Homework 02

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2020 年 10 月 31 日

II-2

(a)

The Lagrangian of the system is written

$$\mathcal{L} = (\nabla\Phi)^*(\nabla\Phi) - V(|\Phi|^2) \tag{1}$$

which gives the Euler-Lagrange equation:

$$\nabla\frac{\delta\mathcal{L}}{\delta(\nabla\Phi)} - \frac{\delta\mathcal{L}}{\delta\Phi} = 0, \qquad \nabla\frac{\delta\mathcal{L}}{\delta(\nabla\Phi^*)} - \frac{\delta\mathcal{L}}{\delta\Phi^*} = 0. \tag{2}$$

The system is invariant under the phase transformation

$$\Phi \to e^{-i\alpha}\Phi, \qquad \Phi^* \to e^{i\alpha}\Phi^*$$

Thus

$$\delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta\Phi}\delta\Phi + \frac{\delta\mathcal{L}}{\delta(\nabla\Phi)}\delta(\nabla\Phi) + \frac{\delta\mathcal{L}}{\delta\Phi^*}\delta\Phi^* + \frac{\delta\mathcal{L}}{\delta(\nabla\Phi^*)}\delta(\nabla\Phi^*) = 0, \tag{3}$$

and according to the Eq.(2),

$$\begin{aligned}\delta\mathcal{L} &= \nabla\frac{\delta\mathcal{L}}{\delta(\nabla\Phi)}\delta\Phi + \frac{\delta\mathcal{L}}{\delta(\nabla\Phi)}\delta(\nabla\Phi) + \nabla\frac{\delta\mathcal{L}}{\delta(\nabla\Phi^*)}\delta\Phi^* + \frac{\delta\mathcal{L}}{\delta(\nabla\Phi^*)}\delta(\nabla\Phi^*) \\ &= -i\nabla(\frac{\delta\mathcal{L}}{\delta(\nabla\Phi)}\Phi)\delta\alpha + i\nabla(\frac{\delta\mathcal{L}}{\delta(\nabla\Phi^*)}\Phi^*)\delta\alpha = 0\end{aligned} \tag{4}$$

The Noether current is defined as

$$\begin{aligned}J_\mu &= -i\frac{\delta\mathcal{L}}{\delta(\partial_\mu\Phi)}\Phi + i\frac{\delta\mathcal{L}}{\delta(\partial_\mu\Phi^*)}\Phi^* \\ &= -i(\partial_\mu\Phi^*)\Phi + i\Phi^*(\partial_\mu\Phi) \\ &= -i[(\partial_\mu\Phi^*)\Phi - \Phi^*(\partial_\mu\Phi)] \\ &= 2\mathrm{Im}[(\partial_\mu\Phi^*)\Phi].\end{aligned} \tag{5}$$

And the Noether charge is

$$
\begin{aligned}
N &= \int d^3x J_0 = 2\int d^3x \mathrm{Im}[(\partial_t \Phi^*)\Phi] \\
&= 2\int d^3x \mathrm{Im}[\pi^*\Phi]
\end{aligned}
\tag{6}
$$

We require the complex scalar field Φ and its conjugate π to be

$$
\Phi = \Phi_1 + i\Phi_2, \qquad \pi = \pi_1 + i\pi_2
$$

The Hamiltonian can be written as

$$
\begin{aligned}
H &= \int d^d x \big[\pi^*\pi + (\nabla\Phi)^*(\nabla\Phi) + V(|\Phi|^2)\big] \\
&= \int d^d x \big[\pi_1^2 + \pi_2^2 + (\nabla\Phi_1)^2 + (\nabla\Phi_2)^2 + V(|\Phi|^2)\big].
\end{aligned}
\tag{7}
$$

And the chemical potential is

$$
\begin{aligned}
-\mu N &= -2\mu \int d^3x \mathrm{Im}[\pi^*\Phi] \\
&= -2\mu \int d^3x \mathrm{Im}[(\pi_1 - i\pi_2)(\Phi_1 + i\Phi_2)] \\
&= -2\mu \int d^3x (\pi_1\Phi_2 - \pi_2\Phi_1)
\end{aligned}
\tag{8}
$$

Then we can write the partition function in path integral form

$$
\begin{aligned}
Z =& \int \mathcal{D}\pi\mathcal{D}\pi^*\mathcal{D}\Phi\mathcal{D}\Phi^* \exp\left[-\int_0^\beta d\tau(H-\mu N) + i\int d^{d+1}x((\pi(\partial_\tau\Phi^*) + \pi^*(\partial_\tau\Phi)))\right] \\
=& \int \mathcal{D}\pi\mathcal{D}\pi^*\mathcal{D}\Phi\mathcal{D}\Phi^* \exp\Bigg\{-\int d^{d+1}x\Big[\pi_1^2 + \pi_2^2 + (\nabla\Phi_1)^2 + (\nabla\Phi_2)^2 + V(|\Phi|^2) \\
&- 2\mu(\pi_1\Phi_2 - \pi_2\Phi_1) - 2i\pi_1\partial_\tau\Phi_1 - 2i\pi_2\partial_\tau\Phi_2\Big]\Bigg\} \\
=& \int \mathcal{D}\pi\mathcal{D}\pi^*\mathcal{D}\Phi\mathcal{D}\Phi^* \exp\Bigg\{-\int d^{d+1}x\Big[\pi_1^2 + 2\pi_1(-i\partial_\tau\Phi_1 - \mu\Phi_2) + \pi_2^2 \\
&+ 2\pi_2(-i\partial_\tau\Phi_2 + \mu\Phi_1) + (\nabla\Phi_1)^2 + (\nabla\Phi_2)^2 + V(|\Phi|^2)\Big]\Bigg\}.
\end{aligned}
\tag{9}
$$

Now carry out the Gaussian integral over π_1 and π_2 in Eq.(9),

$$
\begin{aligned}
&\int d\pi_1 \exp\left\{-\int d^{d+1}x\Big[\pi_1^2 + 2\pi_1(-i\partial_\tau\Phi_1 - \mu\Phi_2)\Big]\right\} \\
=& C\exp\Big[-\int d^{d+1}x(\partial_\tau\Phi_1 - i\mu\Phi_2)^2\Big]
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
&\int d\pi_2 \exp\bigg\{-\int d^{d+1}x\Big[\pi_2^2 + 2\pi_2(-i\partial_\tau\Phi_2 + \mu\Phi_1)\Big]\bigg\}\\
=&C\exp\Big[-\int d^{d+1}x(\partial_\tau\Phi_2 + i\mu\Phi_1)^2\Big]
\end{aligned}
\tag{11}
$$

And the partition function becomes

$$
\begin{aligned}
Z = C\int \mathcal{D}\Phi\mathcal{D}\Phi^* \exp\bigg\{&-\int d^{d+1}x\Big[(\partial_\tau\Phi_1 - i\mu\Phi_2)^2 + (\partial_\tau\Phi_2 + i\mu\Phi_1)^2\\
&+ (\nabla\Phi_1)^2 + (\nabla\Phi_2)^2 + V(|\Phi|^2)\Big]\bigg\}\\
= C\int \mathcal{D}\Phi\mathcal{D}\Phi^* \exp\bigg\{&-\int d^{d+1}x\Big[(\partial_\tau + \mu)\Phi^*(\partial_\tau - \mu)\Phi + (\nabla\Phi)^*(\nabla\Phi) + V(|\Phi|^2)\Big]\bigg\}
\end{aligned}
\tag{12}
$$

(b)

With the field redefinition

$$
\Phi'(x,t) = e^{i\mu t}\Phi \tag{13}
$$

the time derivation of the new field is given by

$$
\begin{aligned}
\partial_\tau\Phi' &= \mu\Phi' + e^{i\mu t}\partial_\tau\Phi\\
\partial_\tau\Phi'^* &= -\mu\Phi'^* + e^{-i\mu t}\partial_\tau\Phi^*
\end{aligned}
\tag{14}
$$

and the modified partition function becomes

$$
\begin{aligned}
Z = C\int \mathcal{D}\Phi'\mathcal{D}\Phi'^* \exp\bigg\{&-\int d^{d+1}x\Big[(\partial_\tau + \mu)\Phi'^*(\partial_\tau - \mu)\Phi' + (\nabla\Phi')^*(\nabla\Phi')\\
&+ V(|\Phi'|^2)\Big]\bigg\}\\
= C\int \mathcal{D}\Phi\mathcal{D}\Phi^* \exp\bigg\{&-\int d^{d+1}x\Big[\partial_\tau\Phi^*\partial_\tau\Phi + (\nabla\Phi)^*(\nabla\Phi) + V(|\Phi|^2)\Big]\bigg\},
\end{aligned}
\tag{15}
$$

which is equivalent to the path integral without the modification.