ELEMENTARY PARTICLES

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September 30, 2025

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1 Introduction

Elementary or fundamental particles are treated as pointlike objects in particle physics. The spatial size of these particles is not observable yet. All elementary particles of the same type are utterly identical. They can be described as excitations of quantum fields in the framework of quantum field theory (QFT). Elementary particles can be associated with 4 fundamental forces or interactions: gravity, electromagnetism, strong, and weak. The laws of physics between elementary particles and fundamental interactions are formulated in the Standard Model (SM).

The SM predictions are all confirmed by experiments, but it is not the whole story of physics. For now, gravity is still incompatible with the SM. Moreover, the SM cannot explain dark matter and dark energy. Figure 1.1 shows the most famous figure of the SM.

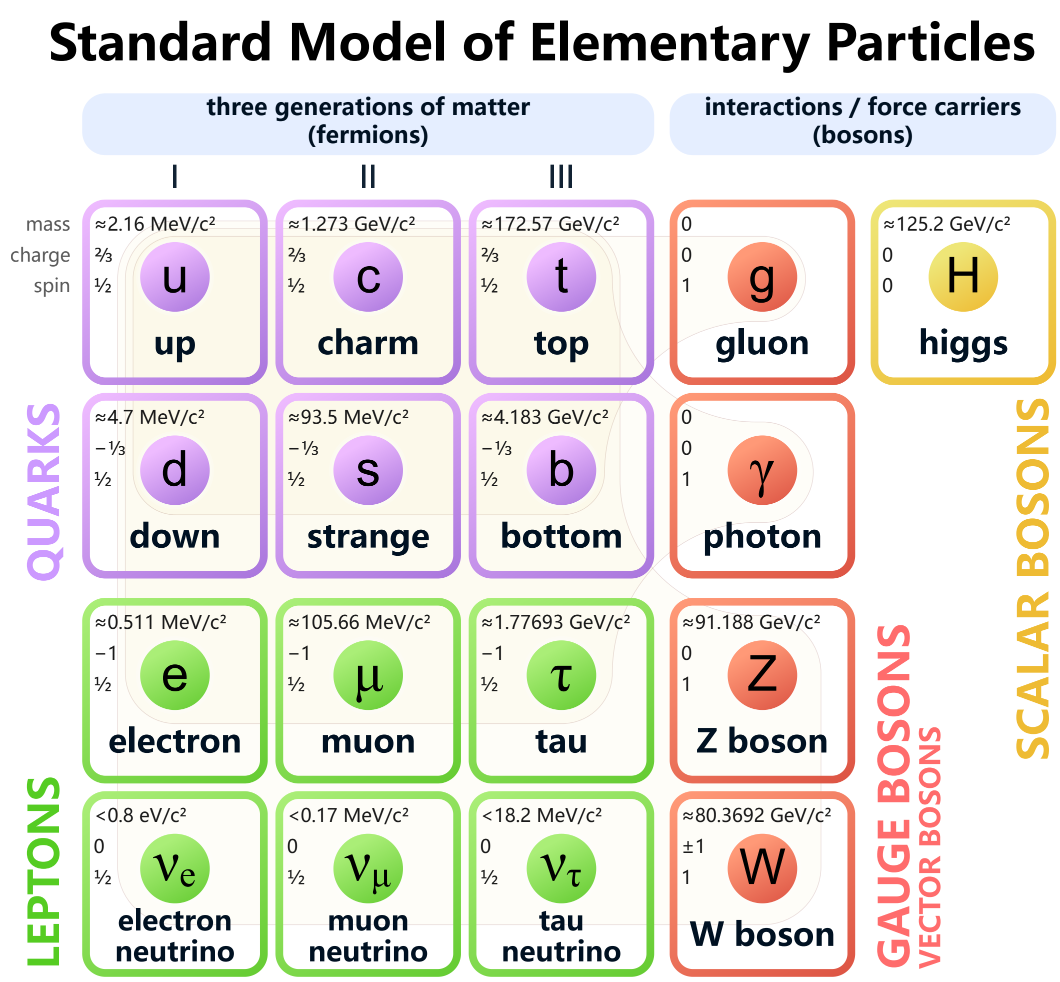


Figure 1.1: The table of the Standard Model.

1.1 Properties of Elementary Particles

Elementary particles can be classified via various properties:

- Mass (m): particles can be massive or massless. Massless particles ($m = 0$) include photons and gluons. Massive particles ($m \neq 0$) include leptons, quarks, the W, Z, and Higgs (H) boson. Electron, muon, and tau are often denoted as ℓ^- altogether, where $\ell = e, \mu, \tau$, and their corresponding neutrinos are denoted as ν_ℓ. Quarks are written as q_i, where $q_i = u, d, s, c, b, t$.
- Spin (J): the spin is an intrinsic angular momentum of a particle that has no classical counterparts. Particles in the SM have

spin $J = 0$, $1/2$, or 1.

Spin (J)	0	1/2	1
Particles	Higgs boson	Leptons and quarks	Gauge bosons
Symbols	H	ν_ℓ, ℓ^-, q_i	$\gamma, g, Z, W^\pm$
Fields	Higgs field	Dirac field	Gauge field

- Charge: charges specify the couplings to quantum fields. For example, an electron has the electric charge subject to electromagnetic interactions, quarks have the color charge associated with the strong force, etc. Charges also specify interactions. The logic of reasoning whether a particle experience a certain interaction is the following: What charges does it have? What do these charges couple to? Then What interactions is associated with those charges? For example, quarks have color charge, electric charge, and weak charge. Thus, it can couple to gluons, photons, and W and Z. It experience strong, EM, and weak interactions. Another example is the photon. Photons do not have electric charges, so they *do not couple to other photons*, so photons do not experience EM interactions (with other photons).
- Lifetime (τ): Most particles will decay into other particles after some characteristic time called the mean lifetime. Particle number is not conserved as a consequence.

According to QFT, for each *type* of particle a, there must be a corresponding antiparticle $\bar{a}$. Compared to a, its antiparticle has exactly the same mass, mean lifetime, spin, but oppositve charge. For example, the antiparticle of the electron e^- is the positron e^+.

1.2 Quantum Mechanics + Relativity

When quantum mechanics meets special relativity, they produce relativistic quantum mechanics and later quantum field theory. In standard quantum mechanics, particles are described by the Schrödinger equation:

$$i\hbar\frac{\partial\Psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\Psi.$$

One can immediately see that the Schrödinger equation is not complete and is incompatible with special relativity. First, it does not describe particles with $m = 0$ such as photons because the $-\hbar^2/2m$ term blows up. Second, the Schrödinger equation predicts nothing about spins. In non-relativistic quantum mechanics, the spin state of a particle looks like an add-on to its spatial wavefunction. Third, the time-derivative and spatial derivatives are not of the same order. One of the most significant implications of special relativity is that time is treated equally as space, known as the spacetime. If the order of spatial derivatives is 2, then the order of time-derivatives should also be 2.

To reach relativistic quantum mechanics, here is an simple and intuitive modification. Physically, the Schrödinger equation is just the *quantized* version of the classical energy-momentum relation:

$$E = \frac{\mathbf{p}^2}{2m} + V \quad \text{with} \quad \mathbf{p} \to -i\hbar\nabla, \quad E \to i\hbar\frac{\partial}{\partial t}.$$

If we stick to free particles for now ($V = 0$), the it just says that $E = \mathbf{p}^2/2m$. In special relativity, a free particle of mass m and 3-momentum $\mathbf{p}$ has the energy-momentum relation

$$E^2 = (mc^2)^2 + (\mathbf{p}c)^2.$$

Substituting $E \to i\hbar\partial_t$ and $\mathbf{p} \to -i\hbar\nabla$, one obtain

$$\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + c^2\nabla^2\right)\phi(\mathbf{x}, t) = \left(\frac{mc^2}{\hbar}\right)^2\phi(\mathbf{x}, t).$$

Note that we use ϕ, a quantum field, instead of ψ. Dividing by c^2 on both sides gives the **Klein-Gordan equation**.

$$\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2\right)\phi = \left(\frac{mc}{\hbar}\right)^2\phi.$$

The constant coefficient $(mc/\hbar)$ has units of Length^{-1}. Its inverse is known as the Compton wavelength of a particle,

$$\lambda_C \equiv \frac{\hbar}{mc}.$$

To study or probe the structure of a target particle on a distance scale Δx, the de Broglie wavelength of the probe particle should satisfy $\lambda_{\mathrm{dB}} \sim \hbar/p \leq \Delta x$. The energy of this particle is at least $E \sim pc \geq \hbar c/\Delta x$. If $\Delta x < \lambda_C = \hbar/mc$, the energy of the probe particle is larger than its rest mass energy, $E \geq mc^2$. That is, the energy is enough to create another particle. This means when the length scale is smaller than the Compton wavelength of a particle, particle creation is inevitable—one enters the regime of relativistic quantum mechanics. Elementary particle physics works under this length scale.

1.3 Mass and Energy Units

In particle physics, it is convenient to use natural units, in which fundamental constants c and $\hbar$ are set to 1. In this case, the rest mass energy is the same as mass: $E_{\mathrm{rest}} = mc^2$. The mass unit in particle physics is often expressed in eV/c^2. For example, $m_p \approx 1.67 \times 10^{-27}\,\mathrm{kg} = 0.938\,\mathrm{GeV}/c^2$. This is saying that the rest mass energy of a proton is about $0.938\,\mathrm{GeV}$. The following table lists the energy scale used in physics.

Scale	Value in eV	Physical systems and processes	Examples
1 eV	1 eV	Atomic physics, molecular physics, chemistry.	Ground state energy of a hydrogen atom: about $-13.6\,\mathrm{eV}$.
1 keV	10^3 eV	X-rays	Transitions between inner-shell atomic states for high Z atoms.
1 MeV	10^6 eV	Nuclear physics, γ-rays.	Nuclear binding energy per nucleon: $\sim 8\,\mathrm{MeV}$.
1 GeV	10^9 eV	Particle physics	Rest mass energy of a proton: $0.938\,\mathrm{GeV}$.
1 TeV	10^{12} eV	Particle physics	Large hadron collider (LHS) can accelerate particles to $6.5\,\mathrm{TeV}$.
-	10^{28} eV	Quantum gravity	Planck mass: $m_{\mathrm{Pl}} \approx 1.2 \times 10^{19}\,\mathrm{GeV}$.

1.4 The History of Elementary Particles

1.4.1 The Classical Era (1897-1932)

Elementary particle physics began with the discovery of electron by J. J. Thomson. Thomson discovered that cathode rays were produced by a hot filament. Using methods in mass spectrometry, Thomson obtained the velocity of the particle and its charge-mass ratio. This ratio was greater than any of the known particles, indicating that it either had a large amount of charge or a very small mass. This particle is later called the **electron**.

Thomson believed that the electron is one component of atoms, but he proposed a wrong model called the **plum-pudding model**, in which electron distributed uniformly in a positively charged paste. Rutherford argued against Thomson's model by his scattering experiment. He showed that the positive charge is concentrated in the **nucleus** at the center of the atom. In his experiment, Rutherford fired an α particle beam to a gold foil. It turned out that most of the α particles passed through the foil while only a few deflected wildly. If the atom was in the plum-pudding model, all α particles should be deflected only a bit. The nucleus of the lightest atom is named as the **proton** by Rutherford.

It is natural to think that for heavier atoms, to support more electrons' orbit, there should be a corresponding number of protons. That is, if there are two electrons, the nucleus should contain two protons and thus be twice as heavier. The truth is helium atom weighs about 4 times heavier than hydrogen nucleus. This was resolved with the discovery of **neutron** by James Chadwick in 1932, and it marked an end to the classical period of elementary particle physics.

1.4.2 The Photon (1900-1924)

In 1900, Max Planck was trying to explain the **blackbody spectrum** for the electromagnetic radiation emitted by a hot object. He came up with an assumption that the electromagnetic radiation is quantized,

$$E = h\nu,$$

where ν is the frequency of the radiation and $h = 6.626 \times 10^{-34}\,\mathrm{J \cdot s}$ is the Planck's constant.

In 1905, Einstein used Planck's idea to explain the **photoelectric effect**: electrons were ejected when electromagnetic radiation shines on a metal surface. A surprising result of the photoelectric effect is that the maximum energy of the electron ejected is independent of the intensity of the light, but on its color (i.e. frequency). Einstein argued that electromagnetic radiation is by its nature quantized, but this received harsh criticism. In 1923, A. H. Compton discovered the famous **compton effect**. He found out that the light scattered from a particle at rest is shifted in wavelength and will scatter of with an angle according to the equation

$$\lambda' = \lambda + \lambda_c(1 - \cos\theta),$$

where λ is the incident wavelength, λ' is the scattered wavelength, θ is the scattering angle, and $\lambda_c = h/mc$ is the **Compton wavelength**. This formula can also be derived by treating light as a particle with zero mass, and invoking conservation of relativistic energy and momentum. It is saying that light behaved like a particle, called the **photon**, on the subatomic scale.

1.4.3 Mesons (1934-1947)

After the discovery of protons and neutrons in the nucleus, there appeared a question: what holds these particles together as there should be proton-proton repulsion inside the nucleus. There must be a force stronger than the Coulomb force. Physicists called this force the **strong force**. There appeared another question: if this force is stronger than electromagnetic force, why it does not appear in everyday life? After all, every force we encounter daily is electromagnetic in origin, with gravity the only exception. The answer to this is that the strong force is a short-range force. Typical forces behaves like $e^{-(r/a)}/r^2$, where a is the range. For Coulomb force and gravity, $a = \infty$, but for strong force, $a \sim 10^{-15}\,\mathrm{m}$.

The first significant theory of the strong force was proposed by Yukawa in 1934. Before this theory, it was known that some particles served as mediators of forces or fields (e.g. electromagnetic force is mediated by photon) using the uncertainty principle. The range of the force is inversely proportional to the mass of the mediators (e.g. photon is massless so electromagnetic force has an infinite range). Yukawa calculated that the mediator of the strong force should be about 300 times massive than the electron, but this mass is smaller than that of the proton. Such particle is called the **meson**, the middle-weight. By the same token, the electron is called a **lepton**, the light weight, while the proton and the neutron are **baryon**, the heavy-weight.

By 1937, middle-weight particles were detected from the cosmic rays, but those particles appeared lighter than Yukawa's prediction and had a wrong lifetime. In 1946, experiments in Rome showed that these particles interact weakly with atomic nuclei. Everything seemed to be out of order. The resolution came in 1947 when Powell found that there are two middle-weight particles: one is called the **pion**, π, and the other is the **muon**, μ. The correct Yukawa meson is the π

1.4.4 Antiparticles (1930-1956)

The complete nonrelativistic quantum mechanics was invented in 1923-1926. The first significant discovery of relativistic quantum mechanics is the Dirac equation (by Paul Dirac) in 1927. The Dirac equation describes free electrons with energy given by the relativistic formula $E^2 - p^2c^2 = m^2c^4$. Interestingly, there exists two solutions $E = \pm\sqrt{p^2c^2 + m^2c^4}$. To account for the negative solution, Dirac proposed an infinite sea, called the **Dirac sea**, of electrons in negative-energy states. Dirac used the Pauli exclusion principle to explain why all electrons we observed are in positive-energy states (because all negative-energy states are fileld). If a negative-energy electron is absence at the expected position in the sea, the "hole" is seen as a particle with a net positive energy and charge. Such particle was confirmed by Anderson's discovery of the **positron**, e^+, in 1931.

Later in the 1940s, the Feynman-Stuckelberg formulation saw the negative solution as a positive-energy states of a different particle. The dualism in Dirac's equation is a profound and universal feature of quantum field theory: every particle has a corresponding antiparticle, with the same mass and opposite electric charge. The antiparticle is denoted by the symbol of its corresponding particle with a bar. The **antiproton**, $\bar{p}$, and **antineutron**, $\bar{n}$, were observed in 1955 and 1956 respectively at Berkeley. Though antineutron is also neutral, it is different from ordinary neutron by other quantum numbers, specifically the change in sign of the baryon number. There also exists a **crossing symmetry** for a reaction: for

$$A + B \to C + D,$$

a particle can be crossed over to the other side by turning it to its antiparticle, e.g.

$$A \to \bar{B} + C + D.$$

One mystery about antiparticles is the matter-antimatter asymmetry. It is observed that all of the observable universe is made of ordinary matter unless some energetic events occur, like the ones in the particle accelerator.

1.4.5 Neutrinos (1930-1962)

A beta decay is a nuclear reaction that emits an electron,

$$A \to B + e^-.$$

In 1930, the study of beta decay indicated a missing energy on the right side of the equation. Pauli suggested that a neutral particle was emitted together with the electron, and he called this particle the neutron. In 1932, the actual neutron was discovered and Chadwick adopted this name. The following year Fermi showed that the neutral particle emitted in beta decay must be extremely light, and he called it the **neutrino**, the little neutral one. In modern terminology, the extra particle emitted in the beta decay is actually the antineutrino,

$$n \to p^+ + e^- + \bar{\nu}.$$

Neutrinos interact extremely weakly with matter as they can penetrate a thousand light years of lead. In every second, hundreds of billions of neutrinos from the sun pass through a human's body. This makes it difficult to detect neutrinos. By 1950, theoretical framework for the existence of neutrinos was established, but its discovery was in the mid-1950s.

Physicists wanted to study the property distinguishing a neutrino from antineutrino. If neutrino is its own antiparticle, then

$$\nu + n \to p^+ + e^- \quad \text{and} \quad \bar{\nu} + N \to p^+ + e^-$$

should have the same footing—they should occur about the same rate. In the late 1950s, Davis and Harmer made an experiment about this and the second reaction did not occur. Hence neutrino and antineutrino are distinct particles. In 1953 Konopinsky and Mahmoud introduced the **lepton number**, L, and its conservation. They assigned $L = +1$ to electron, the muon, and the neutrino, and $L = -1$ to their corresponding antiparticles. Both sides of the beta decay has total lepton number zero.

Theoretically, the decay of a muon into an electron and a photon follows the conservation of lepton number (and other conservation laws). However, this process was never observed:

$$\mu^- \nrightarrow e^- + \gamma.$$

The answer arrived in late 1950s and early 1960s to this is that there should be two different kinds of neutrino, one associated with electron and the other with muon. The **muon number** L_μ and the **electron number** was invented: $L_\mu = +1$ to μ^- and ν_μ, negative for their corresponding antiparticles, and the same went for electron and electron neutrino ν_e. Originally it was thought that two neutrinos were produced in muon decay,

$$\mu^- \to e^- + 2\nu.$$

This is true in some sense, but wrong in detail. The corrected version should be

$$\mu^- \to e^- + \bar{\nu}_e + \nu_\mu.$$

Due to the small mass of neutrinos, in many calculations neutrinos are assumed to be massless. Physically they have mass, but we do not know what their masses are. Neutrinos belong to the lepton family as they do not participate in strong interactions.

1.4.6 Strange Particles (1947-1960)

In December 1947, Rochester and Butler observed a neutral particl from cosmic rays that decays into a π^+ and a π^-. This new particle is called the kaon K^0 and falled into the meson family. Many more mesons such as the η, the ϕ, the ω, the ρ's, were discovered. In 1950, Anderson's group at Caltech observed another new particle, now called Λ, that decays into a p^+ and a π^-. Such particle is heavier than proton, so it belongs to the baryon family. Actually, there exists a question in 1938: why is the proton stable? Stückelberg proposed a law of **conservation of baryon number**, which is assigned $A = +1$ to all baryons and $A = -1$ to all antibaryons. Proton is the lightest baryon, so it cannot spontaneously decay into other particles. Hence for the conservation of baryon number, Λ must be in the baryon family. More heavy baryons, such as the Σ's, the Ξ's, the Δ's, were discovered in the next few years.

The new heavy baryons and mesons were collectively known as "strange" particles. Until now, the only source of strange particles are from cosmic rays. In 1952, the first modern particle accelerator called the Brookhaven Cosmotron started operating and it made producing strange particles in the lab possible. The produced strange particle are "strange" because they are produced in $\sim 10^{-23}$ s, but they decay in $\sim 10^{-10}$ s. This suggested another unknown mechanism in decay process. Indeed, the strange particles are produced by the strong force, but the decays are governed by the weak force. Pais theory of decay required that strange particles come in pairs. In 1953 Gell-Mann and Nishijima proposed the **conservation of strangeness** to account for this pair production. They assigned strangeness to each particle, just like baryon numbers. Strangeness is conserved in any strong interaction, but not in weak interactions. For example: the K has strangeness $S = +1$ while the Σ and Λ has $S = -1$ (ordinary particles like proton or pion have $S = 0$),

$$\pi^- + p^+ \to K^+ + \Sigma^-.$$

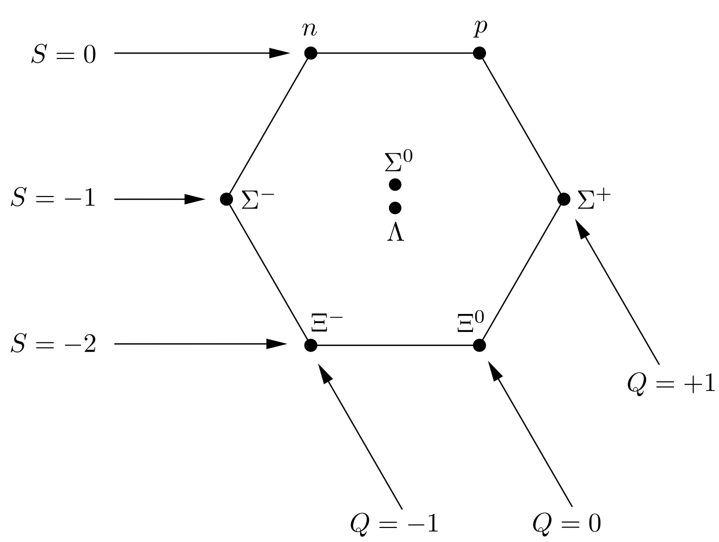
Strangeness is not conserved in decays,

$$\Lambda \to p^+ + \pi^-.$$

Now there were so many particles discovered, but there is no underlying orders in them. It was just like the situation of chemistry when atoms were found but the periodic table was not invented.

1.4.7 The Eightfold Way (1961-1964)

The "periodic table" called **Eightfold Way** for elementary particles was introduced by Murray Gell-Mann in 1961. The baryons and mesons are arranged in geometric patterns according to their charge and strangeness: the following baryon octet has the eight lightest baryons.



There are also meson octet, baryon decuplet, etc.

1.4.8 The Quark Model (1964)

The Eightfold Way still has a bizarre pattern. There must be a simpler explanation to the existence of different baryons and mesons. In 1964, Gell-Mann and Zweig independently proposed that hadrons are composed of smaller particles. Gell-Mann called these elementary particles **quarks**. There are three types of quarks for now: the u (up) quark with charge $\frac{2}{3}$ and strangeness zero; the d (down) quark with charge $-\frac{1}{3}$ and strangeess zero; the s (strange) quark with charge $-\frac{1}{3}$ and strangeness $S = -1$. Antiquarks have opposite charge and strangeness. In addition, there are two composition rules:

- Every baryon is composed of three quarks, and every antibaryon is composed of three antiquarks.
- Every meson is composed of a quark and an antiquark.

For example, the heavy baryons in the baryon decuplet have quark components

qqq	Q	S	Baryon
uuu	2	0	Δ^{++}
uud	1	0	Δ^{+}
udd	0	0	Δ^{0}
ddd	-1	0	Δ^{-}
uus	1	-1	Σ^{*+}
uds	0	-1	Σ^{*0}
dds	-1	-1	Σ^{*-}
uss	0	-2	Ξ^{*0}
dss	-1	-2	Σ^{*-}
sss	-1	-3	Ω^{-}

Table 1.1: The baryon decuplet.

The same quark combination can make different particles, e.g. both the Δ^+ and the proton has uud. This is because quarks can bound in different ways, and their energy difference are very large so that we treat them as different particles.

The quark model is a nice model, but there are some problems with it: there was no observations of an individual quarks. They should be easy to produce by firing particles to protons; they should be easy to identify because of their special fractional charge; and they are easy to store because at least one quark is stable and there is no lighter particle with fractional charge. Yet no one has ever found one. Later physicists who believed in the quark model came up with the **quark confinement**: for some reason quarks are absolutely confined in baryons and mesons. The question becomes: what mechanism does this? Just as what Rutherford did to an atom, experimentalists fired particles to proton. Similarly, most of the incident particles passed through, with only a small fractions scattering sharply. The only difference is that while atoms contains one lumps of charge, the nucleus, the proton contains three.

Consider the particle Δ^{++} with uuu inside. It has three identical particles in it and seems to violate the Pauli exclusion principle. In 1964, O. W. Greenberg suggested that quarks are characterized by another charge called the **color** (red, green, and blue). Then the three u's are no longer identical. Moreover, the concept of color has another nice feature: all naturally occuring particles are colorless. The word "colorless" means that either the total amount of each color is zero or all three colors are present in equal amounts.

1.4.9 The November Revolution and Its Aftermath (1974-1983 and 1995)

The quark model was not rescued by color, quark confinement, or anything similar above. It was the discovery of the ψ meson by C. C. Ting's group and Burton Richter's group both independently in 1974 that did the job. The ψ was a neutral and heavy meson which weighed over 3 times the mass of the proton (since many mesons appeared to be heavier than the proton, the "middle-weight" meaning was no longer available). This particle has an extremely long life time ($\sim 10^{20}$ s) as other particle with the same mass has typical lifetime $\sim 10^{-23}$ s. The discovery of the ψ marked the start of the **November Revolution**.

The quark model was the ultimate winner of this discovery: ψ is a bound state of a new quark called the c, **charm** quark, and its antiquark, $\psi = (c\bar{c})$. Indeedn, Bjorken and Glashow proposed the idea of the fourth quark many years ago because they believed if there were four leptons, there should also be four quarks. In other words, the quark model was waiting for the discovery of the ψ and to provide an explanation. As there are now four quarks, new baryons and mesons carrying different amounts of charm should emerge. The charm of the ψ is hidden as it has a charm quark and an anticharm quark. The first charmed baryons ($\Lambda_c^+ = udc$ and $\Sigma_c^{++} = uuc$) were discovered in 1975. The first charmed mesons ($D^0 = c\bar{u}$ and $D^+ = c\bar{d}$) appeared in 1976, and charmed strange meson $D_s^+ = c\bar{s}$ in 1977.

Aside from the charm quark, a new lepton called the **tau** and its neutrino were found in 1975. Glashow's lepton and quark symmetry seem to be spoiled. However, in 1977 there appeared a new heavy meson $\Upsilon = b\bar{b}$ carrying a fifth quark called b, the **bottom**. Later in the 1980s the first bottom baryon, $\Lambda_b^0 = udb$ were found; the first bottom meson ($\bar{B}^0 = b\bar{d}$ and $B^- = b\bar{u}$) were observed in 1983. The prediction of the sixth quark should be a natural thing here, for there was already six leptons. The t, the **top** quark, appeared to be extraordinarily heavy ($174\,\mathrm{GeV/c^2}$) and its lifetime was too short to form bound states. In 1995, accelerator Tevatron provided the observed reaction $u + \bar{u} \to t + \bar{t}$, from which the top and anti-top decayed immediately.

1.4.10 Intermediate Vector Bosons (1983)

In the theory of beta decay, Fermi thought of the process as a contact interaction. He did a great approximation because the **weak force** was a short-range force. The mediator was named **intermediate vector boson**. The weak force was too weak to bind particles together, so we cannot directly measure the range of it. Later, Glashow, Weinberg, and Salam came up with the electroweak theory. They believed there should be three intermediat vector bosons, $W^\pm$, and Z. Their calculated masses were

$$M_W = (82 \pm 2)\,\mathrm{GeV/c^2}, \quad M_Z = (92 \pm 2)\,\mathrm{GeV/c^2}.$$

In 1983, Carlo Rubbia's group discovered the W and the Z and they measured the masses to be

$$M_W = (80.403 \pm 0.029)\,\mathrm{GeV/c^2}, \quad M_Z = (91.188 \pm 0.002)\,\mathrm{GeV/c^2}.$$

1.4.11 The Standard Model (1978-?)

Now, all matter is constituted of three categories of particles: leptons, quarks, and mediators. The leptons fall into three generations and are classified according to their charge, L_e, L_μ, and L_τ:

ℓ	Q	L_e	L_μ	L_τ
e	-1	1	0	0
ν_e	0	1	0	0
μ	-1	0	1	0
ν_μ	0	0	1	0
τ	-1	0	0	1
ν_τ	0	0	0	1

Table 1.2: Lepton classification

There are also 6 antileptons, which give 12 leptons in total.

There are six flavors of quarks: up, down, strange, charm, bottom, and top, each with associated conservation. Similar to leptons, they fall into three generations (and have corresponding antiquarks):

q	Q	D	U	S	C	B	T
d	$-1/3$	-1	0	0	0	0	0
u	$2/3$	0	1	0	0	0	0
s	$-1/3$	0	0	-1	0	0	0
c	$2/3$	0	0	0	1	0	0
b	$-1/3$	0	0	0	0	-1	0
t	$2/3$	0	0	0	0	0	1

Table 1.3: Quark classification

Finally, there are mediators with corresponding interactions: the photon for the electromagnetic force, the W's and the Z for the weak force, and the (hypothetical) graviton for gravity. There is some problem with the strong force. In Yukawa's theory the mediator was the pion, but now there exists more heavy mesons that could act as mediators. Moreover, with the quark model, protons, neutrons, and mesons are just composites of quarks. Then what is the exchange particle between quarks? The answer is the **gluon**, and there are eight of them in the Standard Model. Just like quarks, gluons have carry color and cannot exist as lone particles.

In all, there are 60 elementary particles: 12 leptons, 36 quarks, 12 mediators (graviton is not counted in the Standard Model). The Glashow-Weinberg-Salam theory calls for one additional particle called the **Higgs boson**, which is discovered recently at CERN in 2012. One shall ask then, who needs the two "extra" generations of particles when ordinary matter is made up of up and down quarks and electrons? Surprisingly, this has to do with the matter-antimatter asymmetry: this asymmetry can be explained by the Standard Model only if there are at least three generations. Nonetheless, even with the Standard Model, there are still many questions for particle physics:

- Why are there only three generations?
- All quark and lepton masses were determined by experiments. How do we calculate their masses, as a mature theory always do?
- The Grand Unified Theories (GUTs) that unify strong, electromagnetic, and weak interactions?
- Supersymmetry: every fermion has an associated boson?
- What about the graviton?

2 Special Relativity

There are two postulates in special relativity:

1. The laws of physics take the same form in all inertial reference frames. All inertial frames are fundamentally indistinguishable based on experiments in that frame only.
2. The speed of light (and other massless particles) is the maximum speed in the universe and has the same value in all inertial reference frames,
$$c = 2.99792458 \times 10^8 \text{ m/s}.$$

Note that this number is exact because modern physics define the "meter" using the speed of light in vacuum. With these assumptions, many odd things would occur: Galilean transformation is no longer true, clocks do not run universally, etc. In this chapter, we will review special relativity.

2.1 Spacetime Geometry

In special relativity, time and space are somewhat mixed up in coordinate transformations. Thus, they are unified into a four-dimensional entity called **spacetime**. To be precise, let's list properties of an inertial frame in special relativity:

1. Clocks that have been synchronized at some time will remain synchronized.
2. The properties of points in *space* is described by Euclidean geometry.
3. The spatial separation between points in space with specified coordinates does not change with time.

When one tranform one inertial frame to another, simultaneous events in one inertial reference frame need not be simultaneous in other frames. Events that occur at the same place need not be at the same place in other frames.

An event in an inertial frame can be labeled with four coordinates known as **Minkowski coordinates**: (ct, x, y, z). (In spacetime coordinates, time is conventionally the zeroth coordinate and is multiplied by a "c" to ensure the dimension is right. Later, we will set $c = 1$ for convenience so that the time coordinate is just t, but for now let's stick with ct.) The geometry of spacetime in an inertial frame is given by the line element
$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2. \tag{2.1}$$
(We will prove it later using Lorentz transformations.) It is the square of the **spacetime interval** ds between two infinitesimally separated events. This spacetime is called **Minkowski spacetime**. The line element is a scalar quantity, invariant under any coordinate transformations. In another frame with Minkowski coordinates (ct', x', y', z'), the line element has exactly the form
$$ds^2 = c^2dt'^2 - dx'^2 - dy'^2 - dz'^2 = c^2dt^2 - dx^2 - dy^2 - dz^2.$$

Note the minus sign in front of $d\mathbf{x}^2$. This indicates that the spacetime is flat, but non-Euclidean. The line element ds^2 can be positive, negative, or zero. We can put any two infinitesimally separated events into these three categories:

1. $ds^2 > 0$: the two events are **timelike** separated. The time separation dominates over the distnace separation. For example, setting $dx = dy = dz = 0$ makes two events occur at the same point in space. Any Lorentz transformation cannot reverse the time ordering of the two events.
2. $ds^2 < 0$: the two events are **spacelike** separated. The distance separation dominates over the time separation. For example, setting $dt = 0$ makes two events occur at the same time, or simultaneously. A Lorentz transformation can reverse the time ordering of the two events, so these events cannot be not causal.
3. $ds^2 = 0$: in this case, $c^2dt^2 = dx^2 + dy^2 + dz^2$, or $d\mathbf{x}^2/dt^2 = c^2$. This describes a photon, so such events are called **lightlike** separated or **null** separated.

Since ds^2 is (Lorentz) invariant, these qualitative result (spacelike, null, or timelike) are also independent of reference frame or coodinate system. In otherwords, if two events are spacelike separated (not necessarily occur at the same time), one can always find a reference frame such that these two events occur simultaneously. Similarly, if two events are timelike separated, one can always find a reference frame such that they occur at the same place.

2.2 The Lorentz Transformation

In Newtonian mechanics, suppose an observer in frame S assigns an event A with coordinates (t, x, y, z) and another observer in a frame S' that has a relative velocity v in x-direction to S. Assume both observers are in inertial frames, and their x-axes are aligned. Also, assume that $t = 0$ represents the point when the origins of S and S' (denoted by $\mathcal{O}$ and $\mathcal{O}'$ respectively) overlap. Then according to Galilean transformation, the coordinates (t', x', y', z') of the event A in S' frame will be

$$\begin{aligned} t' &= t, \\ x' &= x - vt, \\ y' &= y, \\ z' &= z. \end{aligned}$$

This is the usual coordinate transformation between inertial frames in classical mechanics. In special relativity, S' will assign the coordinates of A according to the **Lorentz transformation** from S to S',

$$\boxed{\begin{aligned} ct' &= \gamma(ct - \beta x), \\ x' &= \gamma(x - \beta ct), \\ y' &= y, \\ z' &= z, \end{aligned}} \tag{2.2}$$

where c is the speed of light, $\beta = v/c$, and $\gamma = 1/\sqrt{1-\beta^2}$ is the Lorentz factor. A detailed derivation of the Lorentz transformation is given in Appendix A.1. Taking the non-relativistic limit ($v/c \to 0$) reduces the Lorentz transformation to the Galilean transformation. One can show that the line element satisfies $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 = c^2dt'^2 - dx'^2 - dy'^2 - dz'^2$. The inverse Lorentz transformation (from S' to S) can be obtained by replacing β with $-\beta$ (reversing the velocity).

The **proper time** of an object τ is the time measured by a clock that always travels with that object. In that object's own frame, the proper time is its time coordinate, and its spatial coordinates are all zero. Thus, the proper time can be calculated by the line element

$$c^2d\tau^2 = ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

in some other frame S. In other words, the proper time is the shortest time measured between two events in any frame.

Example 2.1. Time dilation and muon decay

The proper time τ of a clock fixed at $\mathcal{O}'$ is just the time t' measured by that clock in S' frame,

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \equiv \tau.$$

In S frame, the clock is moving at a velocity of v, so its position is $x = vt$. Hence the proper time τ of the clock is related to t by

$$\tau = \gamma\left(t - \frac{v^2t}{c^2}\right) = \gamma\left(1 - \frac{v^2}{c^2}\right)t = \frac{t}{\gamma}.$$

As $\gamma \geq 1$, moving clocks always run slower ($\tau < t$). Time dilation plays a very important role in particle physics. A typical example is the decay of muon. Muons have a mean lifetime (this is the proper time) of $\tau = 2.2\,\mu$s. Suppose a muon travels near the speed of light, $v \lesssim c$. If we ignore relativity, then the mean distance it travels before it decay is

$$\langle \Delta x \rangle < c\tau = 6.6 \times 10^2\,\text{m}.$$

However, we see muons produced by cosmoc-rays protons at an altitude of $\sim 15\,\text{km}$. Muons typically have $\gamma \approx 40$, and $\gamma\langle\Delta x\rangle > 15\,\text{km}$, so most muons do reach the Earth.

From now on, we will in general use the natural units $c = 1$. The Lorentz transformation is given by

$$t' = \gamma(t - vx), \quad x' = \gamma(x - vt), \quad y' = y, \quad z' = z. \tag{2.3}$$

It looks more symmetric for t' and x'. Also, the line element and proper time are essentially the same,

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 = d\tau^2. \tag{2.4}$$

To return to ordinary units, we shall use dimensional analysis to find out where should the c go. For now, let's develop some math tools that are useful for relativity.

2.2.1 Addition of Velocities

Suppose in frame S, an object is traveling at speed u_x in the x-direction. Then in frame S' with a velocity of v relative to S, the object is also traveling in the x-direction, but certainly with some other velocity u'_x. To determine u'_x, we use the definition of velocity and the Lorentz transformation (2.3),

$$u'_x = \frac{dx'}{dt'} = \frac{\gamma(dx - v\,dt)}{\gamma(dt - v\,dx)} = \frac{dx/dt - v}{1 - v\,dx/dt} = \frac{u_x - v}{1 - u_x v}.$$

This is the velocity addition formula. To check that it is valid, we will first consider the classical limit where $u_x, v \ll 1$. Then by Taylor expanding the denominator and keeping the first order in velocities,

$$v' \approx (u_x - v)(1 + u_x v) \approx u_x - v,$$

This is the classical Galilean velocity addition. Next, let's check the constancy of light. If $u_x = 1$ (a photon), then the velocity addition formula says

$$u'_x = \frac{1 - v}{1 - v} = 1.$$

In frame S', the photon also travels at the speed of light as expected.

2.2.2 Relativistic Doppler Shift

Consider a sinusoidal electromagnetic wave of frequency ν in frame S. Its wavelength is given by $\lambda = 1/\nu = T$, where T is the period of this EM wave. In frame S, if we focus on the same crest between a time interval $\Delta t = T$, it will travel by a distance of one wavelength $\Delta x = \lambda = T$. In frame S' that travels along the wave direction with a velocity v will measure a period T' according to the Lorentz transformation (2.3):

$$T' = \gamma(T - vT) = T\frac{1 - v}{\sqrt{1 - v^2}} = T\sqrt{\frac{1 - v}{1 + v}}.$$

The relativistic Doppler shift is then

$$\nu' = \nu\sqrt{\frac{1 + v}{1 - v}}. \tag{2.5}$$

2.3 Vector and Tensor Algebra

2.3.1 4-Vectors

In special relativity, we often use Minkowski coordinates with basis vectors $\{\boldsymbol{e}_\mu\} = \{\boldsymbol{e}_t, \boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z\}$. The basis vectors always have *subscript* indices. Conventionally, Greek indices represent numbers $(0, 1, 2, 3) \to (t, x, y, z)$, while Latin indices represent $(1, 2, 3) \to$

(x, y, z). The vector pointing from the origin to an event in S frame can be written as:

$$\boldsymbol{x} = x^0\boldsymbol{e}_0 + x^1\boldsymbol{e}_1 + x^2\boldsymbol{e}_2 + x^3\boldsymbol{e}_3 = \sum_{\mu=0}^{3} x^\mu \boldsymbol{e}_\mu = (t, x, y, z).$$

Note that all components $\{x^\mu\}$ are labeled with *superscript* indices. These vectors are called **4-vectors**. To avoid writing the summation symbol everywhere, we will adopt the **Einstein summation convention**: repeated subscript and superscript indices are implicitly summed over. They are called dummy indices. Indices that do not repeat are known as free indices. In this notes, I will write 4-vectors bolded-italic (e.g. $\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{u}$), or sometimes just italic (x, p, u), to distinguish with bolded three-vectors (e.g. $\mathbf{x}, \mathbf{p}, \mathbf{u}$). Therefore, a 4-vector can be written compactly as

$$\boldsymbol{x} = x^\mu \boldsymbol{e}_\mu.$$

By definition, the component of any 4-vector $\boldsymbol{a}$ (not just position) transform from one inertial frame to another according to the Lorentz transformation:

$$\begin{aligned} a'^0 &= \gamma(a^0 - va^1), \\ a'^1 &= \gamma(a^1 - va^0), \\ a'^2 &= a^2, \\ a'^3 &= a^3. \end{aligned}$$

A 4-vector have a “length”, or **magnitude**, defined by

$$\boldsymbol{a} \cdot \boldsymbol{a} = a^0a^0 - a^1a^1 - a^2a^2 - a^3a^3 \begin{cases} < 0, & \Longrightarrow \quad \text{timelike vector}, \\ = 0, & \Longrightarrow \quad \text{null/lightlike vector}, \\ > 0, & \Longrightarrow \quad \text{spacelike vector}. \end{cases}$$

The magnitude is an invariant quantity, just like the line element. (However, the position or displacement 4-vectors can only be defined in flat spacetime. In curved spacetime, 4-vectors need to be defined in a local, flat, tangent space to the curved spacetime.) The definition of a magnitude can be generalized to a **scalar product** (or inner product) of 4-vectors:

$$\boldsymbol{a} \cdot \boldsymbol{b} = a^0b^0 - a^1b^1 - a^2b^2 - a^3b^3 = \boldsymbol{b} \cdot \boldsymbol{a} \quad \text{or} \quad \boldsymbol{a} \cdot \boldsymbol{b} = \eta_{\mu\nu} a^\mu b^\nu.$$

Here $\eta_{\mu\nu}$ is called the **Minkowski metric**. The $\{\eta_{\mu\nu}\}$ are components of the **metric tensor** in the (t, x, y, z) Minkowski coordinate basis of an inertial frame in special relativity. It can be represented by a diagonal matrix:

$$[\eta_{\mu\nu}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{2.6}$$

Note that $\eta_{\mu\alpha}\eta_{\alpha\nu} = \delta^\mu{}_\nu$, the identity matrix. The inner product of four vectors can also be written as

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_\nu b^\nu = a^\mu b_\mu$$

where $a_\nu = \eta_{\mu\nu} a^\mu$ and $b_\mu = \eta_{\mu\nu} b^\nu$. Components of vectors with subscript indices, $\{a_\mu\}$, are called **covariant components**. Components of vectors with superscript indices, $\{a^\mu\}$, are called **contravariant components**. In special relativity, they are related by $a^0 = a_0$, and $a^i = -a_i$. Note that now the inner product is explicitly

$$\boldsymbol{a} \cdot \boldsymbol{b} = a^0b^0 + (-a^1)b^1 + (-a^2)b^2 + (-a^3)b^3 = a^0b^0 + a^1(-b^1) + a^2(-b^2) + a^3(-b^3).$$

The metric tensor $\eta_{\mu\nu}$ directly related to basis vectors. We can write $\boldsymbol{a} \cdot \boldsymbol{b} = (a^\mu \boldsymbol{e}_\mu) \cdot (b^\nu \boldsymbol{e}_\nu) = a^\mu b^\mu (\boldsymbol{e}_\mu \cdot \boldsymbol{e}_\nu)$. Hence

$$\eta_{\mu\nu} = \boldsymbol{e}_\mu \cdot \boldsymbol{e}_\nu.$$

This implies that the Minkowski coordinate basis vectors satisfy $\boldsymbol{e}_\mu \cdot \boldsymbol{e}_\nu = 0$ if $\mu \neq \nu$—it is an orthogonal basis. Furthermore, they are also unit vectors because all nonzero components of $\eta_{\mu\nu}$ are only -1 or 1. In conclusion, the Minkowski coordinate basis vectors form an *orthonormal* set of basis vectors. Now with the metric, we can write down the most usual notation of the line element:

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu.$$

This is why $\eta_{\mu\nu}$ is called the metric tensor. It enables one to compute the infinitesimal spacetime interval ds between events with infinitesimal differences in coordinates.

2.3.2 The 4-Derivative

In special relativity, time and space are unified. One may suggest that the derivatives $\partial/\partial t$ and ∇ should also be unified as a 4-vector. This section studies Lorentz-transformation properties of the object $\partial_\mu = \partial/\partial x^\mu$, or explicitly,

$$\{\partial_\mu\} = \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right) = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).$$

Later we will know why ∂_μ has a subscript instead of a superscript. First, looking at the Lorentz transformation equations (2.3),

$$x'^0 = \gamma(x^0 - vx^1), \quad x'^1 = \gamma(x^1 - vx^0), \quad x'^2 = x^2, \quad x'^3 = x^3,$$

we can immediately summarize the four equations into a compact one:

$$x'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} x^\nu.$$

This is because the Lorentz transformation is linear. Now we know that partial derivatives pick out coefficients before x^μ, we can apply this to the chain rule and obtain ∂_μ:

$$\begin{aligned}
\frac{\partial}{\partial t} &= \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} + \frac{\partial x'}{\partial t}\frac{\partial}{\partial x'} + \frac{\partial y'}{\partial t}\frac{\partial}{\partial y'} + \frac{\partial z'}{\partial t}\frac{\partial}{\partial z'}, \\
\frac{\partial}{\partial x} &= \frac{\partial t'}{\partial x}\frac{\partial}{\partial t'} + \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \frac{\partial y'}{\partial x}\frac{\partial}{\partial y'} + \frac{\partial z'}{\partial x}\frac{\partial}{\partial z'}, \\
\frac{\partial}{\partial y} &= \frac{\partial t'}{\partial y}\frac{\partial}{\partial t'} + \frac{\partial x'}{\partial y}\frac{\partial}{\partial x'} + \frac{\partial y'}{\partial y}\frac{\partial}{\partial y'} + \frac{\partial z'}{\partial y}\frac{\partial}{\partial z'}, \\
\frac{\partial}{\partial z} &= \frac{\partial t'}{\partial z}\frac{\partial}{\partial t'} + \frac{\partial x'}{\partial z}\frac{\partial}{\partial x'} + \frac{\partial y'}{\partial z}\frac{\partial}{\partial y'} + \frac{\partial z'}{\partial z}\frac{\partial}{\partial z'}.
\end{aligned}$$

Extracting all partial derivatives from the Lorentz transformation:

$$\begin{aligned}
\partial_t &= \gamma\partial_{t'} - \gamma v\partial_{x'}, & & & t &= \gamma t' + \gamma v x', \\
\partial_x &= -\gamma v\partial_{t'} + \gamma\partial_{x'}, & &\overset{\text{Compare}}{\Longleftrightarrow} & x &= \gamma v t' + \gamma x', \\
\partial_y &= \partial_{y'}, & & & y &= y', \\
\partial_z &= \partial_{z'}. & & & z &= z'.
\end{aligned}$$

The LHS is the transformation of derivatives from S' to S, while the RHS is the *inverse* Lorentz transformation, from S' to S. This tells that the derivatives ∂_μ transform according to the inverse Lorentz transformation. They transform the same way as x_μ does but not x^μ. The derivatives ∂^μ (with superscript) can be obtained by $\eta^{\mu\nu}\partial_\nu$, resulting in $\{\partial^\mu\} = (\partial_t, -\nabla)$. There is a Lorentz-invariant operator, called the **wave operator** or **d'Alembert operator**, denoted as

$$\Box = \partial_\mu\partial^\mu = \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu} = \frac{\partial^2}{\partial t^2} - \nabla^2. \tag{2.7}$$

Theorem 2.1. Lorentz Transformation Rule

Contravariant components of any 4-vector transform the way the coordinates do:

$$dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu \quad \Longleftrightarrow \quad a'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} a^\nu.$$

Covariant components of any 4-vector transform the way the derivatives of the contravariant coordinates do:

$$\frac{\partial}{\partial x^\mu} = \frac{\partial x'^\nu}{\partial x^\mu}\frac{\partial}{\partial x'^\nu} \quad \Longleftrightarrow \quad a_\mu = \frac{\partial x'^\nu}{\partial x^\mu} a'_\nu \quad \text{or} \quad a'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} a_\nu.$$

These rules are how we define a 4-vector. A way to incorporate all these ideas is to verify the invariant inner product,

$$a'^{\mu}b'_{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}}a^{\nu}\frac{\partial x^{\alpha}}{\partial x'^{\mu}}b_{\alpha} = \frac{\partial x'^{\mu}}{\partial x^{\nu}}\frac{\partial x^{\alpha}}{\partial x'^{\mu}}a^{\nu}b_{\alpha} = \frac{\partial x^{\alpha}}{\partial x^{\nu}}a^{\nu}b_{\alpha} = \delta^{\alpha}{}_{\nu}a^{\nu}b_{\alpha} = a^{\nu}b_{\nu}.$$

Another way is to note that the *divergence*:

$$\partial_{\mu}a^{\mu} = \partial_t a^t + \partial_x a^x + \partial_y a^y + \partial_z a^z = \partial_t a^t + \nabla\cdot\mathbf{a}.$$

It does not have a minus sign before ∇!

2.3.3 The Lorentz Group

A **group** G is a set of **group elements**, $\{g_a\}$. The elements can be composed together, or multiplied together, by a **composition**. The composition is usually indicated by a dot: $g_a\cdot g_b$. A composition satisfies four axioms:

1. Closure: given any two elements g_a and g_b in a group G, their product $g_a\cdot g_b = g_c$ is also in G.
2. Associativity: composition is associative, $(g_a\cdot g_b)\cdot g_c = g_a\cdot(g_b\cdot g_c)$.
3. Existence of the identity: there exists a group element called the **identity**, I, such that $I\cdot g_a = g_a\cdot I = g_a$.
4. Existence of the inverse: for every group element g_a, there exists a unique group element called the **inverse** of g_a, denoted by g_a^{-1}, such that $g_a^{-1}\cdot g_a = g_a\cdot g_a^{-1} = I$.

The group theory is the key mathematical tool to study symmetries. A **symmetry** is a feature of a system that is invariant under some transformation. The postulates of special relativity already suggest some symmetry: the laws of physics take the same form in all inertial frames; the spacetime interval Δs^2 between two events is invariant under Lorentz transformations.

The **Lorentz group** is a set of transformations that leave $ds^2 = \eta_{\mu\nu}dx^{\mu}dx^{\nu}$ invariant. In general, we want to find matrix representations of a group to study its properties. The Lorentz transformation (2.3) can be expressed in the matrix form:

$$\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix}. \tag{2.8}$$

The transformation matrix is denoted as Λ. It is specified by the continuous variable v. We can check whether Lorentz transformations satisfy all group axioms:

1. Closure: if Λ_1 and Λ_2 are two Lorentz transformations along the x-axis, then $\Lambda_1\cdot\Lambda_2$ is also a Lorentz transformation along the x-axis. However, the product of two Lorentz transformations along different axes is not a pure Lorentz transformation in general, but a combination of a rotation and a Lorentz transformation. Thus, the Lorentz group should also include rotations.
2. Associativity: matrix multiplication is associative.
3. Identity: the identity matrix which has $v = 0$.
4. Inverse: the inverse of a Lorentz transformation with argument v is a Lorentz transformation with argument $-v$. The existence of inverse requires $\det\Lambda \neq 0$.

As we have mentioned, Lorentz transformations are linear. The matrix element of Λ are denoted as

$$\{\Lambda^{\mu}{}_{\nu}\} \equiv \left\{\frac{\partial x'^{\mu}}{\partial x^{\nu}}\right\} \quad\Longrightarrow\quad x'^{\mu} = \Lambda^{\mu}{}_{\nu}x^{\nu}. \tag{2.9}$$

Note that the superscript index refers to frame S' while the subscript index refers to frame S. To find out what other transformations are in the Lorentz group, we invoke our invariant line element ds^2:

$$ds^2 = \eta'_{\mu\nu}dx'^{\mu}dx'^{\nu} = \eta_{\mu\nu}dx^{\mu}dx^{\nu},$$

where $\eta'_{\mu\nu} = \eta_{\mu\nu}$. The line element in S' frame can be written as $ds^2 = \eta'_{\mu\nu}dx'^\mu dx'^\nu = \eta'_{\mu\nu}(\Lambda^\mu{}_\alpha dx^\alpha)(\Lambda^\nu{}_\beta dx^\beta)$, so

$$\eta'_{\mu\nu}(\Lambda^\mu{}_\alpha dx^\alpha)(\Lambda^\nu{}_\beta dx^\beta) = \eta_{\mu\nu}dx^\mu dx^\nu = \eta_{\alpha\beta}dx^\alpha dx^\beta \quad \Longrightarrow \quad (\eta'_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta - \eta_{\alpha\beta})dx^\alpha dx^\beta = 0.$$

In the last step of the LHS equation, we change the dummy indices (μ, ν) to (α, β). Since $dx^\alpha dx^\beta$ is arbitrary, the terms in the bracket must vanish,

$$\eta_{\alpha\beta} = \eta'_{\mu\nu}\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta. \tag{2.10}$$

This is the defining equation for all matrices Λ that make up the Lorentz group. Note that all of $\eta_{\mu\nu}$, $\Lambda^\mu{}_\alpha$ and $\Lambda^\nu{}_\beta$ are just numbers, so they commute and we can write

$$\eta_{\alpha\beta} = \Lambda^\mu{}_\alpha\eta_{\mu\nu}\Lambda^\nu{}_\beta \quad \Longleftrightarrow \quad \eta = \Lambda^T\eta\Lambda,$$

where Λ^T is the transpose of the matrix Λ. The above two equations are completely equivalent if you match indices according to matrix multiplication $(\eta\Lambda)_{\mu\beta} = \eta_{\mu\nu}\Lambda^\nu{}_\beta$. The first index represents rows, while the second represents columns. In matrix multiplication, the second index of the first matrix always matches the first index of the second matrix. If they do not match but still have the same index, such as $\Lambda^\mu{}_\alpha\eta_{\mu\nu}$, then taking the transpose of Λ interchanges μ and α. In this case, the matrix multiplication is valid.

Taking the determinant,

$$\det\eta = \det\big(\Lambda^T\eta\Lambda\big) = \det\Lambda^T\det\eta\det\Lambda = (\det\Lambda)^2\det\eta.$$

Since $\det\eta = -1 \neq 0$, we must have $(\det\Lambda)^2 = 1$, or $\det\Lambda = \pm 1$. In conclusion, to satisfy the invariance of ds^2, each element Λ in the Lorentz group should have the property $\det\Lambda = \pm 1$. There are four types of transformations that satisfy such a property:

1. Lorentz transformations (boosts): transformations between inertial frames with different velocities, specified by three real parameters: $\mathbf{v} = (v_x, v_y, v_z)$.
2. Rotations: transformations describing rotations of spatial coordinates (time is unaffected), specified by three Euler angles (α, β, γ) about three axes of rotation. Since time is unchanged, rotations preserve $d\ell^2 = dx^2 + dy^2 + dz^2$.
3. The time-reversal transformation T: a discrete transformation that transforms $(t, x, y, z) \to (-t, x, y, z)$. Its matrix form is $\Lambda_T = \mathrm{diag}(-1, 1, 1, 1)$.
4. The parity operation P: a discrete transformation that transforms $(t, x, y, z) \to (t, -x, -y, -z)$. Its matrix form is $\Lambda_P = \mathrm{diag}(1, -1, -1, -1)$.

Lorentz boosts and rotations are continuous transformations with determinant $+1$. The time-reversal transformation and parity are discrete transformations with determinant -1. The latter two will play important roles in understanding the nature of interactions.

2.3.4 Tensors

So far we have encountered four types of mathematical objects in special relativity: scalars, 4-vectors, the metric tensor, and Lorentz transformation matrices. The first three are tensors in special relativity, but the last one is not. A tensor can be specified by its rank: a rank $\binom{m}{n}$ tensor has m contravariant indices and n covariant indices. It transform under the rule

$$(T')^{\alpha_1\alpha_2\ldots\alpha_m}{}_{\beta_1\beta_2\ldots\beta_n} = (\Lambda^{\alpha_1}{}_{\mu_1}\Lambda^{\alpha_2}{}_{\mu_2}\cdots\Lambda^{\alpha_m}{}_{\mu_m})(\Lambda^{\nu_1}{}_{\beta_1}\Lambda^{\nu_2}{}_{\beta_2}\cdots\Lambda^{\nu_n}{}_{\beta_n})T^{\mu_1\mu_2\ldots\mu_m}{}_{\nu_1\nu_2\ldots\nu_n}. \tag{2.11}$$

Each upper index is associated with a contravariant Lorentz transformation, and each lower index is associated with a covariant Lorentz transformation.

Example 2.2. A 4-vector

A 4-vector is a tensor. The transformation rule of a 4-vector is

$$(A')^\alpha = \Lambda^\alpha{}_\mu A^\mu,$$

where the prime is to be written out for clarity. It satisfies (2.9) and (2.11) if we replace T by A.

Example 2.3. The metric tensor

The metric tensor $\eta_{\mu\nu}$ is another good example of a tensor. It has two covariant indices. From (2.10), it follows the rule

$$\eta_{\alpha\beta} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\eta'_{\mu\nu}.$$

But the prime index is on the wrong side of the equation comparing to (2.11). Recall that $\Lambda^{\mu}{}_{\alpha}$ and $\Lambda^{\nu}{}_{\beta}$ comes from $dx'^{\mu} = \Lambda^{\mu}{}_{\alpha}dx^{\alpha}$ and $dx'^{\nu} = \Lambda^{\nu}{}_{\beta}dx^{\beta}$. These two Λ's are for contravariant Lorentz transformation because dx'^{μ}, dx^{α}, dx'^{ν}, and dx^{β} are contravariant components (they have upper indices). We know that the Lorentz transformation for contravariant and covariant components are inverse of each other. Now we can see why $\eta_{\alpha\beta} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\eta'_{\mu\nu}$ has the prime index on the opposite side. $\Lambda^{\mu}{}_{\alpha}$ and $\Lambda^{\nu}{}_{\beta}$ are contravariant transformation, the inverse of covariant transformation, so it is transforming back from the prime coordinates to unprimed coordinates. If $\Lambda^{\mu}{}_{\alpha}$ and $\Lambda^{\nu}{}_{\beta}$ were covariant transformation, then we will have (2.11):

$$\eta'_{\alpha\beta} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\eta_{\mu\nu}, \quad [\Lambda \text{ covariant trans.}] \quad \Longleftrightarrow \quad \eta_{\alpha\beta} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\eta'_{\mu\nu}, \quad [\Lambda \text{ contravariant trans.}].$$

The metric tensor is not only a tensor. It is an **invariant tensor** because $\{g'_{\mu\nu}\} = \{g_{\mu\nu}\}$, i.e. the matrix elements are the same.

One thing to notice is that a tensor must have all indices refer to components measured in one inertial frame. The indices on $\Lambda^{\mu}{}_{\nu} = \partial x'^{\mu}/\partial x^{\nu}$ are different: the upper index is for one frame, and the lower is for the other. Therefore, Lorentz transformation matrices are not tensors.

2.4 Relativistic Dynamics

2.4.1 Proper Time and 4-Velocity

The trajectory of a particle in spacetime is called its **world line**. In general, it can be parametrized by a smooth and differentiable parameter λ,

$$t = t(\lambda), \quad x = x(\lambda), \quad y = y(\lambda), \quad z = z(\lambda).$$

(It is similar to how we parametrize a curve in three-space by a parameter t. But t is now a coordinate, so it is better to use a fifth parameter λ for generality.) Recall that the infinitesimal proper time $d\tau$ is defined by (2.4). We can now write

$$d\tau = \sqrt{\eta_{\mu\nu}dx^{\mu}dx^{\nu}} = \sqrt{(dt^2 - dx^2 - dy^2 - dz^2)}.$$

The proper time of a matter particle is guaranteed to be real because all particles will travel below the speed of light, $dt^2 > dx^2 + dy^2 + dz^2$. The proper time of a particle along the world line is the integral

$$\tau = \int d\tau = \int_{\lambda_{\min}}^{\lambda_{\max}} d\lambda \sqrt{\left(\frac{dt}{d\lambda}\right)^2 - \left(\frac{dx}{d\lambda}\right)^2 + \left(\frac{dy}{d\lambda}\right)^2 + \left(\frac{dz}{d\lambda}\right)^2} = \int_{\lambda_{\min}}^{\lambda_{\max}} d\lambda \sqrt{\eta_{\mu\nu}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda}}.$$

Example 2.4. Muon in a ring

Consider a muon moving with speed $v = \omega R$ on a circular ring, where ω is the angular frequency and R is the radius of the ring. For convenience, we will parametrize the motion by the time t measured in the lab frame:

$$t = t, \quad x = R\cos\omega t, \quad y = R\sin\omega t, \quad z = z_0.$$

We are interested in how much time has passed in one revolution in muon's frame (assuming that it does not decay). The proper time differential is

$$d\tau = \sqrt{dt^2 - dx^2 - dy^2 - dz^2} = \sqrt{dt^2 - (\omega R)^2\,dt^2} = dt\sqrt{1 - (\omega R)^2}.$$

The total proper time is

$$\tau = \int_0^{2\pi R/v} dt\sqrt{1-(\omega R)^2} = \frac{2\pi R}{v}\sqrt{1-v^2} = \frac{t_{\text{lab}}}{\gamma},$$

as expected by time dilation.

The most prototypical 4-vector is the differential displacement $\{dx^\mu\} = (dt, dx, dy, dz)$. From this we are able to generalize other classical three-vectors to their counterparts in special relativity. The first one is the velocity:

$$\mathbf{v} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) \quad \overset{?}{\Longrightarrow} \quad \left\{\frac{dx^\mu}{dt}\right\} = \left(\frac{dt}{dt}, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right).$$

Though (dt, dx, dy, dz) is a 4-vector, when they are divided by dt, the resulting object is no longer a 4-vector because dt is one of the components, and it is *not* Lorentz invariant. To preserve Lorentz invariance of (dt, dx, dy, dz), we want to divide a $d\lambda$ that is also Lorentz invariant. Meanwhile, $d\lambda$ should have units of time, since we want a velocity object. The only Lorentz-invariant measure of time we know is $d\tau$. Thus, we define the **4-velocity** of a particle to be

$$u \equiv \{u^\mu\} = \left\{\frac{dx^\mu}{d\tau}\right\} = \left(\frac{dt}{d\tau}, \frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}\right). \tag{2.12}$$

Using $dt/d\tau = \gamma$, we can relate the 4-velocity to ordinary velocity (v_x, v_y, v_z),

$$u = \left(\frac{dt}{d\tau}, \frac{dx}{dt}\frac{dt}{d\tau}, \frac{dy}{dt}\frac{dt}{d\tau}, \frac{dz}{dt}\frac{dt}{d\tau}\right) = (\gamma, \gamma v_x, \gamma v_y, \gamma v_z).$$

The magnitude of the 4-velocity must be invariant to be a 4-vector. Indeed,

$$u \cdot u = \eta_{\mu\nu}u^\mu u^\nu = \eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = \frac{d\tau^2}{d\tau^2} = 1.$$

This means in ordinary units, $u \cdot u = c^2$, which is of course Lorentz invariant. Interestingly, the result above shows that the 4-velocity is a timelike unit vector. This can be seen in a particle's rest frame, where its spatial velocity is zero. The only nonzero component of the 4-velocity is then its time component, solely along its time-axis. As a consequence, the 4-velocity is always tangent to the world line (because it represents the time-axis) viewed other frame.

2.4.2 4-Momentum and the Invariant Mass

Consider a free particle with $m > 0$ for now. The **4-momentum** of this particle is defined as the product of its 4-velocity and its mass,

$$p \equiv \{p^\mu\} = (E, p_x, p_y, p_z) = (\gamma m, \gamma m v_x, \gamma m v_y, \gamma m v_z).$$

The components are relativistic energy and momentum. In ordinary units, the 4-momentum is

$$p = (\gamma mc^2, \gamma m v_x c, \gamma m v_y c, \gamma m v_z c).$$

We can check both energy and momentum in the non-relativistic limit $v \ll 1$,

$$E = \gamma m = \frac{m}{\sqrt{1-v^2}} = \left(1 + \frac{1}{2}v^2 + \cdots\right)m = m + \frac{1}{2}mv^2 + \mathcal{O}(v^4),$$

$$p_x = \gamma m v = \frac{mv}{\sqrt{1-v^2}} = \left(1 + \frac{1}{2}v^2 + \cdots\right)mv = mv + \mathcal{O}(v^3).$$

The total energy is the rest mass energy plus the classical kinetic energy, and the momentum is the mass times velocity. In the non-relativistic limit, $v \to 0$, we have $E \to m$ and $|\mathbf{p}| \to 0$. In the relativistic limit, $v \to 1$ and $|\mathbf{p}| \to E$.

The associated Lorentz-invariant quantity of the 4-momentum is the invariant mass:

$$p \cdot p = \eta_{\mu\nu}p^\mu p^\nu = m^2\eta_{\mu\nu}u^\mu u^\nu = m^2. \tag{2.13}$$

In terms of energy and spatial momentum, this is saying

$$p_\mu p^\mu = E^2 - \mathbf{p}^2 = m^2 \quad \text{or} \quad E^2 = \mathbf{p}^2c^2 + m^2c^4.$$

the famous relation in relativity. Like the three-momentum in classical mechanics, 4-momentum is conserved for a free particle (or a system of free particles). The conservation of 4-momentum plays an important role in particle physics. It represents four conserved quantities: one for conserved energy, and three for conserved spatial momentum. These conservation laws determine whether a decay/scattering is kinematically forbidden or not.

2.4.3 Action and Lagrangian for a Free Paticle

Some review: in classical mechanics, we define the Lagrangian to be $\mathcal{L} = T - V$, where T and P are the kinetic and potential energy, respectively, of the system. The action along a trajectory from point a to point b is defined as

$$S[q_i(t)] = \int_{t_a}^{t_b} dt\, \mathcal{L}(q_i, \dot{q}_i),$$

where q_i is some generalized coordinates that specify every point on the trajectory. By Hamilton's principle (or principle of stationary action), the action of a actual physical trajectory is extremized among all possible trajectories going from a to b. We then extremize the action using calculus of variations. It helps obtain the equation of motion for each $q_i(t)$, which is the Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = 0.$$

For a non-relativistic free particle of mass m, if q_i represents (x, y, z), then

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2), \quad V = 0.$$

The Euler-Lagrange equation says that

$$0 = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}(m\dot{q}_i) = m\ddot{q}_i.$$

This is just Newton's second law for a free particle $m\mathbf{a} = 0$. It says that a free particle should travel at constant velocity in space.

The Lagrangian/action is the most fundamental one among all physical quantities. It tells which physical path a particle will take. Even though observers in each frame can use different coordinates to represent the trajectory, every one should agree with the actual physical trajectory it takes. Hence the Lagrangian/action should be the "most" invariant quantity, so it needs to be Lorentz invariant in special relativity.

The goal now is to find $\mathcal{L}$ for a relativistic particle. The classical Lagrangian cannot do the job because: 1. it has no c and hence no speed limit; 2. it is not Lorentz invariant. Note that the action has dimensions of angular momentum, $[S] = [E \cdot T]$. We want to have some Lorentz-invariant quantities that make the units right. There are three fundamental constants we can use: the speed of light c, Planck constant $\hbar$, and gravitational constant G. (In this section we will write c's out explicitly.) Planck constant $\hbar$ has the unit of angular momentum, but we cannot use it because there is no quantum mechanics going on yet. Of course G is not useful since there is no gravity. The only available constants now are c and m. We have no choice but to set $\mathcal{L} \propto mc^2$. To get dimensions of T, the only Lorentz-invariant candidate is the proper time τ. In conclusion, the action for a relativistic free particle must be of the form

$$S \propto mc^2 \int_{\text{WL}} d\tau,$$

where WL stands for world line. For future convenience, we will choose the constant of proportionality to be -1. To make it an integral over dt in general reference frame, use the relation $d\tau = dt/\gamma$. Then the action along the world line is

$$S = -mc^2 \int_{t_a}^{t_b} \frac{dt}{\gamma} = -mc^2\sqrt{1-\beta^2}\int_{t_a}^{t_b} dt,$$

where $\beta = v/c$ indicates the speed of this free particle in some inertial frame. We can identify the Lagrangian to be

$$\mathcal{L} = -mc^2\sqrt{1-\beta^2}. \tag{2.14}$$

Checking the non-relativistic limit:

$$\mathcal{L} = -mc^2\left(1 - \frac{1}{2}\beta^2 + \cdots\right) = -mc^2 + \frac{1}{2}mv^2 + \mathcal{O}(v^4).$$

Comparing to $\mathcal{L} = T - V$ in classical mechanics, the rest mass energy acts like a potential energy of the particle.

In the last section we stated the definition of 4-momentum without a reason, and we did not prove its conservation either. Now with the Lagrangian, it is time to compute the conserved quantities. We know that if a coordinate q_i is cyclic (which means it does not appear in the Lagrangian), then its conjugate generalized momentum $\partial\mathcal{L}/\partial\dot{q}_i$ is conserved. In the free-particle case, there is no explicit x, y, z. The conserved momentum in x is

$$p_x = \frac{\partial\mathcal{L}}{\partial\dot{x}} = -mc^2\frac{\partial}{\partial\dot{x}}\left(1 - \frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{c^2}\right)^{-1/2} = -mc^2\left(\frac{\dot{x}/c^2}{\sqrt{1-(v/c)^2}}\right) = mc\frac{\dot{x}/c}{\sqrt{1-(v/c)^2}} = \gamma mc\beta_x.$$

The same holds for p_y and p_z. Note that the Lagrangian has no explicit time-dependence, which means the Hamiltonian/total energy is also conserved. The Hamiltonian is defined as

$$\begin{aligned}
H &= \sum_i p_i\dot{q}_i - \mathcal{L} \\
&= (mc\boldsymbol{\beta}\gamma)(c\boldsymbol{\beta}) - \left(-mc^2\sqrt{1-\beta^2}\right) \\
&= mc^2\left(\gamma\beta^2 + \sqrt{1-\beta^2}\right) \\
&= mc^2\left[\frac{\beta^2}{\sqrt{1-\beta^2}} + \frac{1-\beta^2}{\sqrt{1-\beta^2}}\right] \\
&= \gamma mc^2.
\end{aligned}$$

This is the conserved relativistic energy. We have proved almost everything. The only problem now is the massless particle, for which $\mathcal{L} = 0$. By the invariant mass (2.13),

$$p_\mu p^\mu = E^2 - p^2 = m^2 = 0.$$

This forces $E = p$ for a massless particle. Then how do we know E, for example, for a photon? It is given by Planck's formula $E = \hbar\omega = h\nu$.

2.4.4 Decay and Scattering

In this section, we will look at kinematics behind important processes in particle physics using conservation of 4-momentum. The examples include: two-body decay, ordinary scattering, and Compton scattering.

Two-Body Decay

A decay process involves a parent particle A and more than one daughter particles: $A \to 1 + 2 + \cdots + N$. To analyze a decay, we focus on several questions: Is a given process kinematically allowed or forbidden? What are the energies of te daughter particles? Does the decay violates other conservation laws, such as charge, baryon number, angular momentum, parity, etc.? What is the decay rate?

The simplest decay process is a two-body decay: $A \to 1 + 2$, where A is a parent particle with mass m_A, and 1 and 2 are two daughter particles with mass m_1 and m_2, respectively. Let the particles A, 1, and 2 have 4-momentum p_A, p_1, and p_2, respectively. The invariant mass says that

$$p_A^2 = m_A^2, \quad p_1^2 = m_1^2, \quad p_2^2 = m_2^2.$$

The general conservation of 4-momentum reads

$$p_A^\mu = p_1^\mu + p_2^\mu \quad \Longleftrightarrow \quad p_A = p_1 + p_2.$$

which represents four independent equations. It is most convenient to analyze the problem in the **center-of-momentum** (CM) frame of A. In this frame, the 4-momentum of A is $p_A = (m_A, 0, 0, 0)$. Due to conservation of 3-momentum, this frame forces the daughter particles to travel in opposite direction. Now we will try to determine the energies of the daughter particles. It turns out that we don't even need to worry about the index μ because the invariant mass will do all the job.

First, rewrite the conservation of 4-momentum as

$$p_A - p_1 = p_2.$$

Squaring both sides,

$$p_A^2 + p_1^2 - 2p_A \cdot p_1 = p_2^2.$$

(By squaring a 4-vector p we mean to compute the Lorentz-invariant inner product with itself $p_\mu p^\mu$.) Now using the invariant mass,

$$m_A^2 + m_1^2 - 2p_A \cdot p_1 = m_2^2.$$

In the CM frame, the dot product $p_A \cdot p_1 = (m_A, 0, 0, 0) \cdot (E_1, \mathbf{p}_1) = m_A E_1$. This is why we choose the CM frame, the zeros take care of the unknown $\mathbf{p}_1$. Putting this back into the equation, we have the energy:

$$m_A^2 + m_1^2 - 2m_A E_1 = m_2^2 \quad \Longrightarrow \quad E_1 = \frac{m_A^2 + m_1^2 - m_2^2}{2m_A}. \tag{2.15}$$

Interchanging $1 \leftrightarrow 2$ or using $E_2 = m_A - E_1$ gives

$$E_2 = \frac{m_A^2 + m_2^2 - m_1^2}{2m_A}.$$

After knowing the energies and masses, we can compute the spatial momentum:

$$\begin{aligned}
|\mathbf{p}_1| = |\mathbf{p}_2| = \sqrt{E_1^2 - m_1^2} &= \sqrt{\frac{(m_A^2 + m_1^2 - m_2^2)^2 - 4m_A^2 m_1^2}{4m_A^2}} \\
&= \frac{\sqrt{(m_A^2 + m_1^2 - m_2^2 + 2m_A m_1)(m_A^2 + m_1^2 - m_2^2 - 2m_A m_1)}}{2m_A} \\
&= \frac{\sqrt{[(m_A + m_1)^2 - m_2^2][(m_A - m_1)^2 - m_2^2]}}{2m_A} \\
&= \frac{\sqrt{[m_A^2 - (m_1 + m_2)^2][m_A^2 - (m_1 - m_2)^2]}}{2m_A}.
\end{aligned}$$

The most crucial conclusion from this two-body decay is that the energies of the decay products are uniquely determined by the masses. We say that the decay products are **mono-energetic**. The decay products of a three-body decay is not mono-energetic. For instance, in the beta decay $n \to pe^- \bar{\nu}_e$ (this notation is no different from $n \to p + e^- + \bar{\nu}_e$), the electron in the beta decay has a spectrum of energy. In fact, it was this mysterious spectrum that inspired Wolfgang Pauli to propose that there must be an invisible particle—a neutrino—along with a proton and an electron.

Example 2.5. Kinematically forbidden pair production

The pair production from one photon to an electron-positron pair, $\gamma \to e^+ e^-$, is kinematically not allowed. In principle, we can always find a frame such that the total 3-momentum of the electron-positron pair is zero. But there is no such frame for *one* photon: a photon has a nonzero energy, and its magnitude of momentum is exactly equal to the energy. In other words, the magnitude of its 4-momentum is $p_\gamma^2 = E_\gamma^2 - \mathbf{p}_\gamma^2 = 0$. To have a kinematically allowed pair production, one needs at least two photons.

Scattering Processes

A scattering process is a process in which two particles collide/interact to produce two or more particles: $A + B \to 1 + 2 + \cdots + N$. In the CM frame, particles A and B have 4-momenta

$$p_A = (E_A, \mathbf{p}_A) = \left(\sqrt{m_A^2 + \mathbf{p}_A^2}, \mathbf{p}_A\right), \quad p_B = (E_B, \mathbf{p}_B) = \left(\sqrt{m_B^2 + \mathbf{p}_B^2}, \mathbf{p}_B\right).$$

The total 4-momentum is then

$$p_{\text{tot}} = p_A + p_B = (E_A + E_B, 0, 0, 0).$$

Here we can compute the invariant quantity $\sqrt{s}$:

$$\sqrt{s} \equiv \sqrt{(p_A + p_B)^2} = E_A + E_B = E_{\text{CM}}.$$

Once we know $\sqrt{s}$ in the CM frame, it is the same in every other inertial frames. In experiments with colliding beams, the particles usually have the same mass, e.g. $e^+ + e^- \to 1 + 2 + \cdots + n$. In these cases, $\sqrt{s} = 2E$. We need to note that $\sqrt{s} = E_{\text{CM}}$ is linearly proportional to the beam energy E.

Particle physics always works with E_{CM} because it determines whether a process is allowed. The useful energy for creating particles comes from E_{CM}. As long as E_{CM} exceeds the total rest mass $m_1 + m_2 + \cdots m_N$, the scattering process is energetically allowed.

Now consider a fixed target scattering, where the particle B is at rest in the lab frame. This lab frame is not the CM frame. As usual, first list the 4-momenta of particles A and B:

$$p_A = (E_A, \mathbf{p}_A) = \left(\sqrt{m_A^2 + \mathbf{p}_A^2}, \mathbf{p}_A\right), \quad p_B = (E_B, \mathbf{p}_B) = (m_B, 0, 0, 0).$$

The total 4-momentum is

$$p_{\text{tot}} = p_A + p_B = (E_A + m_B, \mathbf{p}_A).$$

The E_{CM} is given by

$$E_{\text{CM}} = \sqrt{s} = \sqrt{p_A^2 + p_B^2 + 2p_A \cdot p_B} = \sqrt{m_A^2 + m_B^2 + 2E_A m_B}. \tag{2.16}$$

The beam energy E_A is often much larger than the rest mass energy of A or B, so $E_{\text{CM}} \simeq \sqrt{2E_A m_B}$. The total energy in the CM frame is proportional to the *square root* of the beam energy E_A in a fixed target experiment. This is big trouble: much of the energy in the lab frame is used to satisfy conservation of linear momentum. The final-state particles acquire kinetic energies, so there is not enough energy to create additional particles or massive particles we want.

Example 2.6. The trouble

In the LHC, the lab frame is the CM frame of two proton beams, in which each proton has energy $E_p = 6.5\,\text{TeV}$. The total energy is $E_{\text{CM}} = 2E_p = 13\,\text{TeV}$. Let's calculate how much energy E_p^{FT} is required for the proton beam to het a stationary proton target to reach $E_{\text{CM}} = 13\,\text{TeV}$. Using $E_{\text{CM}} \simeq \sqrt{2E_A m_B}$, where $E_A = E_p^{\text{FT}}$ now,

$$E_p^{\text{FT}} = \frac{E_{\text{CM}}^2}{2m_p} = \frac{(13 \times 10^3\,\text{GeV})^2}{2 \times 0.938\,\text{GeV}} \simeq 10^5\,\text{TeV}.$$

Evidently, this is far higher than the energy in the LHC. This tells us it is much better to collide two beams with each other than a to collide with a fixed target.

Sometimes one would like to produce a particular set of final-state particles in a fixed-target experiment. The goal is to find the minimum beam energy needed. The conservation of 4-momentum says

$$p_A + p_B = p_1 + p_2 + \cdots p_N.$$

In the CM frame (primed frame),

$$p_1' + p_2' + \cdots + p_N' = (E_1' + E_2' + \cdots + E_N', 0, 0, 0) = \left(\sqrt{\mathbf{p}_1'^2 + m_1^2} + \sqrt{\mathbf{p}_2'^2 + m_2^2} + \cdots + \sqrt{\mathbf{p}_N^2 + m_N^2}, 0, 0, 0\right).$$

By (2.16), minimizing E_{CM} means minimizing E_A. Remember that $E_{\text{CM}} = \sqrt{s}$ is frame invariant, which means it can also be written as $E_{\text{CM}} = \sqrt{(p_A' + p_B')^2} = \sqrt{(p_1' + p_2' + \cdots + p_N')^2}$. To minimize E_{CM}, we just need all spatial momenta of final-state particles to vanish so that there is no kinetic energy required. Then all energies are for creating particles' rest mass,

$$E_{\text{CM}}^{\text{min}} = m_1 + m_2 + \cdots + m_N.$$

Setting this equal to (2.16),

$$m_A^2 + m_B^2 + 2E_A^{\text{min}} = (m_1 + m_2 + \cdots + m_N)^2.$$

The minimum beam energy required is

$$E_A^{\min} = \frac{(m_1 + m_2 + \cdots + m_N)^2 - m_A^2 - m_B^2}{2m_B}. \tag{2.17}$$

4-Momentm Transfer

A scattering process may have an initial-state particle that survives to the final state. Suppose such a particle has mass m, initial 4-momentum $p_i = (E_i, \mathbf{p}_i)$, and final 4-momentum $p_f = (E_f, \mathbf{p}_f)$. The change in its 4-momentum is

$$q = p_f - p_i = (E_f - E_i, \mathbf{p}_f - \mathbf{p}_i).$$

Since q is also a 4-momentum, its magnitude is Lorentz-invariant,

$$q^2 = (p_f - p_i)^2 = p_i^2 + p_f^2 - 2p_i \cdot p_f = 2m^2 - 2(E_iE_f - \mathbf{p}_i \cdot \mathbf{p}_f).$$

This change in momentum is called the **4-momentum transfer**.

Compton Scattering

Compton scattering is a photon-electron scattering process:

$$\gamma + e^- \to \gamma + e^-.$$

A similar process can be $g + q \to g + q$, where g is a gluon and q is a quark. In Compton's original scattering experiment, the initial electron is actually not a free particle, but instead bounded to an atom. However, the binding energy of the electron $E_{\text{bind}} \ll E_\gamma$ for energetic photons such as X-rays, so the electron can be treated as a free electron. Compton found that when a photon scatters of an electron, its wavelength λ increases by

$$\Delta\lambda = \frac{2\pi\hbar}{m_e c}(1 - \cos\theta),$$

where θ is the scattering angle (the angle between the final 3-momentum and initial 3-momentum of the photon). We will show this by conservation of momentum.

$$p_e + p_\gamma = p'_e + p'_\gamma.$$

where the primes indicate final-state quantities. Since we do not know anything about the final 4-momentum p'_e of the electron, we will move it to one side and all other quantities to the other side:

$$p_e + p_\gamma - p'_\gamma = p'_e,$$

Squaring both sides gives

$$p_e^2 + p_\gamma^2 + p'^2_\gamma + p_e \cdot p_\gamma - p_e \cdot p_\gamma - p_\gamma \cdot p'_\gamma = p'^2_e.$$

By the invariant mass, $p_e^2 = p'^2_e = m_e^2$ and $p_\gamma^2 = p'^2_\gamma = 0$. We see that p'^2_e is eliminated by the invariant mass and we don't have to worry about it anymore:

$$\cancel{m_e^2} + p_e \cdot (p_\gamma - p'_\gamma) - p_\gamma \cdot p'_\gamma = \cancel{m_e^2}.$$

Now using the fact that the electron is initially at rest, so $p_e = (m_e, 0, 0, 0)$. Its dot product with $p_\gamma - p'_\gamma$ is just $m_e(E_\gamma - E'_\gamma)$. Meanwhile,

$$p_\gamma \cdot p'_\gamma = E_\gamma E'_\gamma - |p_\gamma||p'_\gamma| \cos\theta = E_\gamma E'_\gamma(1 - \cos\theta),$$

Putting these back

$$m_e(E - E') - E_\gamma E'_\gamma(1 - \cos\theta) = 0 \quad \Longrightarrow \quad m_e(E - E') = E_\gamma E'_\gamma(1 - \cos\theta).$$

Dividing both sides by $m_e E_\gamma E'_\gamma$, we get the Compton scattering formula:

$$\boxed{\frac{1}{E'_\gamma} - \frac{1}{E_\gamma} = \frac{1}{m_e}(1 - \cos\theta).} \tag{2.18}$$

It related the final energy of the photon E'_γ to the scattering angle θ. Using Planck's formula to connect energy to wavelength,

$$E_\gamma = \frac{hc}{\lambda} = \frac{2\pi\hbar c}{\lambda},$$

the Compton scattering formula becomes

$$\frac{\lambda'}{2\pi\hbar c} - \frac{\lambda}{2\pi\hbar c} = \frac{1}{m_e c^2}(1 - \cos\theta).$$

In terms of $\Delta\lambda = \lambda' - \lambda$, it is

$$\boxed{\Delta\lambda = \frac{2\pi\hbar}{m_e c}(1 - \cos\theta).} \tag{2.19}$$

If $\theta \neq 0$ (the trivial scattering) in the Compton scattering process, the photon wavelength increases because $\cos\theta < 1$, or $1-\cos\theta > 0$. As a result, the photon frequency ν must decrease because of the inverse relation $\nu = c/\lambda$, and so does the energy.

3 Observables

Particle physics require observations. There are many observables and measurements that connects experiments to theory. Most of the measurements serve the following goals:

1. **Determination of the properties of particles.** These properties include masses, lifetimes, spins, charges, magnetic moments, decay rates, branching fractions, etc. Because most particles have finite lifetimes, there appear different methods to measure these quantities experimentally.
2. **Searches for new particles.** Throughout the history, particle physicists are always searching for new particles (e^+, ν's, $W^\pm$ and Z, the Higgs boson H, etc.) to perfect the standard model. There are also searches that seek particles outside the standard model, such as supersymmetric particles and dark matter.
3. **Measurement of cross-sections.** The cross-sections are relevant to scattering processes. They are quantities that help study how particles interact.

Natural units are adopted, $\hbar = 1$, $c = 1$, except when we are dealing with numbers.

3.1 Observables

3.1.1 Why do Particles Decay?

When studying quantum mechanics, we put efforts finding the eigenstates and eigenvalues of the Hamiltonian. These states have perfectly defined energies. For example, the energy levels of the hydrogen atom is

$$E_n = -\frac{13.6\,\mathrm{eV}}{n^2},$$

where n is the principal quantum number, determined by the radial equation of the Schrödinger equation. The time dependence of energy eigenstates are described by a oscillatory phase factor e^{-iEt},

$$\psi(\mathbf{x}, t) = \psi_{n\ell m}(\mathbf{x}) e^{-iE_n t},$$

where $\ell = 0, 1, \ldots, n-1$ is the azimuthal/orbital quantum number, and $m = -\ell, -\ell+1, \cdots -1, 0, 1, \ldots, \ell-1, \ell$ is the magnetic quantum number. The $\psi(\mathbf{x}, t)$ above oscillates forever and hence cannot undergo decay. But we know that excited states of the hydrogen atom *do* decay—they spontaneously emit one or more photons and return to the ground state in a finite amount of time. There appears a contradiction between theory and observations. The resolution is that the Hamiltonian in the non-relativistic Schrödinger equation of the hydrogen atom is not the full Hamiltonian. The eigenstates $\psi_{n\ell m}$ obtained in quantum mechanics are not the "true eigenstates". In fact, for any quantum state that survives for a finite interval Δt, such a state must be made up of a superposition of energy eigenstates with a spread of energy ΔE. These uncertainties are related by the uncertainty principle

$$\Delta E \Delta t \geq \frac{\hbar}{2}.$$

3.1.2 The Exponential Law and Decay observables

The decay of particles follow the exponential decay law. Let $P(t)$ be the probability to survive until time t,

$$P(t) = 1 - \int_0^t \mathcal{P}(t')\, dt',$$

where $\mathcal{P}(t)$ is the probability density to decay at t. Then the probability to survive until $t+dt$ is

$$\begin{aligned}
P(t+dt) &= (\text{prob. to survive until } t)(\text{prob. to then survive from } t \text{ to } t+dt) \\
&= (\text{prob. to survive until } t)(1 - \text{prob. to decay during } dt) \\
&= P(t)(1-\mathcal{P}(t)\,dt).
\end{aligned}$$

For unstable particles, the probability to decay during dt is a constant $\mathcal{P} = K$ that is independent of t. That is, the probability to decay in the next dt does not depend on how long the particle has lived. Then we can write

$$P(t+dt) = P(t)(1-K\,dt).$$

Rearranging some terms,

$$\frac{P(t+dt)-P(t)}{dt} = -KP(t) \quad \Longrightarrow \quad \frac{dP}{dt} = -KP(t).$$

This gives the exponential decay law,

$$P(t) = e^{-Kt}.$$

We define the constant K to be the **total decay rate**,

$$K = R_{\text{tot}} = \sum_i R_i, \tag{3.1}$$

where R_i is the decay rate of the ith decay mode. The **mean proper lifetime** (or mean lifetime or just lifetime), τ, is the inverse of the decay rate

$$\tau \equiv \frac{1}{K} = \frac{1}{R_{\text{tot}}}. \tag{3.2}$$

The exponential decay law can be written in as

$$P(t) = e^{-t/\tau}. \tag{3.3}$$

At each time, there is a probability density $\mathcal{P}(t) = P(t)$ that determines the decay probability within time t and $t+dt$, defined as

$$P(t) = 1 - \int_0^t \mathcal{P}(t')\,dt'.$$

Taking the time-derivative on both sides gives

$$\boxed{\mathcal{P}(t) = \frac{1}{\tau}e^{-t/\tau}.} \tag{3.4}$$

This ensures that the probability density $\mathcal{P}(t)$ is normalized. This probability density should not be confused with the decay rate K: $\mathcal{P}(t)\,dt$ is the probability to first survive to t and decay within time t and $t+dt$. It incorporates the probability of a particle survival to t after it was created. On the other hand, K is the probability to decay within any dt, given that the particle has not decay. It does not care about how much time the particle has lived.

The reason why τ is called the mean lifetime is obvious. It is the *mean* lifetime of a particle in probabilistic sense,

$$\langle t\rangle = \int_0^\infty dt\,t\cdot\mathcal{P}(t) = \int_0^\infty dt\,t\left(\frac{1}{\tau}e^{-t/\tau}\right) = \tau\int_0^\infty \frac{dt}{\tau}\frac{t}{\tau}e^{-t/\tau} = \tau\int_0^\infty dx\,xe^{-x} = \tau.$$

Back to the decay rates R. We can write (3.1) as

$$1 = \sum_i \frac{R_i}{R} = \sum_i B_i \quad \text{where} \quad B_i \equiv \frac{R_i}{R} \tag{3.5}$$

is called the **branching fraction** of the ith decay mode. A branching fraction can be determined by measuring the fractions of decays that lead to a specific decay final state. When one obtain all R_i's, the branching fraction B_i can be easily calculated. Conversely, obtaining all B_i's does not mean knowing decay rates, because we have lost some information. The *rates* is lost in the fraction R_i/R, where s^{-1} are canceled. Hence measuring all branching fraction cannot tell the mean lifetime either.

It is convenient to express the decay rate in energy units, so we define the **decay width** Γ to be

$$\Gamma_i = \hbar R_i, \quad \Gamma = \sum_i \Gamma_i = \sum_i \hbar R_i. \tag{3.6}$$

The quantities Γ_i are called **partial decay widths** or simply partial widths, and Γ is the **total decay width**, or the total width of the particle. The decay width is connected to everything: the decay rate, branching fractions, and mean lifetime. In particular, the total width is related to the mean lifetime by

$$\tau = \frac{\hbar}{R} = \frac{1}{\sum_i R_i} = \frac{\hbar}{\sum_i \Gamma_i} = \frac{\hbar}{\Gamma},$$

and the branching fraction by

$$\Gamma_i = \Gamma \cdot \frac{\hbar \Gamma_i}{\hbar \Gamma} = \Gamma \frac{R_i}{R} = \frac{\hbar}{\tau} B_i.$$

In natural units, $\Gamma_i = R_i$ and $\Gamma = 1/\tau$. This will be useful when we derive the Breit-Wigner line shape.

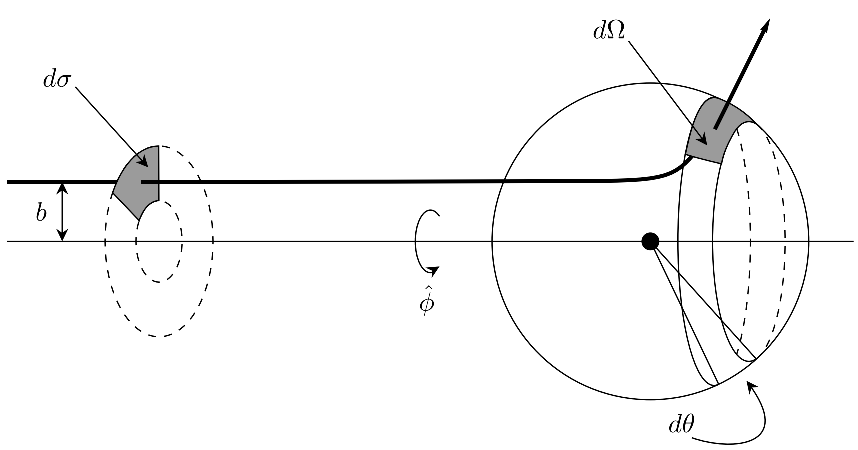
All above above are referred to the rest frame of the particle. In the lab frame, particles are moving at relativistic speeds, so quantities such as the observed mean lifetime will be different from the mean proper lifetime.

3.1.3 Angular Measurements

The parameter to describe a scattering process is the cross-section σ of the incident beam. The cross-section σ has dimensions of an area A. We are interested in the total cross-section

$$\sigma_{\text{tot}} = \sum_{i=1}^{n} \sigma_i.$$

A scattering can be described as the following:



Suppose a particle come across a potential and scatters of at the **scattering angle** θ. The scattering angle is a function of the **impact parameter** b, the distance by which the incident particle would have missed the scattering center. If he incident beam passes through an infinitesimal cross-section $d\sigma$, it will scatter into a solid angle $d\Omega$, and

$$d\sigma = D(\theta)\, d\Omega,$$

where $D(\theta)$ is called the **differential cross-section**. The figure shows that

$$d\sigma = |b\, db\, d\phi| \quad \text{and} \quad d\Omega = |\sin\theta\, d\theta\, d\phi|.$$

Hence the differential cross-section can be expressed as

$$D(\theta) = \frac{d\sigma}{d\Omega} = \left| \frac{b}{\sin\theta} \left(\frac{db}{d\theta} \right) \right|.$$

The total cross-section is evidently

$$\sigma = \int d\sigma = \int D(\theta)\, d\Omega.$$

Experimentally, we can control the **luminosity** L of a beam of incident particles. L is the number of particles passing down the line per unit time, per unit area. $dN = L\, d\sigma$ is the number of particles per unit time passing through area $d\sigma$, and the number per unit time scattered into the solid angle $d\Omega$:

$$dN = L\, d\sigma = LD(\theta)\, d\Omega.$$

If the luminosity is known (controlled), we can measure the number of particles entering the detector over a solid angle, and subsequently determine the differential cross-section,

$$D(\theta) = \frac{dN}{L\, d\Omega}.$$

3.1.4 Breit-Wigner Line Shape and the Decay Width

In non-relativistic quantum mechanics, the time evolution of an eigenstate can be described by

$$|\psi(t)\rangle = e^{-iEt}\,|\psi(0)\rangle\,.$$

In the particle's rest frame, $E = M$, and let $|\psi(0)\rangle$ be normalized,

$$|\psi(t)\rangle = e^{-iMt}\,|\psi(0)\rangle \quad \text{with} \quad \langle\psi(0)|\psi(0)\rangle = 1.$$

We see that once $|\psi\rangle$ is normalized at time $t = 0$, it is normalized forever,

$$\langle\psi(t)|\psi(t)\rangle = \langle\psi(0)|\,(e^{iEt}e^{-iEt})\,|\psi(0)\rangle = \langle\psi(0)|\psi(0)\rangle = 1.$$

Physically, an eigenstate has a definite energy and it lives forever. This does not describe particles that can decay. For an unstable particle, the probability density to decay at time t is given by (3.4),

$$\mathcal{P}(t) = \langle\psi(t)|\psi(t)\rangle = \frac{1}{\tau}e^{-t/\tau},$$

where τ is the mean lifetime. This suggests that if we replace $M \to M - i\Gamma/2$ and multiply a prefactor $\sqrt{\Gamma}$ to $|\psi(0)\rangle$,

$$|\psi(t)\rangle = \sqrt{\Gamma}\,|\psi(0)\rangle\, e^{-i(M-i\Gamma/2)t} \quad \Longrightarrow \quad P(t) = \langle\psi(t)|\psi(t)\rangle = \langle\psi(0)|\psi(0)\rangle\,\Gamma e^{-\Gamma t} = e^{-\Gamma t},$$

This reproduces $\tau = 1/\Gamma$. The time and energy are conjugate variables. We can Fourier transform $|\psi(t)\rangle$ to the energy space $|\tilde{\psi}(E)\rangle$, starting at $t = 0$,

$$|\tilde{\psi}(E)\rangle = \frac{1}{\sqrt{2\pi}}\int_0^\infty dt\, e^{iEt}\,|\psi(t)\rangle = \sqrt{\frac{\Gamma}{2\pi}}\int_0^\infty dt\, e^{[i(E-M)-\Gamma/2]t}\,|\psi(0)\rangle = \sqrt{\frac{\Gamma}{2\pi}}\,\frac{e^{[i(E-M)-\Gamma/2]t}}{i(E-M)-\Gamma/2}\bigg|_0^\infty |\psi(0)\rangle\,.$$

(Here E is not exactly equal to M because there will be a spread in E, resulting from a finite lifetime of the particle.) The evaluation at ∞ vanishes because of the exponential decay term $e^{-\Gamma t/2}$, so

$$|\tilde{\psi}(E)\rangle = -\sqrt{\frac{\Gamma}{2\pi}}\,\frac{1}{i(E-M)-\Gamma/2}\,|\psi(0)\rangle\,.$$

The probability density of finding the particle with energy E is

$$\mathcal{P}(E) = \langle\tilde{\psi}(E)|\tilde{\psi}(E)\rangle = \frac{\Gamma}{2\pi}\,\frac{1}{(E-M)^2 + (\Gamma/2)^2}\,\langle\psi(0)|\psi(0)\rangle = \frac{1}{2\pi}\,\frac{\Gamma}{(E-M)^2 + (\Gamma/2)^2}.$$

This probability density is a function of E, with parameters M and Γ. Meanwhile, E represents the rest mass energy m of a state. In other words,

$$\boxed{\mathcal{P}(m) = \frac{1}{2\pi}\,\frac{\Gamma}{(m-M)^2 + (\Gamma/2)^2}.} \tag{3.7}$$

This is the **Breit-Wigner line shape**, describing the spread of the observed mass m around the most probable mass M. The spread is characterized by the width Γ. Evidently, the maximum of $\mathcal{P}$ is at $m = M$, and $\mathcal{P}(m)$ is symmetric about $m = M$. There exist two points of m at which $\mathcal{P}(m)/\mathcal{P}(M) = 1/2$ (half maximum). One can easily calculate that

$$m_{\text{half max}} = M \pm \frac{1}{2}\Gamma.$$

This is why Γ is called the decay *width*. It is the full width at half maximum of the probability density. If we obtain the distribution of m in an experiment, we can calculate Γ as full width at half maximum. Quantum field theory will improve the formula for the Breit-Wigner line shape, but we shall stick with (3.7) now.

3.2 Particle Detection

3.2.1 Fundamental Principles of Particle Detection

There are three fundamental principles of *direct* particle detection (not decay products).

- **Principle I.** To detect a given particle A, the detection process must involve a type of fundamental interaction that couples to A. The detection process must also involve a type of fundamental interaction that couples to the matter or fields in the detector.

  There is no universal interaction that all particles must possess, except gravity. However, gravity does not play a major role in the Standard Model, so the detection of strong, electromagnetic, and weak interactions should be viewed differently. For example, to detect neutrinos, the detection process must involve weak interaction. This results in a low detection probability of neutrinos. Each interaction process is associated with a cross-section σ that characterizes the strength and range of an interaction between A and a target particle. The probability for A to undergo an interaction also depends on the number density n of target particles. Combining the two gives a characteristic interaction length for each type of interaction process i,

  $$\lambda_i = \frac{1}{n_i \sigma_i}.$$

  The probability for A to interact with the detector by a certain process when traveling a distance Δx through the detector is

  $$P(\Delta x) = 1 - e^{-\Delta x/\lambda_i} = 1 - e^{-n_i \sigma_i \Delta x}.$$

  If $\Delta x/\lambda_i \ll 1$, then $P(\Delta x) \simeq \Delta x/\lambda_i$.

- **Principle II.** The particle interacting with the detector must transfer an amount of energy that is much larger than the background noise in the system. The electronics that read out a detector should have a threshold criterion. This criterion varies from particles to particles, depending on the amount of energy transferred and the form of this energy. For example, a key energy transfer process in charged particle is ionization. It allows us to track the trajectory of a charged particle without much perturbation to its path. On the contrary, the detection of electrically neutral particles are often "destructive"—the particle is typically absorbed in the detection process.

- **Principle III.** A detector's properties and performance must be well understood. The resolution for a certain quantity should be known; all parts of the detector should work properly. This is to ensure that no event is mis-measured.

3.2.2 Classification of Particles by Stability and Interactions

The classification of particles is based on their mean lifetime and the types of interactions that they experiences. There are hundreds of particles in the Standard Model, but only a few of them can be detected directly. Most particles decay so fast that only their decay products are detectable. Almost all particles that decay via strong or electromagnetic interactions can only be detected indirectly. (There are also weak decay particles that are not directly detectable, such as W, Z, H, and the top quark t, mainly because of their very large masses.) To see why, let's calculate how far they can travel in a detector.

Example 3.1. A typical strong interaction decay

Consider a particle with decay width Γ of the order $10 - 100\,\text{MeV}$ (a typical decay via strong interactions). Their mean lifetime, according to (3.6), is about

$$\tau = \frac{\hbar}{\Gamma} \simeq \frac{66 \times 10^{-23}\,\text{MeV s}}{\Gamma\ (\text{MeV})} \sim 10^{-23}\,\text{s}.$$

This is the mean proper lifetime in the particle's rest frame. In the lab frame, there will be relativistic time dilation. The distance it can travel before decay at time Δt_{rest} is

$$\Delta x = v\Delta t = \gamma\beta c \Delta t_{\text{rest}},$$

where $\beta = v/c$. The mean distance traveled is

$$\langle \Delta x \rangle = \gamma\beta c \langle \Delta t_{\text{rest}} \rangle = \gamma\beta c\tau.$$

Recall that the relativistic 3-momentum is given by $|\mathbf{p}| = \gamma m v = \gamma\beta m c$. To achieve the maximum mean distance traveled, we want the largest value of $\gamma\beta = |\mathbf{p}|c/mc^2$. In a typical high energy experiment, a hadron with mass $mc^2 \sim 1\,\text{GeV}$ has momentum $|\mathbf{p}|c \sim 10\,\text{GeV}$, so the mean distance traveled is

$$\Delta x = (10c) \times 10^{-23}\,\text{s} = 3 \times 10^{-14}\,\text{m}.$$

This is only 10 times the size of an atomic nucleus, and is much smaller than the distance of an atom. There is no way to detect it directly with current technologies.

Here we will classify particles into three categories:

- **Detector-stable particles.** These particles usually traverse all the way through a detector. Some of them are *completely* stable (e.g. γ, p, $e^\pm$), and some are not (e.g. n, $\mu^\pm$, $\pi^\pm$).
- **Detector quasi-stable particles.** These particles travel a measurable distance from their production point before decaying via weak interactions. They can produce decay **vertices**, points in space where the measured trajectories of their daughter particles intersect. These vertices are distinctive and can be easily measured.
- **Detector-unstable particles.** These particles decay before traveling any measurable distance from their production point. Most particles are in this category. They can be detected indirectly using the conservation of 4-momentum.

Table 3.1 and 3.2 lists all detector-stable and detector quasi-stable particles, respectively.

Particle	Mass (GeV)	J	τ	$c\tau$ (m)	Ionization track?	EM casc. shower?	Strong casc. shower?	Penetrate had. absorber?
γ	0	1	stable	stable	-	✓	-	-
$e^\pm$	0.00511	1/2	stable	stable	✓	✓	-	-
$\mu^\pm$	0.106	1/2	$2.2\,\mu$s	659	✓	-	-	✓
$\nu_\ell, \bar{\nu}_\ell$	~ 0	1/2	stable	stable	-	-	-	-
$\pi^\pm$	0.140	0	26 ns	7.8	✓	-	✓	-
$K^\pm$	0.494	0	12 ns	3.7	✓	-	✓	-
K^0_L	0.498	0	0.51 ns	0.153	-	-	✓	-
$p, \bar{p}$	0.938	1/2	stable	stable	✓	-	✓	-
$n, \bar{n}$	0.940	1/2	879 s	2.6×10^{11}	-	-	✓	-
α	3.727	0	stable	stable	✓	-	✓	-

Table 3.1: A list of all detector-stable particles.

Particle	Mass (GeV)	J	τ	$c\tau$	Typical decay location
$\tau^\pm$	1.777	1/2	0.3 ps	$87\,\mu$m	inside beam pipe
K^0_S	0.498*	0	0.09 ns	2.7 cm	outside beam pipe
$\Lambda, \bar{\Lambda}$	1.116	1/2	0.26 ns	7.9 cm	outside beam pipe
$D^0, \bar{D}^0$	1.865	0	0.4 ps	$123\,\mu$m	inside beam pipe
$D^\pm$	1.870	0	1.0 ps	$312\,\mu$m	inside beam pipe
$D^\pm_s$	1.968	0	0.5 ps	$151\,\mu$m	inside beam pipe
$B^\pm$	5.279	0	1.6 ps	$491\,\mu$m	inside beam pipe
$B^0, \bar{B}^0$	5.280	0	1.5 ps	$455\,\mu$m	inside beam pipe
$B^0_s, \bar{B}^0_s$	5.367	0	1.5 ps	$454\,\mu$m	inside beam pipe

Table 3.2: A list of all detector quasi-stable particles.

Table 3.1 also lists the signitures for detector-stable particles. These signatures are detected in ionization tracking detectors, electromagnetic calorimeters, hadronic calorimeters, and absorber penetration detectors for muons. Charged particles produce an ionization track from EM interactions with atoms. The photon and electron produce EM cascade showers in dense matter. Detector-stable hadrons produce strong cascade showers in dense matter. The α-particle (helium-4 nucleus) is included because it is important in nuclear transitions. Whatever particles that make up dark matter is not included in Table 3.1, but they are stable particles. They exist in galaxies since the early universe, but are hard to detect.

Table 3.2 consists of all detector quasi-stable particles. All these particles except the τ-lepton are ground-state hadrons containing s, c, or b quarks. They cannot decay via strong or electromagnetic interactions because there are no lighter hadrons containing the same quark content. These particles decay either inside or outside around the beam pipe in a colliding beam experiment.

This kind of classification certainly depends on the detector, and it also depends on the momentum of the particle. (If the particle is produced at rest in the lab frame, we will observe it to decay very quickly.) Nonetheless, they work very well for most particle physics experiments.

3.2.3 Signatures of Particles in Detectors

4 Elementary Particle Interactions

This chapter discusses how elementary particles interact, and the **Feynman diagrams** that represent the allowed interactions. It is a qualitative chapter that shows how each of the relevent forces acts on quarks and leptons.

To be written.

4.1 The Four Interactions

There are four fundamental forces/interactions

Force	Strength	Theory	Mediator
Strong	1	Chromodynamics	Gluon g
Electromagnetic	10^{-2}	Electrodynamics	Photon γ
Weak	10^{-13}	Flavordynamics	W and Z
Gravitational	10^{-42}	Geometrodynamics/Relativity	Graviton (?)

Table 4.1: Four fundamental forces/interactions. The numbers in this table (especially the one for the weak force) should not be taken seriously because the forces depends on the sources and how far away they are.

The classical theory of gravity follows Newton's law of universal gravitation. The relativistic version is the general relativity by Einstein. Yet, there is still no satisfactory quantum theory of gravity. The classical electrodynamics is formulated by Maxwell and is consistent with special relativity. The quantum electrodynamics is perfected by Tomonaga, Feynman, and Schwinger in the 1940s. The first theory of weak forces is by Fermi in 1933, later refined by Lee and Yang, Feynman and Gell-Mann, and put into the modern form by Glashow, Weinberg, and Salam in the 1960s. In this notes, the theory of weak forces is refered to as the Glashow-Weinberg-Salam (GWS) theory. The strong force theory was first presented by Yukawa in 1934, and the chromodynamics appeared in the 1970s.

The hierarchy of interaction strengths is typically Strong $\gg$ Electromagnetic $\gg$ Weak. This hierarchy holds true when comparing cross-sections and decay rates, and its inverse is true when comparing lifetimes. If the CM-frame energy $\sqrt{s}$ is comparable to or larger than the mass of the W and Z, then the weak interaction can have strength comparable to the one of electromagnetism:

$$\begin{aligned}&\text{Strong} \gg \text{Electromagnetic} \gg \text{Weak}(\sqrt{s} \ll m_W, m_Z),\\ &\text{Strong} \gg \text{Electromagnetic} \sim \text{Weak}(\sqrt{s} \geq m_W, m_Z).\end{aligned}$$

4.1.1 Transition Amplitudes

In non-relativistic quantum mechanics, a system of particles at an initial time t_0 is a state $|\psi_i\rangle = |\psi(t_0)\rangle$. Its time evolution is governed by the Schrödinger equation, but here we will use another approach. Define the **time evolution operator** $U(t, t_0)$ such that

$$|\psi(t)\rangle = U(t, t_0)\,|\psi(t_0)\rangle\,.$$

The time evolution operator is usually a function of the Hamiltonian H. The **amplitude** of a final state $\psi(t)_f$ at time t is given by the inner product

$$\mathcal{A}_{fi}(t, t_0) = \langle\psi_f(t)|\psi(t)\rangle = \langle\psi_f|U(t, t_0)|\psi_i\rangle\,.$$

It is simply a projection of the true state $|\psi(t)\rangle$ at time t into the particular final state, $|\psi_f(t)\rangle$, because the true state is often a superposition of $|\psi_f(t)\rangle$ with different f. The square modulus of the amplitude $\mathcal{A}^2$ is the transition probability. To determine $U(t,t_0)$, we write the (total) Hamiltonian as

$$H = H_0 + H',$$

where H_0 is a free-particle Hamiltonian and H' describes the interactions. In many cases, the interaction Hamiltonian H' can be treated as a perturbation, but there are situations where it cannot.

4.1.2 The S-matrix

The study of particle physics requires relativistic quantum mechanics. The free-particle states in relativistic quantum mechanics depend on the spin of the particles. We will focus on spin-0, spin-1/2, and spin-1 particles. Elementary particles in the Standard Model all falls within one the these categories. The particles of those spins are described by the Klein-Gordon, Dirac, and Maxwell-like equations, respectively. For particle interaction processes (such as scattering), in order to have initial-state particles being free particles, they must be far apart so that their interactions are negligible ($H' = 0$) at $t_0 \to -\infty$. Similarly, at $t \to \infty$, the final-state particles must be far apart to be free particles. We assume that both conditions will be true when $t_0 \to -\infty$ and $t \to \infty$. The amplitude is then given by the S-matrix,

$$S_{fi} \equiv \lim_{\substack{t\to+\infty \\ t_0\to-\infty}} \mathcal{A}_{fi}(t,t_0) = \lim_{\substack{t\to+\infty \\ t_0\to-\infty}} \langle\psi_f|U(t,t_0)|\psi_i\rangle\,. \tag{4.1}$$

Both the initial state $|\psi_i\rangle = |\psi(t_0)\rangle$ and final state $|\psi_f(t)\rangle$ are eigenstates of H_0—they are free-particle states. In general, the S-matrix elements are complex-valued function of the particle properties, including masses, spins, charges, 4-momenta, even orbital angular momenta. They also involve quantities that characterize any intermediate-state particles before they evolve into the final-state particles. These intermediate-state particles are usually **virtual particles**. There can be multiple allowed trasition pathways through intermediate states as we will see in Feynman diagrams. This is how we define a *process*: a process should have the same initial state and the same final state, but it does not care about how the initial state become the final state. For example, $e^+ + e^- \to \gamma^* \to \mu^+ + \mu^-$ and $e^+ + e^- \to Z^* \to \mu^+ + \mu^-$ count as the same process. It does not care about intermediate particles, whether it is a photon or a Z-boson. The amplitude should be summed up before squaring to get the probability or probability density,

$$\mathcal{P}_{fi} \propto |\mathcal{A}_{fi,\text{tot}}|^2 = \left|\sum_{j=1}^{n} \mathcal{A}_{fi,j}\right|^2.$$

Note that we do not calculate individual probabilities first and sum them up because the amplitudes interfere with each other, so their relative phases do matter. If the set of final-state particles (or initial-state particles) are different, we should sum up the probabilities as they represent completely different processes. Summing up the amplitude of different processes are meaningless in this case.

4.1.3 Feynman Diagrams

A Feynman diagram represents a transition amplitude of a process, with the intermediate state fully specified. A set of Feynman diagrams describe different transition amplitudes order-by-order in perturbation theory. Though Feynman diagrams look like cartoons, they are extremely precise and rigorous. They are derived from the Lagrangian of the Standard Model

$$\mathcal{L}_{\text{SM}} = \mathcal{L}_{\text{strong}} + \mathcal{L}_{\text{EM}} + \mathcal{L}_{\text{weak}}.$$

The rules to draw Feynman diagrams are different for each type of interactions as we will discuss in the following sections.

A Feynman diagram consists of three components: external lines, internal lines, and fundamental interaction vertices. The external lines represent free-particle momentum eigenstates. The internal lines represent intermediate-state particles. In some processes in which no internal lines are present, there is a direct interaction between the initial-state particles and final-state particles. The interaction vertices specify a particular interaction between particles. In each vertex, the 4-momentum is strictly conserved. We will see many examples below.

There are several conventions to draw Feynman diagrams. In this notes and in most of the books, the time axis runs from left to right. The line-style conventions are listed in Table 4.2. Note that Feynman diagrams with precise Feynman rules work only for

Line style	Description	Particles Represented
	Solid line with no arrow.	Fermion or anntifermion according to label.
	Solid line with right arrow	Fermion traveling to the right in time.
	Solid line with left arrow	Antifermion traveling to the right in time.
	Wiggly line	$J = 1$ colorless boson: $\gamma, Z, W^{\pm}$ in SM.
	Spring	$J = 1$ colored boson: g in SM.
	Dashed line	$J = 0$ boson: H in SM.

Table 4.2: Line-style conventions commonly used for Feynman diagrams.

elementary particles in SM. Hadrons are not elementary particles as they are composed of quarks and gluons. Thus, they cannot be represented by a single line. Sometimes a hadron is represented by explicitly drawing all the quark lines in Feynman diagrams, but because these quarks are confined (they are not free particles), Feynman rules for them should be modified.

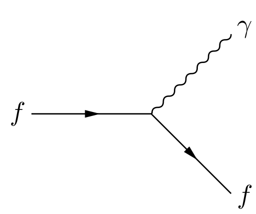
In the following sections, we will discuss how to draw Feynman diagrams in theories describing particular interactions: quantum electrodynamics (QED), quantum chromodynamics (QCD), and weak/GWS theory.

4.2 Quantum Electrodynamics (QED)

In quantum electrodynamics (QED), photons interact with electrically charged particles. This section mainly talks about interactions between photons and spin-1/2 fermions of the SM, i.e. charged leptons and quarks. The charged spin-1/2 fermions are denoted as f, and their antiparticles as $\bar{f}$.

4.2.1 Fundamental Vertices in QED

All electromagnetic phenomena are reducible to the following process:

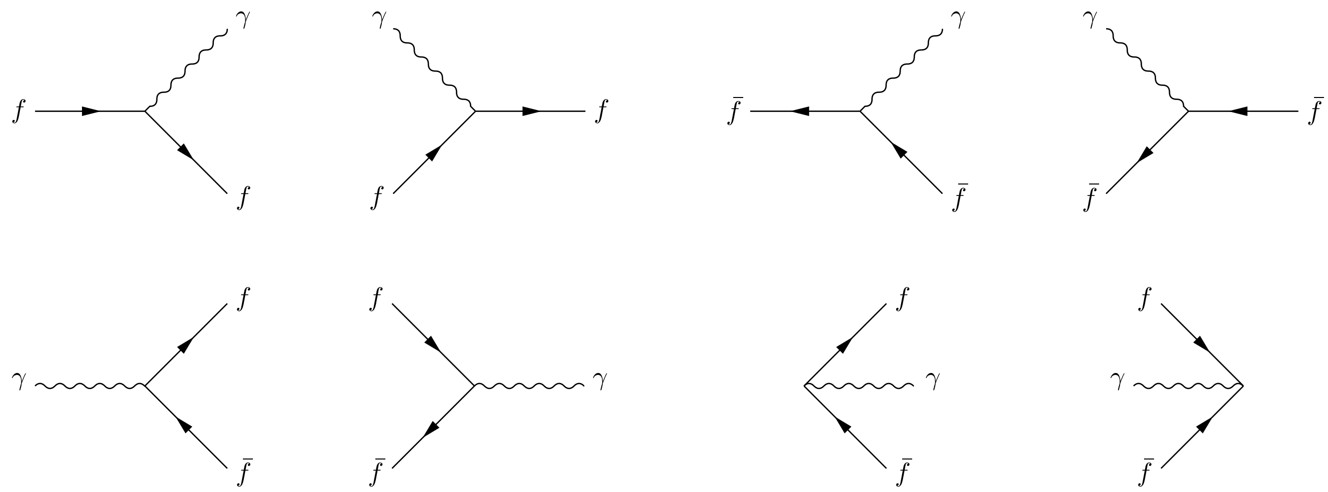


There are several features on this single vertex.

- A photon is involved, so it must be an electromagnetic process.
- The separation of lines in a Feynman diagram does not represent physical separation. The angle between the lines do not matter; they can be whatever they want as long as the time ordering is unchanged.
- The fermion on the LHS of the vertex is an incoming particle, while the fermion and the photon are outgoing particles. This vertex can be written as $f \to f\gamma$, where f is a spin-1/2 fermion of the *same* type. Thus, the process $c \to u\gamma$ is not allowed, even though it conserves electric charge. The electromagnetic vertex simply does not allow the fermion to change type in the process. On the contrary, the 4-momentum and the spin direction of the fermion can change.
- The mathematical factor associated with an electromagnetic vertex is linear in the fermion's charge. In other words, the amplitude for emitting or absorbing a photon is proportional to the charge of the particle. For example, the strange quark has charge $-\frac{1}{3}e$, while the electron has $-e$. The ratio of amplitude for emittnig or absorbing a photon is 1/3, giving a probability of 1/9.
- The 4-momentum is conserved at every vertex in a Feynman diagram. This is a sutble point: if a process only involve one vertex, then the 4-momentum is not conserved. That is, the diagram above is not a physical process. If the 4-momentum were conserved in this process, $p_f = p'_f + p_\gamma$, then squaring both sides gives $m_e^2 = m_e^2 + 2p_\gamma \cdot p'_f$, or $p_\gamma \cdot p'_f$. Let the energy of the

photon be E_γ, so $E_\gamma E'_f - E_\gamma\sqrt{E'^2_f - m_e^2}\cos\theta = 0$, where θ is the angle between the final electron and the photon. The LHS is greater than zero; 4-momentum is not conserved in this process. The resolution is that this vertex is only a part of a Feynman diagram. Later we will see that if this photon is an intermediate particle, then it is in principle not observable.

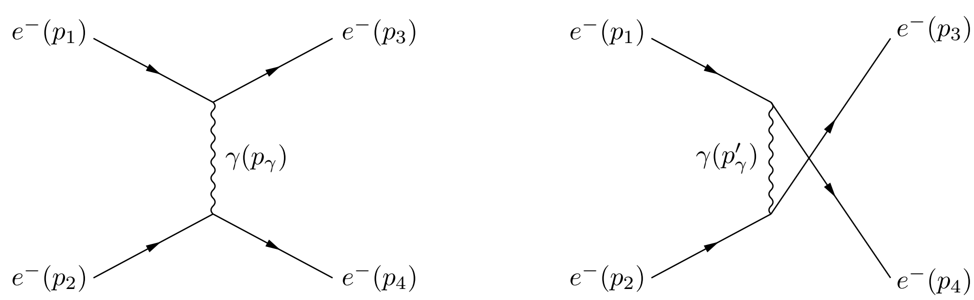
A way to generate alternative vertices that involve antiparticles is known as **crossing**. The rule is that if one switch an incoming particle to an outcoming one, then it changes into its antiparticle. Crossing symmetry can be written compactly: $A + B \to C + D$ implies that $A \to \bar{B} + C + D$ is allowed. There are in total eight possibilities to draw a fundamental vertex:



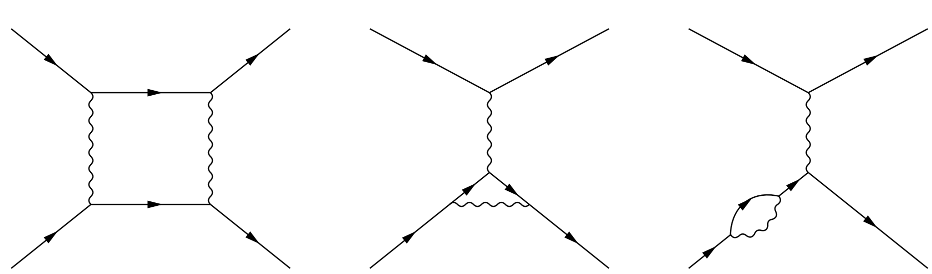
Evidently, the last two are not possible because they either have no incoming or outgoing particles. All the other six vertices are valid. One can quickly find a common feature of these vertices: the arrows on the lines cannot point towards the same vertex. All electromagnetic processes can be represented by combinations of those vertices. The next section introduces some examples.

4.2.2 Feynman Diagrams in QED

Møller Scattering. The first and probably the simplest process is called **Møller scattering**: $e^- + e^- \to e^- + e^-$. It is just an electron scattering from an electron.

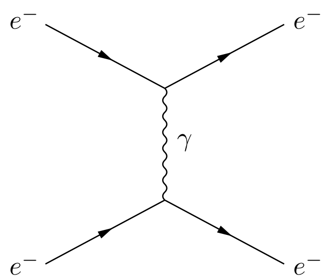


The two Feynman diagrams above are the lowest order diagrams in EM coupling for Møller scattering. The lowest order means that it contains the fewest number of interaction vertices (2 in this case). Note that the amplitude of the two diagrams are different. This is because electrons are identical particles, and it is in principle impossible to distinguish the two identical final states above (if their spin configurations are also the same). However, the intermediate photons have distinct 4-momenta: the left one has $p_\gamma = p_1 - p_3$, but the right one has $p'_\gamma = p_1 - p_4$. Hence they contribute to two amplitudes. This happens when we have two identical outgoing particles. Higher order diagrams for Møller scattering can be

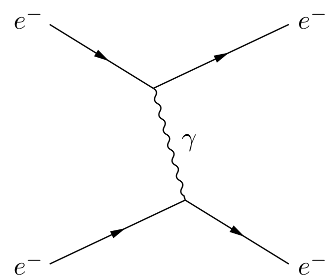


They have more vertices so they are less probable. In most cases, the lowest order Feynman diagrams are good enough. Moreover, we are not considering Feynman diagrams with loops. They are more complicated and advanced diagrams. All we will draw are tree diagrams.

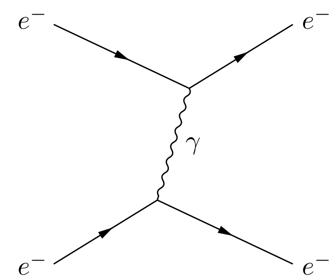
Another thing to notice is that the photon is drawn vertically in these Feynman diagrams. For Feynman diagrams with intermediate particles drawn vertically, it means that both time ordering are included:



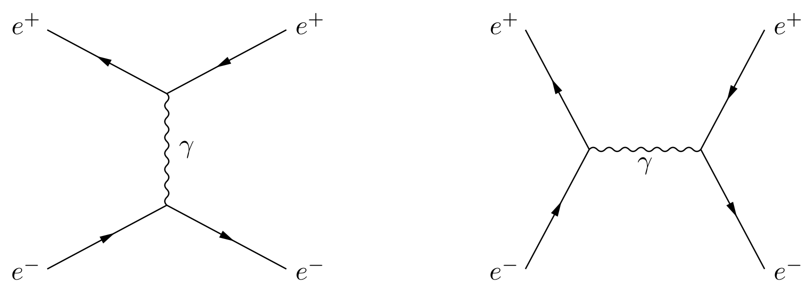
includes



and

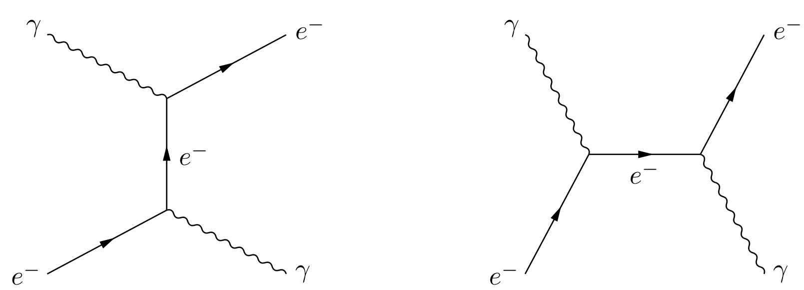


Bhabha Scattering. The second Feynman diagram to study is **Bhabha scattering**, $e^+ + e^- \to e^+ + e^-$, or electron-positron scattering. The lowest order diagrams are



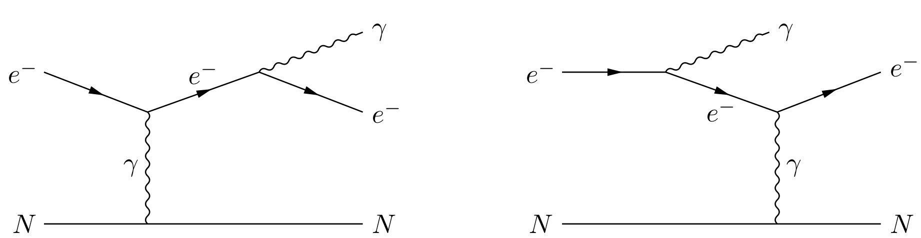
It is one of the most important processes at an e^+e^- collider.

Compton Scattering. The Compton scattering is $\gamma + e^- \to \gamma + e^-$. The lowest order EM coupling diagrams are



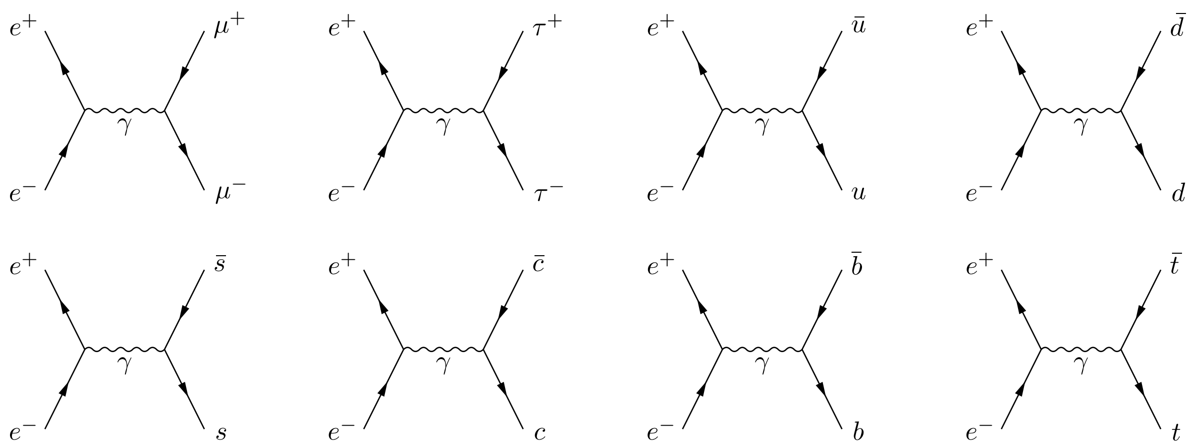
In compton scattering, the intermediate-state particle can no longer be a photon because that will violate charge conservation. The intermediate state for a QED Compton scattering process will be an electron.

Bremsstrahlung. The word **bremsstrahlung** means "braking radiation". We know that accelerating a charged particle will make it radiate. Accelerating an electron can easily achieve such an effect because electrons are very light. The process is described as $e^- + N \to e^- + \gamma + N$, where N is a nucleus with charge $+Ze$. (Here Z is the atomic number, not the Z-boson.) An electron is stopped by a nucleus so it radiates energetic photons. There are two lowest order diagrams for Bremsstrahlung in QED:



The intermediate process are distinct. It is straight forward to find that the 4-momenta of the intermediate electron in the two diagrams are different.

Pair Production and Annihilation If the energy of a e^+e^- pair is high enough, they can produce a pair of charged fermions. The charged fermions need not be leptons; they can also be quarks. They cannot be neutrinos because neutrinos do not couple to photons as they don't have electric charge. In QED, the pair production process for one species of particle in the final state has unique lowest order Feynman diagram (except electron pair production which is mentioned in Bhabha scattering). Here are possible outcomes of e^+e^- annihilation to pair production:

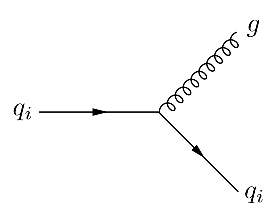


Charges of all kinds are strictly conserved in these processes. All vertices have net charge zero.

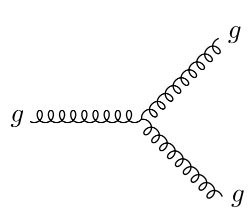
4.3 Quantum Chromodynamics (QCD)

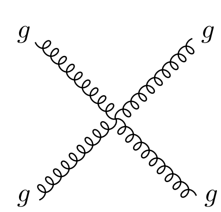
4.3.1 Fundamental Vertices in QCD

Color is the "charge" in chromodynamics, and the fundamental interaction is between quarks and gluons:



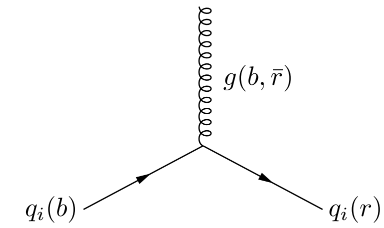
where q_i represents any quark (u, d, s, c, b, t). The force between two quarks is mediated by the exchange of gluons. In QED, photons cannot couple to other photons because they don't possess electric charge. However, in QCD, gluons are able to couple to other gluons because they have color charge. Hence there are two more fundamental vertices involving only gluons:





QCD has three differences from QED:

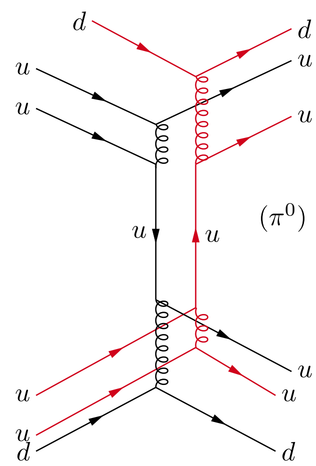
- There are three kinds of color (red, green, blue). The color (not flavor) of the quark may change, but they are always conserved. The gluon must carry away the color difference:



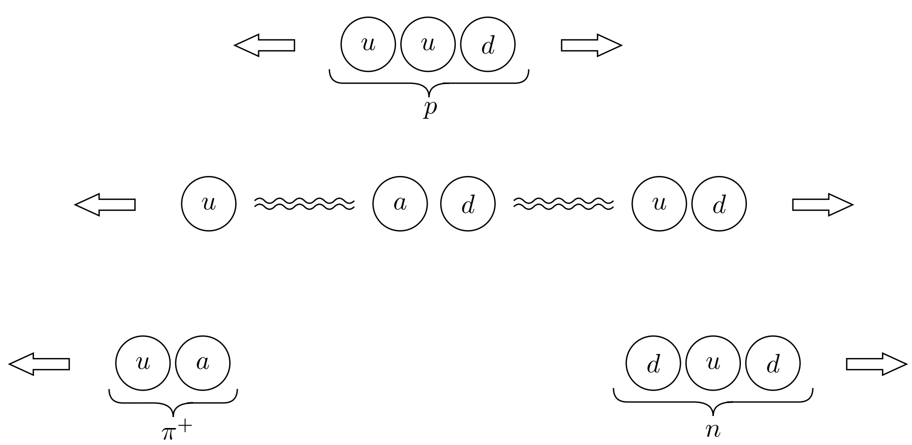
Here b stands for blue, r stands for red, and $\bar{r}$ is anti-red.

- QCD and QED differs from the coupling constant. Recall that in QED, each vertex contribute a factor of $\alpha = 1/137$. However, the coupling constant α_s in QCD is greater than 1. Hence Feynman rules do not work in QCD. In fact, the coupling "constant" depends on the separation distance between particles. At a distance less than the size of a proton the coupling constant becomes small. This phenomenon is called the **asymptotic freedom**.

- The last difference is that while many particles carry charges, no natural particles carry color. The strong force between natural particles is a more complicated process. For example: an interaction between two protons can be



This is the exchange of the pion between protons in Yukawa's theory. If QCD is correct, then it must account for quark confinement. One possible explanation is that the energy required to separate quarks apart is enough to create a new quark-antiquark pairs:



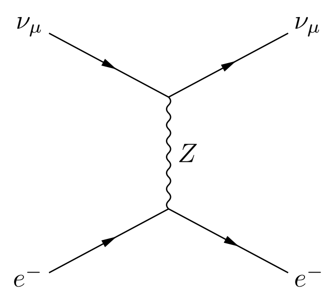
Confinement involves long-range behavior of strong force, and its explanation is hard because this is where Feynman rules fail.

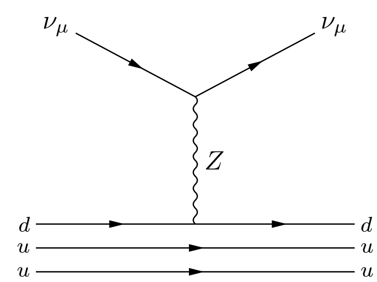
4.4 Weak Interactions

There is no "charge" that produce weak forces. All quarks and leptons can have weak interactions. There are two kinds of weak interactions: the charged mediated by the $W^{\pm}$, and neutral mediated by the Z. There are many fundamental vertices that are associated with the W and the Z. We will put them in Section 4.5.2. Here we will discuss some examples.

4.4.1 Neutral

Neutrino-electron scattering and neutrino-proton scattering:



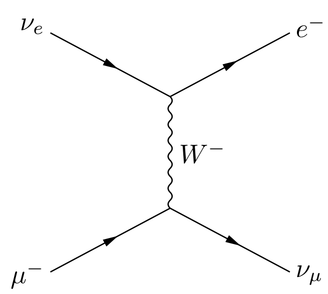


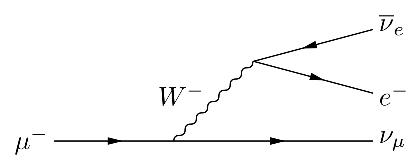
Any process mediated by the photon can also be mediated by the Z. The photon process dominates, but we can still trace the weak process because it violates the conservation of parity. A pure neutral weak interaction occurs only in neutrino scattering.

4.4.2 Charged

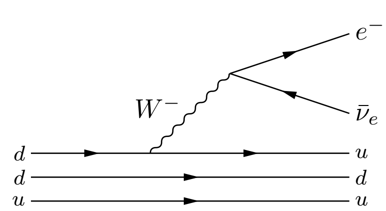
In strong, electromagnetic, and neutral weak interactions, same quarks or leptons come out as they went in (quarks may differ in color but never in flavor). Only the charged weak interaction can change flavor.

For leptonic processes, a lepton is converted into its corresponding neutrino by emitting a W^- or absorbing a W^+. Here is an example of a process with two vertices:





In quark weak interactions, the quark going in must have the same color as the one going out, but a different flavor. At the other end of the W^- line can be leptons, which is called a **semileptonic process**, or it can be quarks, the pure **hadronic process**. Here is a classic example of semileptonic process, the beta decay:



W and Z can directly couple to each other just like gluon-gluon couplings. Additionally, W is charged so it can also couple to the photon.

4.5 Algorithm to Draw Feynman Diagrams

4.5.1 Conservation Laws

This section summarizes an algorithm and several tips to draw Feynman diagrams. We focus mainly on conservation laws. Some conservation laws are universal, such as 4-momentum conservation and charge conservation. Some are true in the Standard Model, but are violated in physics beyond the Standard model, such as baryon number conservation. Some are true for specific interactions or situations, such as lepton number conservation and flavor conservation.

1. **4-momentum conservation.** Before drawing a Feynman diagram, classify the process as a decay or a scattering. For a decay process, check if the mass of the parent particle is higher than the total mass of the daughter particles. If not, the process is forbidden by conservation of energy. For a scattering process, unless $\sqrt{s}$ is specified, one may assume that $\sqrt{s}$ is enough to produce final-state particles. If angular momentum are specified, always check whether it is conserved.

2. **Charge conservation.** Electric charge is conserved locally at every SM vertex and in the overall process. In general, we will not worry about color charge in a Feynman diagram.

3. **Baryon number conservation.** The baryon number B of a particle is defined as $B = N_{\text{quark}} - N_{\text{antiquark}}$. Thus, all baryons (which have 3 quarks) have baryon number 1, and all mesons (which have 1 quark and 1 antiquark) have baryon number 0. Leptons, gauge bosons, and Higgs boson have baryon number 0.

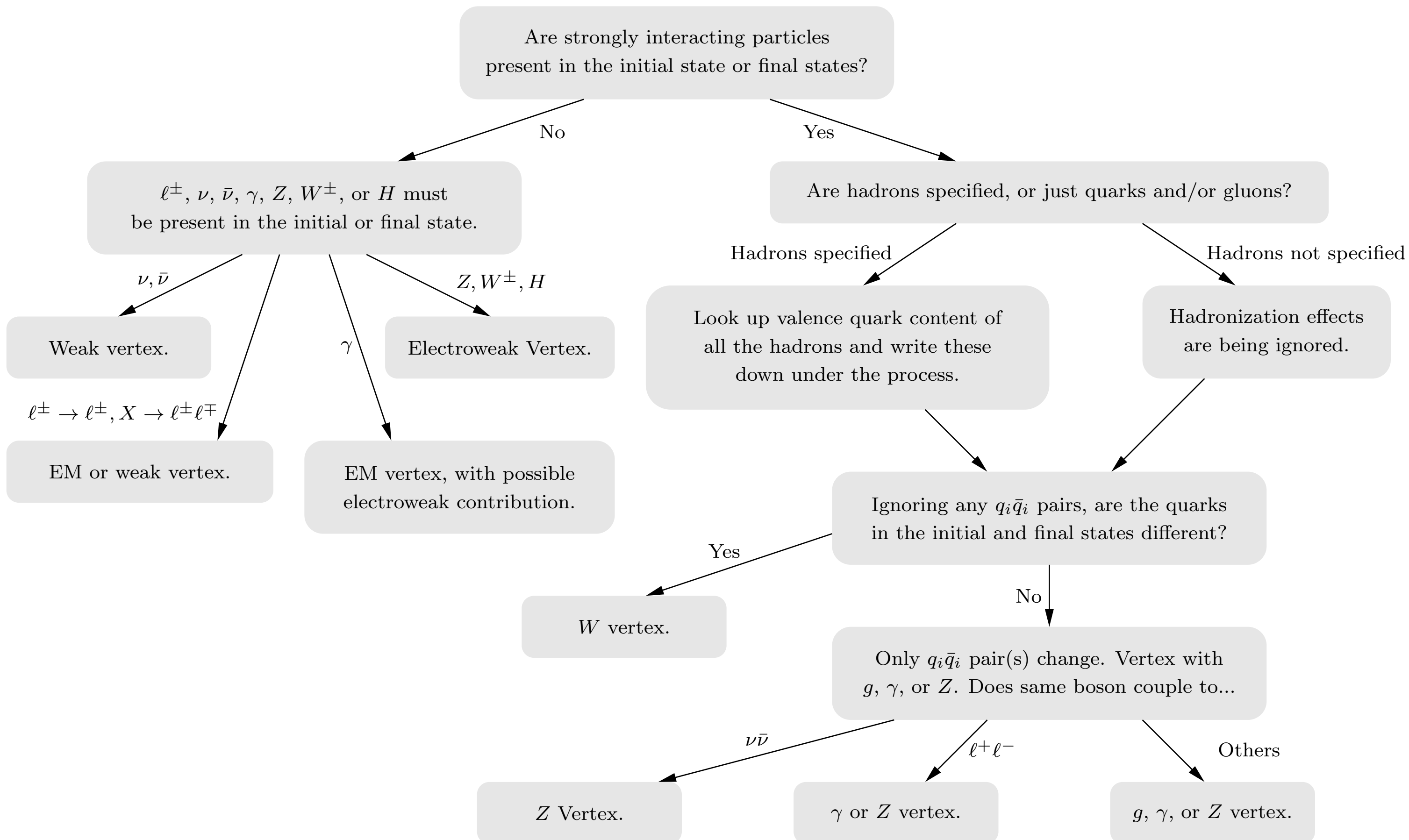
4. **Lepton number conservation.** The ℓ-lepton number L_ℓ (where $\ell = e, \mu, \tau$) is defined as $L_\ell = N(\ell^-, \nu_\ell) - N(\ell^+, \bar{\nu}_\ell)$. For example, e^- has electron-lepton number 1; $\bar{\nu}_\tau$ has τ-lepton number -1. The lepton number of three families are separately conserved. The conservation of lepton number is true in most cases, but neutrino oscillations (such as $\nu_e \leftrightarrow \nu_\mu$) violate such conservations.

5. **Flavor.** The conservation of flavor works in strong and electromagnetic interactions, but not in weak ones. If there is a flavor change, then there must be a W involved in the process.

6. **Other conservations.** A decay is electromagnetic if a photon comes out; it is weak if a neutrino comes out; it is strong if a gluon comes out, If there are only quarks and leptons, other conservation rules should be taking into consideration. These conservation rules include parity, charge conjugation, CP, isospin, etc.

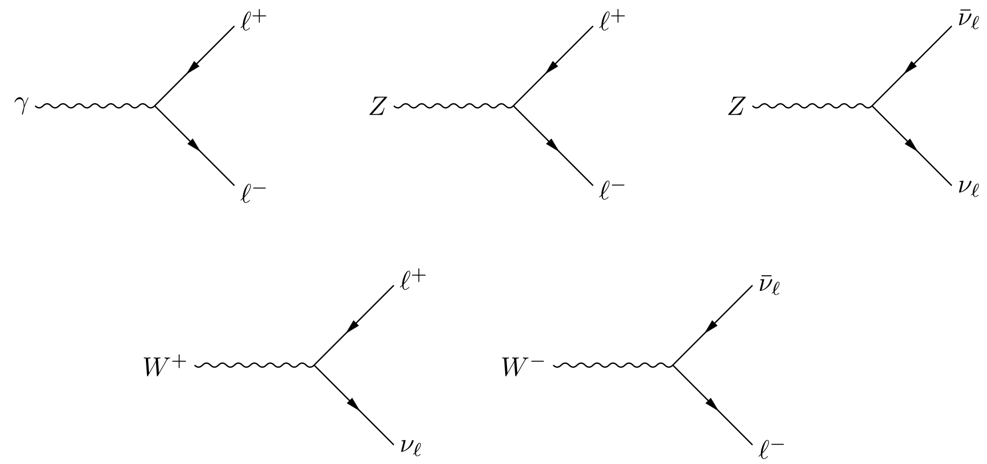
7. **The OZI rule.** as we have mentioned in Chapter 1, the J/ψ $(c\bar{c})$ has a 1000 times longer lifetime than typical strong decays. The explanation is the following: Okubo, Zweig, and Iizuka came up with the **OZI rule** to account for the branching fraction of the ϕ $(s\bar{s})$. The ϕ has two decay modes, two K's decay and three π's decay where the latter is energetically favored because π weights much lower. The puzzle is that the branching fraction is much higher for two K's decay. The OZI rule says that the three π's decay is suppressed because the gluons have high energy in this process. The asymptotc freedom makes the gluons couple weakly at high energy, so the ϕ is more likely to go the two K's path. The OZI rule work for the J/ψ: the J/ψ can decay into three π's or two D's. Again the three π's path is suppressed, but meanwhile, the two D's path is forbidden because they weigh more than the J/ψ, which is kinematically forbidden. Hence the J/ψ will live unusually longer.

Here is a flow chart to draw Feynman diagrams.

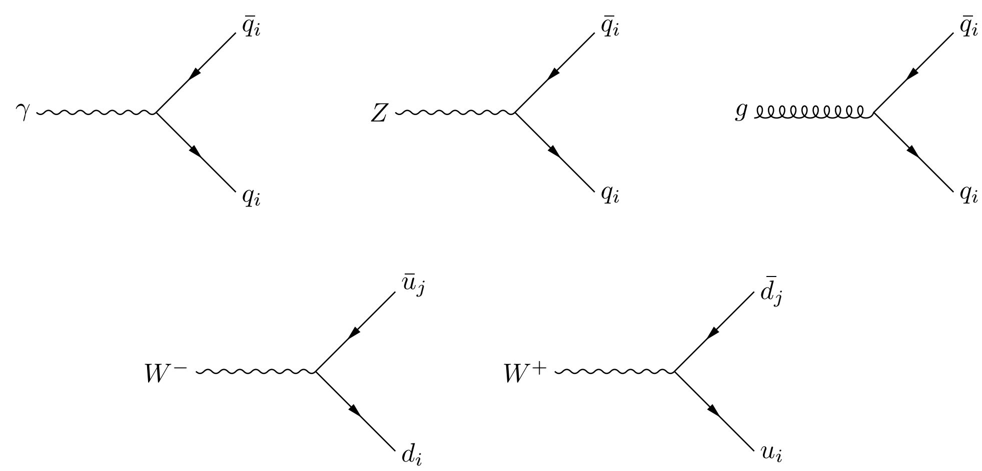

Are strongly interacting particles present in the initial state or final states?
No
Yes
$\ell^\pm$, ν, $\bar{\nu}$, γ, Z, $W^\pm$, or H must be present in the initial or final state.
Are hadrons specified, or just quarks and/or gluons?
$\nu, \bar{\nu}$
$Z, W^\pm, H$
Hadrons specified
Hadrons not specified
Weak vertex.
γ
Electroweak Vertex.
Look up valence quark content of all the hadrons and write these down under the process.
Hadronization effects are being ignored.
$\ell^\pm \to \ell^\pm, X \to \ell^\pm \ell^\mp$
EM or weak vertex.
EM vertex, with possible electroweak contribution.
Ignoring any $q_i\bar{q}_i$ pairs, are the quarks in the initial and final states different?
Yes
No
W vertex.
Only $q_i\bar{q}_i$ pair(s) change. Vertex with g, γ, or Z. Does same boson couple to...
$\nu\bar{\nu}$
$\ell^+\ell^-$
Others
Z Vertex.
γ or Z vertex.
g, γ, or Z vertex.

4.5.2 Allowed Vertices

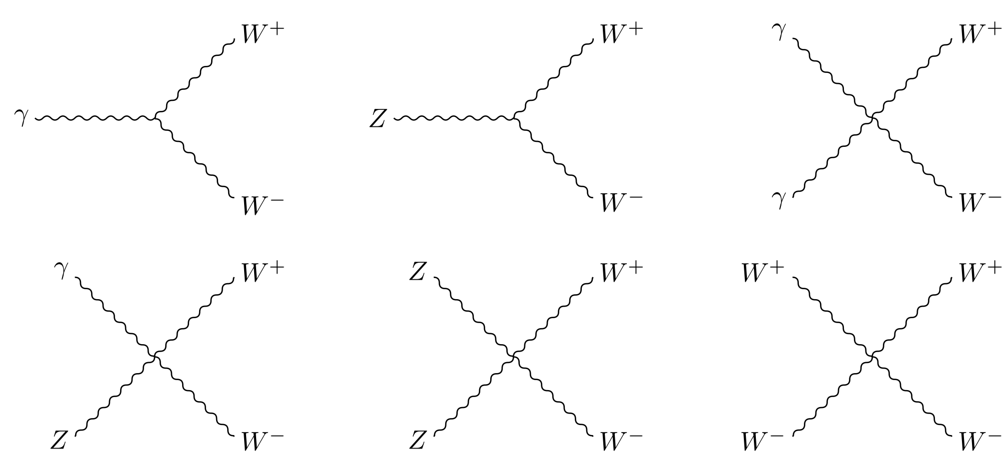
Vertices involving gauge bosons and leptons: $\ell = e, \mu, \tau$.



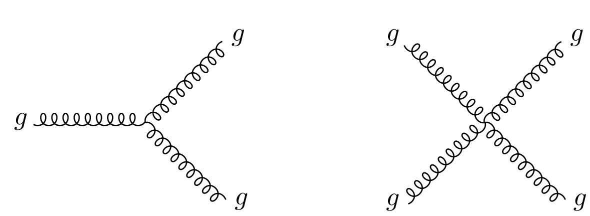
Vertices involving gauge bosons and quarks: $q_i = u, d, s, c, b, t$, $d_i = d, s, b$, $u_i = u, c, t$.



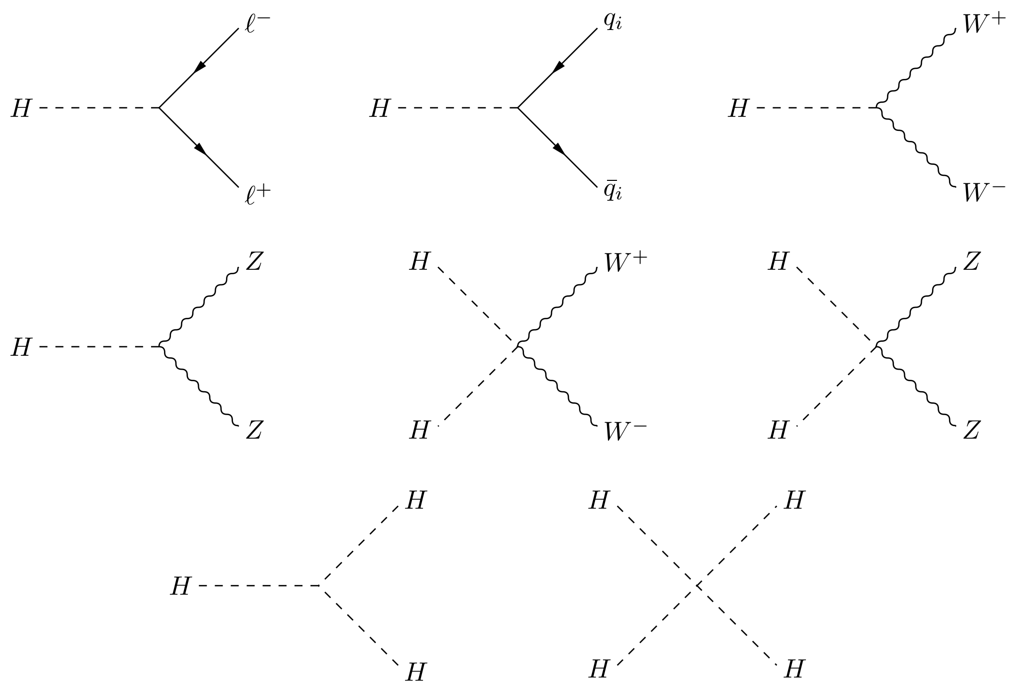
Vertices involving only $J = 1$ bosons (electroweak):



Vertices involving only $J = 1$ bosons (strong):



Vertices involving Higgs bosons:



Each fundamental vertices are not alone. One can always invoke crossing symmetry to produce other vertices, just like we did in Section 4.2.1, as long as 4-momentum is conserved. Many common Feynman diagrams are recorded in Appendix B for reference.

4.6 Phenomenology

In physics, phenomenology is the use of theory to predict results of experiment. In this section, we will talk about how to predict cross-sections and decay rate from Feynman diagrams and some simple results from quantum field theory. The key ideas and methods include:

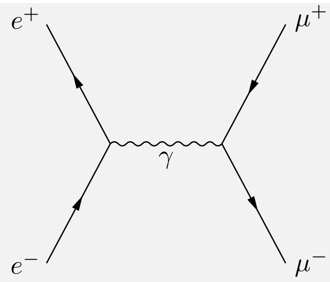
- Drawing relevant Feynman diagrams.
- Identifying relevant energies and masses, and making possible approximations.
- Considering effects of vertex factors on amplitude $\mathcal{A}$ and observables $|\mathcal{A}|^2$.
- Considering effects of propagator factors on amplitudes and observables.
- Using dimensional analysis in the framework of natural units.

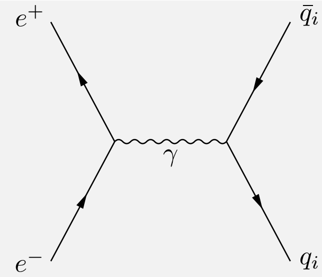
4.6.1 Vertex Factors

We know that the amplitude of Feynman diagrams with more vertices are more suppressed. However, each vertex does not contribute to the same "suppression factor", but according to the **vertex factor**. For an electromagnetic process, the vertex factor is proportional to the electric charge of the particle produced. For example, the vertex $e^+e^- \to \gamma$ contributes a factor of $q_e = -e$, while the vertex $\gamma \to q_i\bar{q}_i$ contributes a factor of q_{q_i}, which can be $\frac{2}{3}e$ or $-\frac{1}{3}e$.

Example 4.1. Cross-section of e^+e^- collisions

We would like to study the cross-sections of $e^+e^- \to \mu^+\mu^-$ and $e^+e^- \to q_i\bar{q}_i$, and their ratio $R(\sqrt{s}) = \sigma_{\text{hadrons}}(\sqrt{s})/\sigma_{\mu\mu}(\sqrt{s})$. Following the steps above, we should first draw Feynman diagrams for the two processes:





The second step is to list out the energies and masses. Suppose we know the center of momentum energy is $\sqrt{s} \ll m_Z$ so that the processes are dominated by electromagnetic interaction. The masses involved in $e^+e^- \to \mu^+\mu^-$ are m_e and m_μ. If we take $\sqrt{s} \gg m_e, m_\mu$, the only energy or mass scale relevant in this process is $\sqrt{s}$. By dimensional analysis (in natural units), the dimensions of a cross-section σ is

$$[\sigma] = L^2 = \left[\frac{\hbar c}{E}\right] \to \frac{1}{E^2} = \frac{1}{M^2}.$$

We arrive at a very important conclusion: the cross-section goes down with increasing $\sqrt{s}$

$$\sigma \propto \frac{1}{E_{\mathrm{CM}}^2} \sim \frac{1}{\sqrt{s}} = \frac{1}{s}.$$

Though we used a lot of approximations, the result is perfectly consistent with experimental results. Now consider the second process $e^+e^- \to q_i\bar{q}_i$. In most cases, the quark and antiquark produced form hadrons with other quarks. However, at resonance $\sqrt{s}$ close to meson with content $(q_i\bar{q}_i)$, they form such a themselves, such as a $J/\psi(c\bar{c})$ when $\sqrt{s} \approx m_{J/\psi}$. We make the same assumption that $\sqrt{s} \gg m_{q_i}$ so that the only relevant energy in the problem is $\sqrt{s}$. Also, we will neglect resonance for simplicity now.

The ratio of cross-sections can be calculated as

$$R(\sqrt{s}) \equiv \frac{\sum_i \sigma(e^+e^- \to q_i\bar{q}_i, \sqrt{s})}{\sigma(e^+e^- \to \mu^+\mu^-, \sqrt{s})} \simeq \frac{\sum_i (E_{\mathrm{CM}})^{-2} \cdot 3 \cdot [(-e) \cdot q_{q_i}]^2}{(E_{\mathrm{CM}})^{-2}(-e)^2} \simeq 3 \sum_i q_{q_i}^2,$$

where i runs over all kinematically accessible quarks. The factor of 3 counts for color charge (R,G,B) the quark can have. That is, $\sigma(e^+e^- \to q_R\bar{q}_{\bar{R}}) = \sigma(e^+e^- \to q_G\bar{q}_{\bar{G}}) = \sigma(e^+e^- \to q_B\bar{q}_{\bar{B}})$. One important thing to notice is that different q_i and colors count for different final states. Their amplitudes need to be square and then add up. Only when the final state is the same but with different intermediate states, the amplitude first add up and then squared.

For concreteness, consider $\sqrt{s} = 27\,\mathrm{GeV}$ as an example. This energy is enough to produce 2 bottom quarks, then

$$R(\sqrt{s} \sim 27\,\mathrm{GeV}) \simeq 3\left[\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2\right] = \frac{11}{3}.$$

In real measurements, when we increase $\sqrt{s}$, we can observe "steps" in $R(\sqrt{s})$. The quark mass threshold perfectly explain this because the amplitudes add discrete steps except at resonance masses of particles like J/ψ, Υ, Z, etc.

For vertices that change the flavor of quarks, the magnitude of each vertex factor is proportional to gV_{ij}. Here i refers to quarks that have charge $\pm 2/3$ and j refers to quarks that have charge $\mp 1/3$. For example, the process $W^- \to u\bar{d}$ has a vertex factor gV_{ud}. The quantities V_{ij} are elements of the 3×3 **Cabibbo-Kobayashi-Maskawa (CKM) quark mixing matrix** of the Standard Model. For most purposes, we only need to know the magnitudes of V_{ij}:

$$\begin{bmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{bmatrix} = \begin{bmatrix} 0.974 & 0.225 & 0.004 \\ 0.23 & 0.96 & 0.041 \\ 0.008 & 0.039 & 1.0 \end{bmatrix}.$$

4.6.2 Propagators

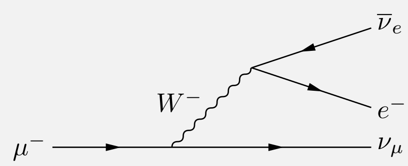
Another factor that contributes to the amplitude of a Feynman diagram is the **propagator**:

$$\Delta = \frac{i}{p^2 - m^2 + i\epsilon}. \tag{4.2}$$

Here p is the momentum of the intermediate particle, which may not equal to m^2. The quantity p^2 is the *off-shell* mass of the intermediate particle, while m^2 is the *on-shell* mass. In general, the term $i\epsilon$ can be neglected unless $p^2 = m^2$.

Example 4.2. Muon decay

Consider a simple decay mediated by a W-boson. $\mu^- \to e^- \bar{\nu}_e \nu_\mu$. As usual, first draw the Feynman diagram for this process.



Since this is a weak process, the vertex factor is no longer the charge of particles, but is related to a weak coupling constant g. Considering both vertex factors and the propagator, the amplitude is proportional to

$$\mathcal{A}(\mu^- \to e^- \bar{\nu}_e \nu_\mu) \propto \frac{g^2}{p^2 - m_W^2}.$$

The decay rate or the width Γ is proportional to the amplitude squared,

$$\Gamma(\mu^- \to e^- \bar{\nu}_e \nu_\mu) \propto |\mathcal{A}|^2 \propto \frac{g^4}{(p^2 - m_W^2)}.$$

Here are some reasonings we can make. First, Γ has dimensions of M, but the RHS has dimensions M^{-4}, so the proportionality factor should have dimensions M^5. The mass of the W-boson does not play a role in the proportionality factor; it only affect the propagator. Hence we need some other mass. The momentum of the intermediate W is negligible compared to m_W. This is because in the rest frame of the initial muon, its 4-momentum is $(m_\mu, 0, 0, 0)$. The transferred momentum to the W is

$$p^2 = (p_\mu^2 - p_{\nu_\mu}^2) = m_\mu^2 + m_{\nu_\mu}^2 - 2m_\mu E_{\nu_\mu}.$$

The maximum p^2 the W can have is $p_{\text{max}}^2 = m_\mu^2 + m_{\nu_\mu}^2 - 2m_\mu m_{\nu_\mu} = (m_\mu - m_{\nu_\mu})^2 < m_\mu^2 \lll m_W^2$ because $m_W = 80.37\,\text{GeV}$! Therefore, p^2 is negligible in the amplitude. Finally, notice that $m_e, m_{\bar{\nu}_e}, m_{\nu_\mu} \ll m_\mu$, we may guess that the proportionality factor depends only on m_μ effectively, so

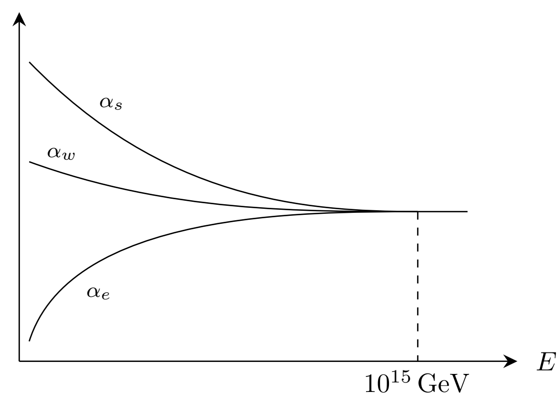
$$\Gamma(\mu^- \to e^- \bar{\nu}_e \nu_\mu) \propto \frac{m_\mu^5 g^4}{m_W^4}.$$

It turns out that this method can be generalized to other processes mediated by a W-boson, so our reasoning works.

4.7 Unification Schemes

Historically, electricity and magnetism were used to be two separate subject. Maxwell unified them into electromagnetism. Einstein wanted to combine gravity with electromagnetism but he failed. On the other way, Glashow, Weinberg, and Salam combined the weak and electromagnetism into the electroweak theory. The theory begins with four massless mediators, but three of them (the $W^\pm$ and the Z) aquires mass via **Higgs mechanism**, the remaining massless one is the photon.

Started from the 1970s, people worked to unify chromodynamics with the electroweak theory into the **Grand Unified Theory** (GUT). This is possible, as the strong coupling constant α_s decreases at short distance and so does the weak α_w, but with a slower rate. The electromagnetic coupling constant α_e (smallest among the three) increases with shorting distance. They may converge to a limiting value at extraodinary high energy:



Another prediction of the GUT is that proton can decay, so that the baryon number and lepton number is no longer conserved:

$$p^+ \to e^+ + \pi^0 \quad \text{or} \quad p^+ \to \bar{\nu}_\mu + \pi^+.$$

Its half-life is at least 10^{19} times the age of the universe, so protons are still very stable. After all the unification of the strong, electromagnetic, and weak forces, the final step is to incorporate gravity.

5 Symmetries

5.1 Symmetries, Groups, and Conservation Laws

A **symmetry** is an operation you can perform on a system that leaves it invariant. For example, for an odd function $f(x)$, changing the sign of the argument $x \to -x$ and multiplying the function by -1, $f(x) \to -f(-x)$, is a symmetry operation. Rotating an equilateral triangle by 120° is also a symmetry operation. The set of all symmetry operations R on a particular system has properties:

- Closure: if R_i and R_j are in the set, then $R_i R_j$ (first perform R_j and then perform R_i) is also in the set.
- Identity: there exists an element I such that $IR_i = R_i I = R_i$ for all elements R_i.
- Inverse: for every element R_i there is an inverse, R_i^{-1}, such that $R_i R_i^{-1} = R_i^{-1} R_i = I$.
- Associativity: $R_i(R_j R_k) = (R_i R_j)R_k$.

Mathematically, this is precisely the definition of a **group**, so group theory studies symmetries. If all elements in a group commute, then it is called **abelian**. For example, translations in space and time are abelian, but 3D rotations are not.

In elementary particle physics, we use groups of matrices:

Group name	Dimension	Matrices in group
$U(n)$	$n \times n$	unitary ($U^\dagger U = 1$)
$SU(n)$	$n \times n$	unitary, determinant 1
$O(n)$	$n \times n$	orthogonal ($O^\mathrm{T} O = 1$)
$SO(n)$	$n \times n$	orthogonal, determinant 1

Table 5.1: Important symmetry groups.

The group $SO(n)$ can be seen as the group of all rotations in an n-dimensional space. For example, $SO(3)$ is the rotational symmetry of our world. Every group G can be represented by a group of matrices: every element a has a corresponding matrix M_a. The group elements multiplication are just represented by multiplication of matrices: if $ab = c$, then $M_a M_b = M_c$.

In 1917, Emmy Noether presented her famous theorem, the **Noether's theorem**, relating symmetries and conservation laws.

Symmetry		Conservation law
Translation in time	$\leftrightarrow$	Energy
Translation in space	$\leftrightarrow$	Momentum
Rotation	$\leftrightarrow$	Angular momentum
Gauge transformation	$\leftrightarrow$	Charge

Table 5.2: Symmetries and conservation laws.

5.2 Angular Momentum

There are two types of angular momentum: **orbital** angular momentum and **spin** angular momentum. Macroscopically, the Earth's motion around the sun has orbital angular momentum $L = rmv$, and Earth's rotation about its axis has spin $L = I\omega$. Microscopically, electron is orbiting the nucleus and has similar orbital angular momentum, but a different type of spin. Since electrons are perceived as point particles, there is no "jell" of electron charges revolving around its axis—the spin of a particle is an intrinsic property of itself.

In quantum mechanics, it is *in principle* impossible to measure three components of orbital angular momentum simultaneously. A measurement in one component will affect the value of other components. However, we can measure the magnitude of orbital angular momentum L^2 with one component L_z (by convention). Moreover, the measurements are quantized: a measurement of L^2 can only take values from

$$L^2 = \ell(\ell+1)\hbar^2, \quad \ell = 0, 1, 2, 3, \dots.$$

For a given value of ℓ, a measurement of L_z will give a quantized value of

$$L_z = m_\ell \hbar, \quad m_\ell = -\ell, -\ell+1, \dots, -1, 0, 1, \dots, \ell-1, \ell.$$

Note that the angular momentum cannot be completely in the z-direction, as the largest m_ℓ is $\ell < \sqrt{\ell(\ell+1)}$.

Spin angular momentum is similar. A measurement of S^2 gives

$$S^2 = s(s+1)\hbar^2,$$

but s may take a half-integer or an integer: $s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$. A measurement of S_z yields

$$S_z = m_s \hbar, \quad m_s = -s, -s+1, \dots, s-1, s.$$

A given particle can have any orbital angular momentum, but the its spin is fixed. For example, every electron or quark carries $s = \frac{1}{2}$. Particles with half-integer spin are called **fermions**. Particles with integer spin are called **bosons**.

5.2.1 Addition of Angular Momenta

The states for angular momentum are represented by kets: $|\ell\ m_\ell\rangle$ or $|s\ m_s\rangle$. For instance, an electron in orbital state $|3\ -1\rangle$ and spin $\left|\frac{1}{2}\ \frac{1}{2}\right\rangle$ has $\ell = 3, m_\ell = -1, s = \frac{1}{2}$ and $m_s = \frac{1}{2}$. Sometimes we may want to add two angular momenta. We use $\mathbf{J}$ to denote the general angular momentum: it could be orbital angular momentum or spin, or the addition of both. The question is: what is total angular momentum state $|j\ m\rangle$ if we combine the two states $|j_1\ m_1\rangle$ and $|j_2\ m_2\rangle$? First, the z components add directly,

$$m = m_1 + m_2.$$

It turns out that j can have values between $(j_1 + j_2)$ and $|j_1 - j_2|$,

$$j = |j_1 - j_2|, |j_1 - j_2| + 1, \dots, (j_1 + j_2) - 1, (j_1 + j_2).$$

Sometimes we want explicit decomposition of $|j_1\ m_1\rangle\,|j_2\ m_2\rangle$ into specific states of possible $|j\ m\rangle$. We use the **Clebsch-Gordan** coefficients:

$$|j_1\ m_1\rangle\,|j_2\ m_2\rangle = \sum_{j=|j_1-j_2|}^{j_1+j_2} C^{j j_1 j_2}_{m m_1 m_2}\,|j\ m\rangle\,, \quad \text{with } m = m_1 + m_2.$$

There are tables of Clebsch-Gordan coefficients that we can look up on the next page. The probability of getting the total angular momentum $J^2 = j(j+1)\hbar^2$ of a specific j is the square of the corresponding Clebsch-Gordan coefficient.

Example 5.1. The electron in a hydrogen atom occupies the orbital state $|2\ -1\rangle$ and the spin state $\left|\frac{1}{2}\ \frac{1}{2}\right\rangle$. If we measure J^2, what values can we get and what are their corresponding probabilities?

Solution: the possible values of j are $l + s = \frac{5}{2}$ and $l - s = \frac{3}{2}$. The z component of J is $m_\ell + m_s = -\frac{1}{2}$, so the two possible states are

$$\left|\tfrac{5}{2}\ -\tfrac{1}{2}\right\rangle \quad \text{and} \quad \left|\tfrac{3}{2}\ -\tfrac{1}{2}\right\rangle.$$

Look up the Clebsch-Gordan table (on the next page) for $2 \times \frac{1}{2}$ (that is, $\ell \times s$), we get

$$|2\ -1\rangle\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle = \sqrt{\frac{2}{5}}\left|\tfrac{5}{2}\ -\tfrac{1}{2}\right\rangle - \sqrt{\frac{3}{5}}\left|\tfrac{3}{2}\ -\tfrac{1}{2}\right\rangle.$$

The probability of getting $j = \frac{5}{2}$ is $\frac{2}{5}$, and the probability of getting $j = \frac{3}{2}$ is $\frac{3}{5}$.

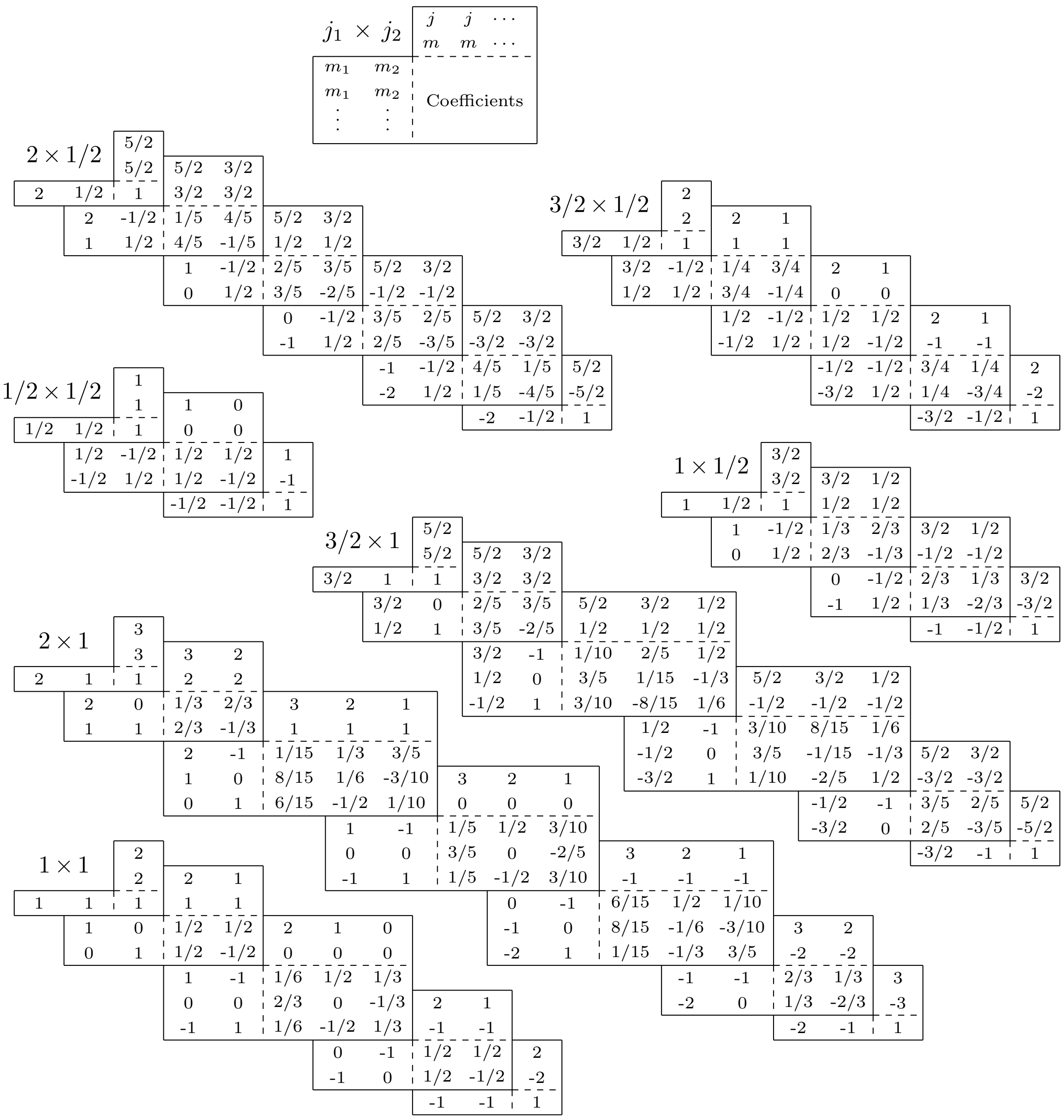


Figure 5.1: Clebsch-Gordan coefficients. A square root is understood on each coefficient, that is, $-1/3$ means $-\sqrt{1/3}$.

5.2.2 Spin $\frac{1}{2}$

A particle with spin $\frac{1}{2}$ can have $m_2 = \frac{1}{2}$ (spin up), or $m_s = \frac{1}{2}$ (spin down). The two states are represented by **spinors**

$$\left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \quad \left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle = \begin{pmatrix}0\\1\end{pmatrix}.$$

The most general state of a spin-$\frac{1}{2}$ particle is the linear combination of spinors,

$$\begin{pmatrix}\alpha\\\beta\end{pmatrix} = \alpha\begin{pmatrix}1\\0\end{pmatrix} + \beta\begin{pmatrix}0\\1\end{pmatrix},$$

where $|\alpha|^2$ is the probability of getting spin $\frac{1}{2}\hbar$ by measuring S_z, and $|\beta|^2$ is the probability of getting $-\frac{1}{2}\hbar$. Normalization condition applies: $|\alpha|^2 + |\beta|^2 = 1$.

Now we want to know the probability of getting spin up/down when measuring S_x and S_y. They should return values $\pm\frac{1}{2}\hbar$ because rotational symmetry indicates that the choice of z-direction is arbitrary. For each component of $\mathbf{S}$, there is a 2×2 matrix whose eigenvalues are $\pm\frac{1}{2}\hbar$:

$$S_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \quad S_y = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\i & 0\end{pmatrix}, \quad S_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}.$$

Take S_x for example. Its normalized eigenvectors are

$$\chi_{\pm} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \pm\frac{1}{\sqrt{2}}\end{pmatrix}.$$

A spinor $\begin{pmatrix}\alpha\\\beta\end{pmatrix}$ can be written as

$$\begin{pmatrix}\alpha\\\beta\end{pmatrix} = a\begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix} + b\begin{pmatrix}\frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}}\end{pmatrix},$$

where

$$a = \frac{1}{\sqrt{2}}(\alpha+\beta), \quad b = \frac{1}{\sqrt{2}}(\alpha-\beta).$$

The probability of getting $\frac{1}{2}\hbar$ is $|a|^2$, and the probability of getting $-\frac{1}{2}\hbar$ is $|b|^2$.

5.3 Flavor Symmetries

Heisenberg noticed that the neutron and the proton are nearly identical apart from their charges. He suggested that we can treat them as a single particle called the **nucleon**, but in two different states. We write the nucleon as

$$N = \begin{pmatrix}\alpha\\\beta\end{pmatrix} \quad \text{where} \quad p = \begin{pmatrix}1\\0\end{pmatrix} \quad \text{and} \quad n = \begin{pmatrix}0\\1\end{pmatrix}.$$

Here we introduce a new concept called the **isospin**, $\mathbf{I}$, analogous to the spin $\mathbf{S}$. The isospin is not a vector in ordinary space. Instead, it is in the "isospin space" with components I_1, I_2, and I_3. The nucleon carries an isospin of $\frac{1}{2}$, and the third component, I_3, has the eigenvalues $+\frac{1}{2}$ (the proton) and $-\frac{1}{2}$ (the neutron):

$$p = \left|\tfrac{1}{2}\ \tfrac{1}{2}\right\rangle, \quad n = \left|\tfrac{1}{2}\ -\tfrac{1}{2}\right\rangle.$$

Heisenberg also proposed that the strong interactions are invariant under rotations in the isospin space, which is called an **internal symmetry**. By Noether's theorem, the internal symmetry indicates that isospin is conserved in all strong interactions. This is just like angular momentum is conserved with rotational invariance in ordinary space. In later studies, it turns out that up and down quarks also have isospins. The third component I_3 is related to the charge of the particle.

5.4 Discrete Symmetries

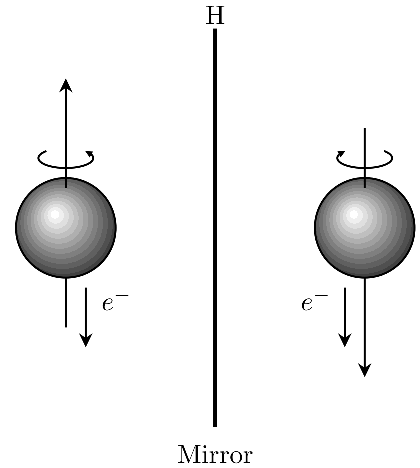


Figure 5.2: Decay of Cobalt 60 (left) and its mirror image (right). In the beta decay of Cobalt 60, most electrons are emitted in the direction opposite to the nuclear spin.

5.4.1 Parity

Before 1956, it was widely believed that the mirror image of any physical process is also a perfectly possible physical process. This is called mirror symmetry, or **parity** invariance. In 1956, Lee and Yang found that there are evidence of parity invariance in strong and electromagnetic processes, but none for weak interactions. They asked Wu for an experiment to test the parity invariance in weak processes. In the experiment, cobalt 60 nuclei were aligned so that their spins point at the z-direction, or upward. These nuclei undergoes beta decay (${}^{60}\mathrm{Co} \to {}^{60}\mathrm{Ni} + e + \bar{\nu}_e$). Most of the emitted electrons (emitted downward) are opposite to the nuclear spin. This concluded the experiment.

Consider the mirror image of this experiment in Figure 5.2: The spins of the nuclei are now downward, while the electrons after mirroring are also downward. Now most of the electrons are emitted in the same direction as the nuclear spin. The mirror image does not occur in nature, and parity is not invariant in weak processes. If parity were invariant, the electrons should come out in equal numbers in both upward and downward direction.

Parity violation is a trace of weak force. It is mostly connected to the behavior of neutrinos. Suppose a particle is traveling with velocity $\mathbf{v}$. We usually pick its direction of motion as the z-axis. The value of m_s/s for this axis is called the **helicity** of the particle. A particle of spin $\frac{1}{2}$ can have a helicity of $+1$ ($m_s = \frac{1}{2}$, right-handed), or -1 ($m_s = -\frac{1}{2}$, left-handed). For electrons, the helicity is not Lorentz-invariant. If you are traveling in the direction of $\mathbf{v}$ in the lab frame, but with a greater speed than $|\mathbf{v}|$, then in your frame, the electron will appear to be traveling opposite to you. The helicity will then flip.

Now we assume that neutrinos are massless so they travel at the speed of light. Then it is impossible to flip its helicity because you cannot travel faster. Thus, for massless particles, the helicity is Lorentz-invariant. It is discovered by experiments that neutrinos are left-handed, while antineutrinos are right handed. There are indirect ways (as neutrinos are hard to detect directly) to determine the helicity of a neutrino using the decay of the pion: $\pi^- \to \mu^- + \bar{\nu}_\mu$ (see Figure 5.3). Using conservation of momentum and angular momentum with the fact that the pion has spin 0, we will have the muon and the antineutrino traveling in opposite direction, with opposite spin. Hence they should have the same helicity in the pion's rest frame, and measuring the muon's helicity will give the antineutrino's.

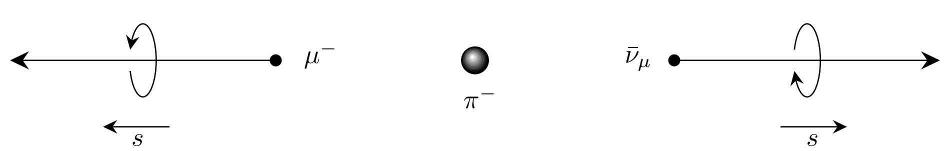


Figure 5.3: Decay of π^- at rest.

On the contrary, the decay of $\pi^0 \to \gamma + \gamma$ is an electromagnetic process in which parity is invariant, so left-handed photon pairs and right-handed photon pairs are in equal amount. This does not work for neutrinos because they only appear in weak interaction. Therefore, every neutrino is left handed. The mirror image of a neutrino does not exist.

Now, we want to develop some formalism instead of using "mirrors", or reflection about a plane, e.g. $(x, y, z) \to (x, -y, z)$. We define the **inversion**, in which every point is reflected about the origin: $(x, y, z) \to (-x, -y, -z)$. We also define the **parity operator** P to denote inversion. When it is applied to a vector $\mathbf{a}$, it produce the opposite vector, $P(\mathbf{a}) = -\mathbf{a}$. However, for the cross product of two vectors, $\mathbf{c} = \mathbf{a} \times \mathbf{b}$, the parity operator does not change its sign $P(\mathbf{c}) = \mathbf{c}$ because it changes both signs of $\mathbf{a}$ and $\mathbf{b}$. Hence obviously there are two types of vectors: the **(polar) vectors** with sign changed, and the **pseudovectors** with sign unchanged under parity operator. The cross product of a polar vector with a pseudovector is a polar vector. There are pseudovectors that we are familiar with: angular momentum, magnetic field, etc. The dot product of two polar vectors does not change sign under P, but the dot product of a polar vector and a pseudovector (such as the triple scalar product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$) does change sign. The former is called the scalar, and the latter is the **pseudoscalar**.

Evidently,

$$P^2 = I,$$

so the parity group has only two elements: I and P. The eigenvalues of P are 1, for scalars and pseudovectors, and -1, for pseudoscalars and vectors. According to QFT, the parity of a fermion is opposite to that of its antiparticle. The parity of a boson is the same as its antiparticle. By convention, we let quarks to have positive intrinsic parity. The parity of a composite system in its ground state is the product of the parities of its constituents, so parity is a "multiplicative" quantum number. Unlike parity, the charge is "additive".

With the formalism of parity, we can see how Lee and Yang doubted the parity invariance in weak processes. The question arose from the "tau-theta puzzle" in the early 1950s. The puzzle is the following: the two mesons τ and θ have the same mass same spin, same charge, and so on, but their decay process are different:

$$\theta^+ \to \pi^+ + \pi^0 \qquad\qquad (P = (-1)^2 = +1),$$

$$\tau^+ \to \begin{cases} \pi^+ + \pi^0 + \pi^0 \\ \pi^+ + \pi^+ + \pi^- \end{cases} \qquad (P = (-1)^3 = -1).$$

Now we know that the τ and the θ are the same particle, the K^+. The parity is just not conserved in the first process.

5.4.2 Charge Conjugation

A change in the sign of all electric charges reverses all fields potentials in classical electrodynamics, but the Lorentz force law $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ does not change. We say that classical electrodynamics is invariant under the sign change of electric charges. In elementary particle physics, the **charge conjugation**, C, changes all the internal quantum numbers: charge, baryon number, lepton number, etc. In other words, it converts particles into their antiparticles:

$$C\,|p\rangle = |\bar{p}\rangle\,.$$

Apparently, $C^2 = I$, so the eigenvalues of C are ± 1. Now, since $|p\rangle$ and $|\bar{p}\rangle$ differs at most by a sign,

$$C\,|p\rangle = \pm\,|p\rangle = |\bar{p}\rangle\,,$$

only the particles that are their own antiparticles can be eigenstates of C.

Photon has a charge conjugation number -1; a system of a spin-$\frac{1}{2}$ particle and its antiparticle, together with orbital angular momentum ℓ and total spin s, is an eigenstate of C with eigenvalue $(-1)^{\ell+s}$. Like parity, charge conjugation is a multiplicative quantum number and is conserved in strong and electromagnetic interactions. For example, for both sides $C = 1$,

$$\pi^0 \to \gamma + \gamma$$

but $\pi^0 \nrightarrow \gamma + \gamma + \gamma$ cannot occur. Charge conjugation is not a symmetry in weak interactions. When C applies to neutrinos, which are always left-handed, they become impossible left-handed (because C does not change spin) antineutrinos.

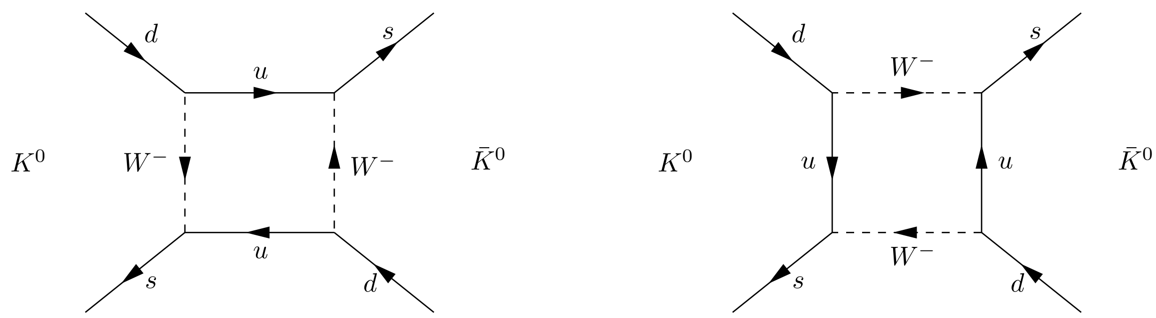
5.4.3 CP

Now we know that the weak interactions are not invariant under the parity operator P and the charge conjugation operator C. Consider the π meson decay and its charge conjugated version:

$$\pi^+ \to \mu^+ + \nu_\mu \quad \text{and} \quad \pi^- \to \mu^- + \bar{\nu}_\mu.$$

Recall that the muon is always right-handed and the antimuon is always left-handed. Hence if the π^+ decay is true (with a left-handed muon), its charge conjugated version will never happen because the muon coming out will be left-handed. However, if we apply the parity operator to the charge conjugated version, we will produce a right-handed muon. It seems like the weak interactions are invariant under the CP, operator.

Neutral Kaons Gell-Mann and Pais noticed that that the K^0 and $\bar{K}^0$ can turn into each other through a second-order weak interaction:



The K's satisfy

$$P\,|K^0\rangle = -\,|K^0\rangle\,, \quad P\,|\bar{K}^0\rangle = -\,|\bar{K}^0\rangle\,, \quad C\,|K^0\rangle = |\bar{K}^0\rangle\,, \quad C\,|\bar{K}^0\rangle = |K^0\rangle\,.$$

Thus, $CP\,|K^0\rangle = -\,|\bar{K}^0\rangle$ and $CP\,|\bar{K}^0\rangle = -\,|K^0\rangle$. The actual particles which are eigenstates of CP that we observe in labs are linear combination of the two particles,

$$|K_1\rangle = \frac{1}{\sqrt{2}}(\,|K^0\rangle - |\bar{K}^0\rangle) \quad \text{and} \quad |K_2\rangle = \frac{1}{\sqrt{2}}(\,|K^0\rangle + |\bar{K}^0\rangle)$$

with

$$CP\,|K_1\rangle = |K_1\rangle \quad \text{and} \quad CP\,|K_2\rangle = -\,|K_2\rangle\,.$$

Note that K_1 and K_2 are not antiparticles of one another, but each is its own antiparticle with $C = -1$ for K_1 and $C = +1$ for K_2.

If we assume CP is conserved in the weak interactions, then K_1 can only decay into a state with $CP = +1$, and K_2 with $CP = -1$. Neutral kaons decay into two or three pions typically. From the tau-theta puzzle, we know that the two-pion system has $P = +1$, three-pion system $P = -1$, and both $C = +1$. Therefore, K_1 decays into two pions while K_2 decays into three pions. The 2π decay is faster than the 3π decay. If the incident beam is

$$|K^0\rangle = \frac{1}{\sqrt{2}}(|K_1\rangle + |K_2\rangle),$$

we should see most 2π events near the source, but 3π events far away.

CP Violation In 1964, Cronin and Fitch found that at the end of a bean 57 feet long, there were 45 two pion events in 22,700 decays. This indicates that the neutraal kaon is not a perfect eigenstate of CP, but an admixture of K_1:

$$|K_L\rangle = \frac{1}{\sqrt{1+|\epsilon|^2}}(|K_2\rangle + \epsilon\,|K_1\rangle),$$

where $\epsilon \approx 2.24 \times 10^{-3}$ is a measure of nature's departure from perfect CP invariance. Unlike parity, which is "maximally" violated in weak interactions (all neutrinos are left-handed, not 50.01% of them), CP violation is a small effect.

5.4.4 Time-Reversal and the TCP Theorem

Imagine a movie of an elastic collision. It will depict a possible physical process when the movie is ran backward. The elementary particle interactions was believed to share this time-reversal invariance. However, parity violation made people wonder whether the time-reversal was invariant.

It is hard to test the invariance of time-reversal. First, no particle is an eigenstate of the time-reversal operator T. Second, time-reversal is invariant in strong and electromagnetic interactions, but it is difficult to do inverse reactions in weak interactions. For example, the inverse reaction of $\Lambda \to p^+ + \pi^-$ would be $p^+ + \pi^- \to \Lambda$, but the strong interaction of the proton and the pion will dominant over the weak interaction. Meanwhile, measurements of neutrinos (which only exist in weak process) are notoriously difficult. For now, no experiments has shown direct evidence of T violation.

However, in quantum field theory, the TCP theorem states that the combined operation of time reversal , charge conjugation, and parity in any order is an exact symmetry of any interaction. In other words, it is impossible to construct a quantum field theory in which TCP is not conserved. We know that CP is violated, so there must be a violation of T.

6 Quantum Electrodynamics

The Klein-Gordan (KG) equation is a fully relativistic equation,

$$(i\partial_\mu)(i\partial^\mu)\phi(x) = m^2\phi(x) \quad \Longleftrightarrow \quad (-\partial_t^2 + \nabla^2)\phi(x) = m^2\phi(x), \tag{6.1}$$

where $\phi(x)$ is a field, and $x = \{x^\mu\}$ is the position 4-vector. The KG equation treats time and space on equally, both in second-derivatives. Also, $\partial_\mu\partial^\mu$ and m^2 are both Lorentz invariant. The KG equation is a second-order wave equation that admits free-particle solutions, $\phi(x) \propto e^{ip\cdot x}$, where $p = \{p^\mu\}$ is the 4-momentum. Plugging in the free-particle solution into the KG equation can recover a fundamental relativistic relation $E^2 - \mathbf{p}^2 = m^2$.

An equation with second-derivatives in spacetime is not the only relativistic equation. Alternatively, there can be a first-order wave equation that also describes relativistic quantum mechanics, and that is the Dirac equation.

6.1 The Dirac Equation

Since the KG equation already shows that free-particle solutions lead to $E^2 - \mathbf{p}^2 = m^2$, the first-order equation we want to find had better automatically satisfy the KG equation at the same time. Suppose the equation of interest is first-order in both ∂_t and ∇:

$$[\boldsymbol{\alpha} \cdot (-i\nabla) + \beta m]\,\psi(x) = i\frac{\partial}{\partial t}\psi(x), \tag{6.2}$$

where the coefficients $\boldsymbol{\alpha} = \{\alpha_i\}$ and β are to be determined. The RHS is the same as in the Schrödinger equation, but the LHS (a Hamiltonian term) is first order in ∇ instead of $-\nabla^2/2m$. Because this equation is linear in momentum, it is natural to involve only one power of mass m. As mentioned above, we want (6.2) to have free-particle solutions that automatically satisfy the KG equation (6.1). This means the following operator equation should hold:

$$(i\partial_t)^2 = [\boldsymbol{\alpha} \cdot (-i\nabla) + \beta m][\boldsymbol{\alpha} \cdot (-i\nabla) + \beta m] = (-i\nabla)^2 + m^2.$$

This will impose some conditions on α_i and β. Compute the LHS explicitly,

$$\begin{aligned}
(i\partial_t)^2 &= [\boldsymbol{\alpha} \cdot (-i\nabla) + \beta m]\,[\boldsymbol{\alpha} \cdot (-i\nabla) + \beta m] \\
&= [\alpha_i\,(-i\partial_i) + \beta m]\,[\alpha_j\,(-i\partial_j) + \beta m] \\
&= [\alpha_i(-i\partial_i)]^2 + (\alpha_i\alpha_j + \alpha_j\alpha_i)(-i)^2\partial_{ij} + (\alpha_i\beta + \beta\alpha_i)(-i\partial_i)m + \beta^2 m^2,
\end{aligned}$$

where repeated indices infer implicit summations, though α_i and ∂_i are both subscripts. (Note that α_i is not the spatial part of a Lorentz 4-vector. It is just a coefficient with a subscript corresponding to partial derivatives ∂_i.) We want the above expression to end up being $(-i\nabla)^2 + m^2$. The requirements are

$$\alpha_i\alpha_j + \alpha_j\alpha_i = 0, \quad \alpha_i\beta + \beta\alpha_i = 0, \quad \text{and} \quad \alpha_1^2 = \alpha_2^2 = \alpha_3^2 = \beta^2 = \mathbb{I},$$

where $\mathbb{I}$ denotes some identity. The first two conditions can be written compactly as **anticommutators**,

$$\{\alpha_i, \alpha_j\} = 0 \quad \text{for} \quad i \neq j, \quad \text{and} \quad \{\alpha_i, \beta\} = 0.$$

These conditions tells us that α_i and β cannot simply be complex numbers. They turn out to be matrices.

6.1.1 The γ-Matrices

Equation (6.2) can be written in a more "relativistic way". First, we want to get rid of the β before m. Multiplying (6.2) by β on both sides,

$$\left[\beta\boldsymbol{\alpha}\cdot(-i\nabla)+\beta^2 m\right]\psi(x)=i\beta\partial_t\psi(x).$$

Now $\beta^2 m$ is just m because $\beta^2=1$. Define

$$\gamma^i\equiv\beta\alpha_i,\quad \gamma^0=\beta. \tag{6.3}$$

It looks like $\gamma^i\equiv\beta\alpha_i$ violates index notation rules, but remember that the index i in α_i is not a Lorentz index as it is just a label. Equation (6.2) can be written as

$$[\boldsymbol{\gamma}\cdot(-i\nabla)+m]\psi=\gamma^0(i\partial_t)\psi \quad\Longrightarrow\quad [\gamma^0(i\partial_t)+\boldsymbol{\gamma}\cdot(i\nabla)]\psi(x)=m\psi(x).$$

Define $\{\gamma^\mu\}=(\gamma^0,\boldsymbol{\gamma})$. Then we can write the above equation as

$$\boxed{i\gamma^\mu\partial_\mu\psi=m\psi.} \tag{6.4}$$

This is the famous **Dirac equation**. It turns out that there will be a lot of terms involving $\gamma^\mu A_\mu$. For any 4-vector A^μ, there is a notation like $\not{A}$ called **Feynman slash notation**. It means $\not{A}=\gamma_\mu A^\mu=\gamma^\mu A_\mu$, to avoid writing $\gamma_\mu A^\mu$ everywhere. Hence the Dirac equation can also be written as

$$i\not{\partial}\psi=m\psi. \tag{6.5}$$

The object γ^μ is independent of x^μ, so it commutes with partial derivatives ∂_μ. Also, because γ^μ are just constant matrices, they do not transform—γ^μ is not a 4-vector. However, we can still raise or lower indices of them: $\eta_{\mu\nu}\gamma^\mu=\gamma_\nu$, $\eta^{\mu\nu}\gamma_\mu=\gamma^\nu$.

6.1.2 Properties of α_i, β, and γ^μ Matrices

Here we list some basic properties in α_i, β, and γ^μ matrices. The proofs and some further results are available in Appendix A.2.

- α_i and β square to the identity: $\alpha_i^2=\beta^2=I$.
- α_i and β anticommute with each other: $\{\alpha_i,\alpha_j\}=0$ for $i\neq j$, and $\{\alpha_i,\beta\}=0$.
- α_i and β are Hermitian because they constitutes the Hamiltonian term in (6.2): $\alpha_i^\dagger=\alpha_i$, $\beta^\dagger=\beta$.
- α_i and β are traceless: $\mathrm{Tr}(\alpha_i)=\mathrm{Tr}(\beta)=0$.
- α_i and β have even dimensions.
- The Hermitian conjugate of γ^μ can be written as $(\gamma^\mu)^\dagger=\gamma^0\gamma^\mu\gamma^0$. In other words, $(\gamma^0)^\dagger=\gamma^0$, and $(\gamma^i)^\dagger=-\gamma^i$.
- Square of γ^μ: $(\gamma^i)^2=-I$, $\gamma^0=I$. This also implies that $\det\gamma^\mu\neq 0$.
- Anticommutator relation of γ^μ:

  $$\boxed{\{\gamma^\mu,\gamma^\nu\}=2\eta^{\mu\nu}I.} \tag{6.6}$$

  Note that γ^μ itself is a matrix, while $\eta^{\mu\nu}$ are elements of the Minkowski metric tensor. This is why there is an identity matrix on the RHS of (6.6).

6.1.3 Representations of α_i, β, and γ^μ

We are looking for α_i and β with the following properties: Hermitian, anticommute with each other, squaring to the identity, traceless, and even dimensionality. We can start with dimensionality and let $N=2$. However, there does not exist four 2×2 Hermitian matrices that anticommute with each other.

Proof. By the definition of a Hermitian matrix, its matrix elements satisfy $H_{mn}=H_{nm}^*$. A 2×2 Hermitian matrix H must have real diagonal entries, and its off-diagonal entries are complex conjugate of each other. We can find that the most general form of any

2×2 Hermitian matrix can be written as

$$H = \begin{bmatrix} a & c - id \\ c + id & b \end{bmatrix}$$

for real numbers a, b, c, d. It is useful to express H in terms of Pauli spin matrices σ_i and the identity:

$$H = \frac{a+b}{2} I + \frac{a-b}{2} \sigma_z + c\sigma_x + d\sigma_y, \quad \text{where} \quad \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Pauli spin matrices have good properties: they anticommute with each other, $\{\sigma_i, \sigma_j\} = 0$ for $i \neq j$, they square to the identity, and they are traceless. So now we already get three anticommuting matrices, we want to find the fourth one.

Suppose a Hermitian matrix A anti-commutes with all of $\sigma_x, \sigma_y, \sigma_z$. Then it cannot be a linear combination of them. (Unless $A = 0$, but that is trivial.) Compute $\{A, \sigma_z\}$:

$$\{A, \sigma_z\} = \frac{a+b}{2} \{I, \sigma_z\} + \frac{a-b}{2} \{\sigma_z, \sigma_z\} + c\{\sigma_x, \sigma_z\} + d\{\sigma_y, \sigma_z\}.$$

The last two anticommutator vanish because Pauli spin matrices anticommute with each other. We are left with

$$\{A, \sigma_z\} = \frac{a+b}{2}(2\sigma_z) + + \frac{a-b}{2}(2I) = (a+b)\sigma_z + (a-b)I = \begin{bmatrix} 2a & 0 \\ 0 & -2b \end{bmatrix}.$$

We want $\{A, \sigma_z\} = 0$, so $a = b = 0$. However, if $a = b = 0$, then A is a linear combination of σ_y and σ_z, which contradicts with our assumption. Hence there does not exist a nontrivial Hermitian matrix that anticommutes with the three Pauli spin matrices. $\square$

$N = 2$ does not work, so we shall turn to $N = 4$. The good properties of Pauli spin matrices may give us some hints. It turns out that it is possible to find four 4×4 Matrices that satisfy all the properties. There are two common representations: the Weyl representation and the Dirac-Pauli representation. The **Weyl representation** has β off-diagonal and α_i diagonal:

$$\beta = \begin{bmatrix} 0 & I_{2\times 2} \\ I_{2\times 2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \qquad \alpha_x = \begin{bmatrix} \sigma_x & 0 \\ 0 & -\sigma_x \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.$$

$$\alpha_y = \begin{bmatrix} \sigma_y & 0 \\ 0 & -\sigma_y \end{bmatrix} = \begin{bmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{bmatrix}, \qquad \alpha_z = \begin{bmatrix} \sigma_z & 0 \\ 0 & -\sigma_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The **Dirac-Pauli representation** has β diagonal and α_i off-diagonal:

$$\beta = \begin{bmatrix} I_{2\times 2} & 0 \\ 0 & -I_{2\times 2} \end{bmatrix}, \quad \alpha_x = \begin{bmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{bmatrix}, \quad \alpha_y = \begin{bmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{bmatrix}, \quad \alpha_z = \begin{bmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{bmatrix}.$$

From the Dirac equation (6.2), we can see that the Weyl representation is more convenient in the relativistic limit where $p \gg m$, and the β term is negligible. The Dirac-Pauli representation is more convenient in the non-relativistic limit where $m \gg p$, and the α_i terms are negligible.

Now with α_i and β, we can compute γ^μ in both representations. In the Weyl representation:

$$\gamma^0 = \begin{bmatrix} 0 & I_{2\times 2} \\ I_{2\times 2} & 0 \end{bmatrix}, \quad \gamma^i = \begin{bmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{bmatrix}.$$

In the Dirac-Pauli representation:

$$\gamma^0 = \begin{bmatrix} I_{2\times 2} & 0 \\ 0 & -I_{2\times 2} \end{bmatrix}, \quad \gamma^i = \begin{bmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{bmatrix}.$$

Even though we know the matrix representations of β, α_i and γ^μ, in most calculates we only need their basic properties such as $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} I$.

6.2 Free-Particle Solutions

If the γ-matrices in the Dirac equation (6.4) are represented by 4×4 matrices. It must act on an object ψ that has 4 components,

$$\psi(x) = \begin{bmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{bmatrix}.$$

This object is called a **Dirac spinor**, which is not a 4-vector because it does not undergo a Lorentz transformation. Hence we will label its components using Latin indices $i = 1, 2, 3, 4$ though it has 4 components. The goal now is to solve the Dirac equation. We will work in the Dirac-Pauli representation for now.

Suppose we want to find the free-particle solution. First, write the spinor $\psi(x)$ as two separate 2-component spinors ϕ and χ:

$$\psi(x) = N \begin{bmatrix} \phi \\ \chi \end{bmatrix} e^{\pm ip\cdot x} = N \begin{bmatrix} \phi \\ \chi \end{bmatrix} e^{\pm i(Et - \mathbf{p}\cdot\mathbf{x})},$$

where N is a normalization factor. Because we assume that $\psi(x)$ is a plane wave solution, ϕ and χ should not have any dependence on x^μ. All of the spacetime dependence is in the exponential factor $e^{\pm ip\cdot x}$. Plug the free-particle solution $\psi(x)$ into the Dirac equation

$$[\gamma^0(i\partial_t) + \boldsymbol{\gamma}\cdot(i\nabla)] \begin{bmatrix} \phi \\ \chi \end{bmatrix} e^{\pm ip\cdot x} = m \begin{bmatrix} \phi \\ \chi \end{bmatrix} e^{ip\cdot x}.$$

The time-derivative extract one factor of $\pm iE$ and the spatial-derivative extract one factor of $\mp \boldsymbol{p}$ from the exponential factor, so

$$[\gamma^0(\mp E) + \boldsymbol{\gamma}\cdot(\pm\mathbf{p})] \begin{bmatrix} \phi \\ \chi \end{bmatrix} e^{\pm ip\cdot x} = m \begin{bmatrix} \phi \\ \chi \end{bmatrix} e^{\pm ip\cdot x}.$$

The exponential factor cancels. Writing out the γ-matrices (in the Dirac-Pauli representation),

$$\left\{ \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} (\mp E) + \begin{bmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{bmatrix} \cdot (\pm \mathbf{p}) \right\} \begin{bmatrix} \phi \\ \chi \end{bmatrix} = m \begin{bmatrix} \phi \\ \chi \end{bmatrix}.$$

Gathering the matrices together,

$$\mp \begin{bmatrix} EI & -\boldsymbol{\sigma}\cdot\mathbf{p} \\ \boldsymbol{\sigma}\cdot\mathbf{p} & -EI \end{bmatrix} \begin{bmatrix} \phi \\ \chi \end{bmatrix} = m \begin{bmatrix} \phi \\ \chi \end{bmatrix}, \tag{6.7}$$

where

$$\boldsymbol{\sigma}\cdot\mathbf{p} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} p_x + \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} p_y + \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} p_z = \begin{bmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{bmatrix}. \tag{6.8}$$

In fact, one can find that

$$\not{p} = \gamma^\mu \partial_\mu = \begin{bmatrix} EI & -\boldsymbol{\sigma}\cdot\mathbf{p} \\ \boldsymbol{\sigma}\cdot\mathbf{p} & -EI \end{bmatrix},$$

which is just the matrix in (6.7). Equation (6.7) is essentially two equations:

$$\begin{bmatrix} \phi \\ \chi \end{bmatrix}_- = \frac{1}{m} \begin{bmatrix} EI & -\boldsymbol{\sigma}\cdot\mathbf{p} \\ \boldsymbol{\sigma}\cdot\mathbf{p} & -EI \end{bmatrix} \begin{bmatrix} \phi \\ \chi \end{bmatrix}, \qquad \begin{bmatrix} \phi \\ \chi \end{bmatrix}_+ = -\frac{1}{m} \begin{bmatrix} EI & -\boldsymbol{\sigma}\cdot\mathbf{p} \\ \boldsymbol{\sigma}\cdot\mathbf{p} & -EI \end{bmatrix} \begin{bmatrix} \phi \\ \chi \end{bmatrix},$$

corresponding to $e^{-ip\cdot x}$ solution and $e^{ip\cdot x}$ solution, respectively. Let's solve the $e^{-ip\cdot x}$ solution by directly multiplying the matrices out,

$$\phi = \frac{1}{m}(E\phi - \boldsymbol{\sigma}\cdot\mathbf{p}\chi),$$
$$\chi = \frac{1}{m}(\boldsymbol{\sigma}\cdot\mathbf{p}\phi - E\chi).$$

Solving for χ in terms of ϕ, we find that

$$\chi = \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m}\phi.$$

One may want to plug this χ into the ϕ-equation, but eventually we will just get $E^2 - \mathbf{p}^2 = m^2$, which we already know. Therefore, the negative free-particle solution is

$$\psi(x) = N\begin{bmatrix} \phi \\ \frac{\boldsymbol{\sigma}\cdot\mathbf{p}}{E+m}\phi \end{bmatrix} e^{-ip\cdot x}. \tag{6.9}$$

There seems to be missing some information as we cannot determine ϕ, but it also means that we have the freedom to choose any ϕ that span the 2-state space for spinors. The simplest basis is

$$\phi_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \phi_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Using (6.8), we can find χ and subsequently, the full Dirac-spinor for free particles:

$$\psi(x) = N\begin{bmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{bmatrix} e^{-ip\cdot x}, \quad \phi_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \psi(x) = N\begin{bmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ -\frac{p_z}{E+m} \end{bmatrix} e^{-ip\cdot x}, \quad \phi_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{6.10}$$

The solution (6.10) is what we called the positive-energy solution. The energy is positive because the energy operator $i\partial_t$ brings down $+E$ from the exponential. It has two spin states ϕ_1 and ϕ_2. For example, (6.10) can represent a free electron with either spin up or spin down. In the rest frame of the electron, $p = 0$ and $E = m$, the solution becomes simpler:

$$\psi(x) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} e^{-imt}, \quad \text{and} \quad \psi(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} e^{-imt}.$$

Going through the whole process again for $e^{+ip\cdot x}$, expressing ϕ in terms of χ in this case, we can find the "negative-energy" solution with $e^{+ip\cdot x}$,

$$\psi(x) = N\begin{bmatrix} \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \\ 1 \\ 0 \end{bmatrix} e^{ip\cdot x}, \quad \chi_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \psi(x) = N\begin{bmatrix} \frac{p_x-ip_y}{E+m} \\ -\frac{p_z}{E+m} \\ 0 \\ 1 \end{bmatrix} e^{ip\cdot x}, \quad \chi_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{6.11}$$

The energy is not truly negative. It is interpreted as the free-antiparticle (e.g. positron) solution of the corresponding free-particle solution. It also has two spin states, just like the electron. In the rest frame of the positron,

$$\psi(x) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} e^{-imt}, \quad \text{and} \quad \psi(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} e^{-imt}.$$

A Proofs and Derivations

A.1 Lorentz Transformation

To derive the Lorentz transformation, we need to use both postulates of special relativity:

1. The laws of physics take the same form in all inertial reference frames. All inertial frames are fundamentally indistinguishable based on experiments in that frame only.
2. The speed of light (and other massless particles) c is the maximum speed in the universe and has the same value in all inertial reference frames.

Consider two frames S and S', moving at a velocity v relative to S, with parallel coordinate axes. The x-axis and x'-axis are aligned. When origins of S and S' coincide (call them $\mathcal{O}$ and $\mathcal{O}'$ respectively), the time in both frames is set to zero, $t = t' = 0$. Since the x-axes are aligned, we will ignore y- and z-components here because they are not interesting. Suppose the Lorentz transformation from inertial frame S to another inertial frame S' takes the form

$$t' = f(x,t), \quad x' = g(x,t),$$

and f and g are linearly independent. Consider a free particle moving with constant velocity $v_x = dx/dt$ in S frame. By postulate 1, this particle observed in S' must also move at a constant velocity $v'_x = dx'/dt'$, which means

$$v'_x = \frac{dx'}{dt'} = \frac{dx'/dt}{dt'/dt} = \frac{\frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial t}}{\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial t}} = \frac{\frac{\partial g}{\partial x}v_x + \frac{\partial g}{\partial t}}{\frac{\partial f}{\partial x}v_x + \frac{\partial f}{\partial t}}.$$

Since v_x and v'_x can be arbitrary constants, all partial derivatives must also be constants. This means the Lorentz transformation must be linear:

$$\begin{aligned} t' &= a_s + a_0 t + a_1 x, \\ x' &= b_s + b_0 t + b_1 x. \end{aligned}$$

The goal is to find the constants a_s, a_0, a_1, b_s, b_0, and b_1. When $\mathcal{O}$ and $\mathcal{O}'$ coincide, we have $x = x' = 0$ at $t' = t = 0$. This synchronization event sets $a_s = b_s = 0$. Now, the frame S' is moving at a velocity v relative to S, then the position of $\mathcal{O}'$ in S is $x = vt$. In S', its origin $\mathcal{O}'$ is always at $x' = 0$, so

$$0 = x' = b_0 t + b_1(vt) \quad \Longrightarrow \quad b_0 = -b_1 v \quad \Longrightarrow \quad x' = b_1(x - vt).$$

Moreover, by symmetry, S is moving at a velocity of $-v$ relative to S'. If the transformation works for all frames (by postulate 1), then we should obtain

$$x = b_1(x' + vt').$$

The transformation equations now look like

$$\begin{aligned} t' &= a_0 t + a_1 x, \\ x' &= b_1(x - vt), \\ x &= b_1(x' + vt'). \end{aligned}$$

It's time to use postulate 2. Suppose a light pulse is emitted at $(t,x) = (0,0)$ and $(t',x') = (0,0)$, i.e., at synchronization. By the constancy of the speed of light, the event $(t,x) = (t,ct)$ in S should have coordinates $(t',x') = (t',ct')$ in S'. Thus,

$$\begin{aligned} x' &= ct' = b_1(x - vt) = b_1(ct - vt) = b_1(1-\beta)ct, \\ x &= ct = b_1(x' + vt') = b_1(ct' + vt') = b_1(1+\beta)ct', \end{aligned}$$

where $\beta \equiv v/c$. Substituting ct into ct', we have

$$ct' = b_1^2(1-\beta^2)ct' \quad \Longrightarrow \quad b_1 = \pm\frac{1}{\sqrt{1-\beta^2}}.$$

We will keep the positive solution as the negative solution does not satisfy $x = x'$ and $t = t'$ at $v = 0$. This b_1 is the Lorentz factor,

$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}.$$

Using $x' = \gamma(x - vt)$ and $x = \gamma(x' + vt')$ to solve for t' and we will get

$$t' = \gamma t - \gamma\beta\,(x/c)\,.$$

The Lorentz transformation is

$$\begin{aligned} ct' &= \gamma(ct - \beta x), \\ x' &= \gamma(x - \beta ct), \\ y' &= y, \\ z' &= z. \end{aligned}$$

A.2 Diracology

A.2.1 Trace of α_i and β

First, we need the cyclic property of the trace from linear algebra:

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA), \quad \mathrm{Tr}(ABC) = \mathrm{Tr}(BCA) = \mathrm{Tr}(CAB), \quad \text{and so on.}$$

Proof. The matrix element of AB can be calculated as

$$(AB)_{ik} = \sum_j A_{ij} B_{jk},$$

by the definition of matrix multiplication. Using this to calculate the trace:

$$\mathrm{Tr}(AB) \equiv \sum_i (AB)_{ii} = \sum_{i,j} A_{ij} B_{ji}.$$

Since A_{ij} and B_{ji} are just numbers, they commute,

$$\mathrm{Tr}(AB) = \sum_{i,j} B_{ji} A_{ij} = \sum_j (BA)_{jj} = \mathrm{Tr}\{BA\}.$$

By this result (and the associativity of matrix multiplication), it is straightforward to show cyclic property of the trace of more products of matrices:

$$\mathrm{Tr}(ABC) = \mathrm{Tr}[(AB)C] = \mathrm{Tr}[C(AB)] = \mathrm{Tr}(CAB), \quad \mathrm{Tr}(ABC) = \mathrm{Tr}[A(BC)] = \mathrm{Tr}[(BC)A] = \mathrm{Tr}[BCA],$$

and so on. □

The trace of α_i: because $\beta\alpha_i = -\alpha_i\beta$,

$$\mathrm{Tr}\big(\alpha_i \beta^2\big) = -\mathrm{Tr}(\beta\alpha_i\beta) = -\mathrm{Tr}\big(\beta^2\alpha_i\big).$$

The second equality is from the cyclic property of the trace. But $\mathrm{Tr}\big(\alpha_i\beta^2\big) = -\mathrm{Tr}\big(\beta^2\alpha_i\big)$ infers that $\mathrm{Tr}(\alpha_i) = -\mathrm{Tr}(\alpha_i)$ because $\beta^2 = I$, so $\mathrm{Tr}(\alpha_i) = 0$. Similarly, by $\alpha_i\beta = -\beta\alpha_i$ and the cyclic property of the trace,

$$\mathrm{Tr}(\alpha_i\beta\alpha_i) = -\mathrm{Tr}\big(\beta\alpha_i^2\big) = -\mathrm{Tr}\big(\alpha_i^2\beta\big),$$

$$\mathrm{Tr}(\alpha_i\beta\alpha_i) = \mathrm{Tr}\big(\alpha_i^2\beta\big),$$

$$\mathrm{Tr}(\beta) = -\mathrm{Tr}(\beta) \quad \Longrightarrow \quad \mathrm{Tr}(\beta) = 0.$$

Hence both α_i and β are traceless.

A.2.2 Determinants

Taking the determinant of $\alpha_i\beta$:

$$\det(\alpha_i\beta) = \det(-\beta\alpha_i).$$

$$\det\alpha_i \det\beta = \det(-I)\det\beta\det\alpha.$$

$$1 = \det(-I) = (-1)^N,$$

where N is the dimension of the identity matrix, α_i and β. In this case N must be even.

A.2.3 Hermitian Conjugate of γ^μ

For $\gamma^0 = \beta$, because β is Hermitian, $(\gamma^0)^\dagger = \gamma^0$. For γ^i,

$$(\gamma^i)^\dagger = (\beta\alpha_i)^\dagger = \alpha_i^\dagger\beta^\dagger = \alpha_i\beta = -\beta\alpha_i = -\gamma^i,$$

where we used the anticommutation relation $\{\alpha_i, \beta\} = 0$, or $\alpha_i\beta = -\beta\alpha_i$.

A.2.4 Anticommutator $\{\gamma^\mu, \gamma^\nu\}$

We can split the anticommutator $\{\gamma^\mu, \gamma^\nu\}$ into three cases: $\{\gamma^0, \gamma^0\}$, $\{\gamma^0, \gamma^i\}$, and $\{\gamma^i, \gamma^j\}$.

$$\begin{aligned}
\{\gamma^0, \gamma^0\} &= 2(\gamma^0)^2 = 2\beta^2 = 2I, \\
\{\gamma^0, \gamma^i\} &= \gamma^0\gamma^i + \gamma^i\gamma^0 = \beta\beta\alpha_i + \beta\alpha_i\beta = \alpha_i - \beta\beta\alpha_i = \alpha_i - \alpha_i = 0, \\
\{\gamma^i, \gamma^j\} &= \beta\alpha_i\beta\alpha_j + \beta\alpha_j\beta\alpha_i = -\beta^2\alpha_i\alpha_j - \beta^2\alpha_j\alpha_i = -(\alpha_i\alpha_j + \alpha_j\alpha_i) = -2\delta_{ij}I.
\end{aligned}$$

They can be summarized into

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}I.$$

A.2.5 More Properties of γ^μ

Let $\gamma'\mu = U^{-1}\gamma^\mu U$, where U is an arbitrary unitary transformation. The anticommutation property of the γ matrices are preserved under these transformations: $\{\gamma'^\mu, \gamma'^\nu\} = 2\eta^{\mu\nu}I$.

Proof.

$$\begin{aligned}
\{\gamma'^\mu, \gamma'^\nu\} &= \{U^{-1}\gamma^\mu U, U^{-1}\gamma^\nu U\} \\
&= U^{-1}\gamma^\mu UU^{-1}\gamma^\nu U + U^{-1}\gamma^\nu UU^{-1}\gamma^\mu U \\
&= U^{-1}\{\gamma^\mu, \gamma^\nu\} U \\
&= U^{-1}(2\eta^{\mu\nu}I)U^{-1} \\
&= 2\eta^{\mu\nu}IU^{-1}U \\
&= 2\eta^{\mu\nu}I.
\end{aligned}$$

□

The contraction of γ^μ with itself is $\gamma_\mu\gamma^\mu = 4I$,

Proof.

$$\gamma_\mu\gamma^\mu = \eta_{\mu\nu}\gamma^\nu\gamma_\mu = \gamma_0^2 - \gamma_1^2 - \gamma_2^2 - \gamma_3^2 = I - (-I) - (-I) - (-I) = 4I.$$

□

The trace of any two slashed 4-vectors is $\mathrm{Tr}(\not{a}\not{b}) = 4a \cdot b$.

Proof. Write

$$\mathrm{Tr}(\not{a}\not{b}) = \mathrm{Tr}(a_\mu\gamma^\mu b_\nu\gamma^\nu) = a_\mu b_\nu \,\mathrm{Tr}(\gamma^\mu\gamma^\nu).$$

We can pull out a_μ and b_ν because they are just numbers. Then write

$$\begin{aligned}
a_\mu b_\nu \,\mathrm{Tr}(\gamma^\mu\gamma^\nu) &= a_\mu b_\nu(\gamma^\nu\gamma^\mu), \\
a_\mu b_\nu \,\mathrm{Tr}(\gamma^\mu\gamma^\nu) &= a_\mu b_\nu[\mathrm{Tr}(2\eta^{\mu\nu}I) - \mathrm{Tr}(\gamma^\nu\gamma^\mu)].
\end{aligned}$$

In the first line we use the cyclic property of the trace. In the second line we use $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}I$. Setting the two RHS's equal gives

$$a_\mu b_\nu \,\mathrm{Tr}(\gamma^\nu\gamma^\mu) = a_\mu b_\nu[\mathrm{Tr}(2\eta^{\mu\nu}I) - \mathrm{Tr}(\gamma^\nu\gamma^\mu)] \quad\Longrightarrow\quad 2a_\mu b_\nu \,\mathrm{Tr}(\gamma^\nu\gamma^\mu) = \eta^{\mu\nu}a_\mu b_\nu \,\mathrm{Tr}(2I).$$

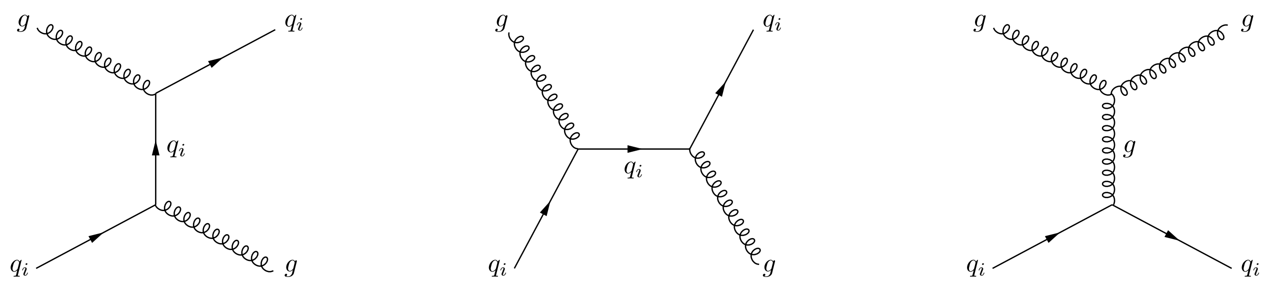
We know that $\mathrm{Tr}(2I) = 8$, so

$$2a_\mu b_\nu \,\mathrm{Tr}(\gamma^\nu \gamma^\mu) = 8\eta^{\mu\nu} a_\mu b_\nu = 8a \cdot b \quad \Longrightarrow \quad \mathrm{Tr}(\not{a}\not{b}) = 4a \cdot b.$$

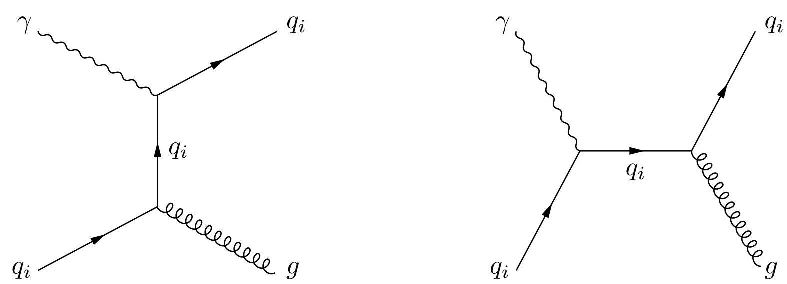
$\square$

B Feynman Diagrams

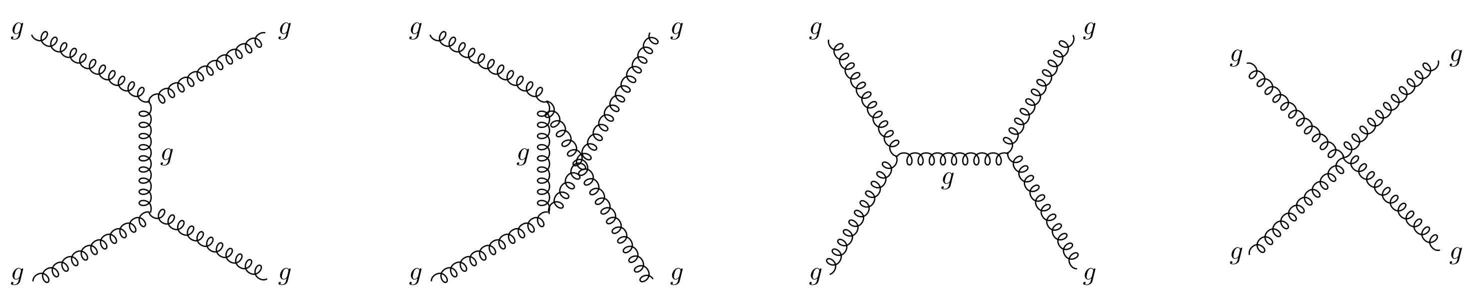
B.1 QED and QCD Feynman Diagrams



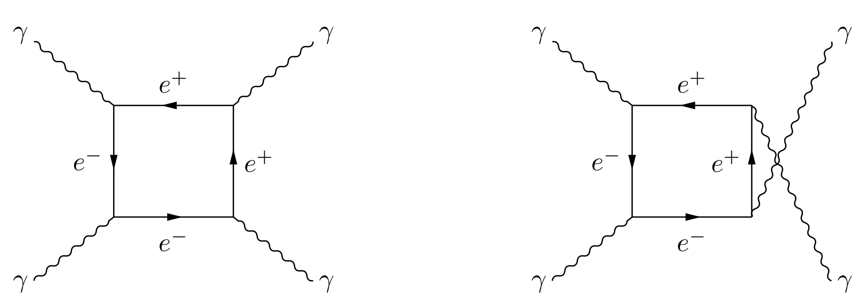
$gq_i \to gq_i$



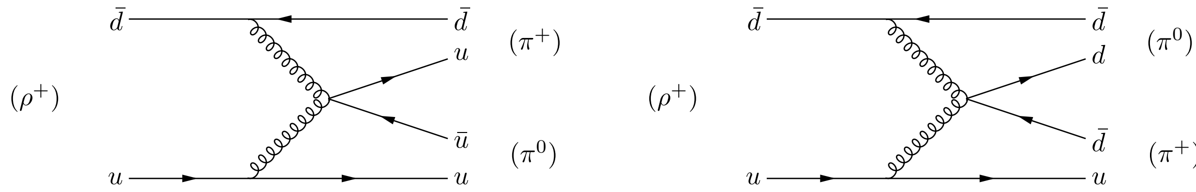
$\gamma q_i \to gq_i$



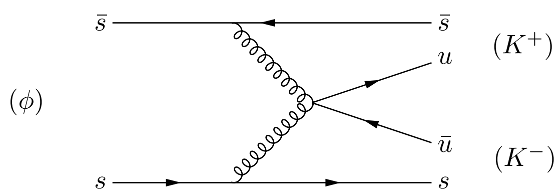
$gg \to gg$



$\gamma\gamma \to \gamma\gamma$



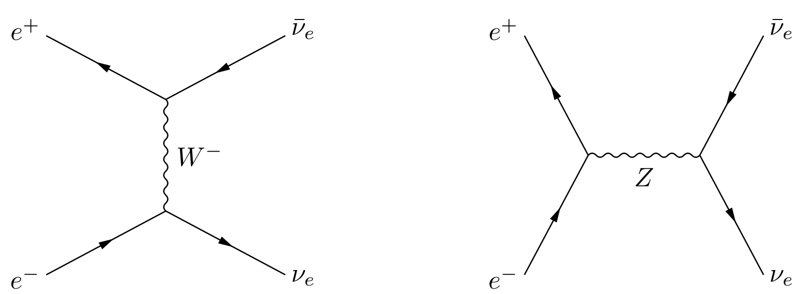
$\rho^+ \to \pi^+\pi^0$



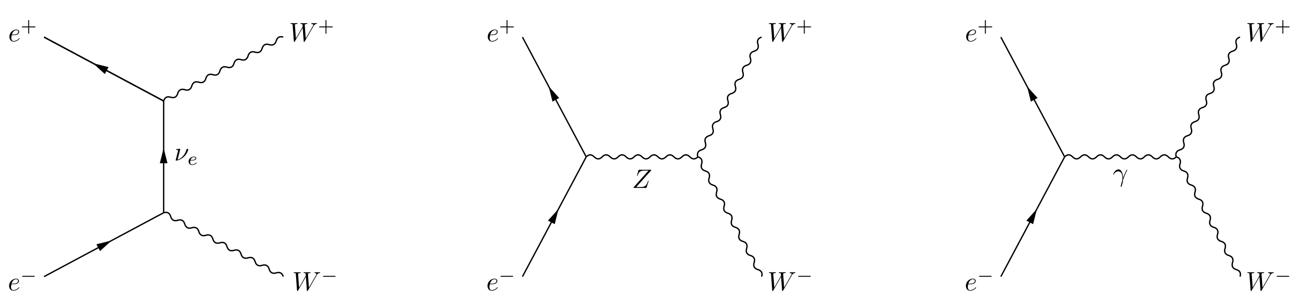
$\phi \to K^+K^-$

B.2 Feynman Diagrams Involving W and Z

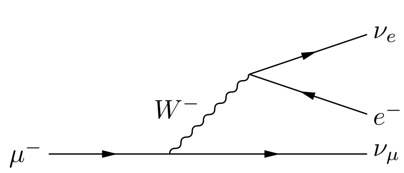
B.2.1 Leptonic



$e^+e^- \to \bar{\nu}_e\nu_e$

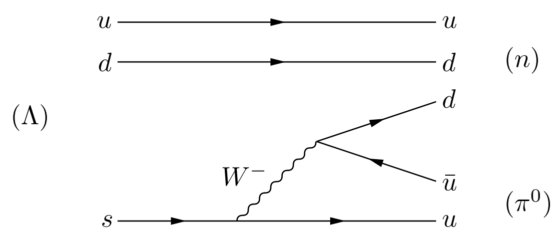


$e^+e^- \to W^+W^-$

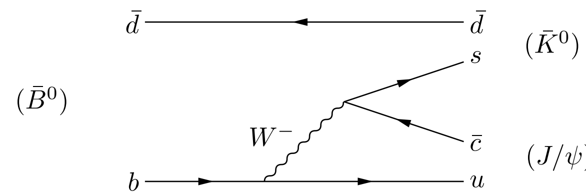


$\mu^- \to e^-\bar{\nu}_e\nu_\mu$

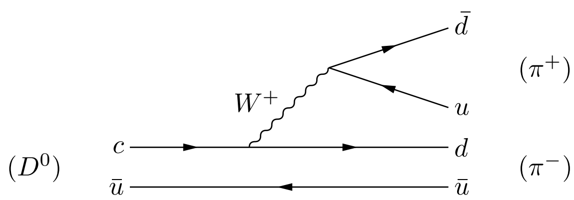
B.2.2 Hadronic



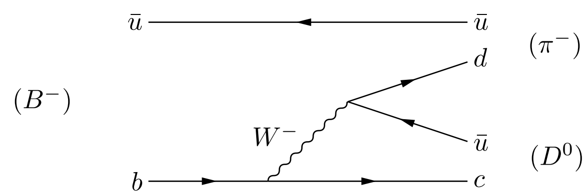
$\Lambda \to n\pi^0$



$\bar{B}^0 \to \bar{K}^0 + J/\psi$

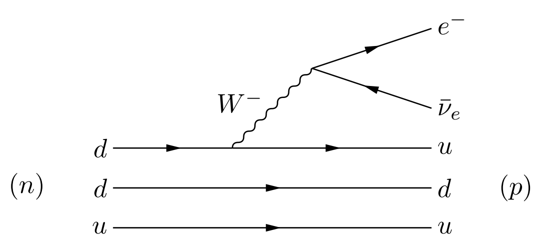


$D^0 \to \pi^+\pi^-$

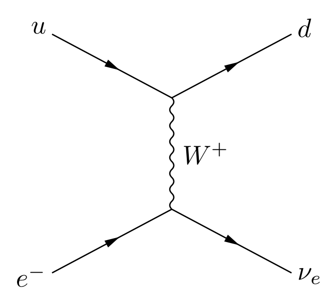


$B^- \to D^0\pi^-$

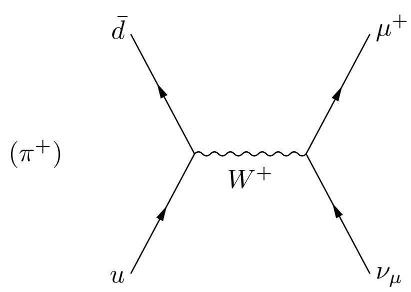
B.2.3 Processes Involving Both Leptons and Quarks



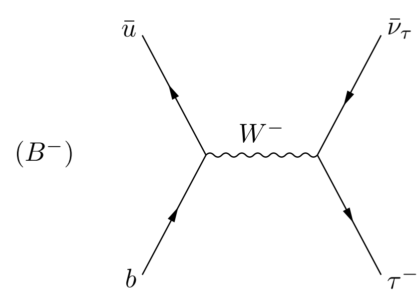
Beta decay: $n \to pe^-\bar{\nu}_e$



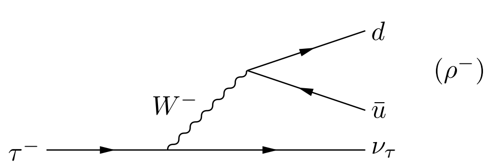
$ue^- \to d\nu_e$



$\pi^+ \to \mu^+\nu_\mu$



$B^- \to \tau^-\bar{\nu}_\tau$



$\tau^- \to \rho^-\nu_\tau$