# Digital Signal Processing

**Formulary**
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# Contents



## Preface

This formulary is based on the formulary of the course Stochastic Signals and Systems, which can be found here `https://github.com/Tyde/stosigsysfs/blob/master/document.pdf?raw=true`. If you find any errors or have any ideas for improvement, mail me at studium@daniel-thiem.de or file an issue at !GITHUBLINK. The LaTeX source code is online on !GITHUBLINK and can be improved and extended.

# 1 Digital Processing of Continous-Time Signals

## 1.1 Periodic Sampling

Basic principles of sampling and transforming signals can be found in the Stochastic Signals and Systems formulary

### 1.1.1 Reconstruction of Band-Limited Signals

Assume, that $H_r(j\Omega)$ and $h_r(t)$ are the frequency and time responses for an ideal low pass filter where $x(n)$ is the input signal. Then the output will be

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(n) h_r(t - nT) \tag{1.1}$$

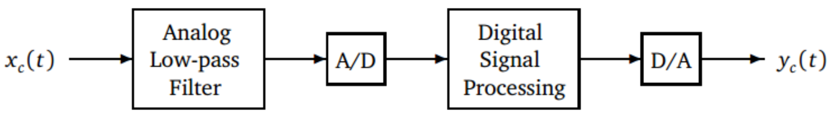
Because the filter is assumed to be ideal, its impulse response is given by:

$$h_r(t) = \frac{\sin(\pi t/T)}{\pi t/T} \tag{1.2}$$

where T is the sampling interval that should fulfil the Nyquist condition $\Omega_s > 2\Omega_B$

### 1.1.2 Discrete-time Processing of Continous-Time Signals

To process a continous-time signal $x_c(t)$ with digital signal processing techniques, the following setup is used:



# 2 Filter Design

## 2.1 Digital Filters

Let $x(n)$ and $y(n), n = 0 \pm 1, \ldots$ be the input and the output of the system. Linear time-invariant discrete-time systems can be characterized by linear constant coefficient difference equations of the following type, where $\max(N-1, M)$ is the order:

$$\sum_{k=0}^{M} a_k y(n-k) = \sum_{r=0}^{N-1} b_r x(n-r) \tag{2.1}$$

### 2.1.1 Finite Impulse Response (FIR) filter

If $M = 0$ then the impulse response of the filter is finite so that:

$$y(n) = \frac{1}{a_0} \sum_{k=-\infty}^{\infty} b_r x(n-k) \tag{2.2a}$$

comparison with the convolution sum yields

$$h(n) = \begin{cases} b_n / a_n, & n = 0, \ldots, N-1 \\ 0, & \text{otherwise} \end{cases} \tag{2.2b}$$

### 2.1.2 Linear Phase Filters

A digital filter can be described as linear phase filter if $H(e^{j\omega})$ can be expressed as:

$$H(e^{j\omega}) = H_M(e^{j\omega})\, e^{-j\omega\alpha} \tag{2.3}$$

with $H_M(e^{j\omega})$ real and $\alpha$(group delay) constant

### Generalized linear phase system

A Linear phase system with:

$$H(e^{j\omega}) = H_M(e^{j\omega})\, e^{-j\omega\alpha + j\beta} \tag{2.4}$$

has to meet the condition:

$$\sum_{n=-\infty}^{\infty} h(n) \sin[\omega(n-\alpha)+\beta] = 0 \forall \omega \in \mathbb{R} \tag{2.5}$$

### 2.1.3 Linear Phase FIR Filters

Applying the condition for generalized linear phase systems (2.1.2) to the FIR Filters yields two possible causal FIR Systems:

$$h(n) = \begin{cases} h(N-1-n) & ,0 \le n \le N-1 \\ 0 & ,\text{otherwise} \end{cases} \tag{2.6a}$$

$$H(e^{j\omega}) = H_e(e^{j\omega})\, e^{-j\omega(N-1)/2} \tag{2.6b}$$

and

$$h(n) = \begin{cases} -h(N-1-n) & ,0 \le n \le N-1 \\ 0 & ,\text{otherwise} \end{cases} \tag{2.7a}$$

$$H(e^{j\omega}) = H_o(e^{j\omega})\, e^{-j\omega(N-1)/2 + j\pi/2} \tag{2.7b}$$

### 2.1.4 FIR Filter Types

The linear phase FIR filters have been classified into 4 Types:

- **Type I** symmetric impulse response and N is odd
- **Type II** symmetric impulse response and N is even
- **Type III** antisymmetric impulse response and N is odd
- **Type IV** antisymmetric impulse response and N is even

## Type I linear phase filter

$N$ is odd and the impuls response is symmetric. $\beta = 0$

$$h(n) = h(N-1-n) \quad 0 \leq n \leq N-1 \tag{2.8a}$$

$$H(e^{j\omega}) = e^{-j\omega\frac{N-1}{2}} \left[ \sum_{k=0}^{(N-1)/2} a(k)\cos(\omega k) \right] \tag{2.8b}$$

where

$$a(0) = h\left(\frac{N-1}{2}\right) \text{ and } a(k) = 2h\left(\frac{N-1}{2} - k\right), \qquad k = 1, 2, \ldots, (N-1)/2$$

## Type II linear phase filter

$N$ is even and the impuls response is symmetric. $\beta = 0$

$$h(n) = h(N-1-n) \quad 0 \leq n \leq N-1 \tag{2.9a}$$

$$H(e^{j\omega}) = e^{-j\omega\frac{N-1}{2}} \left\{ \sum_{k=1}^{N/2} b(k)\cos\left[\omega\left(k - \frac{1}{2}\right)\right] \right\} \tag{2.9b}$$

where

$$b(k) = 2h[N/2 - k], \qquad k = 1, 2, \ldots, N/2$$

## Type III linear phase filter

$N$ is odd and the impuls response is antisymmetric. $\beta = \pi/2$

$$h(n) = -h(N-1-n) \quad 0 \leq n \leq N-1 \tag{2.10a}$$

$$H(e^{j\omega}) = je^{-j\omega\frac{N-1}{2}} \left[ \sum_{k=1}^{(N-1)/2} c(k)\sin(\omega k) \right] \tag{2.10b}$$

where

$$c(k) = 2h[(N-1)/2 - k], \qquad k = 1, 2, \ldots, (N-1)/2$$

### Type IV linear phase filter

$N$ is even and the impuls response is antisymmetric. $\beta = \pi/2$

$$h(n) = -h(N-1-n) \quad 0 \leq n \leq N-1 \tag{2.11a}$$

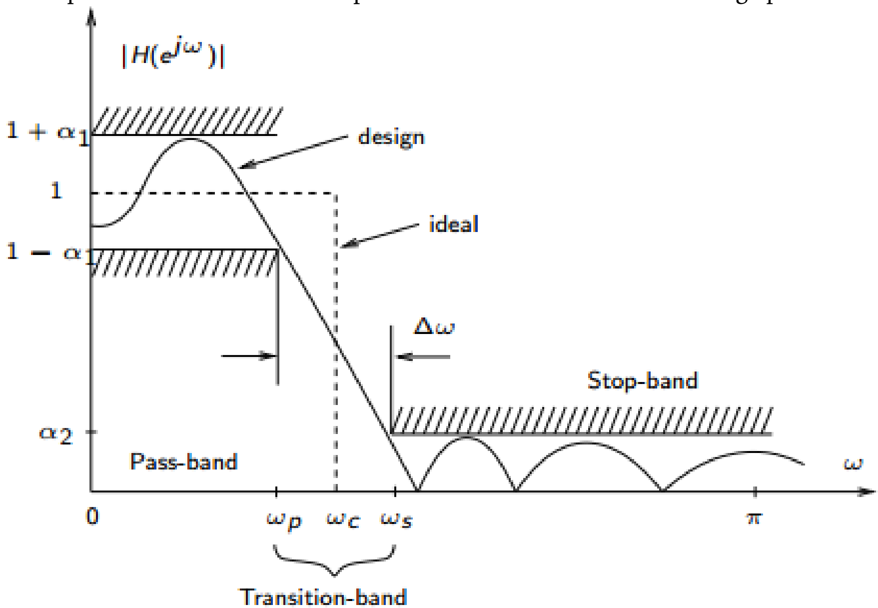
$$H(e^{j\omega}) = je^{-j\omega\frac{N-1}{2}} \left\{ \sum_{k=1}^{N/2} d(k) \sin\left[\omega\left(k - \frac{1}{2}\right)\right] \right\} \tag{2.11b}$$

where

$$d(k) = 2h[N/2-k], \qquad k = 1, 2, \ldots, N/2$$

## 2.2 Filter Specficaion

Non-ideal Filters have one or more pass-bands($\omega_p$), transition-bands(between $\omega_p$ and $\omega_s$) and stop-bands($\omega_s$). For the pass- and the stop- band, a tolerance $\alpha_i$ has to be specified. Therefore a low-pass filter would have the following specification:



# 3 Miscellaneous

## 3.1 Useful mathematical equations

### 3.1.1 Geometric series

$$\sum_{n=0}^{N-1} q^n = \frac{1-q^N}{1-q} \quad \text{for} \quad |q| < 1 \tag{3.1}$$

### 3.1.2 Conversion from Geometric series to trigonometric fraction

Let $\frac{1-q^N}{1-q}$ be with $q = e^{-j\omega}$

$$\frac{1-e^{-jN\omega}}{1-e^{-j\omega}} = e^{-j\omega\frac{N-1}{2}} \frac{\sin\left(\frac{N}{2}\omega\right)}{\sin\left(\frac{\omega}{2}\right)} \tag{3.2}$$