Liu's Chapter 1. Some Topics in Commutative Algebra

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Contents

1 Tensor Products

1.1 Exercise

Let $\{M_i\}_{i\in I}$, $\{N_j\}_{j\in J}$ be two families of modules over a ring A. Show that

$$\left(\bigoplus_{i\in I} M_i\right) \otimes_A \left(\bigoplus_{j\in J} N_j\right) \cong \bigoplus_{(i,j)\in I\times J} (M_i \otimes_A N_j).$$

Proof. By Proposition 1.5, particularly the commutativity and distributivity of the tensor product,

$$\begin{aligned}
\left(\bigoplus_{i\in I} M_i\right) \otimes_A \left(\bigoplus_{j\in J} N_j\right) &\cong \bigoplus_{i\in I} \left(M_i \otimes_A \left(\bigoplus_{j\in J} N_j\right)\right) \\
&\cong \bigoplus_{i\in I} \left(\left(\bigoplus_{j\in J} N_j\right) \otimes_A M_i\right) \\
&\cong \bigoplus_{i\in I} \left(\bigoplus_{j\in J} (N_j \otimes_A M_i)\right) \\
&\cong \bigoplus_{i\in I} \bigoplus_{j\in J} (M_i \otimes_A N_j) \\
&\cong \bigoplus_{(i,j)\in I\times J} (M_i \otimes_A N_j) .
\end{aligned}$$

□

1.2 Exercise

Show that there exists a unique A-linear map

$$f : \mathrm{Hom}_A(M, M') \otimes_A \mathrm{Hom}_A(N, N') \to \mathrm{Hom}_A(M \otimes_A N, M' \otimes_A N')$$

such that $f(u \otimes v) = u \otimes v$.

Proof. Define ψ to be the obvious map

$$\psi : \mathrm{Hom}_A(M, M') \times \mathrm{Hom}_A(N, N') \to \{M \times N \to M' \otimes_A N'\}$$

defined by $(u \times v)(m, n) = u(m) \otimes v(n)$. Since u and v are A-linear, one easily verifies that $u \times v$ is bilinear. By the universal property, the map $u \times v$ factors through $M \otimes_A N$, producing a map

$$\phi : \mathrm{Hom}_A(M, M') \times \mathrm{Hom}_A(N, N') \to \{M \otimes_A N \to M' \otimes_A N'\}.$$

The maps in the image are easily A-linear:

$$\begin{aligned}
(u \times v)(a(m \otimes n)) &= (u \times v)((am) \otimes n) \\
&= u(am) \otimes v(n) \\
&= (a(u(m))) \otimes v(n) \\
&= a(u(m) \otimes v(n)) \\
&= a(u \times v)(m \otimes n).
\end{aligned}$$

Hence,

$$\phi : \mathrm{Hom}_A(M, M') \times \mathrm{Hom}_A(N, N') \to \mathrm{Hom}_A(M \otimes_A N, M' \otimes_A N').$$

Now, we must show the bilinearity of ϕ. By symmetry, we will verify only the key properties in the first variable. We have

$$\phi(au, v)(m \otimes n) = (au)(m) \otimes v(n) = (a(u(m))) \otimes v(n) = a(u(m) \otimes v(n)) = a\phi(u, v)(m \otimes n)$$

so that $\phi(au, v) = a\phi(u, v)$. Similarly,

$$\phi(u_1 + u_2, v)(m \otimes n) = (u_1 + u_2)(m) \otimes v(n) = (u_1(m) + u_2(m)) \otimes v(n)$$

$$= u_1(m) \otimes v(n) + u_2(m) \otimes v(n) = \phi(u_1, v)(m, n) + \phi(u_2, v)(m, n).$$

so that $\phi(u_1 + u_2, v) = \phi(u_1, v) + \phi(u_2, v)$. Since ϕ is bilinear, we invoke the universal property to obtain a unique, A-linear map

$$f : \mathrm{Hom}_A(M, M') \otimes_A \mathrm{Hom}_A(N, N') \to \mathrm{Hom}_A(M \otimes_A N, M' \otimes_A N')$$

in which $f(u \otimes v) = u \otimes v$. $\square$

1.3 Exercise

Let M, N be A-modules, and $i : M' \to M$, $j : N' \to N$ submodules of M and N, respectively. Then there exists a canonical isomorphism

$$(M/M') \otimes_A (N/N') \cong (M \otimes_A N)/(\mathrm{Im}\, i_N + \mathrm{Im}\, j_M).$$

Show that $(\mathbb{Z}/n\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/m\mathbb{Z}) = \mathbb{Z}/l\mathbb{Z}$, where $l = \gcd(m, n)$.

Proof. We'll start with the natural map:

$$\phi : M \times N \to (M \otimes_A N)/(\operatorname{Im} i_N + \operatorname{Im} j_M),$$

which is clearly bilinear. Note that if $m - m' \in M'$, then for any $n \in N$,

$$\phi(m - m', n) = (m - m') \otimes n + (\operatorname{Im} i_N + \operatorname{Im} j_M) = 0 + (\operatorname{Im} i_N + \operatorname{Im} j_M)$$

By bilinearity, $\phi(m, n) = \phi(m', n)$. The same argument can be applied to N. Therefore, we have a well-defined map

$$\phi : (M/M') \times (N/N') \to (M \otimes_A N)/(\operatorname{Im} i_N + \operatorname{Im} j_M).$$

Let $f : (M/M') \times (N/N') \to L$ be a bilinear map into an A-module L. We can pull f back to $g : M \times N \to L$ by setting $g(m, n) = f(m + M', n + N')$; this is clearly also bilinear. By the universal property, there is a unique linear map $h : M \otimes_A N \to L$ so that $h(m \otimes n) = g(m, n)$. If $m' \in M'$ and $n \in N$, then $h(m' \otimes n) = f(m' + M', n + N') = f(0, n + N') = 0$, so $\operatorname{Im} i_N \subset \ker h$. By the same argument, $\operatorname{Im} j_M$ is also in the kernel, so by linearity, h vanishes on $\operatorname{Im} i_N + \operatorname{Im} j_M$. Thus, h projects to a well-defined map $\tilde{f}$ on $(M \otimes_A N)/(\operatorname{Im} i_N + \operatorname{Im} j_M)$ such that $f = \tilde{f} \circ \phi$. Invoking the universal property, ϕ defines an isomorphism

$$\phi : (M/M') \otimes_A (N/N') \cong (M \otimes_A N)/(\operatorname{Im} i_N + \operatorname{Im} j_M).$$

Now, $m\mathbb{Z}$ and $n\mathbb{Z}$ are two $\mathbb{Z}$-submodules of $\mathbb{Z}$. By our result, if $l = \gcd(m, n)$,

$$\begin{aligned}
(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) &\cong (\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z})/((m\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z} + \mathbb{Z} \otimes_{\mathbb{Z}} (n\mathbb{Z})) \\
&\cong (\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z})/((m\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z} + (n\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}) \\
&\cong (\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z})/((m\mathbb{Z} + n\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}) \\
&\cong (\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z})/((l\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}) \\
&\cong \mathbb{Z}/(l\mathbb{Z}).
\end{aligned}$$

At the end, we use the isomorphism $x \otimes y \mapsto xy$ for $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \to \mathbb{Z}$ so that the image of the divisor is $l\mathbb{Z}$. □

1.4 Exercise

Let M, N be A-modules, and let B, C be A-algebras.

(a) If M and N are finitely generated over A, then so is $M \otimes_A N$.

(b) If B and C are finitely generated over A, then so is $B \otimes_A C$.

(c) Taking $A = \mathbb{Z}$, $M = B = \mathbb{Z}/2\mathbb{Z}$, and $N = C = \mathbb{Q}$, show that the converse of (a) and (b) is false.

Proof. (a) By assumption, we can choose a finite generating set $\{e_j\}_{j \in J}$ of M and $\{f_k\}_{k \in K}$ of N over A. Every element $t \in M \otimes_A N$ can be written as finite sum $t = \sum_i m_i \otimes n_i$ for $m_i \in M$, $n_i \in N$. Then,

$$t = \sum_i \left(\sum_j m_{ij} e_j \right) \otimes \left(\sum_k n_{ik} f_k \right) = \sum_i \sum_{(j,k) \in J \times K} m_{ij} n_{ik} e_j \otimes f_k = \sum_{(j,k) \in J \times K} \left(\sum_i m_{ij} n_{ik} \right) e_j \otimes f_k.$$

Hence, $\{e_j \otimes f_k\}_{(j,k) \in J \times K}$ is a finite, generating set of $M \otimes_A N$, as required.

(b) By assumption, we can choose a finite generating set $\{X_j\}_{j \in J}$ of B and $\{Y_k\}_{k \in K}$ of C over A. We want to show that $S = \{X_j \otimes 1\} \cup \{1 \otimes Y_k\}$ is a generating set of $B \otimes_A C$. Now, every element $t \in B \otimes_A C$

can be written $t = \sum_i b_i \otimes c_i$ for $b_i \in B$ and $c_i \in C$. Hence,

$$\begin{aligned}
t &= \sum_i b_i \otimes c_i \\
&= \sum_i \left(\sum_u b_{iu} \prod_j X_j^{u_j} \right) \otimes \left(\sum_v c_{iv} \prod_k Y_k^{v_k} \right) \\
&= \sum_i \sum_{u,v} b_{iu} c_{iv} \left(\prod_j X_j^{u_j} \right) \otimes \left(\prod_k Y_k^{v_k} \right) \\
&= \sum_i \sum_{u,v} b_{iu} c_{iv} \prod_j (X_j \otimes 1)^{u_j} \cdot \prod_k (1 \otimes Y_k)^{v_k},
\end{aligned}$$

proving S is a finite-generating set of $B \otimes_A C$ over A.

(c) First, we will show that $\mathbb{Q}$ is not finitely-generated as $\mathbb{Z}$-algebra, and thus, neither as a $\mathbb{Z}$-module. Suppose we had a finite set of generators $S = \{p_i/q_i\} \subset \mathbb{Q}$. Let $q = \mathrm{lcm}\{q_i\}$ and let ℓ be a prime not dividing q. Now, suppose $\ell^{-1} \in \mathbb{Q}$ can be generated by S. Then,

$$\frac{1}{\ell} = \sum_j z_j \prod_i \left(\frac{p_i}{q_i} \right)^{r_{ij}},$$

where $z_j, p_i, q_i, r_{ij} \in \mathbb{Z}$ and $r_{ij} \geq 0$. Let $r = \max\{r_{ij}\}$ and n be the cardinality of S. Then,

$$\frac{q^{nr}}{\ell} = q^{nr} \sum_j z_j \prod_i \left(\frac{p_i}{q_i} \right)^{r_{ij}} = \sum_j z_j \prod_i q^{r - r_{ij}} \left(p_i \frac{q}{q_i} \right)^{r_{ij}} \in \mathbb{Z}$$

since q_i divides q and $r - r_{ij} \geq 0$. However, since q does not contain the prime ℓ, neither does q^{nr}, leading us to the desired contradiction.

Let's now turn to $(\mathbb{Z}/2\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$. We can represent an arbitrary element as $\sum_i a_i \otimes q_i$, where $a_i \in \mathbb{Z}/2\mathbb{Z}$ and $q_i \in \mathbb{Q}$. We have

$$\sum_i a_i \otimes q_i = \sum_i a_i \otimes (2 \cdot q_i/2) = \sum_i (2 \cdot a_i) \otimes (q_i/2) = \sum_i 0 \otimes (q_i/2) = 0,$$

so $(\mathbb{Z}/2\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$ is the zero ring, which is obviously finitely-generated as a $\mathbb{Z}$-module and as a $\mathbb{Z}$-algebra. Therefore, the converses to (a) and (b) are false. $\square$

1.5 Exercise - In Progress

Let $\rho : A \to B$ be a ring homomorphism, M an A-module, and N a B-module. Show that there exists a canonical isomorphism of A-modules

$$\mathrm{Hom}_A(M, \rho_* N) \cong \mathrm{Hom}_B(\rho^* M, N).$$

1.6 Exercise - In Progress

Let $(N_i)_{i \in I}$ be a direct system of A-modules. Then for any A-module M, there exists a canonical isomorphism

$$\varinjlim (N_i \otimes_A M) \cong (\varinjlim N_i) \otimes_A M.$$

(Hint: show that $\varinjlim (N_i \otimes_A M)$ verifies the universal property of the tensor product $(\varinjlim N_i) \otimes_A M$.)

1.7 Exercise

Let B be an A-algebra, and let M, N be B-modules. Show that there exists a canonical surjective homomorphism

$$M \otimes_A N \to M \otimes_B N.$$

Proof. Consider the natural map

$$\psi : M \times N \to M \otimes_B N,$$

which is clearly surjective and B-bilinear. Since B is an A-algebra, there is a ring homomorphism $\rho : A \to B$. Then, ψ is also A-bilinear by defining $a \cdot m = \rho(a) \cdot m$ for $m \in M$; it's the same construction for N. By the universal property, ψ factors through a map

$$\phi : M \otimes_A N \to M \otimes_B N.$$

This is an A-module homomorphism, as it is A-linear. ϕ is also surjective since ψ factors through ϕ. □

2 Flatness

3 Formal Completion