MATH475b. Homework 2 Due February 22nd

In this homework your goal is to program (and experiment with) two *single-step* numerical methods for solving ODEs. Suppose we want to solve an initial value problem,

$$\dot{x}(t) = f(t,x), \quad t > 0; \qquad x(0) = x_0. \tag{1}$$

Discretize the interval $[0,T]$ into sub-intervals of size $\Delta t = T/N$; denote $t_n = n\Delta t$, $x_n \approx x(t_n)$ — is the numerical approximation of the true solution.

Trapezoidal method is an implicit method in which x_{n+1} may be obtained from

$$x_{n+1} = x_n + \frac{\Delta t}{2}\big[f(t_n,x_n) + f(t_{n+1},x_{n+1})\big]. \tag{2}$$

Observe that this is already in the form $x_{n+1} = F(x_{n+1})$, i.e., it is suitable for functional (fixed point) iterations which you may use to find x_{n+1}.

Q: Give some condition on the size of the time step Δt which guarantees convergence of functional iterations (note, that this is not the same as convergence of the trapezoidal method itself).

Q: What is the region of A-stability of the trapezoidal method? Recall that this is the region in complex λ-plane in which $|x_{n+1}| < |x_n|$, which guarantees that the numerical approximations to the solutions of $\dot{x} = \lambda x$ decay as $t \to \infty$.

Q: Solve the differential equation

$$\dot{x}(t) = \frac{1}{1+x}$$

with $x(0) = 1$ on the interval $[0,16]$. First of all, find the exact solution. Then solve the equation numerically. Note, that unlike in the first assignment, equation (2) will be nonlinear: you need to solve it for x_{n+1} either analytically or numerically. How does the approximation error for $x(16)$ (i.e., at $t = 16$) depend on N? Make plots to support your findings. Deduce the order of accuracy of the trapezoidal method.

Classical Runge-Kutta-4 method. This is the four step method with the following stages:

$$z_1 = f(t_n, x_n); \tag{3a}$$
$$z_2 = f(t_n + \Delta t/2, x_n + z_1\Delta t/2); \tag{3b}$$
$$z_3 = f(t_n + \Delta t/2, x_n + z_2\Delta t/2); \tag{3c}$$
$$z_4 = f(t_{n+1}, x_n + z_3\Delta t); \tag{3d}$$
$$x_{n+1} = x_n + \frac{\Delta t}{6}\big[z_1 + 2z_2 + 2z_3 + z_4\big]. \tag{3e}$$

Q: Solve the same ODE as in the previous problem using this (RK4) method (program it yourself). Again, study the dependence of the approximation error at $t = 16$ on N. What is the order of accuracy of this method?