Gaussians

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The entropy of a probability distribution P over $\mathbb{R}$ is given by

$$H(X) = -\int_{\mathbb{R}} P(x) \log P(x) dx$$

If X is the standard normal distribution, then

$$\begin{aligned} H(X) &= -\int \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \log\left(\frac{1}{\sqrt{2\pi}} e^{-x^2/2}\right) dx \\ &= -\frac{1}{\sqrt{2\pi}} \int e^{-x^2/2} \left(-\frac{1}{2}\log 2\pi - \frac{x^2}{2}\right) dx \\ &= \frac{\log 2\pi}{2} + \frac{1}{2\sqrt{2\pi}} \int x^2 e^{-x^2/2} dx. \end{aligned}$$

Isolating the integral and using integration by parts, we have

$$\int x^2 e^{-x^2/2} dx = \int x(xe^{-x^2/2}) dx = -\int 1 \cdot (-e^{-x^2/2}) dx = \int e^{-x^2/2} dx = \sqrt{2\pi}.$$

Thus,

$$H(X) = \frac{1}{2}(1 + \log 2\pi)$$

Kernel Formula

For n a non-negative integer, define

$$I(n) = \int_{-\infty}^{\infty} x^n e^{-x^2/2} dx.$$

Using polar coordinates, we can obtain the popular result that

$$I(0) = \sqrt{2\pi}.$$

For n odd, the integrand is odd, so $I(n) = 0$.

Now, we can use integration by parts to obtain

$$I(2k) = \int x^{2k} e^{-x^2/2} dx = \int x^{2k-1}(xe^{-x^2/2}) dx$$

$$= -\int (2k-1)x^{2k-2}(-e^{-x^2/2}) dx = (2k-1)\int x^{2(k-1)} e^{-x^2/2} dx = (2k-1)I(2k-2).$$

By induction, we can see that

$$I(2k) = \frac{(2k)!\sqrt{2\pi}}{2^k k!}.$$