\mathrm{LMod}_p(\Sigma_0,\mathcal{B}(m)))$ is isomorphic to a subgroup of S_m. Our next goal is to find the subgroup of S_m

isomorphic to $\widehat{\Psi}_m(\mathrm{LMod}_p(\Sigma_0, \mathcal{B}(m)))$ where p is a balanced superelliptic covering map.

Let $p_{g,k} : \Sigma_g \to \Sigma_0$ be the balanced superelliptic cover. We will denote $\mathrm{LMod}_p(\Sigma_0, \mathcal{B})$ by $\mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B})$. Recall that $\mathcal{B} = \mathcal{B}(2n+2)$ where $n = g/(k-1)$. We will suppress the $2n+2$ in our notation, since the number of branch points of the balanced superelliptic covers is determined by g and k.

Parity of a permutation: Fix an integer $m \geqslant 2$. Let τ be a permutation in S_m. We say that τ *preserves parity* if $\tau(q) = q \mod 2$ for all $q \in \{1, \cdots, m\}$. We say that τ *reverses parity* if $\tau(q) \neq q \mod 2$ for all $q \in \{1, \cdots, m\}$.

Let S_{2l} be the symmetric group on the set $\{1, \ldots, 2l\}$. Let $W_{2l} < S_{2l}$ be the subgroup consisting of permutations that either preserve parity, or reverse parity. Then

$$W_{2l} \cong (S_l \times S_l) \rtimes \mathbb{Z}/2\mathbb{Z}$$

where $\mathbb{Z}/2\mathbb{Z}$ acts on $S_l \times S_l$ by switching the coordinates.

Lemma 3.6. *Let $\overline{p}_{g,k} : S_g \to S_0$ be a balanced superelliptic covering map of degree k. Let $\Psi_{2n+2} : \mathrm{Mod}(\Sigma_0^\circ) \to S_{2n+2}$ be the homomorphism induced from the action of $\mathrm{Mod}(\Sigma_0^\circ)$ on $H_1(\Sigma_0^\circ, \mathbb{Z})$. If $k = 2$, the image $\widehat{\Psi}_{2n+2}(\mathrm{LMod}_{g,2}(\Sigma_0, \mathcal{B})) = S_{2n+2}$. For $k > 2$, $\widehat{\Psi}_{2n+2}(\mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B})) = W_{2n+2}$.*

Lemmas 3.4 and 3.5 characterize the curves in Σ_0° that lift. Let γ be a curve in Σ_0° and let T_γ be a Dehn twist about γ. It is possible for T_γ to lift to a homeomorphism of Σ_g even if γ does not lift.

Proof. Let γ be a curve in Σ_0° and let $[\gamma] = \sum_{i=1}^{2n+1} \gamma_i x_i \in H_1(\Sigma_0^\circ; \mathbb{Z})$. Let $[f] \in \mathrm{Mod}(\Sigma_0^\circ)$ and let $\sigma = \Psi([f]) \in S_{2n+2}$.

The permutation σ fixes the jth puncture if and only if $\sigma(j) = j$. If $\sigma(2n+2) = 2n+2$, then

$$[f(\gamma)] = \sum_{i=1}^{2n+1} \gamma_i x_{\sigma(i)} = \sum_{i=1}^{2n+1} \gamma_{\sigma^{-1}(i)} x_i$$

in homology.

Now consider the case where $\sigma(j) = 2n+2$ for some $j \neq 2n+2$. If δ is a curve homotopic to the $(2n+2)$nd puncture, then $[\delta] = -\sum_{i=1}^{2n+1} x_i$. Therefore $[f(x_j)] = -\sum_{i=1}^{2n+2} x_i$ and

$$\begin{aligned}
[f(\gamma)] &= \sum_{i=1}^{j-1} \gamma_i x_{\sigma(i)} + \sum_{j+1}^{2n+1} \gamma_i x_{\sigma(i)} - \gamma_j \left(\sum_{i=1}^{2n+1} x_i \right) \\
&= \left(\sum_{\substack{i \in \{1, \ldots, 2n+1\} \\ i \neq \sigma(2n+2)}} (\gamma_{\sigma^{-1}(i)} - \gamma_j) x_i \right) - \gamma_j x_{\sigma(2n+2)}.
\end{aligned}$$

In order to simplify the indices, we will include $\gamma_{2n+2} = 0$. The curve γ lifts if and only if $\sum_{i=1}^{2n+2}(-1)^{i+1}\gamma_i \equiv 0 \mod k$. Let f be a homeomorphism of Σ_0° and $\Psi([f]) = \sigma$ with $\sigma(j) = 2n+2$. The image $f(\gamma)$ lifts if and only if $\sum_{i=1}^{2n+2}(-1)^{i+1}(\gamma_{\sigma^{-1}(i)} - \gamma_j) \equiv 0 \mod k$ by lemma 3.4.

Case 1: $k = 2$

Let $[f] \in \mathrm{Mod}(\Sigma_0^\circ)$ with $\Psi([f]) = \sigma$ such that $\sigma(j) = 2n+2$. Observe that in $\mathbb{Z}/2\mathbb{Z}$,

$$\sum_{i=1}^{2n+2}(-1)^{i+1}(\gamma_{\sigma^{-1}(i)} - \gamma_j) = (2n+2)\gamma_j + \sum_{i=1}^{2n+2}(-1)^{i+1}\gamma_{\sigma^{-1}(i)} = \sum_{i=1}^{2n+2}(-1)^{i+1}\gamma_i$$

so γ lifts if and only if $f(\gamma)$ lifts. We can then conclude f lifts. Therefore the image of $[f]$ under the isomorphism $\mathrm{Mod}(\Sigma_0^\circ) \to \mathrm{Mod}(\Sigma_0, \mathcal{B})$ is in $\mathrm{LMod}(\Sigma_0, \mathcal{B})$.

Case 2: $k \geqslant 3$

Let $[f] \in \mathrm{Mod}(\Sigma_0^\circ)$ with $\Psi([f]) = \sigma$ such that $\sigma(j) = 2n+2$ and $\sigma \in W_{2n+2}$. If σ is parity preserving then

$$\sum_{i=1}^{2n+2}(-1)^{i+1}(\gamma_{\sigma^{-1}(i)} - \gamma_j) = \sum_{i=1}^{2n+2}(-1)^{i+1}\gamma_{\sigma^{-1}(i)} = \sum_{i=1}^{2n+2}(-1)^{i+1}\gamma_i$$

so γ lifts if and only if $f(\gamma)$ lifts. If σ is parity reversing,

$$\sum_{i=1}^{2n+2}(-1)^{i+1}(\gamma_{\sigma^{-1}(i)} - \gamma_j) = \sum_{i=1}^{2n+2}(-1)^{i+1}\gamma_{\sigma^{-1}(i)} = -\sum_{i=1}^{2n+2}(-1)^{i+1}\gamma_i$$

so γ lifts if and only if $f(\gamma)$ lifts. If σ is either parity reversing or parity preserveing, then f lifts. Therefore the image of $[f]$ under the isomorphism $\mathrm{Mod}(\Sigma_0^\circ) \to \mathrm{Mod}(\Sigma_0, \mathcal{B})$ is in $\mathrm{LMod}(\Sigma_0, \mathcal{B})$.

Conversely, assume that $\sigma \notin W_{2n+2}$. Then there exist odd integers p and q such that $\sigma(p)$ is odd and $\sigma(q)$ is even. Without loss of generality, we may assume that $\sigma(p) = 1, \sigma(q) = 2$.

Let $f \in \mathrm{Mod}(\Sigma_0, \mathcal{B})$ such that $\widehat{\Psi}_{2n+2}(f) = \sigma^{-1}$. We need to show that there exists some curve η that lifts such that $f(\eta)$ does not lift. Indeed, let $\eta = x_1 + x_2$, then $\hat{i}(\eta, \alpha) = 0$. The homology class of $f(\eta)$ is $x_p + x_q$, but since both p and q are odd, $\hat{i}(f(\eta), \alpha) = 2$. Therefore $f(\eta)$ does not lift and the homeomorphism f does not lift by lemma 2.4. ∎

Let $\overline{p}_{g,k} : S_g \to S_0$ be a balanced superelliptic covering map of degree k. The short exact sequence (2) restricts to a short exact sequence:

$$1 \to \mathrm{PMod}(\Sigma_0, \mathcal{B}) \to \mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B}) \to W_{2n+2} \to 1 \tag{3}$$

for $k \geqslant 3$.

Lemma 3.6 gives us the following result. The case $k = 2$ has already been proven by Birman and Hilden [6] using different methods.

Scholium 3.7. *For $k = 2$, $\mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B}) = \mathrm{Mod}(\Sigma_0, \mathcal{B})$. For $k \geqslant 3$, the index $|\mathrm{Mod}(\Sigma_0, \mathcal{B}) : \mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B})|$ is $\frac{(2n+2)!}{2((n+1)!)^2}$.*

Proof. If $k = 2$, we are in case 1 in the proof of 3.6.

For $k \geqslant 3$,

$$\begin{aligned}
&|\mathrm{Mod}(\Sigma_0, \mathcal{B}) : \mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B})| \\
&\quad = |\mathrm{Mod}(\Sigma_0, \mathcal{B}) / \mathrm{PMod}(\Sigma_0, \mathcal{B}) : \mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B}) / \mathrm{PMod}(\Sigma_0, \mathcal{B})| \\
&\quad = |S_{2n+2} : W_{2n+2}| \,.
\end{aligned}$$

Observing that $|W_{2n+2}| = 2((n+1)!)^2$ completes the proof. ∎

4. Presentations of $\mathrm{PMod}(\Sigma_0, \mathcal{B}(m))$ and W_{2n+2}

As in section 3, $\mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B}(2n+2))$ can be written as a group extension of W_{2n+2} by the pure mapping class group $\mathrm{PMod}(\Sigma_0, \mathcal{B}(2n+2))$. A presentation of $\mathrm{PMod}(\Sigma_0, \mathcal{B}(2n+2))$ is found in lemma 4.1. A presentation of W_{2n+2} is found in lemma 4.2.

4.1. **A presentation of** $\mathrm{PMod}(\Sigma_0, \mathcal{B}(m))$**.** Let D_m be a disk with m marked points. Number the marked points from 1 to m. Let σ_i be the half twist that exchanges the ith and $(i+1)$th marked points where the arc about which σ_i is a half twist in Σ_0 is shown in figure 3. The pure braid group, denoted PB_m is generated by elements $A_{i,j}$ with $1 \leqslant i < j \leqslant m$ of the form:

$$A_{i,j} = (\sigma_{j-1} \cdots \sigma_{i+1}) \sigma_i^2 (\sigma_{j-1} \cdots \sigma_{i+1})^{-1}.$$

An example of of the curve about which $A_{i,j}$ is a twist is in figure 3.

Lemma 4.1. *The group* $\mathrm{PMod}(\Sigma_0, \mathcal{B}(m))$ *is generated by* $A_{i,j}$ *for* $1 \leqslant i < j \leqslant m-1$ *and has relations:*

(1) $[A_{p,q}, A_{r,s}] = 1$ *where* $p < q < r < s$
(2) $[A_{p,s}, A_{q,r}] = 1$ *where* $p < q < r < s$
(3) $A_{p,r} A_{q,r} A_{p,q} = A_{q,r} A_{p,q} A_{p,r} = A_{p,q} A_{p,r} A_{q,r}$ *where* $p < q < r$
(4) $\left[A_{r,s} A_{p,r} A_{r,s}^{-1}, A_{q,s}\right] = 1$ *where* $p < q < r < s$
(5) $(A_{1,2} A_{1,3} \cdots A_{1,m-1}) \cdots (A_{m-3,n-2} A_{m-3,n-1})(A_{m-2,m-1}) = 1$

Proof. Let PB_m be the braid group on m strands, which is isomorphic to the mapping class group of a disk D_m with m marked points.

By the capping homomorphism $Cap : PB_{m-1} \longrightarrow \mathrm{PMod}(\Sigma_0, \mathcal{B}(m))$, there is a short exact sequence:

$$1 \longrightarrow \mathbb{Z} \longrightarrow PB_{m-1} \xrightarrow{Cap} \mathrm{PMod}(\Sigma_0, \mathcal{B}(m)) \longrightarrow 1. \tag{4}$$

Here $\mathbb{Z}$ is generated by the Dehn twist about a curve homotopic to the boundary of D_{m-1}, which we will denote T_β. From [13, page 250] we have

$$T_\beta = (A_{1,2} A_{1,3} \cdots A_{1,m}) \cdots (A_{m-3,m-2} A_{m-3,m-1})(A_{m-2,m-1}).$$

Using the presentation for PB_m in Margalit–McCammond [20, Theorem 2.3] and lemma 2.1, we obtain the desired presentation. ∎

4.2. **A presentation of W_{2n+2}.** As in section 3.3, W_{2n+2} is the subgroup the symmetric group S_{2n+2} given by all permutations of $\{1, \ldots, 2n+2\}$ that either preserve or reverse parity.

The symmetric group S_m admits the presentation:

(5)
$$S_m = \left\langle \tau_1, \ldots, \tau_{m-1} \mid \begin{cases} \tau_i^2 = 1 & \text{for all } i \in \{1, \ldots, m-1\} \\ \tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1} & \text{for all } i \in \{1, \ldots, m-2\} \\ [\tau_i, \tau_j] = 1 & \text{for } |i-j| > 1 \end{cases} \right\rangle$$

where τ_i is the transposition $(i \;\; i+1)$.

Lemma 4.2. *Let S_{2n+2} be the symmetric group on $\{1, \ldots, 2n+2\}$. Let $x_i = (2i-1 \;\; 2i+1)$, $y_i = (2i \;\; 2i+2)$, and $z = (1 \;\; 2) \cdots (2n+1 \;\; 2n+2)$. Then W_{2n+2} admits a presentation with generators $\{x_1, \ldots, x_n, y_1, \ldots, y_n, z\}$ and relations*

1. $[x_i, y_j] = 1$ *for all* $i, j \in \{1, \ldots, n\}$,
2. $x_i^2 = 1$ *and* $y_i^2 = 1$ *for all* $i \in \{1, \ldots, n\}$,
3. $x_i x_{i+1} x_i = x_{i+1} x_i x_{i+1}$ *and* $y_i y_{i+1} y_i = y_{i+1} y_i y_{i+1}$ *for all* $i \in \{1, \ldots, n-1\}$,
4. $[x_i, x_j] = 1$ *and* $[y_i, y_j] = 1$ *for all* $|i-j| \geqslant 2$,
5. $z^2 = 1$, *and*
6. $z x_i z^{-1} = y_i$ *for all* $i \in \{1, \ldots, n\}$.

Proof. We have the short exact sequence

$$1 \longrightarrow S_{n+1} \times S_{n+1} \xrightarrow{\alpha} W_{2n+2} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 1.$$

The homomorphism α maps the first coordinate in $S_{n+1} \times S_{n+1}$ to permutations of $\{1, 3, \ldots, 2n+1\}$ and the second coordinate to permutations of $\{2, 4, \ldots, 2n+2\}$. The map π is given by $\pi(\sigma) = 0$ if σ is parity preserving, and $\pi(\sigma) = 1$ if it is parity reversing.

Using the presentation (5) for S_{n+1} and lemma 2.2, we find the presentation. The details are left as an exercise. ∎

5. The presentation

In this section we compute a presentation for $\mathrm{LMod}_{g,k}(\mathrm{Mod}(\Sigma_0, \mathcal{B})$, which is given in theorem 5.7. Throughout this section, let $p_{g,k} : \Sigma_g \to \Sigma_0$ be the balanced superelliptic cover of degree k. Let $\mathcal{B}$ be the set of $2n+2$ branch points in Σ_0.

We apply lemma 2.2 to the short exact sequence (3) from section 3.3:

$$1 \longrightarrow \mathrm{PMod}(\Sigma_0, \mathcal{B}) \xrightarrow{\iota} \mathrm{LMod}(\Sigma_0, \mathcal{B}) \xrightarrow{\widehat{\Psi}_{2n+2}} W_{2n+2} \longrightarrow 1.$$

The inclusion map $\iota : \mathrm{PMod}(\Sigma_0, \mathcal{B}) \to \mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B})$ maps the generators of $\mathrm{PMod}(\Sigma_0, \mathcal{B})$ to generators of $\mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B})$ by the identity. Thus the generators of $\mathrm{PMod}(\Sigma_0, \mathcal{B}(2n+2))$ comprise the set $\widetilde{S}_K$ from lemma 2.2 in

$\mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B})$. Similarly, the relations of $\mathrm{PMod}(\Sigma_0, \mathcal{B})$ comprise the set $\widetilde{R}_K$ in $\mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B})$.

The generators $\widetilde{S}_H$ are the lifts in $\mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B})$ of the generators of W_{2n+2}. The relations R_1 are the lifts in $\mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B})$ of the relations of W_{2n+2}. We calculate the lifts in $\mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B})$ of both generators of W_{2n+2} and relations of W_{2n+2} in 5.1.

Finally the set R_2 is comprised of relations that come from conjugations of elements of $\widetilde{S}_K$ by elements in $\widetilde{S}_H$. We calculate the conjugation relations in three steps in section 5.2.

5.1. **Lifts of generators and relations.** Let σ_i be the half twist that exchanges the ith and $i+1$st branch points about the arc in Σ_0 as in image figure 3. The mapping class group $\mathrm{Mod}(\Sigma_0, \mathcal{B})$ admits the presentation [13, page 122]

$$\left\langle \sigma_1, \dots, \sigma_{2n+1} \mid \begin{cases} [\sigma_i, \sigma_j] = 1 & |i-j| > 1, \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & i \in \{1, \dots, 2n\}, \\ (\sigma_1\sigma_2\cdots\sigma_{2n+1})^{2n+2} = 1, & \\ (\sigma_1\cdots\sigma_{2n+1}\sigma_{2n+1}\cdots\sigma_1) = 1 & \end{cases} \right\rangle.$$

Since $\mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B})$ is a subgroup of $\mathrm{Mod}(\Sigma_0, \mathcal{B})$, we define the generators of $\mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B})$ in terms of the $\{\sigma_i\}$ in lemma 5.1.

Lemma 5.1. *The group* $\mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B})$ *is generated by*

(1) $\{(\sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1})\sigma_i^2(\sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1})^{-1} : 1 \leqslant i < j \leqslant 2n+1\}$,
(2) $\sigma_1\sigma_3\cdots\sigma_{2n+1}$,
(3) $\{\sigma_{2i}\sigma_{2i-1}\sigma_{2i}^{-1} : i \in \{1, \dots, n\}\}$, *and*
(4) $\{\sigma_{2i+1}\sigma_{2i}\sigma_{2i+1}^{-1} : i \in \{1, \dots, n\}\}$.

Proof. The elements from (1) are exactly the images of the generators for $\mathrm{PMod}(\Sigma_0, \mathcal{B})$ from lemma 4.1 under the inclusion map ι. The elements of (2)-(4) map to generators of W_{2n+2} in lemma 4.2.

Lemma 2.2 tells us that the generators of types (1)-(4) suffice to form a generating set for $\mathrm{LMod}(\Sigma_0, \mathcal{B}(2n+2))$. ∎

We will denote the generators by the following symbols.

(6)
$$\begin{aligned} A_{i,j} &= (\sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1})\sigma_i^2(\sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1})^{-1}, \quad 1 \leqslant i < j \leqslant 2n+1 \\ c &= \sigma_1\sigma_3\cdots\sigma_{2n-1}\sigma_{2n+1} \\ a_i &= \sigma_{2i}\sigma_{2i-1}\sigma_{2i}^{-1}, \quad i \in \{1, \dots, n\} \\ b_i &= \sigma_{2i+1}\sigma_{2i}\sigma_{2i+1}^{-1}, \quad i \in \{1, \dots, n\}. \end{aligned}$$

The generators $A_{i,j}$, a_l, and b_l are all shown in figure 3. The elements a_i exchange odd marked points and the elements b_i exchange even marked points. The generator c is the composition of half twists about the arcs on the right side of figure 3, and c switches all odd marked points with even marked points and vice versa.

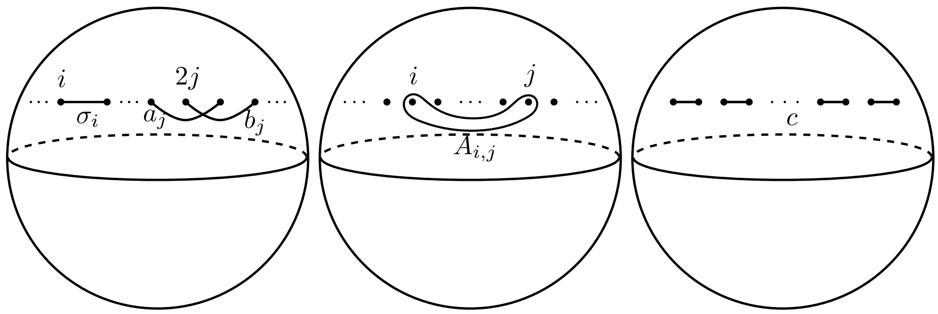


FIGURE 3. *Left to right*: the arcs about which σ_i, a_j, and b_j are half twists, the curve about which $A_{i,j}$ is a Dehn twist, and the collection of arcs about which c is a composition of half twists. The labels above the marked points are to indicate the enumeration.

Although the elements $A_{i,2n+2} = (\sigma_{2n+1} \cdots \sigma_{i+1})\sigma_i^2(\sigma_{2n+1} \cdots \sigma_{i+1})^{-1}$ are in $\mathrm{PMod}(\Sigma_0, \mathcal{B})$, they are not part of the generating set (6). However, it will be useful to use the elements $A_{i,2n+2}$ in the set of relations for our final presentation. The next lemma rewrites the elements $A_{i,2n+2}$ as words in the generators $A_{i,j}$ with $1 \leqslant i < j \leqslant 2n+1$.

Lemma 5.2. *Fix $\ell \in \{1, \dots, 2n+1\}$. Define*

$$\overline{A}_{i,j} := \begin{cases} A_{i,j} & \text{if } j < \ell \\ A_{\ell,j+1}^{-1} A_{i,j+1} A_{\ell,j+1} & \text{if } i < \ell \leqslant j \\ A_{i+1,j+1} & \text{if } \ell \leqslant i. \end{cases}$$

Then $A_{\ell,2n+2} = (\overline{A}_{1,2} \cdots \overline{A}_{1,2n})(\overline{A}_{2,3} \cdots \overline{A}_{2,2n}) \cdots (\overline{A}_{2n-1,2n})$.

To prove lemma 5.2, we use the following facts:

Let $T_{r,s} = \sigma_{r+s}\sigma_{r+s-1} \cdots \sigma_r$, then:

1. $T_{r,2n-r}^{-1} A_{i,j} T_{r,2n-r} = A_{i,j}$ if $j < r$.
2. $T_{r,2n-r}^{-1} A_{i,j} T_{r,2n-r} = A_{r,j+1}^{-1} A_{i,j+1} A_{r,j+1}$ if $i < r \leqslant j$.
3. $T_{r,2n-r}^{-1} A_{i,j} T_{r,2n-r} = A_{i+1,j+1}$ if $r \leqslant i$.

These facts can be checked through routine, but lengthy computation.

Proof. We first show that

$$A_{2n+1,2n+2} = (A_{1,2} \cdots A_{1,2n})(A_{2,3} \cdots A_{2,2n}) \cdots (A_{2n-1,2n}).$$

Indeed, let γ be the curve about which $A_{2n+1,2n+2}$ is a twist. The curve γ is a separating curve in the sphere with $2n+2$ marked points. On one side, γ bounds a disk containing the first $2n$ marked points and on the other side,

γ bounds a disk containing the $n+1$st and $n+2$nd marked points. A Dehn twist about the boundary of the disk with $2n$ marked points can be written as

$$(A_{1,2}\cdots A_{1,2n})(A_{2,3}\cdots A_{2,2n})\cdots(A_{2n-2,2n-1}A_{2n-2,2n})(A_{2n-1,2n}),$$

as seen in Farb and Margalit [13, page 260].

Then let $T = T_{\ell,2n+2}$. Notice that $A_{\ell,2n+2} = T^{-1}A_{2n+1,2n+2}T$. Facts (1), (2), and (3) above show $T^{-1}A_{i,j}T = \overline{A}_{i,j}$ for all $1 \leqslant i < j \leqslant 2n$. Therefore

$$\begin{aligned} A_{\ell,2n+2} &= T^{-1}A_{2n+1,2n+2}T \\ &= T^{-1}A_{1,2}TT^{-1}\cdots TT^{-1}A_{1,2n}TT^{-1}\cdots \\ &\qquad TT^{-1}A_{2n-2,2n-1}TT^{-1}A_{2n-2,2n}TT^{-1}A_{2n-1,2n}T \\ &= (\overline{A}_{1,2}\cdots\overline{A}_{1,2n})(\overline{A}_{2,3}\cdots\overline{A}_{2,2n})\cdots(\overline{A}_{2n-2,2n-1}\overline{A}_{2n-2,2n})(\overline{A}_{2n-1,2n}). \end{aligned}$$

■

The relations of R_1 of lemma 2.2 are given in lemma 5.3. To consolidate the family of commutator relations, let

$$C_{i,j} = \begin{cases} A^{-1}_{2i-1,2i}A^{-1}_{2i+1,2i+2}A_{2i-1,2i+2}A_{2i,2i+1} & \text{if } i = j \\ A^{-1}_{2i+1,2i+2}A^{-1}_{2i,2i+3}A_{2i+2,2i+3}A_{2i,2i+1} & \text{if } i = j+1 \\ 1 & \text{otherwise} \end{cases} \tag{7}$$

for $1 \leqslant i \leqslant j \leqslant n$.

Lemma 5.3. *Let $A_{i,j}, a_i, b_i$, and c be the generators defined in (6). The following relations hold.*

Commutator relations

(1) $[a_i, b_j] = C_{i,j}$ *where $C_{i,j}$ is given by (7)*

Braid relations

(2) $a_ia_{i+1}a_i = a_{i+1}a_ia_{i+1}$ *and* $b_ib_{i+1}b_i = b_{i+1}b_ib_{i+1}$ *for* $i \in \{1,\dots,n-1\}$

(3) $[a_i, a_j] = [b_i, b_j] = 1$ *if* $|j-i| > 1$

Half twists squared are Dehn twists

(4) $a_i^2 = A_{2i-1,2i+1}$ *and* $b_i^2 = A_{2i,2i+2}$ *for* $i \in \{1,\dots,n\}$.

(5) $c^2 = A_{1,2}A_{3,4}\cdots A_{2n+1,2n+2}$

Parity Flip

(6) $ca_ic^{-1}b_i^{-1} = 1$.

Proof. For (1) it suffices to show $[a_i, b_j]C^{-1}_{i,j}$ is the identity. If $i = j$,

$$\begin{aligned} [a_i,b_j]C^{-1}_{i,j} = {}&(\sigma_{2i}\sigma_{2i-1}\sigma^{-1}_{2i})(\sigma_{2i+1}\sigma_{2i}\sigma^{-1}_{2i+1})(\sigma_{2i}\sigma^{-1}_{2i-1}\sigma^{-1}_{2i})(\sigma_{2i+1}\sigma^{-1}_{2i}\sigma^{-1}_{2i+1}) \\ &(\sigma^{-1}_{2i}\sigma^{-1}_{2i})(\sigma_{2i+1}\sigma_{2i}\sigma^{-1}_{2i-1}\sigma^{-1}_{2i-1}\sigma^{-1}_{2i}\sigma^{-1}_{2i+1})(\sigma^2_{2i+1})(\sigma_{2i-1}\sigma_{2i-1}). \end{aligned}$$

Any solution to the word problem for the braid group, for example Dehornoy's handle reduction [12], will reduce the word $[a_i, b_j]C^{-1}_{i,j}$ to the identity.

The remaining cases can be deduced similarly and details are available in [26]. ∎

Topological interpretation: Although the proof of lemma 5.3 is purely algebraic, there are topological interpretations of most of the relations. Let γ_i be the arc about which a_i is a half twist, and δ_i the arc about which b_i is a half twist.

When $i \neq j, j+1$, γ_i and δ_j can be modified by homotopy to be disjoint so the relations $[a_i, b_j] = 1$ in (1) hold. The homeomorphisms $\{a_i\}$ are supported on a closed neighborhood of the union $\gamma_1 \cup \cdots \cup \gamma_n$, which is an embedded disk D_{n+1} with $n+1$ marked points. The mapping class group of D_{n+1} is isomorphic to the braid group B_{n+1}. Embedding D_{n+1} in Σ_0 with $2n+2$ marked points induces a homomorphism $\iota : B_{n+1} \to \mathrm{Mod}(\Sigma_0, \mathcal{B}(2n+2))$. The homomorphism ι maps the standard braid generators to the a_i, and so the braid relations (2) and (3) hold. The same applies to the b_i.

Relations (4) and (5) reflect the fact that squaring a half twist about an arc is homotopic to a Dehn twist about a curve surrounding the arc. Recall that if τ_γ is a half twist about an arc γ in Σ_0 and f is a homeomorphism of Σ_0, then $f^{-1}\tau_\gamma f = \tau_{f(\gamma)}$. We realize $ca_ic^{-1} = b_i$ in (6) as the homeomorphism c^{-1} applied to the arc γ_i about which a_i is a twist.

5.2. **Conjugation relations.** We now shift our attention to finding the relations that comprise R_2 from lemma 2.2. Lemmas 5.4, 5.5, and 5.6 give us the conjugation relations.

First we consider conjugation of the pure braid group generators by c. Let

$$(8)\quad X_{i,j} = \begin{cases} A_{i,j} & \text{for odd } i,\ j = i+1 \\ A_{i+1,j+1} & \text{for odd } i, j \\ (A_{i-1,j}A_{i-1,i}^{-1})^{-1}A_{i-1,j-1}(A_{i-1,j}A_{i-1,i}^{-1}) & \text{for even } i, j \\ A_{i,j+1}^{-1}A_{i-1,j+1}A_{i,j+1} & \text{for even } i, \text{ odd } j \\ A_{j-1,j}A_{i+1,j-1}A_{j-1,j}^{-1} & \text{otherwise.} \end{cases}$$

Lemma 5.4. *For $1 \leqslant i < j \leqslant 2n+1$, let $A_{i,j}$ and c be as above. Then*

$$cA_{i,j}c^{-1} = X_{i,j}$$

where the $X_{i,j}$ are as in (8).

Proof. Recall that $A_{i,j} = (\sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1})\sigma_i^2(\sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1})^{-1}$ for $1 \leqslant i < j \leqslant 2n+1$ and $c = \sigma_1\sigma_3\cdots\sigma_{2n-1}\sigma_{2n+1}$.

Let $\gamma_{i,j}$ be the simple closed curve in Σ_0 about which $A_{i,j}$ is a Dehn twist. Let $T_{\gamma_{i,j}} = A_{i,j}$. Recall that $cT_{\gamma_{i,j}}c^{-1} = T_{c^{-1}(\gamma_{i,j})}$ (where we maintain our convention that we read products from left to right). Therefore to prove the lemma, it suffices to show that $c^{-1}(\gamma_{i,j})$ is the curve about which $X_{i,j}$ is a twist. The homeomorphism c^{-1} is the product (counterclockwise) twists $\sigma_1, \sigma_3, \cdots, \sigma_{2n+1}$. The cases for the image of $c^{-1}(\gamma_{i,j})$ depend on the signs of the intersections of arcs about which c is a twist and $\gamma_{i,j}$.

We first note that when i is odd and $j = i + 1$ the curve $\gamma_{i,j}$ is disjoint from c, therefore $cA_{i,j}c^{-1} = A_{i,j}$. We then consider the remaining cases.

i and j are both even: The curves $\gamma_{i,j}$ and $c^{-1}(\gamma_{i,j})$ where i and j are even are shown in figure 4.

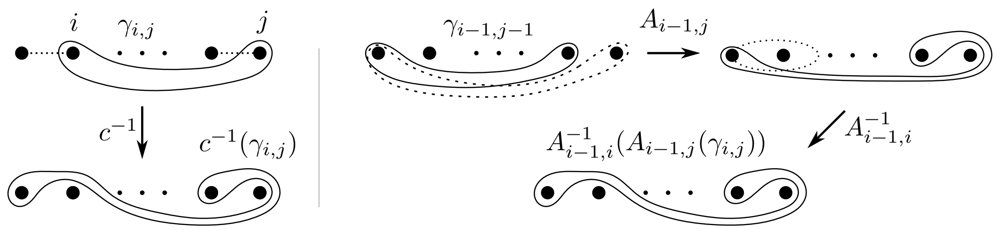


FIGURE 4. *Both i and j are even.* The left figure shows $\gamma_{i,j}$ and its image under c^{-1}, which is a product of half twists about the dashed arcs. Starting at the top left and going clockwise, the right figure shows $\gamma_{i-1,j-1}$, $A_{i-1,j}(\gamma_{i-1,j-1})$, and $A^{-1}_{i-1,i}(A_{i-1,j}(\gamma_{i-1,j-1}))$. The dashed curves indicate the curves about which $A_{i-1,j}$ and $A^{-1}_{i-1,i}$ are Dehn twists.

As seen in figure 4, the curve $c^{-1}(\gamma_{i,j})$ is a conjugate of $\gamma_{i-1,j-1}$ by an element of $\mathrm{PMod}(\Sigma_0, \mathcal{B}(2n+2))$. In figure 4, we see that $A^{-1}_{i-1,j}(A_{i-1,j}(\gamma_{i-1,j-1}))$ (with composition applied as indicated) is isotopic to $c^{-1}(\gamma_{i,j})$.

The remaining cases can be calculated similarly. The details are left as an exercise or are available in [26]. ∎

Next we consider conjugation of the elements $A_{i,j}$ by the generators a_ℓ. The resulting relations (along with the conjugates by b_ℓ in lemma 5.6) correspond to the conjugates of the words in $\widetilde{S}_K$ by the words in $\widetilde{S}_H$. Let

$$Y_{i,j,\ell} = \begin{cases} A_{i,j} & \text{if } i < 2\ell-1, j > 2\ell+1, \\ A_{i,j} & \text{if } i, j > 2\ell+1 \text{ or } i, j < 2\ell-1 \\ A_{i,j} & \text{if } i = 2\ell-1, j = 2\ell+1 \\ A_{i-1,j-1} & \text{if } i = 2\ell, j = 2\ell+1 \\ A_{i,j+1}A_{j,j+1}A^{-1}_{i,j+1} & \text{if } i = 2\ell-1, j = 2\ell \\ A^{-1}_{i,j}A_{i,j-2}A_{i,j} & \text{if } i < 2\ell-1, j = 2\ell+1 \\ A_{i,j+2} & \text{if } i < 2\ell-1, j = 2\ell-1 \\ (A^{-1}_{i,j-1}A_{i,j+1})^{-1}A_{i,j}(A^{-1}_{i,j-1}A_{i,j+1}) & \text{if } i < 2\ell-1, j = 2\ell \\ A_{i+2,j} & \text{if } i = 2\ell-1, j > 2\ell+1 \\ (A^{-1}_{i,i+1}A_{i-1,i})^{-1}A_{i,j}(A^{-1}_{i,i+1}A_{i-1,i}) & \text{if } i = 2\ell, j > 2\ell+1 \\ A^{-1}_{i,j}A_{i-2,j}A_{i,j} & \text{if } i = 2\ell+1, j > 2\ell+1 \end{cases} \tag{9}$$

Lemma 5.5. *For $1 \leqslant i < j \leqslant 2n+1$ and $\ell \in \{1, \ldots, n\}$, let $A_{i,j}$ and a_ℓ be as above. Then*

$$a_\ell A_{i,j} a_\ell^{-1} = Y_{i,j,\ell}$$

where the $Y_{i,j,\ell}$ are as in (9).

Proof. Recall that $A_{i,j} = (\sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1})\sigma_i^2(\sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1})^{-1}$ and $a_\ell = \sigma_{2\ell}\sigma_{2\ell-1}\sigma_{2\ell}^{-1}$. The transpositions σ_p and σ_q commute if $|p-q| \geqslant 2$. Therefore a_ℓ and $A_{i,j}$ commute if both $(2\ell-1)-(j-1) \geqslant 2$ and $2\ell-(i+1) \geqslant 2$, if $i, j > 2\ell+1$, or if $i, j < 2\ell-1$.

Therefore in the first two cases of (9), a_ℓ and $A_{i,j}$ commute. In the remaining cases, at least one of i and j is equal to $2\ell-1, 2\ell$, or $2\ell+1$. The calculations are routine, but lengthy and can be done either algbraically or topologically as in lemma 5.4. Details are also given in [26]. ■

Next we consider conjugation of the elements $A_{i,j}$ by the generators b_ℓ. The resulting relations correspond to the remaining words in the set of conjugates of the words in $\widetilde{S}_K$ by the words in $\widetilde{S}_H$.

Let

(10)

$$Z_{i,j,\ell} = \begin{cases} A_{i,j} & \text{if } i < 2\ell, j > 2\ell+2 \\ A_{i,j} & \text{if } i, j > 2\ell+2 \text{ or } i, j < 2\ell \\ A_{i,j} & \text{if } i = 2\ell, j = 2\ell+2 \\ A_{i-1,j-1} & \text{if } i = 2\ell+1, j = 2\ell+2 \\ A_{i,j+1}A_{j,j+1}A_{i,j+1}^{-1} & \text{if } i = 2\ell, j = 2\ell+1 \\ A_{i,j}^{-1}A_{i,j-2}A_{i,j} & \text{if } i < 2\ell, j = 2\ell+2 \\ A_{i,j+2} & \text{if } i < 2\ell, j = 2\ell \\ (A_{i,j-1}^{-1}A_{i,j+1})^{-1}A_{i,j}(A_{i,j-1}^{-1}A_{i,j+1}) & \text{if } i < 2\ell, j = 2\ell+1 \\ A_{i+2,j} & \text{if } i = 2\ell, j > 2\ell+2 \\ (A_{i,i+1}^{-1}A_{i-1,i})^{-1}A_{i,j}(A_{i,i+1}^{-1}A_{i-1,i}) & \text{if } i = 2\ell+1, j > 2\ell+2 \\ A_{i,j}^{-1}A_{i-2,j}A_{i,j} & \text{if } i = 2\ell+2, j > 2\ell+2. \end{cases}$$

Lemma 5.6. *For $1 \leqslant i < j \leqslant 2n+1$ and $\ell \in \{1, \ldots, n\}$, let $A_{i,j}$ and b_ℓ be as above. Then*

$$b_\ell A_{i,j} b_\ell^{-1} = Z_{i,j,\ell}$$

where the $Z_{i,j,\ell}$ are as in (10). ■

The proof of lemma 5.6 is the same as the proof of lemma 5.5 with an increase in index by 1.

5.3. **Proof of the presentation.** We are now ready to write down a presentation for $\mathrm{LMod}_{g,k}(\mathrm{Mod}(\Sigma_0, \mathcal{B})$.

Theorem 5.7. *Let Σ_g be a surface of genus $g \geqslant 2$. Let $\Sigma_g \to \Sigma_0$ be a balanced superelliptic cover of degree $k \geqslant 3$ with set of branch points $\mathcal{B} =$*

$\mathcal{B}(2n+2)$. The subgroup $\mathrm{LMod}_{g,k}(\Sigma_0,\mathcal{B})$ *is generated by*

$$\begin{aligned} A_{i,j} &= (\sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1})\sigma_i^2(\sigma_{j-1}\sigma_{j-2}\cdots\sigma_{i+1})^{-1}, 1 \leqslant i < j \leqslant 2n+1 \\ c &= \sigma_1\sigma_3\cdots\sigma_{2n-1}\sigma_{2n+1} \\ a_i &= \sigma_{2i}\sigma_{2i-1}\sigma_{2i}^{-1},\ i \in \{1,\dots,n\} \\ b_i &= \sigma_{2i+1}\sigma_{2i}\sigma_{2i+1}^{-1},\ i \in \{1,\dots,n\}. \end{aligned}$$

For $\ell \in \{1,\dots,2n+1\}$, let $A_{\ell,2n+2}$ be defined as in lemma 5.2. Then $\mathrm{LMod}_{g,k}(\Sigma_0,\mathcal{B})$ *has defining relations:*

Commutator relations

(1) $[A_{i,j}, A_{p,q}] = 1$ *where* $1 \leqslant i < j < p < q \leqslant 2n+1$.
(2) $[A_{i,p}, A_{j,q}] = 1$ *where* $1 \leqslant i < j < p < q \leqslant 2n+1$.
(3) $\left[A_{p,q}A_{i,p}A_{p,q}^{-1}, A_{j,q}\right] = 1$ *where* $1 \leqslant i < j < p < q \leqslant 2n+1$.
(4) $[a_i, b_j] = C_{i,j}$ *where $C_{i,j}$ are as in (7).*

Braid relations

(5) $A_{i,p}A_{j,p}A_{i,j} = A_{j,p}A_{i,j}A_{i,p} = A_{i,j}A_{i,p}A_{j,p}$ *where* $1 \leqslant i < j < p \leqslant 2n+1$.
(6) $a_ia_{i+1}a_i = a_{i+1}a_ia_{i+1}$ *and* $b_ib_{i+1}b_i = b_{i+1}b_ib_{i+1}$ *for* $i \in \{1,\dots,n-1\}$.
(7) $[a_i, a_j] = [b_i, b_j] = 1$ *if* $|j-i| > 1$.

Subsurface support

(8) $(A_{1,2}A_{1,3}\cdots A_{1,m-1})\cdots(A_{m-3,n-2}A_{m-3,n-1})(A_{m-2,m-1}) = 1$ *for* $m = 2n+2$.

Half twists squared are Dehn twists

(9) $a_i^2 = A_{2i-1,2i+1}$ *and* $b_i^2 = A_{2i,2i+2}$ *for* $i \in \{1,\dots,n\}$.
(10) $c^2 = A_{1,2}A_{3,4}\cdots A_{2n+1,2n+2}$.

Parity Flip

(11) $ca_ic^{-1}b_i^{-1} = 1$

Conjugation relations

(12) $cA_{i,j}c^{-1} = X_{i,j}$ *where the $X_{i,j}$ are as in (8).*
(13) $a_\ell A_{i,j}a_\ell^{-1} = Y_{i,j,\ell}$ *where the $Y_{i,j,\ell}$ are as in (9).*
(14) $b_\ell A_{i,j}b_\ell^{-1} = Z_{i,j,\ell}$ *where the $Z_{i,j,\ell}$ are as in (10).*

Proof. We prove the elements in (6) are the generators of $\mathrm{LMod}_{p,k}(\Sigma_0,\mathcal{B})$ in lemma 5.1.

Let $\widetilde{R}_K$ denote the image of the relations of $\mathrm{PMod}(\Sigma_0,\mathcal{B}(2n+2))$ in $\mathrm{LMod}_{g,k}(\Sigma_0,\mathcal{B}(2n+2))$. Then $\widetilde{R}_K$ consists of the relations

- $[A_{i,j}, A_{p,q}] = 1$ where $i < j < p < q$,
- $[A_{i,p}, A_{j,q}] = 1$ where $i < j < p < q$,
- $A_{i,p}A_{j,p}A_{i,j} = A_{j,p}A_{i,j}A_{i,p} = A_{i,j}A_{i,p}A_{j,p}$ where $i < j < p$,
- $\left[A_{p,q}A_{i,p}A_{p,q}^{-1}, A_{j,q}\right] = 1$ where $i < j < p < q$, and
- $(A_{1,2}A_{1,3}\cdots A_{1,2n+1})\cdots(A_{2n-1,2n}A_{2n-1,2n+1})(A_{2n,2n+1}) = 1$

by lemma 4.1.

Let R_1 denote the lifts in $\mathrm{LMod}_{p,k}(\Sigma_0,\mathcal{B})$ of the relations of W_{2n+2}. We prove that relations (1), (2), (3), (5), and (8) come from lemma 4.1. The relations (4), (6), (7), (9), (10), and (11) are the relations of R_1 in lemma 5.3.

Finally, the set R_2 in lemma 2.2 consists of the relations (12)-(14) as proved in lemmas 5.4, 5.5, and 5.6.

By lemma 2.2 the sets $\widetilde{R}_K, R_1$, and R_2 comprise all of the relations of $\mathrm{LMod}_{p,k}(\Sigma_0, \mathcal{B})$. ■

The strategy we employed to find this presentation can be used to find a presentation for $\mathrm{LMod}_p(\Sigma_0, \mathcal{B})$ where $p : \Sigma_g \to \Sigma_0$ is any abelian branched cover of the sphere. Indeed, $\mathrm{LMod}_p(\Sigma_0, \mathcal{B})$ can be written as a group extension of $\widehat{\Psi}(\mathrm{LMod}_p(\Sigma_0, \mathcal{B}))$ by $\mathrm{PMod}(\Sigma_0, \mathcal{B})$. If one can compute a presentation for $\widehat{\Psi}(\mathrm{LMod}_p(\Sigma_0, \mathcal{B}))$, which is a subgroup of the symmetric group $S_{|\mathcal{B}|}$, then the generators of $\mathrm{LMod}_p(\Sigma_0, \mathcal{B})$ will be the lifts of the generators of $\widehat{\Psi}(\mathrm{LMod}_p(\Sigma_0, \mathcal{B}))$ and the Dehn twists $A_{i,j}$ in theorem 5.7. The relations can then be found by performing the analogous computations to lemmas 5.3, 5.4, 5.5, and 5.6 and applying lemma 2.2.

6. Abelianization

In this section we will prove theorems 1.1 and 1.2. Recall that for any group G, $H_1(G; \mathbb{Z}) \cong G/[G, G]$. For this section, fix $k \geqslant 3$ and let $p_{g,k} : \Sigma_g \to \Sigma_0$ be the balanced superelliptic cover of degree k. Recall that there are $2n + 2$ branch points where $n = g/(k-1)$. The abelianization of $\mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B})$ depends on n. For ease of notation, let $G_n = \mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B})$ for the remainder of this section. Let $\phi : G_n \to G_n/[G_n, G_n]$ be the abelianization map. Note that if $a, b \in G_n$ are in the same conjugacy class of G_n, then $\phi(a) = \phi(b)$.

A presentation for $G_n/[G_n, G_n]$ is given by taking a presentation for G_n and adding the set of all commutators to the set of defining relations. We begin with the presentation given in theorem 5.7.

Performing Tietze transformations we may add the generators $A_{\ell,2n+2}$ for $\ell \in \{1, \ldots, 2n+1\}$ along with the relations

$$A_{\ell,2n+2} = (\overline{A}_{1,2} \cdots \overline{A}_{1,2n})(\overline{A}_{2,3} \cdots \overline{A}_{2,2n}) \cdots (\overline{A}_{2n-2,2n-1}\overline{A}_{2n-2,2n})(\overline{A}_{2n-1,2n})$$

where the $\overline{A}_{i,j}$ are as in lemma 5.2.

Lemma 6.1. *If $j - i \equiv t - s \mod 2$, then $A_{i,j}$ is conjugate to $A_{s,t}$ in G_n.*

Proof. We consider two cases: either $j - i \equiv t - s \equiv 0 \mod 2$ or $j - i \equiv t - s \equiv 1 \mod 2$.

Case 1: $j - i \equiv t - s \equiv 0 \mod 2$.

Let i and j be even. Recall conjugation relations

$$b_\ell A_{i,j} b_\ell^{-1} = A_{i,j}^{-1} A_{i,j-2} A_{i,j}$$

for $i < 2\ell$ and $j = 2n + 2$, and

$$b_\ell A_{i,j} b_\ell^{-1} = A_{i,j+2}$$

for $i < 2\ell$ and $j = 2\ell$. Therefore for any even i, all generators $A_{i,j}$ with even j are in the same conjugacy class of G_n. We also have the conjugation relations

$$b_\ell A_{i,j} b_\ell^{-1} = A_{i+2,j}$$

for $i = 2\ell$ and $j > 2\ell + 2$, and

$$b_\ell A_{i,j} b_\ell^{-1} = A_{i,j}^{-1} A_{i-2,j} A_{i,j}$$

for $i = 2\ell + 2$ and $j > 2\ell + 2$. Therefore for any fixed even j, all the $A_{i,j}$ such that i is even are in the same conjugacy class of G_n. Then by varying j, we conclude that if i, j, s, t are all even, then $A_{i,j}$ and $A_{s,t}$ are conjugate.

Similarly we can consider the conjugacy relations $a_\ell A_{i,j} a_\ell^{-1} = Y_{i,j,\ell}$ to conclude that if i, j, s, t are all odd, then $A_{i,j}$ is conjugate to $A_{s,t}$ in G_n.

Observe that $cA_{1,3}c^{-1} = A_{2,4}$. We may finally conclude that if $j - i \equiv t - s \equiv 0 \mod 2$, then $A_{i,j}$ is conjugate to $A_{s,t}$ in G_n.

Case 2: $j - i \equiv t - s \equiv 1 \mod 2$**.**
Similar to the proof of case 1 above, we use relations from the family of relations $a_\ell A_{i,j} a_\ell^{-1} = Y_{i,j,\ell}$ to conclude that for any fixed even i, all the $A_{i,j}$ for any odd j are in the same conjugacy class of G. Using relations of the form $b_\ell A_{i,j} b_\ell^{-1} = Z_{i,j,\ell}$ gives us that for any fixed odd j, all the $A_{i,j}$ for any even i are in the same conjugacy class of G_n. Therefore if i and s are even and j and t are odd, then $A_{i,j}$ and $A_{s,t}$ are conjugate in G_n.

Similarly, if i and s are odd and j and t are even, then $A_{i,j}$ and $A_{s,t}$ are conjugate in G_n.

Finally, the relation $cA_{2,3}c^{-1} = A_{2,4}^{-1} A_{1,4} A_{2,4}$ allows us to conclude that if $j - i \equiv t - s \equiv 1 \mod 2$, then $A_{i,j}$ is conjugate to $A_{s,t}$ in G_n, completing the proof. ∎

From now on, let $A = \phi(A_{1,2})$ and $B = \phi(A_{1,3})$.

Lemma 6.2. *For each $\ell \in \{1, \ldots, 2n+1\}$, consider the relation*

$$A_{\ell,2n+2} = (\overline{A}_{1,2} \cdots \overline{A}_{1,2n})(\overline{A}_{2,3} \cdots \overline{A}_{2,2n}) \cdots (\overline{A}_{2n-2,2n-1} \overline{A}_{2n-2,2n})(\overline{A}_{2n-1,2n})$$

where the $\overline{A}_{i,j}$ are as in lemma 5.2. Applying ϕ to each of these relations gives the relation $B^{n^2-n} = A^{1-n^2}$ in $G_n/[G_n, G_n]$.

Proof. Fix $\ell \in \{1, \ldots, 2n+1\}$ and let

$$\begin{aligned}
\overline{W} &= (\overline{A}_{1,2} \cdots \overline{A}_{1,2n})(\overline{A}_{2,3} \cdots \overline{A}_{2,2n}) \cdots (\overline{A}_{2n-2,2n-1} \overline{A}_{2n-2,2n})(\overline{A}_{2n-1,2n}) \\
W &= (A_{1,2} \cdots A_{1,2n+1})(A_{2,3} \cdots A_{2,2n+1}) \cdots (A_{2n-1,2n} A_{2n-1,2n+1})(A_{2n,2n+1}) \\
L &= \prod_{\substack{1 \leqslant i < j \leqslant 2n+1 \\ i = \ell \text{ or } j = \ell}} A_{i,j}.
\end{aligned}$$

Observe that $\phi(\overline{W}) = \phi(W)\phi(L)^{-1}$. By lemma 6.1 we have

$$\begin{aligned}\phi(W) &= ((AB)^n)((AB)^{n-1}A)((AB)^{n-1})\cdots(AB)(A)\\ &= A^{2n}A^{2(n-1)}\cdots A^2B^nB^{2(n-1)}B^{2(n-2)}\cdots B^2\\ &= A^{n(n+1)}B^{n^2}\end{aligned}$$

since $\sum_{i=1}^{n-1} 2i = n(n-1)$.

If ℓ is even, $\phi(L) = A^{n+1}B^{n-1}$. Applying ϕ to the relation above gives

$$B = \phi(W) = A^{n(n+1)}B^{n^2}A^{-n-1}B^{1-n}.$$

This rearranges to $B^{n^2-n} = A^{1-n^2}$.

If ℓ is odd, $\phi(L) = A^nB^n$. Applying ϕ to the relation above gives $B^{n^2-n} = A^{1-n^2}$. ∎

Lemma 6.3. *In the abelianization of G_n, $B^{n^2} = A^{-n^2-1}$.*

Proof. Consider the subsurface support relation,

$$(A_{1,2}\cdots A_{1,2n+1})(A_{2,3}\cdots A_{2,2n+1})\cdots(A_{2n-1,2n}A_{2n-1,2n+1})(A_{2n+1,2n}) = 1.$$

Applying ϕ to both sides gives $1 = A^{n(n+1)}B^{n^2}$ by the computation of $\phi(W)$ in the proof of lemma 6.2. ∎

Lemma 6.4. *For all $1 \leqslant i, j \leqslant n$, $\phi(a_i) = \phi(b_j)$.*

Proof. By lemma 5.3, we have the braid relations $(a_{i+1}^{-1}a_i)a_{i+1}(a_{i+1}^{-1}a_i)^{-1} = a_i$ for $i \in \{1,\ldots,n-1\}$ and $(b_{i+1}^{-1}b_i)b_{i+1}(b_{i+1}^{-1}b_i)^{-1} = b_i$ for all $i \in \{1,\ldots,n-1\}$. Therefore all $\phi(a_i) = \phi(a_j)$ and $\phi(b_i) = \phi(b_j)$ for all $i, j \in \{1,\ldots,n-1\}$. The parity flip relation $ca_1c^{-1} = b_1$ allows us to deduce that a_i and b_j are conjugate for all $1 \leqslant i, j \leqslant n$ and $\phi(a_i) = \phi(b_j)$. ∎

Lemma 6.5. *The abelianization $G_n/[G_n, G_n]$ admits the presentation*

$$\langle a, d, A, B \mid B^{n^2-n} = A^{1-n^2}, B^{n^2} = A^{-n^2-1}, a^2 = B, d^2 = A^{n+1}, \mathcal{T}\rangle$$

where $a = \phi(a_1)$, $d = \phi(c)$, $A = \phi(A_{1,2})$, $B = \phi(A_{1,3})$, and $\mathcal{T}$ is the set of all commutators.

Proof. Lemmas 6.1 and 6.4 show that the elements $\phi(a_1), \phi(c), \phi(A_{1,2})$ and $\phi(A_{1,3})$ form a generating set for $G_n/[G_n, G_n]$.

Lemmas 6.2 and 6.3 show that the relations $B^{n^2-n} = A^{1-n^2}$ and $B^{n^2} = A^{-n^2-1}$ hold in $G_n/[G_n, G_n]$. Applying ϕ to the relation $a_1^2 = A_{1,3}$ shows that $a^2 = B$. Applying ϕ to the relation $c^2 = A_{1,2}A_{3,4}\cdots A_{2n+1,2n+2}$ gives the relation $d^2 = A^{n+1}$.

Lemma 6.2 shows that for all $\ell \in \{1,\ldots,2n+1\}$, the relation

$$A_{\ell,2n+2} = (\overline{A}_{1,2}\cdots\overline{A}_{1,2n})(\overline{A}_{2,3}\cdots\overline{A}_{2,2n})\cdots(\overline{A}_{2n-2,2n-1}\overline{A}_{2n-2,2n})(\overline{A}_{2n-1,2n})$$

is derivable from $\mathcal{T}$ and $B^{n^2-n} = A^{1-n^2}$.

It remains to show that in the abelianization, the relations from the presentation of G_n in theorem 5.7 can be derived from the proposed defining relations.

The commutator relations (1)-(4) of theorem 5.7 all map to the identity under ϕ. The braid relations (5) and (7) of theorem 5.7 are derivable from $\mathcal{T}$. The braid relation (6) is also derivable from $\mathcal{T}$ since all relations in this family take the form $a = a$ in the abelianization. Relation (8) is derivable from $B^{n^2} = A^{-n^2-1}$ by lemma 6.3. Relations (9) and (10) are derivable from $a^2 = B$ and $d^2 = A^{n+1}$ respectively. The image $\phi(ca_ic^{-1}b_i^{-1})$ is the identity by lemma 6.4. Finally, the conjugation relations (12)-(14) are all of the form $A = A$ or $B = B$ in the abelianization, so they are all derivable from $\mathcal{T}$. ∎

We now have everything needed to prove theorem 1.1.

Proof of theorem 1.1. Recall $H_1(\mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B}); \mathbb{Z}) = G_n/[G_n, G_n]$. We will start with the presentation from lemma 6.5 and perform Tietze transformations to simplify it.

Starting with $B^{n^2-n} = A^{1-n^2}$, we may substitute in the relation $B^{n^2} = A^{-n^2-1}$ to obtain $A^2 = B^{-n}$. Thus we may add the relation $A^2 = B^{-n}$ to the set of defining relations. Observe $B^{n^2-n} = A^{1-n^2}$ is derivable from $A^2 = B^{-n}$ and $B^{n^2} = A^{-n^2-1}$ so we may delete the relation $B^{n^2-n} = A^{1-n^2}$.

Similarly, we may add the relation $A^{(n-1)^2} = 1$ and delete the relation $B^{n^2} = A^{-n^2-1}$. Deleting the generator B and replacing it with a^2 then gives the presentation

$$G_n/[G_n, G_n] \cong \langle a, d, A \mid A^2 = a^{-2n}, A^{(n-1)^2} = 1, d^2 = A^{n+1}, \mathcal{T} \rangle. \tag{11}$$

This presentation has presentation matrix $\begin{bmatrix} 2n & 0 & 2 \\ 0 & 0 & (n-1)^2 \\ 0 & 2 & -1-n \end{bmatrix}$.

If n is odd, this matrix has Smith normal form $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & n(n-1)^2 \end{bmatrix}$. Therefore $H_1(\mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B}); \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/(n(n-1)^2)\mathbb{Z}$.

If n is even, the presentation matrix has Smith normal form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2n(n-1)^2 \end{bmatrix}$, so $H_1(\mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B}); \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/(2n(n-1)^2)\mathbb{Z}$. ∎

The first betti number of a group G is the rank of the abelian group $H_1(G; \mathbb{Z}) = G/[G, G]$. We have the following corollary.

Let $\hat{D}$ be the image of the deck group in $\mathrm{Mod}(\Sigma_g)$. Recall that $\mathrm{SMod}_{g,k}(\Sigma_g)$ is the normalizer of $\hat{D}$ in $\mathrm{Mod}(\Sigma_g)$.

Proof of theorem 1.2. A result of Birman and Hilden in [8] gives a short exact sequence

$$1 \longrightarrow \mathbb{Z}/k\mathbb{Z} \longrightarrow \mathrm{SMod}_{g,k}(\Sigma_g) \longrightarrow \mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B}) \longrightarrow 1.$$

Since the abelianization functor is right exact, we have the exact sequence

$$\mathbb{Z}/k\mathbb{Z} \longrightarrow H_1(\mathrm{SMod}_{g,k}(\Sigma_g); \mathbb{Z}) \longrightarrow H_1(\mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B}); \mathbb{Z}) \longrightarrow 1.$$

Since $\mathbb{Z}/k\mathbb{Z}$ and $H_1(\mathrm{LMod}_{g,k}(\Sigma_0, \mathcal{B}); \mathbb{Z})$ are both finite, so is $H_1(\mathrm{SMod}_{g,k}(\Sigma_g); \mathbb{Z})$ and the result follows. ■

It is known that theorem 1.2 is also true for the hyperelliptic mapping class group $\mathrm{SMod}_{g,2}(\Sigma_g)$.

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