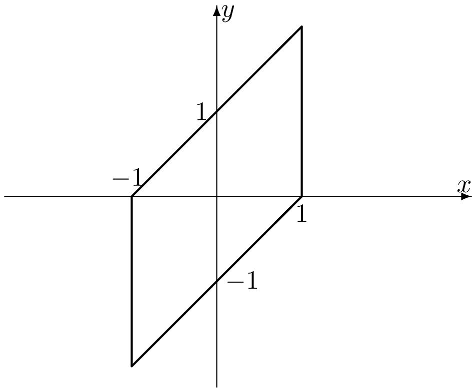
CORO-IMARO

CORO-SIP, CORO-EPICO, STATES exam
16 December 2020

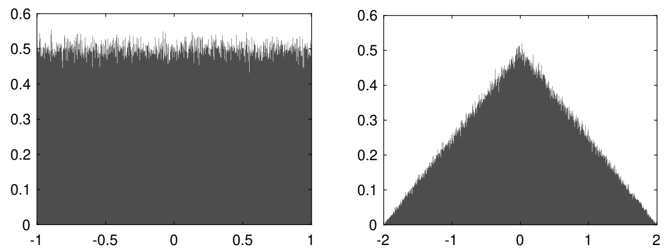
Exercise 1: Probability and estimation theory

A mobile robot moves at random on a given area, so that its position (x, y) is a pair of continuous valued r.v. uniformly distributed over the support below. We observe the y coordinate, and we want to estimate the x coordinate.



For information:

- the empirical estimation of $\mathrm{Var}\left(\left[\begin{smallmatrix}\mathsf{x}\\ \mathsf{y}\end{smallmatrix}\right]\right)$ from one million realizations of (x, y) with Matlab gave the matrix $\left[\begin{smallmatrix}0.3331 & 0.3333\\ 0.3333 & 0.6673\end{smallmatrix}\right]$;
- the normalized histogram of the realizations of x gave the left figure below, the normalized histogram of the realizations of y gave the right one:



a) What is the support area. So what is the constant value of the joint PDF over the support?

b) What are the empirical values of the variance of x, of the variance of y, of the covariance of x and y produced by Matlab?

c) Give and plot the PDF of x.

d) Give its mean and its variance.

e) Give and plot the PDF of y.

f) Give its mean and its variance.

g) Give the conditional PDF of $\mathsf{x} \mid \mathsf{y}$.

h) Give its mean and its variance (a good idea is to make appear the absolute value of y in the variance formula).

i) We remind that $\mathrm{Cov}\left(\mathrm{E}\left(\mathsf{x} \mid \mathsf{y}\right), \mathsf{y}\right) = \mathrm{Cov}\left(\mathsf{x}, \mathsf{y}\right)$. Give the covariance of x et y.

j) Give the exact value of $\mathrm{Var}\left(\left[\begin{smallmatrix}\mathsf{x}\\ \mathsf{y}\end{smallmatrix}\right]\right)$.

k) From y, we estimate x. Give the LMMSE estimator $\mathrm{E}\left(\mathsf{x}\right) + \frac{\mathrm{Cov}(\mathsf{x},\mathsf{y})}{\mathrm{Var}(\mathsf{y})}\left(\mathsf{y} - \mathrm{E}\left(\mathsf{y}\right)\right)$ and its variance $\mathrm{Var}\left(\mathsf{x}\right) - \frac{\left(\mathrm{Cov}(\mathsf{x},\mathsf{y})\right)^2}{\mathrm{Var}(\mathsf{y})}$.

l) From y, we estimate x. Give the MMSE estimator and its variance.

1 Exercise 2: Gain estimation

A discrete time univariate signal $\mathsf{y} = (\mathsf{y}[n])_{1 \leqslant n \leqslant N}$ is supposed to be proportional to a univariate signal $(\mathsf{u}[n])_{1 \leqslant n \leqslant N}$, up to an error $\mathsf{w} = (\mathsf{w}[n])_{1 \leqslant n \leqslant N}$; for all $n \in \{1, \ldots, N\}$:

$$\mathsf{y}[n] = \mathsf{u}[n]\,\mathsf{x} + \mathsf{w}[n]$$

The sequence w is supposed to be independent and identically distributed given the unknown parameter x, with zero mean and variance σ^2 independent of x.

We define the vectors $\mathsf{Y} = \begin{bmatrix} \mathsf{y}[1] \\ \vdots \\ \mathsf{y}[N] \end{bmatrix}$, $\mathsf{U} = \begin{bmatrix} \mathsf{u}[1] \\ \vdots \\ \mathsf{u}[N] \end{bmatrix}$ and $\mathsf{W} = \begin{bmatrix} \mathsf{w}[1] \\ \vdots \\ \mathsf{w}[N] \end{bmatrix}$.

a) What are the mean and the variance of W?

b) Write the relation between Y, W, U and x.

c) Write the GLS estimator $\hat{x}(\mathsf{Y}, \mathsf{U})$ of x from the observations Y and U, as if U was deterministic (thus, the solution is not linear with respect to U).

d) Write the bias and the variance of this estimator given the parameter, σ^2 and U.

e) Actually, U is random. Write the bias and the variance of this estimator given the parameter, in function of σ^2 and the expectation of a function of U.