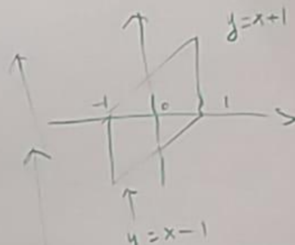
a) support area $(2\times 2\times \frac{1}{2})\times 2 = 4$

joint PDF calculation $\iint P_{x,y}(x,y)\,dx\,dy = 1$

$4\,P_{x,y}(x,y) = 1$

$P_{x,y}(x,y) = \frac{1}{4}$



b) var

c) PDF of x

$x<-1$ $\quad \int P_{x,y}(x,y)\,dy = 0$

$-1<x<0$ $\quad \int_{x-1}^{x+1} P_{x,y}(x,y)\,dy = \frac{1}{4}(x+1-x+1) = \frac{1}{2}$

$\frac{1}{4}x\Big|_{x-1}^{x+1}$

$0<x<1$ $\quad \int_{x-1}^{x+1} P_{x,y}(x,y)\,dy = \frac{1}{4}[x+1-(x-1)] = \frac{1}{2}$

$x>1$ $\quad \int P_{x,y}(x,y)\,dy = 0$

$\frac{(1+1)^2}{12} = \frac{1}{3}$

Variance

$\boxed{\frac{(b-a)^2}{12}}$

d) $E(x) = 0$ $\quad Var(x) = E(x^2) - (E(x))^2$

$= E(x^2) - 0$

$= \int_{-1}^{1} x^2 \frac{1}{2}\,dx = \frac{1}{6}x^3\Big|_{-1}^{1} = \frac{1}{3}$

last year exam

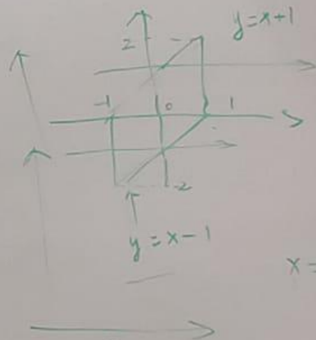
e) PDF of y

$y < -2, \int P_{x,y}(x,y)\cdot dx = 0$

$-2 < y < 0, \int_{-1}^{y+1} P_{x,y}(x,y)\cdot dx = \int_{-1}^{y+1} \frac{1}{4}\cdot dx = \frac{y+2}{4}$

$0 < y < 2, \int_{y-1}^{1} P_{x,y}(x,y) dx = \int_{y-1}^{1} \frac{1}{4}\cdot dx = \frac{2-y}{4}$

$y > 2, \int P_{x,y}(x,y)\cdot dx = 0$



x = y+1

f) $E(y) = 0$

$Var(y) = E(y^2) - (E(y))^2$

$= E(y^2)$

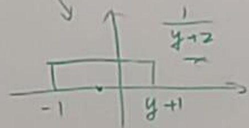
$= \int_{-2}^{0} \frac{y+2}{4}\cdot y^2 dy + \int_{0}^{2} \frac{2-y}{4} y^2 dy$

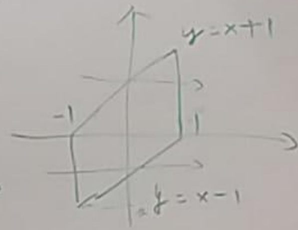
$= \frac{2}{3}$

last year exam part2

g) $p(x|y) =$ For $y \in [0, 2]$, $x \in [y-1, 1]$

For $y \in [-2, 0]$, $x \in [-1, y+1]$





h) $E(x|y) = \frac{y}{2}$

$\mathrm{Var}(x|y) = \frac{(2-|y|)^2}{12}$

① $\frac{(1-y+1)^2}{12}$

② $\frac{(y+1+1)^2}{12}$

$|\;| < 1$

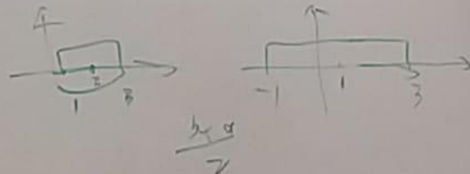
$\frac{3-(-1)}{2}$

i) $\mathrm{Cov}(x,y) = \mathrm{Cov}(E(x|y), y)$

$= \mathrm{Cov}(\frac{y}{2}, y)$

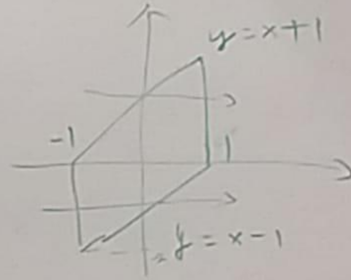
$= \frac{1}{2}\mathrm{Cov}(y, y)$

$= \frac{1}{2}\mathrm{Var}(y) = \frac{1}{3}$



last year exam part3

j) $var\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} C_{xx} & C_{xy} \\ C_{xy} & C_{yy} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$



k) $\hat{x}_{LMMSE} = E(x) + \frac{Cov(x,y)}{Var(y)}(y - E(y))$

$= 0 + \frac{\frac{1}{3}}{\frac{2}{3}}(y-0) = \frac{y}{2}$

$var(x) - \frac{(Cov(x,y))^2}{Var(y)} = \frac{1}{3} - \frac{(\frac{1}{3})^2}{\frac{2}{3}} = \frac{1}{3} - \frac{1}{6} = \frac{1}{6}$

l) $\hat{x}_{MMSE} = E(x|y) = \frac{y}{2}$ P32

$Var(E(x|y) - x) = Var(\frac{y}{2} - x) = var\left(\begin{bmatrix} -1 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}\right)$

$= \begin{bmatrix} -1 & \frac{1}{2} \end{bmatrix} var\begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} -1 \\ \frac{1}{2} \end{bmatrix}$

$= \frac{1}{3}\begin{bmatrix} -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ \frac{1}{2} \end{bmatrix}$

$= \frac{1}{3}\begin{bmatrix} -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} -1 \\ \frac{1}{2} \end{bmatrix} = \frac{1}{6}$