Solution to Problem V-1

Zhang Tingyu 35206402

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(a)

For Dirac Fermion with Lagrangian density

$$\mathcal{L} = \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi, \tag{1}$$

the canonical momentums are derived as

$$\begin{aligned} \Pi &= \frac{\partial\mathcal{L}}{\partial\dot{\Psi}} = -i\bar{\Psi}\gamma^0 \\ \bar{\Pi} &= \frac{\partial\mathcal{L}}{\partial\dot{\bar{\Psi}}} = 0. \end{aligned} \tag{2}$$

Thus the primary constraints are given by

$$\chi_1 = \Pi + i\bar{\Psi}\gamma^0, \qquad \chi_2 = \bar{\Pi}. \tag{3}$$

Notice that

$$[\Psi(\mathbf{x}), \Pi(\mathbf{y})]_P = \left[\bar{\Psi}(\mathbf{x}), \bar{\Pi}(\mathbf{y})\right]_P = \delta^3(\mathbf{x}-\mathbf{y}), \tag{4}$$

and $[\chi_1(\mathbf{x}), \chi_2(\mathbf{y})] = i\gamma^0\delta^3(\mathbf{x}-\mathbf{y}) \neq 0$, which shows that the constraints χ_1 and χ_2 are second class constraints.

$$C_{1x,2y} = -C_{2y,1x} = i\gamma^0\delta^3(\mathbf{x}-\mathbf{y}). \tag{5}$$

$$(C^{-1})^{1x,2y} = -(C^{-1})^{2y,1x} = -i\gamma^0 \int \frac{d^3k}{(2\pi)^3} e^{\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \tag{6}$$

Then the Dirac bracket can be derived by Poission bracket

$$[A, B]_D = [A, B]_P - [A, \chi_N]_P \left(C^{-1}\right)^{NM} [\chi_M, B]_P . \tag{7}$$

According to this equation, we can write these Dirac bracket:

$$
\begin{aligned}
&\left[\Psi(\mathbf{x}), \bar{\Psi}(\mathbf{y})\right]_D = -i\gamma^0\delta^3(\mathbf{x}-\mathbf{y}) \\
&[\Psi(\mathbf{x}), \Pi(\mathbf{y})]_D = \delta^3(\mathbf{x}-\mathbf{y}) \\
&\left[\bar{\Psi}(\mathbf{x}), \Pi(\mathbf{y})\right]_D = 0 \\
&\left[\bar{\Psi}(\mathbf{x}), \bar{\Pi}(\mathbf{y})\right]_D = 0.
\end{aligned} \tag{8}
$$

(b)

For photon with Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{9}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{10}$$

the canonical momentum fields are derived as

$$\Pi^\mu = \frac{\delta\mathcal{L}}{\delta(\partial_0 A_\mu)} = -F^{0\mu}. \tag{11}$$

The primary constraint is given by

$$\chi_1 = \Pi^0. \tag{12}$$

The motion equation gives the secondary constriant

$$\partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_0 A_\mu)}\right) = \partial_\mu\Pi^\mu = \partial_i\Pi^i = 0 \tag{13}$$

$$\Rightarrow \chi_2 = \partial_i\Pi^i. \tag{14}$$

Obviously, $[\chi_1, \chi_2]_P = 0$, which means they are the first class constraints. We'll have to fix a gauge to eliminate the corresponding degrees of freedom. In Coulomb gauge, we take

$$\chi_3 = \partial_i A^i = 0. \tag{15}$$

Notice that $[\chi_2, \chi_3] \neq 0$. In this system, the current $J_0 = 0$, thus we have the secondary constraint in Coulomb gauge

$$\chi_4 = \partial^i\partial_i A_0 = 0. \tag{16}$$

We can see that $[\chi_2, \chi_3] \neq 0$, which means χ_2 and χ_3 are the second class constraints. Meanwhile, χ_1 and χ_4 are first class constraints. We set

$$\chi_1' = \partial_i A^i, \qquad \chi_2' = \partial_i\Pi^i. \tag{17}$$

The elements of C matrix can be written

$$C_{1x,2y} = -C_{2y,1x} = -\partial^i \partial_i \delta^3(\mathbf{x}-\mathbf{y}), \tag{18}$$

$$(C^{-1})_{1x,2y} = -(C^{-1})_{2y,1x} = -\int \frac{d^3k}{(2\pi)^3} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}}{\mathbf{y}^2} = -\frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}. \tag{19}$$

$$\begin{aligned} [A^i(\boldsymbol{x}), \chi_{2\boldsymbol{y}}]_P &= -\frac{\partial}{\partial x^i}\delta^3(\boldsymbol{x}-\boldsymbol{y}) \\ [\Pi_i(\boldsymbol{x}), \chi_{1\boldsymbol{y}}]_P &= \frac{\partial}{\partial x^i}\delta^3(\boldsymbol{x}-\boldsymbol{y}). \end{aligned} \tag{20}$$

According to Eq.(7), we can write these Dirac brackets

$$\begin{aligned} &\left[A^\mu(\mathbf{x}), \Pi_\nu(\mathbf{y})\right]_D = (\delta^\mu{}_\nu - g_\nu{}^0 \delta^\mu{}_0)\delta^3(\mathbf{x}-\mathbf{y}) - \partial^\mu \partial_\nu \frac{1}{4\pi|\mathbf{x}-\mathbf{y}|} \\ &\left[A^\mu(\mathbf{x}), A_\nu(\mathbf{y})\right]_D = 0 \\ &\left[\Pi^\mu(\mathbf{x}), \Pi_\nu(\mathbf{y})\right]_D = 0. \end{aligned} \tag{21}$$