Automation and Control Chair, Kiel University

Reduced Order Observer Design for Nonlinear Systems

Sundarapandian, V. (2006)

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Content

Problem statement

To design a reduced order observer for nonlinear systems

- **The form of nonlinear plants to be considered**

$$\begin{bmatrix} \dot{x}_m \\ \dot{x}_u \end{bmatrix} = \begin{bmatrix} F_1(x_m, x_u) \\ F_2(x_m, x_u) \end{bmatrix} \tag{1}$$

$$y = x_m \tag{2}$$

where $x_m \in \mathbb{R}^p$ and $x_u \in \mathbb{R}^{n-p}$

- **Reduced order exponential observer**

$$\dot{z}_u = G(z_u, y) \tag{3}$$

where $z_u \in \mathbb{R}^{n-p}$

Following conditions should be satisfied:

(O1) If $z_u(0) = x_u(0)$, then $z_u(t) = x_u(t)$ for all $t \geq 0$.
(O2) If $z_u(0) - x_u(0) \in B_\delta(0)$, then $z_u(t) - x_u(t) \in B_\epsilon(0)$ for all $t \geq 0$.
(O3) $\|z_u(t) - x_u(t)\| \leq M \exp(-\alpha t) \|z_u(0) - x_u(0)\|$, $M, \alpha > 0$ for all $t \geq 0$.

Reduced order observer design

Linearization

- **Linearize at equilibrium x=0**

$$\begin{bmatrix} \dot{x}_m \\ \dot{x}_u \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_m \\ x_u \end{bmatrix} + \begin{bmatrix} \phi(x_m, x_u) \\ \psi(x_m, x_u) \end{bmatrix} \tag{4}$$

$$y = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} x_m \\ x_u \end{bmatrix} \tag{5}$$

- **Preconditions for the existence of reduced order exponential observers**[1]

(H1) The equilibrium $x = 0$ of the system (4) is Lyapunov stable.
(H2) The pair (A_{22}, A_{12}) is detectable.

Remark for **(H1)**: see the discussion regarding the difference between linear and nonlinear systems in [1, p.20].
Remark for **(H2)**: see Theorem 1.9 in [1, p.14].

[1] A. Schaum. *Selected topics in Systems and Control*. `https://www.control.tf.uni-kiel.de/en/teaching/winter-term/seminar-selected-topic-in-automatic-control`.

Reduced order observer design

Correction term

- **Full order Luenberger observer**

$$\begin{aligned}\dot{z} = G(z, y) &\triangleq F(z) + L[\mathrm{CorrTerm}] \\ &= F(z) + L[y - z]\end{aligned}$$

- **Candidate reduced order observer a**

$[\mathrm{CorrTerm}] = y - z_m$
then $\dot{z}_u = F_2(y, z_u) + L[y - z_m]$ **?**

- **Candidate reduced order observer b**

Note that $\dot{y} = \dot{x}_m = F_1(x_m, x_u)$
$[\mathrm{CorrTerm}] = \dot{y} - \hat{\dot{x}}_m$
then

$$\dot{z}_u = F_2(y, z_u) + L[\dot{y} - F_1(y, z_u)] \tag{6}$$

Reduced order observer design

Error dynamics

- **Candidate reduced order observer b**

$\dot{z}_u = F_2(y, z_u) + L[\dot{y} - F_1(y, z_u)]$
After linearization,
$$\dot{z}_u = A_{21}y + A_{22}z_u + \psi(y, z_u) + L\left[\dot{y} - A_{11}y - A_{12}z_u - \phi(y, z_u)\right] \tag{7}$$

In the plant (4), $\dot{x}_u = A_{21}y + A_{22}x_u + \psi(y, x_u)$

- **Error dynamics**

The estimation error e is defined by $e \triangleq z_u - x_u$,
then it follows that
$$\dot{e} = (A_{22} - LA_{12})\,e + \psi(y, e + x_u) - \psi(y, x_u) - L\left[\phi(y, x_u + e) - \phi(y, x_u)\right] \tag{8}$$

Note that the linearization matrix $(A_{22} - LA_{12})$ should be Hurwitz.

(H2) The pair (A_{22}, A_{12}) is detectable.

Problem extension i

Remove the derivative of the measurement vector y

- **Candidate reduced order observer b**

$$\dot{z}_u = F_2(y, z_u) + L[\dot{y} - F_1(y, z_u)]$$

- **Construct a new estimator state**

$\zeta_u = z_u - Ly$

then

$$\dot{\zeta}_u = F_2(y, \zeta_u + Ly) - LF_1(y, \zeta_u + Ly) \quad (9)$$

$$z_u = \zeta_u + Ly \quad (10)$$

- **Theorem**

Consider a nonlinear plant (1) that satisfies **(H1)** and **(H2)**. Linearize the plant equations at the origin, we obtain the equivalent form for the plant given by (4). Let observer gain L be any matrix such that $A_{22} - LA_{12}$ is Hurwitz. Then we can construct a reduced order observer of the form (9) and (10), with the estimator state z_u. If the pair (A_{22}, A_{12}) is observable, the convergence speed can be assigned arbitrarily.

Problem extension ii

A generalized form of nonlinear plants

- **The form to be considered**

$\dot{x} = f(x)$

$y = Cx$

where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^p$

- **Introduce a coordinate transformation**

$$\xi = \begin{bmatrix} \xi_m \\ \xi_u \end{bmatrix} = Tx = \begin{bmatrix} C \\ Q \end{bmatrix} x$$

where $\xi_m \in \mathbb{R}^p$ and $\xi_u \in \mathbb{R}^{n-p}$

- **Get a new state space**

$$\begin{bmatrix} \dot{\xi}_m \\ \dot{\xi}_u \end{bmatrix} = \begin{bmatrix} F_1(\xi_m, \xi_u) \\ F_2(\xi_m, \xi_u) \end{bmatrix} \tag{11}$$

$$y = \xi_m \tag{12}$$

where $F(\xi) = f\left(T^{-1}\xi\right)$

An example

A nonlinear pendulum

- **Plant equations**

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\sin\left(\omega_0^2 x_1\right)$$
$$y = x_1$$

- **Solution**
  - Verify **(H1)**
    Consider the total energy function as a candidate Lyapunov function:
    $V\left(x_1, x_2\right) = \frac{1}{\omega_0^2}\left[1 - \cos\left(\omega_0^2 x_1\right)\right] + \frac{1}{2}x_2^2 \rightarrow$ P.D.
    $\dot{V} \equiv 0 \rightarrow$ S.N.D.
    $\Rightarrow$ The origin is a Lyapunov stable equilibrium.
  - Verify **(H2)**
    By linearizing around the origin, one gets:
    $$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \omega_0^2 & 0 \end{bmatrix}.$$
    $\Rightarrow$ The pair $\left(A_{22}, A_{12}\right)$ is observable.

An example

A nonlinear pendulum

- **Solution**
  - Verify **(H1)**
  - Verify **(H2)**
  - Observer design

    The linearized error dynamics is given by:

    $\dot{e} = (A_{22} - LA_{12})\, e.$

    To place the error pole, we get the characteristic equation based on the linearization matrix:

    $A_{22} - LA_{12} = -L \rightarrow sI_{1\times 1} - (-L) = 0$

    $\Rightarrow s = -L$

    One can take the observer gain as $L = 5\omega_0$,

    then the reduced order exponential observer is given by:

    $\dot{\zeta}_2 = -\sin\left(\omega_0^2 y\right) - 5\omega_0\left(\zeta_2 + 5\omega_0 y\right),$

    $z_2 = \zeta_2 + 5\omega_0 y.$

A nonlinear pendulum

- **Simulation**

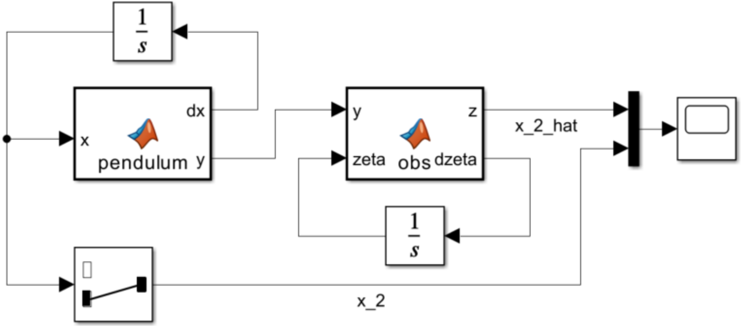

Figure 1: Simulation diagram

An example

A nonlinear pendulum

- Simulation

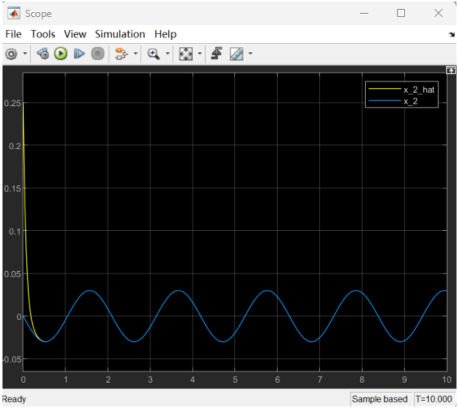

Figure 2: The convergence process in the observer

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Contact data

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