Proof of commutativity

commutativity.

Hypothesis :
operator $\diamond : \mathbb{R}^2 \to \mathbb{R}$
abstraction $\diamond^{\Delta} \subseteq \Delta^3$
$(d_1, d_2) \in \mathbb{R}^2$
$(d_1', d_2') \in \mathbb{R}^2$
$d_1 \diamond d_2 = d_2 \diamond d_1$
$d_1' \diamond d_2' = d_2' \diamond d_1'$
$(d_1, d_1') \in \delta_1$
$(d_2', d_2') \in \delta_2$

If $(\delta_1, \delta_2, \delta_3) \in \diamond^{\Delta}$
then $\exists(d_1, d_1') \in \delta_2, \exists(d_2, d_2') \in \delta_2, \exists(d_3, d_3') \in \delta_3$:
$\Rightarrow d_1 \diamond d_2 = d_3$ *and* $d_1' \diamond d_2' = d_3'$
Moreover : $d_2 \diamond d_1 = d_3$ *and* $d_2' \diamond d_1' = d_3'$
So $\exists(d_2, d_2') \in \delta_2, \exists(d_1, d_1') \in \delta_1, \exists(d_3, d_3') \in \delta_3$:
$d_2 \diamond d_1 = d_3$ *and* $d_2' \diamond d_1' = d_3'$
Therefore :$(\delta_2, \delta_1, \delta_3) \in \diamond^{\Delta}$

Finally : $\diamond^{\Delta}(d_1, d_2) = \diamond^{\Delta}(d_2, d_1)$
$\Rightarrow$ Abstract operators are commutatif. $\square$

Proof of associativity

associativity.

Hypothesis :
operator $\diamond : \mathbb{R}^2 \to \mathbb{R}$
abstraction $\diamond^{\Delta} \subseteq \Delta^3$
$(d_1, d_2) \in \mathbb{R}^3$
$(d_1', d_2') \in \mathbb{R}^3$
$d_1(\diamond d_2) \diamond \dot{3} = (d_1 \diamond d_2) \diamond d_3$
Let's define : $d_4 = d_1 \diamond d_2, d_5 = d_2 \diamond d_3, d_4' = d_1' \diamond d_2', d_5' = d_2' \diamond d_3'$

If $(\delta_1, \delta_5, \delta_6) \in \diamond^{\Delta}$
Then $\exists(d_1, d_1') \in \delta_1, \exists(d_5, d_5') \in \delta_5, \exists(d_6, d_6') \in \delta_6$:
$d_1 \diamond d_5 = d_6$ *and* $d_1' \diamond d_5' = d_6'$
Moreover, if $(\delta_2, \delta_3, \delta_5) \in \diamond^{\Delta}$
Then, $\exists(d_2, d_2') \in \delta_2, \exists(d_3, d_3') \in \delta_3$:
$d_2 \diamond d_3 = d_5$ *and* $d_2' \diamond d_3' = d_5'$
That's why : $d_1 \diamond (d_2 \diamond d_3) = d_6$ *and* $d_1' \diamond (d_2' \diamond d_3') = d_6'$

From Hypothesis : $(d_1 \diamond d_2) \diamond d_3 = d_6$ *and* $(d_1' \diamond d_2') \diamond d_3' = d_6'$
$\Rightarrow d_4 \diamond d_3 = d_6$ *and* $d_4' \diamond d_3' = d_6'$
So $\exists(d_4, d_4') \in \delta_4, \exists(d_3, d_3') \in \delta_3, \exists(d_6, d_6') \in \delta_6$:
$d_4 \diamond d_3 = d_6$ *and* $d_4' \diamond d_3' = d_6'$

And $\exists(\boldsymbol{d_1, d'_1}) \in \boldsymbol{\delta_1}, \exists(\boldsymbol{d_2, d'_2}) \in \boldsymbol{\delta_2}$
$\boldsymbol{d_1 \diamond d_2 = d_4 \ and \ d'_1 \diamond d'_2 = d'_4}$
Therefore :$(\boldsymbol{\delta_1, \delta_2, \delta_4}) \in \diamond^{\boldsymbol{\Delta}} \ \boldsymbol{and} \ (\boldsymbol{\delta_4, \delta_3, \delta_6}) \in \diamond^{\boldsymbol{\Delta}}$

To conclude :$(\boldsymbol{\delta_1, \delta_5, \delta_6}) \in \diamond^{\boldsymbol{\Delta}} \ \boldsymbol{and} \ (\boldsymbol{\delta_4, \delta_3, \delta_6}) \in \diamond^{\boldsymbol{\Delta}}$
$\Rightarrow$ Abstract operators are associatif. $\square$