HW02 Analysis

Tuesday, February 20, 2018 11:14 AM

Trapezoidal method is an implicit method in which x_{n+1} may be obtained from

$$x_{n+1} = x_n + \frac{\Delta t}{2}\left[f(t_n, x_n) + f(t_{n+1}, x_{n+1})\right].$$

Q: Give some condition on the size of the time step Δt which guarantees convergence of functional iterations (note, that this is not the same as convergence of the trapezoidal method itself).

?

Q: What is the region of A-stability of the trapezoidal method? Recall that this is the region in complex λ-plane in which $|x_{n+1}| < |x_n|$, which guarantees that the numerical approximations to the solutions of $\dot{x} = \lambda x$ decay as $t \to \infty$.

A-stable, assume $f_n = \lambda X_n$

then $$X_{n+1} = X_n + \frac{\Delta t}{2}\left(f(t_n, X_n) + f(t_{n+1}, X_{n+1})\right)$$

$$X_{n+1} = X_n + \frac{\Delta}{2}\left(f_n + f_{n+1}\right)$$

$$= X_n + \frac{\Delta}{2}\left(\lambda X_n + \lambda X_{n+1}\right)$$

$$= X_n + \frac{\Delta\lambda}{2} X_n + \frac{\Delta\lambda}{2} X_{n+1}$$

$$X_{n+1}\left(1 - \frac{\Delta\lambda}{2}\right) = X_n\left(1 + \frac{\Delta\lambda}{2}\right)$$

$$X_{n+1} \quad 2+\Delta\lambda \qquad \left|\frac{X_{n+1}}{}\right| = \left|\frac{2+\lambda\Delta}{}\right| < 1$$

$$\frac{X_{n+1}}{X_n} = \frac{2+\Delta\lambda}{2-\Delta\lambda} \longrightarrow \left|\frac{X_{n+1}}{X_n}\right| = \underbrace{\left|\frac{2+\lambda\Delta}{2-\lambda\Delta}\right| < 1}_{\text{A-stable}}$$

$$\longrightarrow |2+\lambda\Delta| < |2-\lambda\Delta|$$

using the fact

$$|x-n| = |n-x| \quad \boxed{\forall n \in \overset{?}{\mathbb{C}} \cup \mathbb{R}}$$

we can say

let $\lambda\Delta = z$ ...

$\boxed{\text{if } Re(z) < 0, \text{ then}}$

$|2+z| < |2-z|$ is true,

if $R(z) \geq 0$, then

$|2+z| \not< |2-z|$

so

$Re(\lambda\Delta) < 0$

$$\boxed{Re(\lambda) < 0}$$

Q: Solve the differential equation

$$\dot{x}(t) = \frac{1}{1+x}$$

with $x(0) = 1$ on the interval $[0, 16]$. First of all, find the exact solution. Then solve the equation numerically. Note, that unlike in the first assignment, equation (2) will be nonlinear: you need to solve it for x_{n+1} either analytically or numerically. How does the approximation error for $x(16)$ (i.e., at $t = 16$) depend on N? Make plots to support your findings. Deduce the order of accuracy of the trapezoidal method.

$\dot{x}$ 1

$$= \dot{x} \frac{1}{1+x}$$

$$\frac{dx}{dt}(1+x) = 1 \longrightarrow t = x + \frac{x^2}{2} + C$$

$$0 = x(0) + \frac{x(0)^2}{2} + C = 3/2 + C = 0$$

$$\boxed{C = -3/2}$$

$$\frac{x^2}{2} + x - (3/2 + t) = 0$$

$$x(t) = -1 \pm \sqrt{4+2t}$$

try to satisfy init cond

$$\boxed{x(0) = -1 + \sqrt{4} = 1}$$ ✓

$$x(0) = -1 - \sqrt{4} = -3$$ X

$$\Rightarrow \boxed{x(t) = \sqrt{4+2t} - 1}$$

OBTAIN EXPLICIT TRAP

Trapezoidal method is an implicit method in which x_{n+1} may be obtained from

$$x_{n+1} = x_n + \frac{\Delta t}{2}\left[f(t_n, x_n) + f(t_{n+1}, x_{n+1})\right].$$

let $f_n = \dfrac{1}{1+x_n}$

$$x_{n+1} = x_n + \frac{\Delta}{2}\left(f_n + f_{n+1}\right)$$

$$x_{n+1} = x_n + \frac{\Delta}{2}\left(\frac{1}{1+x_n} + \frac{1}{1+x_{n+1}}\right)$$

$$\left(x_n - x_{n+1}\right) + \frac{\Delta}{2}\left(\frac{1}{1+x_n} + \frac{1}{1+x_{n+1}}\right) = g(x_{n+1}, x_n)$$

find when $g(\overline{x}_n, x_{n+1}) = 0$ for a given $\overline{x}_n$

using scipy.optimize.fsolve.