Homework 03

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(a)

$$\begin{aligned}\int d^3x\psi^\dagger\psi &= \int d^3x\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3q}{(2\pi)^3}\sum_{r,s}\xi_r^T\xi_s a^\dagger_{\vec{p},r}a_{\vec{q},s}e^{i(E_{\vec{p}}-E_{\vec{q}})t-i(\vec{p}-\vec{q})\cdot\vec{x}}\\&=\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3q}{(2\pi)^3}\sum_{r,s}\xi_r^T\xi_s a^\dagger_{\vec{p},r}a_{\vec{q},s}e^{i(E_{\vec{p}}-E_{\vec{q}})t}(2\pi)^3\delta^3(\vec{p}-\vec{q})+\text{const}\\&=\int\frac{d^3p}{(2\pi)^3}\sum_{r,s}\xi_r^T\xi_s a^\dagger_{\vec{p},r}a_{\vec{p},s}+\text{const}\\&=\int\frac{d^3p}{(2\pi)^3}\sum_{r}a^\dagger_{\vec{p},r}a_{\vec{p},r}+\text{const.}\end{aligned}$$

Similarly, we get

$$\begin{aligned}H&=\int d^3x\psi^\dagger\Big[-\frac{1}{2m}\vec{\partial}\cdot\vec{\partial}\Big]\psi\\&=\int\frac{d^3p}{(2\pi)^3}\frac{|\vec{p}|^2}{2m}\sum_r a^\dagger_{\vec{p},r}a_{\vec{p},r}+\text{const.}\end{aligned}$$

Thus

$$H'\equiv H-\epsilon_F\int d^3x\psi^\dagger\psi=\int\frac{d^3p}{(2\pi)^3}\sum_r(E_{\vec{p}}-\epsilon_F)a^\dagger_{\vec{p},r}a_{\vec{p},r}+\text{const}$$

(b)

The Noether current J^μ is

$$J^\mu=-i\frac{\delta\mathcal{L}}{\delta(\partial_\mu\psi)}\psi+i\frac{\delta\mathcal{L}}{\delta(\partial_\mu\psi^\dagger)}\psi^\dagger$$

and the $\mu = 0$ component is

$$J^0 = -i\frac{\delta\mathcal{L}}{\delta(\partial_t\psi)}\psi + i\frac{\delta\mathcal{L}}{\delta(\partial_t\psi^\dagger)}\psi^\dagger = \psi^\dagger\psi.$$

The Noether charge, the electron number is $N_e = \int d^3x\psi^\dagger\psi$.

(d)

$$\begin{aligned}
H' &= \int \frac{d^3p}{(2\pi)^3}\sum_r (E_{\vec{p}} - E_{p_F})a^\dagger_{\vec{p},r}a_{\vec{p},r} + \text{const} \\
&= -\int\limits_{|\vec{p}|<|\vec{p}_F|} \frac{d^3p}{(2\pi)^3}\sum_r (E_{p_F} - E_{\vec{p}})a^\dagger_{\vec{p},r}a_{\vec{p},r} \\
&\quad + \int\limits_{|\vec{p}|>|\vec{p}_F|} \frac{d^3p}{(2\pi)^3}\sum_r (E_{\vec{p}} - E_{p_F})a^\dagger_{\vec{p},r}a_{\vec{p},r} + \text{const.}
\end{aligned}$$

Notice that $a^\dagger_{\vec{p},r}a_{\vec{p},r} = 1 - b^\dagger_{\vec{p},r}b_{\vec{p},r}$, and the Hamiltonian H' may be rewritten:

$$\begin{aligned}
H' = &-\int\limits_{|\vec{p}|<|\vec{p}_F|} \frac{d^3p}{(2\pi)^3}\sum_r (E_{p_F} - E_{\vec{p}}) + \int\limits_{|\vec{p}|<|\vec{p}_F|} \frac{d^3p}{(2\pi)^3}\sum_r (E_{p_F} - E_{\vec{p}})b^\dagger_{\vec{p},r}b_{\vec{p},r} \\
&+ \int\limits_{|\vec{p}|>|\vec{p}_F|} \frac{d^3p}{(2\pi)^3}\sum_r (E_{\vec{p}} - E_{p_F})a^\dagger_{\vec{p},r}a_{\vec{p},r} + \text{const} \\
= &\int\limits_{|\vec{p}|<|\vec{p}_F|} \frac{d^3p}{(2\pi)^3}\sum_r (E_{p_F} - E_{\vec{p}})b^\dagger_{\vec{p},r}b_{\vec{p},r} \\
&+ \int\limits_{|\vec{p}|>|\vec{p}_F|} \frac{d^3p}{(2\pi)^3}\sum_r (E_{\vec{p}} - E_{p_F})a^\dagger_{\vec{p},r}a_{\vec{p},r} + \text{const.}
\end{aligned}$$

When $b^\dagger_{\vec{p},r}b_{\vec{p},r} = 0$ for each $|\vec{p}| < |\vec{p}_F|$, the Hamiltonian reaches minimum, the system is at ground state, in which all the levels below the Fermi surface are filled.

$$\begin{aligned}
&T\{\psi(\vec{x},t)\psi^\dagger_I(\vec{y},t')\} \\
=&\Theta(t-t')\int \frac{d^3p}{(2\pi)^3}\sum_{r,s}\xi_r\xi_s^T a_{\vec{p},r}e^{-iE_{\vec{p}}t+i\vec{p}\cdot\vec{x}}\int \frac{d^3q}{(2\pi)^3}a^\dagger_{\vec{q},s}e^{iE_{\vec{q}}t'-i\vec{q}\cdot\vec{y}} \\
&-\Theta(t-t')\int \frac{d^3q}{(2\pi)^3}\sum_{r,s}\xi_r\xi_s^T a^\dagger_{\vec{q},s}e^{iE_{\vec{q}}t'-i\vec{q}\cdot\vec{y}}\int \frac{d^3p}{(2\pi)^3}a_{\vec{p},r}e^{-iE_{\vec{p}}t+i\vec{p}\cdot\vec{x}}
\end{aligned}$$

Notice that

$$\langle 0| a^\dagger_{\vec{p},r} a_{\vec{q},s} |0\rangle = 0, \qquad \langle 0| a_{\vec{p},r} a^\dagger_{\vec{q},s} |0\rangle = (2\pi)^3 \delta_{r,s} \delta^3(\vec{p}-\vec{q})$$

Thus we can write

$$\begin{aligned}
&\langle 0| T\left\{\psi(\vec{x},t)\psi_I^\dagger(\vec{y},t')\right\} |0\rangle \\
=&\Theta(t-t')\Theta(|\vec{p}|-p_F)\int \frac{d^3p}{(2\pi)^3}\sum_{r,s}\xi_r\xi_s^T e^{-i(E_{\vec{p}}-\epsilon_F)t+i\vec{p}\cdot\vec{x}} \\
&\times \int \frac{d^3q}{(2\pi)^3} e^{i(E_{\vec{q}}-\epsilon_F)t'-i\vec{q}\cdot\vec{y}}(2\pi)^3\delta_{r,s}\delta(\vec{p}-\vec{q}) \\
&-\Theta(t'-t)\Theta(p_F-|\vec{p}|)\int \frac{d^3q}{(2\pi)^3}\sum_{r,s}\xi_r\xi_s^T e^{i(E_{\vec{q}}-\epsilon_F)t'-i\vec{q}\cdot\vec{y}} \\
&\times \int \frac{d^3p}{(2\pi)^3} e^{-i(E_{\vec{p}}-\epsilon_F)t+i\vec{p}\cdot\vec{x}}(2\pi)^3\delta_{r,s}\delta(\vec{p}-\vec{q}) \\
=&\mathbf{1}_{2\times 2}\int \frac{d^3p}{(2\pi)^3}\Big(\Theta\left(t-t'\right)\Theta\left(|p|-p_F\right)e^{-i(E_p-\epsilon_F)(t-t')+i\vec{p}\cdot(\vec{x}-\vec{y})} \\
&-\Theta\left(t'-t\right)\Theta\left(p_F-|p|\right)e^{-i(E_p-\epsilon_F)(t-t')-i\vec{p}\cdot(\vec{x}-\vec{y})}\Big)
\end{aligned}$$