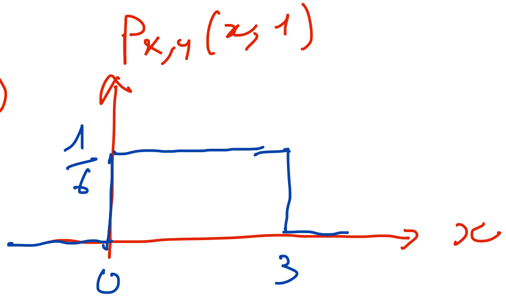
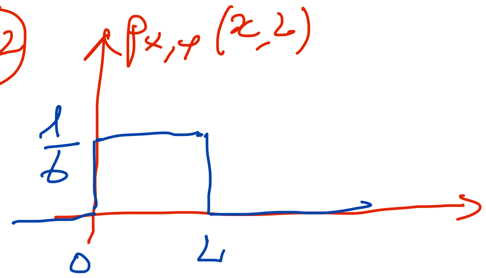
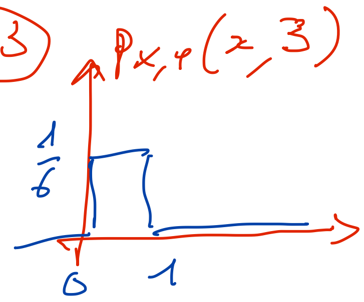
a) $y=1$



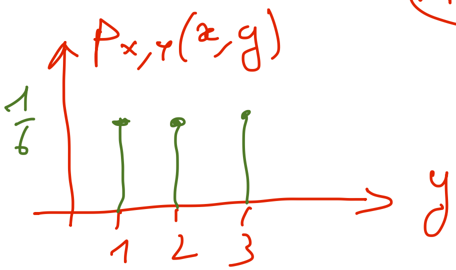
$y=2$



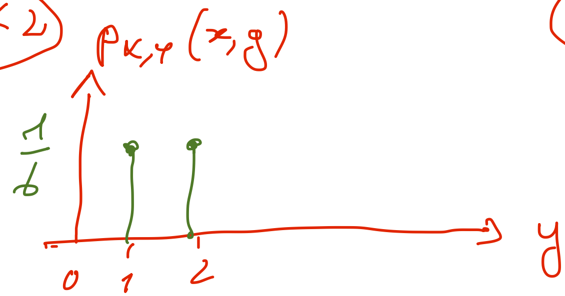
$y=3$



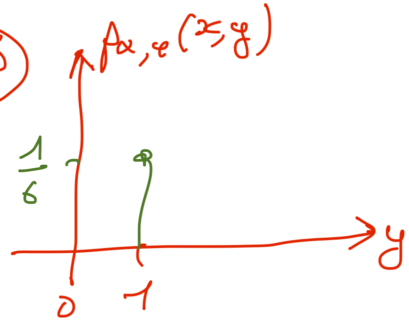
$0<x<1$



$1<x<2$

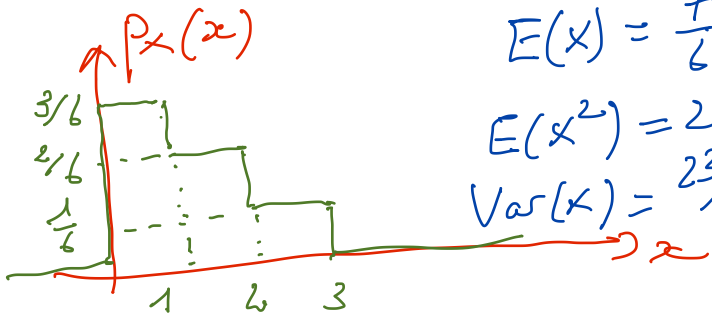


$2<x<3$



b) $$p_X(x) = \sum_y p_{X,Y}(x,y) = p_{X,Y}(x,1) + p_{X,Y}(x,2) + p_{X,Y}(x,3)$$

$$= \begin{cases} \frac{3}{6} & \text{if } 0<x<1 \\ \frac{2}{6} & \text{if } 1<x<2 \\ \frac{1}{6} & \text{if } 2<x<3 \end{cases}$$

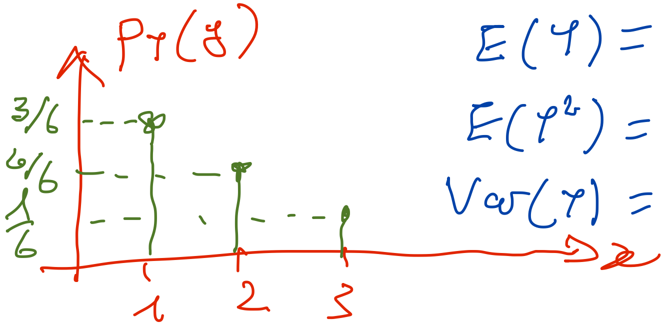


$E(X) = \frac{7}{6}$

$E(X^2) = 2$

$Var(X) = 23/36$

c) $$p_Y(y) = \int p_{X,Y}(x,y)\, dx$$



$E(Y) = 5/3$

$E(Y^2) = 10/3$

$Var(Y) = 5/9$

Classical estimation $\quad p_{Y|X}$ likelihood

Bayesian estimation $\quad \underbrace{p_{Y|X} \text{ and } p_X}_{\text{that is } p_{X,Y}}$ $\quad p_X$ is the prior distribution

Classical estimation : maximum likelihood $\quad \hat{x}_{MLE}(y) = \underset{x}{\arg\max}\, p_{Y|X}(y,x)$

Bayesian estimation : MMSE $\quad \hat{x}_{MMSE}(y) = E(X|Y)$

MAP $\quad \hat{x}_{MAP}(y) = \underset{x}{\arg\max}\, p_{X|Y}(x)$

$$= \underset{x}{\arg\max}\, p_{Y|X}(y,x)\, p_X(x)$$

LMMSE $\quad \hat{x}_{LMMSE}(y) = m_X + C_{XY} C_{YY}^{-1}(y - m_Y)$

Specific case : (X,Y) is gaussian : LMMSE = MMSE = MAP

Is an estimator «good»?

Error : $\hat{x}(Y) - X$ ~~bias~~

Bias : $E(\hat{x}(Y) - X)$ in bayesian estimation

$E(\hat{x}(Y) - X \mid X)$ in classical estimation

Error Variance : $Var(\hat{x}(Y) - X)$ in bayesian estimation

$Var(\hat{x}(Y) - X \mid X)$ in classical estimation

MSE : $E(\|\hat{x}(Y) - X\|^2)$ in bayesian estimation

$E(\|\hat{x}(Y) - X\|^2 \mid X)$ in classical estimation

$$\|X\|^2 = X^T X$$

n is the day index. $x[n]$ is the longitude

At $n = 1$, $x[n] = 0$ for sure (we know where is the harbour, it is the departure day)

The ship makes a jump of longitude D, up to an error $v[n]$ of variance q, every day

Every day, the GPS provides a measure $y[n]$ of $x[n]$, up to an error $w[n]$ of variance r

* HMModel with state $x[n]$ and observation $y[n]$
* Kalman filter
* Awful weather $(q = +\infty)$
* No GPS $(r = +\infty)$

$$\begin{cases} y[n] = x[n] + w[n] \\ x[n+1] = x[n] + D + v[n] \end{cases}$$

$$x[1] = 0$$

$H_n = 1$ $\quad h_n = 0$

$F_n = 1$ $\quad f_n = D$

Init

$$x^{|0}[1] = 0$$

$$p^{|0}[1] = 0$$

Loop

$$\hat{y}^{|n-1}[n] = \hat{x}^{|n-1}[n]$$

$$C_{xy}[n] = p^{|n-1}[n]$$

$$C_{yy}[n] = p^{|n-1}[n] + r$$

$y[n]$ is observed

$$\hat{x}^{|n}[n] = x^{|n-1}[n] + \underbrace{\frac{p^{|n-1}[n]}{p^{|n-1}[n]+r}}_{\text{Kalman gain}} \left(y[n] - \hat{x}^{|n-1}[n]\right)$$

$$p^{|n}[n] = \frac{p^{|n-1}[n]\, r}{p^{|n-1}[n] + r}$$

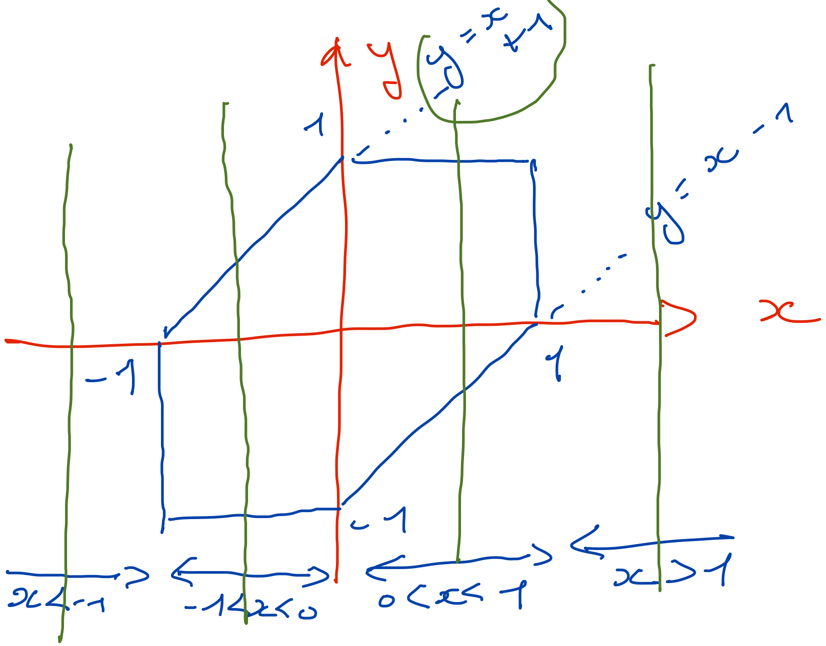
$$\hat{x}^{|n}[n+1] = \hat{x}^{|n}[n] + D$$

$$p^{|n}[n+1] = p^{|n}[n] + q$$

Special case (awful weather) : $q = +\infty$ $\quad x^{|n}[n] = y[n]$, $\quad p^{|n}[n] = r$

Special case (no GPS) : $r = +\infty$ $\quad \hat{x}^{|n}[n] = (n-1)D$ $\quad p^{|n}[n] = (n-1)q$

Exam
15/12/2021



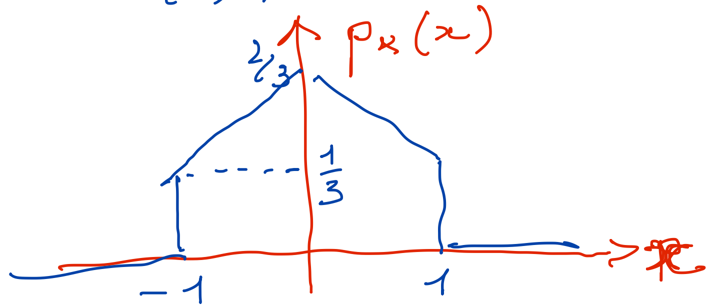
a) $p_{X,Y}(x,y) = \frac{1}{3}$ if $(x,y) \in S(X,Y)$

b) $\begin{bmatrix} Var(X) & Cov(X,Y) \\ Cov(X,Y) & Var(Y) \end{bmatrix}$

$Var(X) \simeq 0.2779$
$Var(Y) \simeq 0.2778$
$Cov(X,Y) \simeq 0.1391$

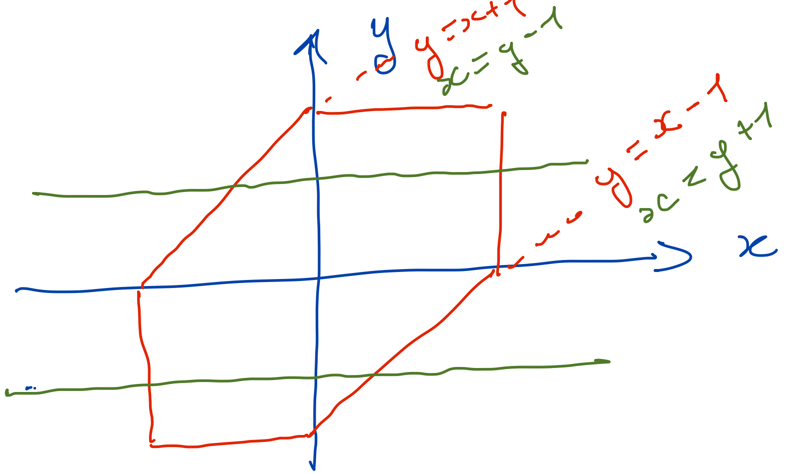
c) $p_X(x) =$

$$= \int p_{X,Y}(x,y)\,dy = \begin{cases} 0 & \text{if } x < -1 \\ \frac{x+2}{3} & \text{if } -1 < x < 0 \\ \frac{2-x}{3} & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$$



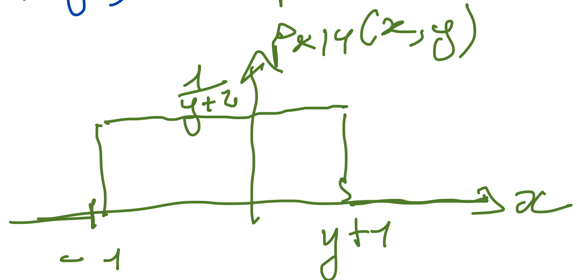
d) $E(X) = 0$

$$Var(X) = E(X^2) = \int x^2 p_X(x)\,dx = 2\int_0^1 x^2 \frac{2-x}{3}\,dx = \frac{5}{18}$$

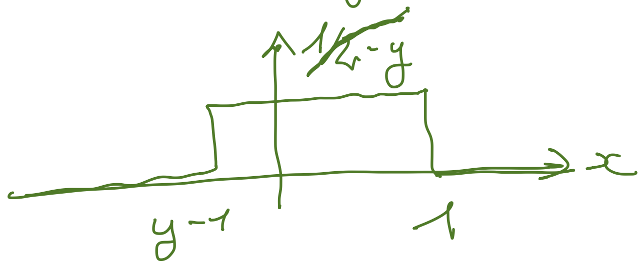


e) $p_{X|Y}(x,y) \propto p_{X,Y}(x,y)$

$-1 < y < 0$



$0 < y < 1$



length of the interval: $2-|y|$

center of the interval: $\frac{y}{2}$

$$E(X|Y=y) = \frac{y}{2}$$

$$E(X|Y) = \frac{Y}{2}$$

$$Var(X|Y) = \frac{(2-|Y|)^2}{12}$$

g) $Cov(X,Y) = Cov(E(X|Y),Y) = Cov\left(\frac{Y}{2}, Y\right) = \frac{1}{2} Cov(Y,Y)$

$$= \frac{1}{2} Var(Y) = \frac{5}{36}$$

i) $Var\left(\begin{bmatrix} X \\ Y \end{bmatrix}\right) = \begin{pmatrix} Var(X) & Cov(X,Y) \\ Cov(X,Y) & Var(Y) \end{pmatrix} = \frac{1}{36}\begin{bmatrix} 10 & 5 \\ 5 & 10 \end{bmatrix}$

$\forall y \quad \int p_{X|Y}(x,y)\, dx = 1$

i) $C_{xx} - C_{xy} C_{yy}^{-1} C_{yx} = Var(x) - \frac{Cov(x,y)^2}{Var(y)} = \frac{5}{24}$

j) $Var\left(\frac{1}{2} - x\right) = Var\left(\begin{bmatrix} -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}\right)$

$$= \begin{bmatrix} -1 & \frac{1}{2} \end{bmatrix} Var\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) \begin{bmatrix} -1 \\ \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{10}{36} & \frac{5}{36} \\ \frac{5}{36} & \frac{10}{36} \end{bmatrix} \begin{bmatrix} -1 \\ \frac{1}{2} \end{bmatrix} = \frac{5}{24}$$

k) $\hat{x}_{LMMSE}(y) = \cancel{E(x)} + \underbrace{Cov(x,y)}_{C_{xy}} \underbrace{\left[Var(y)\right]^{-1}}_{C_{yy}^{-1}} \left(y - \cancel{E(y)}\right)$

$$= \frac{y}{2}$$

$ErrVar_{LMMSE} = 5/24$

l) $\hat{x}_{MMSE}(y) = E(x|y) = \frac{y}{2}$ $\qquad ErrVar_{MMSE} = \frac{5}{24}$