MATH475b. Assignment 4. Due April 1st

In this assignment your task is to use finite difference methods to solve the heat (diffusion) equation,

$$c_t(x,t) = c_{xx}; \qquad x \in (0,1), \quad t > 0. \tag{1}$$

Assume that the initial datum is given by

$$c(x,0) = x\sin(\pi x), \tag{2}$$

Discretize the domain $[0,1]$ using $N = 100$ equal intervals of size $\Delta = 0.01$. Denote the time step by δ and let c_n^m be the numerical approximations to $c(n\Delta, m\delta)$; $n = 0,\ldots,N$, $m = 0,1,2,\ldots$

Forward vs Backward Euler methods. Consider the problem with periodic boundary: $c(0,t) = c(1,t)$; $c_x(0,t) = c_x(1,t)$. This is equivalent to letting n vary over $\mathbb{Z}$ and demanding that $c_{n+N}^m = c_n^m$. As we discussed in class, the forward (explicit) and backward (implicit) Euler methods may be written, respectively, as

$$\text{(FE)} \quad \boldsymbol{c}^{m+1} = (I + \delta D)\boldsymbol{c}^m; \qquad\qquad \text{(BE)} \quad \boldsymbol{c}^{m+1} = (I - \delta D)^{-1}\boldsymbol{c}^m.$$

Here the vector $\boldsymbol{c}^m$ contains all the c_n^m, $n = 0,\ldots,N-1$, and the second difference matrix D for the periodic boundary conditions is given by

$$D = \frac{1}{\Delta^2}\begin{pmatrix} -2 & 1 & 0 & \cdots & & & 1 \\ 1 & -2 & 1 & 0 & \cdots & & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ 0 & \cdots & 0 & 1 & -2 & 1 & 0 \\ 0 & & \cdots & 0 & 1 & -2 & 1 \\ 1 & & & \cdots & 0 & 1 & -2 \end{pmatrix}.$$

Propagate the initial condition up to time $t = 2$ using both methods and plot your solution (as a function of both x and t). Experiment with different time steps δ. For which values of δ does it look like the backward Euler method has converged in the "eye-ball norm," (i.e., when the plots seem the same)? What about the forward Euler method? Explain your findings.

Symmetric vs non-symmetric second difference operator. Now consider the Neumann boundary, $c_x(0,t) = c_x(1,t) = 0$. As we discussed in class, there are various ways to implement these boundary conditions. In particular, we have symmetric and non-symmetric versions of the second difference operator (note that in this case n should vary from 0 to N rather than to $N-1$ as above):

$$D_{\text{sym}} = \frac{1}{\Delta^2}\begin{pmatrix} -1 & 1 & 0 & \cdots & & & 0 \\ 1 & -2 & 1 & 0 & \cdots & & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ 0 & \cdots & 0 & 1 & -2 & 1 & 0 \\ 0 & & \cdots & 0 & 1 & -2 & 1 \\ 0 & & & \cdots & 0 & 1 & -1 \end{pmatrix}; \quad D_{\text{non}} = \frac{1}{\Delta^2}\begin{pmatrix} -2 & 2 & 0 & \cdots & & & 0 \\ 1 & -2 & 1 & 0 & \cdots & & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & \vdots \\ 0 & \cdots & 0 & 1 & -2 & 1 & 0 \\ 0 & & \cdots & 0 & 1 & -2 & 1 \\ 0 & & & \cdots & 0 & 2 & -2 \end{pmatrix}.$$

Use the backward Euler method with these two matrices D to solve the heat equation up to time $t = 1$. Set $\delta = \Delta = 1/N$, vary N, and study convergence of these methods (pick your own convergence criterion). Explain your findings.