Queen's University
Faculty of Arts and Science
Department of Mathematics and Statistics

MTHE 224 - T. Meadows
December 8th, 2023

HAND IN
Answers recorded
on exam paper

- No aids other than your calculator (Casio 991 series) and an 8.5"×11" formula sheet (double sided) are allowed.
- For full marks, you must show all your work and explain how you arrived at your answers, unless explicitly told to do otherwise.
- **Please note:** Proctors are unable to respond to queries about the interpretation of exam questions. Do your best to answer exam questions as written.

Question 1 (6 pts).

a. Show that $\{1, t, |t|\}$ is a linearly dependent set on the interval $(0, \infty)$.

b. Show that $\{1, t, |t|\}$ is a linearly independent set on the interval $(-\infty, \infty)$.

Question 2 (10 pts)**.** Rockets propel themselves forward by burning fuel, and shooting the rapidly expanding gases and byproducts created by burning the fuel out in a controlled manner, as a result the mass of the rocket changes over time.

a. Suppose the mass of the rocket changes according to

$$\frac{dm}{dt} = r(M - m),$$

where r is the rate at which the burnt fuel is ejected. Find $m(t)$ assuming that $m(0) = 4M$.

b. What happens as the mass as $t \to \infty$? Use your answer to interpret the meaning of M in the previous equation.

c. Suppose the only force acting on the rocket is due to gravity $F = -mg$. The velocity of the rocket can be described by

$$\frac{dv}{dt}m = F - \frac{dm}{dt}v,$$

where m was found in part a.

Find an expression for the velocity of the rocket $v(t)$ at time t.

Question 3 (6 pts)**.** Consider the following initial value problem:

$$\frac{dy}{dt} = -\sqrt{y} - t \qquad\qquad y(0) = 1 \tag{1}$$

Use Euler's method with a step size of $1/4$ to approximate $y(1)$

Question 4 (10 pts)**.** Consider the differential equation

$$t\frac{d^2y}{dt^2} - (1+t^2)\frac{dy}{dt} - 6t^3y = 0 \tag{2}$$

a. Verify that $y_1(t) = e^{-t^2}$ is a fundamental solution.

b. Find a second, linearly independent fundamental solution.

Question 5 (8pts)**.** Find the general solution to the Cauchy-Euler equation

$$t^2\frac{d^2y}{dt^2}+t\frac{dy}{dt}+y=0 \tag{3}$$

for $t \in (0,\infty)$. **Hint:** You can write $t^{ai}=e^{i\ln(t^a)}$

Question 6 (8pts)**.** Find the general solution to the ordinary differential equation

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = t^3 + e^t \tag{4}$$

Question 7 (8pts)**.** Find the solution to the initial value problem

$$\frac{d}{dt}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{5}$$

Room for extra work

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