QFT II/QFT   homework I-1 (Sept. 28, 2020)

- **Be aware that this "sample solution" may be wrong or contain error.** If you find an error, TW will appreciate very much if you let him know.
- Please be considerate not to post a copy online, or to pass it on to your friends.

1. **Harmonic oscillator energy levels in path integral formulation** [B]
   In a harmonic oscillator system with mass m and frequency ω (see the lecture note for more about the convention), path-integral can be used to derive

$$\langle q_f|e^{-iHT}|q_i\rangle = \sqrt{\frac{m\omega}{2\pi i\sin(\omega T)}}e^{iS_{\rm cl}}, \tag{1}$$

$$S_{\rm cl} = \frac{m\omega}{2}\frac{1}{\sin(\omega T)}\left[\cos(\omega T)(q_f^2+q_i^2)-2q_iq_f\right]. \tag{2}$$

Derivation of the result above is found in the lecture note, and also in section 8? of the textbook by Feynman and Hibbs. Now, use the result above, and expand it into the form of

$$\langle q_f|e^{-iHT}|q_i\rangle = \sum_n \psi_n(q_f)\psi_n^*(q_i)e^{-iE_nT}, \tag{3}$$

to find out the energy eigenvalues $\{E_n\}$ and their corresponding wavefunctions, for the ground state and the first excited state. [Derivation will be found in the Feynman Hibbs textbook, but it will be fun if you do it on your own.]

....

Sample Solution

The plan is just to do the Taylor series expansion with respect to $e^{-i\omega T}$. So, of course,

$$2i\sin(\omega T) = e^{i\omega T}\left(1-e^{-2i\omega T}\right), \tag{4}$$

$$2\cos(\omega T) = e^{i\omega T}\left(1+e^{-2i\omega T}\right). \tag{5}$$

Now,

$$\sqrt{\frac{m\omega}{\pi}}\frac{1}{\sqrt{2i\sin(\omega T)}} = \sqrt{\frac{m\omega}{\pi}}e^{-i\frac{\omega}{2}T}\left(1+\frac{1}{2}e^{-2i\omega T}+\frac{3}{8}e^{-4i\omega T}+\cdots\right), \tag{6}$$

and

$$iS_{\rm cl} \;= -\frac{m\omega}{2}(q_i^2+q_f^2)\left(1+\frac{2e^{-i2\omega T}}{1-e^{-2i\omega T}}\right)+2m\omega q_i q_f\left(e^{-i\omega T}+e^{-3i\omega T}+\cdots\right). \tag{7}$$

So,

$$\begin{aligned}
&\langle q_f@T|e^{-iHT}\;|q_i@0\rangle\times\sqrt{\frac{\pi}{m\omega}} \\
&\qquad = e^{-i\frac{\omega}{2}T}e^{-\frac{m\omega}{2}q_f^2}e^{-\frac{m\omega}{2}q_i^2} \\
&\qquad\quad + e^{-i\frac{3}{2}\omega T}e^{-\frac{m\omega}{2}q_f^2}e^{-\frac{m\omega}{2}q_i^2}(\sqrt{2m\omega}q_f)(\sqrt{m\omega}q_i) \\
&\qquad\quad + e^{-i\frac{3}{2}\omega T}e^{-\frac{m\omega}{2}q_f^2}e^{-\frac{m\omega}{2}q_i^2}\left(\frac{1}{2}-m\omega(q_i^2+q_f^2)+2(m\omega)^2q_i^2q_f^2\right).
\end{aligned} \tag{8}$$

Note that

$$\left(\frac{1}{2}-m\omega(q_i^2+q_f^2)+2(m\omega)^2q_i^2q_f^2\right)=\left(\frac{1}{\sqrt{2}}-\sqrt{2}m\omega q_i^2\right)\left(\frac{1}{\sqrt{2}}-\sqrt{2}m\omega q_f^2\right). \tag{9}$$

We have thus managed to learn that the energy eigenvalues and the corresponding wavefunctions of this system are

$$E_0=\frac{\omega}{2} \qquad \psi_0(q)=\left(\frac{m\omega}{\pi}\right)^{\frac{1}{4}}e^{-\frac{m\omega}{2}q^2}, \tag{10}$$

$$E_1=\frac{3}{2}\omega \qquad \psi_1(q)=\left(\frac{m\omega}{\pi}\right)^{\frac{1}{4}}e^{-\frac{m\omega}{2}q^2}\;(\sqrt{2m\omega}q), \tag{11}$$

$$E_2=\frac{5}{2}\omega \qquad \psi_2(q)=\left(\frac{m\omega}{\pi}\right)^{\frac{1}{4}}e^{-\frac{m\omega}{2}q^2}\;\frac{2m\omega q^2-1}{\sqrt{2}}. \tag{12}$$

Hermite polynomials are $H_0(x)\propto 1$, $H_1(x)\propto x$ and $H_2(x)\propto x^2-1$.

So, without solving the Schroedinger equation, we get the right answer including the zero-point oscillation energy and the wavefunction normalization. It is a wow experience, isn't it?