Math475a, Fall 2017. Assignment 3. Due September 24

What if your computer didn't have a built-in function to compute square roots? Could you program a function which computes it using only basic arithmetic operations? The problem: given $X > 0$, find $x > 0$, such that

$$x^2 = X.$$

Program all three methods below and compare their convergence rates. Try $X = 0.01,\ 4,\ 1000$.

Bisection. You must first transform this problem into a problem of finding roots of some equation, $f(x) = 0$. (Don't use the square root function!) Here is one possibility: $f(x) = x^2 - X$; can you think of any other? To set up the bisection algorithm, you must first find an initial interval $[a, b]$ which contains $\sqrt{X}$. *Hint: you may use that* $X \leq \sqrt{X}$ *for* $X \leq 1$ *and* $\sqrt{X} \leq X$ *for* $X \geq 1$.

Newton's method. Show that Newton's method (with $f(x)$ as above) amounts to iterating

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{X}{x_n}\right). \tag{1}$$

What will you pick as the initial guess, x_0?

Functional Iterations. Here you need to transform the relation, $x^2 = X$ into $x = f(x)$ (again, without explicitly using the square root function). Here is one way to do this:

$$x = X/x, \tag{2}$$

Is this relation adequate to compute $\sqrt{X}$ for any X?

More generally, we can set up the functional iterations, e.g., via

$$x = (1-\alpha)x + \alpha X/x, \quad \alpha \neq 0 \quad \text{is an arbitrary number.} \tag{3}$$

Do you spot similarity with Newton's method? Is there a way to pick *the optimal* value of the parameter, α? Try $\alpha = 1/4,\ 1/2,\ 3/4$.

Comparison. Compute $\sqrt{2}$ using these three methods up to some given precision, i.e., stop the iterations when the difference between the two most recent approximations is less than 10^{-13} (or some other small number). Compare the number of iterations needed to achieve the chosen precision in these methods.

I recommend checking out the Wikipedia page on "Methods of computing square roots" — you may find lots of useful information.