Kinematics Of a Two-tiered linkage robot for Lumbar Spinal Injection

Tyler Lehrfeld

February 2025

1 Introduction

This report will detail the forward and inverse calculations of the kinematics of this robot. The hope is to generalize the math behind the robot and to develop fast code in order to adjust the robot to a new position.

2 Robot design

There is a layered architecture with each layer having 2 degrees of freedom on the x-y plane. When the two layers are connected to a needle, the line formed by extending the vector from the end effector of the top layer to the end effector of the bottom layer forms a line determining the of approach for the needle to take.

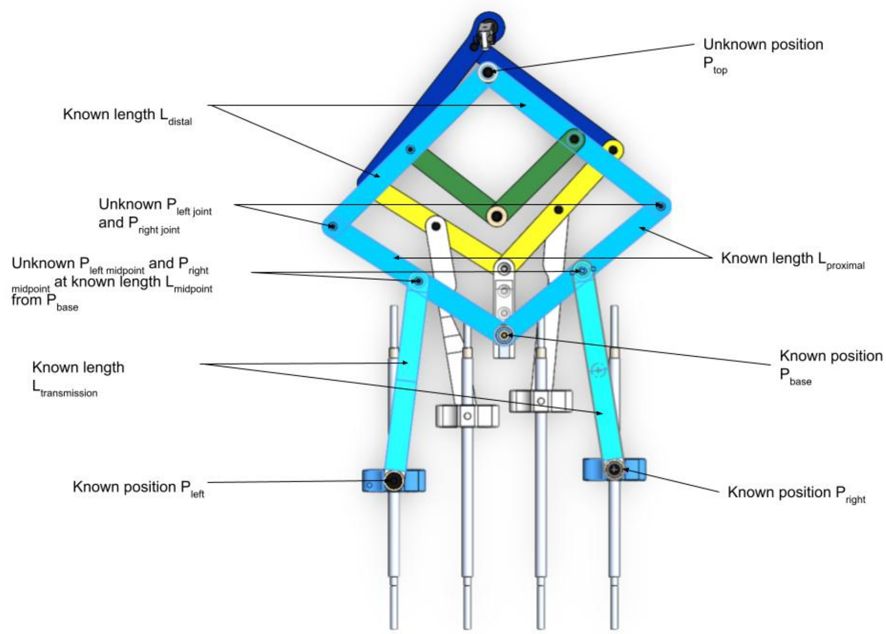

Figure 1: Linkage Labels

3 Top layer kinematics

3.1 Joint calculation

$\vec{P}_{\text{left midpoint}}$ can be determined using the known positions of $\vec{P}_{\text{base}}$ and $\vec{P}_{\text{left}}$ and the distances of $L_{\text{transmission}}$ and L_{midpoint}. We will treat these as 2D coordinates because the z of the end effector will be known beforehand (assuming sufficiently rigid construction).

We will first form a triangle such that we can use trigonometry to get our unknown variables. We have a triangle with points $\vec{P}_{\text{left}}$, $\vec{P}_{\text{base}}$, and unknown $\vec{P}_{\text{left midpoint}}$. We have length L_{midpoint}, which is the distance between $\vec{P}_{\text{base}}$ and $\vec{P}_{\text{left midpoint}}$ and we have $L_{\text{transmission}}$, which is the distance between $\vec{P}_{\text{left}}$ and $\vec{P}_{\text{left midpoint}}$ as drawn below.

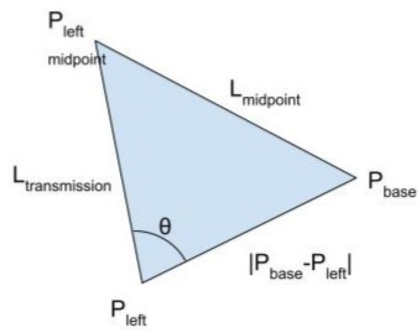

Figure 2: Slider, Midpoint, and Base Linkage Triangle

Now we can use the law of cosines to get
$L^2_{\text{midpoint}} = L^2_{\text{transmission}} + |\vec{P}_{\text{base}} - \vec{P}_{\text{left}}|^2 - 2L_{\text{transmission}}|\vec{P}_{\text{base}} - \vec{P}_{\text{left}}|cos(\theta)$
$L^2_{\text{midpoint}} - L^2_{\text{transmission}} - |\vec{P}_{\text{base}} - \vec{P}_{\text{left}}|^2 = -2L_{\text{transmission}}|\vec{P}_{\text{base}} - \vec{P}_{\text{left}}|cos(\theta)$
$\frac{L^2_{\text{midpoint}} - L^2_{\text{transmission}} - |\vec{P}_{\text{base}} - \vec{P}_{\text{left}}|^2}{-2L_{\text{transmission}}|\vec{P}_{\text{base}} - \vec{P}_{\text{left}}|} = cos(\theta)$
$\arccos\left(\frac{L^2_{\text{midpoint}} - L^2_{\text{transmission}} - |\vec{P}_{\text{base}} - \vec{P}_{\text{left}}|^2}{-2L_{\text{transmission}}|\vec{P}_{\text{base}} - \vec{P}_{\text{left}}|}\right) = \theta$
Now that we have θ, we can rotate $\frac{\vec{P}_{\text{base}} - \vec{P}_{\text{left}}}{|\vec{P}_{\text{base}} - \vec{P}_{\text{left}}|}$ counter clockwise by θ, then multiply it by $L_{\text{transmission}}$ and add it to $\vec{P}_{\text{left}}$.
$L_{\text{transmission}} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \frac{\vec{P}_{\text{base}} - \vec{P}_{\text{left}}}{|\vec{P}_{\text{base}} - \vec{P}_{\text{left}}|} + \vec{P}_{\text{left}} = \vec{P}_{\text{left midpoint}}$
$\vec{P}_{\text{left joint}} = L_{\text{proximal}} \frac{\vec{P}_{\text{left midpoint}} - \vec{P}_{\text{base}}}{|\vec{P}_{\text{left midpoint}} - \vec{P}_{\text{base}}|} + \vec{P}_{\text{base}}$
Next, we will get the right joint. The only difference between the right and left joint is that theta will be negative and we will rotate the opposite direction.
$\arccos\left(\frac{L^2_{\text{midpoint}} - L^2_{\text{transmission}} - |\vec{P}_{\text{base}} - \vec{P}_{\text{right}}|^2}{-2L_{\text{transmission}}|\vec{P}_{\text{base}} - \vec{P}_{\text{right}}|}\right) = \theta$
$L_{\text{transmission}} \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} \frac{\vec{P}_{\text{base}} - \vec{P}_{\text{right}}}{|\vec{P}_{\text{base}} - \vec{P}_{\text{right}}|} + \vec{P}_{\text{right}} = \vec{P}_{\text{right midpoint}}$
$\vec{P}_{\text{right joint}} = L_{\text{proximal}} \frac{\vec{P}_{\text{right midpoint}} - \vec{P}_{\text{base}}}{|\vec{P}_{\text{right midpoint}} - \vec{P}_{\text{base}}|} + \vec{P}_{\text{base}}$

3.2 Top End Effector

Once the left and right joint are calculated, there is an isosceles triangle formed with a base between the two joints and the two equal sides being the distal links. We get a triangle like the one pictured below.

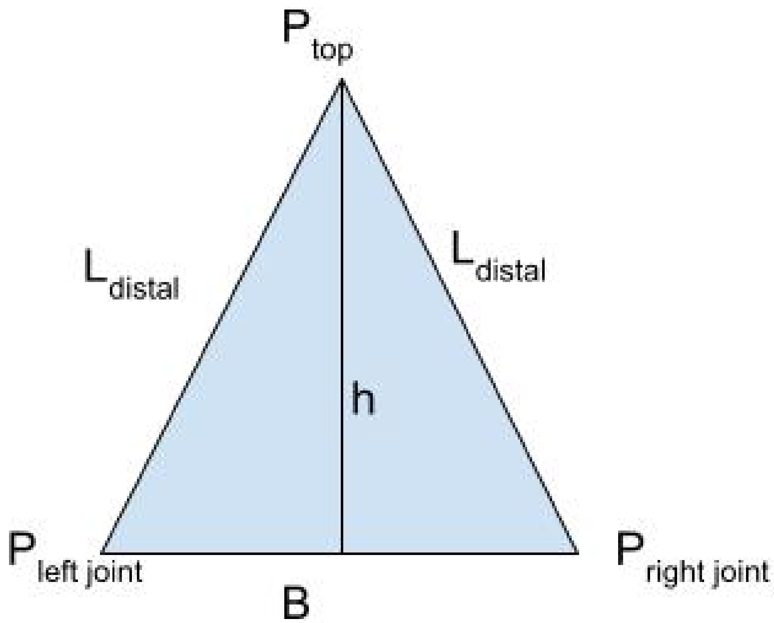

Figure 3: Isosceles Triangle Of Distal Linkage

The approach we will take to get $\vec{P}_{\text{top}}$ is to find h and add a vector perpendicular to the base of length h to the midpoint of that base.

$h = \sqrt{L_{\text{distal}}^2 - \frac{|\vec{P}_{\text{left joint}} - \vec{P}_{\text{right joint}}|^2}{4}}$

$\vec{P}_{\text{left joint}} - \vec{P}_{\text{right joint}} = \begin{bmatrix} X \\ Y \end{bmatrix}$

$\vec{P}_{\text{perpendicular}} = \begin{bmatrix} -Y \\ X \end{bmatrix}$

$\vec{P}_{\text{top}} = \frac{\vec{P}_{\text{right joint}} + \vec{P}_{\text{left joint}}}{2} + h \frac{\vec{P}_{\text{perpendicular}}}{|\vec{P}_{\text{perpendicular}}|}$

Both upper and lower linkages can be calculated the same way.

3.2.1 Extensions

The robot has different extensions for the upper and lower linkages. The upper linkage essentially extends h in order to accommodate the needle driver adapter extension that allows for rotation of the needle. We can change the equation of $\vec{P}_{\text{top}}$ to be $\vec{P'}_{\text{top}} = \frac{\vec{P}_{\text{right joint}} + \vec{P}_{\text{left joint}}}{2} + (h + L_{\text{extension}}) \frac{\vec{P}_{\text{perpendicular}}}{|\vec{P}_{\text{perpendicular}}|}$

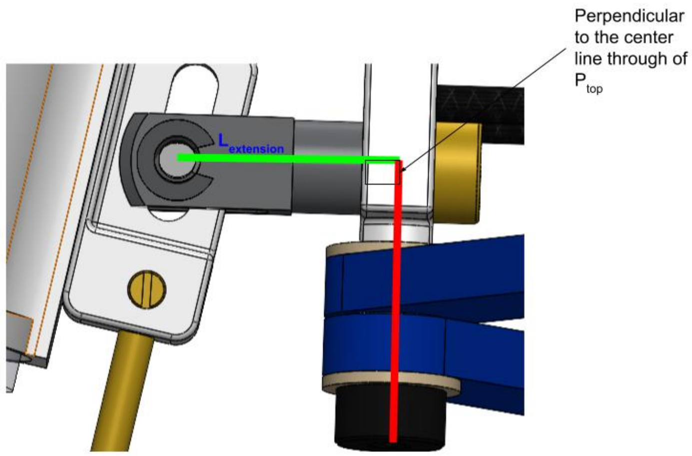

Figure 4: Upper Linkage $L_{\text{extension}}$

$L_{\text{extension}}$ is the length of the green line above extending from the center of where the links connect to make $\vec{P}_{\text{top}}$. Remember, we are disregarding Z until the end because it is irrelevant to our calculations until we have to find the z distance between where the upper and lower linkages connect to the needle holder/driver.

For the lower extension, we extend by $L_{\text{extension}}$ in the direction from $\vec{P}_{\text{left joint}}$ to $\vec{P}_{\text{top}}$. Therefore $\vec{P'}_{\text{top}} = \vec{P}_{\text{left joint}} + \frac{L_{\text{distal}}+L_{\text{extension}}}{L_{\text{distal}}}(\vec{P}_{\text{top}} - \vec{P}_{\text{left joint}})$

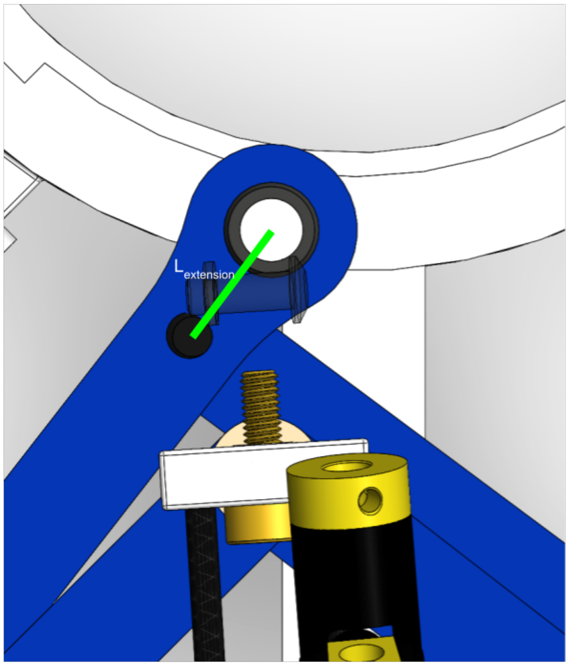

Figure 5: Lower linkage $L_{\text{extension}}$

3.3 Needle Point from upper and lower end effector positions

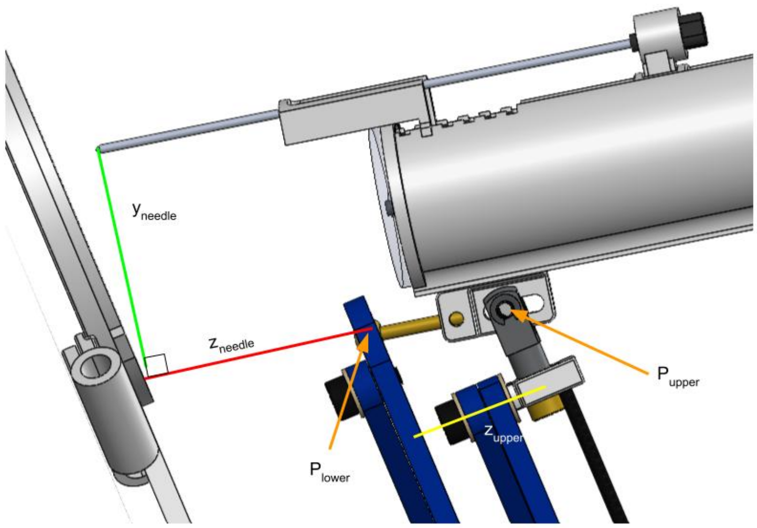

Figure 6: Needle Tip Transform

To get the needle end point, we need to find the pose of the needle with origin at $\vec{P}_{\text{lower}}$. To do this we will find the coordinate system based on the orientation of the lower end effector joy stick. First we will find the z axis, then get the x axis using the $\vec{P}_{\text{perpendicular}}$ from the top linkage, through a cross product. Then we will get x through cross a product. We can then put these into a transformation matrix and calculate the needle tip in the robot pose.

$\vec{z}_{\text{transform}} = \frac{\vec{P}_{\text{upper}} - \vec{P}_{\text{lower}}}{|\vec{P}_{\text{upper}} - \vec{P}_{\text{lower}}|}$

$\vec{x}_{\text{transform}} = \frac{\vec{z}_{\text{transform}} \times \vec{P}_{\text{perpendicular}}}{|\vec{z}_{\text{transform}} \times \vec{P}_{\text{perpendicular}}|}$

This works because the upper linkage end effector extends through the x axis towards the needle, putting it in the y-z plane. Therefore it can be crossed with z to get a vector in x.

$\vec{y}_{\text{transform}} = \vec{x}_{\text{transform}} \times \vec{z}_{\text{transform}}$

Then,

$T_{\text{needle}} = \begin{bmatrix} \vec{x}_{\text{transform}} & \vec{y}_{\text{transform}} & \vec{z}_{\text{transform}} & \vec{P}_{\text{lower}} \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$\vec{P}_{\text{needle tip}} = T_{\text{needle}} \begin{bmatrix} 0 \\ y_{\text{needle}} \\ z_{\text{needle}} \\ 1 \end{bmatrix}$$

4 Edge cases

TODO

5 Inverse Kinematics

For inverse kinematics we will tackle it first from a target and angle approach, then we will translate the problem into a target and injection point approach. The first will choose a target point and an approach angle. The second will take two points: an entry point and a target point.

5.1 Target and Angle approach

We must first recognize that it would be best to define an approach angle from the frame of the patient's anatomy rather than the robot. We want to approach from a certain angle to avoid or hit critical anatomy. This means a target point and injection point will require a frame transformation into the robot's frame. Let us assume this is given as $\mathbf{F}_{\text{RP}}$ for "robot frame from patient."
For the target and angle approach, we will first take an input of a point $\vec{P}_{\text{target}}$ and two angles θ, and ϕ. The needle tip would be at $\vec{P}_{\text{target}}$ and it would extend, and it would extend upwards at an angle ϕ from the z axis and angle θ from the robot frame's x axis.

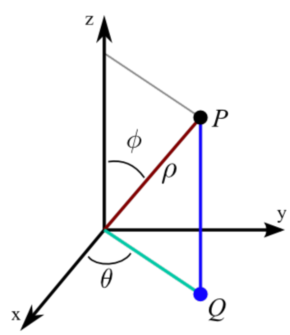

Figure 7: Inverse Kinematic Angles

We can form a parametric equation of the needle line like so:

$$\vec{P}_{\text{injection}} = \vec{P}_{\text{target}} + t \begin{bmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{bmatrix}$$

Where t is a parameter greater than 0. To translate into a target and injection point problem, we can just say $t = 1$ and treat this new point as our injection point.

5.2 Target and Injection Point approach

Say we have a target point $\vec{P}_{\text{target}}$ and an injection point $\vec{P}_{\text{injection}}$. We can multiply by $\mathbf{F}_{\text{RP}}$ to get $\vec{P}_{\text{robot target}} = \mathbf{F}_{\text{RP}}\vec{P}_{\text{target}}$ and $\vec{P}_{\text{robot injection}} = \mathbf{F}_{\text{RP}}\vec{P}_{\text{injection}}$. Now we can use these points to find the end effectors of the robot.
Now we can form a line extending from $\vec{P}_{\text{robot target}}$ to $\vec{P}_{\text{robot injection}}$ with the equation $\vec{P}_{\text{needle}} = \vec{P}_{\text{robot target}} + t(\vec{P}_{\text{robot injection}} - \vec{P}_{\text{robot target}})$. We also know that the lower end effector is a known distance x_{needle} and y_{needle} from the upper and lower end effector and an unknown distance z_{needle} (See figure 6).
We will try and reverse our equation from the previous forward kinematic section:

$$\vec{P}_{\text{robot target}} = \begin{bmatrix} x' & y' & z' & \vec{P}_{\text{lower}} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{\text{needle}} \\ y_{\text{needle}} \\ z_{\text{needle}} \\ 1 \end{bmatrix}$$

We know $z' = \frac{\vec{P}_{\text{robot injection}} - \vec{P}_{\text{robot target}}}{|\vec{P}_{\text{robot injection}} - \vec{P}_{\text{robot target}}|}$
We can get x' by getting a vector in the z-y plane and crossing it with z'. The vector from the upper base to the needle point at the same height as the end effector satisfies this condition. This is because the needle driver is held so that it cannot twist. We can get this point on the needle by solving for when t gives us a $\vec{P}_{\text{needle}}.z$ of $z_{\text{upper end effector}}$

$\vec{P}_{\text{needle}} = \vec{P}_{\text{robot target}} + t(\vec{P}_{\text{robot injection}} - \vec{P}_{\text{robot target}})$

$\vec{P}_{\text{needle}}.z = \vec{P}_{\text{robot target}}.z + t(\vec{P}_{\text{robot injection}} - \vec{P}_{\text{robot target}}).z$

Set $\vec{P}_{\text{needle}}.z$ to the z of the upper linkage's end effector as shown in figure 4: $z_{\text{upper end effector}}$

$z_{\text{upper end effector}} = \vec{P}_{\text{robot target}}.z + t(\vec{P}_{\text{robot injection}} - \vec{P}_{\text{robot target}}).z$

$\frac{z_{\text{upper end effector}} - \vec{P}_{\text{robot target}}.z}{(\vec{P}_{\text{robot injection}} - \vec{P}_{\text{robot target}}).z} = t$

$\vec{P}_{\text{needle}} = \vec{P}_{\text{robot target}} + (\frac{z_{\text{upper end effector}} - \vec{P}_{\text{robot target}}.z}{(\vec{P}_{\text{robot injection}} - \vec{P}_{\text{robot target}}).z})(\vec{P}_{\text{robot injection}} - \vec{P}_{\text{robot target}})$

Assuming that $\vec{P}_{\text{upper base}}$ is on the same z level as the end effector (which we have been assuming for simplicity the whole time):

$x' = \frac{\vec{P}_{\text{needle}} - \vec{P}_{\text{upper base}}}{|\vec{P}_{\text{needle}} - \vec{P}_{\text{upper base}}|} \times z'$

Then,

$y' = z' \times x'$

Our only unknown left is z_{needle}

$\vec{0} = \vec{P}_{\text{robot target}} + z_{\text{needle}} \frac{\vec{P}_{\text{robot injection}} - \vec{P}_{\text{robot target}}}{|\vec{P}_{\text{robot injection}} - \vec{P}_{\text{robot target}}|} + y_{\text{needle}} * y'$

$0 = \vec{P}_{\text{robot target}}.z + z_{\text{needle}}(\frac{\vec{P}_{\text{robot injection}} - \vec{P}_{\text{robot target}}}{|\vec{P}_{\text{robot injection}} - \vec{P}_{\text{robot target}}|}).z + y_{\text{needle}} * y'.z$

$\frac{-\vec{P}_{\text{robot target}}.z - y_{\text{needle}} * y'.z}{(\frac{\vec{P}_{\text{robot injection}} - \vec{P}_{\text{robot target}}}{|\vec{P}_{\text{robot injection}} - \vec{P}_{\text{robot target}}|}).z} = z_{\text{needle}}$

Now we have all of our variables to solve the equation for our lower end effector:

$$\vec{P}_{\text{lower}} = \vec{P}_{\text{robot target}} - \begin{bmatrix} x' & y' & z' \end{bmatrix} \begin{bmatrix} x_{\text{needle}} \\ y_{\text{needle}} \\ z_{\text{needle}} \end{bmatrix}$$

We now can get the needle extension distance required by subtracting z_{needle_0} (the default z needle distance) from z_{needle}:

needle extension $= z_{\text{needle}} - z_{\text{needle}_0}$

We need to solve for $\vec{P}_{\text{upper}}$. To do this we need to find a t where $\vec{P}_{\text{lower}} + t * z' = \vec{P}_{\text{upper}}$. We also know $z_{\text{upper end effector}}$ so we get the equation $z_{\text{upper end effector}} = \vec{P}_{\text{lower}}.z + t * z'.z$

$t = \frac{z_{\text{upper end effector}} - \vec{P}_{\text{lower}}.z}{z'.z}$

$\vec{P}_{\text{upper}} = \vec{P}_{\text{lower}} + \frac{z_{\text{upper end effector}} - \vec{P}_{\text{lower}}.z}{z'.z} z'$

5.2.1 Reversing the Upper Extension

To get the end effector minus the upper extension, we have to subtract the length of $L_{\text{extension}}$ from the end effector towards the base.

$\vec{P'}_{\text{upper end effector}} = (|\vec{P}_{\text{upper}} - \vec{P}_{\text{upper base}}| - L_{\text{extension}}) \frac{\vec{P}_{\text{upper}} - \vec{P}_{\text{upper base}}}{|\vec{P}_{\text{upper}} - \vec{P}_{\text{upper base}}|}$

5.2.2 Reversing the Lower Extension

To reverse the lower extension, we have to find the left joint and extend it in the direction of the lower end effector the length of the distal joint. To get the left joint we make a circle of length $L_{\text{extension}} + L_{\text{distal}}$ around our original $\vec{P}_{\text{lower}}$ and find the intersection of the circle of length L_{proximal} around $\vec{P}_{\text{lower base}}$. I use this algorithm: Intersection of two circles to find that intersection.

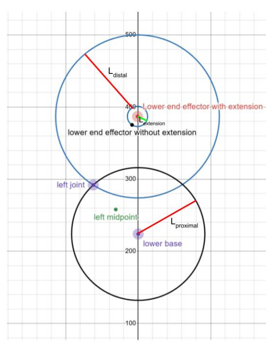

Figure 8: Using the intersection of two circles to find the left joint

After we get the left joint, we know that $\vec{P'}_{\text{lower end effector}} = L_{\text{distal}} \frac{\vec{P}_{\text{lower}} - \vec{P}_{\text{left joint}}}{|\vec{P}_{\text{lower}} - \vec{P}_{\text{left joint}}|}$

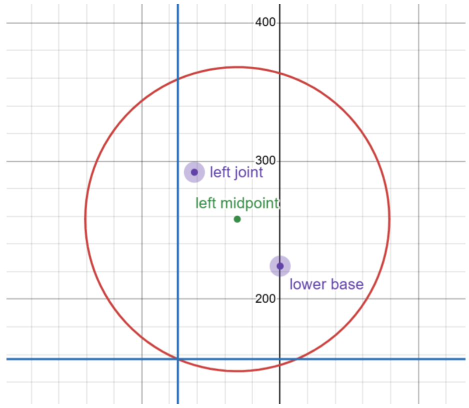

Figure 9: Slider positions on a circle around the midpoint of radius $L_{\text{transmission}}$

5.3 Getting sliders from symmetrical links

After we get rid of the extensions, we can treat each linkage similarly. To get the left and right joints $\vec{P}_{\text{left joint}}$ and $\vec{P}_{\text{right joint}}$, we take the intersection of the circle around the $\vec{P}_{\text{end effector}}$ of radius L_{distal} and the circle around $\vec{P}_{\text{base}}$ of radius L_{proximal}. See figure 8 for what this might look like. Getting the midpoint is just a reversal of the forward kinematic process.

$\vec{P}_{\text{left midpoint}} = \vec{P}_{\text{base}} + L_{\text{midpoint}} \frac{\vec{P}_{\text{left joint}} - \vec{P}_{\text{base}}}{|\vec{P}_{\text{left joint}} - \vec{P}_{\text{base}}|}$

$\vec{P}_{\text{right midpoint}} = \vec{P}_{\text{base}} + L_{\text{midpoint}} \frac{\vec{P}_{\text{right joint}} - \vec{P}_{\text{base}}}{|\vec{P}_{\text{right joint}} - \vec{P}_{\text{base}}|}$

Now let us introduce two variables: $x_{\text{left slider}}$ and $x_{\text{right slider}}$. We are solving for $y_{\text{left slider}}$ and $y_{\text{right slider}}$. (We are at this point ignoring z as if all were on the same plane. We only care about these y values to control the robot). We can create a circle of radius $L_{\text{transmission}}$ around the left joint. We can see where the left joint circle intersects with the line $x = x_{\text{left slider}}$. Writing out the equation of a circle we get:

$(x - \vec{P}_{\text{midpoint}}.x)^2 + (y - \vec{P}_{\text{midpoint}}.y)^2 = L^2_{\text{transmission}}$

To get the y of slider positions as a function of $x_{\text{left slider}}$, we can use this equation:

$$y_{\text{left slider}} = -\sqrt{\left(L^2_{\text{transmission}} - \left(x_{\text{left slider}} - \vec{P}_{\text{left midpoint}}.x\right)^2\right)} + \vec{P}_{\text{left midpoint}}.y$$

Similarly,

$$y_{\text{right slider}} = -\sqrt{\left(L^2_{\text{transmission}} - \left(x_{\text{right slider}} - \vec{P}_{\text{right midpoint}}.x\right)^2\right)} + \vec{P}_{\text{right midpoint}}.y$$

6 Kinematic Error Analysis