1. Hybrid modeling and properties

* what is a hybrid model?

A system which involves the interaction between different kinds of dynamics- continuous dynamics and discrete dynamics.

Hybrid automata = (Autonomous) hybrid systems. 完成

$H = (Q, X, Init, f, Dom, E, G, R)$

Q : finite set of discrete states

$X \subseteq \mathbb{R}^n$: set of continous states

$Init \subseteq Q \times X$: set of admissible initial states

$f: Q \times X \to TX$: the continous flow in each discrete state $q \in Q$: $\dot{x} = f(q, x)$

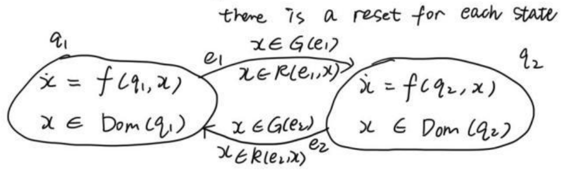
$Dom: Q \to 2^X$: for each $q \in Q$, there is a domin for x : $x \in Dom(q)$

$E \subseteq Q \times Q$: finite set of discrete transitions.

$G: E \to 2^X$: guard conditions at each discrete transition $e = (q, q') \in E$

$R: E \times X \to 2^X$: for each discrete transition $e = (q, q') \in E$,

there is a reset for each state satisifies guard condition $x \in G(e)$



2^X : set of subsets of X

* def of execution

Execution (or evolution) of a hybrid automaton $H = (Q, X, Init, f, Dom, E, G, R)$

is a trajectory $\mathcal{X} = (\tau, q, x)$, $q = \{q_i\}_{i=0}^N$ $x = \{x^i(\cdot)\}_{i=0}^N$

with $\tau \in \mathcal{T}$, $q: I_i \to Q$, $x: I_i \to x^i(t)$ $I_i = [\tau_i, \tau_i']$

such that

- $(q_0(0), x^0(0)) \in Init$ define a initial set
- Discrete evolution

  $\forall i$ $(q_i(\tau_i'), q_{i+1}(\tau_{i+1})) \in E$ q changes

  $x^i(\tau_i') \in G(\underbrace{q_i(\tau_i'), q_{i+1}(\tau_{i+1})}_{e_i})$ still need to check guard condition

  and $x^{i+1}(\tau_{i+1}) \in R(\underbrace{q_i(\tau_i'), q_{i+1}(\tau_{i+1})}_{e_i}, x^i(\tau_i'))$ x resets
- Continuous evolution

  $q_i(\cdot): I_i \to Q$ is constant over $t \in I_i$: $q_i(t) = q_i(\tau_i)$ q constant

  $x^i(\cdot): I_i \to X$ solution of $\frac{dx^i}{dt} = f(q_i(t), x^i(t))$ $\dot{x} = f()$

  over I_i, starting from $x^i(\tau_i)$ need an initial value

  $x^i(t) \in Dom(q_i(t))$ $\forall t \in [\tau_i, \tau_i')$ still need to check domain condition.

* transition states

$\Omega_H(q_0, x_0)$: set of all trajectories of H with $(q_0, x_0) \in Init$ 完成

$\Omega_H^\infty(q_0, x_0)$: set of all infinite trajectories of H with $(q_0, x_0) \in Init$

$\Omega_H = \bigcup_{(q_0, x_0) \in Init} \Omega_H(q_0, x_0)$; $\Omega_H^\infty = \bigcup_{(q_0, x_0) \in Init} \Omega_H^\infty(q_0, x_0)$

Reachable states

$Reach_H = \{(q, x) \in Q \times X : \exists \underset{\text{trajectory}}{\mathcal{X}} \in \Omega_H, (q(\tau_N'), x(\tau_N')) = (q, x), N < \infty\}$

Transition states

when the solution $x(t)$ has jumped from $Dom(q)$

$Trans_H = \{(q, x) \in Q \times X : \forall \varepsilon > 0, \exists t \in [0, \varepsilon) : \phi(t, x) \notin Dom(q)\}$

solution of the differential equation associated with q and initial condition x.

$\bigcup_{q \in Q} \{q\} \times Dom(q)^c \subseteq Trans_H$

Q X complement set

if $Dom(q)$ is a closed set, Trans may also contain pieces of the boundary.

x(t) is about to jump

* def of non-blocking
- what is blocking?
* def of determination

Def: H is non-blocking if for $\forall$ initial states $(q_0, x_0) \in Init$, s.t.
$\Omega_H^{\infty}(q_0, x_0) \neq \phi$, i.e. $\exists$ an infinite execution stating at (q_0, x_0)

Def: H is deterministic if for $\forall$ initial states $(q_0, x_0) \in Init$, s.t.
there $\exists$ at most one maximal execution $\mathcal{X}$

Theorem: H is non-blocking if
(#) $\forall (\hat{q}, \hat{x}) \in Reach \cap Trans$, $\exists \hat{q}' \in Q : (\hat{q}, \hat{q}') \in E$ and $\hat{x} \in G(\hat{q}, \hat{q}')$
explanation: for $\forall$ reachable transition states $(\hat{q}, \hat{x})$, $\exists$ an exection from it to $(\hat{q}', \hat{x}')$ and $\hat{x}$ satisfies the guard condition for $e = (\hat{q}, \hat{q}')$

if H is deterministic, then it is non-blocking iff condition (#) holds.

Proof: blabla

Theorem: H is deterministic iff
- if $(q, q') \in E$ and $x \in G(q, q')$ then $(q, x) \in Trans_H$ ②
- if $(q, q') \in E$ and $(q, q'') \in E$ with $q' \neq q''$ ②
  then $x \notin G(q, q') \cap G(q, q'')$
- if $(q, q') \in E$ and $x \in G(q, q')$ then $R(q, q', x)$ returns only one element. ①

Hint: nondeterminism is due to ① non-deterministic resets of x
② non-deterministic transitions

* execution
- existence of execution
- unique of execution

H accepts a unique infinite execution for $\forall$ inital state
if H is non-blocking & deterministic. ☆

Theorem: Existence of solutions
If $f: \mathbb{R}^n \to \mathbb{R}^n$ is continuous
$\Rightarrow \forall x_0 \in \mathbb{R}^n$, $\exists$ at least one solution with $x(0) = x_0$ in $[0, \varepsilon)$

Theorem: Uniqueness
If $f: \mathbb{R}^n \to \mathbb{R}^n$ is lipschitz continuous
$\Rightarrow \forall x_0 \in \mathbb{R}^n$, $\exists$ unique solution with $x(0) = x_0$ in $[0, \varepsilon)$

If $f: \mathbb{R}^n \to \mathbb{R}^n$ is globally lipschitz continuous,
$\Rightarrow \forall x_0 \in \mathbb{R}^n$, $\exists$ unique solution with $x(0) = x_0$ in $[0, \infty)$

* what is model checking

Automatically analyze the properties of a system by exploring the state space. Properties: stability, safety, liveness, reachability, non-blocking, deterministic, etc.

* def of safety
* def of liveness

Safety property.
the state (q, x) always remains in a set of states $F \subset Q \times X$ "not bad set"
temporal logic $\square (q, x) \in F$

Liveness property.
the state (q, x) certainly reaches a set of states $\bar{F} \subset Q \times X$ eventually "good set"
temporal logic $\lozenge (q, x) \in \bar{F}$

- safety checking

iff Reach ⊂ F (F is a safety specification: (q,x) is always belongs to F)

* zeno

An execution time $\tau_\infty(\mathcal{X})$ of a trajectory $\mathcal{X}$ is

$\tau_\infty(\mathcal{X}) = \sum_{i=0}^{N} (\tau_i' - \tau_i)$

a trajectory $\mathcal{X}$ is

Finite if $N<\infty$ and $I_N = [\tau_N, \tau_N']$

Infinite if $N=\infty$ or $\tau_\infty(\mathcal{X})=\infty$

Zeno if $N=\infty$ but $\tau_\infty(\mathcal{X})<\infty$

Maximal if it is not the strict prefix of any other execution.

example: bouncing ball

2. Symbolic model

* what is symbolic system?

Definition 1. *[3] A transition system is a tuple*

$$T = (X, X_0, U, \longrightarrow, X_m, Y, H),$$

consisting of

- *a set of states X;* ✓
- *a set of initial states $X_0 \subseteq X$;* ✓
- *a set of inputs U;* ☆
- *a transition relation $\longrightarrow \subseteq X \times U \times X$;* ✓
- *a set of marked states $X_m \subseteq X$;* ✓
- *a set of outputs Y and* ☆
- *an output function $H : X \to Y$.* ☆

– *symbolic/finite, if X and U are finite sets;* number of elements is finite.

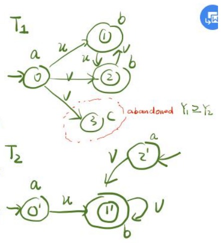
* def of bisimulation

Definition 1. *[3, 4] Let $T_i = (X_i, X_{0,i}, U_i, \underset{i}{\longrightarrow}, X_{m,i}, Y_i, H_i)$ $(i = 1, 2)$ be transition systems ~~with the same output sets $Y_1 = Y_2$~~. A relation*

$Y_2 \subseteq Y_1$

$$\mathcal{R} \subseteq X_1 \times X_2$$

is said to be a simulation relation from T_1 to T_2 if it satisfies the following conditions:



i) for every $x_1 \in X_{0,1}$ and $x_2 \in X_2$, such that $(x_1, x_2) \in \mathcal{R}$, it holds that $x_2 \in X_{0,2}$; 如果x_1在$X_{0,1}$里，$(x_1,x_2)\in R$，那么x_2在$X_{0,2}$里.

ii) for every $x_1 \in X_{m,1}$ and $x_2 \in X_2$, such that $(x_1, x_2) \in \mathcal{R}$, it holds that $x_2 \in X_{m,2}$; 如果x_1在$X_{m,1}$里，$(x_1,x_2)\in R$，那么x_2在$X_{0,m}$里.

iii) $\forall (x_1, x_2) \in \mathcal{R}$, $H_1(x_1) = H_2(x_2)$; 输出相等.

iv) $\forall (x_1, x_2) \in \mathcal{R}$ if $x_1 \xrightarrow[1]{u_1} x_1'$ then there exists $x_2 \xrightarrow[2]{u_2} x_2'$ such that $(x_1', x_2') \in \mathcal{R}$. 如果$(x_1,x_2)\in R$，$x_1 \to x_1'$，那么$\exists\, x_2 \to x_2'$，$(x_1',x_2')\in R$.

$R = \{(0',0),(2',0),(1',1),(1',2)\}$

Transition system T_1 is simulated by transition system T_2, denoted 注意：输入可以不同.

$$T_1 \preceq T_2,$$

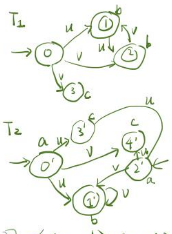
if there exists a simulation relation from T_1 to T_2.

Intuitively, if T_2 simulates T_1 then the behavior of T_2 contains the behavior of T_1. Moreover,

$H_1(x_1) = H_2(x_2)$

The converse implication in the result above is clearly not true in general. We now introduce bisimulation equivalence:

Definition 2. *[3, 4] Let $T_i = (X_i, X_{0,i}, U_i, \underset{i}{\longrightarrow}, X_{m,i}, Y_i, H_i)$ $(i = 1, 2)$ be transition systems with the same output sets $Y_1 = Y_2$. A relation*



$$\mathcal{R} \subseteq X_1 \times X_2$$

is said to be a bisimulation relation between T_1 and T_2 if it satisfies the following conditions:

- *$\mathcal{R}$ is simulation relation from T_1 to T_2;*
- *$\mathcal{R}^{-1}$ is a simulation relation from T_2 to T_1, where $\mathcal{R}^{-1} \subseteq X_2 \times X_1$ is the inverse relation of $\mathcal{R}$, defined by*

$$(x_2, x_1) \in \mathcal{R}^{-1} \iff (x_1, x_2) \in \mathcal{R}.$$

$R = \{(0,0'),(0,2'),(1,1'),(3,3'),(3,4'),(2,1')\}$

Transition systems T_1 and T_2 are bisimilar, denoted

$$T_1 \cong T_2,$$

if there exists a bisimulation relation $\mathcal{R}$ between T_1 and T_2.

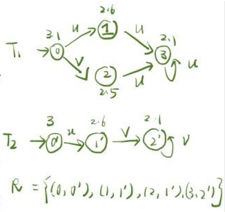
Intuitively, T_1 and T_2 are bisimilar if the behavior of T_1 is the same as the behavior of T_2. Moreover,

- why bisimulation is important

Bisimulations preserve all properties that can be expressed in temporal logic, such like reachability, non-blocking, aliveness

* approximated bisimulation

Proposition 7. *If T_1 and T_2 are deterministic then $T_1 \preceq T_2$ if and only if $T_1 \preceq^{\text{alt}} T_2$.*



The notion of simulation and bisimulation relations and its alternating variants, we have introduced so far, are also called 'exact' because they require the outputs of two states x_1 and x_2 in the relation to be exactly the same, see condition iii) of Definition 1. We now extend the notion above to an approximating setting where condition

$$H_1(x_1) = H_2(x_2)$$

is replaced by

$$\mathbf{d}(H_1(x_1), H_2(x_2)) \leq \mu,$$ it is an approximate simulation relation from T_1 to T_2 with accuracy μ.

where $\mathbf{d}$ is a metric placed on the output sets of the transition systems involved and $\mu \in \mathbb{R}_0^+$ is a desired accuracy.

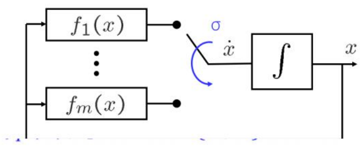
3. Stability of switching system

* what is switching system

- a family of systems

$$\dot{x} = f_p(x), \ p \in \mathcal{P}$$

- a rule that orchestrates the switching between them



σ: a rule that defines ~~the~~ switching between them.

- what is the difference from hybrid automaton

1.switched systems can be seen as a higher-level abstraction of hybrid automata (details of the discrete behavior neglected)

2.simpler to describe but with more solutions than the original hybrid automata (conservative analysis results)

* G.U.A.S.

GLOBAL UNIFORM ASYMPTOTIC STABILITY (GUAS)

$$\boxed{\dot{x} = f_\sigma(x)}$$

$$f_p(0) = 0, \ p \in \mathcal{P} = \{1, 2, \ldots, m\}$$

The equilibrium x=0 is GUAS if

function decreasing to zero

$$\|x(t)\| \leq \beta(\|x(0)\|, t), \forall t \geq 0, \forall x(0), \forall \sigma$$

increasing function

Uniform w.r.t. σ

Remark:
if the equilibrium x=0 is not GAS for one of the systems, then it cannot be GUAS for the switched system

- for linear systems, the N.S. condition of G.E.S

Therom: the switched linear system is GUES (GUAS) if and only if it is locally attractive (all trajectories → 0 with $t \to \infty$)

for linear system: locally attractive ⟺ GUES.

* σ(t) σ(x)

$\sigma : [0, \infty) \to \mathcal{P}$ 外生的. (exogenous) switching signal

- piecewise constant function of time
- σ(t) specifies the system that is active at time t

$\sigma : X \to \mathcal{P}$ 内生的 (endogenous) switching signal

- the state space X is partitioned into operating regions, each one associated to a system
- σ(x) specifies the which system that is active when the state is x

* common Lyapunov function

The family of systems

$$\dot{x} = f_p(x), \ p \in \mathcal{P} = \{1, 2, \ldots, m\}$$

share a radially unbounded common Lyapunov function at x=0 if there exists a continuously differentiable $V \in C^1$ function V such that

$$V(x) > 0, \forall x \neq 0, V(0) = 0$$

$$\|x\| \to \infty \Rightarrow V(x) \to \infty$$

$$\boxed{\frac{dV}{dx}(x) f_p(x) < 0, \forall x \neq 0, \ \forall p \in \mathcal{P}}$$

If all systems in the family

$$\dot{x} = f_p(x), \ p \in \{1, 2, \ldots, m\}$$

share a radially unbounded common Lyapunov function at x=0, then, the equilibrium x=0 is GUAS.

- common quadratic Lyapunov function

$$\boxed{\dot{x} = A_\sigma x}$$ Linear

If there exists $P = P^T > 0$ such that

$$PA_p + A_p^T P < 0, \forall p \in \mathcal{P} = \{1, 2, \ldots, m\}$$

then, the equilibrium x=0 is GUAS.

$PA_p + A_p^T P = -Q.$

Proof.

$V(x) = x^T P x$ is a radially unbounded common Lyapunov function at x=0.

* commuting matrix

COMMUTING STABLE MATRICES => GUAS

$$\boxed{\dot{x} = A_\sigma x}$$ switched linear system

$$\mathcal{P} = \{1, 2\} \ A_1 A_2 = A_2 A_1$$ (A_1 and A_2 commute)

$[A_1, A_2]$ lie bracket
$= A_1A_2 - A_2A_1$

$\sigma = 1$ | $\sigma = 2$ | $\sigma = 1$ | $\sigma = 2$ | $\cdots$ $\to t$
s_1 | t_1 | s_2 | t_2

① ⇄ ②

$$x(t) = e^{A_2 t_k} e^{A_1 s_k} \cdots e^{A_2 t_1} e^{A_1 s_1} x(0)$$

$$= e^{A_2 (t_k + \ldots + t_1)} e^{A_1 (s_k + \ldots + s_1)} x(0) \to 0$$

$$\mathcal{P} = \{1, 2\} \ A_1 A_2 = A_2 A_1$$

$\exists$quadratic common Lyapunov function: $V(x) = x^T P_2 x$

$$P_1 A_1 + A_1^T P_1 = -I$$

$$P_2 A_2 + A_2^T P_2 = -P_1$$

* triangular form

$$\boxed{\dot{x} = A_\sigma x}$$

$$\mathcal{P} = \{1, 2\}, \ X = \Re^2$$

$$\dot{x}_1 = \lambda_{1,\sigma} x_1 + b_\sigma x_2$$
$$\dot{x}_2 = \qquad \lambda_{2,\sigma} x_2$$

$\lambda_{1,\sigma}, \lambda_{2,\sigma} < 0$

$$\dot{x}_2 = \lambda_{2,\sigma} x_2 \to |x_2(t)| \leq e^{\max_p \lambda_{2,p} t} |x_2(0)|$$

the eigenvalue closest to origin.

$$\boxed{\dot{x}_1 = \lambda_{1,\sigma} x_1} + \boxed{b_\sigma x_2}$$

exponentially stable system

exponentially decaying perturbation

衰减 $x_2(t) \to 0$ as $t \to \infty$

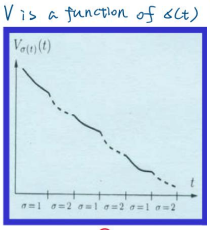
* multiple Lyapunov function

when there is no common V

P={1,2}

① If V_σ is continuous then asy. stability for switched system.

V is a function of $\sigma(t)$



①

②

Theorem: Let the systems of the family be GAS and let V_p *with* $p \epsilon P$, *the corresponding family of radially unbounded Lyapunov functions .*

If for all pairs of switching times (t_i, t_j), $i<j$, *such that* $\sigma(t_i)=\sigma(t_j)=p \epsilon P$ *and* $\sigma(t_k) \neq p$ *for* $t_i<t_k<t_j$,

$\sigma=1$ 跳 $\sigma=1$ 时

$$V_p(x(t_j)) - V_p(x(t_i)) < 0$$

⇓

then the switching system is GUAS.

* dwell time

'Dwell time', is the minimum value of the time intervals between consecutive time instances in which switching occurs. It is shown that a sufficiently large dwell time can guarantee the stability of the switched system.

STABILITY UNDER SLOW SWITCHING

$$\boxed{\dot{x} = A_\sigma x}$$

Hurwitz matrices $A_q,\ q \in Q = \{1, 2\}$

$\sigma = 1 \quad \sigma = 2 \quad \sigma = 1 \quad \sigma = 2 \quad \cdots$

$s_1 \quad t_1 \quad s_2 \quad t_2 \quad t$

The switching intervals satisfy $t_i, s_i \geq \tau_D$

$$\left.\begin{array}{l} x(t) = e^{A_2 t_k} e^{A_1 s_k} \ldots e^{A_2 t_1} e^{A_1 s_1} x(0) \\ \| e^{A_i \Delta t} \| \leq e^{-\lambda_0 \tau_D + \log\mu} \leq e^{-\lambda \tau_D} < 1 \end{array}\right\} \quad \boxed{\tau_D \geq \frac{\log\mu}{\lambda_0 - \lambda}}$$

τ_D取的越大，
$\lambda_0 - \lambda$ 越小
λ 越大 → 切换系统收敛的越好。

$\exists \lambda \in (0, \lambda_0)$

if $-\lambda\tau_D \geq -\lambda_0\tau_D + \log\mu$

Remark: in switched observers, we also have to take min. dwell time into account, in order to make sure estimation error converges to zero for any e(0) or σ(t).

4. Observers

* current location observability

A FSM is alive if it has no blocking states

$\Leftrightarrow \forall q \in Q,\ \exists \delta \in \phi(q)$ and $q' \in Q$, $q \in \varphi(q, \delta)$

An alive FSM is said to be current-state observable if $\exists K$ s.t.

for $\forall i > K$

– for $\forall$ unknown $q(0)$

– and $\forall$ input sequence $\delta(1), \delta(2), \cdots \delta(i)$ can be unavailable

the state $q(i)$ can be determined $\hat{q}(i) = q(i)$

from the observation (output) sequence $\psi(1), \psi(2), \cdots \psi(i)$

Theorem: an alive FSM is current state observable iff there exists a non-empty subset E_0 of singletons in the observer s.t.

- E_0 is invariant
- all cycles are contained in E_0

valid only if ψ does not contain ε.
↳ can be associated to every discrete transition.

* persistent states
* design procedure

Definition 4. *[1] A control system Σ is incrementally globally asymptotically stable (δ–GAS) if it is forward complete and there exist a $\mathcal{KL}$ function β such that for any $t \in \mathbb{R}_0^+$, any $x, y \in \mathbb{R}^n$ and any $\mathbf{u} \in \mathcal{U}$ the following condition is satisfied:*

$$|\mathbf{x}(t, x, \mathbf{u}) - \mathbf{x}(t, y, \mathbf{u})| \leq \beta(|x - y|, t). \tag{1}$$

Function V is called a δ–GAS Lyapunov function for Σ, if there exist $\kappa \in \mathbb{R}^+$ and $\mathcal{K}_\infty$ functions α_1 and α_2 such that:

(i) for any $x, y \in \mathbb{R}^n$

$$\alpha_1(|x - y|) \leq V(x, y) \leq \alpha_2(|x - y|);$$

(ii') for any $x, y \in \mathbb{R}^n$ and any $u \in U$

$$\frac{\partial V}{\partial x} f(x, u) + \frac{\partial V}{\partial y} f(y, u) < -\alpha_3(|x - y|),$$