2. Stabilize to the origin the system

$$\begin{aligned}
\dot{x}_1 &= x_3^2 + 4u \\
\dot{x}_2 &= x_2 x_3 + (x_3^2 + 4)x_1 \\
\dot{x}_3 &= x_3^2 + x_2 \\
y &= x_2
\end{aligned}$$

via a state–feedback. State if the stability is local, global, asymptotic or exponential.
[8 points]

2a Using the state feedback linearization, one checks that the relative degree in the origin is $r = 2$, but the zero dynamics correspond to the unstable dynamics $\dot{x}_3 = x_3^2$. Therefore, this technique can not be used to stabilize the system to the origin. The backstepping procedure can be used instead, since the system is in the strict feedback form

$$V_1 = \frac{1}{2}x_3^2 \quad \Rightarrow \quad \dot{V}_1 = -k_3x_3^2 + \underbrace{(x_2 + x_3^2 + k_3x_3)}_{x_2 - x_{2,r}}x_3 = -k_3x_3^2 + (x_2 - x_{2,r})x_3, \quad \boxed{k_3 > 0}$$

with $x_{2,r} = -(x_3 + k_3)x_3$. At the next step one takes

$$V_2 = V_1 + \frac{1}{2}(x_2 - x_{2,r})^2$$

obtaining

$$
\begin{aligned}
\dot{V}_2 &= -k_3 x_3^2 - k_2(x_2 - x_{2,r})^2 \\
&\quad + (x_2 + x_3^2 + k_3 x_3)\Big[x_3 + x_2 x_3 + (x_3^2+4)x_1 + (2x_3 + k_3)(x_3^2 + x_2) + k_2(x_2 + x_3^2 + k_3 x_3)\Big], \qquad \boxed{k_2 > 0} \\
&= -k_3 x_3^2 - k_2(x_2 - x_{2,r})^2 + (x_1 - x_{1,r})(x_2 + x_3^2 + k_3 x_3)(x_3^2 + 4)
\end{aligned}
$$

$$
x_{1,r} = -\frac{1}{x_3^2 + 4}\Big(x_3 + x_2 x_3 + (2x_3 + k_3)(x_3^2 + x_2) + k_2(x_2 + x_3^2 + k_3 x_3)\Big).
$$

Note that $x_3^2 + 4 \neq 0$. Finally, the Lyapunov function is

$$
V = V_2 + \frac{1}{2}(x_1 - x_{1,r})^2
$$

with

$$
\dot{V} = -k_3 x_3^2 - k_2(x_2 - x_{2,r})^2 + (x_1 - x_{1,r})\Big[(x_2 + x_3^2 + k_3 x_3)(x_3^2 + 4) + x_3^2 + 4u - \dot{x}_{1,r}\Big]
$$

(the expression of $\dot{x}_{1,r}$ is left to the reader). Hence the control

$$
u = \frac{1}{4}\Big[-(x_2 + x_3^2 + k_3 x_3)(x_3^2 + 4) - x_3^2 + \dot{x}_{1,r} - k_1(x_1 - x_{1,r})\Big]
$$

globally exponentially stabilizes the system to the origin

$$
\dot{V} = -k_3 x_3^2 - k_2(x_2 - x_{2,r})^2 - k_1(x_1 - x_{1,r})^2
$$

since $x_3 \to 0$, $x_2 \to x_{2,r} \to 0$, $x_1 \to x_{1,r} \to 0$.