Digital Signal Processing

Formulary

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Preface

This formulary is based on the formulary of the course Stochastic Signals and Systems, which can be found here `https://github.com/Tyde/stosigsysfs/blob/`

`master/document.pdf?raw=true`. If you find any errors or have any ideas for improvement, mail me at studium@daniel-thiem.de or file an issue at !GITHUBLINK. The LaTeX source code is online on !GITHUBLINK and can be improved and extended.

1 Digital Processing of Continous-Time Signals

1.1 Periodic Sampling

Basic principles of sampling and transforming signals can be found in the Stochastic Signals and Systems formulary

1.1.1 Reconstruction of Band-Limited Signals

Assume, that $H_r(j\Omega)$ and $h_r(t)$ are the frequency and time responses for an ideal low pass filter where $x(n)$ is the input signal. Then the output will be

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(n) h_r(t - nT) \tag{1.1}$$

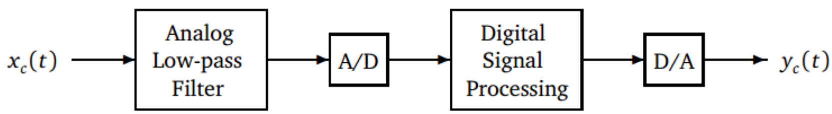
Because the filter is assumed to be ideal, its impulse response is given by:

$$h_r(t) = \frac{\sin(\pi t/T)}{\pi t/T} \tag{1.2}$$

where T is the sampling interval that should fulfil the Nyquist condition $\Omega_s > 2\Omega_B$

1.1.2 Discrete-time Processing of Continous-Time Signals

To process a continous-time signal $x_c(t)$ with digital signal processing techniques, the following setup is used:



2 Digital Filter Design

Let $x(n)$ and $y(n), n = 0 \pm 1, \ldots$ be the input and the output of the system. Linear time-invariant discrete-time systems can be characterized by linear constant coefficient difference equations of the following type, where $\max(N-1, M)$ is the order:

$$\sum_{k=0}^{M} a_k y(n-k) = \sum_{r=0}^{N-1} b_r x(n-r) \tag{2.1}$$

2.1 Finite Impulse Response (FIR) filter

If $M = 0$ then the impulse response of the filter is finite so that:

$$y(n) = \frac{1}{a_0} \sum_{k=-\infty}^{\infty} b_r x(n-k) \tag{2.2a}$$

comparison with the convolution sum yields

$$h(n) = \begin{cases} b_n/a_n, & n = 0, \ldots, N-1 \\ 0, & \text{otherwise} \end{cases} \tag{2.2b}$$

2.2 Linear Phase Filters

A digital filter can be described as linear phase filter if $H(e^{j\omega})$ can be expressed as:

$$H(e^{j\omega}) = H_M(e^{j\omega})\, e^{-j\omega\alpha} \tag{2.3}$$

with $H_M(e^{j\omega})$ real and α(group delay) constant

2.2.1 Generalized linear phase system

A Linear phase system with:

$$H(e^{j\omega}) = H_M(e^{j\omega})\, e^{-j\omega\alpha + j\beta} \tag{2.4}$$

has to meet the condition:

$$\sum_{n=-\infty}^{\infty} h(n) \sin[\omega(n-\alpha) + \beta] = 0 \forall \omega \in \mathbb{R} \tag{2.5}$$

2.3 Linear Phase FIR filters

Applying the condition for generalized linear phase systems (2.2.1) to the FIR Filters yields two possible causal FIR Systems:

$$h(n) = \begin{cases} h(N-1-n) & ,0 \le n \le N-1 \\ 0 & ,\text{otherwise} \end{cases} \tag{2.6a}$$

$$H(e^{j\omega}) = H_e(e^{j\omega})\, e^{-j\omega(N-1)/2} \tag{2.6b}$$

and

$$h(n) = \begin{cases} -h(N-1-n) & ,0 \le n \le N-1 \\ 0 & ,\text{otherwise} \end{cases} \tag{2.7a}$$

$$H(e^{j\omega}) = H_o(e^{j\omega})\, e^{-j\omega(N-1)/2 + j\pi/2} \tag{2.7b}$$

2.4 Linear Phase FIR filter Types

The linear phase FIR filters have been classified into 4 Types:

- **Type I** symmetric impulse response and N is odd
- **Type II** symmetric impulse response and N is even
- **Type III** antisymmetric impulse response and N is odd
- **Type IV** antisymmetric impulse response and N is even

2.4.1 Type I FIR linear phase filter

N is odd and the impuls response is symmetric. $\beta = 0$

$$h(n) = h(N-1-n) \quad 0 \le n \le N-1 \tag{2.8a}$$

$$H(e^{j\omega}) = e^{-j\omega\frac{N-1}{2}} \left[\sum_{k=0}^{(N-1)/2} a(k)\cos(\omega k) \right] \tag{2.8b}$$

where

$$a(0) = h\left(\frac{N-1}{2}\right) \text{ and } a(k) = 2h\left(\frac{N-1}{2} - k\right), \qquad k = 1, 2, \ldots, (N-1)/2$$

2.4.2 Type II FIR linear phase filter

N is even and the impuls response is symmetric. $\beta = 0$

$$h(n) = h(N-1-n) \quad 0 \le n \le N-1 \tag{2.9a}$$

$$H(e^{j\omega}) = e^{-j\omega\frac{N-1}{2}} \left\{ \sum_{k=1}^{N/2} b(k) \cos\left[\omega\left(k - \frac{1}{2}\right)\right]\right\} \tag{2.9b}$$

where

$$b(k) = 2h[N/2 - k], \qquad k = 1, 2, \ldots, N/2$$

2.4.3 Type III FIR linear phase filter

N is odd and the impuls response is antisymmetric. $\beta = \pi/2$

$$h(n) = -h(N-1-n) \quad 0 \le n \le N-1 \tag{2.10a}$$

$$H(e^{j\omega}) = je^{-j\omega\frac{N-1}{2}} \left[\sum_{k=1}^{(N-1)/2} c(k) \sin(\omega k) \right] \tag{2.10b}$$

where

$$c(k) = 2h[(N-1)/2 - k], \qquad k = 1, 2, \ldots, (N-1)/2$$

2.4.4 Type IV FIR linear phase filter

N is even and the impuls response is antisymmetric. $\beta = \pi/2$

$$h(n) = -h(N-1-n) \quad 0 \le n \le N-1 \tag{2.11a}$$

$$H(e^{j\omega}) = je^{-j\omega\frac{N-1}{2}} \left\{ \sum_{k=1}^{N/2} d(k) \sin\left[\omega\left(k - \frac{1}{2}\right)\right]\right\} \tag{2.11b}$$

where

$$d(k) = 2h[N/2 - k], \qquad k = 1, 2, \ldots, N/2$$

2.5 FIR filter design

2.5.1 FIR filter type choice

Not every FIR filter type fits on every filter.

Type	Low Pass	High Pass	Band Pass	Band Stop
I				
II		Not suitable		Not suitable
III	Not suitable	Not suitable		Not suitable
IV	Not suitable			Not suitable

2.5.2 FIR filter design by Windowing

Most idealized systems are non causal and have infinite impulse responses. To achieve finite impulse responses and causality, truncating the ideal response is the most straightforward approach.

2.6 Infinite Impulse Response (IIR) filter

IIR filters are systems, which follow the following Restrictions:
$h(n)$ is

- real
- causal
- satifies stability

$h(n)$ possesses a rational z-transform with $a_0 = 1$, i.e.

$$H(z) = \frac{\sum_{k=0}^{N-1} b_k z^{-k}}{1 - \sum_{k=0}^{N-1} a_k z^{-k}} \tag{2.12}$$

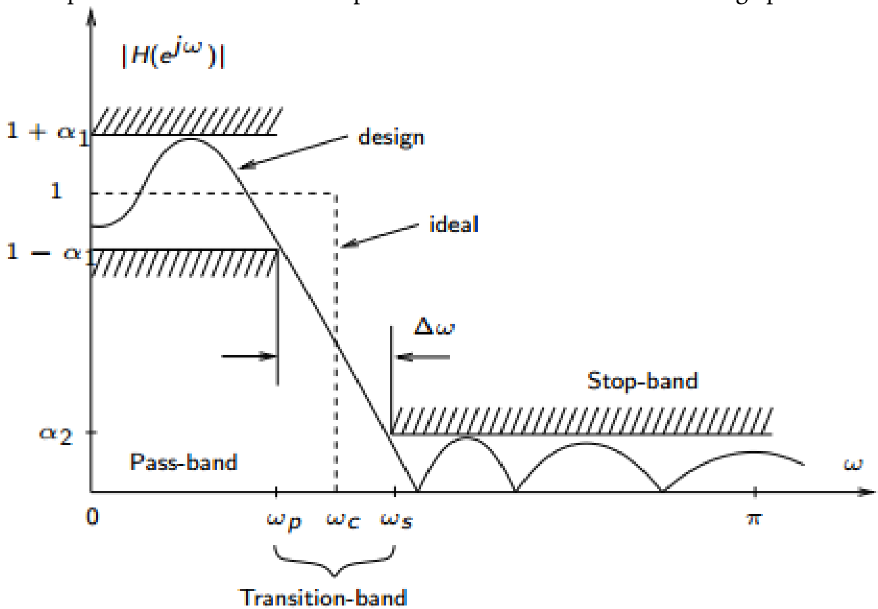
2.6.1 Properties of IIR filters

- **Stability**
  All poles of $H(z)$ must lie inside the unit disk

- **Linear Phase**
  IIR filters do not have linear phase. Only the magnitude response is specified.

2.7 Filter Specficaion

Non-ideal Filters have one or more pass-bands(ω_p), transition-bands(between ω_p and ω_s) and stop-bands(ω_s). For the pass- and the stop- band, a tolerance α_i has to be specified. Therefore a low-pass filter would have the following specification:



3 Miscellaneous

3.1 Useful mathematical equations

3.1.1 Geometric series

$$\sum_{n=0}^{N-1} q^n = \frac{1-q^N}{1-q} \quad \text{for} \quad |q| < 1 \tag{3.1}$$

3.1.2 Conversion from Geometric series to trigonometric fraction

Let $\frac{1-q^N}{1-q}$ be with $q = e^{-j\omega}$

$$\frac{1-e^{-jN\omega}}{1-e^{-j\omega}} = e^{-j\omega\frac{N-1}{2}} \frac{\sin\left(\frac{N}{2}\omega\right)}{\sin\left(\frac{\omega}{2}\right)} \tag{3.2}$$