Surface Classification

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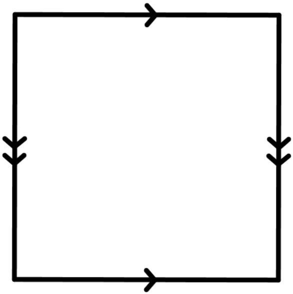
Overview

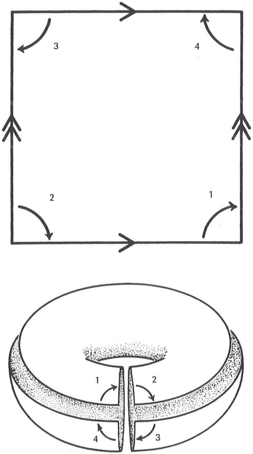
1. Plane Models
2. Orientability
3. Homogeneous Geometry
4. Euler Characteristic
5. Gauss-Bonnet Formula
6. Surface Classification Theorem

Plane Model of a Torus



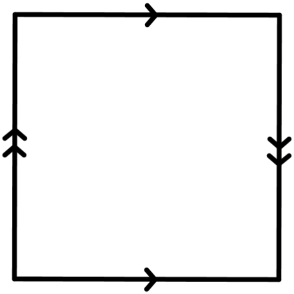
Model of a torus with edges identified.

Plane Model of a Torus



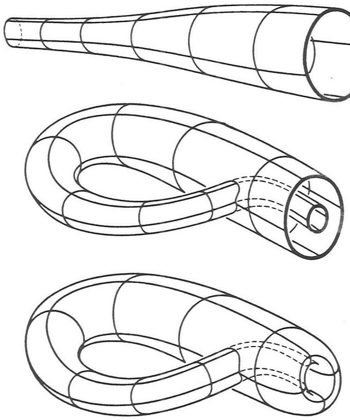
Identifying the edges of a torus through the third dimension.

Plane Model of a Klein Bottle



Model of a Klein Bottle with edges identified.

Plane Model of a Klein Bottle

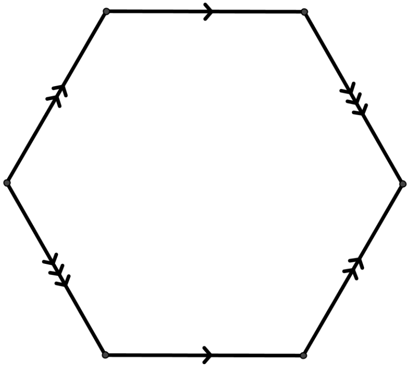


Identifying the edges of a Klein Bottle through the third dimension.

2n-gons

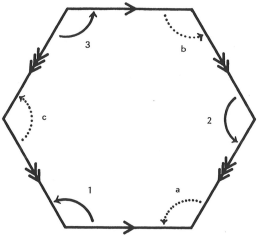
What about other 2n-gons?

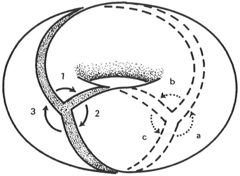
2n-gons



Will a hexagon work?

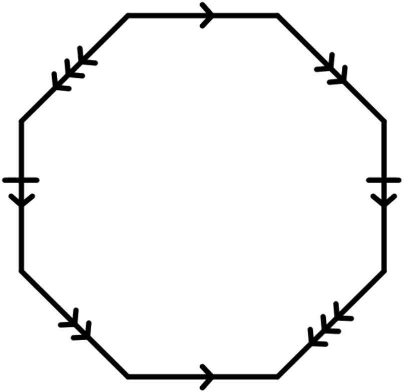
2n-gons





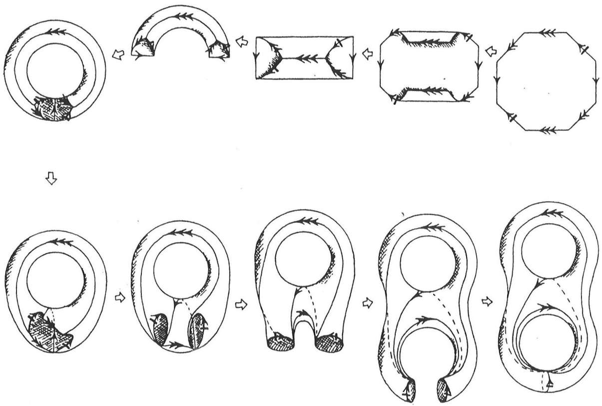
Identifying the edges of a hexagonal torus through the third dimension.

2n-gons



Will an octagon work?

2n-gons



We get a 2-holed torus!

Orientability

What about different identifications?

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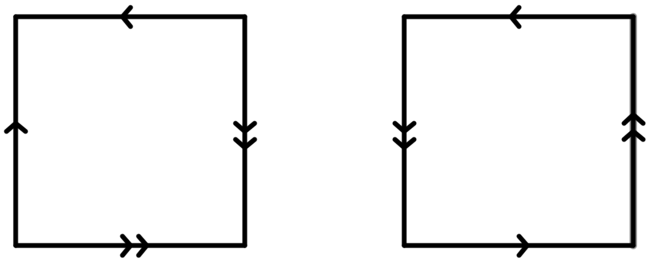
Orientability

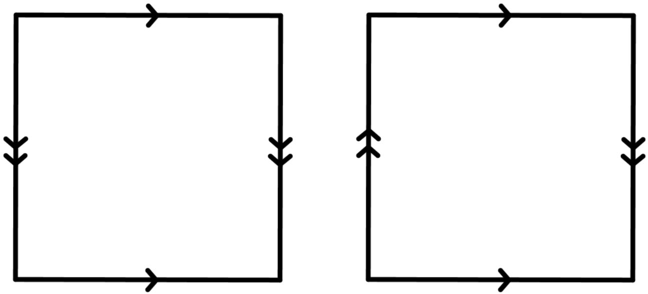


Do these make manifolds? How are they different?

Determaning Orientability

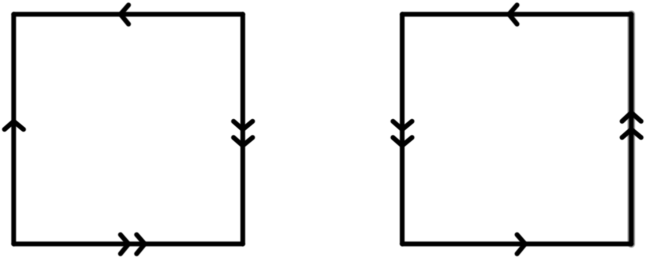
How can we determine orientability based on a plane model?

Determaning Orientability



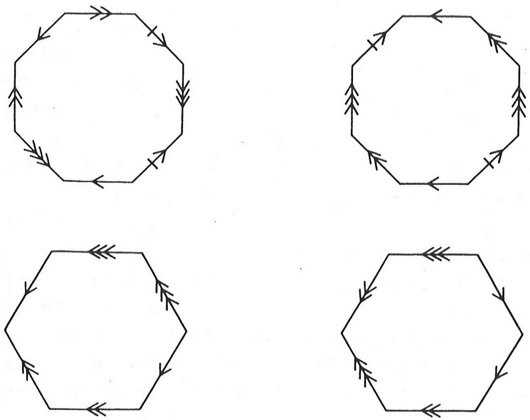
Let's look at a torus and a Klein Bottle.

Determaning Orientability



Are these orientable or not?

Determaning Orientability



What about hexagons or octagons with the edges identified? Are these orientable or not?

We can classify based on orientability. How else can we classify a manifold?

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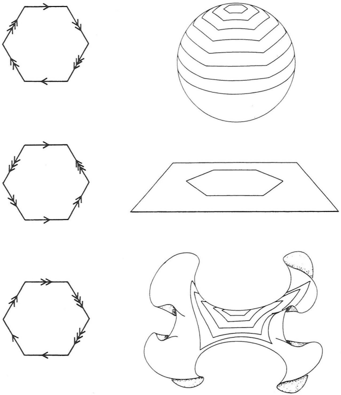
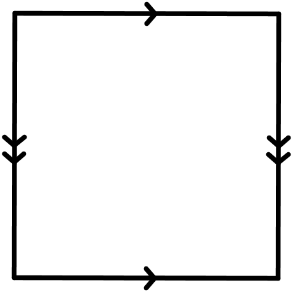
Homogeneous Geometry



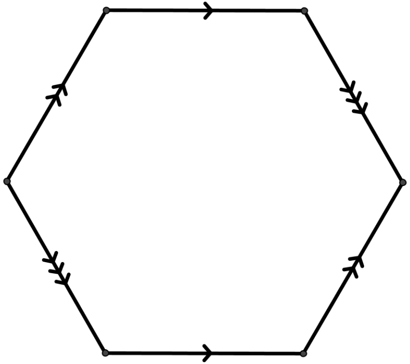
We can classify based on the geometry the surface admits.

Homogeneous Geometry



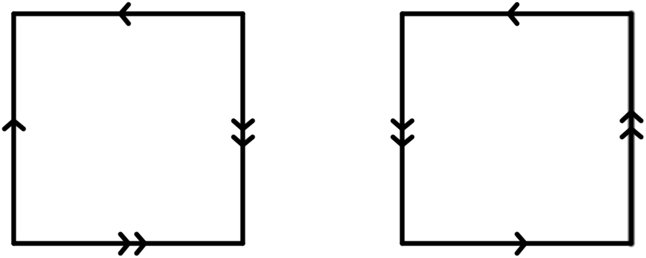
What geometry does a torus have?

Homogeneous Geometry



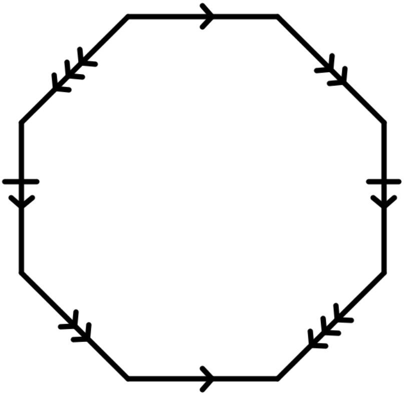
Does this work with a hexagon model of a torus?

Homogeneous Geometry



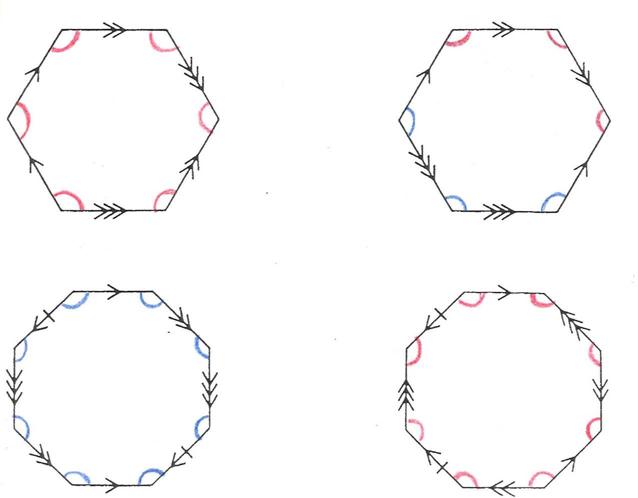
We already know the orientability of these manifolds, what about the geometry they admit?

Homogeneous Geometry



What geometry does a 2-holed torus emit?

Homogeneous Geometry



Let's check the geometry of these surfaces.

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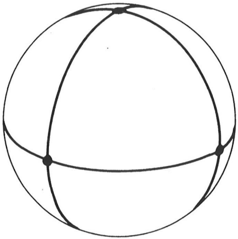
Euler Characteristic

Given a cell division, the Euler Characteristic of a surface is

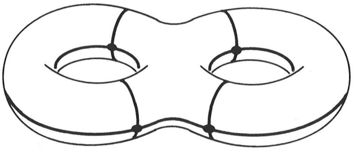
$$\chi = v - e + f$$

where v is the number of vertices, e is the number of edges, and f is the number of faces.

Euler Characteristic



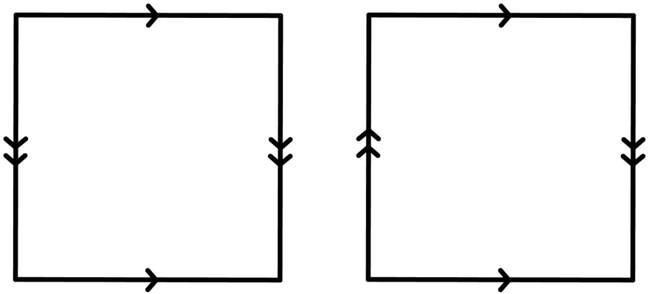
(a) a cell-division of S^2



(b) a cell-division of $T^2 \# T^2$

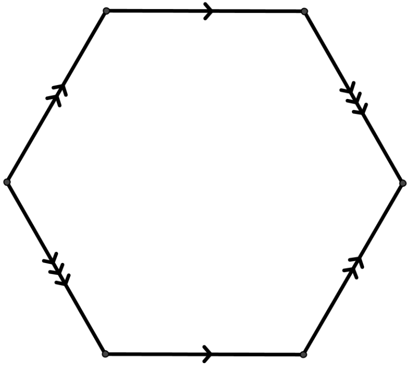
Let's talk about cell divisions.

Euler Characteristic



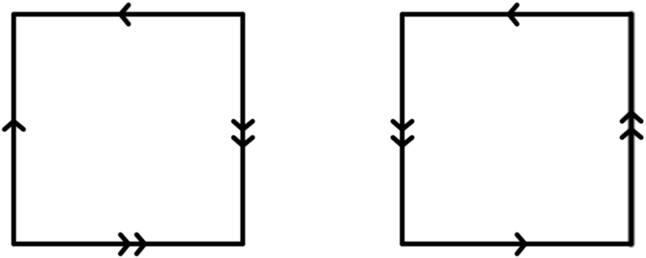
A cell division of a torus and a Klein Bottle.

Euler Characteristic



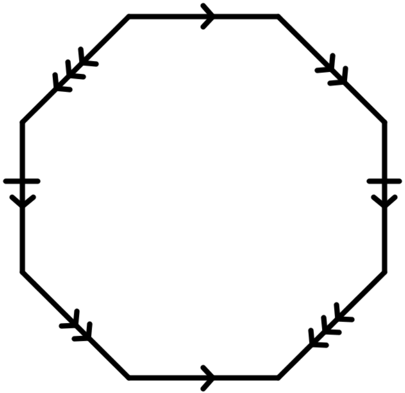
The Euler Characteristic is invariant.

Euler Characteristic



What is the Euler Characteristic of a sphere and projective plane?

Euler Characteristic



Let's check the Euler Characteristic of one more manifold.

Euler Characteristic Observations

For all χ:

- $\chi > 0$ will have elliptical geometry.
- $\chi < 0$ will have hyperbolic geometry.
- $\chi = 0$ will Euclidean geometry.

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Gauss-Bonnet Formula

Recall that elliptical geometry has positive curvature, Euclidean geometry has zero curvature, and hyperbolic geometry has negative curvature.

Area of a Triangle

Recall the area of a triangle in non-Euclidean geometries.

- Elliptical: $A = (\alpha + \beta + \gamma) - \pi$
- Hyperbolic: $A = \pi - (\alpha + \beta + \gamma)$

where α, β, and γ are the measurements of each angle of the triangle.

This can be generalized to $A = \frac{1}{k}(\alpha + \beta + \gamma - \pi)$ where k is the curvature of the triangle.

Divide a unit sphere into n triangles.

$$\begin{aligned}
A &= a_1 + a_2 + \cdots + a_n \\
&= \sum_{i=1}^{n} (\alpha_i + \beta_i + \gamma_i - \pi) \\
&= \sum_{i=1}^{n} (\alpha_i + \beta_i + \gamma_i) - n\pi \\
&= \sum_{i=1}^{n} (\alpha_i + \beta_i + \gamma_i) - 3n\pi + 2n\pi \\
&= 2\pi v - 2\pi e + 2\pi f \\
&= 2\pi(v - e + f) = 2\pi\chi
\end{aligned}$$

Gauss-Bonnet Formula

$$kA = 2\pi\chi$$

where k is curvature, A is the area, and χ is the Euler Characteristic of the manifold.

General Gauss-Bonnet Formula

$$\int k \cdot dA = 2\pi\chi$$

General Gauss-Bonnet Formula

$$
\begin{aligned}
\int k \cdot dA &= \lim_{n\to\infty} \sum_{i=1}^{n} (k_i A_i) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} (\alpha_i + \beta_i + \gamma_i - \pi) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} (\alpha_i + \beta_i + \gamma_i) - n\pi \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} (\alpha_i + \beta_i + \gamma_i) - 3n\pi + 2n\pi \\
&= \lim_{n\to\infty} (2\pi v - 2\pi e + 2\pi f) = 2\pi\chi
\end{aligned}
$$

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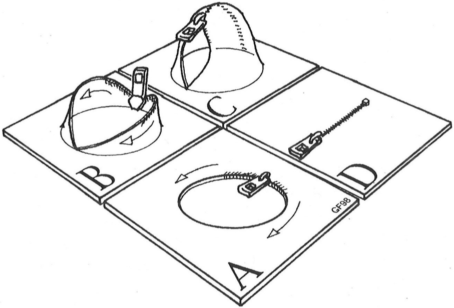
Connected Sums

Some connected sums:

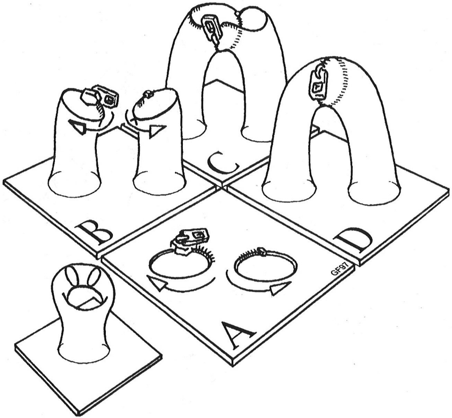
1. Sphere
2. 2 Holed Torus
3. Klein Bottle

Caps



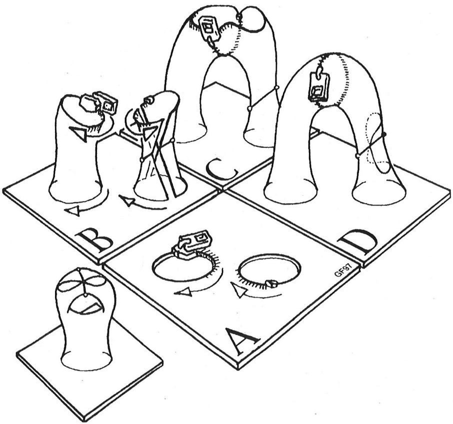
The formation of a cap.

Handles



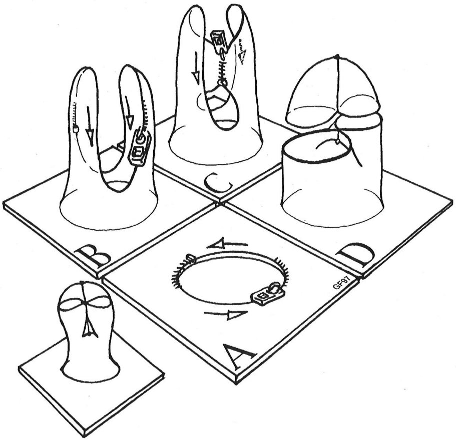
The formation of a handle.

Crosshandles



The formation of a crosshandle.

Crosscaps



The formation of a crosscap.

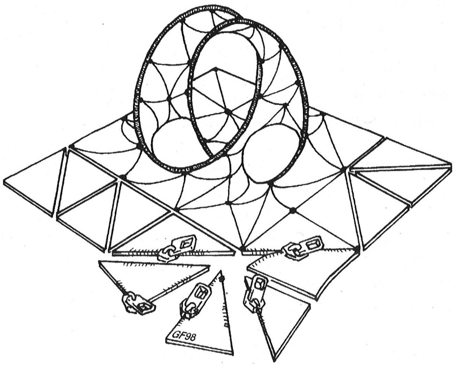
Ordinary Manifolds

A surface is ordinary if it is homeomorphic to a finite collection of spheres, each with a finite number of handles, crosshandles, crosscaps, and perferations.

Ordinary Manifolds

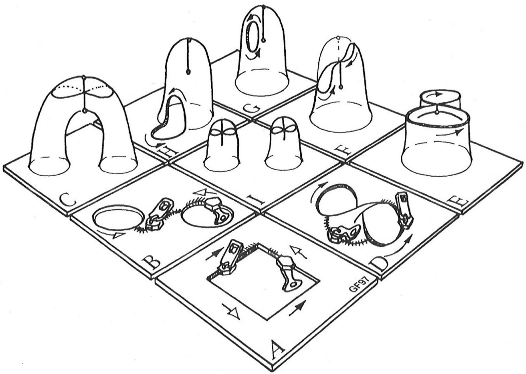
Claim: All manifolds are ordinary.

Ordinary Manifolds



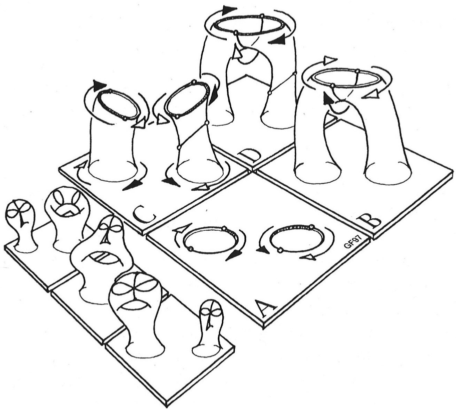
Start with an arbitrary manifold, triangulate it, and unzip all the triangle.

Crosshandles and Crosscaps



A crosshandle is equivelent to two crosscaps.

Handles and Crosshandles with a Crosscap



If there is a crosscap, a handle is equivlent to a crosshandle.

Suface Classification Theorem

Any compact and connected 2-Manifold is either a sphere or a connected sum of a finite number or tori or a finite number of projective planes.

Surface Classification Theorem Proof

Proof by Cases:

1. The manifold has no handles, crosshandles, or crosscaps.
2. The manifold has a finite number of handles only.
3. The manifold has a finite number of crosscaps only.
4. The manifold has a finite number of crosshandles only.
5. The manifold has a finite number of handles and crosscaps.

Thank you!