Linear Algebra: Linear Transformations

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1 Introduction to Linear Transformations

A linear transformation is a function $T : V \to W$ between two vector spaces that preserves vector addition and scalar multiplication:

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all vectors $\mathbf{u}, \mathbf{v} \in V$
2. $T(c\mathbf{v}) = cT(\mathbf{v})$ for all vectors $\mathbf{v} \in V$ and all scalars c

Linear transformations are foundational in linear algebra because they preserve the structure of vector spaces. They appear throughout mathematics, physics, computer science, and engineering.

Example 1

Consider the function $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by:

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x \\ 3y \end{bmatrix}$$

This is a linear transformation because:

- $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
- $T(c\mathbf{v}) = cT(\mathbf{v})$

Geometrically, this transformation scales the x-coordinate by 2 and the y-coordinate by 3.

2 Matrix Representation of Linear Transformations

For finite-dimensional vector spaces, every linear transformation can be represented by a matrix. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then there exists a unique $m \times n$ matrix A such that:

$$T(\mathbf{x}) = A\mathbf{x}$$

for all $\mathbf{x} \in \mathbb{R}^n$.

Finding the Matrix of a Linear Transformation

To find the matrix A representing a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$:

1. Let $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be the standard basis for $\mathbb{R}^n$.
2. Compute $T(\mathbf{e}_1), T(\mathbf{e}_2), \ldots, T(\mathbf{e}_n)$.
3. The matrix A has columns $T(\mathbf{e}_1), T(\mathbf{e}_2), \ldots, T(\mathbf{e}_n)$.

Example 2

For the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ where $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x \\ 3y \end{bmatrix}$:

1. Compute $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$
2. Compute $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$
3. The matrix is $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

3 Kernel and Image of Linear Transformations

Two important subspaces associated with a linear transformation $T : V \to W$ are:

Kernel (Null Space)

The kernel of T, denoted $\ker(T)$ or $\text{null}(T)$, is the set of all vectors in V that map to the zero vector in W:

$$\ker(T) = \{\mathbf{v} \in V : T(\mathbf{v}) = \mathbf{0}\}$$

The kernel is a subspace of the domain V.

Image (Range)

The image of T, denoted $\text{im}(T)$ or $\text{range}(T)$, is the set of all possible outputs:

$$\text{im}(T) = \{T(\mathbf{v}) : \mathbf{v} \in V\}$$

The image is a subspace of the codomain W.

The Rank-Nullity Theorem

For a linear transformation $T : V \to W$ where V is finite-dimensional:

$$\dim(V) = \dim(\ker(T)) + \dim(\text{im}(T))$$

This theorem relates the dimension of the domain, kernel, and image.

4 Important Classes of Linear Transformations

Injective Linear Transformations

A linear transformation $T : V \to W$ is injective (one-to-one) if and only if $\ker(T) = \{\mathbf{0}\}$.

Surjective Linear Transformations

A linear transformation $T : V \to W$ is surjective (onto) if and only if $\text{im}(T) = W$.

Bijective Linear Transformations

A linear transformation is bijective if and only if it is both injective and surjective.

Isomorphisms

A bijective linear transformation $T : V \to W$ is called an isomorphism. Two vector spaces are isomorphic if there exists an isomorphism between them.

5 Common Examples of Linear Transformations

Rotation in $\mathbb{R}^2$

A rotation by angle θ counterclockwise around the origin:

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Reflection in $\mathbb{R}^2$

A reflection across a line through the origin with angle θ from the positive x-axis:

$$A = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$$

Projection in $\mathbb{R}^n$

Projection onto a subspace W with orthonormal basis $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k\}$:

$$\text{proj}_W(\mathbf{v}) = \sum_{i=1}^{k} (\mathbf{v} \cdot \mathbf{w}_i)\mathbf{w}_i$$

6 Composition of Linear Transformations

If $S : U \to V$ and $T : V \to W$ are linear transformations, then their composition $T \circ S : U \to W$ is also a linear transformation.

If S and T are represented by matrices A and B respectively, then $T \circ S$ is represented by the matrix product BA.

Practice Problems

1. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x + y \\ y + z \end{bmatrix}$. Find the matrix representation of T.

2. Find the kernel and image of the linear transformation represented by the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$.

3. Determine whether the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x - y \\ x + y \end{bmatrix}$ is injective, surjective, or bijective.