Sample 1

Ex 1

a) $$\boxed{\hat{x}[n] = x_H + z_H\, y[n]}$$

b) $$x[n] = x_H + z_H\, y_{true}[n]$$

$$\hat{x}[n] = x_H + z_H\, y[n]$$

$$= x_H + z_H\left(y_{true}[n] + w[n]\right)$$

$$\hat{x}[n] - x[n] = z_H\, w[n]$$

$$\boxed{\mathrm{Var}\left(\hat{x}[n] - x[n]\right) = z_H^2\, r}$$

c) $$x[n] = x_H + z_H\left(y[n] - w[n]\right)$$

$$\boxed{y[n] = \frac{1}{z_H}\, x[n] - \frac{x_H}{z_H} + w[n]}$$ observation equation

d) $$\boxed{x[n+1] = x[n] + D + v[n]}$$ state equation

e) $$\boxed{\begin{aligned} x^{|0}[1] &= 0 \\ p^{|0}[1] &= 0 \end{aligned}}$$ (since $x[1]=0$ for sure)

f) $$\hat{x}^{|n}[n] = \hat{x}^{|n-1}[n] + \frac{p^{|n-1}[n]\, z_H}{p^{|n-1}[n] + r z_H^2}\left(y[n] - \frac{1}{z_H}\hat{x}^{|n-1}[n] + \frac{x_H}{z_H}\right)$$

$$p^{|n}[n] = \frac{p^{|n-1}[n]\, r z_H^2}{p^{|n-1}[n] + r z_H^2}$$

g) $$\hat{x}^{|n}[n+1] = x^{|n}[n] + D$$

$$p^{|n}[n+1] = p^{|n}[n] + q$$

h) $r = +\infty$ $$\boxed{\begin{aligned} \hat{x}^{|n}[n] &= (n-1)D \\ p^{|n}[n] &= (n-1)\, q \end{aligned}}$$

i) $q = +\infty$ $$\boxed{\begin{aligned} \hat{x}^{|n}[n] &= x_H + z_H\, y[n] \\ p^{|n}[n] &= r z_H^2 \end{aligned}}$$ see a & b

Sample 1

Ex 2 $\quad \mathrm{Prob}(T=t\,|\,x) = e^{-x}\dfrac{x^t}{t!} \qquad \forall t\in\mathbb{N}$

a) $E(T|x) = \sum_{t\geq 1} t\,\mathrm{Prob}(T=t|x) \qquad$ (0 in $t=0$)

$$= \sum_{t\geq 1} e^{-x}\frac{x^t}{(t-1)!}$$

$$= e^{-x}\,x\,\underbrace{\sum_{t\geq 1}\frac{x^{t-1}}{(t-1)!}}_{e^x}$$

$$= x$$

b) $E(T(T-1)|x) = \sum_{t\geq 2} t(t-1)\,\mathrm{Prob}(T=t|x)$

$$= \sum_{t\geq 2} t(t-1)\,e^{-x}\frac{x^t}{t!}$$

$$= e^{-x}x^2\underbrace{\sum_{t\geq 2}\frac{x^{t-2}}{(t-2)!}}_{e^x}$$

$$= x^2$$

$$E(T^2|x) = E(T(T-1)|x) + E(T|x)$$

$$= x^2 + x$$

$$\mathrm{Var}(T|x) = E(T^2|x) - [E(T|x)]^2$$

$$= x^2 + x - x^2$$

$$= x$$

c) $L_{\vec{t}}(x) = \sum_{i=1}^{n} \ln \mathrm{Prob}(T=t_i|x)$

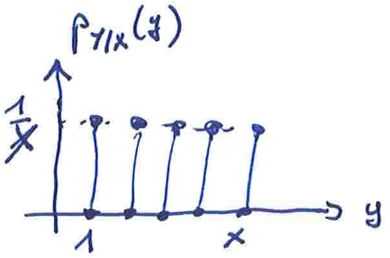
$$= \sum_i -x + t_i\ln x - \ln t_i!$$

$$= -nx + \ln x\sum_i t_i - \sum_i \ln t_i!$$

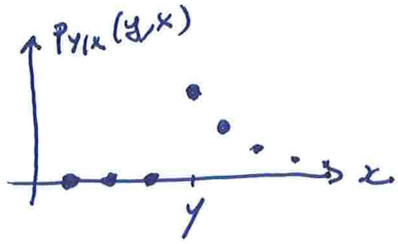
d) $\dfrac{dL_{\vec{t}}}{dx}(x) = -\dfrac{n}{x^2} + \dfrac{1}{x^2}\sum_i t_i \quad$ which is 0 for $\hat{x}_{MLE}(\vec{t}) = \frac{1}{n}\sum_{i=1}^{n} t_i$

Sample 2

Ex ① a) $\text{Prob}(Y=y\mid X) = \dfrac{1}{X}$ for all $y \in \{1,\dots,x\}$



b) $$E(Y\mid X) = \sum_{y=1}^{x} y \,\text{Prob}(Y=y\mid X)$$
$$= \frac{1}{X}\sum_{y=1}^{x} y$$
$$= \frac{1}{x}\,\frac{X(X+1)}{2}$$
$$= \frac{X+1}{2}$$



c) $$E(Y^2\mid X) = \sum_{y=1}^{x} \frac{y^2}{X}$$
$$= \frac{1}{X}\,\frac{X(X+1)(2X+1)}{6}$$
$$= \frac{(X+1)(2X+1)}{6}$$

$$\text{Var}(Y\mid X) = \frac{(X+1)(2X+1)}{6} - \frac{(X+1)^2}{4}$$
$$= \frac{(X+1)(X-1)}{12}$$

d) $$\hat{x}(y) = \arg\max_{x} \text{Prob}(Y=y\mid X=x)$$
$$= \arg\max_{x\geq y} \frac{1}{x}$$
$$= y$$

$$\hat{x}(Y) = Y$$

e) $\text{Prob}(Y=y\mid X) = \dfrac{1}{X}$ if $y \in \{1,\dots,X\}$

Likelihood $\begin{cases} \dfrac{1}{X^n} & \text{if } Y_i \leq X \;\forall i \\ 0 & \text{elsewhere} \end{cases}$

$$\hat{x}_{MLE}(Y_1,\dots,Y_n) = \max_{i=1\dots n}(Y_i)$$

Sample 2

Ex 2 Almost done during the lecture sessions

Model

$$y[n] = x[n] + w[n] \qquad var(w[n]) = r$$
$$x[n+1] = x[n] + u[n] + v[n] \qquad var(v[n]) = q$$

KF init

$$\hat{x}^{|0}[1] = 0$$
$$p^{|0}[1] = 0$$

KF loop

$$\hat{x}^{|n}[n] = \hat{x}^{|n-1}[n] + \frac{p^{|n-1}[n]}{p^{|n-1}[n] + r}\left(y[n] - \hat{x}^{|n-1}[n]\right)$$

$$p^{|n}[n] = \frac{p^{|n-1}[n]\, r}{p^{|n-1}[n] + r}$$

$$\hat{x}^{|n+1}[n] = \hat{x}^{|n}[n] + u[n]$$
$$p^{|n+1}[n] = p^{|n}[n] + q$$

$r = +\infty$ Dead reckoning

$$\hat{x}^{|n}[n] = \sum_{k=1}^{n-1} u[k]$$
$$p^{|n}[n] = (n-1)\, q$$

$q = +\infty$ GPS only

$$\hat{x}^{|n}[n] = y[n]$$
$$p^{|n}[n] = r$$