QFT II/QFT homework II-2 (Oct 05, 2020)

- **Be aware that this "solution" may be wrong.** If you find an error, TW will appreciate very much if you let him know.
- Please be considerate not to post a copy online, or to pass it on to your friends.

2. **Chemical potential** [B]
One can switch from the canonical ensemble of a quantum system to its grand canonical ensemble by modifying the Hamiltonian H to $H - \sum_i \mu_i N_i$, where the label i runs over (a subset of) all the conserved numbers (charges) of the system of one's interest. N_i is the Noether charge of a U(1) symmetry, and μ_i the corresponding chemical potential. In the case of the Standard Model of particle physics, for example, the lepton number and the baryon number are conserved charges.[1] When an effective theory with much lower energy scale is considered, the number of atoms of various kinds (labeled by i) may be conserved separately.

(a) **relativisistic boson case** Suppose that an effective theory with a relativistic complex boson is given by the Hamiltonian:

$$H = \int d^d x \, \left(\pi^* \pi + (\nabla\Phi)^*(\nabla\Phi) + V(|\Phi|^2)\right), \tag{1}$$

where π is the canonical conjugate momentum of a complex scalar field Φ^*. First, write down the Noether charge $N = \int d^3x J^0$ in terms of π and Φ. Secondly, carry out the Gaussian integral with respect to π and π^* in

$$Z = \int \mathcal{D}\pi\mathcal{D}\pi^*\mathcal{D}\Phi\mathcal{D}\Phi^* \exp\left[-i\int dt(H-\mu N) + i\int d^{d+1}x\,(\pi(\partial_t\Phi^*) + \pi^*(\partial_t\Phi))\right], \tag{2}$$

to see how the Lagrangian is modified.

(a) A: First,

$$N = \int d^3x J^0 = \int d^3x \left(i\Phi^*(\partial^0\Phi) - i(\partial^0\Phi^*)\Phi\right) = \int d^3x\,(i\Phi^*\pi - i\pi^*\Phi)\,. \tag{3}$$

[1] when the tiny neutrino masses and non-perturbative electroweak effects are ignored, to be more precise.

Second,

$$
\begin{aligned}
Z &= \int \mathcal{D}\pi\mathcal{D}\pi^*\mathcal{D}\Phi\mathcal{D}\Phi^* \exp\left[-i\int dtd^3x\right.\\
&\qquad\qquad (\pi^*\pi - (\partial_t\Phi^*)\pi - \pi^*(\partial_t\Phi) - i\mu\Phi^*\pi + i\mu\pi^*\Phi\\
&\qquad\qquad \left.+|\nabla\Phi|^2 + V(|\Phi|)\big)\right], \qquad (4)\\
&= \int \mathcal{D}\pi\mathcal{D}\pi^*\mathcal{D}\Phi\mathcal{D}\Phi^* \exp\left[-i\int dtd^3x\right.\\
&\qquad\qquad ((\pi^* - \partial_t\Phi^* - i\mu\Phi^*)(\pi - \partial_t\Phi + i\mu\Phi)\\
&\qquad\qquad \left.-[(\partial_t + i\mu)\Phi^*][(\partial_t - i\mu)\Phi] + |\nabla\Phi|^2 + V(|\Phi|)\big)\right], \qquad (5)\\
&\propto \int \mathcal{D}\Phi\mathcal{D}\Phi^* \exp\left[+i\int dtd^3x\left([(\partial_t + i\mu)\Phi^*][(\partial_t - i\mu)\Phi] - |\nabla\Phi|^2 - V(|\Phi|)\right)\right]. \qquad (6)
\end{aligned}
$$

So, the modified Lagrangian density is read out to be

$$\mathcal{L} = (D_\mu\Phi)^*(D^\mu\Phi) - V(|\Phi|), \qquad (7)$$

where the covariant derivative D_0 is $D_0 = \partial_t - i\mu$ on Φ and $D_0 = \partial_0 + i\mu$ on Φ^*.

(b) Verify that the path integral with the $e^{\mu N\beta} = e^{i\int dt\mu N}$ factor is equivalent to the path integral without the modification, but with the field redefinition $\Phi_{\text{orignl}}(x,t) = e^{-i\mu t}\Phi_{\text{new}}(x,t)$.

(b) A: under the field redefinition $\Phi_{\text{original}} = e^{-i\mu t}\Phi_{\text{new}}$, we have

$$\partial_t\Phi_{\text{org}} = e^{-i\mu t}(\partial_t - i\mu)\Phi_{\text{new}}, \qquad \partial_t\Phi^*_{\text{org}} = e^{i\mu t}(\partial_t + i\mu)\Phi^*_{\text{new}}. \qquad (8)$$