5

Flatness-Based 2DOF Controller Design for a Rotary Flexible Joint

M.Sc. Laboratory Advanced Control (WS 22/23)

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The objective of this lab exercise is a steady state (swing up) transition for the vertically-mounted rotary flexible joint shown (in its horizontal setup) in Fig. 5.1. To this end, **two** mathematical models are derived from Lagrangian mechanics. The first model takes into account the individual translational spring deflections and is used as a *simulation model*. The second model, namely, the control model, approximates the spring-beam system as one single rotational spring, which greatly simplifies the mathematical model. Have a look at rfj_swing_up.avi for a simulative illustration of the swing up to get an idea of the behavior of the RFJ.

Therefore, we will

(i) study the relationship between the the two models,

(ii) design a flatness-based feedforward controller based on the control model that realizes the transition between steady states

(iii) design a pre-defined polynomial target trajectory for the transition,

(iv) implement a trajectory tracking state-feedback controller, which ensures that disturbances and model errors are compensated,

(v) implement a nonlinear Luenberger observer to estimate the state of the system needed for the state-feedback controller.

This document is part of the zip-archive ex5_A.zip with the following content:

- rfj_A.pdf
- rfj_init.m
- rfj_impulse_response.mat
- rfj_swing_up.avi
- param.mat

If you have any questions or suggestions regarding this experiment, please contact:

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Figure 5.1: Picture of the rotary flexible joint [4].

5.1 Mathematical Model

The rotary flexible joint setup, its geometric relations and coordinate conventions are shown in Fig. 5.2. A spring-loaded beam B is mounted on top of a platform P, where the springs connect both elements. The beam consists of a main arm and a load arm, where $m_{B,i}$, $l_{B,i}$, $\quad i = 1,2$ are their respective mass and length. Furthermore, $l_{M,2}$ denotes the length to the middle of the load arm ($i = 2$) measured from the entire beam's joint to the platform, and l_G is the length to the center of gravity (COG) of the beam. The platform and beam deflection are given by the angles φ_P and φ_B, respectively. The platform P is actuated by a DC motor, which is controlled by an input voltage. The deflection angle of the beam w.r.t. the inertial coordinate system, i.e., $\varphi_P + \varphi_B$ can be measured.

The vectors from the inertial coordinate frame to the springs' fix-points are given by

$$\boldsymbol{p}_0^a = R_0^P \boldsymbol{p}_P^a, \quad \boldsymbol{p}_0^b = R_0^P \boldsymbol{p}_P^b, \boldsymbol{p}_0^c = R_0^B \boldsymbol{p}_B^c, \tag{5.1}$$

with vectors $\boldsymbol{p}_P^a = [l_x^a, -l_y^a, 0]^\top$, $\boldsymbol{p}_P^b = [l_x^a, l_y^a, 0]^\top$, and $\boldsymbol{p}_B^c = [l^c, 0, 0]^\top$. The coordinate system attached to the base of the platform is given by $(0x_P, y_P, z_P)$ and the one attached to the beam by $(0x_B, y_B, z_B)$. The rotation matrices read $R_0^P = R_z(\varphi_P)$, $R_P^B = R_z(\varphi_B)$, $R_0^B = R_z(\varphi_P + \varphi_B)$, where

$$R_z(\phi) = \begin{bmatrix} \cos(\phi) & -\sin(\phi) & 0 \\ \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The system's potential energy for

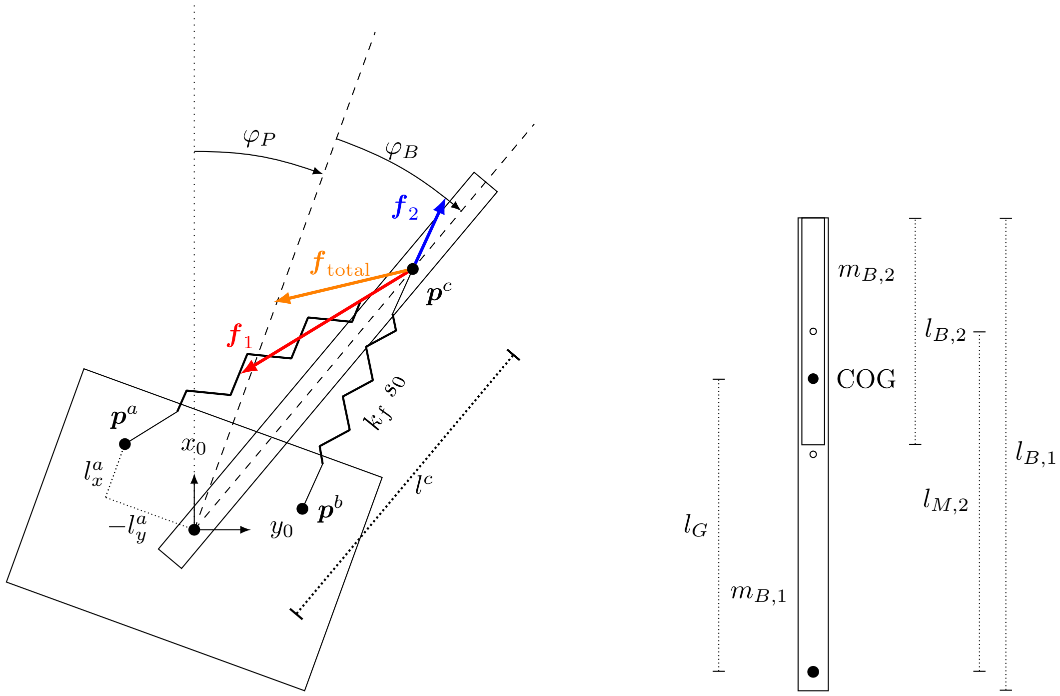
a) the simulation model is given by

$$E_{\text{pot}}^{\text{sim}} = \frac{1}{2} k_f \left((\Delta s_1)^2 + (\Delta s_2)^2 \right) + m g l_G \cos(\varphi_P + \varphi_B) \tag{5.2}$$

where k_f is the spring constant, and

$$\Delta s_1 = ||\boldsymbol{p}_P^c - \boldsymbol{p}_P^a||_2 - s_0 = \sqrt{(\boldsymbol{p}_P^c - \boldsymbol{p}_P^a)^\top (\boldsymbol{p}_P^c - \boldsymbol{p}_P^a)} - s_0 \tag{5.3a}$$

$$\Delta s_2 = ||\boldsymbol{p}_P^c - \boldsymbol{p}_P^b||_2 - s_0 = \sqrt{(\boldsymbol{p}_P^c - \boldsymbol{p}_P^b)^\top (\boldsymbol{p}_P^c - \boldsymbol{p}_P^b)} - s_0 \tag{5.3b}$$



(a) Rotary flexible joint quantities.   **(b)** Beam parameters.

Figure 5.2: Schematic overview of geometric relations and conventions for the rotary flexible joint with an example configuration of forces acting due to the first (red) and second (blue) spring with total force (orange) acting on the beam.

are the extensions of first and second spring, respectively. Therein, s_0 denotes the length of a relaxed spring.

b) the control model is given by

$$E_{\text{pot}}^{\text{ctrl}} = \frac{1}{2} k_r \varphi_B^2 + m g l_G \cos(\varphi_P + \varphi_B), \tag{5.4}$$

i.e., the spring-beam system is modeled as a rotational spring, where $k_f \neq k_r$.

The kinetic energy consists of the rotational energy of the platform and beam. In view of the coordinate system $(0x_0, y_0, z_0)$ the kinetic energy reads

$$E_{kin} = \frac{1}{2} J_{P,zz} \left(\dot{\varphi}_P\right)^2 + \frac{1}{2} J_{B,zz} \left(\dot{\varphi}_P + \dot{\varphi}_B\right)^2 \tag{5.5}$$

with the moments of inertia $J_{P,zz}$ and $J_{B,zz}$ for both the simulation and the control model. The energy dissipation is given by

$$R = \frac{1}{2} d_P \left(\dot{\varphi}_P\right)^2, \tag{5.6}$$

where d_P is a positive damping constant. Note that damping for the beam is assumed to be negligible[1]. The DC motor's momentum reads

$$\boldsymbol{\tau} = \begin{bmatrix} 0 & 0 & K_g \tau_m \end{bmatrix}^\top,$$

[1] This assumption also ensures that the system is/remains differentially flat.

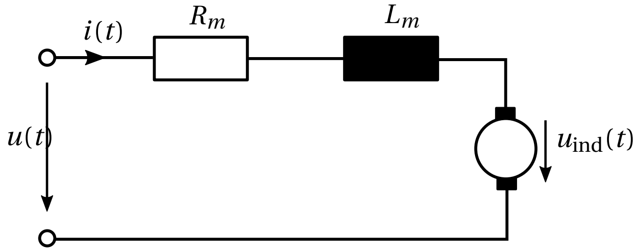


Figure 5.3: Electrical circuit diagram of the motor.

Group	**Name**	**Symbol**	**Value**
Spring(s)	fix-point(s)	l^c	0.0768 m
		l_x^a	0.0251 m
		l_y^a	0.0316 m
	translational stiffness	k_t	112 N/m
	rotational stiffness	k_r	Ex. 5.1 (iii)
		k_r	Ex. 5.1 (iv)
	relaxed length	s_0	0.03 m
Beam	length from joint to middle of load arm	$l_{M,2}$	0.2206 m
	length from joint to center of gravity	l_G	Ex. 5.1 (i)
	moment of inertia	$J_{B,zz}$	Ex. 5.1 (i)
	mass of main arm	$m_{B,1}$	0.064 kg
	mass of load arm	$m_{B,2}$	0.030 kg
	length of main arm	$l_{B,1}$	0.298 m
	length of load arm	$l_{B,2}$	0.156 m
Platform	moment of inertia	$J_{P,zz}$	0.0019 kg m^2
	damping	d_P	0.015 kg m^2/s
Motor	resistance	R_m	2.53 Ω
	current-torque	k_b	0.0077 kg m^2/(A s^2)
	back-emf	k_m	0.0077 V/(rad s)
	gear ratio	K_g	70
	gearbox efficiency	η_g	0.895
	motor efficiency	η_m	0.69

Table 5.1: Parameters of the rotary flexible joint and the DC motor.

where K_g is the gear ratio and

$$\tau_m = k_b \eta_m \eta_g i \tag{5.7}$$

with the motor constant k_b, the motor efficiency coefficient η_m, the gear efficiency coefficient η_g and the motor current i. Considering the mesh equation in Fig. 5.3, Kirchhoff's voltage law yields

$$u = R_m i + L_m \frac{\mathrm{d} i}{\mathrm{d} t} + u_{\text{ind}}. \tag{5.8}$$

The induced motor voltage reads

$$u_{\text{ind}} = k_m \dot{\varphi}_P \tag{5.9}$$

with the constant k_m. The nonlinear dynamics given by the Euler-Lagrange equations read

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_j}L - \frac{\partial}{\partial q_j}L + \frac{\partial}{\partial \dot{q}_j}R = Q_j^{nc}, \qquad j = 1,\dots,n \tag{5.10}$$

with the generalized coordinates $\boldsymbol{q} = [\varphi_P, \varphi_B]^\top$ [3]. Herein, $\dot{\boldsymbol{q}} = [\dot{\varphi}_P \;\; \dot{\varphi}_B]^\top$ are the generalized velocities, R is the Rayleigh dissipation function, $\boldsymbol{Q}^{nc} = [Q_1^{nc},\dots,Q_n^{nc}]^\top$ are the generalized, non-conservative forces, and $L = E_{\mathrm{kin}} - E_{\mathrm{pot}}$ is the Lagrange function. Considering that the movements of the mechanical system are very slow compared to the rate of change of the electrical current, the simplification $\frac{\mathrm{d}}{\mathrm{d}t} i = 0$ is justified [1]. As a result, the electrical current is given by

$$\boxed{i = \frac{u - k_m \dot{\varphi}_P}{R_m}} \tag{5.11}$$

and the generalized, non-conservative forces induced by the DC motor read

$$\boldsymbol{Q}^{nc} = \left(J(\boldsymbol{q})\right)^\top \boldsymbol{\tau} = \begin{bmatrix} \frac{\nu(u - k_m \dot{\varphi}_P)}{R_m} \\ 0 \end{bmatrix} \tag{5.12}$$

with the geometric manipulator Jacobian $J(\boldsymbol{q}) = \frac{\partial \boldsymbol{v}}{\partial \dot{\boldsymbol{q}}}$ [3, Rem. 3.4] and $\nu = K_g k_b \eta_m \eta_g$.

Exercise 5.1 (Mathematical Model).

(i) *Determine the moment of inertia of the entire beam $J_{B,zz}$, which can be approximated as a rod, based on the data given in Tab. 5.1. Use the parallel axis theorem. Furthermore, calculate the value l_G.*

(ii) *Determine the rotary flexible joint's two sets of equations of motion using the Lagrange formalism and the `Matlab` symbolic toolbox. Describe the system(s) in the form*

$$\Sigma : \begin{cases} \dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{g}(\boldsymbol{x})u, & t > t_0, \quad \boldsymbol{x}(t_0) = \hat{\boldsymbol{x}}_0 \\ y = h(\boldsymbol{x}) \end{cases}$$

with the state vector $\boldsymbol{x} = [\varphi_P, \dot{\varphi}_P, \varphi_B, \dot{\varphi}_B]^\top$ with

a) *the potential energy given by* (5.2) *(simulation model).*

b) *the potential energy given by* (5.4) *(control model).*

(iii) *Find an analytic relationship between k_f and k_r. To this end, assume that the RFJ is set up horizontally and express the total (vector-valued) force $\boldsymbol{f}_{total}$ in terms of the points $\boldsymbol{p}_P^a$, $\boldsymbol{p}_P^b$, $\boldsymbol{p}_P^c$, k_f and Δs_i, $i = 1,2$, see also Fig. 5.2. Subsequently, investigate the mapping of this resulting force to the generalized force*

$$Q(q) = J^\top(q)\boldsymbol{f}_{total}$$

acting on the generalized coordinate $q = \varphi_B$, i.e., assume that the generalized force is independent of the platform angle. The geometric manipulator Jacobian is given by $J(q) = \frac{\partial \boldsymbol{v}}{\partial \dot{q}}$, where $\boldsymbol{v} = \dot{\boldsymbol{s}}$ and $\boldsymbol{s} = [x, y, z]^\top$. Since this relationship is nonlinear, linearize $Q(q)$ around $q = \varphi_B = 0$, which gives $Q \approx -k_r\varphi_B$. Is this a conservative or non-conservative generalized force?

(iv) *Compare the value obtained using the analytic relationship from the previous task to an empirically-obtained value of k_r. To this end, calculate k_r using the relationship $k_r \approx \omega_0^2 J_{B,zz}$, where the natural frequency ω_0 can be obtained by means of the power spectral density*

$$S_{yy} = \frac{|FFT(\varphi_B(t))|}{N} \tag{5.13}$$

of the impulse response provided in the file rfj_ impulse_ response.mat.

(v) *Show that $\lambda(\boldsymbol{x}) = y = \varphi_P + \varphi_B$ is a flat output of the system by investigating the relative degree r of the* ***control*** *model [2]. Specify the transformations $\boldsymbol{x} = \boldsymbol{\theta}_x(\lambda, \dot{\lambda}, \ldots, \lambda^{(\beta-1)})$, and $u = \theta_u(\lambda, \dot{\lambda}, \ldots, \lambda^{(\beta-1)}, \lambda^{(\beta)})$.*

(vi) *Implement a Simulink model that compares the simulation and control model of the rotary flexible joint. Use the input*

$$u = \begin{cases} 5\,\mathrm{V}, & \text{for } t \in [t_0, t_0 + 0.2\,\mathrm{s}] \\ 0\,\mathrm{V}, & \text{otherwise,} \end{cases} \tag{5.14}$$

where $t_0 = 1\,\mathrm{s}$. Compare the output trajectories of the simulation and control the model. The required parameters are listed in Table 5.1 and are stored in the param struct.

References

[1] Petar V. Kokotovic, Hassan K. Khalil, and John O'Reilly. *Singular Perturbation Methods in Control: Analysis and Design.* Classics in Applied Mathematics 25. SIAM, 1986 (cit. on p. 5).

[2] T. Meurer. „Nonlinear Control Systems". In: `https://www.control.tf.uni-kiel.de/en/teaching/summer-term/nonlinear-control-systems` (2020) (cit. on pp. 6, 7, 9).

[3] T. Meurer. „Rigid Body Dynamics and Robotics". In: `https://www.control.tf.uni-kiel.de/en/teaching/winter-term/rigid-body-dynamics-robotics` (2020) (cit. on p. 5).

[4] Quanser. *Quansers's Rotary Flexible Joint.* `http://www.quanser.com/Images/Products/494-200-600.jpg`. [Online; accessed 02-December-2016]. 2016 (cit. on p. 2).