COSC242 - Assignment 1

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1 Proof by Induction

Show that $2^n = O(n!)$, i.e. show that

$$\exists C, n_0 : |2^n| \leq C\,|n!|\,\forall n > n_0$$

First, show that it is true for an atom, in this case $n = 4$.

$$\left|2^4\right| \leq C\,|4!| \Rightarrow |8| \leq C\,|24|\,,$$

which is true for (say) $C = 1$.
Therefore, $n_0 = 4$ has the required property.
Assuming that this is true for $k \geq n_0$, show that this is true for $k + 1$.
Assume $2^k \leq C \cdot k!$ for $k \geq n_0$. Then

$$\frac{2^k}{k!} \leq C.$$

Taking $n = k + 1$, we can re-write $\frac{2^{k+1}}{(k+1)!}$as

$$\frac{2 \cdot 2^k}{(k+1)\,k!} = \frac{2}{k+1} \cdot \frac{2^k}{k!}$$

Since $\frac{2^k}{k!} \leq C$ by assumption,

$$\frac{2}{k+1} \cdot \frac{2^k}{k!} \leq C,$$

for all $k \geq 1$. Therefore,

$$\left|2^{k+1}\right| \leq C\,|(k+1)!|$$

and by induction,

$$|2^n| \leq C\,|n!|\,\forall n > n_0.$$

Therefore,

$$2^n = O(n!)\,.$$

2 Proof by Contradiction

Show that $2^n \neq O(n!)$
Assume

$$n! \leq C \cdot 2^n$$

for all $n \geq n_0$ and some real number C.
Let P be the set of all C_i, such that

$$n! \leq C_i \cdot 2^n$$

for i in some index set $I \subseteq \mathbb{R}$.
P is a partially ordered set, and by the well-ordering principle, has a least element, C_0.
Take $n \geq 3$. We can write

$$\frac{n!}{2^n} \leq C_0$$

for all $n \geq n_0$, and also

$$\frac{(n+1)!}{2^{n+1}} \leq C_0$$

by the assumption. But

$$\frac{(n+1)!}{2^{n+1}} = \frac{(n+1)n!}{2 \cdot 2^n} = \frac{n!}{2^n} \cdot \frac{n+1}{2} \leq C_0 \Rightarrow \frac{n!}{2} \leq \frac{2}{n+1} \cdot C_0.$$

Since $n \geq 3$,

$$\frac{2}{n+1} \cdot C_0 < C_0.$$

However, C_0 was defined as the smallest possible number such that $\frac{n!}{2^n} \leq C_0$, a contradiction.
Therefore the assumption is false, and $2^n \neq O(n!)$ is proven true by contradiction.

3 Proof by Iteration

Solve the recurrence equations by iteration:

$$g(1) = 10$$

$$g(n) = 2g(n-1)$$

This implies that

$$g(n) = 2g(n-1) = 2 \cdot 2g(n-2) = 2 \cdot 2 \cdot 2g(n-3) = ... = 2^k \cdot g(n-k).$$

Let $n = k$, then

$$g(n) = 2^k \cdot g(n-k) \Rightarrow g(n) = 2^n \cdot g\left(0\right).$$

$$g(0) = g(1-1) = 2^{-1} \cdot g(1) = \frac{1}{2} \cdot 10 = 5$$

Therefore

$$g(n) = 5 \cdot 2^n.$$