

Proof of commutativity

commutativity. Hypothesis :

operator $\diamond : \mathbb{R}^2 \rightarrow \mathbb{R}$

abstraction $\diamond^\Delta \subseteq \Delta^3$

$(d_1, d_2) \in \mathbb{R}^2$

$(d'_1, d'_2) \in \mathbb{R}^2$

$d_1 \diamond d_2 = d_2 \diamond d_1$

$d'_1 \diamond d'_2 = d'_2 \diamond d'_1$

$(d_1, d'_1) \in \delta_1$

$(d'_2, d'_2) \in \delta_2$

If $(\delta_1, \delta_2, \delta_3) \in \diamond$

then $\forall (d_1, d'_1) \in \delta_2, (d_2, d'_2) \in \delta_2, (d_3, d'_3) \in \delta_3 :$

$d_1 \diamond d_2 = d_3$ and $d'_1 \diamond d'_2 = d'_3$

Moreover : $d_2 \diamond d_1 = d_3$ and $d'_2 \diamond d'_1 = d'_3$

So $\forall (d_2, d'_2) \in \delta_2, (d_1, d'_1) \in \delta_1, (d_3, d'_3) \in \delta_3 :$

$d_2 \diamond d_1 = d_3$ and $d'_2 \diamond d'_1 = d'_3$

Therefore : $(\delta_2, \delta_1, \delta_3) \in \diamond^\Delta$

Finally : $\diamond^\Delta(d_1, d_2) = \diamond^\Delta(d_2, d_1)$

→ Abstract operator are commutatif. □

Proof of associativity

associativity. Hypothesis :

operator $\diamond : \mathbb{R}^2 \rightarrow \mathbb{R}$

abstraction $\diamond^\Delta \subseteq \Delta^3$

$(d_1, d_2) \in \mathbb{R}^3$

$(d'_1, d'_2) \in \mathbb{R}^3$

$d_1(\diamond d_2) \diamond d_3 = (d_1 \diamond d_2) \diamond d_3$

Let's define : $d_4 = d_1 \diamond d_2, d_5 = d_2 \diamond d_3, d'_4 = d'_1 \diamond d'_2, d'_5 = d'_2 \diamond d'_3$

If $(\delta_1, \delta_5, \delta_6) \in \diamond^\Delta$

Then $\exists (d_1, d'_1) \in \delta_1, \exists (d_5, d'_5) \in \delta_5, \exists (d_6, d'_6) \in \delta_6 :$

$d_1 \diamond d_5 = d_6$ and $d'_1 \diamond d'_5 = d'_6$

Moreover, if $(\delta_2, \delta_3, \delta_5) \in \diamond^\Delta$

Then, $\exists (d_2, d'_2) \in \delta_2, \exists (d_3, d'_3) \in \delta_3 :$

$d_2 \diamond d_3 = d_5$ and $d'_2 \diamond d'_3 = d'_5$

That's why : $d_1 \diamond (d_2 \diamond d_3) = d_6$ and $d'_1 \diamond (d'_2 \diamond d'_3) = d'_6$

From Hypothesis : $(d_1 \diamond d_2) \diamond d_3 = d_6$ and $(d'_1 \diamond d'_2) \diamond d'_3 = d'_6$

→ $d_4 \diamond d_3 = d_6$ and $d'_4 \diamond d'_3 = d'_6$

So $\exists (d_4, d'_4) \in \delta_4, \exists (d_3, d'_3) \in \delta_3, \exists (d_6, d'_6) \in \delta_6 :$

$d_4 \diamond d_3 = d_6$ and $d'_4 \diamond d'_3 = d'_6$

And $\exists (d_1, d'_1) \in \delta_1, \exists (d_2, d'_2) \in \delta_2$

$d_1 \diamond d_2 = d_4$ and $d'_1 \diamond d'_2 = d'_4$

Therefore : $(\delta_1, \delta_2, \delta_4) \in \diamond^\Delta$ and $(\delta_4, \delta_3, \delta_6) \in \diamond^\Delta$

To conclude : $(\delta_1, \delta_5, \delta_6) \in \diamond^\Delta$ *and* $(\delta_4, \delta_3, \delta_6) \in \diamond^\Delta$
 \longrightarrow Abstract operator are associatif.

□