Instructions

(1) This assignment is due on Friday November 11th.
(2) Please submit your written solutions to crowdmark with each problem started on a separate page.
(3) Please list your collaborators on your assignment. It's important to give credit to those you have worked with.

Question 1 (Categorizing Differential Equations)**.** State the order, independent variable, state variable, and parameters (if there are any) for each differential equation. Determine whether or not the differential equation is linear.

a.

$$\frac{d^3y}{dt^3} = \left(\frac{dy}{dt}\right)^2 + t^3 + y$$

b.

$$x^2\frac{d^2u}{dx^2} + bx\frac{du}{dx} + au = \ln(x)$$

c.

$$\sqrt{y'(x) + my(x)} = \sqrt{x^3}$$

Question 2 (Solutions to Differential Equations)**.** Anna drops a ball of mass m off of a 200m high cliff. Due to air resistance, the height h of the ball is given by the following differential equation

$$m\frac{d^2h}{dt} = k\left(\frac{dh}{dt}\right)^2 - mg \qquad h'(0) = 0,\ h(0) = 200. \tag{1}$$

Show that

$$h(t) = 200 - \frac{m}{k}\ln\left(\cosh\left(\sqrt{\frac{kg}{m}}t\right)\right),$$

is a solution to this initial value problem, where $\cosh(x) = \frac{e^{-x}+e^{x}}{2}$.

Question 3 (First Order Linear Differential Equations)**.** Find a solution to each initial value problem

a.

$$\frac{dy}{dt} + \frac{1}{t}y = 2e^{-t}, \qquad y(1) = 1$$

b.

$$\sin(x)\frac{du}{dx} + \cos(x)u = x\cos(x), \qquad u(\pi/2) = 1$$

c.

$$\frac{dy}{dt} + 2ty = e^{-t^2}, \qquad y(0) = 0$$

Question 4 (First Order Separable Differential Equations)**.** Find a one-parameter family of solutions for each differential equation.

a.
$$\frac{dy}{dt} = \frac{(ty^2 + y^2 + t + 1)}{y}$$

b.
$$\frac{dy}{dt} = 2e^{t+y}$$

c.
$$\frac{du}{dt} = -u^3$$

Question 5 (First Order Autonomous Differential Equations)**.** Bioreactors use bacteria to convert complex sugars into ethanol and other biofuels. Due to how the bacteria grow on the sugars in the reactor, the concentration of sugar can be modeled by the differential equation

$$\frac{dS}{dt} = C - S - \frac{\mu S}{K + S}, \tag{2}$$

where $S(t)$ is the concentration of sugar at time t, C is the input concentration of sugar, μ is the growth rate of the bacteria, and K is a parameter called the half-saturation constant. For simplicity assume that $K = 2C$, $\mu = C$, and $C > 0$.

a. Find the positive equilibrium solutions and determine their stability.
b. Draw a phase-diagram for this differential equation.
c. According to this model, is it possible for the concentration of sugar to become negative? Explain why or why not.