Homework 01

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$$\langle q_f| e^{-HT} |q_i\rangle = \sqrt{\frac{m\omega}{2\pi i \sin(\omega T)}} \exp\left\{\frac{im\omega}{2\sin(\omega T)}[\cos(\omega T)(q_f^2+q_i^2)-2q_iq_f]\right\}$$

Notice that

$$i\sin(\omega T) = \frac{1}{2}(e^{i\omega T}-e^{-i\omega T})$$
$$\cos(\omega T) = \frac{1}{2}(e^{i\omega T}+e^{-i\omega T})$$

then we can expand the expression in powers of $e^{-i\omega T}$, to the second order:

$$\begin{aligned}
\langle q_f| e^{-HT} |q_i\rangle &= \sqrt{\frac{m\omega}{\pi}} e^{-i\frac{\omega}{2}T}(1-e^{-i2\omega T})^{-\frac{1}{2}} \\
&\times \exp\left\{-m\omega\Big[\frac{e^{i\omega T}+e^{-i\omega T}}{2(e^{i\omega T}-e^{-i\omega T})}(q_f^2+q_i^2) - \frac{2q_iq_f}{e^{i\omega T}-e^{-i\omega T}}\Big]\right\} \\
&= \sqrt{\frac{m\omega}{\pi}} e^{-i\frac{\omega}{2}T}(1+\frac{1}{2}e^{-i2\omega T}+\dots) \\
&\times \exp\{-\frac{m\omega}{2}[(q_f^2+q_i^2)(1+2e^{-i2\omega T}+\dots)-4q_iq_fe^{-i\omega T}(1+e^{-i2\omega T}+\dots)]\} \\
&= \sqrt{\frac{m\omega}{\pi}} e^{-i\frac{\omega}{2}T}(1+\frac{1}{2}e^{-i2\omega T}+\dots)e^{-\frac{m\omega}{2}(q_f^2+q_i^2)} \\
&\times [1-m\omega e^{-i2\omega T}+2m\omega q_iq_fe^{-i\omega T}+2m^2\omega^2q_i^2q_f^2e^{-i2\omega T}+\dots]
\end{aligned}$$

From the equation above, we can find out the energy eigenvalues $\{E_n\}$

$$E_n = \omega(\frac{1}{2}+n)$$

The lowest order term is

$$\sqrt{\frac{m\omega}{\pi}} e^{-i\frac{\omega}{2}T} e^{-\frac{m\omega}{2}(q_f^2+q_i^2)}$$

thus the wave function for the ground state is

$$\psi_0(q_i) = (\frac{m\omega}{\pi})^{\frac{1}{4}} e^{-\frac{m\omega}{2}q_i^2}$$
$$\psi_0(q_i) = (\frac{m\omega}{\pi})^{\frac{1}{4}} e^{-\frac{m\omega}{2}q_f^2}$$

The next term is

$$2m\omega\sqrt{\frac{m\omega}{\pi}} e^{-i\frac{3}{2}\omega T} q_i q_f e^{-\frac{m\omega}{2}(q_f^2+q_i^2)}$$

the wave function for the first excited state is

$$\psi_1(q_i) = (m\omega)^{\frac{3}{4}}(\frac{4}{\pi})^{\frac{1}{4}} q_i e^{-\frac{m\omega}{2}q_i^2}$$
$$\psi_1(q_f) = (m\omega)^{\frac{3}{4}}(\frac{4}{\pi})^{\frac{1}{4}} q_f e^{-\frac{m\omega}{2}q_f^2}$$

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(a)

$$\begin{aligned}
\langle 0| e^{-\beta H} |0\rangle &= \int d\bar{\theta}_N d\theta_N \langle 0| (|0\rangle + \theta_N |1\rangle)\Psi_{fin}(\bar{\theta}_N) e^{-\bar{\theta}_N\theta_N} \\
&= \int d\bar{\theta}_N d\theta_N d\bar{\theta}_{N-1} d\theta_{N-1} \dots d\bar{\theta}_0 d\theta_0 e^{[\dots]} \Psi_{in}(\bar{\theta}_0)(1 - \bar{\theta}_N\theta_N) \\
&= \int d\bar{\theta}_N d\theta_N d\bar{\theta}_{N-1} d\theta_{N-1} \dots d\bar{\theta}_0 d\theta_0 e^{[\dots]} \bar{\theta}_N\theta_N \\
&= \int d\bar{\theta}_N d\bar{\theta}_{N-1} d\theta_{N-1} \dots d\bar{\theta}_0 d\theta_0 e^{[\dots]} \bar{\theta}_N
\end{aligned}$$

$$\begin{aligned}
\langle 1| e^{-\beta H} |1\rangle &= \int d\bar{\theta}_N d\theta_N \langle 1| (|0\rangle + \theta_N |1\rangle)\Psi_{fin}(\bar{\theta}_N) e^{-\bar{\theta}_N\theta_N} \\
&= \int d\bar{\theta}_N d\theta_N \theta_N d\bar{\theta}_{N-1} d\theta_{N-1} \dots d\bar{\theta}_0 d\theta_0 e^{[\dots]} \Psi_{in}(\bar{\theta}_0)(1 - \bar{\theta}_N\theta_N) \\
&= \int d\bar{\theta}_N d\theta_N d\bar{\theta}_{N-1} d\theta_{N-1} \dots d\bar{\theta}_0 d\theta_0 e^{[\dots]} \Psi_{in}(\bar{\theta}_0)\theta_N\bar{\theta}_0 \\
&= \int d\bar{\theta}_N d\bar{\theta}_{N-1} d\theta_{N-1} \dots d\bar{\theta}_0 d\theta_0 e^{[\dots]} \bar{\theta}_0
\end{aligned}$$

Thus the partition function Z is written

$$
\begin{aligned}
Z = \mathrm{tr}\left[e^{-\beta H}\right] &= \langle 0| \, e^{-\beta H} \, |0\rangle + \langle 1| \, e^{-\beta H} \, |1\rangle \\
&= \int d\bar{\theta}_N d\bar{\theta}_{N-1} d\theta_{N-1} \ldots d\bar{\theta}_0 d\theta_0 e^{[\cdots]} (\bar{\theta}_N + \bar{\theta}_0)
\end{aligned}
$$

(b)

We suppose the Grassmann function $f(\bar{\theta}_N, \bar{\theta}_0) = c_{00} + c_{10}\bar{\theta}_N + c_{01}\bar{\theta}_0 + c_{11}\bar{\theta}_N\bar{\theta}_0$, then

$$
\begin{aligned}
&\int d\bar{\theta}_N (\bar{\theta}_N + \bar{\theta}_0) f(\bar{\theta}_N, \bar{\theta}_0) \\
=&\int d\bar{\theta}_N (\bar{\theta}_N + \bar{\theta}_0)(c_{00} + c_{10}\bar{\theta}_N + c_{01}\bar{\theta}_0 + c_{11}\bar{\theta}_N\bar{\theta}_0) \\
=&\int d\bar{\theta}_N (c_{00}\bar{\theta}_N - c_{10}\bar{\theta}_N\bar{\theta}_0 + c_{01}\bar{\theta}_N\bar{\theta}_0) \\
=&c_{00} - c_{10}\bar{\theta}_0 + c_{01}\bar{\theta}_0 = f(-\bar{\theta}_0, \bar{\theta}_0),
\end{aligned}
$$

which means that the factor $(\bar{\theta}_N + \bar{\theta}_0)$ inserted in a Grassmann integral can be regarded as something like a delta function $\delta(\bar{\theta}_N + \bar{\theta}_0)$.

(d)

$$
\begin{aligned}
&\int d\theta d\bar{\theta} d\theta' d\bar{\theta}' \exp\left[(\bar{\theta}, \bar{\theta}') \begin{pmatrix} -m & p \\ p & -m \end{pmatrix} \begin{pmatrix} \theta \\ \theta' \end{pmatrix}\right] \\
=&\int d\theta d\bar{\theta} d\theta' d\bar{\theta}' \exp\left(-\bar{\theta}\theta m + \bar{\theta}'\theta p + \bar{\theta}\theta' p - \bar{\theta}'\theta' m\right) \\
=&\int d\theta d\bar{\theta} d\theta' d\bar{\theta}' [1 - \bar{\theta}\theta m + \bar{\theta}'\theta p + \bar{\theta}\theta' p - \bar{\theta}'\theta' m + \frac{1}{2}(\bar{\theta}\theta\bar{\theta}'\theta' m^2 + \bar{\theta}'\theta\bar{\theta}\theta' p^2)] \\
=&\frac{1}{2}\int d\theta d\bar{\theta} d\theta' d\bar{\theta}' (\bar{\theta}'\theta'\bar{\theta}\theta m^2 - \bar{\theta}'\theta'\bar{\theta}\theta p^2) \\
=&\frac{1}{2}(m^2 - p^2)
\end{aligned}
$$

(e)

The algebra of creation and annihilation operators in fermion case is

$$
\{\hat{a}, \hat{a}\} = \{\hat{a}^\dagger, \hat{a}^\dagger\} = 0, \qquad \{\hat{a}, \hat{a}^\dagger\} = 1.
$$

The Hamiltonian can be written analogous to that of bosons

$$\hat{H} = \frac{\hbar\omega}{2}(\hat{a}^\dagger \hat{a} - \hat{a}\hat{a}^\dagger) = \hbar\omega\Big(\hat{a}^\dagger \hat{a} - \frac{1}{2}\Big)$$

The partition function can be written

$$\begin{aligned} Z &= \mathrm{tr}\big[e^{-\beta H}\big] = \ \langle 0|\, e^{-\beta H}\, |0\rangle + \ \langle 1|\, e^{-\beta H}\, |1\rangle \\ &= e^{\frac{\beta\hbar\omega}{2}}\Big[\ \langle 0|0\rangle + \sum_{n=0}^{\infty} \frac{(-\beta\hbar\omega)^n}{n!}\ \langle 1|\, (\hat{a}^\dagger \hat{a})^n\, |1\rangle\ \Big] \end{aligned}$$

Notice that $\langle 1|\, (\hat{a}^\dagger \hat{a})^n\, |1\rangle = 1$, and the expression can be simplified

$$Z = e^{\frac{\beta\hbar\omega}{2}}(1 + e^{-\beta\hbar\omega}).$$

The corresponding free energy is

$$F = -T\ln Z = T\Big(- \frac{\beta\hbar\omega}{2} - \ln\big(1 + e^{-\beta\hbar\omega}\big)\Big)$$