

# Liu's Chapter 1. Some Topics in Commutative Algebra

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## Contents

# 1 Tensor Products

## 1.1 Exercise

Let  $\{M_i\}_{i \in I}$ ,  $\{N_j\}_{j \in J}$  be two families of modules over a ring  $A$ . Show that

$$\left( \bigoplus_{i \in I} M_i \right) \otimes_A \left( \bigoplus_{j \in J} N_j \right) \cong \bigoplus_{(i,j) \in I \times J} (M_i \otimes_A N_j).$$

*Proof.* By Proposition 1.5, particularly the commutativity and distributivity of the tensor product,

$$\begin{aligned} \left( \bigoplus_{i \in I} M_i \right) \otimes_A \left( \bigoplus_{j \in J} N_j \right) &\cong \bigoplus_{i \in I} \left( M_i \otimes_A \left( \bigoplus_{j \in J} N_j \right) \right) \\ &\cong \bigoplus_{i \in I} \left( \left( \bigoplus_{j \in J} N_j \right) \otimes_A M_i \right) \\ &\cong \bigoplus_{i \in I} \left( \bigoplus_{j \in J} (N_j \otimes_A M_i) \right) \\ &\cong \bigoplus_{i \in I} \bigoplus_{j \in J} (M_i \otimes_A N_j) \\ &\cong \bigoplus_{(i,j) \in I \times J} (M_i \otimes_A N_j). \end{aligned}$$

□

## 1.2 Exercise

Show that there exists a unique  $A$ -linear map

$$f : \text{Hom}_A(M, M') \otimes_A \text{Hom}_A(N, N') \rightarrow \text{Hom}_A(M \otimes_A N, M' \otimes_A N')$$

such that  $f(u \otimes v) = u \otimes v$ .

*Proof.* Define  $\psi$  to be the obvious map

$$\psi : \text{Hom}_A(M, M') \times \text{Hom}_A(N, N') \rightarrow \{M \times N \rightarrow M' \otimes_A N'\}$$

defined by  $(u \times v)(m, n) = u(m) \otimes v(n)$ . Since  $u$  and  $v$  are  $A$ -linear, one easily verifies that  $u \times v$  is bilinear. By the universal property, the map  $u \times v$  factors through  $M \otimes_A N$ , producing a map

$$\phi : \text{Hom}_A(M, M') \times \text{Hom}_A(N, N') \rightarrow \{M \otimes_A N \rightarrow M' \otimes_A N'\}.$$

The maps in the image are easily  $A$ -linear:

$$\begin{aligned} (u \times v)(a(m \otimes n)) &= (u \times v)((am) \otimes n) \\ &= u(am) \otimes v(n) \\ &= (a(u(m))) \otimes v(n) \\ &= a(u(m) \otimes v(n)) \\ &= a(u \times v)(m \otimes n). \end{aligned}$$

Hence,

$$\phi : \text{Hom}_A(M, M') \times \text{Hom}_A(N, N') \rightarrow \text{Hom}_A(M \otimes_A N, M' \otimes_A N').$$

Now, we must show the bilinearity of  $\phi$ . By symmetry, we will verify only the key properties in the first variable. We have

$$\phi(au, v)(m \otimes n) = (au)(m) \otimes v(n) = (a(u(m))) \otimes v(n) = a(u(m) \otimes v(n)) = a\phi(u, v)(m \otimes n)$$

so that  $\phi(au, v) = a\phi(u, v)$ . Similarly,

$$\begin{aligned} \phi(u_1 + u_2, v)(m \otimes n) &= (u_1 + u_2)(m) \otimes v(n) = (u_1(m) + u_2(m)) \otimes v(n) \\ &= u_1(m) \otimes v(n) + u_2(m) \otimes v(n) = \phi(u_1, v)(m, n) + \phi(u_2, v)(m, n). \end{aligned}$$

so that  $\phi(u_1 + u_2, v) = \phi(u_1, v) + \phi(u_2, v)$ . Since  $\phi$  is bilinear, we invoke the universal property to obtain a unique,  $A$ -linear map

$$f : \text{Hom}_A(M, M') \otimes_A \text{Hom}_A(N, N') \rightarrow \text{Hom}_A(M \otimes_A N, M' \otimes_A N')$$

in which  $f(u \otimes v) = u \otimes v$ . □

### 1.3 Exercise

Let  $M, N$  be  $A$ -modules, and  $i : M' \rightarrow M$ ,  $j : N' \rightarrow N$  submodules of  $M$  and  $N$ , respectively. Then there exists a canonical isomorphism

$$(M/M') \otimes_A (N/N') \cong (M \otimes_A N)/(\text{Im } i_N + \text{Im } j_M).$$

Show that  $(\mathbb{Z}/n\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/m\mathbb{Z}) = \mathbb{Z}/l\mathbb{Z}$ , where  $l = \gcd(m, n)$ .

*Proof.* We'll start with the natural map:

$$\phi : M \times N \rightarrow (M \otimes_A N)/(\text{Im } i_N + \text{Im } j_M),$$

which is clearly bilinear. Note that if  $m - m' \in M'$ , then for any  $n \in N$ ,

$$\phi(m - m', n) = (m - m') \otimes n + (\text{Im } i_N + \text{Im } j_M) = 0 + (\text{Im } i_N + \text{Im } j_M)$$

By bilinearity,  $\phi(m, n) = \phi(m', n)$ . The same argument can be applied to  $N$ . Therefore, we have a well-defined map

$$\phi : (M/M') \times (N/N') \rightarrow (M \otimes_A N)/(\text{Im } i_N + \text{Im } j_M).$$

Let  $f : (M/M') \times (N/N') \rightarrow L$  be a bilinear map into an  $A$ -module  $L$ . We can pull  $f$  back to  $g : M \times N \rightarrow L$  by setting  $g(m, n) = f(m + M', n + N')$ ; this is clearly also bilinear. By the universal property, there is a unique linear map  $h : M \otimes_A N \rightarrow L$  so that  $h(m \otimes n) = g(m, n)$ . If  $m' \in M'$  and  $n \in N$ , then  $h(m' \otimes n) = f(m' + M', n + N') = f(0, n + N') = 0$ , so  $\text{Im } i_N \subset \ker h$ . By the same argument,  $\text{Im } j_M$  is also in the kernel, so by linearity,  $h$  vanishes on  $\text{Im } i_N + \text{Im } j_M$ . Thus,  $h$  projects to a well-defined map  $\tilde{f}$  on  $(M \otimes_A N)/(\text{Im } i_N + \text{Im } j_M)$  such that  $f = \tilde{f} \circ \phi$ . Invoking the universal property,  $\phi$  defines an isomorphism

$$\phi : (M/M') \otimes_A (N/N') \cong (M \otimes_A N)/(\text{Im } i_N + \text{Im } j_M).$$

Now,  $m\mathbb{Z}$  and  $n\mathbb{Z}$  are two  $\mathbb{Z}$ -submodules of  $\mathbb{Z}$ . By our result, if  $l = \gcd(m, n)$ ,

$$\begin{aligned} (\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) &\cong (\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}) / ((m\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z} + \mathbb{Z} \otimes_{\mathbb{Z}} (n\mathbb{Z})) \\ &\cong (\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}) / ((m\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z} + (n\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}) \\ &\cong (\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}) / ((m\mathbb{Z} + n\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}) \\ &\cong (\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}) / ((l\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}) \\ &\cong \mathbb{Z}/(l\mathbb{Z}). \end{aligned}$$

At the end, we use the isomorphism  $x \otimes y \mapsto xy$  for  $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \rightarrow \mathbb{Z}$  so that the image of the divisor is  $l\mathbb{Z}$ . □

## 1.4 Exercise

Let  $M, N$  be  $A$ -modules, and let  $B, C$  be  $A$ -algebras.

- (a) If  $M$  and  $N$  are finitely generated over  $A$ , then so is  $M \otimes_A N$ .
- (b) If  $B$  and  $C$  are finitely generated over  $A$ , then so is  $B \otimes_A C$ .
- (c) Taking  $A = \mathbb{Z}$ ,  $M = B = \mathbb{Z}/2\mathbb{Z}$ , and  $N = C = \mathbb{Q}$ , show that the converse of (a) and (b) is false.

*Proof.* (a) By assumption, we can choose a finite generating set  $\{e_j\}_{j \in J}$  of  $M$  and  $\{f_k\}_{k \in K}$  of  $N$  over  $A$ . Every element  $t \in M \otimes_A N$  can be written as finite sum  $t = \sum_i m_i \otimes n_i$  for  $m_i \in M$ ,  $n_i \in N$ . Then,

$$t = \sum_i \left( \sum_j m_{ij} e_j \right) \otimes \left( \sum_k n_{ik} f_k \right) = \sum_i \sum_{(j,k) \in J \times K} m_{ij} n_{ik} e_j \otimes f_k = \sum_{(j,k) \in J \times K} \left( \sum_i m_{ij} n_{ik} \right) e_j \otimes f_k.$$

Hence,  $\{e_j \otimes f_k\}_{(j,k) \in J \times K}$  is a finite, generating set of  $M \otimes_A N$ , as required.

(b) By assumption, we can choose a finite generating set  $\{X_j\}_{j \in J}$  of  $B$  and  $\{Y_k\}_{k \in K}$  of  $C$  over  $A$ . We want to show that  $S = \{X_j \otimes 1\} \cup \{1 \otimes Y_k\}$  is a generating set of  $B \otimes_A C$ . Now, every element  $t \in B \otimes_A C$  can be written  $t = \sum_i b_i \otimes c_i$  for  $b_i \in B$  and  $c_i \in C$ . Hence,

$$\begin{aligned} t &= \sum_i b_i \otimes c_i \\ &= \sum_i \left( \sum_u b_{iu} \prod_j X_j^{u_j} \right) \otimes \left( \sum_v c_{iv} \prod_k Y_k^{v_k} \right) \\ &= \sum_i \sum_{u,v} b_{iu} c_{iv} \left( \prod_j X_j^{u_j} \right) \otimes \left( \prod_k Y_k^{v_k} \right) \\ &= \sum_i \sum_{u,v} b_{iu} c_{iv} \prod_j (X_j \otimes 1)^{u_j} \cdot \prod_k (1 \otimes Y_k)^{v_k}, \end{aligned}$$

proving  $S$  is a finite-generating set of  $B \otimes_A C$  over  $A$ .

(c) First, we will show that  $\mathbb{Q}$  is not finitely-generated as a  $\mathbb{Z}$ -algebra, and thus, neither as a  $\mathbb{Z}$ -module. Suppose we had a finite set of generators  $S = \{p_i/q_i\} \subset \mathbb{Q}$ . Let  $q = \text{lcm}\{q_i\}$  and let  $\ell$  be a prime not dividing  $q$ . Now, suppose  $\ell^{-1} \in \mathbb{Q}$  can be generated by  $S$ . Then,

$$\frac{1}{\ell} = \sum_j z_j \prod_i \left( \frac{p_i}{q_i} \right)^{r_{ij}},$$

where  $z_j, p_i, q_i, r_{ij} \in \mathbb{Z}$  and  $r_{ij} \geq 0$ . Let  $r = \max\{r_{ij}\}$  and  $n$  be the cardinality of  $S$ . Then,

$$\frac{q^{nr}}{\ell} = q^{nr} \sum_j z_j \prod_i \left( \frac{p_i}{q_i} \right)^{r_{ij}} = \sum_j z_j \prod_i q^{r-r_{ij}} \left( p_i \frac{q}{q_i} \right)^{r_{ij}} \in \mathbb{Z}$$

since  $q_i$  divides  $q$  and  $r - r_{ij} \geq 0$ . However, since  $q$  does not contain the prime  $\ell$ , neither does  $q^{nr}$ , leading us to the desired contradiction.

Let's now turn to  $(\mathbb{Z}/2\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$ . We can represent an arbitrary element as  $\sum_i a_i \otimes q_i$ , where  $a_i \in \mathbb{Z}/2\mathbb{Z}$  and  $q_i \in \mathbb{Q}$ . We have

$$\sum_i a_i \otimes q_i = \sum_i a_i \otimes (2 \cdot q_i / 2) = \sum_i (2 \cdot a_i) \otimes (q_i / 2) = \sum_i 0 \otimes (q_i / 2) = 0,$$

so  $(\mathbb{Z}/2\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$  is the zero ring, which is obviously finitely-generated as a  $\mathbb{Z}$ -module and as a  $\mathbb{Z}$ -algebra. Therefore, the converses to (a) and (b) are false.  $\square$

## 1.5 Exercise - In Progress

Let  $\rho : A \rightarrow B$  be a ring homomorphism,  $M$  an  $A$ -module, and  $N$  a  $B$ -module. Show that there exists a canonical isomorphism of  $A$ -modules

$$\mathrm{Hom}_A(M, \rho_* N) \cong \mathrm{Hom}_B(\rho^* M, N).$$

## 1.6 Exercise - In Progress

Let  $(N_i)_{i \in I}$  be a direct system of  $A$ -modules. Then for any  $A$ -module  $M$ , there exists a canonical isomorphism

$$\varinjlim(N_i \otimes_A M) \cong (\varinjlim N_i) \otimes_A M.$$

(Hint: show that  $\varinjlim(N_i \otimes_A M)$  verifies the universal property of the tensor product  $(\varinjlim N_i) \otimes_A M$ .)

## 1.7 Exercise

Let  $B$  be an  $A$ -algebra, and let  $M, N$  be  $B$ -modules. Show that there exists a canonical surjective homomorphism

$$M \otimes_A N \rightarrow M \otimes_B N.$$

*Proof.* Consider the natural map

$$\psi : M \times N \rightarrow M \otimes_B N,$$

which is clearly surjective and  $B$ -bilinear. Since  $B$  is an  $A$ -algebra, there is a ring homomorphism  $\rho : A \rightarrow B$ . Then,  $\psi$  is also  $A$ -bilinear by defining  $a \cdot m = \rho(a) \cdot m$  for  $m \in M$ ; it's the same construction for  $N$ . By the universal property,  $\psi$  factors through a map

$$\phi : M \otimes_A N \rightarrow M \otimes_B N.$$

This is an  $A$ -module homomorphism, as it is  $A$ -linear.  $\phi$  is also surjective since  $\psi$  factors through  $\phi$ .  $\square$

## 2 Flatness

## 3 Formal Completion