Atiyah-Macdonald Chapter 1

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Exercises

Exercise 1.

Proof. By assumption, $x^m = 0$ for some m. Then,

$$(1+x)\sum_{i=0}^{m-1}(-1)^i x^i = 1 + (-1)^{m-1}x^m = 1,$$

so $1 + x$ is a unit. $\square$

Proof. Let x be nilpotent and u be a unit. Then, $u + x = u(1 + u^{-1}x)$. But $u^{-1}x$ is nilpotent since x is. By the previous result, $1 + u^{-1}x$ is a unit and hence $u(1 + u^{-1}x) = u + x$ is also a unit. $\square$

Exercise 2. Unfinished

Proof. ($\implies$) Let $g = b_0 + b_1 x + \cdots + b_m x^m$ be the inverse of f. Then, for $k \in \{0, \ldots, n+m\}$... $\square$

Exercise 11. Unfinished (iii).

Proof. For every $x \in A$,

$$\begin{aligned} (1+x)^2 &= 1 + x \\ 1 + 2x + x^2 &= 1 + x \\ 1 + 2x + x &= 1 + x \\ 2x &= 0. \end{aligned}$$

$\square$

Proof. Suppose $x \in A$ represents the class $\bar{x} \in A/\mathfrak{p}$. Then,

$$\bar{x}^2 = (\mathfrak{p} + x)(\mathfrak{p} + x) = \mathfrak{p} + x^2 = \mathfrak{p} + x = \bar{x}.$$

Since $\mathfrak{p}$ is prime, $A/\mathfrak{p}$ is an integral domain. Hence, $\bar{x}^2 = \bar{x} \implies \bar{x}(\bar{x} - 1) = 0 \implies \bar{x} \in \{0, 1\}$. Thus, $A/\mathfrak{p}$ is the field of size 2, making $\mathfrak{p}$ a maximal ideal. $\square$

Exercise 12.

Proof. If x is a unit, $x^2 = x \implies x^2x^{-1} = xx^{-1} \implies x = 1$. If x is not a unit, then $x \in \mathfrak{m}$. That means that $1 - x$ is not in $\mathfrak{m}$ (otherwise $x + (1 - x) = 1 \in \mathfrak{m}$) and consequently that $1 - x$ is a unit. $x^2 = x \implies x(1 - x) = 0$, but since $1 - x$ is a unit, we have $x = 0$. $\square$