Digital Signal Processing

Formulary

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TECHNISCHE
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Contents

Preface

This formulary is based on the formulary of the course Stochastic Signals and Systems, which can be found here `https://github.com/Tyde/stosigsysfs/blob/master/document.pdf?raw=true`. If you find any errors or have any ideas for improvement, mail me at studium@daniel-thiem.de or file an issue at `https://github.com/Tyde/dspformulary/issues`. The LaTeX source code is online on `https://github.com/Tyde/dspformulary` and can be improved and extended.

1 Digital Processing of Continous-Time Signals

1.1 Periodic Sampling

Basic principles of sampling and transforming signals can be found in the Stochastic Signals and Systems formulary

1.1.1 Reconstruction of Band-Limited Signals

Assume, that $H_r(j\Omega)$ and $h_r(t)$ are the frequency and time responses for an ideal low pass filter where $x(n)$ is the input signal. Then the output will be

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(n) h_r(t - nT) \tag{1.1}$$

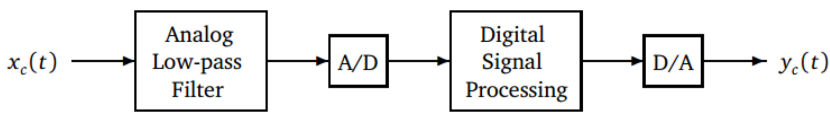
Because the filter is assumed to be ideal, its impulse response is given by:

$$h_r(t) = \frac{\sin(\pi t/T)}{\pi t/T} \tag{1.2}$$

where T is the sampling interval that should fulfil the Nyquist condition $\Omega_s > 2\Omega_B$

1.1.2 Discrete-time Processing of Continous-Time Signals

To process a continous-time signal $x_c(t)$ with digital signal processing techniques, the following setup is used:



2 Digital Filter Design

Let $x(n)$ and $y(n), n = 0 \pm 1, \dots$ be the input and the output of the system. Linear time-invariant discrete-time systems can be characterized by linear constant coefficient difference equations of the following type, where $\max(N-1, M)$ is the order:

$$\sum_{k=0}^{M} a_k y(n-k) = \sum_{r=0}^{N-1} b_r x(n-r) \tag{2.1}$$

2.1 Finite Impulse Response (FIR) filter

If $M = 0$ then the impulse response of the filter is finite so that:

$$y(n) = \frac{1}{a_0} \sum_{k=-\infty}^{\infty} b_r x(n-k) \tag{2.2a}$$

comparison with the convolution sum yields

$$h(n) = \begin{cases} b_n/a_n, & n = 0, \dots, N-1 \\ 0, & \text{otherwise} \end{cases} \tag{2.2b}$$

2.2 Linear Phase Filters

A digital filter can be described as linear phase filter if $H\left(e^{j\omega}\right)$ can be expressed as:

$$H\left(e^{j\omega}\right) = H_M\left(e^{j\omega}\right) e^{-j\omega\alpha} \tag{2.3}$$

with $H_M\left(e^{j\omega}\right)$ real and α(group delay) constant

2.2.1 Generalized linear phase system

A Linear phase system with:

$$H\left(e^{j\omega}\right) = H_M\left(e^{j\omega}\right) e^{-j\omega\alpha + j\beta} \tag{2.4}$$

has to meet the condition:

$$\sum_{n=-\infty}^{\infty} h(n) \sin[\omega(n-\alpha) + \beta] = 0 \forall \omega \in \mathbb{R} \tag{2.5}$$

2.3 Linear Phase FIR filters

Applying the condition for generalized linear phase systems (2.2.1) to the FIR Filters yields two possible causal FIR Systems:

$$h(n) = \begin{cases} h(N-1-n) & ,0 \le n \le N-1 \\ 0 & ,\text{otherwise} \end{cases} \tag{2.6a}$$

$$H\left(e^{j\omega}\right) = H_e\left(e^{j\omega}\right)e^{-j\omega(N-1)/2} \tag{2.6b}$$

and

$$h(n) = \begin{cases} -h(N-1-n) & ,0 \le n \le N-1 \\ 0 & ,\text{otherwise} \end{cases} \tag{2.7a}$$

$$H\left(e^{j\omega}\right) = H_o\left(e^{j\omega}\right)e^{-j\omega(N-1)/2+j\pi/2} \tag{2.7b}$$

2.4 Linear Phase FIR filter Types

The linear phase FIR filters have been classified into 4 Types:

- **Type I** symmetric impulse response and N is odd
- **Type II** symmetric impulse response and N is even
- **Type III** antisymmetric impulse response and N is odd
- **Type IV** antisymmetric impulse response and N is even

2.4.1 Type I FIR linear phase filter

N is odd and the impuls response is symmetric. $\beta = 0$

$$h(n) = h(N-1-n) \quad 0 \le n \le N-1 \tag{2.8a}$$

$$H\left(e^{j\omega}\right) = e^{-j\omega\frac{N-1}{2}}\left[\sum_{k=0}^{(N-1)/2} a(k)\cos(\omega k)\right] \tag{2.8b}$$

where

$$a(0) = h\left(\frac{N-1}{2}\right) \text{ and } a(k) = 2h\left(\frac{N-1}{2} - k\right), \qquad k = 1, 2, \ldots, (N-1)/2$$

2.4.2 Type II FIR linear phase filter

N is even and the impuls response is symmetric. $\beta = 0$

$$h(n) = h(N-1-n) \quad 0 \le n \le N-1 \tag{2.9a}$$

$$H\left(e^{j\omega}\right) = e^{-j\omega\frac{N-1}{2}} \left\{ \sum_{k=1}^{N/2} b(k)\cos\left[\omega\left(k-\frac{1}{2}\right)\right]\right\} \tag{2.9b}$$

where

$$b(k) = 2h[N/2-k], \qquad k = 1,2,\ldots,N/2$$

2.4.3 Type III FIR linear phase filter

N is odd and the impuls response is antisymmetric. $\beta = \pi/2$

$$h(n) = -h(N-1-n) \quad 0 \le n \le N-1 \tag{2.10a}$$

$$H\left(e^{j\omega}\right) = je^{-j\omega\frac{N-1}{2}} \left[\sum_{k=1}^{(N-1)/2} c(k)\sin(\omega k)\right] \tag{2.10b}$$

where

$$c(k) = 2h[(N-1)/2-k], \qquad k = 1,2,\ldots,(N-1)/2$$

2.4.4 Type IV FIR linear phase filter

N is even and the impuls response is antisymmetric. $\beta = \pi/2$

$$h(n) = -h(N-1-n) \quad 0 \le n \le N-1 \tag{2.11a}$$

$$H\left(e^{j\omega}\right) = je^{-j\omega\frac{N-1}{2}} \left\{ \sum_{k=1}^{N/2} d(k)\sin\left[\omega\left(k-\frac{1}{2}\right)\right]\right\} \tag{2.11b}$$

where

$$d(k) = 2h[N/2-k], \qquad k = 1,2,\ldots,N/2$$

2.5 FIR filter design

2.5.1 FIR filter type choice

Not every FIR filter type fits on every filter.

Type	Low Pass	High Pass	Band Pass	Band Stop
I				
II		Not suitable		Not suitable
III	Not suitable	Not suitable		Not suitable
IV	Not suitable			Not suitable

2.5.2 FIR filter design by Windowing

Most idealized systems are non causal and have infinite impulse responses. To achieve finite impulse responses and causality, truncating the ideal response is the most straightforward approach.
For the ideal impulse response

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d\left(e^{j\omega}\right) e^{j\omega n} d\omega \tag{2.12}$$

the truncated response would be

$$h(n) = \begin{cases} h_d(n) & ,0 \leq n \leq N-1 \\ 0 & , \text{otherwise} \end{cases} \tag{2.13a}$$

or

$$h(n) = h_d(n) \cdot w(n) \tag{2.13b}$$

with

$$w(n) = \begin{cases} 1 & ,0 \leq n \leq N-1 \\ 0 & , \text{otherwise} \end{cases}$$

Properties of a rectangular Window

If the window-funciton is a rectangle, the frequency response of the filter has to be multiplied with the fourier transform of the rectangle:

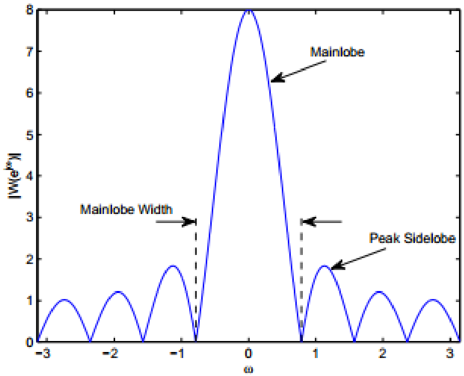
$$H\left(e^{j\omega}\right) = H_d\left(e^{j\omega}\right) \circledast W\left(e^{j\omega}\right) \tag{2.14a}$$

$$= H_d\left(e^{j\omega}\right) \circledast \left[e^{-j\omega\frac{N-1}{2}} \cdot \frac{\sin(\omega N/2)}{\omega/2} \right] \tag{2.14b}$$

This, if $H_d\left(e^{j\omega}\right)$ should be closest to $H\left(e^{j\omega}\right)$, N has to go to infinity

Main and Sidelobes

With increasing N, the 'main-lobe' width increases.



Peak approxomation error (PAE)

$$PAE = \gamma \cdot \max_{\omega\in\mathbb{R}} \left| H_d\left(e^{j\omega}\right) - H\left(e^{j\omega}\right) \right| \tag{2.15}$$

The PAE is also directly related to the maximum allowable specification tolerance. With ω_c as the discontinuity of $H_d\left(e^{j\omega}\right)$:

$$\Delta H_d\left(e^{j\omega_c}\right) \cdot \gamma \leq \min(\alpha_1, \alpha_2) \tag{2.16}$$

Rectangular

$$w(n) = \begin{cases} 1 & ,0 \leq n \leq N-1 \\ 0 & , \text{otherwise} \end{cases} \tag{2.17a}$$

Barlett

$$w(n) = \begin{cases} \frac{2n}{N-1} & ,0 \leq n \leq \frac{N-1}{2} \\ 2-\frac{2n}{N-1} & ,\frac{N-1}{2} \leq n \leq N-1 \end{cases} \tag{2.17b}$$

Hanning

$$w(n) = \frac{1}{2}\left[1-\cos\left(\frac{2\pi n}{N-1}\right)\right] \quad 0 \leq n \leq N-1 \tag{2.17c}$$

Hamming

$$w(n) = 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right) \quad 0 \leq n \leq N-1 \tag{2.17d}$$

Blackman

$$w(n) = 0.42 - 0.5\cos\left(\frac{2\pi n}{N-1}\right) + 0.08\cos\left(\frac{4\pi n}{N-1}\right) \quad 0 \leq n \leq N-1 \tag{2.17e}$$

These windows have different Properties:

Window Type	**Peak Amplitude of Side-Lobe (dB) (relative)**	**Approximate Width (rad) of Main-Lobe** $\Delta\omega$	**Peak Approximation Error (dB)** $20\log_{10}\gamma$
Rectangular	-13	$4\pi/N$	-21
Bartlett	-25	$8\pi/(N-1)$	-25
Hanning	-31	$8\pi/(N-1)$	-44
Hamming	-41	$8\pi/(N-1)$	-53
Blackman	-57	$12\pi/(N-1)$	-74

2.5.3 Kaiser Window Filter Design

The Kaiser window function adapts to the specifications. Let $\alpha = \frac{N-1}{2}$ and I_0 be the zeroth order modified Bessel function of the first kind:

$$w(n) = \begin{cases} \frac{I_0\left\{\beta\left[1-\left(\frac{n-\alpha}{\alpha}\right)^2\right]^{\frac{1}{2}}\right\}}{I_0(\beta)} & , 0 \leq n \leq N-1 \\ 0 & , \text{otherwise} \end{cases} \tag{2.18}$$

Desgin a filter with Kaiser window

Assuming $\alpha_1 = \alpha_2$ and the transition region is $\Delta\omega = \omega_s - \omega_p$. With $A = -20\log_{10}\alpha_1$

$$\beta = \begin{cases} 0.1102(A-8.7) & A > 50 \\ 0.5842(A-21)^{0.4} + 0.07866(A-21) & 21 \leq A \leq 50 \\ 0 & A < 21 \end{cases} \tag{2.19}$$

and

$$N = \left[\frac{A-8}{2.285\Delta\omega} + 1\right] \tag{2.20}$$

Integrated Square Error

To optimize a windowed filter, one has to attempt to minimize the integrated squared error

$$\epsilon^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|H_d\left(e^{j\omega}\right) - H\left(e^{j\omega}\right)\right|^2 d\omega \tag{2.21}$$

2.5.4 Optimal Filter Design

Optimal filter design does not use the integrated square error but instead the maximum error, which occurs at the discontinuities of $H_d\left(e^{j\omega}\right)$. This error gets distributed equally across the frequency band. Please use the script pages 33-36 for information about optimal filter design. ,

2.6 Infinite Impulse Response (IIR) filter

IIR filters are systems, which follow the following Restrictions:
$h(n)$ is

- real
- causal
- satifies stability

$h(n)$ possesses a rational z-transform with $a_0 = 1$, i.e.

$$H(z) = \frac{\sum_{k=0}^{N-1} b_k z^{-k}}{1 - \sum_{k=0}^{N-1} a_k z^{-k}} \tag{2.22}$$

2.6.1 Properties of IIR filters

- **Stability**
  All poles of $H(z)$ must lie inside the unit disk
- **Linear Phase**
  IIR filters do not have linear phase. Only the magnitude response is specified.

2.6.2 IIR filter design from an analog or continous-time filter

Steps to design:

- **Step 1**: Specification of the digital filter
- **Step 2**: Translation to a continuous-time filter specification
- **Step 3**: Determination of the continuous-time filter system function $H_c(s)$
- **Step 4**: Transformation of $H_c(s)$ to a digital filter system function $H(z)$

Step 1

Filter parameters like $\alpha_1, \alpha_2, \omega_p, \omega_s$ and $_d$ have to be set.

Step 2

If a Filter $H_c(\Omega)$ is given, applying the relation $\Omega = \omega/T_d$ leads to the continuous-time filter specifications, where T_d is the design sampling interval. Then the filter has to be checked to fulfil the Specifications given in Step 1

Step 4

Steps for the transformation of $H_c(s)$ to $H(z)$

1. From $H_c(s)$ and $H_c(j\Omega)$, we determin $h_c(t)$
2. Optain the digital sequence by $h(n) = T_d \cdot h_c(nT_d)$ where T_d is the design sampling interval
3. $H(z)$ is obtained from $h(n)$

2.6.3 Bilinear Transformation

The bilinear transformation is used to map the imaginary axis of the s-plane to the unit circle. Now let

$$s = \frac{2}{T_d}\left[\frac{1-z^{-1}}{1+z^{-1}}\right] \tag{2.23a}$$

which leads to

$$H(z) = H_c\left[\frac{2}{T_d}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right] \tag{2.23b}$$

with $z = e^{j\omega}$:

$$s = \frac{2}{T_d}\left[\frac{1-e^{-j\omega}}{1+e^{-j\omega}}\right] = j\frac{2}{T_d}\tan(\omega/2) \tag{2.23c}$$

with $s = \sigma + j\Omega$ follows:

$$\Omega = \frac{2}{T_d}\tan(\omega/2) \text{ and } \sigma = 0 \tag{2.23d}$$

Remarks on the bilinear transformation

- The frequency deformation $\Omega = \frac{2}{T_d}\tan(\omega/2)$ comes to a deviation in the detailed shape of $H_c(j\Omega)$
- The delay time is also modified: $\tau_d = \tau_c[1+((\omega_c T_d)/2)^2]$
- The bilinear transformation has no aliasing problems.
- This transformation is the most used in practise

2.6.4 Implementation of IIR Filters

The output of the filter can be described as

$$y(n) = \sum_{k=1}^{M} a_k y(n-k) + \sum_{r=0}^{N-1} b_r x(n-r) \tag{2.24}$$

Direct Form I

A Signal flow graph can be seen at Figure 5.8 on page 47 in the DSP script.

$$H(z) = H_2(z)H_1(z) = \left(\frac{1}{\sum_{k=1}^{M} a_k z^{-k}}\right)\left(\sum_{r=0}^{N-1} b_r z^{-r}\right) \tag{2.25}$$

Direct Form II

The direct form II can change the order of $H_1(z)$ and $H_2(z)$. A Signal flow graph can be seen at Figure 5.9 on page 48 in the DSP script.

$$H_1(z)H_2(z) = H(z) = H_2(z)H_1(z) \tag{2.26}$$

Cascade Form

A Signal flow graph can be seen at Figure 5.10 on page 48 in the DSP script.

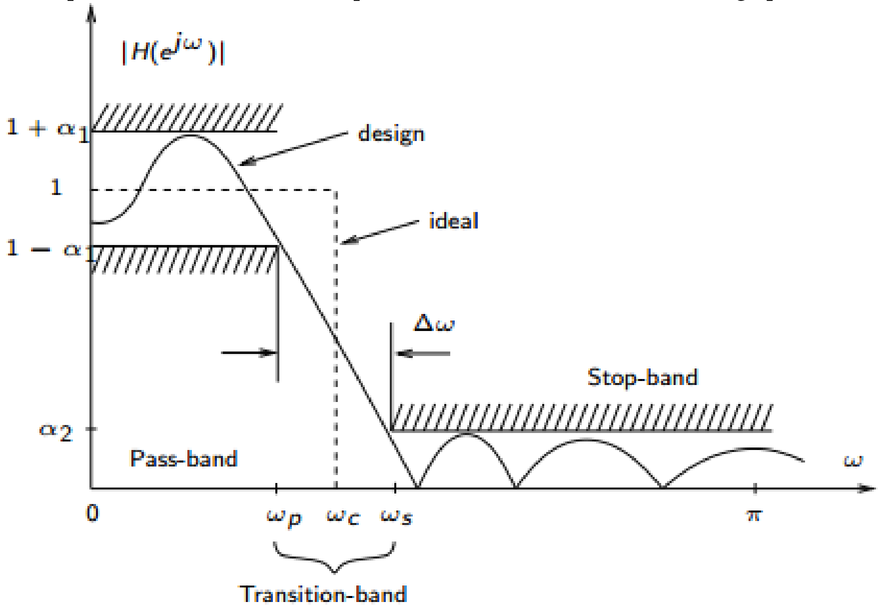
$$H(z) = \frac{\sum_{r=0}^{N-1} b_r z^{-r}}{1 - \sum_{k=1}^{M} a_k z^{-k}} = A \prod_k H_k(z) \tag{2.27a}$$

with

$$H_k(z) = \frac{1 + \beta_{1k} z^{-1} + \beta_{2k} z^{-2}}{1 - \alpha_{1k} z^{-1} - \alpha_{2k} z^{-2}} \tag{2.27b}$$

2.7 Filter Specficaion

Non-ideal Filters have one or more pass-bands(ω_p), transition-bands(between ω_p and ω_s) and stop-bands(ω_s). For the pass- and the stop- band, a tolerance α_i has to be specified. Therefore a low-pass filter would have the following specification:



3 Random Variables and Stochastic Processes

This is a revision of the Stochastic Signals and Systems course, which has an own Formulary which can be found at: `https://github.com/Tyde/stosigsysfs/blob/master/document.pdf?raw=true`

4 The Finite Fourier Transform

4.1 Definition

Let $x(0), \ldots, x(N-1)$ be realisiations of a stationary random process $X(n), n = 0, 1, 2, \ldots, N-1$

$$X_N\left(e^{j\omega}\right) = \sum_{n=0}^{N-1} X(n)e^{-j\omega n}, \quad -\infty < \omega\infty \tag{4.1}$$

and inverse:

$$X(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_N\left(e^{j\omega}\right)e^{j\omega n}d\omega, \quad n = 0, 1, \ldots, N-1 \tag{4.2}$$

4.1.1 Properties

- **Periodicity** $X_N\left(e^{j(\omega+k2\pi}\right) = X_N\left(e^{j\omega}\right) \quad \forall k \in \mathbb{N}$
- **Symmetry** $X_N\left(e^{-j\omega}\right) = X_N\left(e^{j\omega}\right)^* \quad \forall X(n) \in \mathbb{R}$
- **Linearity** $\mathscr{F}\{aX(n) + bY(n)\} = aX_N\left(e^{j\omega}\right) + bY_N\left(e^{j\omega}\right)$

4.1.2 Discrete Finite Fourier Transform

With $\omega_k = 2\pi k/N, k = 0, \ldots, N-1$

$$X_N(e^{j\frac{2\pi k}{N}}) = \sum_{n=0}^{N-1} X(n)e^{-j2\pi kn/N} \quad, k = 0, 1, \ldots, N-1 \tag{4.3}$$

4.1.3 Statistical Properties

White Gaussian Process

Let $X(0),\dots,X(N-1)$ be real valued random variables with $X(n) \sim \mathcal{N}(0,1)$ and $N = 2^r$ with $r \in \mathbb{N}$ Then:

$$\mathfrak{Re}\left\{X_N(e^{j2\pi k/N})\right\} = \sum_{n=0}^{N-1} X(N)\cos(2\pi kn/N) \tag{4.4a}$$

$$\mathfrak{Im}\left\{X_N(e^{j2\pi k/N})\right\} = -\sum_{n=0}^{N-1} X(N)\sin(2\pi kn/N) \tag{4.4b}$$

The Mean and Variance can be determined to:

$$\begin{bmatrix} \mathfrak{Re}\left\{X_N(e^{j2\pi k/N})\right\} \\ \mathfrak{Im}\left\{X_N(e^{j2\pi k/N})\right\} \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \frac{N}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \tag{4.4c}$$

Real Stationary Random Process

If the process has an arbitray $\mu_x, c_{XX}(\kappa)$ and $C_{XX}\left(e^{j\omega}\right)$ for $N \to \infty$ the asymptotic distribution is:

$$\begin{bmatrix} \mathfrak{Re}\left\{X_N\left(e^{j\omega}\right)\right\} \\ \mathfrak{Im}\left\{X_N\left(e^{j\omega}\right)\right\} \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \frac{N}{2}C_{XX}\left(e^{j\omega}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \quad \omega \in (0,\pi) \tag{4.5a}$$

For $\omega = 0$ and $\omega = \pi$ The Finite Fourier Transform is purely real:

$$\mathfrak{Re}\left\{X_N\left(e^{j\omega}\right)\right\} \sim \mathcal{N}\left(N\mu_x\delta(\omega), NC_{XX}\left(e^{j\omega}\right)\right) \tag{4.5b}$$

For fixed frequencies $0 \leq \omega_{(1)} < \dots < \omega_{(M)} \leq \pi$ the random variables $X_N(e^{j\omega(1)}), X_N(e^{j\omega(2)}),\dots,X_N(e^{j\omega(M)})$ are asomptotically for $N \to \infty$ **indepently distributed**

4.1.4 Segmentation of the Finite Fourier Transform

Division of random process $X(n), n = 0,\dots,N-1$ into L segments of length M with $N = ML$ will yield a finite fourier transform for each segment l which has the same asymptotic distribution as for the non-segmented process

$$X_M(e^{j\omega}, l) = \sum_{n=0}^{M-1} X(n-(l-1)M)e^{-j\omega n}, \quad l = 1,\dots,L \tag{4.6}$$

The random variables $X_M(e^{j\omega}, l)$ arre for $l = 1,\dots,L$ asymptotically as $N \to \infty$ indepently distributed

4.1.5 Windowing of the process

5 Miscellaneous

5.1 Useful mathematical equations

5.1.1 Geometric series

$$\sum_{n=0}^{N-1} q^n = \frac{1-q^N}{1-q} \quad \text{for} \quad |q| < 1 \tag{5.1}$$

5.1.2 Conversion from Geometric series to trigonometric fraction

Let $\frac{1-q^N}{1-q}$ be with $q = e^{-j\omega}$

$$\frac{1-e^{-jN\omega}}{1-e^{-j\omega}} = e^{-j\omega\frac{N-1}{2}} \frac{\sin\left(\frac{N}{2}\omega\right)}{\sin\left(\frac{\omega}{2}\right)} \tag{5.2}$$