Some special cases

1. when a part of inversed transformation function is not easy to obtain. one can just write the equation to express the relation between x_i & z

   $x_1 = z_1$

   x_2 such that $x_2^2 + 2x_2 + 1 + z_3 - z_1 z_4 = 0$

   $x_3 = \cdots$

   thus when calculating the normal form, one doesn't need to substitute x_2

   $\dot{z}_3 = z_1 z_4^2 + a z_1 z_4^3 (1 + z_1) - 2(1+x_2) z_1 x_2 + \cdots\cdots$

2. when checking the stability of zero dynamics, if the structure is nonlinear ~~and the dimension is greater than 1.~~ There is a way to judge the local stability, which is calculating the Jacobian linearization matrix and to check it is Hurwitz or not. (according to Routh's criteria). And because of the linearization, one can only confirm the stability around origin (local).

Indirect method

and the zero dynamics are

$$A_L = \frac{\partial f}{\partial z}\Big|_{z=0} = \begin{pmatrix} -3z_4^2 & 1 \\ a & b \end{pmatrix}\Big|_{z_4=0} = \begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix} \qquad \begin{aligned} \dot{z}_4 &= z_5 - z_4^3 \\ \dot{z}_5 &= a z_4 + b z_5. \end{aligned} \qquad \begin{vmatrix} \lambda - 0 & -1 \\ -a & \lambda - b \end{vmatrix} = 0 \Rightarrow \lambda^2 - b\lambda - a = 0$$

Routh's criteria $b > 0$; $a > b$

The zero dynamics are (at least) locally exponentially stable iff the dynamic matrix $A_\ell = \begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix}$ is Hurwitz, i.e. iff $a, b < 0$, as can be checked with the indirect Lyapunov's method.

3. there is a case in 05/07/2021 which introduces external signals w to influence the unobservable dynamics and the calculation of u.
   firstly we have to analyze how do these external signals influence the performance of corresponding state. So one can calculate the expression of w. In our case it can concluded w_1 is a bounded signal which is multiplied by an exponential decreasing in time, i.e. $e^{-t} w_1$ term tends to zero when $t \to \infty$. Then one can analyze the stability and range of x_1 to confirm decoupling matrix $A(x)$ is always nonsingular. And if w_1 is measurable, one can just design u as a function of w_1; if w_1 is not measurable. one can simply neglect w_1 since $e^{-t} w_1 \xrightarrow{t\to\infty} 0$. Finally the zero dynamic's stability can also be analyzed by the same idea.

4. Since inside the controller u there is a term $\tan x_1$ which cannot be defined in $x_1 \in R$ ($\tan x_1 \xrightarrow{x_1 \to \pm\frac{\pi}{2}} \infty$). So the system is simply L.A.S.

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} -2x_2(x_1x_3 - x_2^3 + x_4) - x_2 \tan x_1 - k_{10}x_1 - k_{11}(x_2^2 + x_3) \\ -(x_2^2 + x_3 + x_3^2)(x_1 + x_4) \end{pmatrix}$$

5. if one need globally stabilize a system. Another thing need to note is to make the decoupling matrix nonsingular with all $x \in \mathbb{R}^n$

$\ddot{y} = b(x) + (\alpha + \sin x_1) u.$

$\left.\begin{array}{l} \sin x_1 \in [-1, 1] \Rightarrow \alpha + \sin x_1 \in [\alpha - 1, \alpha + 1] \\ + \quad 0 \notin [\alpha - 1, \alpha + 1] \end{array}\right\} \Rightarrow \begin{array}{l} \alpha > 1 \\ \text{or } \alpha < -1 \end{array}$

6. if the sum of relative degree is greater than n. Then it means there is a redundancy in diffeomosphism, i.e. there are some useless y_i.

$y_1 = x_1 - x_2$

$\dot{y}_1 = x_1 x_3 + (1 + \omega) u_1 - x_2^2 - x_3 \qquad r_1 = 1$

$y_2 = x_2$

$\dot{y}_2 = x_2^2 + x_3$

$\ddot{y}_2 = 2x_2(x_2^2 + x_3) + x_1^2 x_3 + (1 - 2x_1) u_2 \qquad r_2 = 2$

$y_3 = x_3$

$\dot{y}_3 = x_1^2 x_3 + (1 - 2x_1) u_2 \qquad r_3 = 1$

$z = \Phi(x) = \begin{pmatrix} x_1 - x_2 \\ x_2 \\ x_2^2 + x_3 \\ x_3 \end{pmatrix}$ dependance, y_3 is useless.

and then one can reconstruct the feedback linearization based only on y_1, y_2.

Procedure

$\dot{x}_1 = x_1^2 + 3u_1$
$\dot{x}_2 = x_4 + x_3 u_1$
$\dot{x}_3 = \lambda x_3^m + x_4^2$
$\dot{x}_4 = u_2$
$y_1 = x_1$
$y_2 = x_2$

solution:

$y_1 = x_1$
$\dot{y}_1 = x_1^2 + 3u_1$
$y_2 = x_2$
$\dot{y}_2 = x_4 + x_3 \cdot u_1$

in this case, decoupling matrix $A(x) = \begin{pmatrix} 3 & 0 \\ x_3 & 0 \end{pmatrix}$ is singular, the relative degree is not defined. Considering a dynamic extension:

$$\begin{cases} x_5 = u_1 \\ \dot{x}_5 = w_1 \end{cases}$$ then one can rewrite the system:

$\dot{x}_1 = x_1^2 + 3x_5$
$\dot{x}_2 = x_4 + x_3 x_5$
$\dot{x}_3 = \lambda x_3^m + x_4^2$
$\dot{x}_4 = u_2$
$\dot{x}_5 = u_1$
$y_1 = x_1$
$y_2 = x_2$

one gets:

$y_1 = x_1 = h_1(x)$
$\dot{y}_1 = x_1^2 + 3x_5 = L_f h_1$; $L_g h_1 = (0 \ \ 0)$
$\ddot{y}_1 = \underbrace{2x_1(x_1^2 + 3x_5)}_{L_f^2 h_1} + 3w_1 \Rightarrow r_1 = 2$

$y_2 = x_2 = h_2(x)$
$\dot{y}_2 = x_4 + x_3 x_5 = L_f h_2$; $L_g h_2 = (0 \ \ 0)$
$\ddot{y}_2 = u_2 + x_3 w_1 + \underbrace{x_5(\lambda x_3^m + x_4^2)}_{L_f^2 h_2} \Rightarrow r_2 = 2$

i.e.

$$\begin{pmatrix}\ddot{y}_1\\ \ddot{y}_2\end{pmatrix} = \underbrace{\begin{pmatrix}2x_1(x_1^2+3x_5)\\ x_5(\lambda x_3^m + x_4^2)\end{pmatrix}}_{B(x)} + \underbrace{\begin{pmatrix}3 & 0\\ x_3 & 1\end{pmatrix}}_{A(x)}\begin{pmatrix}w_1\\ u_2\end{pmatrix} \qquad A(x) = \begin{pmatrix}L_gL_fh_1\\ L_gL_fh_2\end{pmatrix}$$

thus $|A(x)| = 3 \neq 0$, it is nonsingular $\left(|A(x)|_0 = 3 \neq 0,\ r \text{ is defined in the origin} \right)$

$\Rightarrow$ relative degree $r = \begin{pmatrix}2\\2\end{pmatrix}$ is defined.

the state feedback control which decouples and linearizes the I-O dynamics is:

$$\begin{pmatrix}w_1\\ u_2\end{pmatrix} = \begin{pmatrix}3 & 0\\ x_3 & 1\end{pmatrix}^{-1}\begin{pmatrix}-2x_1(x_1^2+3x_5)+v_1\\ -x_5(\lambda x_3^m + x_4^2)+v_2\end{pmatrix}$$

with:

$$\begin{pmatrix}v_1\\ v_2\end{pmatrix} = \begin{pmatrix}-k_{10}y_1 - k_{11}\dot{y}_1\\ -k_{20}y_2 - k_{21}\dot{y}_2\end{pmatrix} = \begin{pmatrix}\ddot{y}_1\\ \ddot{y}_2\end{pmatrix} \qquad k_{10}, k_{11}, k_{20}, k_{21} > 0$$

The diffeomosphism that shows the linear structure of this feedback system is:

$$\begin{pmatrix}\xi\\ \hline \eta\end{pmatrix} = \begin{pmatrix}z_1\\ z_2\\ z_3\\ z_4\\ \hline z_5\end{pmatrix} = \Phi(x) = \begin{pmatrix}y_1\\ \dot{y}_1\\ y_2\\ \dot{y}_2\\ \hline \phi_5(x)\end{pmatrix} = \begin{pmatrix}x_1\\ x_1^2+3x_5\\ x_2\\ x_4+x_3x_5\\ \phi_5(x)\end{pmatrix}$$

Jacobian matrix

$$\frac{\partial \Phi}{\partial x} = \begin{pmatrix}1 & 0 & 0 & 0 & 0\\ 2x_1 & 0 & 0 & 0 & 3\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & x_5 & 1 & x_3\\ \frac{\partial\phi_5}{\partial x_1} & \frac{\partial\phi_5}{\partial x_2} & \frac{\partial\phi_5}{\partial x_3} & \frac{\partial\phi_5}{\partial x_4} & \frac{\partial\phi_5}{\partial x_5}\end{pmatrix}$$

Consider $\frac{\partial \phi_5(x)}{\partial x_3} = 1$ to make $\left|\frac{\partial\Phi}{\partial x}\right|_0 \neq 0$

thus $\phi_5(x) = x_3 + \varphi(x_1, x_2, x_4, x_5)$

to make $\begin{pmatrix}w_1\\ u_2\end{pmatrix}$ absent in unobservable dynamics.

one has:

$$L_g\phi_5 = (0 \quad 0)$$

$\Downarrow$

$$\begin{pmatrix}\frac{\partial\phi_5}{\partial x_1} & \frac{\partial\phi_5}{\partial x_2} & \frac{\partial\phi_5}{\partial x_3} & \frac{\partial\phi_5}{\partial x_4} & \frac{\partial\phi_5}{\partial x_5}\end{pmatrix}\begin{pmatrix}0 & 0\\ 0 & 0\\ 0 & 0\\ 0 & 1\\ 1 & 0\end{pmatrix} = (0 \quad 0)$$

$\phi_5(x)$ can be simply set as $\phi_5(x) = x_3$

therefore:

$$z_1 = x_1 \qquad\qquad x_1 = z_1$$
$$z_2 = x_1^2 + 3x_5 \quad \xrightarrow{\Phi^{-1}(z)} \quad x_2 = z_3$$
$$z_3 = x_2 \qquad\qquad x_3 = z_5$$
$$z_4 = x_4 + x_3x_5 \qquad\qquad x_4 = z_4 - \frac{z_5}{3}(z_2 - z_1^2)$$
$$z_5 = x_3 \qquad\qquad x_5 = \frac{1}{3}(z_2 - z_1^2)$$

unobservable dynamics:

$$\dot{z}_5 = \lambda x_3^m + x_4^2$$

zero dynamics:

$$\dot{z}_5 = \lambda x_3^m + x_4^2 \Big|_{\xi=0}$$
$$= \lambda x_3^m$$

which is G.A.S. for m odd, $\lambda < 0$

normal form:

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = \ddot{y}_1 \Big|_{\Phi^{-1}(z)}$$
$$\dot{z}_3 = z_4$$
$$\dot{z}_4 = \ddot{y}_2 \Big|_{\Phi^{-1}(z)}$$
$$\dot{z}_5 = \lambda x_3^m + x_4^2$$