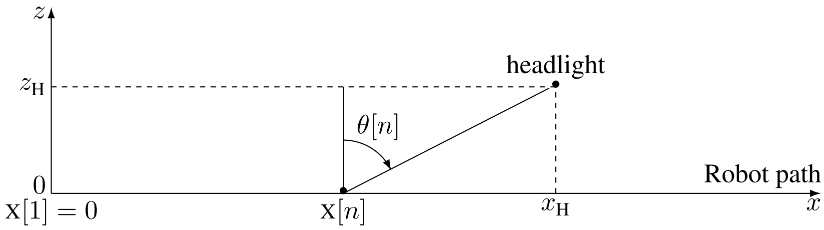
CORO

AV (first part) & STATES: sample exam 1

1 State space modeling and Kalman filter implementation

In the plane (x, z), a mobile robot moves along the straight line $z = 0$.
Its abscissa at discrete time n is $\mathrm{X}[n]$.
At time $n = 1$, $\mathrm{X}[1] = 0$ *for sure*.
To locate the robot, a headlight is available at known position $(x_{\mathrm{H}}, z_{\mathrm{H}})$.
We measure the angle $\theta[n] \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ as represented on the figure.
In the sequel, we will use the observation $\mathrm{Y}[n] = \tan\theta[n]$.



a) Using simple geometry, give the natural estimation of $\mathrm{X}[n]$ in function of $\mathrm{Y}[n]$, x_{H} and z_{H}.

b) If $\mathrm{Y}[n]$ includes a zero-mean noise $\mathrm{W}[n]$ with variance r, what is the variance of the estimator?

In the sequel, in order to improve the localization, we had a new information: the robot has an almost constant velocity, so that the distance browsed between time n and time $n+1$ is a known constant D plus a zero-mean noise $\mathrm{V}[n]$ with variance q. We have to build a state space representation of the system with state $\mathrm{X}[n]$ and output $\mathrm{Y}[n]$, in order to apply the Kalman filter.

c) Use the previous question to obtain the observation equation (that is $\mathrm{Y}[n]$ from $\mathrm{X}[n]$).

d) Write the state equation (that is $\mathrm{X}[n+1]$ from $\mathrm{X}[n]$).

e) Propose a value for the initial prediction $\hat{\mathrm{X}}^{|0}[1]$ and its variance $P^{|0}[1]$.

f) Give the update equations of the Kalman filter ($\hat{\mathrm{X}}^{|n}[n]$ and $P^{|n}[n]$ in function of $\hat{\mathrm{X}}^{|n-1}[n]$, $P^{|n-1}[n]$, r, x_{H}, z_{H} and the new observation $\mathrm{Y}[n]$).

g) Give the prediction equations ($\hat{\mathrm{X}}^{|n}[n+1]$ and $P^{|n}[n+1]$ in function of $\hat{\mathrm{X}}^{|n}[n]$, $P^{|n}[n]$, D and q).

h) The headlight is out of order ($r = +\infty$). Give $\hat{\mathrm{X}}^{|n}[n]$ and $P^{|n}[n]$ in function of n, D and q.

i) The speed varies a lot from its nominal value ($q = +\infty$). Give $\hat{\mathrm{X}}^{|n}[n]$ and $P^{|n}[n]$ in function of $\mathrm{Y}[n]$, r, x_{H}, z_{H}.

2 Probability and estimation (Poisson distribution)

A discrete random variable T is Poisson-distributed with parameter X if its probability mass function writes:

$$\mathrm{Prob}\,(\mathrm{T} = t \mid \mathrm{X}) = e^{-\mathrm{X}}\,\frac{\mathrm{X}^t}{t!} \quad \forall t \in \mathbb{N}$$

Reminder: $e^x = \sum_{t=0}^{+\infty} \frac{x^t}{t!}$.

a) What is the mean value of $\mathrm{T} \mid \mathrm{X}$.

b) What is the variance of $\mathrm{T} \mid \mathrm{X}$ (trick: first calculate $\mathrm{E}\,(\mathrm{T}\,(\mathrm{T}-1) \mid \mathrm{X})$).

c) $\vec{\mathrm{T}} = (\mathrm{T}_1, \ldots, \mathrm{T}_n)$ is a vector of n random variables independent and identically distributed given X, each of them is Poisson-distributed with parameter X. Write the log-likelihood $L_{\vec{t}}(\mathrm{X}) = \ln \mathrm{Prob}\left(\vec{\mathrm{T}} = \vec{t} \mid \mathrm{X}\right)$, for all $\vec{t} = (t_1, \ldots, t_n)$.

d) Give the maximum likelihood estimation $\hat{x}(\vec{\mathrm{T}})$.