Complex Analysis: Key Concepts

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Complex Numbers & Functions

- Complex number: $z = a + bi$ where $a, b \in \mathbb{R}$ and $i^2 = -1$
- Polar form: $z = re^{i\theta}$ where $r = |z| = \sqrt{a^2 + b^2}$ and $\theta = \arg(z)$
- Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$
- Complex function: $f(z) = u(x, y) + iv(x, y)$ where $z = x + iy$
- Example: $f(z) = z^2 = (x + iy)^2 = x^2 - y^2 + i(2xy)$

Analytic Functions

- Complex derivative: $f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$
- Analytic function: Complex differentiable in a region
- Cauchy-Riemann equations: If $f(z) = u(x, y) + iv(x, y)$ is differentiable, then:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

- Example: For $f(z) = z^2 = x^2 - y^2 + i(2xy)$

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}$$

So $f(z) = z^2$ is analytic everywhere

Contour Integrals & Cauchy's Theorems

- Contour integral: $\int_C f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt$
- Cauchy's Integral Theorem: If f is analytic in a simply connected region D and C is a simple closed contour in D, then:

$$\oint_C f(z)\,dz = 0$$

- Cauchy's Integral Formula: If f is analytic in a region containing a simple closed contour C (counterclockwise), then for any point a inside C:

$$f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz$$

- For derivatives: $f^{(n)}(a) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz$

Residues & Applications

- Laurent series: $f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n$ around isolated singularity
- Residue at a: Coefficient c_{-1} in Laurent series
- Residue Theorem: If f is analytic except at isolated singularities $a_1, a_2, \ldots, a_n$ inside a simple closed contour C, then:

$$\oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f, a_k)$$

- Applications:
  - Computing difficult real integrals: $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$
  - Conformal mapping: Analytic functions preserve angles
  - Physics: Fluid dynamics, electrostatics
  - Number theory: Prime number theory