时域响应.

$sX(s)-x_0 = AX(s)+BU(s)$

$X(s) = (sI-A)^{-1}x_0+(sI-A)^{-1}BU(s)$

自由响应 free response

$Y(s) = C(sI-A)^{-1}x_0$

强迫响应 forced response

$Y(s) = C(sI-A)^{-1}BU(s)+DU(s)$

$\quad = [C(sI-A)^{-1}B+D]U(s)$

时域:

$x(t) = e^{A(t)}x(0)+\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$

$\quad = e^{A(t-t_0)}x(t_0)+\int_{t_0}^t e^{A(t-\tau)}Bu(\tau)d\tau$

$y(t) = Ce^{A(t-t_0)}x(t_0)+\int_{t_0}^t Ce^{A(t-\tau)}Bu(\tau)d\tau+Du(t)$

$\quad = Ce^{A(t-t_0)}x(t_0)+\int_{t_0}^t [Ce^{A(t-\tau)}B+D\delta(t-\tau)]u(\tau)d\tau$

冲激响应 impulse response

$h(t) = Ce^{At}B+D\delta(t)$

$H(s) = \mathcal{L}[h(t)] = C(sI-A)^{-1}B+D$

$\quad = \dfrac{C\,\mathrm{Adj}(sI-A)B+D\det(sI-A)}{\det(sI-A)}$

☆ {poles} $\subset$ {eigenvalues of $A$}

{zeros} = {roots of numerator} (SISO sys only)

right eigenvectors: $Av=\lambda v$, $v\neq 0$ 右特征向量

left eigenvectors: $w^TA=w^T\lambda$, $w\neq 0$ 左特征向量.

$A$ is diagonalizable

$\Leftrightarrow$ the dim. of right eigenspace = algebraic multeiply of this eigenvalue

$T=[v_1 \cdots v_n]$ $\quad T^{-1}=\begin{bmatrix} w_1^T \\ \vdots \\ w_n^T \end{bmatrix}$

$T^{-1}AT=\Lambda=\mathrm{diag}\{\lambda_1,\cdots,\lambda_n\}$

$T^{-1}T=I \Rightarrow w_i^Tv_j=\delta_{ij}\begin{cases}1 & i=j\\ 0 & i\neq j\end{cases}$

$y(t)=\sum_{i=1}^n e^{\lambda_i(t-t_0)}(w_i^Tx(t_0))Cv_i$

$\quad +\sum_{i=1}^n Cv_i\int_{t_0}^t e^{\lambda_i(t-\tau)}w_i^TBu(\tau)d\tau+Du(t)$

$\begin{cases}\text{if } Cv_i=0 \Rightarrow \text{unobservable mode}\\ \text{if } w_i^TB=0 \Rightarrow \text{uncontrollable mode}\end{cases}$

$H(s)=CT(sI-\Lambda)^{-1}T^{-1}B+D=\sum_{i=1}^n \dfrac{Cv_iw_i^TB}{s-\lambda_i}+D$

If $A$ is not diagonalizable, it is triangularisable

$T^{-1}AT=J=\begin{bmatrix}J_1 & & \\ & \ddots & \\ & & J_q\end{bmatrix}$ $\quad J_i=\begin{bmatrix}\lambda_i & 1 & \\ & \ddots & 1\\ & & \lambda_i\end{bmatrix}$

$H(s)=\sum_{i=1}^q\sum_{k\geq j=1}^{n_i}\dfrac{Cv_{ij}w_{ik}^TB}{(s-\lambda_i)^{k-j-1}}+D$

$\Sigma$ is internally stable if and only if $\mathrm{Re}(\lambda_i)<0$

$\Sigma$ is (external) input-output stable if and only if ① $h(t)\xrightarrow[t\to\infty]{}0$

或 ② $\mathrm{Re}(p_i)<0$

负反馈：$CL(s) = \dfrac{G(s)}{1+G(s)H(s)}$

最小实现：没有零极点对消的传递函数的实现

$\downarrow\uparrow$

既能控又能观

开环传递函数 $G(s)H(s)$ 是基于负反馈模型

$m_g = 1/g$, when phase $= -180°$

$m_\varphi = 180 + \arg(L(j\omega_p))$, when gain $= 1$

$m_d = \dfrac{m_\varphi}{\omega_p}$

$\mathcal{L}(e^{-\alpha t}) = \dfrac{1}{s+\alpha}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

能控性矩阵

$\mathcal{C}=[B\ \ AB\ \ A^2B \cdots A^{n-1}B]$ 判据: $\mathrm{rank}(\mathcal{C})=n$?

部分可控 $\mathrm{rank}(\mathcal{C})=r<n$

$T^{-1}AT=\begin{bmatrix}A_{11}&A_{12}\\0&A_{22}\end{bmatrix}$; $T^{-1}B=\begin{bmatrix}B_1\\0\end{bmatrix}$

the pair $(A_{11},B_1)$ is controllable.

$\mathrm{rank}(\mathcal{C}_1)=\mathrm{rank}([B_1\ \ A_{11}B_1 \cdots A_{11}^{r-1}B_1])$
$=\mathrm{rank}(\mathcal{C})=r$

$H(s)=C_1(sI_1-A_{11})^{-1}B_1+D$

$(A,B)$不可控的充要条件:

$\exists w,\ w^TA=w^T\lambda$ & $w^TB=0$ (左特征~~根~~向量)

此时, $(w^T,\lambda)$ is an uncontrollable mode

$\bar{A}=T^{-1}AT$; $\bar{B}=T^{-1}BG \Rightarrow \bar{\mathcal{C}}=T^{-1}\mathcal{C}G$

$\mathrm{rank}(\bar{\mathcal{C}})=\mathrm{rank}(\mathcal{C})$

the controllability property is independent of the state space basis and the input space basis.

Controllability decomposition ($\lambda_2$ 不可控)

T = [contMat(:,[1 2 3]), [0;1;0;0]]

for $H(s)=\dfrac{b_1s^{n-1}+b_2s^{n-2}+\cdots+b_n}{s^n+a_1s^{n-1}+\cdots+a_{n-1}s+a_n}+d$

Controllable canonical form

$\bar{A}=\begin{bmatrix}-a_1&-a_2&\cdots&-a_n\\1&0&\cdots&0\\&\ddots&&\vdots\\&&1&0\end{bmatrix}$ $\bar{b}=\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}$

$\bar{C}=[b_1\ \ b_2\ \cdots\ b_n]$ $\bar{d}$

$T=[e_1\ e_2\ e_3 \cdots] \Rightarrow \bar{A}=T^{-1}AT$

$e_1=b$; $e_2=Ab+a_1b$; $e_3=A^2b+a_1Ab+a_2b \cdots$

状态反馈

① controllability is invariant

② the eigenvalues of $A_{22}$ are invariant

$(A,B)$ is stabilizable with a static state feedback if and only if the unstable modes are controllable $\Rightarrow$ $A_{22}$ must be stable

$\boxed{\pi^d_{A+BF}(s)}=s^n+a_1^ds^{n-1}+a_2^ds^{n-2}+\cdots+a_n^d$

Ackermann formula

$F=-e_n^T\mathcal{C}^{-1}\pi^d_{A+BF}(A)$ 注意: $a_n^d \to a_n^d I$

$e_n^T=[0\ \cdots\ 0\ 1]$

对于规范型:

$\bar{F}=[a_1-a_1^d\ \ a_2-a_2^d\ \cdots\ a_n-a_n^d]$ 唯一, if SISO

$F=\bar{F}T^{-1}$

结构:

$\left.\begin{aligned}u&=Fx+v\\ \dot{x}&=Ax+Bu\end{aligned}\right\} \to \dot{x}=(A+BF)x+Bv$

能观性矩阵

$\mathcal{O}=[C\ \ CA\ \ CA^2 \cdots CA^{n-1}]^T$ 判据: $\mathrm{rank}(\mathcal{O})=n$?

部分能观 $\mathrm{rank}(\mathcal{O})=\cancel{r<n}\ n-r<n$

$T^{-1}AT=\begin{bmatrix}A_{11}&0\\A_{21}&A_{22}\end{bmatrix}$; $CT=[C_1\ \ 0]$

the pair $(C_1,A_{11})$ is observable.

$\mathrm{rank}(\mathcal{O}_1)=\mathrm{rank}([C_1\ \ C_1A_{11} \cdots C_1A_{11}^{n-r-1}]^T)$
$=\mathrm{rank}(\mathcal{O})=n-r$

$H(s)=C_1(sI_1-A_{11})^{-1}B_1+D$

$(C,A)$ 不能观的充要条件:

$\exists v,\ Av=\lambda v$ & $Cv=0$ (右特征向量)

此时, $(\lambda,v)$ is an unobservable mode

$\bar{A}=T^{-1}AT$; $\bar{C}=SCT \Rightarrow \bar{\mathcal{O}}=S\mathcal{O}T$

$\mathrm{rank}(\bar{\mathcal{O}})=\mathrm{rank}(\mathcal{O})$

the observability property is independent of the state space basis and the output space basis.

Observability decomposition ($\lambda_3$ 不能观)

T = [ObsvMat(:,[1 2 3]), $[1;0;\underbrace{-1;1]}_{v_3}$]

Observable canonical form (duality)

$\bar{A}=\begin{bmatrix}-a_1&1&&\\-a_2&0&\ddots&\\\vdots&\vdots&&1\\-a_n&0&\cdots&0\end{bmatrix}$ $\bar{b}=\begin{bmatrix}b_1\\b_2\\\vdots\\b_n\end{bmatrix}$

$\bar{c}=[1\ \ 0 \cdots 0]$ $\bar{d}$

$T^{-1}=[r_1\ r_2\ r_3 \cdots]^T \Rightarrow \bar{A}=T^{-1}AT$

$r_1=c$; $r_2=cA+a_1c$; $r_3=cA^2+a_1cA+a_2c \cdots$

状态观测器

① observable subspace is invariant

② the eigenvalues of $A_{22}$ are invariant.

$(C,A)$ is detectable with a static output injection if and only if the unstable modes are observable. $\Rightarrow$ $A_{22}$ must be stable.

$\pi^d_{A-KC}(s)=\cdots\cdots$

对偶性:

$(A^d,B^d,C^d,D^d,F^d)$
$\| \quad \Updownarrow$ duality
$(A^T,C^T,B^T,D^T,-K^T)$

$K=\pi^d_{A-KC}(A)\mathcal{O}^{-1}e_n$

$e_n=[0\ \cdots\ 0\ 1]^T$

$\bar{K}=[-a_1+a_1^d\ \ -a_2+a_2^d\ \cdots\ -a_n+a_n^d]^T$

$K=T\bar{K}$

$\dot{\hat{x}}=A\hat{x}+Bu+K(y-\hat{y})$

$\Downarrow$

$\dot{\hat{x}}=(A-KC)\hat{x}+Bu+Ky$

$\begin{cases}\dot{\tilde{x}}=(A-KC)\tilde{x}\\ \tilde{x}(0)=\tilde{x}_0=x_0-\hat{x}_0\end{cases}$

a sys is observable if and only if its dual sys is controllable.

## 非最小实现 Non-minimal realization

$$T^{-1}AT = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & A_{22} & 0 & A_{24} \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{bmatrix} \quad T^{-1}B = \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix}$$

$$CT = [\,0 \quad C_2 \quad 0 \quad C_4\,] \quad D$$

$$H(s) = C_2(sI_{n_2\times n_2} - A_{22})^{-1}B_2 + D$$

最小实现

$$\begin{cases} \dot{x}_2 = A_{22}x_2 + B_2u \\ y = C_2x_2 + Du \end{cases}$$

Controllable subspace

$$\mathcal{R}_0 = \sum_{i=1}^{n} \operatorname{im}(A^{i-1}B) \qquad \text{判据} \quad \mathcal{R}_0 = \mathcal{X}$$

Unobservable subspace

$$\mathcal{N}_0 = \bigcap_{i=1}^{n} \ker CA^{i-1} \qquad \mathcal{N}_0 = 0$$

## State estimate feedback

$$\begin{cases} \begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & BF \\ KC & A-KC+BF \end{bmatrix}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B \\ B \end{bmatrix}v \\ y = [\,C \quad 0\,]\begin{bmatrix} x \\ \hat{x} \end{bmatrix} + Dv \end{cases}$$

or

controllable part

$$\begin{cases} \begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \end{bmatrix} = \begin{bmatrix} A+BF & -BF \\ 0 & A-KC \end{bmatrix}\begin{bmatrix} x \\ \tilde{x} \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix}v \\ y = [\,C \quad 0\,]\begin{bmatrix} x \\ \tilde{x} \end{bmatrix} + Dv \end{cases}$$

$$\sigma\left(\begin{bmatrix} A+BF & -BF \\ 0 & A-KC \end{bmatrix}\right) = \sigma(A+BF) \cup \sigma(A-KC)$$

$\tilde{x}$ is not controllable.

$$\frac{Y(s)}{V(s)} = C(sI-(A+BF))^{-1}B$$

$$y(t) = C(sI-(A+BF))^{-1}BV(t) + C(sI-(A+BF))^{-1}[x(0) - BF(sI-(A-KC))^{-1}\tilde{x}(0)]$$

for a unobservable sys : $\begin{cases} \text{CLO for observable part} \\ \text{OLO for non-obse---} \end{cases}$

理想观测器：

$$\hat{x}(t) = O^{-1}\left(\begin{bmatrix} y(t) \\ y'(t) \\ \vdots \\ y^{(n-1)}(t) \end{bmatrix} - T\begin{bmatrix} u(t) \\ u'(t) \\ \vdots \\ u^{(n-1)}(t) \end{bmatrix}\right)$$

$$O = \begin{bmatrix} CA \\ CA \\ \vdots \\ C^{n-1} \end{bmatrix} \qquad T = \begin{bmatrix} 0 & & & \\ CB & 0 & & \\ CAB & CB & \ddots & \\ \vdots & & \ddots & \\ CA^{n-2}B & \cdots & CB & 0 \end{bmatrix}$$