MATH475b. Homework 1 Due February 1st

In this homework you will study two *multistep* methods for solving ODEs. Consider an initial value problem,

$$\dot{x}(t) = x - \sin t, \quad t \in (0, \pi); \qquad x(0) = 0. \tag{1}$$

Discretize the interval $[0, \pi]$ into sub-intervals of size $\Delta = \pi/N$; denote $t_n = n\Delta$, $x_n \approx x(t_n)$ — is the numerical approximation of the true solution.

Q: Find the exact solution, $x(t)$, analytically.

The error of the numerical approximation here will be quantified via the *max-norm*, i.e., given by

$$E = \max_{n \in \{0 \ldots N\}} |x(t_n) - x_n|. \tag{2}$$

Two step Adams-Bashforth (AB2) method prescribes,

$$x_{n+1} = x_n + \frac{\Delta}{2}\big[3f(t_n, x_n) - f(t_{n-1}, x_{n-1})\big]. \tag{3}$$

In the beginning of your calculation, you will only have the initial condition, $x_0 = 0$, thus in order to start up the AB2 method, you need to somehow obtain x_1.

Q: First try using the Forward Euler method to get x_1 (this would correspond to $x_1 = x_0 = 0$). Compute the solution over the remaining part of $[0, \pi]$ using the AB2 method and evaluate the error as N increases (plot a graph). What is the order of the method?

Q: Approximate x_1 using the trapezoidal method. (The ODE is linear, so even though the trapezoidal method is implicit, you won't need to solve any nonlinear equations here.) Plot the graph illustrating convergence of the method; what is its order now?

Q: Finally, approximate x_1 using the Forward Euler method again, but rather than using a single step of size Δ, to get x_1, split the very first sub-interval $[0, \Delta]$ into N even smaller intervals (size π/N^2) and make N steps. How does the error depend on N now?

Three step Adams-Moulton (AM3) method is given by

$$x_{n+1} = x_n + \Delta\big[b_{-1} f(t_{n+1}, x_{n+1}) + b_0 f(t_n, x_n) + b_1 f(t_{n-1}, x_{n-1})\big]. \tag{4}$$

In class we found the coefficients b_{-1}, b_0, and b_1 using Lagrange polynomials to interpolate $f(x, t)$ from the values at three consecutive grid points. Here is another methods to obtain these coefficients. Recall that the ODE, $\dot{x} = f(t, x)$, is equivalent to an integral equation,

$$x(t_{n+1}) = x(t_n) + \int_{t_n}^{t_{n+1}} F(s)\,\mathrm{d}s, \quad \text{where} \quad F(s) = f(s, x(s)). \tag{5}$$

If we Taylor expand $F(s)$ around t_{n+1}, t_n, and t_{n-1}, and integrate, we get,

$$x(t_{n+1}) = x(t_n) + \Delta F(t_{n+1}) - \frac{\Delta^2}{2} F'(t_{n+1}) + \frac{\Delta^3}{6} F''(t_{n+1}) + \mathcal{O}(\Delta^4); \tag{6a}$$

$$x(t_{n+1}) = x(t_n) + \Delta F(t_n) + \frac{\Delta^2}{2} F'(t_n) + \frac{\Delta^3}{6} F''(t_n) + \mathcal{O}(\Delta^4); \tag{6b}$$

$$x(t_{n+1}) = x(t_n) + \Delta F(t_{n-1}) + \frac{3\Delta^2}{2} F'(t_{n-1}) + \frac{7\Delta^3}{6} F''(t_{n-1}) + \mathcal{O}(\Delta^4). \tag{6c}$$

Now we add these equations with weights b_{-1}, b_0, and b_1 respectively, and match the result with the numerical method prescribed by (4). This gives us, first of all,

$$b_{-1} + b_0 + b_1 = 1.$$

Further on, we want to make sure that all additional terms of as many orders as possible cancel out. This means that in the second order (with respect to Δ), we must have,

$$-b_{-1} F'(t_{n+1}) + b_0 F'(t_n) + 3b_1 F'(t_{n-1}) = \mathcal{O}(\Delta),$$

which forces

$$-b_{-1} + b_0 + 3b_1 = 0.$$

Similarly, in the third order (with respect to Δ), we get

$$-2b_{-1} + b_0 - 2b_1 = 0.$$

We have three equations for three unknowns, thus we won't be able to cancel any higher-order terms. (You may get a different set of equations if you expand slightly differently, but the solution must be the same.)

Q: Solve equations for b_{-1}, b_0, and b_1 and program the three step Adams-Moulton method.

Q: We should expect a higher order of accuracy from this method than from the two-step method. (Otherwise, why bother, right?) Study how the error depends on the number of discretization points, N, for this method. You will also need to devise a proper way to start the method, i.e., find x_1.