Liu's Chapter 1. Some Topics in Commutative Algebra

Tyler Feemster

Last Updated: **January 11, 2026**

1.1. Tensor Products

Exercise 1.1.

Let $\{M_i\}_{i\in I}$, $\{N_j\}_{j\in J}$ be two families of modules over a ring A. Show that

$$\left(\bigoplus_{i\in I} M_i\right) \otimes_A \left(\bigoplus_{j\in J} N_j\right) \cong \bigoplus_{(i,j)\in I\times J} (M_i \otimes_A N_j).$$

Proof. By Proposition 1.5, particularly the commutativity and distributivity of the tensor product,

$$\begin{aligned}
\left(\bigoplus_{i\in I} M_i\right) \otimes_A \left(\bigoplus_{j\in J} N_j\right) &\cong \bigoplus_{i\in I}\left(M_i \otimes_A \left(\bigoplus_{j\in J} N_j\right)\right) \\
&\cong \bigoplus_{i\in I}\left(\left(\bigoplus_{j\in J} N_j\right) \otimes_A M_i\right) \\
&\cong \bigoplus_{i\in I}\left(\bigoplus_{j\in J} (N_j \otimes_A M_i)\right) \\
&\cong \bigoplus_{i\in I}\bigoplus_{j\in J} (M_i \otimes_A N_j) \\
&\cong \bigoplus_{(i,j)\in I\times J} \left(M_i \otimes_A N_j\right).
\end{aligned}$$

□

Exercise 1.2.

Show that there exists a unique A-linear map

$$f : \mathrm{Hom}_A(M, M') \otimes_A \mathrm{Hom}_A(N, N') \to \mathrm{Hom}_A(M \otimes_A N, M' \otimes_A N')$$

such that $f(u \otimes v) = u \otimes v$.

Proof. Define ψ to be the obvious map

$$\psi : \mathrm{Hom}_A(M, M') \times \mathrm{Hom}_A(N, N') \to \left\{M \times N \to M' \otimes_A N'\right\}$$

defined by $(u \times v)(m, n) = u(m) \otimes v(n)$. Since u and v are A-linear, one easily verifies that $u \times v$ is bilinear. By the universal property, the map $u \times v$ factors through $M \otimes_A N$, producing a map

$$\phi : \mathrm{Hom}_A(M, M') \times \mathrm{Hom}_A(N, N') \to \left\{M \otimes_A N \to M' \otimes_A N'\right\}.$$

The maps in the image are easily A-linear:

$$\begin{aligned}
(u \times v)(a(m \otimes n)) &= (u \times v)((am) \otimes n) \\
&= u(am) \otimes v(n) \\
&= (a(u(m))) \otimes v(n) \\
&= a(u(m) \otimes v(n)) \\
&= a(u \times v)(m \otimes n).
\end{aligned}$$

Hence,

$$\phi : \operatorname{Hom}_A(M, M') \times \operatorname{Hom}_A(N, N') \to \operatorname{Hom}_A(M \otimes_A N, M' \otimes_A N').$$

Now, we must show the bilinearity of ϕ. By symmetry, we will verify only the key properties in the first variable. We have

$$\phi(au, v)(m \otimes n) = (au)(m) \otimes v(n) = (a(u(m))) \otimes v(n) = a(u(m) \otimes v(n)) = a\phi(u, v)(m \otimes n)$$

so that $\phi(au, v) = a\phi(u, v)$. Similarly,

$$\phi(u_1 + u_2, v)(m \otimes n) = (u_1 + u_2)(m) \otimes v(n) = (u_1(m) + u_2(m)) \otimes v(n)$$

$$= u_1(m) \otimes v(n) + u_2(m) \otimes v(n) = \phi(u_1, v)(m, n) + \phi(u_2, v)(m, n).$$

so that $\phi(u_1 + u_2, v) = \phi(u_1, v) + \phi(u_2, v)$. Since ϕ is bilinear, we invoke the universal property to obtain a unique, A-linear map

$$f : \operatorname{Hom}_A(M, M') \otimes_A \operatorname{Hom}_A(N, N') \to \operatorname{Hom}_A(M \otimes_A N, M' \otimes_A N')$$

in which $f(u \otimes v) = u \otimes v$. □

Exercise 1.3.

Let M, N be A-modules, and $i : M' \to M$, $j : N' \to N$ submodules of M and N, respectively. Then there exists a canonical isomorphism

$$(M/M') \otimes_A (N/N') \cong (M \otimes_A N)/(\operatorname{Im} i_N + \operatorname{Im} j_M).$$

Show that $(\mathbb{Z}/n\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/m\mathbb{Z}) = \mathbb{Z}/l\mathbb{Z}$, where $l = \gcd(m, n)$.

Proof. □

Exercise 1.4.

Let M, N be A-modules, and let B, C be A-algebras.

(a) If M and N are finitely generated over A, then so is $M \otimes_A N$.

(b) If B and C are finitely generated over A, then so is $B \otimes_A C$.

(c) Taking $A = \mathbb{Z}$, $M = B = \mathbb{Z}/2\mathbb{Z}$, and $N = C = \mathbb{Q}$, show that the converse of (a) and (b) is false.

Exercise 1.5.

Let $\rho : A \to B$ be a ring homomorphism, M an A-module, and N a B-module. Show that there exists a canonical isomorphism of A-modules

$$\operatorname{Hom}_A(M, \rho_* N) \cong \operatorname{Hom}_B(\rho^* M, N).$$

Exercise 1.6.

Let $(N_i)_{i \in I}$ be a direct system of A-modules. Then for any A-module M, there exists a canonical isomorphism

$$\varinjlim (N_i \otimes_A M) \cong (\varinjlim N_i) \otimes_A M.$$

(Hint: show that $\varinjlim (N_i \otimes_A M)$ verifies the universal property of the tensor product $(\varinjlim N_i) \otimes_A M$.)

Exercise 1.7.

Let B be an A-algebra, and let M, N be B-modules. Show that there exists a canonical surjective homomorphism

$$M \otimes_A N \to M \otimes_B N.$$