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An adaptive observer for joint estimation of states and parameters in both state and output equations

Xiuliang Li[1], Qinghua Zhang[2,*,†] and Hongye Su[1]

[1]*Institute of Advanced Process Control, Zhejiang University, 310027 Zhejiang, People's Republic of China*
[2]*INRIA, Campus de Beaulieu, 35042 Rennes, France*

SUMMARY

An adaptive observer is a recursive algorithm for joint state–parameter estimation of parameterized state-space systems. Previous works on globally convergent adaptive observers consider unknown parameters either in state equations or in output equations, but not in both of them. In this paper, a new adaptive observer is designed for linear time-varying systems with unknown parameters in both state and output equations. Its global convergence for simultaneous estimation of states and parameters is formally established under appropriate assumptions. A numerical example is presented to illustrate the performance of this adaptive observer. Copyright © 2011 John Wiley & Sons, Ltd.

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KEY WORDS: adaptive observer; state and parameter estimation; linear time-varying system; fault detection and isolation

1. INTRODUCTION

Joint estimation of states and parameters in state-space systems is of practical importance for fault diagnosis and for fault tolerant control. Recursive algorithms designed for this purpose are usually known as *adaptive observers*. Some early works on this subject can be found in [1–4]. These results are essentially for linear time invariant (LTI) systems, though some of them have been presented for nonlinear systems that can be exactly linearized by coordinate change and output injection. More recently, adaptive observers for multi-input-multi-output (MIMO) linear time-varying (LTV) systems have been developed in [5, 6]. Some results on nonlinear systems have also been reported in [5, 7–11]. In these cited references, the unknown parameters are all assumed to be located in state equations. As a matter of fact, there is an extra difficulty for estimating parameters in output equations, since the output feedback term used in most adaptive observers then depends on unknown parameters in this case. Recently, adaptive observers for unknown parameters in output equations have been studied, both for linear systems [12] and for nonlinear systems [13]. Nevertheless, all these works consider unknown parameters *either* in state equations *or* in output equations. To our knowledge, no globally convergent algorithm has been developed for joint estimation of states and parameters in *both* state and output equations. The purpose of this paper is thus to fill this gap, by presenting such an adaptive observer for MIMO LTV systems.

The design of the new adaptive observer of this paper follows to some extent the approach already used in [6, 12]. Although only linear systems are considered in this paper, this new result

*Correspondence to: Qinghua Zhang, INRIA, Campus de Beaulieu, 35042 Rennes, France.
†E-mail: zhang@irisa.fr

may serve as the basis for the study of nonlinear systems, like the extension of the results of [6, 12] to nonlinear systems as reported in [10, 11, 13].

The paper is organized as follows. The considered problem is formulated in Section 2. The new adaptive observer is designed and analyzed in Section 3. A numerical example is given in Section 4. Finally, some concluding remarks are drawn in Section 5.

2. THE CONSIDERED JOINT ESTIMATION PROBLEM

The class of systems considered in this paper for joint estimation of states and parameters is in the form of

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + \Phi(t)\theta \tag{1a}$$

$$y(t) = C(t)x(t) + \Psi(t)\theta \tag{1b}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^l$ and $y(t) \in \mathbb{R}^m$ are, respectively, the state, input and output of the system, $A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times l}$, $C(t) \in \mathbb{R}^{m\times n}$, $\Phi(t) \in \mathbb{R}^{n\times p}$ and $\Psi(t) \in \mathbb{R}^{m\times p}$ are known time-varying matrices, $\theta \in \mathbb{R}^p$ is an unknown constant parameter vector (unless otherwise specified in this paper), and the overhead dot denotes the derivative with respect to the time t. The adaptive observer proposed in this paper is for the purpose of joint estimation of the state vector $x(t)$ and the parameter vector θ, from the known time-varying matrices $A(t), B(t), C(t), \Phi(t), \Psi(t)$, and from measured input–output variables $u(t), y(t)$.

An obviously easy approach to the joint estimation of $x(t)$ and θ for system (1) is to consider the *extended system*

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\theta}(t) \end{bmatrix} = \begin{bmatrix} A(t) & \Phi(t) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \theta(t) \end{bmatrix} + \begin{bmatrix} B(t) \\ 0 \end{bmatrix} u(t) \tag{2a}$$

$$y(t) = [C(t) \;\; \Psi(t)] \begin{bmatrix} x(t) \\ \theta(t) \end{bmatrix} \tag{2b}$$

Indeed this is still a linear system, the *Kalman filter* is thus applicable to the joint estimation of $x(t)$ and θ. Compared to this approach, the advantages of adaptive observers in such situations reside both in their simpler convergence condition and in their more efficient numerical implementation, as explained in Appendix A1 of this paper. Another interest of such adaptive observers is their possible extensions to nonlinear systems. The *globally* convergent *nonlinear* adaptive observers presented in [10, 11, 13] are essentially extensions of the linear results presented in [6, 12].

By considering the class of parametric state-space systems (1), it is possible to design globally convergent adaptive observers based on *linear* adaptive estimation techniques only. When unknown parameters are involved in the terms $A(t)x(t)$ and $C(t)x(t)$, unless the system model can be rewritten in the form of (1) after some transformations, it is generally necessary to resort to nonlinear adaptive estimation techniques, like those of [11, 14], in order to design globally convergent algorithms. Nevertheless, the result of this paper may serve as the basis for the study of more general adaptive estimation problems.

The case without the term $\Psi(t)\theta$ in the output equation has been addressed in [6], so has been the case without the term $\Phi(t)\theta$ in the state equation in [12]. Most other similar works consider unknown parameters in state equations only, as in [2–5].

For applications to fault diagnosis, typically the term $\Phi(t)\theta$ in the state equation and the term $\Psi(t)\theta$ in the output equation are used to model actuator faults and sensor faults, respectively. The adaptive observer presented in this paper extends the results of previous works by simultaneously taking into account both actuator and sensor faults.

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Remark 1

In system (1), apparently the *same* parameter vector θ appears in *both* state and output equations. In fact, this formulation can also cover *different* parameters in the two equations. Let

$$\theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix}, \quad \Phi = [\Phi_1 \ \ \Phi_2 \ \ 0], \quad \Psi = [0 \ \ \Psi_2 \ \ \Psi_3]$$

with appropriate sizes of $\theta_1, \theta_2, \theta_3, \Phi_1, \Phi_2, \Psi_2, \Psi_3$ and of the zero blocks, so that system (1) becomes

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + \Phi_1(t)\theta_1 + \Phi_2(t)\theta_2 \tag{3a}$$

$$y(t) = C(t)x(t) + \Psi_2(t)\theta_2 + \Psi_3(t)\theta_3 \tag{3b}$$

It is thus possible, within the formulation of system (1), to deal with different parameter vectors θ_1 and θ_3 in the two equations, as well as the common parameter vector θ_2 shared by the two equations. For notation simplicity, the adaptive observer of this paper will be formulated for system (1), though it can also be applied to system (3) following this remark.

Remark 2

For the design of adaptive observers, the so-called *state affine* systems [15, 16] can be treated as LTV systems. Such systems correspond to system (1) where the time-dependent matrix $A(t)$ is replaced by $A(u(t), y(t))$. As a matter of fact, replacing each time-dependent matrix in system (1) by a matrix depending on any known variables does not increase any difficulty to the application of the adaptive observer designed in this paper.

3. THE ADAPTIVE OBSERVER AND ITS CONVERGENCE ANALYSIS

In this section, the new adaptive observer is first derived following an heuristic procedure, before formally establishing its global convergence and analyzing its robustness to noises. A variant algorithm involving an exponential forgetting factor is also introduced.

3.1. Deriving the adaptive observer

Instead of directly presenting the equations of the adaptive observer for system (1), we choose to first present the heuristic steps which have been followed to derive this adaptive observer, starting from the classical state observer.

For notation simplicity, we do not explicitly write the dependence on t of the variables, unless for emphasizing this dependence. Nevertheless, it can be checked that all the results derived below are valid for the time-varying system (1).

First assume that in system (1) the true parameter vector θ is known and let us consider the simpler state estimation problem. In this case the Luenberg-like state observer is

$$\dot{\hat{x}} = A\hat{x} + Bu + \Phi\theta + K(y - C\hat{x} - \Psi\theta) \tag{4}$$

where $\hat{x}(t)$ is the state estimate, and it is a classical problem to design the (time varying) gain matrix $K(t)$ to ensure the convergence of this observer.

Now, consider again system (1), but with an unknown parameter vector θ. Then in the state estimation equation (4) the *unknown* θ has to be replaced by its estimate. The actual algorithm for the computation of the parameter estimate will be derived later. For the moment, let the parameter estimate be denoted by $\hat{\theta}(t)$. The state estimation equation then becomes

$$\dot{\hat{x}} = A\hat{x} + Bu + \Phi\hat{\theta} + K(y - C\hat{x} - \Psi\hat{\theta}) + \omega(t) \tag{5}$$

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where an extra term $\omega(t)$ has been added in order to compensate the difference between the true parameter vector θ and its estimate $\hat{\theta}(t)$. The detail of $\omega(t)$ and its importance will become clear in the following steps.

Define the state and parameter estimation errors

$$\tilde{x}(t) = x - \hat{x}(t)$$

$$\tilde{\theta}(t) = \theta - \hat{\theta}(t)$$

and notice that, for the constant parameter vector θ, its derivative $\dot{\theta} = 0$. Then Equations (1) and (5) lead to

$$\dot{\tilde{x}} = (A - KC)\tilde{x} + \Phi\tilde{\theta} - K\Psi\tilde{\theta} - \omega \tag{6}$$

Define the linear combination of $\tilde{x}(t)$ and $\tilde{\theta}(t)$

$$\boxed{\tilde{z}(t) = \tilde{x}(t) - \Upsilon(t)\tilde{\theta}(t)} \tag{7}$$

where the time-varying matrix $\Upsilon(t) \in \mathbb{R}^{n\times p}$ remains to be specified. Then it is straightforward to compute

$$\dot{\tilde{z}} = (A - KC)\tilde{z} + [(A - KC)\Upsilon + \Phi - K\Psi - \dot{\Upsilon}]\tilde{\theta} - \Upsilon\dot{\tilde{\theta}} - \omega \tag{8}$$

Now, the two variables $\Upsilon(t)$ and $\omega(t)$ can be chosen to simplify the error equation (8). One obvious choice is

$$\dot{\Upsilon} = (A - KC)\Upsilon + \Phi - K\Psi \tag{9}$$

and

$$\begin{aligned} \omega &= -\Upsilon\dot{\tilde{\theta}} \\ &= \Upsilon\dot{\hat{\theta}} \end{aligned} \tag{10}$$

so that Equation (8) is simplified to

$$\dot{\tilde{z}} = (A - KC)\tilde{z} \tag{11}$$

After inserting the choice of $\omega(t)$ made in (10) into (5), the state estimation equation becomes

$$\dot{\hat{x}} = A\hat{x} + Bu + \Phi\hat{\theta} + K(y - C\hat{x} - \Psi\hat{\theta}) + \Upsilon\dot{\hat{\theta}} \tag{12}$$

If the gain matrix $K(t)$ has been designed to stabilize Equation (11) so that $\tilde{z}(t) \to 0$ as $t \to \infty$, it is then clear that a linear combination of the two estimation errors $\tilde{x}$ and $\tilde{\theta}$ tends to zero. In order to further ensure that both $\tilde{x}$ and $\tilde{\theta}$ tend to zero, the parameter estimation equation needs to be detailed.

Let

$$\hat{y}(t) = C(t)\hat{x}(t) + \Psi(t)\hat{\theta}(t) \tag{13}$$

be the output estimate based on the state and parameter estimates. Assume that the parameter estimation equation is linear in the output error $(y - \hat{y})$, then it may be in the form of

$$\dot{\hat{\theta}} = M(t)(y - \hat{y}) \tag{14}$$

where $M(t) \in \mathbb{R}^{p\times m}$ is a time-varying matrix to be specified.

Remind that $\tilde{\theta}(t) = \theta - \hat{\theta}(t)$ and $\dot{\theta} = 0$, then

$$\begin{aligned} \dot{\tilde{\theta}} &= -\dot{\hat{\theta}} \\ &= -M(y - \hat{y}) \end{aligned}$$

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Following Equations (1b) and (13), this error equation can be written as

$$\begin{aligned}\dot{\tilde{\theta}} &= -M(Cx+\Psi\theta-C\hat{x}-\Psi\hat{\theta})\\ &= -M(C\tilde{x}+\Psi\tilde{\theta})\end{aligned}$$

Substitute $\tilde{x}$ with the equality

$$\tilde{x}=\tilde{z}+\Upsilon\tilde{\theta} \tag{15}$$

which comes from (7), then

$$\dot{\tilde{\theta}}=-MC\tilde{z}-M(C\Upsilon+\Psi)\tilde{\theta} \tag{16}$$

As the previous steps have been made such that $\tilde{z}\to 0$, now it is essential to choose $M(t)$ ensuring the stability of the homogeneous part of Equation (16), so that the parameter estimation error $\tilde{\theta}$ governed by Equation (16) tends to zero. Such a choice may be

$$M(t)=\Gamma(C\Upsilon+\Psi)^{\mathrm{T}} \tag{17}$$

with some *constant* positive-definite matrix $\Gamma\in\mathbb{R}^{p\times p}$, as the homogeneous part of the error equation (16) then writes

$$\dot{\xi}=-\Gamma(C\Upsilon+\Psi)^{\mathrm{T}}(C\Upsilon+\Psi)\xi \tag{18}$$

and its stability can be ensured by assuming some appropriate persistent excitation of the considered system. Equation (16) then becomes

$$\dot{\tilde{\theta}}=-\Gamma(C\Upsilon+\Psi)^{\mathrm{T}}[C\tilde{z}+(C\Upsilon+\Psi)\tilde{\theta}] \tag{19}$$

Insert the choice of $M(t)$ made in (17) into (14) and substitute $\hat{y}$ with (13), the parameter estimation equation then becomes

$$\dot{\hat{\theta}}=\Gamma(C\Upsilon+\Psi)^{\mathrm{T}}(y-C\hat{x}-\Psi\hat{\theta}) \tag{20}$$

The adaptive observer composed of the auxiliary equation (9), the state estimation equation (12) and the parameter estimation equation (20) is then completed. For a better readability, let us assemble these equations of the new adaptive observer

$$\dot{\Upsilon}=(A-KC)\Upsilon+\Phi-K\Psi \tag{21a}$$

$$\dot{\hat{x}}=A\hat{x}+Bu+\Phi\hat{\theta}+[K+\Upsilon\Gamma(C\Upsilon+\Psi)^{\mathrm{T}}](y-C\hat{x}-\Psi\hat{\theta}) \tag{21b}$$

$$\dot{\hat{\theta}}=\Gamma(C\Upsilon+\Psi)^{\mathrm{T}}(y-C\hat{x}-\Psi\hat{\theta}) \tag{21c}$$

where it is easy to recognize that part of the last term of (21b) corresponds to the term $\Upsilon\dot{\hat{\theta}}$ of (12).

3.2. Assumptions and convergence analysis

The global convergence analysis of the adaptive observer (21) for system (1) will be based on the following assumptions.

Assumption 1
The matrices $A(t), B(t), C(t), \Phi(t), \Psi(t)$ in system (1) are all bounded and piecewise continuous.

Assumption 2
Assume that the matrix pair $(A(t), C(t))$ in system (1) is such that a bounded time-varying matrix $K(t)\in\mathbb{R}^{n\times m}$ can be designed such that the homogeneous system

$$\dot{\tilde{z}}(t)=[A(t)-K(t)C(t)]\tilde{z}(t) \tag{22}$$

is exponentially stable.

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Assumption 3
Let $\Omega(t)\in\mathbb{R}^{m\times p}$ be a matrix of signals generated by linearly filtering $\Phi(t)$ and $\Psi(t)$ through the state-space equations

$$\begin{aligned}\dot{\Upsilon}(t) &= [A(t)-K(t)C(t)]\Upsilon(t)+\Phi(t)-K(t)\Psi(t)\\ \Omega(t) &= C(t)\Upsilon(t)+\Psi(t)\end{aligned} \tag{23}$$

where $\Upsilon(t)\in\mathbb{R}^{n\times p}$ and $\Omega(t)\in\mathbb{R}^{m\times p}$ are, respectively, the state and the output of the filter. Assume that $\Phi(t)$ and $\Psi(t)$ are persistently exciting, so that the filtered matrix of signals $\Omega(t)$ satisfies, for some positive constants α, T and for all $t\geqslant t_0$, the inequality

$$\int_t^{t+T} \Omega^T(\tau)\Omega(\tau)\mathrm{d}\tau\geqslant\alpha I_p \tag{24}$$

where I_p is the $p\times p$ identity matrix.

Assumption 1 guarantees well-defined trajectories of system (1).

Assumption 2 ensures the existence of an exponentially convergent state observer (4), characterized by the gain matrix $K(t)$, for system (1) when the parameter vector θ is known. If the matrix pair $(A(t), C(t))$ is *uniformly completely observable*, then the Kalman gain $K(t)$ can be used for this purpose [17]. More precisely, let $P(t)\in\mathbb{R}^{n\times n}$ be the solution of the Riccati equation

$$\dot{P}(t)=A(t)P(t)+P(t)A^{\mathrm{T}}(t)-P(t)C^{\mathrm{T}}(t)R^{-1}(t)C(t)P(t)+Q(t) \tag{25}$$

where $Q(t)\in\mathbb{R}^{n\times n}$ and $R(t)\in\mathbb{R}^{m\times m}$ are two positive-definite design matrices (usually interpreted as the covariance matrices of the noises in the state and output equations), then

$$K(t)=P(t)C^{\mathrm{T}}(t)R^{-1}(t) \tag{26}$$

Assumption 3 is a persistent excitation condition, typically required for parameter estimation problems.

Now, the main convergence result of this paper can be stated.

Theorem 1
Let $\Gamma\in\mathbb{R}^{p\times p}$ be any symmetric positive-definite matrix. Under Assumptions 1–3, the system of ordinary differential equations (21) constitutes a global exponential adaptive observer for system (1). In other words, for any initial values $x(t_0), \hat{x}(t_0), \hat{\theta}(t_0), \Upsilon(t_0)$ and any constant value of θ, the state and parameter estimation errors $x(t)-\hat{x}(t)$ and $\theta-\hat{\theta}(t)$ tend to zero exponentially fast when $t\to\infty$.

Proof of Theorem 1
The main elements for the proof of this result have already been given in Section 3.1 without formally mentioning Assumptions 1–3. Let us shortly summarize the proof for more clarity. Assumption 2 ensures the exponential convergence of (11), thus $\tilde{z}(t)\to 0$ exponentially. This stability and the boundedness of $\Phi(t), \Psi(t)$ and $K(t)$ (under Assumptions 1 and 2) guarantee the boundedness of $\Upsilon(t)$ generated by (9). By applying Lemma B1 of Appendix B with Assumption 3, the exponential stability of (18) (which is the homogeneous part of (19)) is ensured. Then, by applying Lemma B2 of Appendix B, the parameter estimation error $\tilde{\theta}(t)$ driven by (19) tends exponentially to zero. Finally, it follows from (15) and the boundedness of $\Upsilon(t)$ that $\tilde{x}(t)\to 0$ exponentially. □

3.3. Robustness to bounded uncertainties

Although the content of this paper is focused on deterministic systems, it is useful, from the practical point of view, to study the behavior of the adaptive observer designed in this paper when

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modeling errors and measurement uncertainties are present in the studied system. For this purpose, consider systems in the form of

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + \Phi(t)\theta(t) + w(t) \tag{27a}$$

$$\dot{\theta}(t) = q(t) \tag{27b}$$

$$y(t) = C(t)x(t) + \Psi(t)\theta(t) + v(t) \tag{27c}$$

where $w(t) \in \mathbb{R}^n, q(t) \in \mathbb{R}^p$ and $v(t) \in \mathbb{R}^m$ are bounded errors or uncertainties disturbing the state, parameter and output equations.

Theorem 2
If, in addition to Assumptions 1–3, the disturbing terms $w(t), v(t)$ and $q(t)$ in system (27) are bounded, then the state and parameter estimation errors of the adaptive observer (21) applied to system (27), namely $\tilde{x}(t) = x(t) - \hat{x}(t)$ and $\tilde{\theta}(t) = \theta(t) - \hat{\theta}(t)$, are also bounded.

Proof of Theorem 2
Like in the proof of Theorem 1, define the linear combination of state and parameter estimation errors $\tilde{z} = \tilde{x} - \Upsilon\tilde{\theta}$, then the error equations write

$$\dot{\tilde{z}} = (A - KC)\tilde{z} + w - Kv - \Upsilon q \tag{28a}$$

$$\dot{\tilde{\theta}} = -\Gamma(C\Upsilon + \Psi)^{\mathrm{T}}[C\tilde{z} + (C\Upsilon + \Psi)\tilde{\theta}] - \Gamma(C\Upsilon + \Psi)^{\mathrm{T}} v + q \tag{28b}$$

The disturbance-free counterparts of these error equations, namely (11) and (19), have been shown to be exponentially stable in the proof of Theorem 1. Hence the homogeneous part of (28) is exponentially stable. Again as in the disturbance-free case, the matrices $K(t), \Upsilon(t), C(t), \Psi(t)$ are all bounded. The terms in (28) involving $w(t), q(t)$ and $v(t)$ are then all bounded. By applying the Theorem 1 on p. 196 of [18], $\tilde{z}(t)$ and $\tilde{\theta}(t)$ are also bounded, so is $\tilde{x}(t) = \tilde{z}(t) + \Upsilon(t)\tilde{\theta}(t)$. □

Notice that assuming bounded $q(t)$ in (27b) does not imply bounded $\theta(t)$. The above result thus applies even if $\theta(t)$ is not bounded, but the speed of its variations is assumed limited.

3.4. Using an exponential forgetting factor

In practice, it may be difficult to tune the *constant* gain matrix $\Gamma \in \mathbb{R}^{p \times p}$ of the adaptive observer (21) to get a satisfactory convergence behavior, though the convergence is guaranteed for any positive-definite Γ. In order to improve this aspect, the constant gain can be replaced by a time-varying matrix $\Gamma(t) \in \mathbb{R}^{p \times p}$ that is computed in a way similar to the classical recursive least-squares (RLS) algorithm with an exponential forgetting factor. The resulting adaptive observer, which differs from (21) only in the gain matrix $\Gamma(t)$, also improves the behavior regarding random noises. This variant form of the adaptive observer writes

$$\dot{\Upsilon} = (A - KC)\Upsilon + \Phi - K\Psi \tag{29a}$$

$$\dot{\Gamma} = -\Gamma(C\Upsilon + \Psi)^{\mathrm{T}}(C\Upsilon + \Psi)\Gamma + \lambda\Gamma \tag{29b}$$

$$\dot{\hat{x}} = A\hat{x} + Bu + \Phi\hat{\theta} + [K + \Upsilon\Gamma(C\Upsilon + \Psi)^{\mathrm{T}}](y - C\hat{x} - \Psi\hat{\theta}) \tag{29c}$$

$$\dot{\hat{\theta}} = \Gamma(C\Upsilon + \Psi)^{\mathrm{T}}(y - C\hat{x} - \Psi\hat{\theta}) \tag{29d}$$

where $\lambda > 0$ is a chosen forgetting factor and the initial gain $\Gamma(t_0) \in \mathbb{R}^{p \times p}$ is a symmetric positive-definite matrix. Convergence results similar to those of the adaptive observer (21) can be shown by adapting the analysis made in [19] which was made for unknown parameters in state equations only.

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4. NUMERICAL EXAMPLE

In order to illustrate the proposed algorithm, let us consider a simulated flight control system. The linearized lateral dynamics of a remotely piloted aircraft (see [20, p. 188]) is modeled as

$$\begin{aligned}\dot{x}(t) &= Ax(t)+Bu(t)\\ y(t) &= Cx(t)\end{aligned} \tag{30}$$

$$A=\begin{bmatrix}-0.277 & 0 & -32.9 & 9.81 & 0\\ -0.1033 & -8.525 & 3.75 & 0 & 0\\ 0.3649 & 0 & -0.639 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\end{bmatrix}$$

$$B=\begin{bmatrix}-5.432 & 0\\ 0 & -28.64\\ -9.49 & 0\\ 0 & 0\\ 0 & 0\end{bmatrix},\quad C=\begin{bmatrix}0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\end{bmatrix}$$

$$x(t)=\begin{bmatrix}\text{side slip}\\ \text{roll rate}\\ \text{yaw rate}\\ \text{bank angle}\\ \text{yaw angle}\end{bmatrix},\quad u(t)=\begin{bmatrix}\text{rudder}\\ \text{aileron}\end{bmatrix}$$

In our simulation, a saturated Gaussian noise with standard deviation equal to 0.1 is added to each output. The input signals $u(t)$ are generated by a simple proportional controller. Assume that the input vector $u(t)$ is affected by multiplicative faults corresponding to applying scalar coefficients θ_1 and θ_2 to the two components of $u(t)$. Assume also that the sensors measuring $y_1(t)$ and $y_2(t)$ are affected by bias faults θ_3, θ_4. A third bias effecting $y_3(t)$ would not be identifiable. To understand this fact, replace the state variable x_5 by $x_5'=x_5+c$ with an arbitrary constant c. Because x_5 is simply the integral of x_3, it is easy to check that the state equation remains unchanged with the new state variable x'. The system model is thus invariant to any translation of x_5. It is then impossible to estimate the bias affecting the measurement of this state.

The system model is then rewritten as

$$\begin{aligned}\dot{x}(t) &= Ax(t)+B\begin{bmatrix}\theta_1 u_1(t)\\ \theta_2 u_2(t)\end{bmatrix}\\ y(t) &= Cx(t)+\begin{bmatrix}\theta_3\\ \theta_4\\ 0\end{bmatrix}\end{aligned} \tag{31}$$

The initial values and the parameters of the adaptive observer (29) are $x(0)=\hat{x}(0)=[1,1,1,1,1]^{\mathrm{T}}$, $\hat{\theta}(0)=[1,1,0,0]^{\mathrm{T}}$, $\Gamma(0)=I_4$, $\Upsilon(0)=0_{5\times 4}$ and $\lambda=0.6$. The gain matrix $K(t)$ is computed from (26) with $Q=0.02I_5$, $R=0.01I_3$.

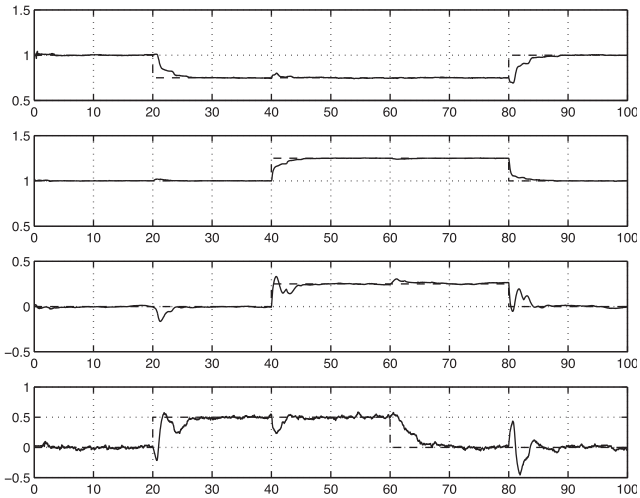

Figure 1. Parameter estimates $\hat{\theta}_1$, $\hat{\theta}_2$, $\hat{\theta}_3$ $\hat{\theta}_4$ ordered from top to bottom. The true parameter values are shown by the dashed lines.

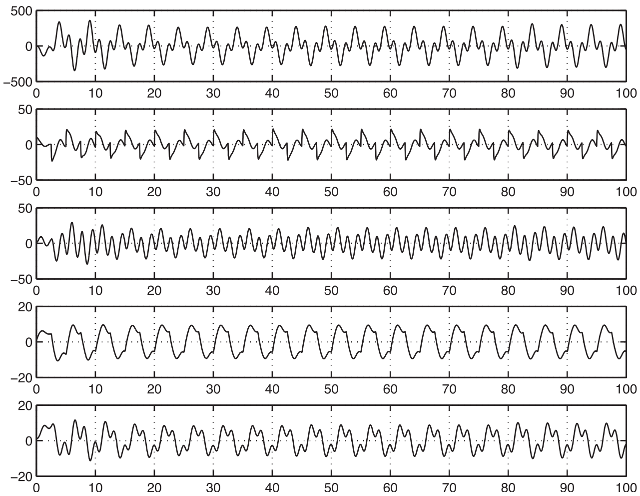

Figure 2. Simulated states $x_1(t)$, $x_2(t)$, $x_3(t)$, $x_4(t)$ and $x_5(t)$.

In Figures 1–3 are, respectively, plotted the parameter estimates, the simulated state variables and the states estimation errors. It can be observed that, for each parameter change, the convergence of the parameter estimation errors is re-established after a transient less than 10 s.

5. CONCLUSION

In this paper an adaptive observer for linear time-varying systems with unknown parameters in both state and output equations has been designed and its global convergence analyzed. Although adaptive observers had been studied separately for unknown parameters either in state equations or in output equations before this work, the extension to unknown parameters in both state and output equations has not been a trivial task. In order to make easier its tuning and to improve its behavior in the presence of noises, this adaptive observer is then amended with an exponential forgetting factor. A numerical example has been presented to illustrate its performance.

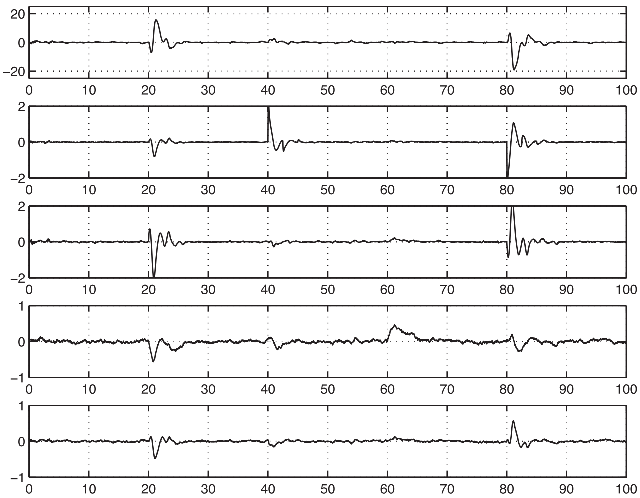

Figure 3. State estimation errors $\tilde{x}_1(t)$, $\tilde{x}_2(t)$, $\tilde{x}_3(t)$, $\tilde{x}_4(t)$ and $\tilde{x}_5(t)$.

APPENDIX A: KALMAN FILTER VERSUS ADAPTIVE OBSERVER

A natural approach to the joint estimation of $x(t)$ and θ for system (1) is to consider the extended system

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\theta}(t) \end{bmatrix} = \begin{bmatrix} A(t) & \Phi(t) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ \theta(t) \end{bmatrix} + \begin{bmatrix} B(t) \\ 0 \end{bmatrix} u(t) \tag{A1a}$$

$$y(t) = [C(t) \;\; \Psi(t)] \begin{bmatrix} x(t) \\ \theta(t) \end{bmatrix} \tag{A1b}$$

This is still a linear system, thus the Kalman filter is applicable to the joint estimation of $x(t)$ and θ. Compared to the Kalman filter applied to this extended system, the advantages of the adaptive observer are as follows.

- To ensure the convergence of the Kalman filter, the *uniform* observability of the *extended* system (A1) should be checked. Notice that even in the case constant matrices A, B and C, the *extended* system (A1) is time varying, because $\Phi(t)$ and $\Psi(t)$ are time-varying matrices in most adaptive estimation problems. The *uniform* observability of a time-varying system is defined through its Gramian observability matrix [17]. For the *extended* system (A1), the Gramian observability matrix has the size $(n+p)\times(n+p)$. The convergence of the Kalman filter is based on the uniform positiveness of this $(n+p)\times(n+p)$ Gramian matrix. In contrast, the adaptive observer presented in this paper requires a simpler condition: the uniform positiveness of two main diagonal blocks (of sizes $n\times n$ and $p\times p$) of this Gramian matrix, equivalent to the Gramian observability matrix of the *non* extended system (determined by the matrices $A(t), C(t)$ and ensuring the exponential stability of (22)), and to the persistent excitation condition (24). See [21] for more details about a similar case.
- For numerical implementation, the Kalman filter for the extended system (A1) requires the numerical solution of an $(n+p)\times(n+p)$ Riccati equation, whereas the adaptive observer requires the numerical solution of two $n\times n$ and $p\times p$ Riccati equations, namely Equations (25) and (29b) (the second one is not necessary when a constant gain Γ is used as in (21)). The numerical advantage of the adaptive observer is obvious, especially for large values of n and p.

- Another interest for studying linear adaptive observers is their possible extensions to nonlinear systems. The *globally convergent* nonlinear adaptive observers presented in [10, 11, 13] are essentially extensions of the linear results presented in [6, 12].

APPENDIX B: TWO LEMMAS ABOUT STABILITY

Lemma B1

Let $\Omega(t)\in\mathbb{R}^{m\times p}$ be a bounded and piecewise continuous matrix and $\Gamma\in\mathbb{R}^{p\times p}$ be any symmetric positive-definite matrix. If there exist positive constants T,α such that, for all $t\geqslant t_0$,

$$\int_t^{t+T}\Omega^T(\tau)\Omega(\tau)\geqslant\alpha I_p$$

then the system

$$\dot{\xi}(t)=-\Gamma\Omega^T(t)\Omega(t)\xi$$

is exponentially stable.

See the Theorem 2.16 of [22] for a proof of this classical result.

Lemma B2

If the autonomous linear time-varying system

$$\dot{\zeta}(t)=F(t)\zeta(t)$$

is exponentially stable, $u(t)$ is bounded and integrable, and $u(t)\to 0$ when $t\to\infty$, then $z(t)$ driven by $u(t)$ through the system

$$\dot{z}(t)=F(t)z(t)+u(t)$$

is bounded and also converges to zero. If moreover $u(t)$ vanishes exponentially fast, then $z(t)$ also vanishes exponentially fast.

This classical result can be proved by directly integrating the differential equation with the transition matrix related to $F(t)$. See also [18, p. 196].

REFERENCES

1. Luders G, Narendra KS. An adaptive observer and identifier for a linear system. *IEEE Transactions on Automatic Control* 1973; **18**:496–499.
2. Kreisselmeier G. Adaptive observers with exponential rate of convergence. *IEEE Transactions on Automatic Control* 1977; **22**(1):2–8.
3. Bastin G, Gevers M. Stable adaptive observers for nonlinear time varying systems. *IEEE Transactions on Automatic Control* 1988; **33**(7):650–658.
4. Marino R, Tomei P. Nonlinear control design. *Information and System Sciences*. Prentice-Hall: London, New York, 1995.
5. Besançon G. Remarks on nonlinear adaptive observer design. *Systems and Control Letters* 2000; **41**(4):271–280.
6. Zhang Q. Adaptive observer for multiple-input-multiple-output (MIMO) linear time varying systems. *IEEE Transactions on Automatic Control* 2002; **47**(3):525–529.
7. Rajamani R, Hedrick K. Adaptive observer for active automotive suspensions—theory and experiment. *IEEE Transactions on Control Systems Technology* 1995; **3**(1):86–93.
8. Cho YM, Rajamani R. A systematic approach to adaptive observer synthesis for nonlinear systems. *IEEE Transactions on Automatic Control* 1997; **42**(4):534–537.
9. Zhang Q, Xu A. Implicit adaptive observers for a class of nonlinear systems. *ACC'2001*, Arlington, 2001; 1551–1556.
10. Xu A, Zhang Q. Nonlinear system fault diagnosis based on adaptive estimation. *Automatica* 2004; **40**(7): 1181–1193.
11. Farza M, M'Saada M, Maatouga T, Kamounb M. Adaptive observers for nonlinearly parameterized class of nonlinear systems. *Automatica* 2009; **45**(10):2292–2299.

DOI: 10.1002/acs

12. Zhang Q. An adaptive observer for sensor fault estimation in linear time varying systems. *IFAC World Congress*, Prague, 2005.
13. Zhang Q, Besançon G. An adaptive observer for sensor fault estimation in a class of uniformly observable non-linear systems. *International Journal of Modelling*, *Identification and Control* 2008; **4**(1):37–43.
14. Besançon G, Zhang Q, Hammouri H. High-gain observer based state and parameter estimation in nonlinear systems. *IFAC Symposium on Nonlinear Control Systems* (*NOLCOS*), Stuttgart, 2004.
15. Nadri M, Hammouri H. Design of a continuous-discrete observer for state affine systems. *Applied Mathematics Letters* 2003; **16**(6):967–974.
16. Besançon G, De Leon Morales J, Huerta Guevara O. On adaptive observers for state affine systems. *International Journal of Control* 2006; **79**(6):581–591.
17. Jazwinski AH. *Stochastic Processes and Filtering Theory*. Mathematics in Science and Engineering, vol. 64. Academic Press: New York, 1970.
18. Brockett RW. *Finite Dimensional Linear Systems*. Wiley: New York, 1970.
19. Zhang Q, Clavel A. Adaptive observer with exponential forgetting factor for linear time varying systems. *IEEE Conference on Decision and Control* (*CDC'01*), Orlando, U.S.A., 2001.
20. Chen J, Patton RJ. *Robust Model-based Fault Diagnosis for Dynamic Systems*. Kluwer Academic Publishers: Boston, Dordrecht, London, 1999.
21. Zhang Q. Revisiting different adaptive observers through a unified formulation. *IEEE Conference on Decision and Control and European Control Conference* (*CDC-ECC*), Seville, Spain, 2005.
22. Narendra KS, Annaswamy AM. *Stable Adaptive Systems*. Prentice-Hall: Boston, 1989.

DOI: 10.1002/acs