Nonlinear Control Systems

A. MODEL OF THE SYSTEM

The process under interest is an inverted pendulum and is described in the following figure.

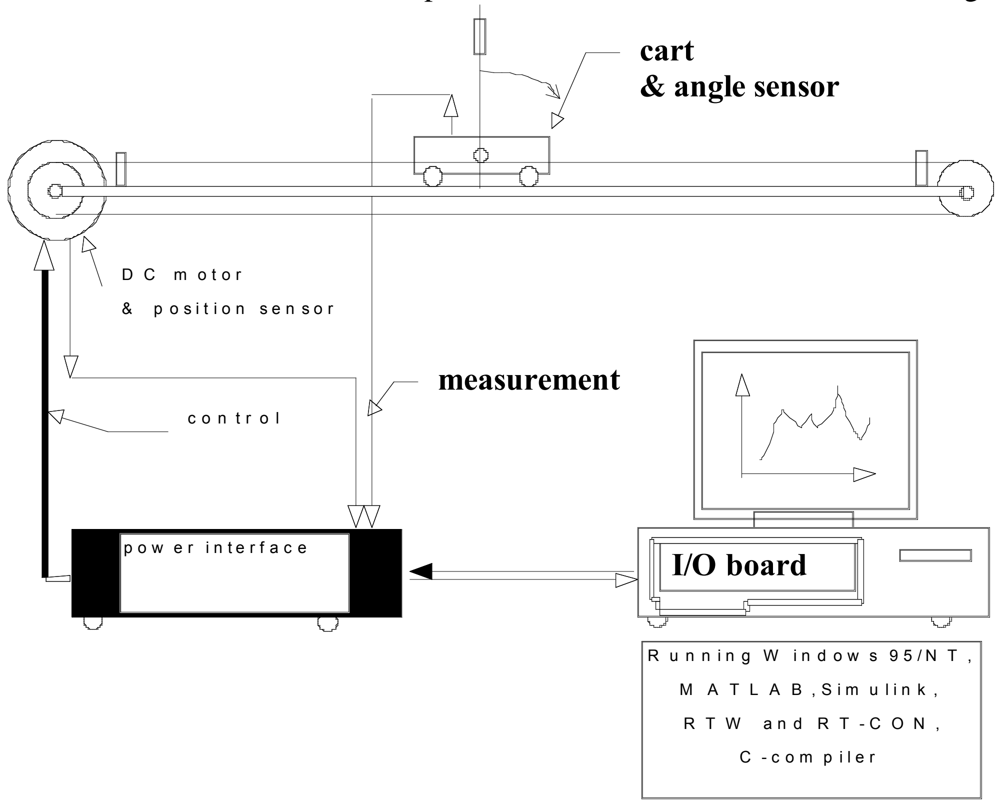


Figure 1 – Scheme of the inverted pendulum

The process consists of a crane driven by a motor, and moving along a guide rail with a length of 1m. On this crane, is placed a pendulum with two rods rotating freely, equipped with weights at their tips. Two incremental encoders allow to know the position of the crane and the angle of the pendulum. Safety stops are located at the end of the way on which the crane is moving.

B. MODELLING OF THE INVERTED PENDULUM

The diagram just below (Figure 2) details the physical parameters of the system. In the sequel, several models will be used, depending on modeling assumptions. Let θ the angle between the pendulum and the vertical, and x the position of the crane. The parameters of the pendulum are the mass of the crane M (=2.4kg), the mass centered at the end of the pendulum m (=0.23kg) and the length of the pendulum L (= 0.36m).

- **Model 1** : It is assumed that the system has 2 inputs, there is no friction (dry or viscous) and one neglects the inertia of the pendulum. The control inputs are the driving force of the carriage u_1 and the torque of the pendulum u_2. It should be noted that in the real application, the control input u_2 does not exists, and should therefore be considered equal to 0.

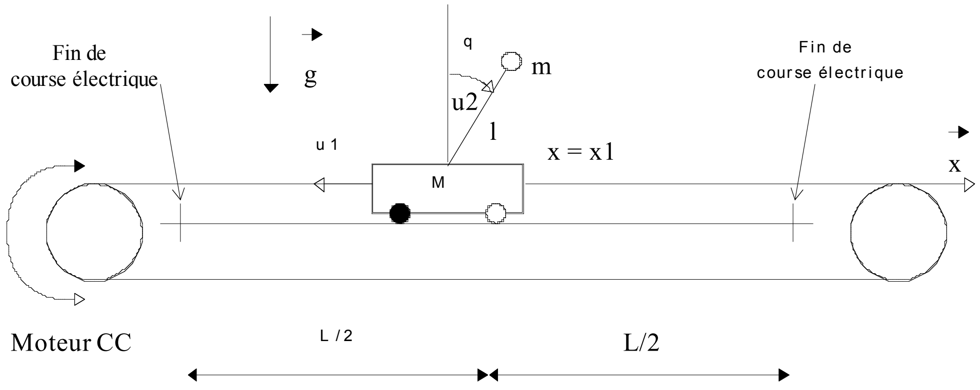


Figure 2 – Scheme of the inverted pendulum

The dynamics reads as

$$(M+m)\ddot{x} + m\,l\cos(\theta)\,\ddot{\theta} - m\,l\sin(\theta)\,\dot{\theta}^2 = u_1$$

$$m\,l\cos(\theta)\ddot{x} + m\,l^2\,\ddot{\theta} - m\,g\,l\sin(\theta) = u_2$$

☞ Write the model under nonlinear state system (with $x_1 = x$, $x_2 = dx/dt$, $x_3 = \theta$, $x_4 = d\theta/dt$.)

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases}$$

- **<u>Model 2</u>** : One suppose now that the system has a single input ($u_2 = 0$) , that there is no friction (dry or viscous) and the pendulum inertia is supposed negligible.

☞ Write the corresponding nonlinear state system.

- **<u>Model 3</u>** : Assume that the system has only one input ($u_2 = 0$), and taking into account the friction and inertia of the pendulum.

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{1}{M + m - \dfrac{m^2 l^2}{J + ml^2}\cos^2(x_3)}\left[m\sin(x_3)\left(l x_4^2 - \frac{ml^2 g}{J + ml^2}\cos(x_3) \right) + u_1 - f_c x_2 \right]$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = \frac{1}{\dfrac{J + ml^2}{ml} - al\cos^2(x_3)}\left[-al\sin(x_3)\cos(x_3)x_4^2 + g\sin(x_3) - \frac{a}{m}\cos(x_3)\left(u_1 - f_c x_2\right) \right]$$

with $[\, x_1\ x_2\ x_3\ x_4]^T$ defined as previously, and $a = \dfrac{m}{M+m}$.

C. PROBLEM STATEMENT

The objective consists in controlling the motion of the inverted pendulum. The controller needs to control the crane position and the angle of the pendulum (for example, one wants to move the carriage bearing the pendulum at a constant angle).

D. Design of a decoupling/linearizing control law for models 1 and 2

1. Design a control law based on a decoupling/linearizing control coupled with a state feedback. Is it possible to get linear dynamics for the position of the carriage and the angle of the pendulum? If yes, tune the coefficients in order to obtain aperiodic dynamics with response times of about 1 *sec*.

   Remark : the control law reads as $u = F(x) + G(x)\, v$ with $v = [v_1\ v_2]^T$. Depending on the chosen state feedback, these new control inputs allow a pole placement.

2. Simulate the close-loop system with Simulink software (the **CIPENDCO_STUDENT**.m file has to used to initialize the simulation (button in the Simulink window)).

3. What is the consequence of the closed-loop system behavior when the control input u_2 is limited by +/- 24 *N.m* (saturation) ? How to manage this technological constraint ?

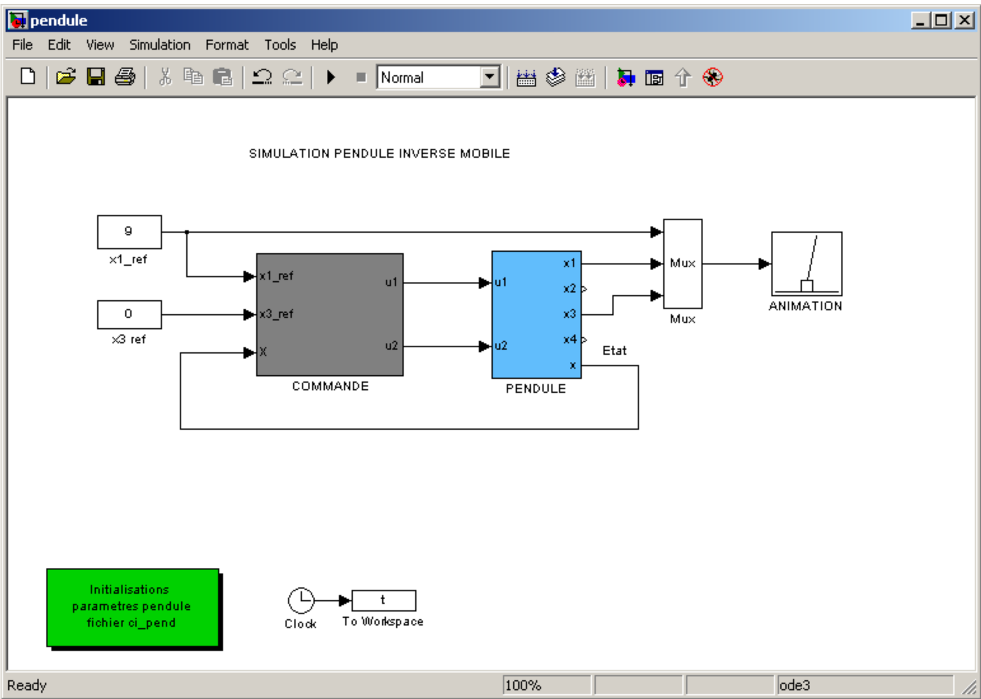
4. For the **model 2**, compute the control law based on input-output linearization/decoupling allowing to maintain the pendulum in the vertical position? Simulate this new control law. What is the behavior of the crane position? Explain theorically the phenomenon described by the simulations.

E. Design of a control law for model 3

In the case of the **model 3**, design a control law based on a linearizing strategy, allowing to stabilize the both variables, crane position and pendulum angle. Tune the parameters of the controller in order to satisfy the control saturations.

F. Appendix

A1 <u>File **Pendule.mdl**</u> developped under Simulink and allowing to simulate the closed-loop system. The user has to modify this file in order to check his own controllers.



A2 <u>File **CIPENDCO_STUDENT.m**</u> used for initialization of the simulator : physical and control law parameters.

% Initialization of the inverted pendulum model

```
%
% File : CI_PEND.m
% State variable x1 crane position
% x2 crane velocity, x3 pendulum angle
% x4 angular velocity of the pendulum
%
format long e;
%
% Crane mass
M=2.4 ;
% Pendulum crane
m=0.23 ;
% Distance between the rotation axis and the center of mass of the pendulum
l=0.36 ;
% Gravity
g=9.81 ;
% Viscous friction
fc = 7.96 ;
% Inertia
J = 0.099 ;
% Ratio a
a = m/(M+m) ;
%
%
%
```