Proof of commutativity

commutativity. Hypothesis :

operator $\diamond : \mathbb{R}^2 \rightarrow \mathbb{R}$
abstraction $\diamond^{\Delta} \subseteq Delta^3$
$(d_1, d_2) \in \mathbb{R}^2$
$d'_1, d'_2) \in \mathbb{R}^2$
$d_1 \diamond d_2 = d_2 \diamond d_1$
$d'_1 \diamond d'_2 = d'_2 \diamond d'_1$
$(d_1, d'_1) \in \delta_1$
$(d'_2, d'_2) \in \delta_2$
If $(\delta_1, \delta_2, \delta_3) \in \diamond$
then $\forall (d_1, d'_1) \in \delta_2, (d_2, d'_2) \in \delta_2, (d_3, d'_3) \in \delta_3 :$
$d_1 \diamond d_2 = d_3 \ and \ d'_1 \diamond d'_2 = d'_3$
Moreover : $d_2 \diamond d_1 = d_3 \ and \ d'_2 \diamond d'_1 = d'_3$
So $\forall (d_2, d'_2) \in \delta_2, (d_1, d'_1) \in \delta_1, (d_3, d'_3) \in \delta_3 :$
$d_2 \diamond d_1 = d_3 \ and \ d'_2 \diamond d'_1 = d'_3$
Therefore :$(\delta_2, \delta_1, \delta_3) \in \diamond^{\Delta}$
Finally : $\diamond^{\Delta}(d_1, d_2) = \diamond^{\Delta}(d_2, d_1)$
$\longrightarrow$ Abstract operator are commutatif. □

Proof of associativity

associativity. Hypothesis :
operator $\diamond : \mathbb{R}^2 \rightarrow \mathbb{R}$
abstraction $\diamond^{\Delta} \subseteq Delta^3$
$(d_1, d_2) \in \mathbb{R}^3$
$d'_1, d'_2) \in \mathbb{R}^3$
$d_1(\diamond d_2) \diamond 3 = (d_1 \diamond d_2) \diamond d_3$
Let's define : $d_4 = d_1 \diamond d_2, d_5 = d_2 \diamond d_3, d'_4 = d'_1 \diamond d'_2, d'_5 = d'_2 \diamond d'_3$

If $(\delta_1, \delta_5, \delta_6) \in \diamond^{\Delta}$
Then $\exists (d_1, d'_1) \in \delta_1, \exists (d_5, d'_5) \in \delta_5, \exists (d_6, d'_6) \in \delta_6 :$
$d_1 \diamond d_5 = d_6 \ and \ d'_1 \diamond d'_5 = d'_6$
Moreover, if $(\delta_2, \delta_3, \delta_5) \in \diamond^{\Delta}$
Then, $\exists (d_2, d'_2) \in \delta_2, \exists (d_3, d'_3) \in \delta_3 :$
$d_2 \diamond d_3 = d_5 \ and \ d'_2 \diamond d'_3 = d'_5$
That's why : $d_1 \diamond (d_2 \diamond d_3) = d_6 \ and \ d'_1 \diamond (d'_2 \diamond d'_3) = d'_6$

From Hypothesis : $(d_1 \diamond d_2) \diamond d_3 = d_6 \ and \ (d'_1 \diamond d'_2) \diamond d'_3 = d'_6$
$\longrightarrow d_4 \diamond d_3 = d_6 \ and \ d'_4 \diamond d'_3 = d'_6$
So $\exists (d_4, d'_4) \in \delta_4, \exists (d_3, d'_3) \in \delta_3, \exists (d_6, d'_6) \in \delta_6 :$
$d_4 \diamond d_3 = d_6 \ and \ d'_4 \diamond d'_3 = d'_6$
And $\exists (d_1, d'_1) \in \delta_1, \exists (d_2, d'_2) \in \delta_2$
$d_1 \diamond d_2 = d_4 \ and \ d'_1 \diamond d'_2 = d'_4$
Therefore :$(\delta_1, \delta_2, \delta_4) \in \diamond^{\Delta} \ and \ (\delta_4, \delta_3, \delta_6) \in \diamond^{\Delta}$

To conclude :$(\boldsymbol{\delta_1, \delta_5, \delta_6}) \in \diamond^{\boldsymbol{\Delta}}$ $\boldsymbol{and}$ $(\boldsymbol{\delta_4, \delta_3, \delta_6}) \in \diamond^{\boldsymbol{\Delta}}$
$\longrightarrow$ Abstract operator are associatif. $\square$