COFINAL ELEMENTS AND FRACTIONAL DEHN TWIST COEFFICIENTS

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Abstract. We show that for a surface S with positive genus and one boundary component, the mapping class of a Dehn twist along a curve parallel to the boundary is cofinal in every left ordering of the mapping class group $\mathrm{Mod}(S)$. We apply this result to show that one of the usual definitions of the fractional Dehn twist coefficient—via translation numbers of a particular action of $\mathrm{Mod}(S)$ on $\mathbb{R}$—is in fact independent of the underlying action when S has genus larger than one. As an algebraic counterpart to this, we provide a formula that recovers the fractional Dehn twist of a homeomorphism of S from an arbitrary left ordering of $\mathrm{Mod}(S)$.

1. Introduction

Braid groups, and more generally mapping class groups of hyperbolic surfaces with nonempty boundary, are left-orderable. There are many techniques for producing explicit examples of such orderings, ranging from combinatorial conditions on representative words relative to a certain generating set, to conditions on arc diagrams in the surface, to hyperbolic geometry (See e.g. [14, 27]). In fact there are uncountably many ways to left order any left-orderable mapping class group, aside from $B_2 \cong \mathbb{Z}$ which only admits two left orderings. Despite this flexibility in creating left orderings of mapping class groups, the left orderings all display a type of algebraic rigidity that is also reflected in the dynamics of their actions on $\mathbb{R}$.

Given a left-ordered group $(G, <)$, an element $g \in G$ is called *cofinal* relative to the ordering $<$ of G if

$$G = \{h \in G \mid \exists k \in \mathbb{Z} \text{ such that } g^{-k} < h < g^k\}.$$

In terms of dynamics, this means that for every action of G on $\mathbb{R}$ by orientation-preserving homeomorphisms, the element g will act without fixed points.

Using $\mathrm{Mod}(\Sigma_g^1)$ to denote the mapping class group of a surface of genus g having one boundary component and T_d to denote the Dehn twist about a curve isotopic to the boundary of Σ_g^1, we prove:

Theorem 1. *For all $g > 0$, the element $T_d \in \mathrm{Mod}(\Sigma_g^1)$ is cofinal in every left ordering of $\mathrm{Mod}(\Sigma_g^1)$.*

For the case of $g = 0$, the braid groups, the above result also holds and is well-known [14, Proposition 3.6]. Aside from the dynamical consequences explored in this paper, it is also worth noting that Theorem 1 implies that every positive cone of $\mathrm{Mod}(\Sigma_g^1)$ is a coarsely connected subset the Cayley graph of $\mathrm{Mod}(\Sigma_g^1)$ [1, Lemma 4.14]. In the language of [1], this means that $\mathrm{Mod}(\Sigma_g^1)$ for $g > 0$ is an example of a *Prieto group*.

There is a classical correspondence between elements of $H^2(G; \mathbb{Z})$ and equivalence classes of central extensions of G by $\mathbb{Z}$ (see [9, Chapter 4]). In the setting of ordered groups, this plays out as a correspondence between circularly-ordered groups (or groups admitting an action S^1 by orientation-preserving homeomorphisms) and left-ordered central extensions with cofinal central elements (or central extensions admitting an action on $\mathbb{R}$ by orientation-preserving homeomorphisms, for which the central element acts as translation by one).

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We can apply this to mapping class groups via the well-known central extension given by the "capping homomorphism," using Theorem 1 to establish a correspondence between left orderings of $\mathrm{Mod}(\Sigma_g^1)$ and circular orderings of $\mathrm{Mod}(\Sigma_{g,1})$ (the mapping class group of a surface with one marked point). By combining this correspondence with a rigidity result of Mann and Wolff [25], we are able to conclude that all left orderings of $\mathrm{Mod}(\Sigma_g^1)$ have certain dynamical properties in common, and apply this to fractional Dehn twist coefficients as follows.

We denote the fractional Dehn twist coefficient of $h \in \mathrm{Mod}(\Sigma_g^1)$ by $c(h)$, and first remark that although this quantity is often defined in terms of singular foliations, it can equally be defined as the translation number of h under a particular action of $\mathrm{Mod}(\Sigma_g^1)$ on $\mathbb{R}$ (E.g. see [21] or [24]). By an application of Theorem 1 and the result of Mann and Wolff, we show that this definition is independent of the choice of action.

Theorem 2. *Suppose that $g \geq 2$ and let $\rho : \mathrm{Mod}(\Sigma_g^1) \to \mathrm{Homeo}_+(\mathbb{R})$ be an injective homomorphism such that the action of $\mathrm{Mod}(\Sigma_g^1)$ on $\mathbb{R}$ is without global fixed points. Then, up to reversing orientation, ρ is conjugate to a representation $\rho' : \mathrm{Mod}(\Sigma_g^1) \to \widetilde{\mathrm{Homeo}_+}(S^1)$ such that $\rho'(T_d)(x) = x + 1$ for all $x \in \mathbb{R}$, and for every $h \in \mathrm{Mod}(\Sigma_g^1)$ the fractional Dehn twist coefficient $c(h)$ is given by the translation number of h as computed from the ρ'-action on $\mathbb{R}$.*

Morally, Theorem 2 tells us that the translation number of an element of $\mathrm{Mod}(\Sigma_g^1)$ is intrinsic to the element itself, and does not depend on any particular choice of action. As a surprising consequence, we also see that since the fractional Dehn twist coefficient is a rational number [20, Section 3], it follows that for every action of $\mathrm{Mod}(\Sigma_g^1)$ on $\mathbb{R}$ with T_d acting as translation by one, all translation numbers are rational.

One weakness of Theorem 2, however, is that in order to compute the fractional Dehn twist coefficient from a representation $\rho : \mathrm{Mod}(\Sigma_g^1) \to \mathrm{Homeo}_+(\mathbb{R})$, one must first normalise so that the Dehn twist T_d is sent to translation by one. This normalisation issue disappears when we re-cast the previous theorem in terms of left orderings.

Given a left ordering $<$ of $\mathrm{Mod}(\Sigma_g^1)$ for which $T_d > id$ and $h \in \mathrm{Mod}(\Sigma_g^1)$, we use $[h]_<$ to denote the unique power of $T_d \in \mathrm{Mod}(\Sigma_g^1)$ such that $T_d^{[h]_<} \leq h < T_d^{[h]_< +1}$. Such a power exists by Theorem 1.

Theorem 3. *Suppose that $g \geq 2$ and $h \in \mathrm{Mod}(\Sigma_g^1)$. Denote the fractional Dehn twist coefficient of h by $c(h)$. For every left ordering $<$ of $\mathrm{Mod}(\Sigma_g^1)$ for which $T_d > id$, we have*

$$c(h) = \lim_{n \to \infty} \frac{[h^n]_<}{n}.$$

This theorem allows for quick estimations of fractional Dehn twist coefficients (Proposition 21), and in some cases allows one to compute the precise value of the fractional Dehn twist coefficient of a particular homeomorphism from only two inequalities (Corollary 22).

1.1. **Outline of the paper.** In Section 2 we review left orderings of groups, their relationship to circular orderings, as well as translation number, rotation number and semiconjugacy. In Section 3 we introduce mapping class groups, prepare several lemmas and prove Theorem 1. In Section 5 we define the fractional Dehn twist coefficient, prove Theorems 2 and Theorem 3, and provide some basic tools for estimating fractional Dehn twist coefficients from these results. In Section 6 we provide examples that show the limitations of these theorems.

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2. Orders, dynamics and semiconjugacy

2.1. **Left-orderable and circularly-orderable groups.** A *left ordering* of a group G is a strict total ordering $<$ of the elements of G such that $g < h$ implies $fg < fh$ for all $f, g, h \in G$. Equivalently, G admits a *positive cone* $P \subset G$, which is a semigroup satisfying $G \setminus \{id\} = P \cup P^{-1}$. We can pass from an ordering $<$ to a positive cone $P_<$ by setting $P_< = \{g \in G \mid g > id\}$, and from a positive cone P to an ordering $<_P$ by declaring $g < h \iff g^{-1}h \in P$. It is straightforward to check that this correspondence is a bijection. When G admits a left ordering it will be called a *left-orderable* group. When G comes equipped with a prescribed left ordering we will refer to it as a *left-ordered group*, and we will write such objects as a pair $(G, <)$.

Given a left-ordered group, a $<$-*cofinal* set $S \subset G$ (or simply "cofinal" if the ordering of G is understood) is a subset of G satisfying

$$G = \{g \in G \mid \exists s, t \in S \text{ such that } s < g < t\}.$$

An element $g \in G$ is called $<$-cofinal if the cyclic subgroup $\langle g \rangle$ is $<$-cofinal.

A *circular ordering* of a group G is typically defined to be a function $c : G^3 \to \{0, \pm 1\}$ that satisfies:

(1) $c(g_1, g_2, g_3) = 0$ if and only if $g_i = g_j$ for some $i \neq j$;
(2) c is a homogeneous cocycle, meaning

$$c(g_1, g_2, g_3) - c(g_1, g_2, g_4) + c(g_1, g_3, g_4) - c(g_2, g_3, g_4) = 0$$

for all $g_1, g_2, g_3, g_4 \in G$;
(3) $c(g_1, g_2, g_3) = c(gg_1, gg_2, gg_3)$ for all $g, g_1, g_2, g_3 \in G$.

However, for ease of exposition in cohomological arguments it is often more straightforward to define circular orderings in terms of inhomogeneous cocycles. That is, one can alternatively define a circular ordering of a group G to be a function $f : G^2 \to \{0, 1\}$ satisfying:

(i) $f(g, g^{-1}) = 1$ for all $g \in G \setminus \{id\}$;
(ii) $f(id, g) = f(g, id) = 0$ for all $g \in G$;
(iii) f is an inhomogeneous cocycle, that is

$$f(g_2, g_3) - f(g_1g_2, g_3) + f(g_1, g_2g_3) - f(g_1, g_2) = 0$$

for all $g_1, g_2, g_3 \in G$.

That these definitions are equivalent follows from the existence of a bijection between the functions $c : G^3 \to \{0, \pm 1\}$ satisfying (1)–(3) and the functions $f : G^2 \to \{0, 1\}$ satisfying (i)–(iii). Explicitly, given a function c satisfying (1)–(3), one defines a function $f^{(c)} : G^2 \to \{0, 1\}$ satisfying (i)–(iii) according to:

$$f^{(c)}(g, h) = \begin{cases} 0 & \text{if } g = id \text{ or } h = id, \\ 1 & \text{if } gh = id \text{ and } g \neq id, \\ \frac{1}{2}(1 - c(id, g, gh)) & \text{otherwise.} \end{cases}$$

Whereas conversely, a function $f : G^2 \to \{0, 1\}$ satisfying (i)–(iii) gives rise to a function $c^{(f)}$ satisfying (1)–(3) via:

$$c^{(f)}(g_1, g_2, g_3) = \begin{cases} 0 & \text{if } g_i = g_j \text{ for some } i \neq j, \\ 1 - 2f(g_1^{-1}g_2, g_2^{-1}g_3) & \text{otherwise.} \end{cases}$$

The complete details of this bijection can be found in [11, Proposition 2.3]. As our main arguments are cohomological, in this paper we will prefer to represent our circular orderings as inhomogeneous cocycles. With this definition of circular ordering in hand, we define *circularly-orderable* groups and *circularly-ordered* groups in the obvious way.

Using this language, it is easy to construct a left-ordered central extension $(\widetilde{G}_f, <_f)$ of any circularly-ordered group (G, f). We take $\widetilde{G}_f = G \times \mathbb{Z}$ as a set, and equip it with the operation

$$(g, n)(h, m) = (gh, n + m + f(g, h)).$$

This group comes equipped with a left ordering $<_f$ whose positive cone is $\{(g, n) \mid n \geq 0\}$, and a canonical positive, central cofinal element $z_f = (id, 1)$. To our knowledge, this construction first appears in [28], however it happens to be nothing more than an application of the usual correspondence between elements of $H^2(G; \mathbb{Z})$ and equivalence classes of central extensions

$$0 \to \mathbb{Z} \to \widetilde{G} \to G \to \{id\}.$$

For details of this correspondence, see [9, Chapter 4].

It is similarly possible to begin with a left-ordered group G admitting a left ordering $<$ and a positive, cofinal, central element $z \in G$, and to construct a circular ordering $f_<$ on $G/\langle z \rangle$ according to the following rule. Given $g \in G$, define $\{g\}$ to be the unique coset representative of $g\langle z \rangle$ satisfying $id \leq \{g\} < z$, and define $f_<$ according to $\{g\}\{h\} = \{gh\}z^{f_<(g\langle z \rangle, h\langle z \rangle)}$. That this defines a circular ordering of $G/\langle z \rangle$ can be checked from the definition.

These two constructions are not inverse to one another, though applying the lift and quotient operations successively does yield a group that is naturally isomorphic to the original for categorical reasons (similarly when one applies the quotient and lift operations successively). See [10, Proposition 2.9] for details, and Lemma 18 below.

2.2. **Dynamic realisations and tight embeddings.** Let $(G, <)$ be a countable left-ordered group, we follow the development of dynamical realisations presented in [8].

A *gap* in $(G, <)$ is a pair of elements $g, h \in G$ with $g < h$ such that no element $f \in G$ satisfies $g < f < h$. An order-preserving embedding $t : G \to \mathbb{R}$ is called a *tight embedding* of $(G, <)$ if whenever $(a, b) \subset \mathbb{R} \setminus t(G)$, there exists a gap $g, h \in G$ such that $(a, b) \subset (t(g), t(h))$. We further require, for ease of exposition, that our tight embeddings satisfy $t(id) = 0$. One can check that the usual construction of $t : G \to \mathbb{R}$ in the definition of the dynamic realisation, using countability of G to inductively construct an embedding by choosing midpoints of previously defined intervals as in [12, Section 2.4], provides an example of a tight embedding.

Define the *dynamic realisation* $\rho_< : G \to \mathrm{Homeo}_+(\mathbb{R})$ of $(G, <)$ as follows. First, fix a tight embedding $t : G \to \mathbb{R}$, and then for each $g \in G$ and $x \in \mathbb{R}$ define $\rho_<(g)(x)$ according to one of the following three mutually exclusive possibilities:

- If $x \in t(G)$, then write $x = t(h)$ for some $h \in G$ and define $\rho_<(g)(t(h)) = t(gh)$.
- If $x \in \overline{t(G)}$, then choose a sequence $\{t(g_i)\} \subset t(G)$ converging to x, and define $\rho_<(g)(x) = \lim \rho_<(g)(t(g_i))$.
- If $x \in \mathbb{R} \setminus \overline{t(G)}$, then there exists a gap h, k such that $x \in (t(h), t(k))$. Write

  $$x = (1 - s)t(h) + st(k)$$

  for some $s \in (0, 1)$ and define

  $$\rho_<(g)(x) = (1 - s)t(gh) + st(gk).$$

That this defines a homeomorphism, and that the assignment $g \mapsto \rho_<(g)(x)$ is a homomorphism $\rho_< : G \to \mathrm{Homeo}_+(\mathbb{R})$, is a rather lengthly check which is sketched in [8]. We call the resulting homomorphism a *dynamic realisation* of $(G, <)$. This construction is also independent of choice of tight embedding, in the following sense.

Proposition 4. [8, Proposition 3.1] *Suppose that $(G, <)$ is countable left-ordered group, and that $t, t' : G \to \mathbb{R}$ are tight embeddings used to construct dynamic realisations $\rho_<, \rho'_< : G \to \mathrm{Homeo}_+(\mathbb{R})$ as above. Then there exists an order-preserving homeomorphism $\phi : \mathbb{R} \to \mathbb{R}$ such that*

$$\rho_<(g) = \phi \circ \rho'_<(g) \circ \phi^{-1}$$

for all $g \in G$.

The essential property of dynamic realisations is that they allow one to recover the ordering $<$ of G by examining the orbit of $t(id) = 0$:

$$(\forall g, h \in G)[g < h \iff t(g) < t(h) \iff \rho_<(g)(0) < \rho_<(h)(0)].$$

2.3. **Dynamic realisations of circular orderings.** We now extend this to the realm of circular orderings. Given a circular ordering f of a countable group G, define the *dynamic realisation* $\rho_f : G \to \mathrm{Homeo}_+(S^1)$ as follows.

Choose a tight embedding $\tilde{t} : \widetilde{G}_f \to \mathbb{R}$, with associated dynamic realisation $\widetilde{\rho}_f : \widetilde{G}_f \to \mathrm{Homeo}_+(\mathbb{R})$. Note that since $\langle z_f \rangle$ is unbounded in $\widetilde{G}_f$, the image $\tilde{t}(\langle z_f \rangle)$ is similarly unbounded in $\mathbb{R}$. As such, the map $\widetilde{\rho}_f(z_f) : \mathbb{R} \to \mathbb{R}$ acts without fixed points, since $\widetilde{\rho}_f(z_f)(\tilde{t}(z_f^k)) = \tilde{t}(z_f^{k+1})$ for all $k \in \mathbb{Z}$. Consequently this map is conjugate to one of $sh(\pm 1) : \mathbb{R} \to \mathbb{R}$, where $sh(k)(x) = x + k$ for all $x \in \mathbb{R}$ and $k \in \mathbb{Z}$. Since z_f is positive in the left ordering of $\widetilde{G}_f$, it follows that $\widetilde{\rho}_f(z_f)$ is conjugate to $sh(1)$. As such, we may assume (by applying the appropriate conjugation) that the tight embedding $\tilde{t} : \widetilde{G}_f \to \mathbb{R}$ satisfies $\tilde{t}(z_f^k) = k$ and $\tilde{t}(1) = 0$, and consequently that $\widetilde{\rho}_f(z_f) = sh(1)$. Therefore we may assume that $\widetilde{\rho}_f : \widetilde{G}_f \to \widetilde{\mathrm{Homeo}_+}(S^1)$, where

$$\widetilde{\mathrm{Homeo}_+}(S^1) = \{f \in \mathrm{Homeo}_+(\mathbb{R}) \mid f(x+1) = f(x) + 1\}.$$

Let $q : \widetilde{\mathrm{Homeo}_+}(S^1) \to \mathrm{Homeo}_+(S^1)$ denote the quotient map whose kernel consists of integral translations, and for arbitrary $g \in G$ let $\widetilde{g} = (g, 0) \in \widetilde{G}_f$. Define $\rho_f : G \to \mathrm{Homeo}_+(S^1)$ by $\rho_f(g)(x) = q(\widetilde{\rho}_f(\widetilde{g}))(x)$. Note that

$$\rho_f(g) \circ \rho_f(h)(x) = q(\widetilde{\rho}_f(\widetilde{g}\widetilde{h}))(x),$$

and that $\widetilde{g}\widetilde{h} = (g, 0)(h, 0) = (gh, f(g, h)) = (gh, 0)(1, f(g, h))$. As such,

$$q(\widetilde{\rho}_f(\widetilde{g}\widetilde{h}))(x) = q(\widetilde{\rho}_f(\widetilde{gh}) \circ sh(f(g, h)))(x) = q(\widetilde{\rho}_f(\widetilde{gh}))(x),$$

meaning $\rho_f : G \to \mathrm{Homeo}_+(S^1)$ is a homomorphism. As before, this construction is defined up to conjugation by a homeomorphism.

Proposition 5. *Suppose that (G, f) is countable circularly-ordered group, and that $t, t' : \widetilde{G}_f \to \mathbb{R}$ are tight embeddings satisfying $t(z_f^k) = t'(z_f^k) = k$, and that these embeddings are used to construct dynamic realisations $\rho_f, \rho'_f : G \to \widetilde{\mathrm{Homeo}_+}(S^1)$ as above. Then there exists an order-preserving homeomorphism $\phi : S^1 \to S^1$ such that*

$$\rho_f(g) = \phi \circ \rho'_f(g) \circ \phi^{-1}$$

for all $g \in G$.

Proof. It follows from Proposition 4 that $\widetilde{\rho}_f$ and $\widetilde{\rho}'_f$ are conjugate via some order-preserving homeomorphism $\psi : \mathbb{R} \to \mathbb{R}$. Note further that ψ satisfies

$$\widetilde{\rho}_f(z_f) \circ \psi = \psi \circ \widetilde{\rho}'_f(z_f).$$

Here $\widetilde{\rho}_f(z_f) = \widetilde{\rho}'_f(z_f) = sh(1)$, so it follows that ψ descends to the required order-preserving homeomorphism $\phi : S^1 \to S^1$. ∎

Moreover, the map $t := q \circ \tilde{t}$ provides an embedding $t : G \to S^1$; having fixed $\tilde{t}(id) = 0$ ensures that $\ker(q) \cap \tilde{t}(\widetilde{G}_f) = \langle z_f \rangle$. This allows us to make a similar observation as in the case of dynamic realisations of left orderings: Identify S^1 with $[0, 1) \cong \mathbb{R}/\mathbb{Z}$, and let $f_{S^1} : (S^1)^2 \to \{0, 1\}$ denote the standard circular ordering of S^1. Then we have

$$(\forall g, h \in G)[f(g, h) = f_{S^1}(t(g), t(h)) = f_{S^1}(\rho_f(g)(0), \rho_f(h)(0))].$$

2.4. **Semi-conjugacy and bounded cohomology.** It is well-known that representations $\rho : G \to \mathrm{Homeo}_+(S^1)$ are classified up to semiconjugacy by their corresponding Euler classes $eu(\rho) \in H_b^2(G;\mathbb{Z})$. We can relate this established notion of semiconjugacy to circular orderings via dynamic realisations.

Given a representation $\rho : G \to \mathrm{Homeo}_+(S^1)$, we can explicitly describe a representative $\omega : G^2 \to \mathbb{Z}$ of the bounded Euler class $eu(\rho)$ as follows. For each $g \in G$, all the choices of lifts $\widetilde{\rho(g)} : \mathbb{R} \to \mathbb{R}$ differ by an integral translation, so we can choose for each $g \in G$ a lift satisfying $\widetilde{\rho(g)}(0) \in [0, 1)$. Then a bounded representative of $eu(\rho)$ is given by:

$$\omega(g, h) = \widetilde{\rho(g)}(\widetilde{\rho(h)}(0)) - \widetilde{\rho(gh)}(0),$$

which is an element of $\mathbb{Z}$ (See, e.g. [18, Lemma 6.3]).

Proposition 6. *If G is a countable group and f is a circular ordering of G, then $[f] = eu(\rho_f) \in H_b^2(G;\mathbb{R})$.*

Proof. Let $\tilde{t} : \widetilde{G}_f \to \mathbb{R}$ denote the tight embedding used in constructing the dynamical realisation $\rho_f : G \to \mathrm{Homeo}_+(S^1)$, recall that $\tilde{t}$ satisfies $\tilde{t}(z_f^k) = k$ and is order-preserving with respect to the orderings $<_f$ of $\widetilde{G}_f$ and $<$ of $\mathbb{R}$.

For each $g \in G$ our choice of lift $\widetilde{\rho_f(g)}$ used to compute the bounded Euler class will be $\widetilde{\rho}_f(\widetilde{g})$, recall that $\widetilde{g} = (g, 0)$. Note that it satisfies $\widetilde{\rho}_f(\widetilde{g})(0) = t(g, 0)$, and since $(id, 0) \leq_f (g, 0) <_f z_f$ we have $0 \leq t(g, 0) < 1$. Then with these choices we are able to compute the following representative of $eu(\rho_f)$:

$$\omega(g, h) = \widetilde{\rho}_f(\widetilde{g})(\widetilde{\rho}_f(\widetilde{h})(\tilde{t}(1))) - \widetilde{\rho}_f(\widetilde{gh})(\tilde{t}(1)) = \tilde{t}(\widetilde{g}\widetilde{h}) - \tilde{t}(\widetilde{gh}).$$

Since $\widetilde{g}\widetilde{h} = (g, 0)(h, g) = (gh, f(g, h))$ then $\tilde{t}(\widetilde{g}\widetilde{h}) = \tilde{t}(\widetilde{gh}z_f^{f(g,h)}) = \tilde{t}(\widetilde{gh}) + f(g, h)$. This yields $\omega(g, h) = f(g, h)$. ■

Proposition 7. *Suppose f_1 and f_2 are two circular orderings on a countable group G such that the dynamic realisations ρ_{f_1}, ρ_{f_2} are semiconjugate. Then $[f_1] = [f_2] \in H_b^2(G;\mathbb{Z})$.*

Proof. This follows immediately from the fact that $eu(\rho_{f_1}) = eu(\rho_{f_2})$ whenever ρ_{f_1} and ρ_{f_2} are semiconjugate ([18, Theorem 6.6], or [17]). ■

Owing to this Proposition, we can say that *two circular orderings f_1, f_2 are semiconjugate* if and only if $[f_1] = [f_2] \in H_b^2(G;\mathbb{Z})$, which happens if and only if ρ_{f_1}, ρ_{f_2} are semi-conjugate in the classical sense.

2.5. **Rotation and translation numbers, algebraically and dynamically.** This section prepares the necessary background to discuss fractional Dehn twist coefficients from a dynamical perspective, but also introduces several algebraic definitions of classical objects required to connect our algebraic arguments to their dynamical counterparts. Background on the classic dynamical development of these ideas can be found in [18, 19], these algebraic ideas appear also in [4].

If $G \subset \widetilde{\mathrm{Homeo}}_+(S^1)$, we define the dynamical translation number (i.e., the classical translation number due to Poincaré [26]) of an element $g \in G$ as

$$\tau^D(g) = \lim_{n\to\infty} \frac{g^n(x) - x}{n},$$

where $x \in \mathbb{R}$ is arbitrary, a quantity that exists and is independent of $x \in \mathbb{R}$ [19, Proposition 2.3]. From here we arrive at rotation numbers as follows: Recall that $q : \widetilde{\mathrm{Homeo}}_+(S^1) \to \mathrm{Homeo}_+(S^1)$ is the quotient map, and if $G \subset \mathrm{Homeo}_+(S^1)$ we use $\widetilde{g} \in q^{-1}(g)$ to denote an arbitrary choice of preimage. Then we define the *dynamic rotation number* of g to be

$$\mathrm{rot}^D(g) = \tau^D(\widetilde{g}) \mod \mathbb{Z}.$$

This definition is independent of the choice of $\widetilde{g}$.

On the other hand, these definitions can also be described algebraically. Given a left-ordered group $(G, <)$ and a positive, cofinal central element $z \in G$, we define the floor of g relative to $<$ to be the unique integer $[g]_<$ such that $z^{[g]_<} \leq g < z^{[g]_< + 1}$. Then for every $g \in G$, we can define the *algebraic translation number* of $g \in G$ relative to $<$ to be

$$\tau^A_<(g) = \lim_{n \to \infty} \frac{[g^n]_<}{n}.$$

This limit always exists as the sequence $\{[g^n]_<\}_{n \geq 0}$ satisfies $[g^n]_< + [g^m]_< \leq [g^{m+n}]_<$ and is therefore superadditive, so we may apply Fekete's lemma.

There is a special circumstance where we have already seen that left orderings with cofinal central elements arise naturally. Given a circular ordering f of a group G, recall that the left-ordered lift $(\widetilde{G}_f, <_f)$ comes equipped with a positive, cofinal central element z_f. To simplify notation, in this setting we will write $[g]_f$ in place of $[g]_{<_f}$ and $\tau^A_f(g)$ in place of $\tau^A_{f_<}(g)$ for all $g \in \widetilde{G}_f$. Then for every $g \in G$ we define the *algebraic rotation number* of g to be

$$\mathrm{rot}^A_f(g) = \tau^A_f(g, k) \mod \mathbb{Z}$$

where $(g, k) \in \widetilde{G}_f$ and $k \in \mathbb{Z}$ is arbitrary. We observe that this definition is independent of k by noting that $(g, k)^n = (g^n, \sum_{i=1}^{n-1} f(g^i, g) + nk)$, so that $[(g, k)^n]_f = \sum_{i=1}^{n-1} f(g^i, g) + nk$. It follows that for $k, k' \in \mathbb{Z}$ we have $\tau^A_f(g, k) - \tau^A_f(g, k') = k - k'$, so that $\mathrm{rot}^A_f(g)$ is well-defined.

Proposition 8. *Let $(G, <)$ be a countable left-ordered group with positive, cofinal central element z, and $\rho : G \to \widetilde{\mathrm{Homeo}}_+(S^1)$ a dynamical realisation of $<$ satisfying $\rho(z)(x) = x + 1$ for all $x \in \mathbb{R}$. Then $\tau^A_<(g) = \tau^D(\rho(g))$ for all $g \in G$.*

Proof. Let $t : G \to \mathbb{R}$ denote the tight embedding used to construct ρ. Observe that for all $g \in G$ and for all $n \in \mathbb{Z}$ we have $\rho(g^n)(0) = \rho(g^n)(t(id)) = t(g^n)$.

Then as $t(z^k) = k$, note that $[g]_< = \lfloor t(g) \rfloor$. Consequently

$$\tau^A_<(g) = \lim_{n \to \infty} \frac{[g^n]_<}{n} = \lim_{n \to \infty} \frac{\lfloor \rho(g^n)(0) \rfloor}{n} = \lim_{n \to \infty} \frac{\rho(g^n)(0)}{n} = \tau^D(\rho(g)).$$

∎

Proposition 8 begins with an ordered group and shows that the canonical representation corresponding to the order allows one to recover the algebraic translation numbers from the dynamics of the action. On the other hand, the next proposition starts with a representation and builds an ordering of G whose algebraic translation numbers agree with those arising from the given dynamics.

Proposition 9. *Suppose that $\rho : G \to \widetilde{\mathrm{Homeo}}_+(S^1)$ is an injective homomorphism and that $z \in G$ satisfies $\rho(z)(x) = x + 1$ for all $x \in \mathbb{R}$. Fix a left ordering $\prec$ of G, and define a new left ordering $<$ of G according to the rule $g < h$ if and only if $\rho(g)(0) < \rho(h)(0)$ or $\rho(g)(0) = \rho(h)(0)$ and $g \prec h$. Then z is positive and cofinal relative to the ordering $<$ of G, and $\tau^D(\rho(g)) = \tau^A_<(g)$ for all $g \in G$.*

Proof. That z is positive and $<$-cofinal follows from the definition of $<$. Next, note that $\lfloor \rho(g)(0) \rfloor = [g]_<$ for all $g \in G$. Therefore

$$\tau^D(\rho(g)) = \lim_{n \to \infty} \frac{\rho(g^n)(0)}{n} = \lim_{n \to \infty} \frac{\lfloor \rho(g^n)(0) \rfloor}{n} = \lim_{n \to \infty} \frac{[g^n]_<}{n} = \tau^A_<(g).$$

∎

Corollary 10. *If (G, f) is a countable circularly-ordered group and $\rho_f : G \to \mathrm{Homeo}_+(S^1)$ the corresponding dynamic realisation, then for every $g \in G$, $\mathrm{rot}^A_f(g) = \mathrm{rot}^D(\rho_f(g))$.*

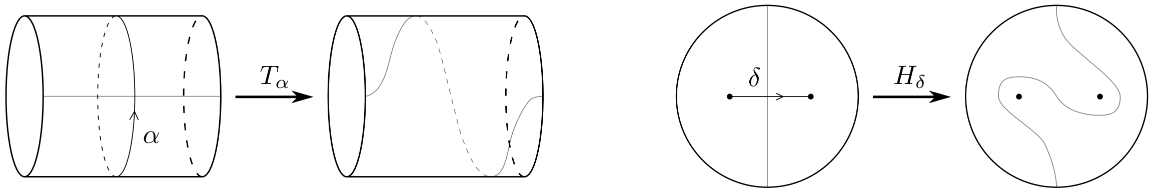

FIGURE 1. A Dehn twist about the curve α and a half twist about the arc δ.

Last, we observe that these quantities are conjugation-invariant, which is an easy observation from the classical definitions. We highlight this fact here as it will be needed in the proof of Theorem 19.

Proposition 11. *If (G, f) is a circularly-ordered group, then $\tau_f^A(g) = \tau_f^A(hgh^{-1})$ for all $g, h \in \widetilde{G}_f$ and $\mathrm{rot}_f^A(g) = \mathrm{rot}_f^A(hgh^{-1})$ for all $g, h \in G$.*

Proof. From $z_f^{[g^n]_f} \leq g^n < z_f^{[g^n]_f+1}$ and $z_f^{[h]_f} \leq h < z_f^{[h]_f+1}$ one finds $z_f^{[g^n]_f-1} \leq hg^nh^{-1} < z_f^{[g^n]_f+2}$, so that $[g^n]_f - 1 \leq [(hgh^{-1})^n]_f \leq [g^n]_f + 1$. It follows that

$$\tau_f^A(g) = \lim_{n\to\infty} \frac{[g^n]_f}{n} = \lim_{n\to\infty} \frac{[(hgh^{-1})^n]_f}{n} = \tau_f^A(hgh^{-1}).$$

It follows that rotation number is also conjugation-invariant. ■

3. Braid groups, Mapping class groups and left orderings

Let $\Sigma_{g,n}^b$ be a compact orientable surface of genus g with b boundary components and n marked points disjoint from $\partial\Sigma_{g,n}^b$. Denote the set of marked points by $\mathcal{P}$. If n or b is 0 we will omit the subscript or superscript. Let $\mathrm{Homeo}^+(\Sigma_{g,n}^b, \mathcal{P}, \partial\Sigma_{g,n}^b)$ be the set of orientation-preserving homeomorphisms f of $\Sigma_{g,n}^b$ so that $f(\mathcal{P}) = \mathcal{P}$ and $f|_{\partial\Sigma_{g,n}^b} = \mathrm{Id}$. Note that if $\partial\Sigma_{g,n}^b \neq \emptyset$, then all homeomorphisms fixing the boundary pointwise are orientation-preserving.

The *mapping class group* is the quotient group

$$\mathrm{Mod}(\Sigma_{g,n}^b) = \mathrm{Homeo}^+(\Sigma_{g,n}^b, \mathcal{P}, \partial\Sigma_{g,n}^b) / \mathrm{Homeo}_0(\Sigma_{g,n}^b, \mathcal{P}, \partial\Sigma_{g,n}^b),$$

where $\mathrm{Homeo}_0(\Sigma_{g,n}^b, \mathcal{P}, \partial\Sigma_{g,n}^b)$ is the subgroup of homeomorphisms isotopic to the identity. The isotopies must be via elements of $\mathrm{Homeo}^+(\Sigma_{g,n}^b, \mathcal{P}, \partial\Sigma_{g,n}^b)$. Elements of the mapping class group are referred to as *mapping classes*.

3.1. Dehn twists and half twists.

We now briefly recall the definitions of Dehn twists and half twists. See Figure 1 for pictures of representative elements of these mapping classes.

Let α be a simple closed curve disjoint from $\partial\Sigma_{g,n}^b$ and $\mathcal{P}$. Choose a regular neighbourhood of α that is homeomorphic, via an orientation-preserving homeomorphism, to an annulus $A \simeq S^1 \times I$ disjoint from $\partial\Sigma_{g,n}^b$ and $\mathcal{P}$. Define the *Dehn twist about* α, denoted T_α, to be the homeomorphism of $\Sigma_{g,n}^b$ given by $(s, t) \mapsto (se^{-2\pi it}, t)$ on A, and the identity outside of A.

Similarly, to define a half twist, consider a simple arc $\delta : [0, 1] \to \Sigma\backslash\partial\Sigma$ such that $\delta^{-1}(\mathcal{P}) = \{0, 1\}$. Choose a regular neighbourhood F of δ such that $F \cap \mathcal{P} = \{\delta(0), \delta(1)\}$ and $F \cap \partial\Sigma_{g,n}^b = \emptyset$. Choose an orientation preserving homeomorphism $\phi : F \to D^2$, where D^2 is the unit disk $D^2 \subset \mathbb{C}$ such that $\phi\delta(t) = t - \frac{1}{2}$. Define the *half twist about* δ, denoted H_δ, to be the homeomorphism of $\Sigma_{g,n}^b$ given by $z \mapsto e^{-2\pi i|z|}z$ on F, and the identity outside of F.

Once an orientation on $\Sigma_{g,n}^b$ has been fixed, the isotopy class of a Dehn twist T_α or half twist H_δ depends only on the unoriented isotopy class of α or δ. Therefore if a is the unoriented isotopy

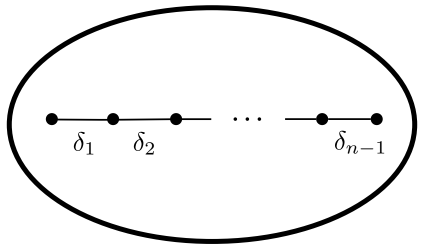

Figure 2. The simple arcs defining the standard generators of the braid group.

class of α, we may abuse notation and write the mapping class $[T_\alpha] \in \mathrm{Mod}(\Sigma^b_{g,n})$ as T_α or T_a. Similarly, if d is the isotopy class of a simple arc δ, we may write H_δ or H_d to denote the mapping class $[H_\delta] \in \mathrm{Mod}(\Sigma^b_{g,n})$.

We have the following important fact, a proof of which can be found in [15, Chapter 4].

Theorem 12. *The mapping class group* $\mathrm{Mod}(\Sigma^b_{g,n})$ *for* $g \geq 2$*,* $b, n \geq 0$*, is generated by Dehn twists about non-separating simple closed curves and half twists.*

3.2. **Braid groups and the Birman-Hilden correspondence.** The *braid group on* n *strands* B_n is the mapping class group $\mathrm{Mod}(\Sigma^1_{0,n})$. Consider the $n-1$ simple arcs $\delta_1, \dots, \delta_{n-1}$ as indicated in Figure 2. Let $\sigma_i = H_{\delta_i}$. Then B_n admits the group presentation

$$\left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{cases} [\sigma_i, \sigma_j] = 1 & |i-j| > 1 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & i \in \{1, \dots, n-2\} \end{cases} \right\rangle .$$

The σ_i are called the *standard generators* (or sometimes the *Artin generators*) for B_n.

Remark. The group B_n is more commonly encountered as a group whose elements are strand diagrams with n strands, which explains the name "braid group" somewhat more than the definition we have given. The braid group was first introduced by Artin [3], from an intuitive perspective with strand diagrams, and then more formally later, again by Artin [2]. For beautiful introductions and surveys of the theory of braids their connections to mapping class groups, see the works of Birman [5], Brendle-Birman [6], and Farb-Margalit [15, Chapter 9], to name only a few.

For $b \in \{1, 2\}$, the surface Σ^b_g admits a double branched cover over the disk, branched at $2g+1$ points if $b = 1$, and $2g+2$ points if $b = 2$. The non-trivial element of the deck group is a hyperelliptic involution (see Figure 3). Under the branched covering map, the curve a_i maps to the arc δ_i. Furthermore, the half twist $\sigma_i = H_{\delta_i}$ lifts to the Dehn twist T_{a_i}. Note the half twist σ_i is an element of $B_n = \mathrm{Mod}(\Sigma^1_{0,n})$, where the n marked points are the branch points of the cover. Lifting homeomorphisms gives an injection

$$\phi : B_n \hookrightarrow \mathrm{Mod}(\Sigma^b_{g,n})$$

given by $\sigma_i \mapsto T_{a_i}$.

Let c be the isotopy class of a curve isotopic to the boundary in $\Sigma^b_{0,n}$. When $b = 1$, let d be the isotopy class of a curve isotopic to the boundary in Σ^1_g. When $b = 2$, let d_1 and d_2 be the isotopy classes of curves isotopic to the two boundary components in Σ^2_g. Note that when $b = 1$, $\phi(T_c^2) = T_d$. When $b = 2$, $\phi(T_c) = T_{d_1}T_{d_2}$.

Obtaining a homomorphism between mapping class groups by lifting homeomorphisms is often referred to as the *Birman-Hilden theorem*, and was first introduced by Birman and Hilden [7]. For the particular injective homomorphism above, see [16, Section 2] or [15, Section 9.4].

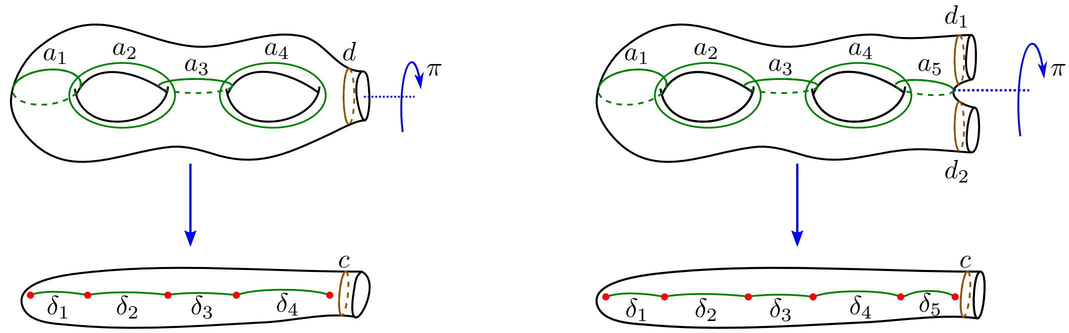

FIGURE 3. The branched covers of the disk induced by a hyperelliptic involution. The case $n = 5$ on the left, and $n = 6$ on the right.

3.3. **Inner automorphisms and capping.** By gluing on a disk with one marked point to the boundary of Σ_g^1 we obtain $\Sigma_{g,1}$. By extending homeomorphisms of Σ_g^1 by the identity on the marked disk, we obtain the central extension

$$1 \longrightarrow \langle T_d \rangle \longrightarrow \mathrm{Mod}(\Sigma_g^1) \longrightarrow \mathrm{Mod}(\Sigma_{g,1}) \longrightarrow 1,$$

where d is the isotopy class of the boundary curve. The process of obtaining $\Sigma_{g,1}$ from Σ_g^1 like this is commonly called *capping the boundary*, and the surjective map $\mathrm{Mod}(\Sigma_g^1) \to \mathrm{Mod}(\Sigma_{g,1})$ is the *capping homomorphism*. Note that a similar capping homomorphism exists for any surface with at least 1 boundary component (see [15, Section 3.6.2]), but we will not need the full generality.

Our goal in this section is to prove Lemma 14 below, which is needed for a key step in the proof of Theorem 19. First we will need the following general lemma about automorphisms of central extensions.

Lemma 13. *Let $1 \to A \hookrightarrow G \to H \to 1$ be a central extension of H by A. Let $\phi, \psi \in \mathrm{Aut}(G)$ be such that $\phi(A) = \psi(A) = A$, and such that the induced automorphisms $\overline{\phi}, \overline{\psi} \in \mathrm{Aut}(H)$ satisfy $\overline{\phi} = \overline{\psi}$. Then $\eta : G \to A$ given by $\eta(g) = \phi(g)\psi(g)^{-1}$ is a homomorphism.*

Proof. Note that since $\overline{\phi} = \overline{\psi}$, $\eta(g)$ is indeed an element of A. For $g, h \in G$ we have

$$\eta(gh) = \phi(gh)\psi(gh)^{-1} = \phi(g)\phi(h)\psi(h)^{-1}\psi(g)^{-1} = \phi(g)\eta(h)\psi(g)^{-1} = \phi(g)\psi(g)^{-1}\eta(h) = \eta(g)\eta(h),$$

completing the proof. ∎

For two isotopy classes a, b of simple closed curves on a surface, denote the *geometric intersection number* by $i(a, b)$, that is, the minimum number of intersection points between any representatives of a and b. It is a useful fact that $T_aT_bT_a = T_bT_aT_b$ if and only if $i(a, b) = 1$ [15, Propositions 3.11 and 3.13].

A *k-chain* on a surface is a set $\{a_1, \ldots, a_k\}$ of isotopy classes of simple closed curves such that $i(a_j, a_{j+1}) = 1$ for all $j \in \{1, \ldots, k-1\}$, and $i(a_j, a_l) = 0$ otherwise (see the top left of Figure 3 for an example of a 4-chain). Choose representatives α_i of a_i so that the α_i are in minimal position. If k is even, a regular neighbourhood of $\cup_{i=1}^k \alpha_i$ is homeomorphic to a genus $\frac{k}{2}$ surface with 1 boundary component. Let e be the isotopy class of the boundary component. The relation

$$(T_{a_1}T_{a_2}\cdots T_{a_k})^{2k+2} = T_e$$

holds, and is known as the *chain relation* [15, Proposition 4.12]. It follows that if $\{a_1, \ldots, a_{2g}\}$ is any $2g$-chain on Σ_g^1,

$$(T_{a_1}T_{a_2}\cdots T_{a_{2g}})^{4g+2} = T_d$$

where d is the isotopy class of the boundary of Σ_g^1.

Before embarking on the proof of Lemma 14, we must recall an important result due to Ivanov [22, Theorem 2]. Let $g \geq 2, n \geq 1$, and denote the *extended mapping class group* of $\Sigma_{g,n}$ (that is, the mapping class group where we allow orientation-reversing homeomorphisms) by $\mathrm{Mod}^{\pm}(\Sigma_{g,n})$. Note that $\mathrm{Mod}(\Sigma_{g,n})$ is an index-2 subgroup of $\mathrm{Mod}^{\pm}(\Sigma_{g,n})$. Ivanov's theorem states that the map $\mathrm{Mod}^{\pm}(\Sigma_{g,n}) \to \mathrm{Aut}(\mathrm{Mod}(\Sigma_{g,n}))$ given by $\gamma \mapsto (f \mapsto \gamma f \gamma^{-1})$ is an isomorphism. It follows that for all isotopy classes of simple closed curves a on $\Sigma_{g,n}$, if $\gamma \in \mathrm{Mod}(\Sigma_{g,n})$, then $\gamma T_a \gamma^{-1} = T_{\gamma(a)}$, and if $\gamma \notin \mathrm{Mod}(\Sigma_{g,n})$, then $\gamma T_a \gamma^{-1} = T^{-1}_{\gamma(a)}$. In particular, we can identify whether or not an automorphism of $\mathrm{Mod}(\Sigma_{g,n})$ is inner by simply observing whether a Dehn twist is sent to a Dehn twist, or the inverse of a Dehn twist.

In preparation for the next lemma, let $\mathcal{C} : \mathrm{Mod}(\Sigma_g^1) \to \mathrm{Mod}(\Sigma_{g,1})$ be the capping homomorphism. For each isotopy class a of a simple closed curve on Σ_g^1, let $\hat{a}$ be the isotopy class that is the image of a under the inclusion $\Sigma_g^1 \hookrightarrow \Sigma_{g,1}$. Note that every isotopy class of simple closed curves on $\Sigma_{g,1}$ is of the form $\hat{a}$ for some isotopy class a on Σ_g^1.

Lemma 14. *Let $g \geq 2$, and let $\varphi \in \mathrm{Aut}(\mathrm{Mod}(\Sigma_g^1))$ be such that $\varphi(T_d) = T_d$, where d is the isotopy class of the boundary curve. Then φ is an inner automorphism.*

Proof. Since $\varphi(T_d) = T_d$, we have an induced automorphism $\overline{\varphi} \in \mathrm{Aut}(\mathrm{Mod}(\Sigma_{g,1}))$. Then by [22, Theorem 2], there exists $\epsilon \in \{\pm 1\}$ so that for all isotopy classes of simple closed curves $\hat{a}$ on $\Sigma_{g,1}$, $\overline{\varphi}(T_{\hat{a}}) = T^{\epsilon}_{\hat{b}}$ where $\hat{b}$ is the image of $\hat{a}$ under an appropriately chosen element of $\mathrm{Mod}^{\pm}(\Sigma_{g,1})$. We will first show that $\epsilon = 1$, with the aim of concluding that $\overline{\varphi}$ is an inner automorphism.

Let $\{\hat{a_1}, \dots, \hat{a_{2g}}\}$ be a $2g$-chain and suppose that for each $i \in \{1, \dots, 2g\}$, $\overline{\varphi}(T_{\hat{a_i}}) = T^{\epsilon}_{\hat{b_i}}$. Then $\varphi(T_{a_i}) = T_d^{s_i} T^{\epsilon}_{b_i}$ for some $s_i \in \mathbb{Z}$. Note $\{b_1, \dots, b_{2g}\}$ is a $2g$-chain. In particular, for $i < 2g$, we have $i(b_i, b_{i+1}) = 1$ so

$$\begin{aligned} 1 &= \varphi(T_{a_i} T_{a_{i+1}} T_{a_i} T^{-1}_{a_{i+1}} T^{-1}_{a_i} T^{-1}_{a_{i+1}}) \\ &= T_d^{s_i - s_{i+1}} T^{\epsilon}_{b_i} T^{\epsilon}_{b_{i+1}} T^{\epsilon}_{b_i} T^{-\epsilon}_{b_{i+1}} T^{-\epsilon}_{b_i} T^{-\epsilon}_{b_{i+1}} \\ &= T_d^{s_i - s_{i+1}} \end{aligned}$$

Therefore $s_i = s_{i+1}$ for all $i < 2g$. Let $s = s_i$. The chain relation gives

$$\begin{aligned} T_d = \varphi(T_d) &= \varphi((T_{a_1} \cdots T_{a_{2g}})^{4g+2}) \\ &= T_d^{s(2g)(4g+2)} (T^{\epsilon}_{b_1} \cdots T^{\epsilon}_{b_{2g}})^{4g+2} \\ &= T_d^{s(2g)(4g+2)} T_d^{\epsilon}. \end{aligned}$$

Therefore $1 = 2gs(4g+2) + \epsilon$. Since $g \geq 2$, we must have $s = 0$ and $\epsilon = 1$. Therefore $\overline{\varphi}$ is an inner automorphism.

Let $\nu \in \mathrm{Mod}(\Sigma_{g,1})$ be such that $\overline{\varphi}(f) = \nu f \nu^{-1}$ for all $f \in \mathrm{Mod}(\Sigma_{g,1})$, and choose $\widetilde{\nu} \in \mathcal{C}^{-1}(\nu)$. Let $\theta \in \mathrm{Aut}(\mathrm{Mod}(\Sigma_g^1))$ be the inner automorphism given by conjugation by $\widetilde{\nu}$, that is $\theta(f) = \widetilde{\nu} f \widetilde{\nu}^{-1}$ for all $f \in \mathrm{Mod}(\Sigma_g^1)$. Then $\theta(T_d) = T_d$ and $\overline{\theta} = \overline{\varphi}$. Now, the only homomorphism $\mathrm{Mod}(\Sigma_g^1) \to \mathbb{Z}$ is trivial [15, Theorem 5.2]. Therefore by Lemma 13, $\varphi(f)\theta(f)^{-1} = 1$ for all $f \in \mathrm{Mod}(\Sigma_g^1)$. We may now conclude $\varphi(f) = \theta(f)$, and so $\varphi(f)$ is an inner automorphism. ∎

4. Cofinality of boundary Dehn twists

The goal of this section is to prove that in the mapping class group of Σ_g^1, an orientable surface of genus g with 1 boundary component, the Dehn twist about a curve isotopic to the boundary component is $<$-cofinal for *every* left ordering on $\mathrm{Mod}(\Sigma_g^1)$ (Theorem 1). This result will imply that Theorem 19 applies to all the actions of $\mathrm{Mod}(\Sigma_g^1)$ on $\mathbb{R}$, up to conjugation (Proposition 20).

We begin with a general lemma concerning left-orderable groups.

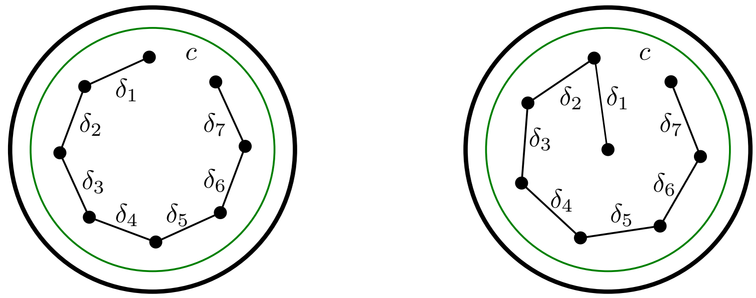

FIGURE 4. For $n = 8$, the arrangement of marked points on the disk showing $(\sigma_1\sigma_2\cdots\sigma_7)^8 = T_c$ on the left, and $(\sigma_1^2\sigma_2\cdots\sigma_7)^7 = T_c$ on the right.

Lemma 15. *Let G be a left-orderable group, and $z \in G$ a central element. Suppose there is a generating set $\{g_i\}_{i\in I}$ such that for each $i \in I$, there exist $n_i, m_i \in \mathbb{Z}\setminus\{0\}$ such that $g_i^{n_i} = z^{m_i}$. Then z is $<$-cofinal for every left ordering $<$ of G.*

Proof. Consider the set $H = \{g \in G \mid \exists n \in \mathbb{Z} \text{ such that } z^{-n} < g < z^n\}$. It suffices to show $H = G$. We first show that H is a subgroup. Note $z \neq 1$ since some power of z is a power of every generator, and G is torsion free. Therefore $z < 1 < z^{-1}$ or $z^{-1} < 1 < z$, so $1 \in H$. Next, if $z^{-n} < g < z^n$ for some $g \in G$ then $z^n < g^{-1} < z^{-n}$. Finally, if $z^{-n} < g < z^n$ and $z^{-m} < h < z^m$, then $gh > gz^{-m} = z^{-m}g > z^{-m}z^{-n} = z^{-(m+n)}$ and similarly $gh < z^{m+n}$. Thus H is a subgroup of G.

By possibly replacing z and each g_i with its inverse, we may assume $z > 1$ and $g_i > 1$ for all $i \in I$, and that $n_i, m_i > 0$. Then

$$z^{-m_i-1} < 1 < g_i < g_i^{n_i} = z^{m_i} < z^{m_i+1}.$$

Therefore H contains a generating set for G, so that $H = G$. ∎

Our goal now is to apply Lemma 15 to the mapping class group of a genus $g > 0$ surface with 1 boundary component, Σ_g^1, proving Theorem 1.

Proof of Theorem 1. Let B_n be the braid group on n strands, viewed as the mapping class group of a disk with n marked points (see Section 3.2). Arranging the marked points and arcs defining the standard half-twist generators $\sigma_1, \ldots, \sigma_{n-1}$ as in the left of Figure 4, one can see the element $X = \sigma_1\sigma_2\cdots\sigma_{n-1}$ is obtained by shifting each marked point one position clockwise. Therefore we obtain the relation $X^n = T_c$, where c is the curve isotopic to the boundary of the disk. Similarly, if the marked points and arcs are arranged as in the right of Figure 4, one sees $Y = \sigma_1^2\sigma_2\cdots\sigma_{n-1}$ is obtained by shifting the outermost $n-1$ marked points one position clockwise, so $Y^{n-1} = T_c$.

Now let $n \geq 3$ be odd and $g = \frac{1}{2}(n-1)$. As in Section 3.2, there is an embedding $\phi : B_n \to \mathrm{Mod}(\Sigma_g^1)$ such that for all $i \in \{1, \ldots, n-1\}$, $\phi(\sigma_i)$ is a Dehn twist about a non-separating curve, and $\phi(T_c)^2 = T_d$. Therefore $\phi(X)^{2n} = \phi(Y)^{2n-2} = T_d$ (this relation is the so-called *chain relation*, see [15, Section 4.4.1]), so in particular, $\phi(X)$ and $\phi(Y)$ are roots of T_d. Furthermore, $YX^{-1} = \sigma_1$, so $\phi(Y)\phi(X)^{-1}$ is a Dehn twist about a non-separating curve. Since T_d is central [15, Fact 3.8], all conjugates of $\phi(X)$ and $\phi(Y)$ are roots of T_d. Dehn twists about non-separating curves generate $\mathrm{Mod}(\Sigma_g^1)$, and since all Dehn twists about non-separating curves are conjugate [15, Section 1.3.1 and Fact 3.8], $\mathrm{Mod}(\Sigma_g^1)$ is generated by roots of T_d. The result follows from Lemma 15. ∎

By allowing n to be even and using the Birman-Hilden correspondence from Section 3.2, one obtains the result that for surfaces Σ_g^2 with two boundary components, the product of the Dehn twists about curves isotopic to the boundary components is cofinal and central in every left ordering of $\mathrm{Mod}(\Sigma_g^2)$. We conjecture something stronger is true.

Conjecture 16. *Let $g \geq 2$ and let $b_1, \ldots, b_n$ be curves isotopic to the boundary components of Σ_g^n. Any element of the form $\Pi_{i=1}^n T_{b_i}^{k_i}$ for any positive exponents $k_1, \ldots, k_n$ is cofinal in every left ordering of* $\mathrm{Mod}(\Sigma_g^n)$.

5. Fractional Dehn twist coefficients and actions on $\mathbb{R}$

5.1. **Fractional Dehn twist coefficients.** Recall that if $\Sigma_{g,n}^b$ is a hyperbolic surface with $b > 0$, there is a "standard action" of $\mathrm{Mod}(\Sigma_{g,n}^b)$ on $\mathbb{R}$ that is constructed as follows.

First, we construct the universal cover $p : \widetilde{\Sigma_{g,n}^b} \to \Sigma_{g,n}^b$, and note that we can think of $\widetilde{\Sigma_{g,n}^b}$ as a closed subset of $\mathbb{H}^2$. Fix a point $x_0 \in \partial\Sigma_{g,n}^b$, say in a component C of the boundary, and a point $\tilde{x_0} \in \widetilde{C} \subset \partial\widetilde{\Sigma_{g,n}^b}$ with $p(\tilde{x_0}) = x_0$. Now for each $h \in \mathrm{Mod}(\Sigma_{g,n}^b)$, there is a unique lift of h satisfying $h(\tilde{x_0}) = \tilde{x_0}$ yielding an action of $\mathrm{Mod}(\Sigma_{g,n}^b)$ on $\widetilde{\Sigma_{g,n}^b}$ fixing $\tilde{x_0}$ and thus fixing $\widetilde{C}$.

Now we can identify $\partial\widetilde{\Sigma_{g,n}^b}$ with the interval $(0, \pi)$, and thus with $\mathbb{R}$, by identifying each point y on the boundary with the unique geodesic from x_0 to y. Then observe that the action of $\mathrm{Mod}(\Sigma_{g,n}^b)$ extends to an action on $\partial\widetilde{\Sigma_{g,n}^b}$ by orientation-preserving homeomorphisms, which is homeomorphic to $\mathbb{R}$. We orient the boundary and parameterise it so that the action of the boundary Dehn twist T_C satisfies $T_C(x) = x + 1$ for all $x \in \mathbb{R}$. This defines a representation

$$\rho_{s,C} : \mathrm{Mod}(\Sigma_{g,n}^b) \to \widetilde{\mathrm{Homeo}_+}(S^1)$$

which we call the *standard representation with respect to C*. The *fractional Dehn twist coefficient* of $h \in \mathrm{Mod}(\Sigma_{g,n}^b)$ can be defined as

$$c(h, C) = \tau_{\rho_{s,C}}^D(h).$$

While this is not the usual definition of the fractional Dehn twist coefficient, that this is equivalent to it appears in [21, Theorem 4.16], and for the special case of $\mathrm{Mod}(\Sigma_{0,n}^1)$ (i.e. for the braid groups) in [24]. When $b = 1$, we will simplify our notation and use ρ_s to denote the standard representation, and $c(h)$ to denote the fractional Dehn twist coefficient.

5.2. **Actions of** $\mathrm{Mod}(\Sigma_{g,n}^b)$ **on** $\mathbb{R}$**.** In this section we prove Theorem 20, which is Theorem 2 from the introduction, from which Theorem 3 follows. We begin with some preparatory lemmas.

Lemma 17. *Suppose that f_1, f_2 are semiconjugate circular orderings of a group G. Then there exists an isomorphism $\phi : \widetilde{G}_{f_1} \to \widetilde{G}_{f_2}$ satisfying $\phi(z_{f_1}) = z_{f_2}$ such that $\tau_{f_1}^A(g) = \tau_{f_2}^A(\phi(g))$ for all $g \in \widetilde{G}_{f_1}$.*

Proof. If f_1, f_2 are semiconjugate then $[f_1] = [f_2] \in H_b^2(G; \mathbb{Z})$, and consequently there exists a bounded function $d : G \to \mathbb{Z}$ satisfying $f_1(g, h) - f_2(g, h) = d(gh) - d(g) - d(h)$ for all $g, h \in G$. The function d necessarily satisfies $d(id) = 0$, and allows us to define an isomorphism $\phi : \widetilde{G}_{f_1} \to \widetilde{G}_{f_2}$ given by $\phi(g, n) = (g, n - d(g))$. The isomorphism satisfies $\phi(z_{f_1}) = \phi(id, 0) = (id, 0)$.

Now since d is bounded, set $M = \max\{|d(g)| \mid g \in G\}$. Observe that for any $g = (h, m) \in \widetilde{G}_{f_1}$, we have

$$|[(h, m)]_{f_1} - [\phi(h, m)]_{f_2}| = |m - (m - d(h))| \leq M.$$

Therefore for every $g \in \widetilde{G}_{f_1}$ we compute

$$\tau_{f_1}^A(g) - \tau_{f_2}^A(\phi(g)) = \lim_{n \to \infty} \frac{[g^n]_{f_1} - [\phi(g)^n]_{f_2}}{n},$$

but since $-M \leq [g^n]_{f_1} - [\phi(g)^n]_{f_2} \leq M$ for all n, we obtain $\tau_{f_1}^A(g) - \tau_{f_2}^A(\phi(g)) = 0$. ■

Recall from Section 2.1 that if $(G,<)$ is a left-ordered group with cofinal, positive central element z, then $\{g\}$ is the unique coset representative of $g\langle z\rangle$ satisfying $id \leq \{g\} < z$, and $f_<$ is the circular ordering of $G/\langle z\rangle$ defined by $\{g\}\{h\} = \{gh\}z^{f_<(g\langle z\rangle, h\langle z\rangle)}$.

Lemma 18. *Suppose that $(G,<)$ is a left-ordered group admitting a positive, cofinal central element $z \in G$. Then there exists an order-preserving isomorphism $\phi : G \to \widetilde{(G/\langle z\rangle)}_{f_<}$ defined by $\{g\}z^{[g]_<} \mapsto (g\langle z\rangle, [g]_<)$ for all $g \in G$.*

Proof. Note that by the definition of $f_<$, we have $\{g\}z^{[g]_<} \cdot \{h\}z^{[h]_<} = \{gh\}z^{[g]_<+[h]_<+f_<(g\langle z\rangle,h\langle z\rangle)}$ while by definition of $\widetilde{(G/\langle z\rangle)}_{f_<}$ we have $(g\langle z\rangle, [g]_<)(h\langle z\rangle, [h]_<) = (gh\langle z\rangle, [g]_<+[h]_<+f_<(g\langle z\rangle, h\langle z\rangle))$, so that the map is a homomorphism. As it is clearly surjective and injective, it is an isomorphism. Last we observe that $id < g$ if and only if $[g]_< \geq 0$, while $(g\langle z\rangle, k)$ is positive in the left ordering of $\widetilde{(G/\langle z\rangle)}_{f_<}$ if and only if $k \geq 0$; so the given map is order-preserving. ∎

For the statement and proof of the next theorem, recall that T_d denotes the Dehn twist around a simple closed curve d that is parallel to $\partial\Sigma_g^1$.

Theorem 19. *Suppose that $g \geq 2$ and that $\rho_i : \mathrm{Mod}(\Sigma_g^1) \to \widetilde{\mathrm{Homeo}}_+(S^1)$ is an injective homomorphism satisfying $\rho_i(T_d)(x) = x+1$ for $i = 1,2$ and for all $x \in \mathbb{R}$. Then $\tau^D_{\rho_1}(h) = \tau^D_{\rho_2}(h)$ for all $h \in \mathrm{Mod}(\Sigma_g^1)$.*

Proof. Fix a left ordering $\prec$ of $\mathrm{Mod}(\Sigma_g^1)$. Then, associated to each homomorphism ρ_i, there is an ordering $<_i$ defined as in Proposition 9 that satisfies $\tau^D(\rho_i(h)) = \tau^A_{<_i}(h)$ for $i = 1,2$ and for all $h \in \mathrm{Mod}(\Sigma_g^1)$. Then for $i = 1,2$ let $\{h\}_i$ denote the unique coset representative of $h\langle T_d\rangle$ satisfying $id \leq_i \{h\}_i <_i T_d$, and f_i the circular ordering of $\mathrm{Mod}(\Sigma_g^1)/\langle T_d\rangle = \mathrm{Mod}(\Sigma_{g,1})$ defined by $\{\alpha\}_i\{\delta\}_i = \{\alpha\delta\}_i T_d^{f_i(\alpha\langle T_d\rangle,\delta\langle T_d\rangle)}$ as in Lemma 18. Then every element $h \in \mathrm{Mod}(\Sigma_g^1)$ can be written uniquely as $\{h\}_i T_d^{[h]_{<_i}}$, and by Lemma 18 the map $\{h\}_i T_d^{[h]_{<_i}} \mapsto (h\langle T_d\rangle, [h]_{<_i})$ is an order-preserving isomorphism between $(\mathrm{Mod}(\Sigma_g^1), <_i)$ and $(\widetilde{(\mathrm{Mod}(\Sigma_g^1)/\langle T_d\rangle)}_{f_i}, <_{f_i})$. As such, $\tau^A_{<_i}(h) = \tau^A_{f_i}(h\langle T_d\rangle, [h]_{<_i})$ for all $h \in \mathrm{Mod}(\Sigma_g^1)$. Owing to these equalities, in order to show that $\tau^D_{\rho_1}(h) = \tau^D_{\rho_2}(h)$ for all $h \in \mathrm{Mod}(\Sigma_g^1)$, it suffices to show that $\tau^A_{f_1}(h\langle T_d\rangle, [h]_{<_i}) = \tau^A_{f_2}(h\langle T_d\rangle, [h]_{<_i})$ for all $h \in \mathrm{Mod}(\Sigma_g^1)$.

To do this, note that f_1 and f_2 are semiconjugate by Proposition 7 and [25]. Therefore by Lemma 17, there exists an isomorphism $\psi : \mathrm{Mod}(\Sigma_g^1) \to \mathrm{Mod}(\Sigma_g^1)$ such that

$$\tau^A_{f_1}(g\langle T_d\rangle, [g]_{<_i}) = \tau^A_{f_2}(\psi(g\langle T_d\rangle, [g]_{<_i})),$$

which by Lemma 14 must be an inner automorphism. Thus $\tau^A_{f_1}(g\langle T_d\rangle, [g]_{<_i}) = \tau^A_{f_2}(g\langle T_d\rangle, [g]_{<_i})$ follows from Proposition 11, completing the proof. ∎

For the next proof, recall that a subgroup C of a left-ordered group $(G,<)$ is called *convex* if, whenever $c,d \in C$ and $g \in G$ then $c < g < d$ implies $g \in C$. The next theorem is Theorem 2 from the introduction.

Theorem 20. *Suppose that $g \geq 2$ and let $\rho : \mathrm{Mod}(\Sigma_g^1) \to \mathrm{Homeo}_+(\mathbb{R})$ be an injective homomorphism such that the action of $\mathrm{Mod}(\Sigma_g^1)$ on $\mathbb{R}$ is without global fixed points. Then, up to reversing orientation, ρ is conjugate to a representation $\rho' : \mathrm{Mod}(\Sigma_g^1) \to \widetilde{\mathrm{Homeo}}_+(S^1)$ such that $\rho'(T_d)(x) = x+1$ for all $x \in \mathbb{R}$ and $c(h) = \tau^D_{\rho'}(h)$ for every $h \in \mathrm{Mod}(\Sigma_g^1)$.*

Proof. Suppose that $\rho : \mathrm{Mod}(\Sigma_g^1) \to \mathrm{Homeo}_+(\mathbb{R})$ is a homomorphism for which the corresponding action on $\mathbb{R}$ has no global fixed point and such that $\rho(T_d)$ is not conjugate to shift by ± 1. Then

$\rho(T_d)$ must have a fixed point, say x_0. By ordering the cosets of the stabilizer $\mathrm{Stab}_\rho(x_0)$ in $\mathrm{Mod}(\Sigma_g^1)$ according to the orbit of x_0, and ordering $\mathrm{Stab}_\rho(x_0)$ however we please, we obtain a contradiction to Theorem 1 since $T_d \in \mathrm{Stab}_\rho(x_0)$, which is a convex subgroup in the resulting ordering. Therefore, after fixing an appropriate orientation of $\mathbb{R}$ we may choose $\rho' : \mathrm{Mod}(\Sigma_g^1) \to \widetilde{\mathrm{Homeo}}_+(S^1)$ satisfying $\rho'(T_d)(x) = x + 1$ for all $x \in \mathbb{R}$. Now by Theorem 19, for every $h \in \mathrm{Mod}(\Sigma_g^1)$ we have $\tau_{\rho'}^D(h) = \tau_{\rho_s}^D(h) = c(h)$. ■

In particular, this means that the fractional Dehn twist coefficient of any element of $\mathrm{Mod}(\Sigma_g^1)$ can be computed directly from an arbitrary left ordering of $\mathrm{Mod}(\Sigma_g^1)$ (See also [21], where this result appears for the special case of the braid groups equipped with the Dehornoy ordering), as in the introduction. In particular, the proof of Theorem 3 in the introduction now follows immediately from Theorem 20 and Proposition 8.

5.3. **Estimating fractional Dehn twist coefficients using left orderings.** In light of Proposition 3, every left ordering of $\mathrm{Mod}(\Sigma_g^1)$ gives rise to some easy techniques for estimating fractional Dehn twists.

Proposition 21. *Suppose that $g \geq 2$ and fix a left ordering $<$ of $\mathrm{Mod}(\Sigma_g^1)$ for which $T_d > id$. If $T_d^k \leq h^m < T_d^\ell$ then $\frac{k}{m} \leq c(h) \leq \frac{\ell}{m}$.*

Proof. Because T_d is central, the inequality $T_d^k \leq h^n < T_d^\ell$ implies that $T_d^{nk} \leq h^{nm} < T_d^{n\ell}$ for all $n > 0$. Therefore $nk \leq [h^{nm}]_< < n\ell$, and so

$$\frac{k}{m} \leq \lim_{n\to\infty} \frac{[h^{nm}]_<}{nm} \leq \frac{\ell}{m},$$

but the central term is clearly equal to $c(h)$. ■

Aside from yielding quick estimates of fractional Dehn twist coefficients, the fact that the previous proposition holds for every left ordering of $\mathrm{Mod}(\Sigma_g^1)$ allows for a new methods of computing fractional Dehn twist coefficients.

Corollary 22. *Suppose that $g \geq 2$, and let $<_1, <_2$ be left orderings of $\mathrm{Mod}(\Sigma_g^1)$ for which $T_d >_i id$ for $i = 1, 2$. If $h^n \geq_1 T_d^k$ and there exists f such that $fh^nf^{-1} \leq_2 T_d^k$, then $c(h) = \frac{k}{n}$.*

Proof. This is a direct consequence of Propositions 21 and 11. ■

In particular, this corollary implies that if there exists a left ordering $<$ of $\mathrm{Mod}(\Sigma_g^1)$ and $g, h \in \mathrm{Mod}(\Sigma_g^1)$ and $n \in \mathbb{Z}$, $n > 0$ such that $[h^n]_< \neq [gh^ng^{-1}]_<$, then we can quickly determine the fractional Dehn twist coefficient of h.

For if $T_d^k \leq h^n < T_d^{k+1}$, then it follows that $T_d^{k-1} \leq gh^ng^{-1} < T_d^{k+2}$. So if $[h^n]_< \neq [gh^ng^{-1}]_<$ then it must be that either $T_d^{k-1} \leq gh^ng^{-1} < T_d^k$ or $T_d^{k+1} \leq gh^ng^{-1} < T_d^{k+2}$. In the former case, $c(h) = \frac{k}{n}$, and in the latter, $c(h) = \frac{k+1}{n}$.

6. Surfaces with many boundary components, low genus and marked points

In this section, we provide examples that show Theorem 20 and its left-orderability counterpart Corollary 3 cannot hold for any surface $\Sigma_{g,n}^b$ with $b > 1$, nor for surfaces Σ_g^1 when $g < 2$. Whether or not our results hold for $\Sigma_{g,n}^1$ when $n > 0$ and $g > 1$ remains open.

6.1. **Surfaces with more than one boundary component.** Suppose that $b > 1$ and choose distinct boundary components $C, C' \subset \partial\Sigma^b_{g,n}$. Construct an action of $\mathrm{Mod}(\Sigma^b_{g,n})$ on $\mathbb{R}$ as in Section 5.1, and consider the action of T'_C on $\mathbb{R}$.

Fix a geodesic γ in $\Sigma^b_{g,n}$ beginning at $x_0 \in C$ and not ending in C'. Then the lift $\tilde{\gamma}$ in the universal cover ends at a point in $\partial\widetilde{\Sigma^b_{g,n}}$ which is fixed by the action of $T_{C'}$. Then $c(T_{C'}, C') = 1$ while the translation number of $T_{C'}$ is zero, so that Theorem 20 does not hold.

6.2. $\mathrm{Mod}(\Sigma^1_{0,n})$**, the braid groups.** We define the *Dehornoy ordering* of the braid group B_n as follows, referring to the presentation with generators $\sigma_1, \dots, \sigma_{n-1}$ from Section 3.2. For an exposition of this ordering and related ideas, see [14].

A word w in the generators $\sigma_1, \dots, \sigma_{n-1}$ is said to be i-positive if there are no occurrences of σ_j in w for all $j < i$, and σ_i occurs in w with only positive powers. So, for example, the word $w = \sigma_3\sigma_2\sigma_3^{-1}\sigma_4^{-1}$ is 2-positive. A braid $\beta \in B_n$ is said to be i-positive if β admits a representative word w in the generators $\sigma_1, \dots, \sigma_{n-1}$ that is i-positive.

Theorem 23. [13] *The set*

$$P = \{\beta \in B_n \mid \beta \textit{ is } i\textit{-positive for some } i\}$$

is the positive cone of a left ordering $<_D$ of B_n.

It is not difficult to check that σ_{n-1} is the least positive element of the ordering $<_D$, for if β is positive then $id < \sigma_{n-1} < \beta$ if and only if $\sigma_{n-1}^{-1}\beta$ admits an i-positive representative word. However, since β is positive it already admits an i-positive representative word w, and so $\sigma_{n-1}^{-1}w$ is an i-positive representative word for $\sigma_{n-1}^{-1}\beta$.

In particular, recall from the proof of Theorem 1 that $(\sigma_1\sigma_2\cdots\sigma_{n-1})^n = T_d$ generates the centre of B_n and corresponds to a Dehn twist about the boundary of the disk (when B_n is viewed as $\mathrm{Mod}(\Sigma^1_{0,n})$). Thus $T_d >_D id$. Supposing that Corollary 3 were to hold for B_n, the inequalities $\sigma_{n-1}^k < T_d$ for all n would imply $c(\sigma_{n-1}) = 0$ by Proposition 21.

On the other hand, suppose we order B_n lexicographically according to the short exact sequence

$$\{id\} \to [B_n, B_n] \to B_n \xrightarrow{\phi} \mathbb{Z} \to \{id\}$$

where the ordering of $[B_n, B_n]$ is arbitrary, and $\mathbb{Z}$ is ordered so that $\phi(\sigma_i) > 0$ for all i. Then $\phi(T_d) = n(n-1)$, $\phi(\sigma_{n-1}) = 1$, and it follows that the resulting lexicographic left ordering satisfies $T_d < \sigma_{n-1}^{n(n-1)+1}$. Assuming Corollary 3 holds for B_n yields $c(\sigma_{n-1}) > \frac{1}{n(n-1)+1}$ by an application of Proposition 21, contradicting our earlier conclusion that $c(\sigma_{n-1}) = 0$.

This shows that Theorem 20 and its restatement Corollary 3 in terms of left orderings cannot hold for B_n when $n \geq 3$.

6.2.1. $\mathrm{Mod}(\Sigma^1_{1,n})$. Let $T_\alpha \in \mathrm{Mod}(\Sigma^1_{1,n})$ denote the class of a Dehn twist along a nonseparating simple closed curve α in $\Sigma^1_{1,n}$. Similar to above, it is possible to build a left ordering of $\mathrm{Mod}(\Sigma^1_{1,n})$ such that powers of T_α are bounded above by T_d, by virtue of the fact that T_α is contained in a proper convex subgroup $C \subset \mathrm{Mod}(\Sigma^1_{1,n})$. For the construction below we follow the exposition of [27].

An *ideal arc* is the image of a map

$$(I, \partial I, \mathrm{int}I) \to (\Sigma^1_{1,n}, \partial\Sigma^1_{1,n} \cup \mathcal{P}, \Sigma^1_{1,n} \setminus (\partial\Sigma^1_{1,n} \cup \mathcal{P})),$$

that is injective on the interior of I. A collection of ideal arcs $\{\gamma_1, \dots, \gamma_k\}$ is a *curve diagram* if $\Sigma^1_{1,n} \setminus \bigcup_{i=1}^k \gamma_i$ is homeomorphic to a disk, ideal arcs are embedded and disjoint, and all ideal arcs are oriented. Moreover the marked points $\mathcal{P} = \{p_1, \dots, p_n\}$ of $\Sigma^1_{1,n}$, if there are any, are connected by oriented ideal arcs such that p_i is connected to p_{i+1} by an oriented ideal arc for all $i = 1, \dots, n-1$. Given two curve diagrams E_0, E_1 in $\Sigma^1_{1,n}$, they can be put in "tight position" relative to one another,

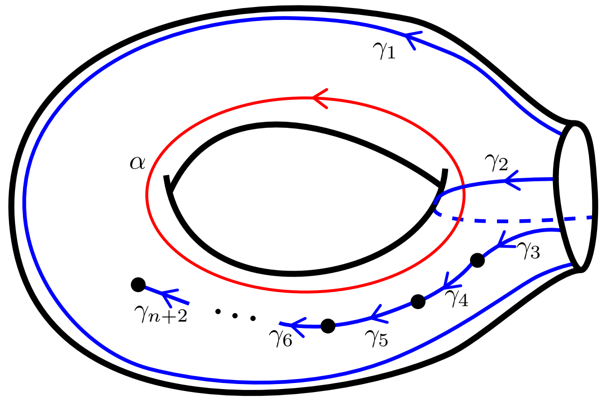

FIGURE 5. The curve diagram used to construct an ordering of $\mathrm{Mod}(\Sigma^1_{1,n})$ and the curve α in red.

which guarantees that arcs of E_0 and E_1 either intersect transversely or coincide. For details, see [14, Chapter X].

Now, given a curve diagram E in $\Sigma^1_{1,n}$, one can define a left ordering of $\mathrm{Mod}(\Sigma^1_{1,n})$ according to the rule: $\phi \in \mathrm{Mod}(\Sigma^1_{1,n})$ is positive if and only if when $\phi(E)$ and E are in tight position relative to one another, the curve diagram $\phi(E)$ branches off E to the left. By this, we mean that after being put in tight position, there exists $j \leq k$ such that $\gamma_i = \phi(\gamma_i)$ for $i = 1, \dots, j-1$ while $\gamma_j \neq \phi(\gamma_j)$ and $\phi(\gamma_j)$ branches to the left of γ_j. Then for $j = 1, \dots, k$, the set

$$C_j = \{\phi \in \mathrm{Mod}(\Sigma^1_{1,n}) \mid \phi(\gamma_i) \text{ and } \gamma_i \text{ coincide for } i = 1, \dots, j\}$$

is a convex subgroup of $\mathrm{Mod}(\Sigma^1_{1,n})$ relative to the left ordering so defined.

Now, using the curve diagram in Figure 5, we can construct a left ordering $<$ of $\mathrm{Mod}(\Sigma^1_{1,n})$ such that $T_\alpha \in C_1$ and $T_d > id$, clearly $T_d \notin C_1$. It follows that $T_\alpha^k < T_d$ for all k, and assuming Corollary 3 holds, Proposition 21 yields $c(T_\alpha) = 0$.

On the other hand, the abelianisation homomorphism provides a map $\mathrm{Mod}(\Sigma^1_{1,n}) \to \mathbb{Z}$ such that $T_\alpha \mapsto 1$ [23, Section 5]. So if we construct a left ordering of $\mathrm{Mod}(\Sigma^1_{1,n})$ using the short exact sequence

$$\{id\} \to [\mathrm{Mod}(\Sigma^1_{1,n}), \mathrm{Mod}(\Sigma^1_{1,n})] \to \mathrm{Mod}(\Sigma^1_{1,n}) \to \mathbb{Z} \to \{id\}$$

then an argument identical to the case of B_n will yield $c(T_\alpha) > 0$ by an application of Proposition 21.

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