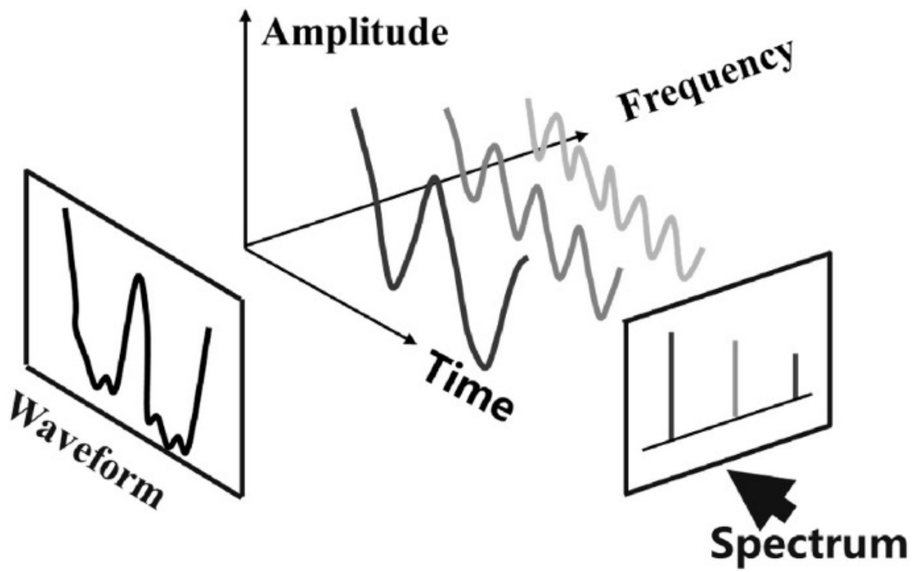


Frequency-domain Operators

Fall 2024

Yi-Ting Chen

1-D Linear Time Invariant System

Linear System

Additivity: $T\{x_1(t) + x_2(t)\} = T\{x_1(t)\} + T\{x_2(t)\} = y_1(t) + y_2(t)$

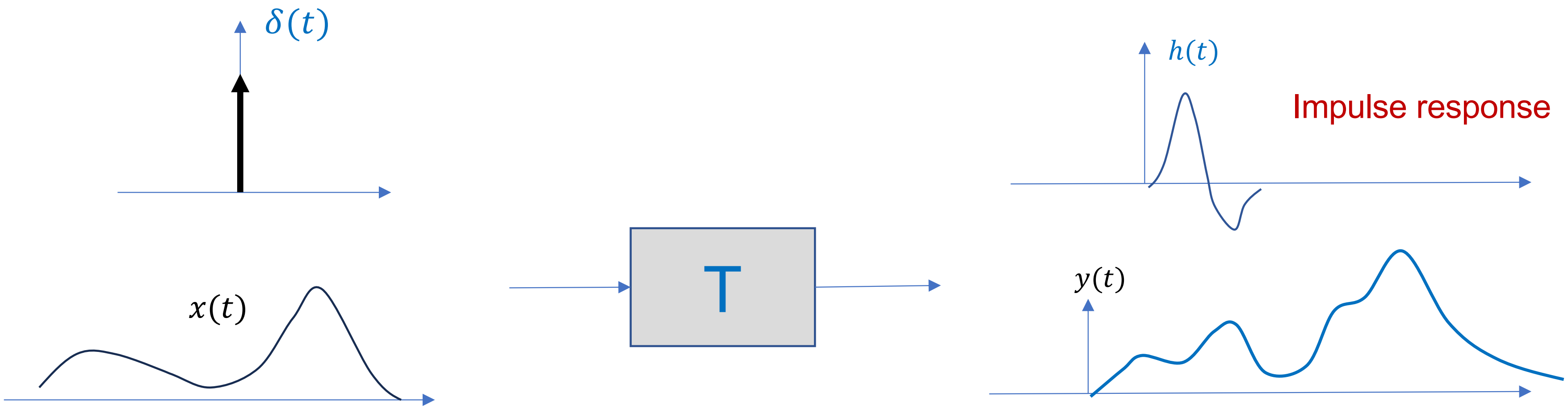
Homogeneity: $T\{ax(t)\} = aT\{x(t)\} = ay(t)$

Time-Invariant System

If $y(t) = T\{x(t)\}$, then $y(t - t_0) = T\{x(t - t_0)\}$.

Summary

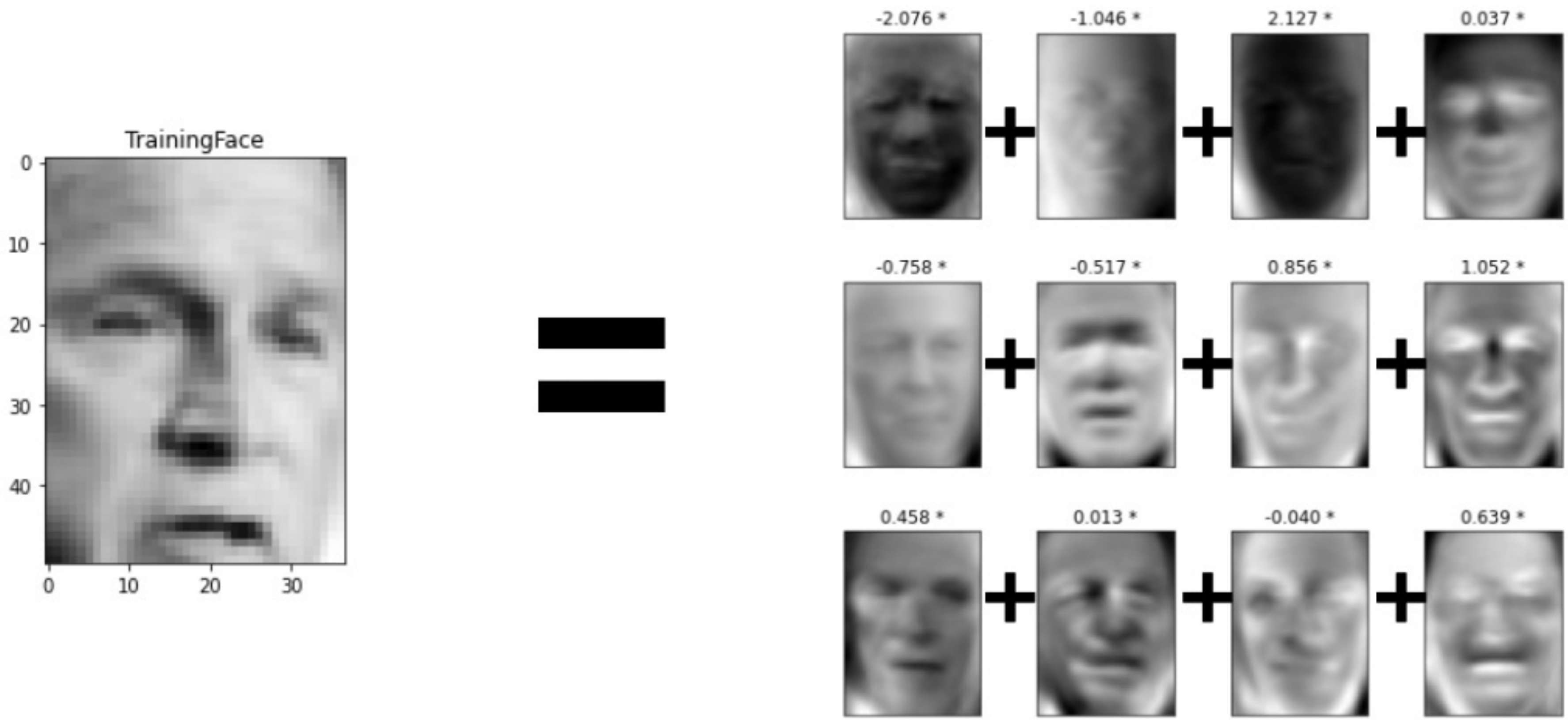
$$h(t) = T\{\delta(t)\}$$



$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

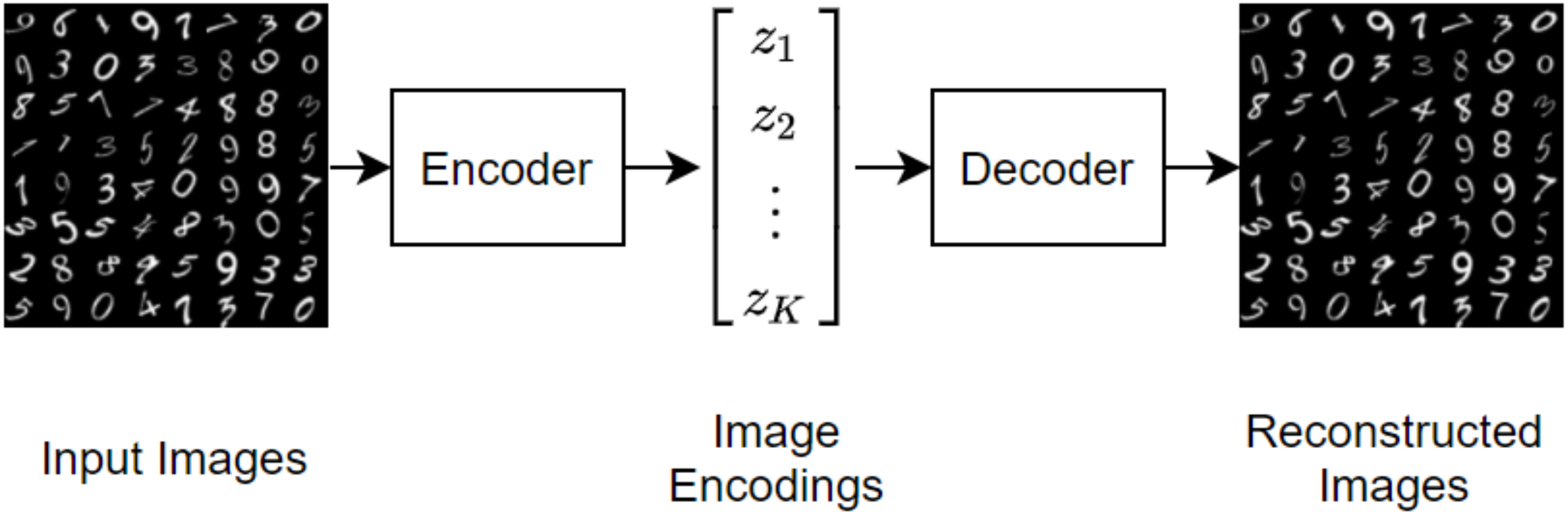
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$ Convolution

Eigenfaces



https://en.wikipedia.org/wiki/Eigenface#:~:text=An%20eigenface%20(%2F%CB%88a%C9%AA%C9%A1,Alex%20Pentland%20in%20face%20classification.

Variational Autoencoder (VAE)



Frequency-Domain Analysis (1/3)

$$e^{j2\pi ft} = \cos(2\pi ft) + j\sin(2\pi ft)$$

$$x(t) = e^{j2\pi ft} \longrightarrow \boxed{T} \longrightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$= \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$$

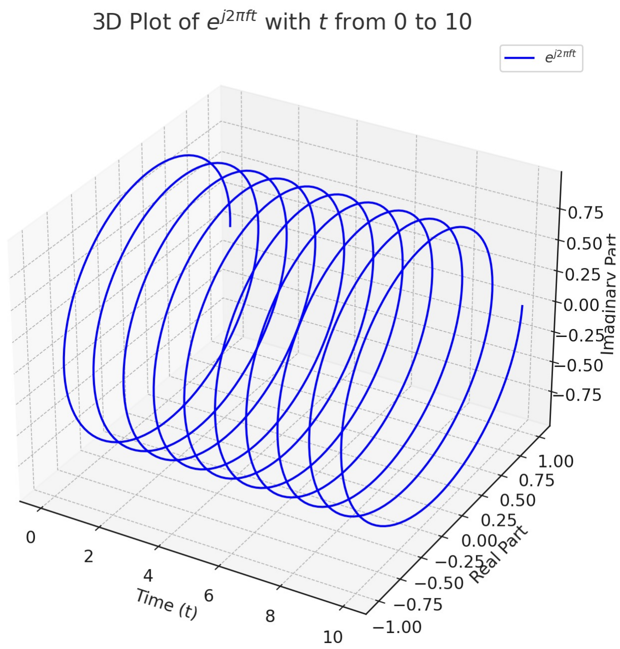
$$= \int_{-\infty}^{\infty} e^{j2\pi f(t-\tau)}h(\tau)d\tau$$

$$= e^{j2\pi ft}\int_{-\infty}^{\infty} e^{-j2\pi f\tau}h(\tau)d\tau$$

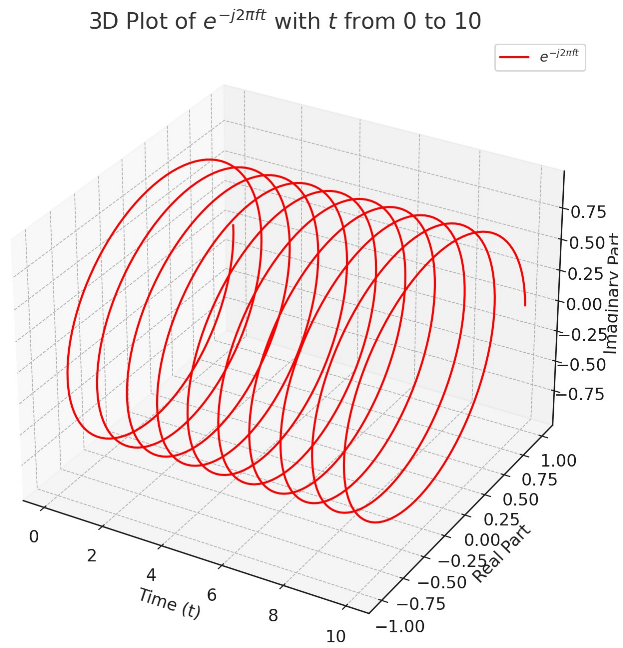
$$\equiv H(f)e^{j2\pi ft}$$

Frequency-Domain Analysis (1/3)

$$e^{j2\pi ft} = \cos(2\pi ft) + j\sin(2\pi ft)$$



$$e^{-j2\pi ft} = \cos(2\pi ft) - j\sin(2\pi ft)$$



Frequency-Domain Analysis (1/3)

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$
$$\equiv H(f)e^{j2\pi f t}$$

H(f): Frequency Response

This is a complex-valued function that characterizes how the system responds to a sinusoid of frequency f. It typically includes both a gain (magnitude) and a phase shift.

$$y(t) = |H(f)|e^{j(2\pi f t + \arg(H(f)))}$$

Can we use this charateristic to decompose our signal?

Frequency-Domain Analysis (2/3)

Fourier Transform Pair

spectrum $X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$ where $e^{-j2\pi ft} = \cos(2\pi ft) - j sin(2\pi ft)$

Signal Decomposition $x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$ where $e^{j2\pi ft} = \cos(2\pi ft) + j sin(2\pi ft)$

$$X(f) = Re\{X(f)\} + jIm\{X(f)\} = |X(f)| e^{j\angle X(f)}$$

$|X(f)|$ Magnitude

$\angle X(f)$ Phase

Frequency-Domain Analysis (3/3)

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df \longrightarrow \boxed{T} \longrightarrow y(t) = \int_{-\infty}^{\infty} X(f)T\{e^{j2\pi ft}\}df$$

$$= \int_{-\infty}^{\infty} X(f)H(f)e^{j2\pi ft}df$$

$$\Rightarrow \quad Y(f) = X(f)H(f)$$

2-D Linear Shift-Invariant System

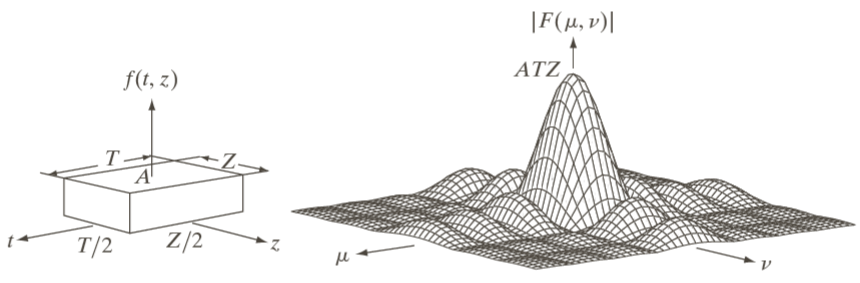
$$
\begin{tikzcd}
I(x,y) \arrow[r] \arrow[d, leftrightarrow, "\text{2-D FT}"] & \boxed{h(x,y)} \arrow[r] \arrow[d, leftrightarrow, "\text{2-D FT}"] & O(x,y) = h(x,y) * I(x,y) \arrow[d, leftrightarrow, "\text{2-D FT}"] \\
I(u,v) & H(u,v) & O(u,v) = H(u,v)I(u,v)
\end{tikzcd}
$$

h(x,y): Point Spread Function (PSF)
H(u,v): optical transfer function (OTF)

2-D Fourier Transform

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-j2\pi(ux+vy)} dxdy$$

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\, e^{j2\pi(ux+vy)} dudv$$

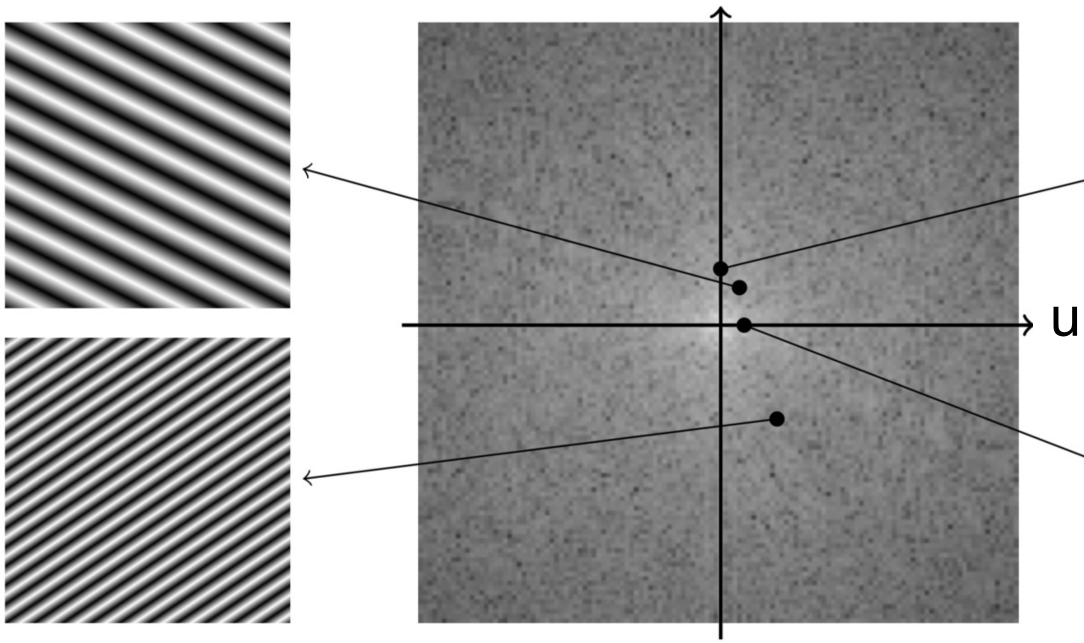


a b

FIGURE 4.13 (a) A 2-D function, and (b) a section of its spectrum (not to scale). The block is longer along the t-axis, so the spectrum is more "contracted" along the μ-axis. Compare with Fig. 4.4.

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\, e^{j2\pi(ux+vy)} du dv$$

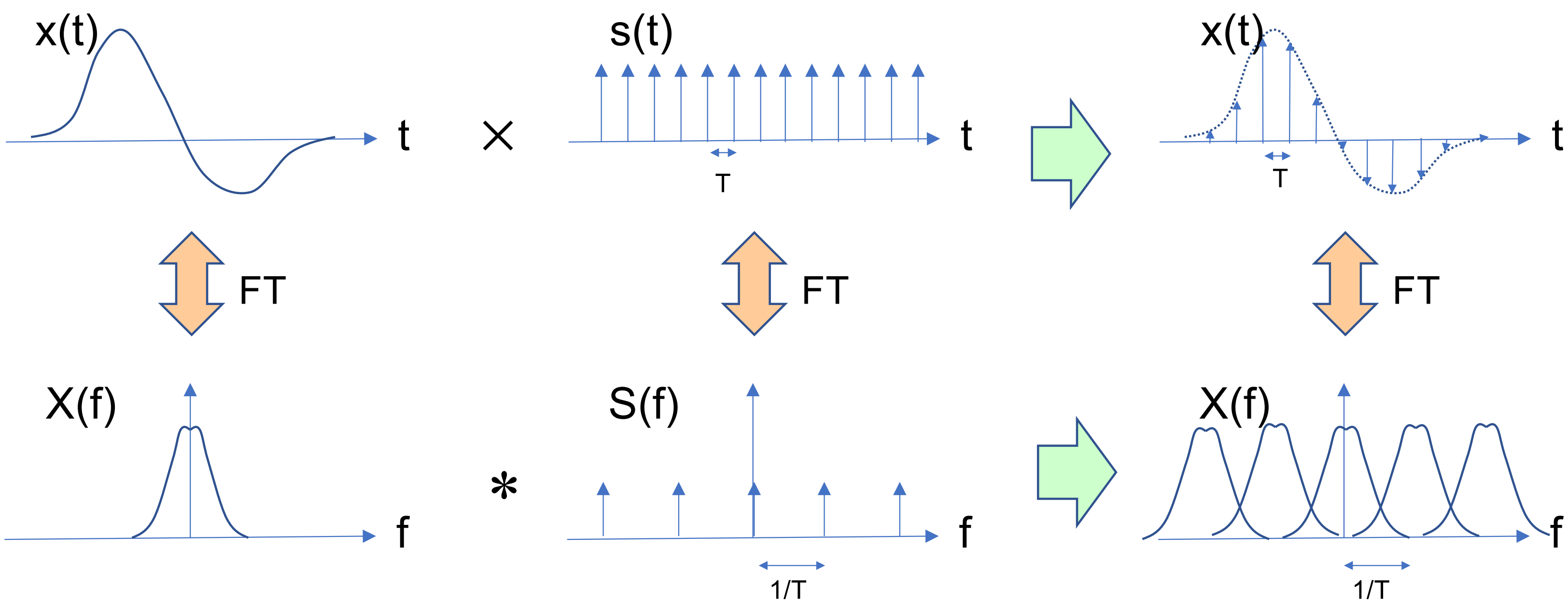
$F(u,v)$



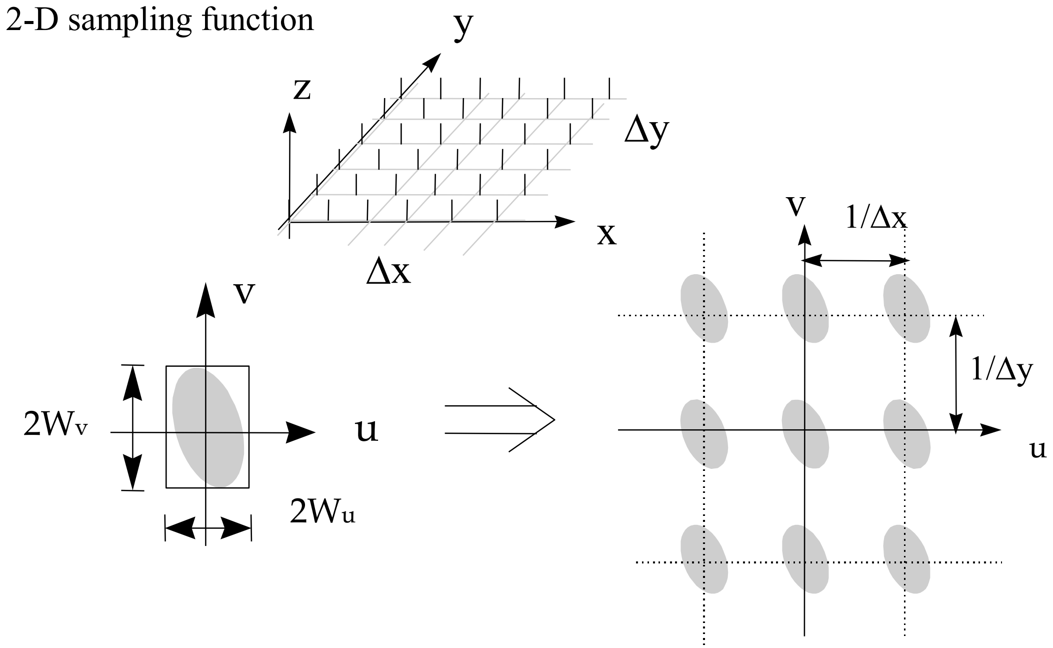
https://commons.wikimedia.org/wiki/File:2D_Fourier_Transform_and_Base_Images.png

Sampling

Whittaker-Shannon Sampling Theorem



2-D Sampling

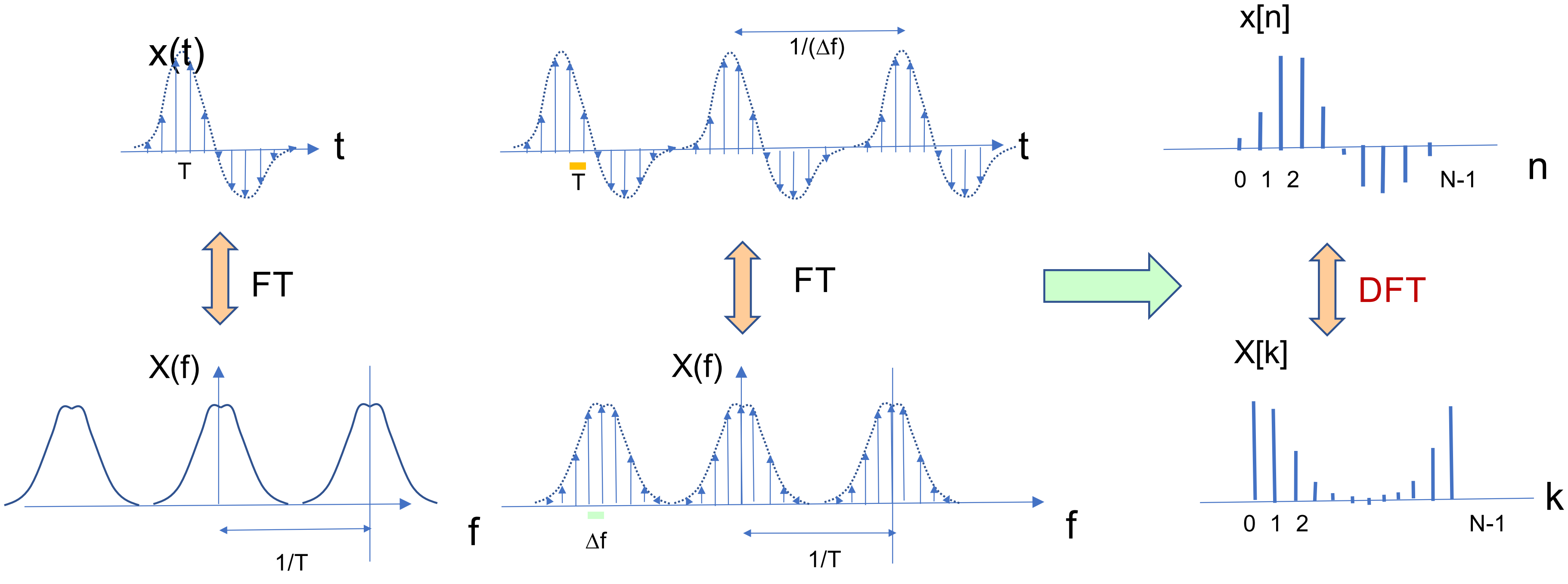


Aliasing

a b c

FIGURE 4.17 Illustration of aliasing on resampled images. (a) A digital image with negligible visual aliasing. (b) Result of resizing the image to 50% of its original size by pixel deletion. Aliasing is clearly visible. (c) Result of blurring the image in (a) with a 3×3 averaging filter prior to resizing. The image is slightly more blurred than (b), but aliasing is not longer objectionable. (Original image courtesy of the Signal Compression Laboratory, University of California, Santa Barbara.)

Discrete Fourier Transform (1/2)



Discrete Fourier Transform (2/2)

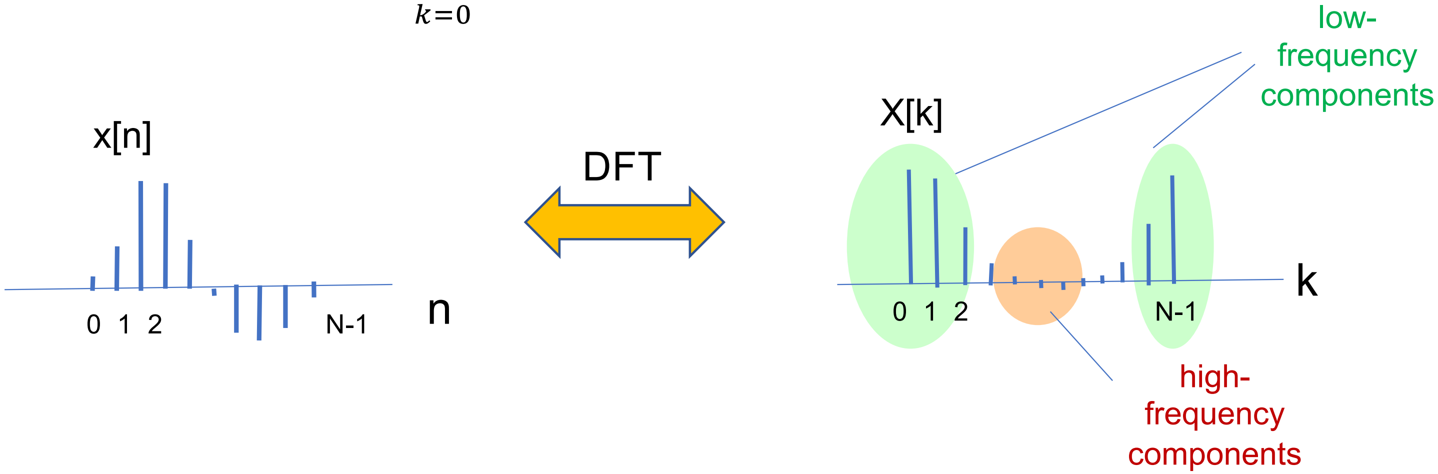
Fourier Transform Pair

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df$$

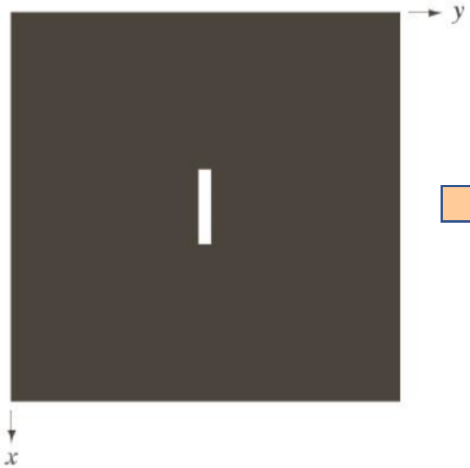
$$X[k] = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi kn}{N}} \qquad k = 0,1,2,..,N-1$$

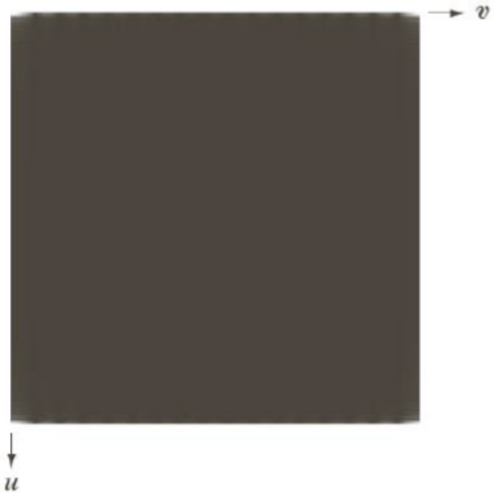
$$x[n] = \sum_{k=0}^{N-1} X[k] e^{j\frac{2\pi kn}{N}} \qquad n = 0,1,2,..,N-1$$

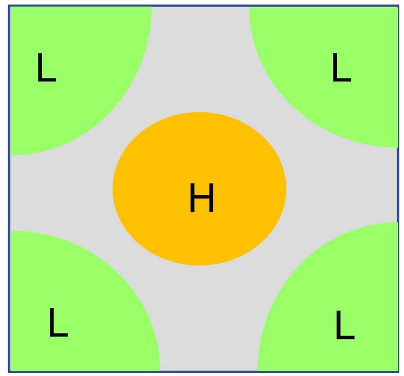


2-D DFT

$$\mathrm{F}[u,v] = \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f[m,n]\, e^{-j2\pi\left(\frac{um}{M}+\frac{vn}{N}\right)} \qquad \mathrm{f}[m,n] = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F[u,v]\, e^{j2\pi\left(\frac{um}{M}+\frac{vn}{N}\right)}$$



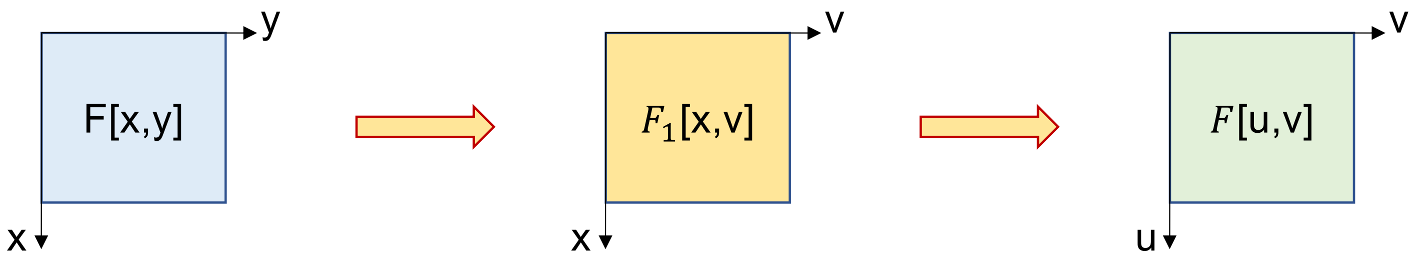




Properties of DFT - Separability

$$F(u,v) = \frac{1}{N^2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi\frac{ux+vy}{N}}$$

$$= \frac{1}{N}\sum_{x=0}^{N-1} e^{-j2\pi\frac{ux}{N}}\{\sum_{y=0}^{N-1}\frac{1}{N} f(x,y)e^{-j2\pi\frac{vy}{N}}\}$$



$$F[u,v] = \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f[m,n]\, e^{-j2\pi(\frac{um}{M}+\frac{vn}{N})}$$

$$f[m,n] = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F[u,v]\, e^{j2\pi(\frac{um}{M}+\frac{vn}{N})}$$

- **Average Value** $$\bar{f}[x,y] = \frac{1}{N^2}\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f[x,y] = F[0,0]$$

- **Periodicity**

$$f[x,y] = f[x+kN, y+lN]$$

$$F[u,v] = F[u+kN, v+lN]$$

$$k, l = 0,1,2,\ldots$$

- **Translation**

$$f[x-x_0, y-y_0] \Leftrightarrow F[u,v]e^{-j2\pi(\frac{ux_0+vy_0}{N})}$$

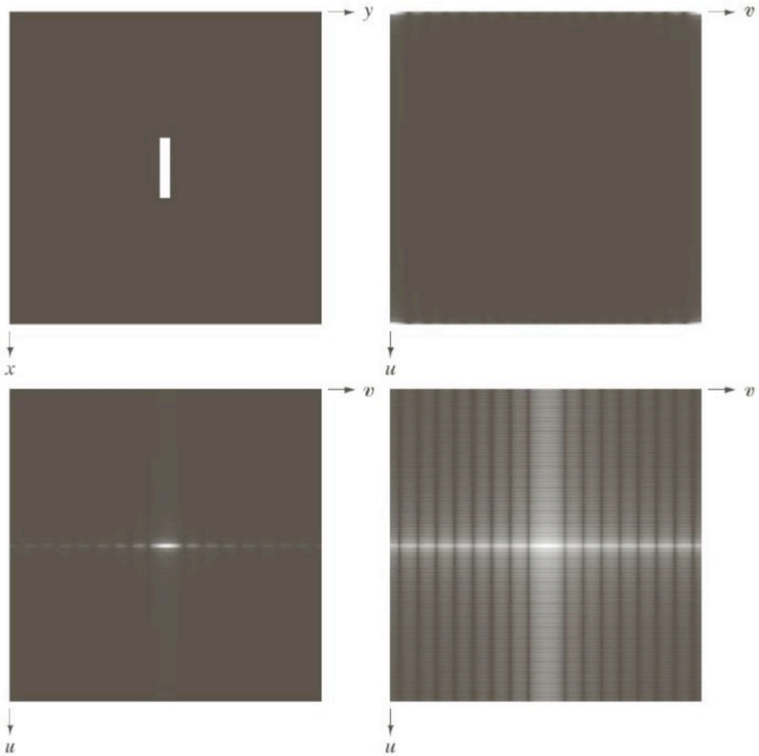
$$f[x,y]e^{j2\pi(\frac{u_0x+v_0y}{N})} \Leftrightarrow F[u-u_0, v-v_0]$$

If $u_0 = v_0 = N/2$

$$f[x, y](-1)^{(x+y)}$$

$$\Big\downarrow \text{DFT}$$

$$F[u - \frac{N}{2}, v - \frac{N}{2}]$$

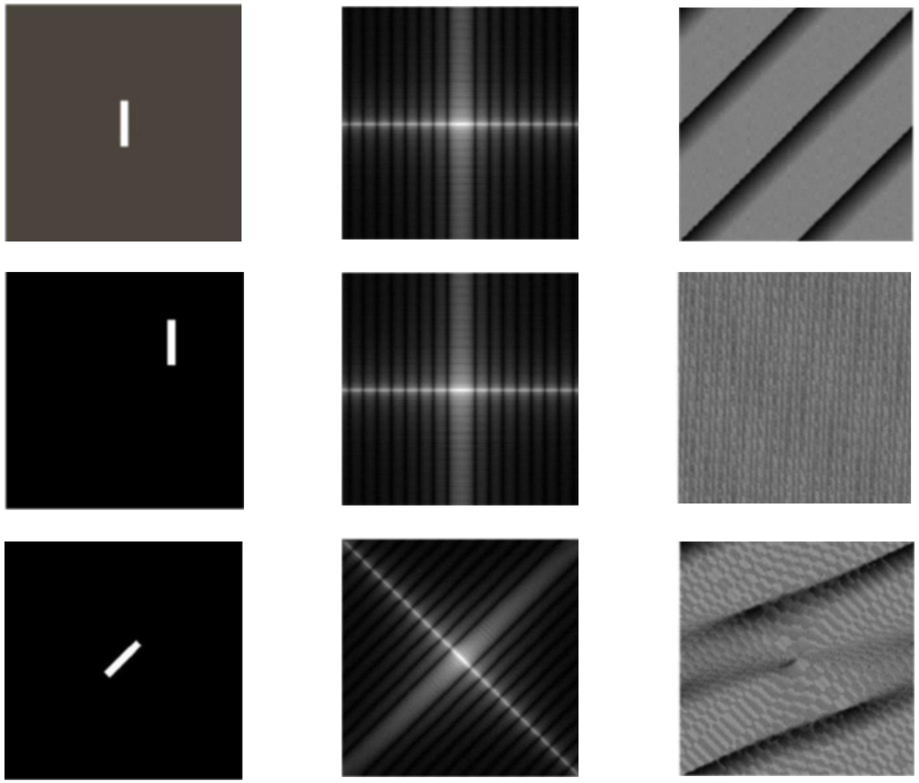


Log function

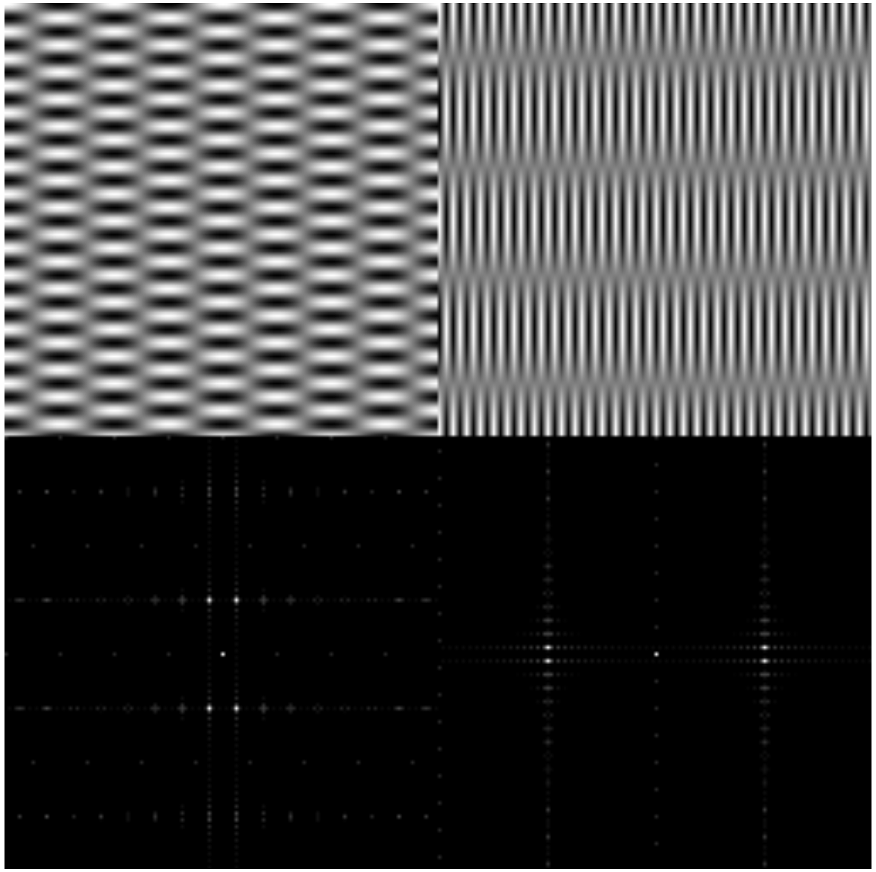
Ref: Gonzalez and Woods, “Digital Image Processing”

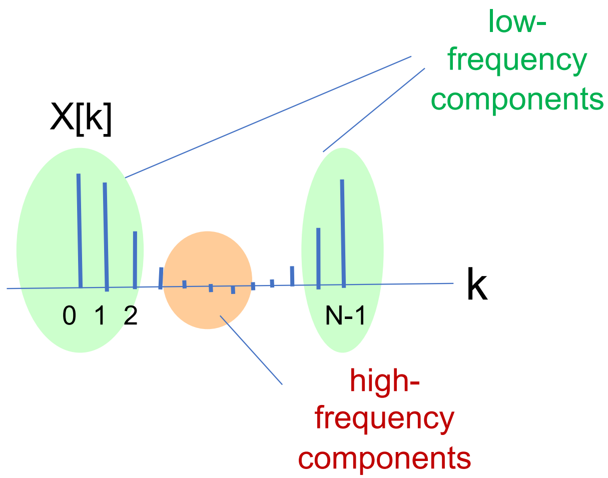
Rotation

In principle, $f[r, \theta + \theta_0] \leftrightarrow F[w, \phi + \theta_0]$

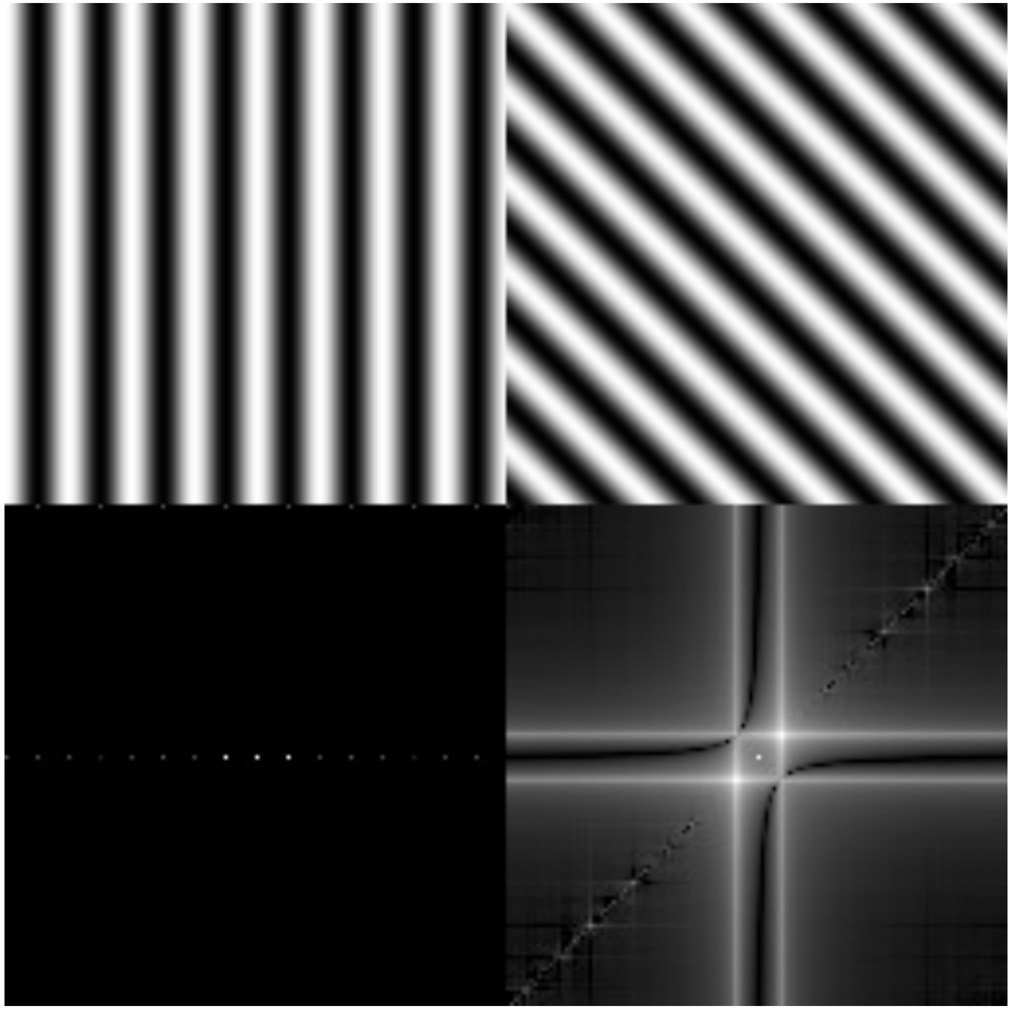


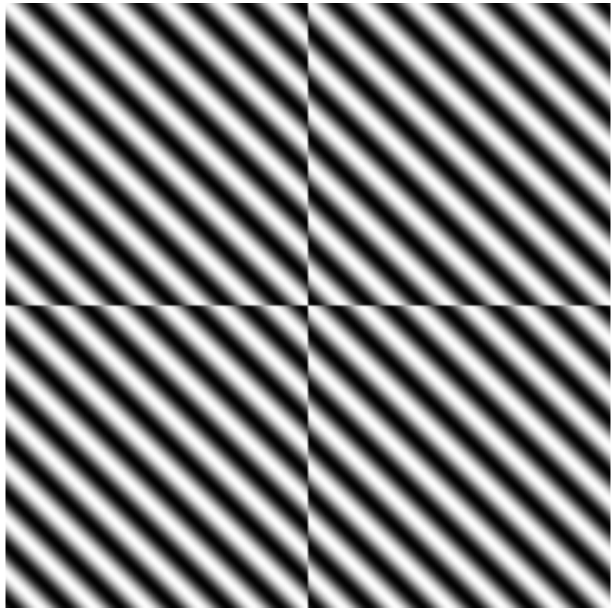
Example



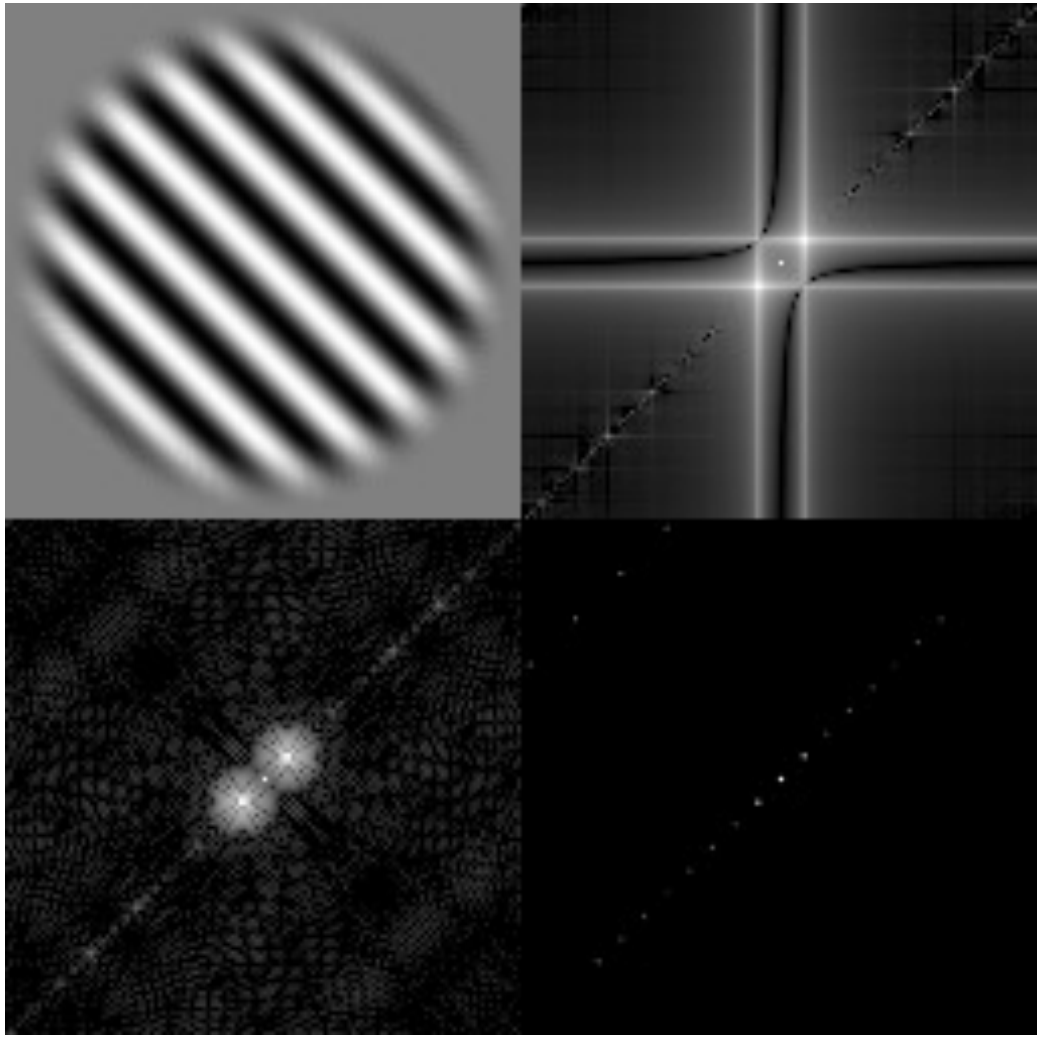


Ref: http://www.cs.unm.edu/~brayer/vision/fourier.html





Ref: http://www.cs.unm.edu/~brayer/vision/fourier.html



Ref: http://www.cs.unm.edu/~brayer/vision/fourier.html

Frequency-domain Filtering

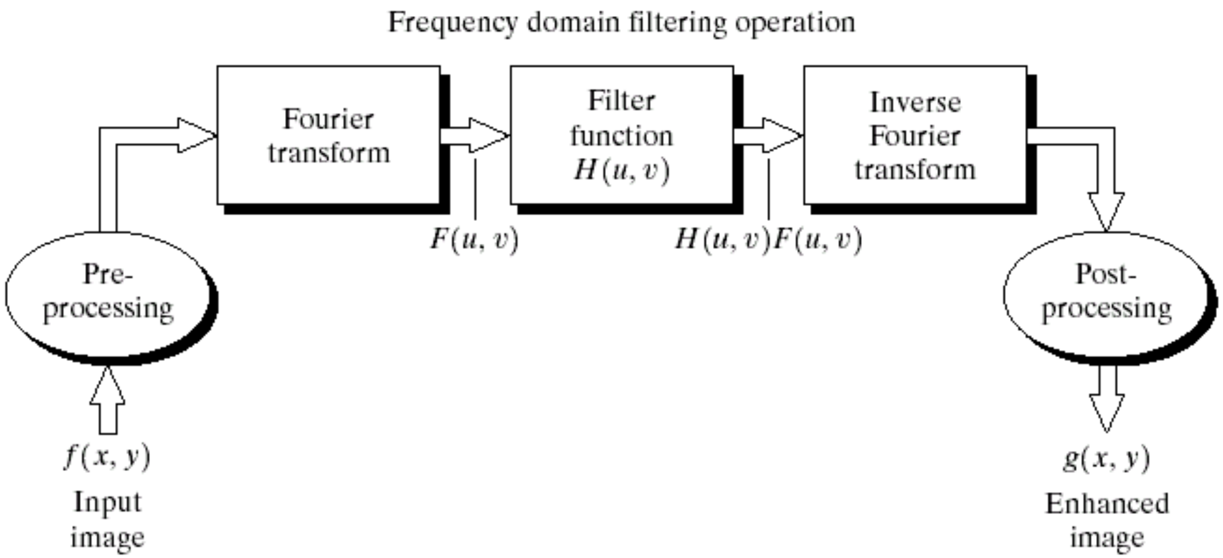
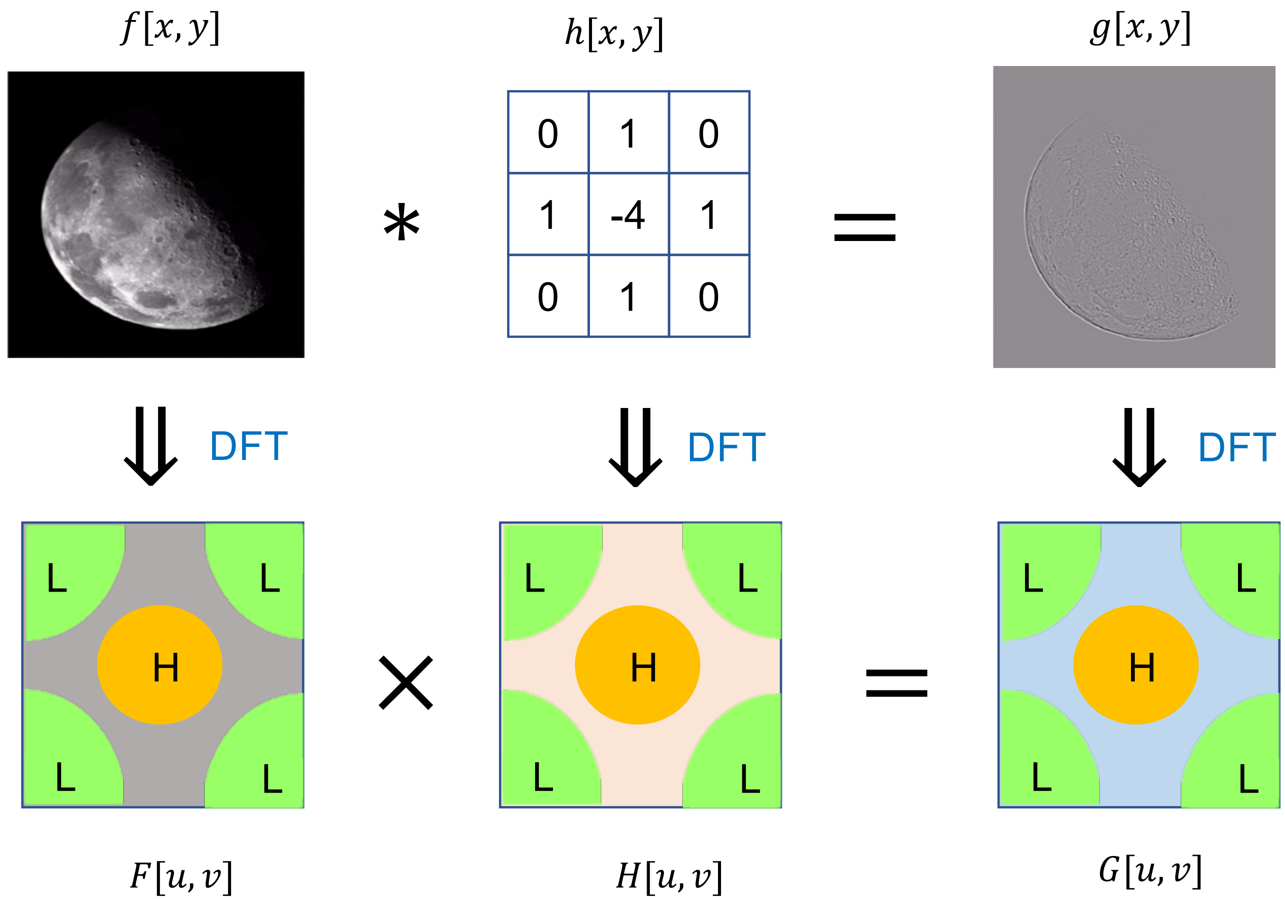
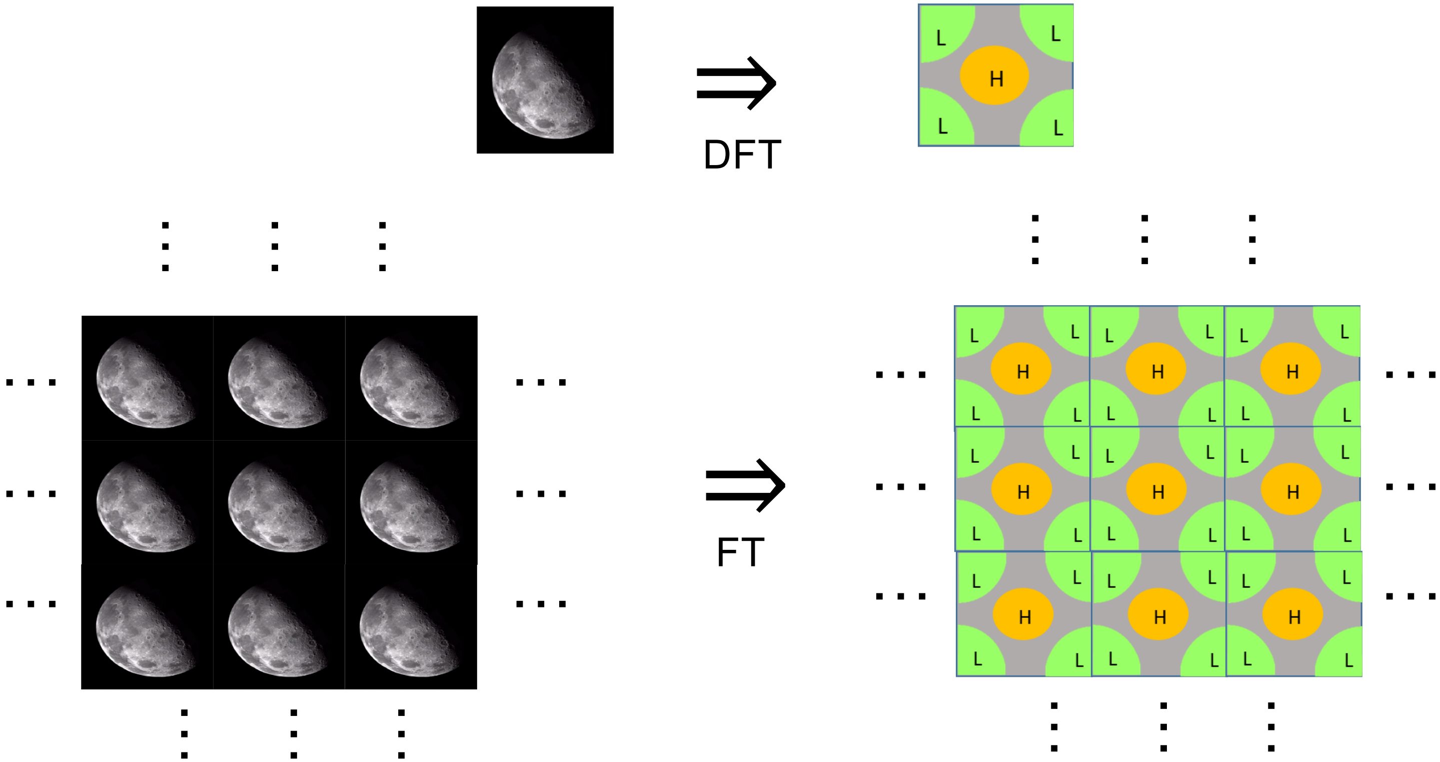


FIGURE 4.5 Basic steps for filtering in the frequency domain.

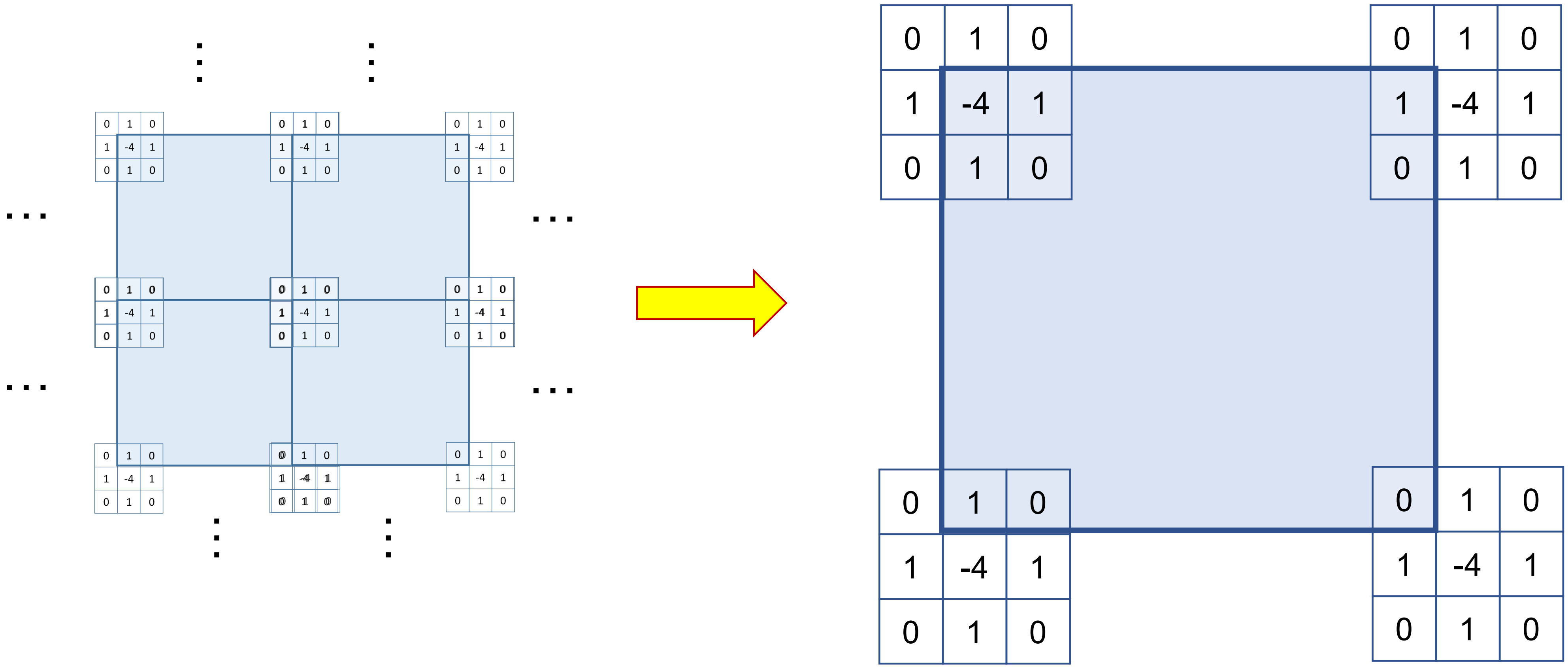
Linear Filtering in Frequency-domain (1/4)



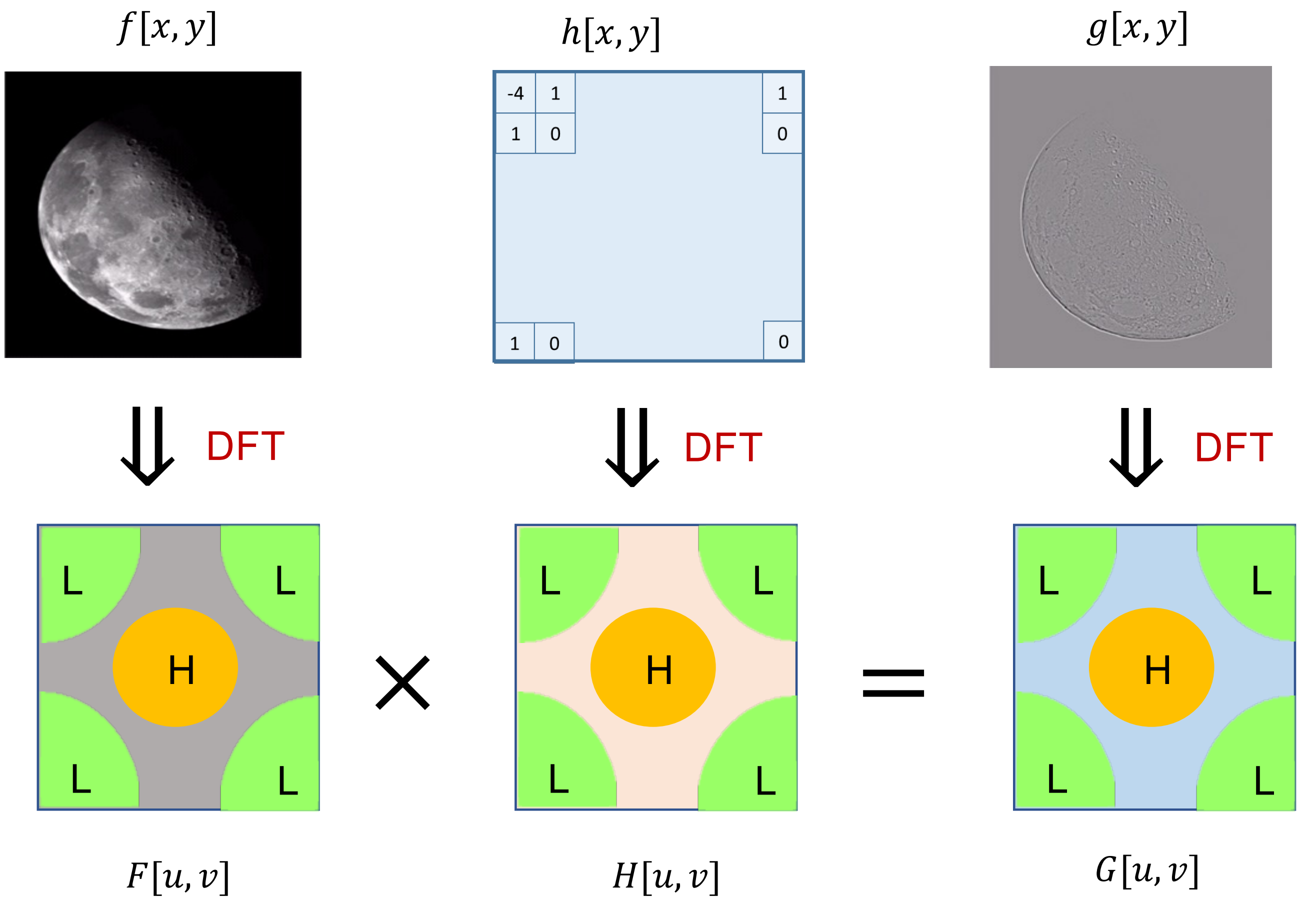
Linear Filtering in Frequency-domain (2/4)



Linear Filtering in Frequency-domain (3/4)



Linear Filtering in Frequency-domain (4/4)



$$\mathrm{F}[u,v] = \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f[m,n]\, e^{-j2\pi\left(\frac{um}{M}+\frac{vn}{N}\right)}$$

Next: Image Enhancement via Spatial-domain and Frequency-domain Operators