MATH475b. Assignment 3 — Due March 15th

In this homework we study Fourier series and Discrete Fourier Tranform (DFT). Recall the definitions: for a function $x(t)$ on $[0,T]$, we define the Fourier coefficients,

$$\tilde{x}_k = \frac{1}{T}\int_0^T e^{-2\pi i k t/T}\, x(t)\, dt; \qquad k \in \mathbb{Z}. \tag{1}$$

The Fourier series allows us to reconstruct the function $x(t)$ from the set of its Fourier coefficients: for all points of continuity of $x(t)$, we have,

$$x(t) = \sum_{k=-\infty}^{\infty} \tilde{x}_k\, e^{2\pi i k t/T}. \tag{2}$$

The DFT is defined by

$$\hat{x}_k = \frac{1}{N}\sum_{n=0}^{N-1} e^{-2\pi i k n/N}\, x_n, \qquad \text{where} \qquad x_n = x(t_n), \quad t_n = \frac{nT}{N}. \tag{3}$$

Usually k varies from 0 to $N-1$, however $\hat{x}_k$-s are well-defined for any k and are N-periodic, i.e., $\hat{x}_k = \hat{x}_{k+N}$. The inverse DFT is given by

$$x_n = \sum_{n=0}^{N-1} e^{2\pi i k n/N}\, \hat{x}_k. \tag{4}$$

(Beware that if you use some numerical package to carry out the DFT, the definitions may slightly vary, e.g., have a different sign in the exponent, or a different normalizing factor.)

Fourier approximation of discontinuous functions. Let $T = 2\pi$. Pick some $t_0 \in (0, 2\pi)$, and consider the indicator function of the interval $[0, t_0]$, $x(t) = \mathbb{1}_{[0,t_0]}(t)$. Fourier coefficients of $x(t)$ are given by

$$\tilde{x}_k = \frac{1}{2\pi}\int_0^{t_0} e^{-ikt}\, dt = \frac{1}{2\pi i k}\left(1 - e^{-ikt_0}\right), \quad k \neq 0; \qquad \tilde{x}_0 = \frac{t_0}{2\pi}. \tag{5}$$

Construct an approximation to $x(t)$ by summing only a finite number (N) of terms in formula (2):

$$x_N(t) = \sum_{k=-N}^{N} \tilde{x}_k\, e^{2\pi i k t/T} = \frac{t_0}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{N} \frac{1}{k}\big(\sin kt - \sin k(t - t_0)\big). \tag{6}$$

Q: Verify formula (6) and study how $x_N(t)$ converges to $x(t)$ as N grows. Assign some value to t_0 to define your $x(t)$; plot a few graphs of $x_N(t)$ for several values of N next to $x(t)$. What do you observe?

Q: Study behavior of the max-norm error: $\|x - x_N\|_\infty = \max_{t\in[0,2\pi]} |x(t) - x_N(t)|$, and the L_2-error:

$$\|x - x_N\|_2 = \left(\int_0^{2\pi} |x(t) - x_N(t)|^2\, dt\right)^{1/2}.$$

Discuss your findings. *(You may carry out this part of the assignment either numerically or analytically.)*

Fourier series and discrete Fourier transform. Study how well the DFT approximates the Fourier series: use the function, $x(t)$ that you used in the first part of the assignment. Compare the *power spectrum* of the Fourier series,

$$\tilde{p}_k = |\tilde{x}_k|^2 = \frac{1 - \cos kt_0}{2\pi^2 k^2}, \quad k \neq 0; \qquad \tilde{p}_0 = \frac{t_0^2}{4\pi^2}, \tag{7}$$

with the power spectrum obtained from the DFT, $\hat{p}_k = |\hat{x}_k|^2$. In order to do this, first sample the vector of $x_n = x(nT/N)$, $n = 0 \ldots N-1$, then evaluate the DFT as given by formula (3). *(Using formula* (3) *directly may become slow when N is large. You can utilize some version of fast Fourier transform provided by your programming environment, or write your own.)*

Q: Plot the $\tilde{p}_k$-s and $\hat{p}_k$-s in the same graph for various values of N, e.g., 10, 100, 1000 (use appropriate scale and periodicity property to be able to see the difference). What do you observe? For which k-s do $\hat{p}_k$ seem to approximate $\tilde{p}_k$ well?