Trapezoidal Derivation...

$$y_{n+1} = y_n + \tfrac{1}{2}h\Big(f(t_n, y_n) + f(t_{n+1}, y_{n+1})\Big),$$

$$f(t, x) = x - \sin(t)$$

$$x_{n+1} = x_n + \frac{\Delta}{2}\Big(x_n - \sin(t_n) + x_{n+1} - \sin(t_{n+1})\Big)$$

$$x_{n+1} = x_n\left(1 + \frac{\Delta}{2}\right) + \frac{\Delta}{2}x_{n+1} - \frac{\Delta}{2}\Big(\sin(t_n) + \sin(t_{n+1})\Big)$$

$$\left(1 - \frac{\Delta}{2}\right)x_{n+1} = x_n\left(1 + \frac{\Delta}{2}\right) - \frac{\Delta}{2}\Big(\sin(t_n) + \sin(t_{n+1})\Big)$$

Taylor expansion for b_{-1}, b_0, b_1

$$F(s) = x - \sin(t)$$

expansion about t_{n+1}

$$\int F(s) \approx \int_{t_n}^{t_{n+1}} \left(F(t_{n+1}) + F'(t_{n+1})(s - t_{n+1}) + \frac{F''(t_{n+1})(s - t_{n+1})^2}{2!} + O(\Delta^3)\right) ds$$

$$= F(t_{n+1})\Delta + F'(t_{n+1})\int_{t_n}^{t_{n+1}}(s - t_{n+1})\,ds + \frac{F''(t_{n+1})}{2}\int_{t_n}^{t_{n+1}}(s - t_{n+1})^2\,ds + O(\Delta^4)$$

$$= \Delta F(t_{n+1}) - \frac{\Delta^2}{2}F'(t_{n+1}) + \frac{\Delta^3}{2 * 3}F''(t_{n+1}) + O(\Delta^4) \quad (1)$$

about t_n

$$\int F(s) \cong \int_{t_n}^{t_{n+1}} \left(F(t_n) + F'(t_n)(s - t_n) + \frac{F''(t_n)(s - t_n)^2}{2} + O(\Delta^3)\right) ds$$

$$\int^{t_{n+1}} (s - t_n)\,ds + F''(t_n)\int^{t_{n+1}} (s - t_n)^2\,ds + O(\Delta^4)$$

t_n

$$= \Delta F(t_n) + F'(t_n)\int_{t_n}^{t_{n+1}}(s-t_n)\,ds + \frac{F''(t_n)}{2}\int_{t_n}^{t_{n+1}}(s-t_n)^2\,ds + O(\Delta^4)$$

$$= \Delta F_n + \frac{\Delta^2}{2}F_n' + \frac{\Delta^3}{6}F_n'' + O(\Delta^4) \quad (2)$$

about t_{n-1}

$$\int_{t_n}^{t_{n+1}} F(s) \cong \int_{t_n}^{t_{n+1}}\left(F(t_{n-1}) + F'(t_{n-1})(s-t_{n-1}) + \frac{F''(t_{n-1})(s-t_{n-1})^2}{2}\right)ds + O(\Delta^4)$$

$$= \Delta F_{n-1} + F_{n-1}'\int_{\Delta}^{2\Delta} u\,du + \frac{F_{n-1}''}{2}\int_{\Delta}^{2\Delta} u^2\,du + O(\Delta^4)$$

$$= \Delta F_{n-1} + \frac{3\Delta^2}{2}F_{n-1}' + \frac{7\Delta^3}{6}F_{n-1}'' + O(\Delta^4)$$

So my equations are:

(1) $X_{n+1} = X_n + \Delta F_{n+1} - \frac{\Delta^2}{2}F_{n+1}' + \frac{\Delta^3}{6}F_{n+1}'' + O(\Delta^4)$

(2) $X_{n+1} = X_n + \Delta F_n + \frac{\Delta^2}{2}F_n' + \frac{\Delta^3}{6}F_n'' + O(\Delta^4)$

(3) $X_{n+1} = X_n + \Delta F_{n-1} + \frac{3\Delta^2}{2}F_{n-1}' + \frac{7\Delta^3}{6}F_{n-1}'' + O(\Delta^4)$

$$\boxed{X_{n+1}(b_{-1}+b_0+b_1) = X_n(b_{-1}+b_0+b_1) +} \longrightarrow \sum_{-1}^{1} b_k = 1 \quad \boxed{C}$$

$$\boxed{\Delta(b_{-1}F_{n+1} + b_0F_n + b_1F_{n-1}) +} \quad \text{we want this}$$

$$\boxed{\frac{\Delta^2}{2}(-b_{-1}F_{n+1}' + b_0F_n' + 3b_1F_{n-1}') + \frac{\Delta^3}{6}(b_{-1}F_{n+1}'' + b_0F_n'' + 7b_1F_{n-1}'')} + O(\Delta^4)$$

Taylor expand to write in terms of F_n and pull F_n out

$$F_{n+1}' = F_n' + \Delta F_n'' + \cancel{O(\Delta^2)} \qquad F_{n+1}'' = F_n'' + \Delta F_n''' + \cancel{O(\Delta^2)}$$

$$F_{n-1}' = F_n' - \Delta F_n'' + \cancel{O(\Delta^2)} \qquad F_{n-1}'' = F_n'' - \Delta F_n''' + \cancel{O(\Delta^2)}$$

$$-\frac{\Delta^2}{2}b_{-1}(F_n' + \Delta F_n'') + \frac{\Delta^2}{2}b_0F_n' + \frac{3\Delta^2}{2}b_1(F_n' - \Delta F_n'')$$

$$-\frac{\Delta}{2} b_{-1}(F_n + \Delta F_n) + \frac{\Delta}{2} b_0 F_n + \frac{\Delta}{2} b_1 (F_n - \Delta F_n)$$

$$= -\frac{\Delta^3}{2} b_{-1} F_n'' - \frac{3b_1 \Delta^3}{2} F_n'' + \boxed{\frac{\Delta^2}{2}\left(-b_{-1}F_n' + b_0 F_n' + 3b_1 F_n'\right)} \boxed{A}$$

ignore $\quad \frac{\Delta^3}{6}(b_{-1}F_{n+1}'' + b_0 F_n'' + 7b_1 F_{n-1}'')$ $\quad$ ignore

$$\frac{\Delta^3}{6} b_{-1}\left(F_n'' + \cancel{\Delta F_n'''}\right) + \frac{\Delta^3}{6} b_0 F_n'' + 7\frac{\Delta^3}{6} b_1 \left(F_n'' - \cancel{\Delta F_n'''}\right)$$

$$\frac{\Delta^3}{6} b_{-1} F_n'' + \frac{\Delta^3}{6} b_0 F_n'' + \frac{7\Delta^3}{6} b_1 F_n'' - \frac{\Delta^3}{2} b_{-1} F_n'' - \frac{3 b_1 \Delta^3}{2} F_n'' + \sim\sim$$

$$\boxed{\frac{\Delta^3}{6} F_n''\left(-2b_{-1} + b_0 - 2b_1\right)} \boxed{B} + \sim\sim$$

Via $\boxed{A}$ and $\boxed{B}$ and $\boxed{C}$ I can conclude

$$b_{-1} + b_0 + b_1 = 1$$
$$-b_{-1} + b_0 + 3b_1 = 0$$
$$-2b_{-1} + b_0 - 2b_1 = 0$$

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 3 \\ -2 & 1 & -2 \end{pmatrix} \vec{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\Longrightarrow \vec{b} = \left(5/12,\ 2/3,\ -1/12\right)$$

$$x_{n+1} = x_n + \Delta\left[b_{-1} f(t_{n+1}, x_{n+1}) + b_0 f(t_n, x_n) + b_1 f(t_{n-1}, x_{n-1})\right].$$

$$f(t,x) = x - \sin(t)$$

$$x_{n+1} = x_n + \Delta b_{-1}\left(x_{n+1} - \sin(t_{n+1})\right) + \Delta b_0 f_n + \Delta b_1 f_{n-1}$$

$$x_n + \Delta b_{-1} x_{n+1} - \Delta b_{-1} \sin(t_{n+1}) + \sim$$

$$x_{n+1}(1 - \Delta b_{-1}) = x_n + \Delta\left(-b_{-1}\sin(t_{n+1}) + b_0 f_n + b_1 f_{n-1}\right)$$

$$x_{n+1}\left(1 - \frac{\Delta 5}{12}\right) = x_n + \Delta\left(\frac{-5}{12}\sin(t_{n+1}) + \frac{2}{3} f_n - \frac{1}{12} f_{n-1}\right)$$

$$\boxed{X_{n+1}\left(1-\frac{\Delta 5}{12}\right) = X_n + \Delta\left(\frac{-5}{12}\sin\left(t_{n+1}\right) + \frac{2}{3} f_n - \frac{1}{12} f_{n-1}\right)}$$