Numerical Research on Gaussian Prime Cycles

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A Gaussian integer is a complex number restricted to integer real and complex components. For example, $5 + 6i$ is a Gaussian integer, while $5.5 + \pi i$ is not. A Gaussian prime, for the purposes of this paper, is a Gaussian integer z such that one of the following is true [3]:

$$\text{im}(z) \text{ is prime and } \text{real}(z) = 0$$

$$\text{real}(z) \text{ is prime and } \text{im}(z) = 0$$

$$\sqrt{\text{real}(z)^2 + \text{im}(z)^2} \text{ is prime and } \text{real}(z) \neq 0 \text{ and } \text{im}(z) \neq 0$$

A Gaussian prime cycle, is a path formed by starting at an initial point (e.g. $(5, 7)$), we will call x_0, with an initial direction (eg $(1, 0)$), we will call d, and stepping right (that is $x_{n+1} = x_n + d$) until x_n is a Gaussian prime. If x_n is a Gaussian prime, we rotate d $90°$, and continue in that direction until another Gaussian prime is encountered; then we rotate again. This process continues until we return to our initial point and have our initial direction. The cycles are not proven to exist for all initial points, and as a new problem, it may be a while until the result is known. [1, 2]

For our purposes, an equivalence relation is a relation that partitions a set. Equivalence classes are sets of elements that are all equal under a given equivalence relation. It is a known property of equivalence relations that every element belongs to exactly one class. An example would be the natural numbers with the equivalence relation being equivalence modulus 2. If n, a natural number, is even, it belongs to the class of even numbers which are all equal mod 2. And vice versa if it is odd. Every number is either even or odd, so every number is in at least one class. No number is both even and odd, so no number can be in both classes. Thus every element is in exactly one class. We use this concept to define an equivalence relation on the Gaussian prime cycles. G, a Gaussian prime cycle, is considered equivalent to H, another Gaussian prime cycle, if G can be rotated or translated such that the are the same shape. More verbosely, they are equal if there exists a bijective translation-rotation transformation $T_M : G \to H$. *Aside: Should I prove T_M is an equivalence relation here?*

We plotted the number of classes in an initial point in a $l \times l$ grid as a function of l, with the grid being from $(0, 0)$ to $(l-1, l-1)$ inclusively. Stated more simply: the number of unique shapes starting in the square and the size

of the square. This allows us to see what the likely trend is as to the rate of discovering new shapes as one searches farther from $(0, 0)$.

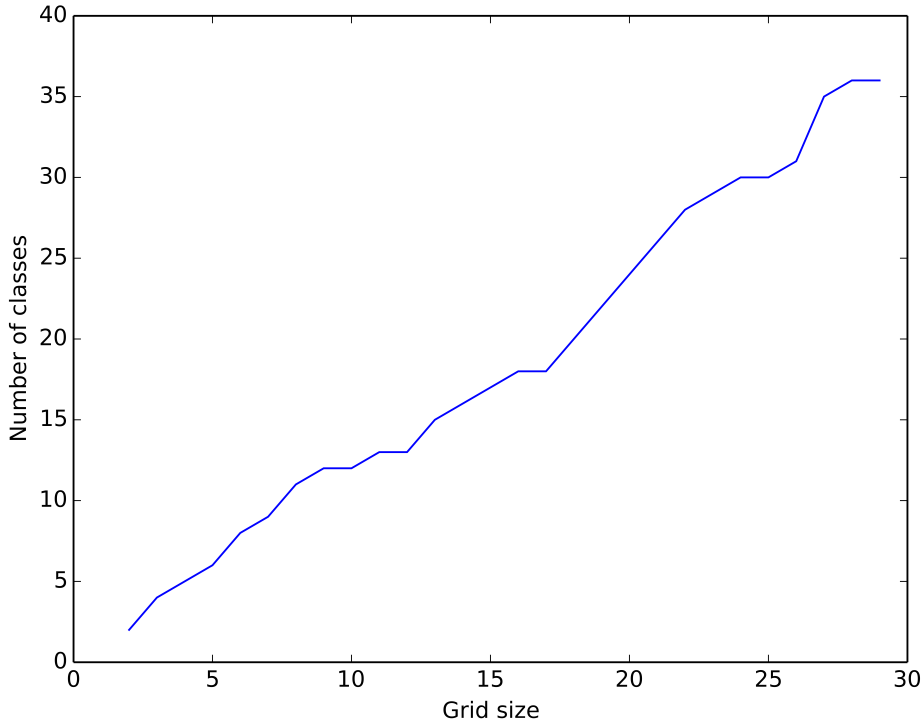
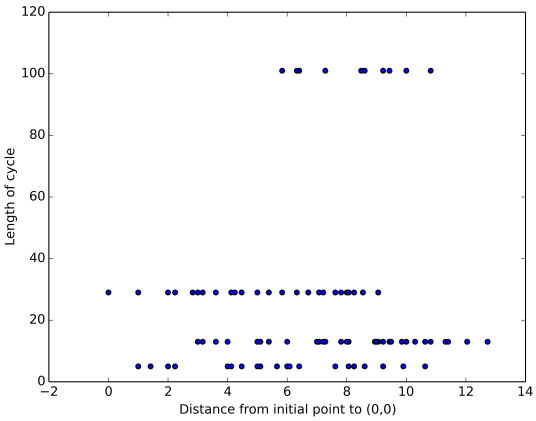
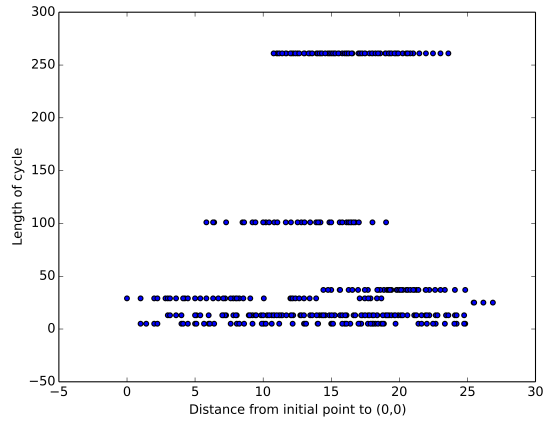


Figure 1: Number of classes as a function of grid size. Interestingly, the relation appears to be approximately linear.

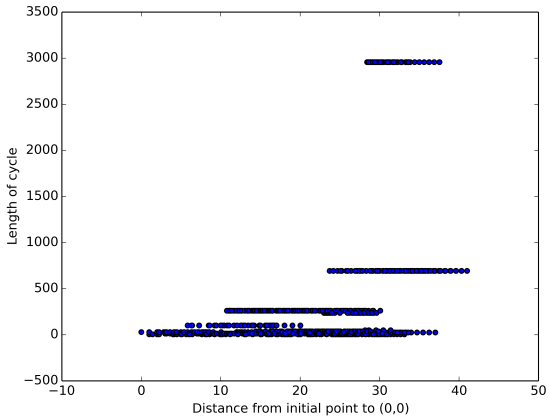
We also plotted the length of a cycle as a (multi-valued) function of the distance from $(0, 0)$. A trend in this would indicate that the farther one is from 0, the longer the cycle one is likely to find. This was accomplished by plotting all cycles in a given grid for multiple grid sizes. Interestingly, the grouping and relative shape of the scatter plot is similar for many sizes of grid, even though the absolute difference is large. This may imply a golden-ratio-like relation between the sizes of paths when the sizes are sorted into a monotonically increasing sequence. This may be so, because the Fibonacci sequence, whose consecutive ratios approximate the golden ratio, exhibits a similar scale-invariance.



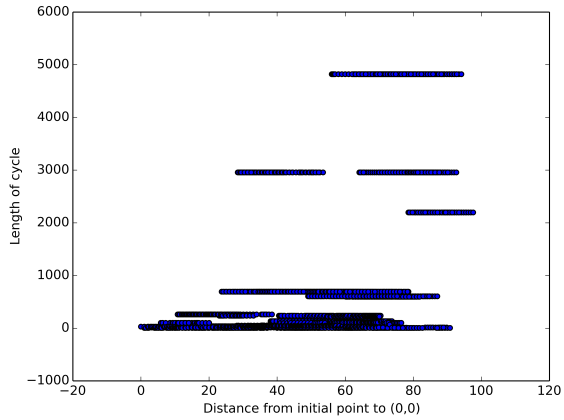
(a) All paths in grid of size 10.



(b) All paths in grid of size 20.



(c) All paths in grid of size 30.



(d) All paths in grid of size 70.

Figure 2: Number of classes as a function of grid size. Notice how, in both, there are essentially three clusters.

Finally, we graphed a 2D scatter plot colored with each point (x_i, y_j) colored by the class to which the cycle with initial point (x_i, y_j) belongs. Unfortunately, there are no obvious patterns to the graph, so there isn't much to discuss in terms of the possible implications of this visualization.

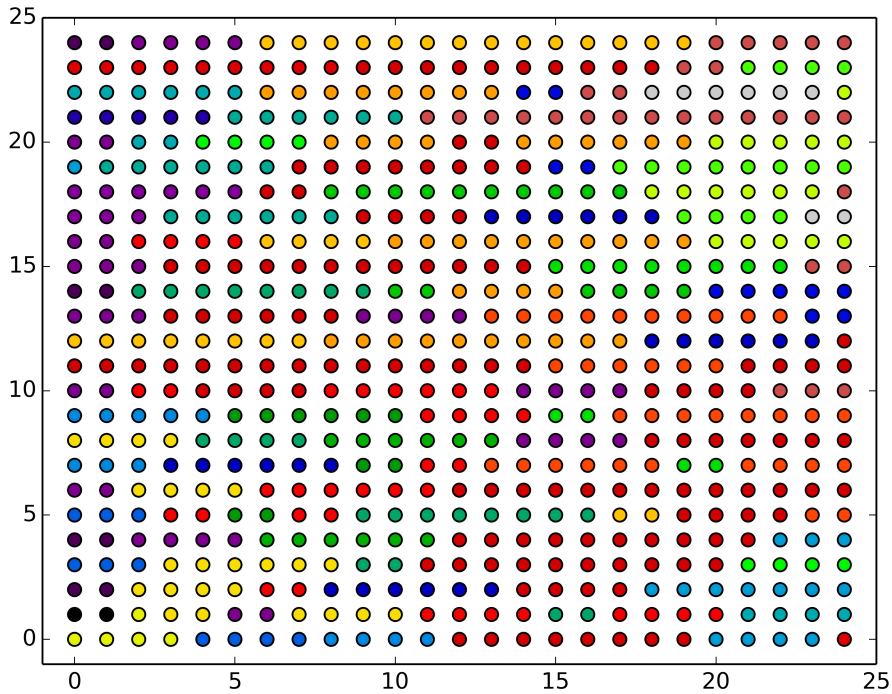


Figure 3: Points colored by the class to which they (i.e. their path) belong.

Unfortunately, due to the deep nature of the Gaussian prime cycle problem with its relation to primes and recursive processes, there isn't much we can recommend in terms of the further, more theoretical research based on our results. It could be interesting to try to prove the correlations we describe, but likely well out of reach. In terms of further numerical research, the (possible) correlation visualized the first graph would benefit more from parallelization and more computational power than the second would, due to the algorithmic complexity of the equivalence class enumeration code.

References

[1] Joseph O'Rourke. Gaussian prime spirals, Mar 2012. [Online; accessed 10-November-2014].

[2] Joseph O'Rourke and Stan Wagon. Gaussian prime spirals, Apr 2012. [Online; accessed 10-November-2014].

[3] John Stillwell. *Elements of Number Theory*. Springer, 2003. [Online; accessed 10-November-2014].